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The mixed Ax-Lindemann theorem and its applications to the Zilber-Pink conjecture

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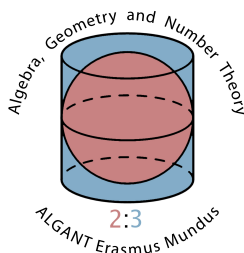
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Le théorème d'Ax-Lindemann mixte et ses applications à la conjecture de Zilber-Pink

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