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Author: Brau Avila, Julio
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Chapter 2

Entanglement correction factors as character sums

2.1 Introduction

The motivation for this chapter comes from the classical conjecture of Artin from 1927 which predicts the density of primes p for which a given rational number is a primitive root modulo p. More precisely, let g be an integer different from ± 1, and let h be the largest integer such that $g = g_0^h$ with $g_0 \in \mathbb{Z}$. The heuristic reasoning described by Artin was the following. If p is a prime number coprime to g, then g is a primitive root modulo p if and only if there is no prime ℓ dividing $p-1$ such that $g \equiv y^\ell \pmod{p}$ for some y. Note that this congruence condition can be given as a splitting condition on the prime p in the field $F_\ell := \mathbb{Q}(\zeta_\ell, \sqrt[\ell]{g})$. Indeed, the condition on p is equivalent to p not splitting completely in the aforementioned field. In other words, g is a primitive root modulo p if and only if for every prime $\ell < p$ we have that Frob_p is not the identity element in $\mathrm{Gal}(F_\ell/\mathbb{Q})$.

For a fixed ℓ, the density of primes which do not split completely in F_ℓ

is equal to

$$\delta_\ell := 1 - \frac{1}{[F_\ell : \mathbb{Q}]},$$

and this equals $1 - \frac{1}{\ell-1}$ for $\ell \mid h$ and $1 - \frac{1}{\ell(\ell-1)}$ otherwise. If we assume the splitting conditions in the various fields F_ℓ to be independent, then it is reasonable to expect that the density of primes p for which g is a primitive root modulo p is equal to $\prod_\ell \delta_\ell$. This was the density originally conjectured by Artin, however years later (see [Ste03]) he noticed that this assumption of independence is not correct, as the fields F_ℓ can have non-trivial intersections. If $F_2 = \mathbb{Q}(\sqrt{g})$ has discriminant $D \equiv 1 \pmod 4$, then F_2 is contained in the compositum of the fields F_ℓ with $\ell \mid D$. The corrected version of the conjecture was proven by Hooley under the assumption of the Generalized Riemann Hypothesis (GRH). He showed in [Hoo67] that, conditional on GRH, the density of primes such that g is a primitive root modulo p equals

$$C_g = \sum_{n=1}^{\infty} \frac{\mu(n)}{[F_n : \mathbb{Q}]} \tag{2.1.1}$$

where $F_n = \mathbb{Q}(\zeta_n, \sqrt[n]{g})$ and μ is the Möbius function. In the same paper Hooley shows that (2.1.1) can be rewritten as

$$C_g = \mathfrak{C}_g \prod_{\ell \mid h} \Big(1 - \frac{1}{\ell - 1}\Big) \prod_{\ell \nmid h} \Big(1 - \frac{1}{\ell(\ell-1)}\Big), \tag{2.1.2}$$

where $\mathfrak{C}_g$ is an *entanglement correction factor*, a rational number which depends on g. In fact it is given explicitly by

$$\mathfrak{C}_g := 1 - \prod_{\substack{\ell \mid D \\ \ell \mid h}} \frac{-1}{\ell - 2} \cdot \prod_{\substack{\ell \mid D \\ \ell \nmid h}} \frac{-1}{\ell^2 - \ell - 1}.$$

One advantage of having C_g in the form given by (2.1.2) is that it makes it easy to see when the density C_g vanishes. Vanishing of C_g implies that, con-

jecturally, there exist only finitely many primes p such that g is a primitive root modulo p, and the multiplicative structure of C_g and $\mathfrak{C}_g$ allows one to identify precisely what are the obstructions to this.

There are many interesting generalisations to Artin's conjecture on primitive roots. For instance, one could consider only primes p which lie in a prescribed congruence class modulo some integer f. One could also study the set of primes p such that g generates a subgroup of a given index in $(\mathbb{Z}/p\mathbb{Z})^\times$. As is shown in [Len77], in both of these cases one can again obtain a density under GRH via a formula similar to (2.1.1). However, it is not clear how to describe the non-vanishing criteria of such densities from such a sum.

In [LMS14], the authors develop an efficient method to compute entanglement correction factors $\mathfrak{C}_g$ for Artin's original conjecture and several of its generalisations. Their method consists in expressing $\mathfrak{C}_g$ as a sum of quadratic characters. More precisely, they show that $\mathfrak{C}_g$ has the form

$$\mathfrak{C}_g = 1 + \prod_{\ell} E_\ell$$

where each E_ℓ is the average value of a character χ_ℓ over an explicit set. One crucial fact used to arrive at this form is that when $D \equiv 1 \pmod 4$, then for n divisible by $2D$ we have that the subgroup

$$\mathrm{Gal}(F_n/\mathbb{Q}) \hookrightarrow \prod_{\ell \mid n} \mathrm{Gal}(F_\ell/\mathbb{Q})$$

is cut out by a quadratic character χ measuring the nature of the entanglement. The structure of C_g as an Euler product and the description of $\mathfrak{C}_g$ naturally lead to non-vanishing criteria.

In this chapter we attempt to generalize this method to the setting of elliptic curves. There are many problems concerning the study of the set of primes p such that the reduced curve $\tilde{E}(\mathbb{F}_p)$ satisfies a certain condition. One of these arises as a natural analogue of Artin's conjecture on

primitive roots. Namely, given an elliptic curve E over $\mathbb{Q}$, the problem is to determine the density of primes p such that $\tilde{E}(\mathbb{F}_p)$ is cyclic. The first thing to note is that the condition of $\tilde{E}(\mathbb{F}_p)$ being cyclic is completely determined by the splitting behaviour of p in the various torsion fields $\mathbb{Q}(E[\ell])$ for different ℓ. Given this, we can proceed similarly by defining local densities δ_ℓ and attempting to find the entanglement correction factor $\mathfrak{C}_E$, however one quickly runs into various difficulties which were not present in the case of classical Artin. One of these is that it is not necessarily true that $\mathrm{Gal}(\mathbb{Q}(E[m])/\mathbb{Q}) \hookrightarrow \prod_{\ell|m} \mathrm{Gal}(\mathbb{Q}(E[\ell])/\mathbb{Q})$ is a normal subgroup and even if so, the quotient need not be $\{\pm 1\}$ or even abelian for that matter.

This leads us to the study in Section 2.2 of so called *abelian entanglements*. If G is a subgroup of $G_1 \times \cdots \times G_n$ such that the projection maps $\pi_i : G \to G_i$ are surjective for $1 \leqslant i \leqslant n$, then we give a necessary and sufficient condition for G being normal in $G_1 \times \cdots \times G_n$ with abelian quotient.

In Section 2.3 we define elliptic curves with abelian entanglements to be those elliptic curves with the property that $G(m_E)$ has abelian entanglements in the sense of Section 2.2. We show that this definition is equivalent to $\mathbb{Q}(E[m_1]) \cap \mathbb{Q}(E[m_2])$ being an abelian extension of $\mathbb{Q}$ for every coprime m_1, m_2. It is for this class of curves that we will be able to apply our character sum method, with Theorem 2.3.4 being a crucial ingredient.

Section 2.4 applies Theorem 2.3.4 to the aforementioned problem of cyclic reduction of elliptic curves. We explicitly evaluate the density C_E as an Euler product $\prod_\ell \delta_\ell$ times an entanglement correction factor $\mathfrak{C}_E$. We then compute $\mathfrak{C}_E$ in the case of Serre curves and give examples of a few other elliptic curves with more complicated Galois Theory, as well as establishing non-vanishing criteria for these conjectural densities.

In Section 2.5 we study a variant of the problem of cyclic reduction on elliptic curves. Namely, we impose the additional condition that p lie in a prescribed congruence class modulo some integer f. This introduces new difficulties as the splitting conditions on p become more complicated, but it

also illustrates the way in which our method can be used to handle a variety of different scenarios. In the end the computation of $\mathfrak{C}_E$ is again reduced to fairly mechanical local computations. Again Serre curves and several other examples are treated in detail.

Section 2.6 we study a different type of problem. We look at a classical conjecture of Koblitz on the asymptotic behaviour of the number of primes p for which the cardinality of $\tilde{E}(\mathbb{F}_p)$ is prime. We see that the character sum approach can also be applied to describe the constant appearing in this asymptotic. In this case there are not even conditional results, and the constant computed is purely conjectural. However the constant we compute has previously been described via different methods by Zywina in [Zyw11c], where he provides some convincing numerical evidence for it.

The study of conjectural constants led us to investigate the class of elliptic curves with abelian entanglements, and naturally leads to the question of whether there exist elliptic curves with non-abelian entanglements. To be precise, can one classify the triples (E, m_1, m_2) with E an elliptic curve over $\mathbb{Q}$ and m_1, m_2 a pair of coprime integers for which the entanglement field $\mathbb{Q}(E[m_1]) \cap \mathbb{Q}(E[m_2])$ is non-abelian over $\mathbb{Q}$? In Chapter 3 we exhibit an infinite family of elliptic curves for which this is the case, and in doing so we obtain a complete set of modular curves which parametrize non-Serre curves.

2.2 Abelian entanglements

In this section we study the following question: if G is a subgroup of $G_1 \times \cdots \times G_n$ such that the projection maps $\pi_i : G \to G_i$ are surjective for $1 \leqslant i \leqslant n$, when does it happen that G is normal in $G_1 \times \cdots \times G_n$ with abelian quotient?

For a group G, we will denote by G' the commutator subgroup of G, and for $x, y \in G$, $[x, y] = x^{-1}y^{-1}xy$ will denote the commutator of x and y. For

a non-empty subset $S \subset \{1, \ldots, n\}$ we write π_S for the projection map

$$\pi_S : G_1 \times \cdots \times G_n \longrightarrow \prod_{i \in S} G_i$$

and let G_S denote the image of G under this projection map. Note that for each partition $\sqcup_j T_j = \{1, \ldots, n\}$ we have a canonical inclusion

$$G \hookrightarrow \prod_j G_{T_j}.$$

Let $\mathcal{P} := \{S, T\}$ be a partition of $\{1, \ldots, n\}$, so that $S \sqcup T = \{1, \ldots, n\}$. Then G is a subdirect product of $G_S \times G_T$ so by Goursat's lemma there is a group $Q_\mathcal{P}$ and a pair of homomorphisms $\psi_\mathcal{P} := (\psi_\mathcal{P}^{(1)}, \psi_\mathcal{P}^{(2)})$ with

$$\psi_\mathcal{P}^{(1)} : G_S \longrightarrow Q_\mathcal{P}$$
$$\psi_\mathcal{P}^{(2)} : G_T \longrightarrow Q_\mathcal{P}$$

such that $G = G_S \times_{\psi_\mathcal{P}} G_T$. We say that G has *abelian entanglements with respect to* $G_1 \times \cdots \times G_n$ if $Q_\mathcal{P}$ is abelian for each two-set partition $\mathcal{P}$ of $\{1, \ldots, n\}$. We will often write only that G has abelian entanglements, omitting with respect to which direct product of groups if this is clear from the context. The following proposition is the main result of this section and provides an answer to the question posed at the start.

Proposition 2.2.1. *Keeping the notation as above, G is a normal subgroup of $G_1 \times \cdots \times G_n$ if and only if G has abelian entanglements.*

The proof will use the following proposition, which is the case $n = 2$.

Proposition 2.2.2. *Let G be a subgroup of $G_1 \times G_2$ such that the projection maps $\pi_1 : G \to G_1$ and $\pi_2 : G \to G_2$ are surjective. Then $G \trianglelefteq G_1 \times G_2$ if and only if G has abelian entanglements.*

Proof. Suppose first that G has abelian entanglements, and let $x := (x_1, x_2) \in G$. We will show that for any $a \in G_1 \times \{1\}$ one has $axa^{-1} \in G$, and similarly for every $b \in \{1\} \times G_2$. The result will then follow. So take $a := (a_1, 1) \in G_1 \times \{1\}$. Let N_1 and N_2 be the corresponding Goursat subgroups associated to G, that is, $N_1 = (G_1 \times \{1\}) \cap G$ and $N_2 = (\{1\} \times G_2) \cap G$. Then because G has abelian entanglements we have that $(G_1 \times \{1\})/N_1$ is abelian, or equivalently $(G_1 \times \{1\})' \leqslant N_1$. It follows that $[(a_1, 1), (x_1, 1)] \in G$, however

$$\begin{aligned}[(a_1, 1), (x_1, 1)] &= (a_1, 1)(x_1, 1)(a_1, 1)^{-1}(x_1, 1)^{-1} \\ &= (a_1, 1)(x_1, x_2)(a_1, 1)^{-1}(x_1, x_2)^{-1}\end{aligned}$$

and $(x_1, x_2)^{-1}$ is in G, hence $(a_1, 1)(x_1, x_2)(a_1, 1)^{-1}$ is also in G, as claimed. Similarly one can show $(1, b_2)(x_1, x_2)(1, b_2)^{-1} \in G$ for any $b_2 \in G_2$, and we conclude G is normal in $G_1 \times G_2$.

For the converse, suppose that $G \trianglelefteq G_1 \times G_2$. We will show that $(G_1 \times \{1\})' \leqslant N_1$, from which it follows that G has abelian entanglements. Let $(x_1, 1)$ and $(y_1, 1)$ be arbitrary elements of $G_1 \times \{1\}$. Because $\pi_1 : G \to G_1$ is surjective, there exists $z \in G_2$ such that $(y_1, z) \in G$. As $G \trianglelefteq G_1 \times G_2$, we have $(x_1, 1)(y_1, z)(x_1, 1)^{-1}$ is in G and hence so is $[(x_1, 1), (y_1, z)]$. Using the fact that $[(x_1, 1), (y_1, 1)] = [(x_1, 1), (y_1, z)]$, we obtain $[(x_1, 1), (y_1, 1)] \in G$. However $[(x_1, 1), (y_1, 1)] = ([x_1, y_1], 1) \in G_1 \times \{1\}$, hence the result. □

Proof of Proposition 2.2.1. Again we suppose first that G has abelian entanglements, and we proceed similarly as in the case $n = 2$. Let $x := (x_1, \ldots, x_n) \in G$, and for $j \in \{1, \ldots, n\}$ let $a := (1, \ldots, 1, a_j, 1, \ldots, 1) \in \{1\} \times \cdots \times \{1\} \times G_j \times \{1\} \times \cdots \times \{1\}$ where the a_j is in the j-th position. Let $S_j := \{1, \ldots, n\} \backslash \{j\}$. Then $G \leqslant G_j \times G_{S_j}$ with surjective projection maps and the corresponding quotient $(G_j \times \{1\})/N_j$ is abelian. By Proposition 2.2.2, G is a normal subgroup of $G_j \times G_{S_j}$. But a is certainly an element of $G_j \times G_{S_j}$, hence $axa^{-1} \in G$. Since j was chosen arbitrarily we conclude

$G \trianglelefteq G_1 \times \cdots \times G_n$.

Conversely, suppose $G \trianglelefteq G_1 \times \cdots \times G_n$, and let $\mathcal{P} := \{S, T\}$ be a partition of $\{1, \ldots, n\}$. Then note that $G_S \times G_T$ may be viewed as a subgroup of $G_1 \times \cdots \times G_n$ and so $G \trianglelefteq G_S \times G_T$. By Proposition 2.2.2 the corresponding Goursat quotient $Q_{\mathcal{P}}$ is abelian, hence G has abelian entanglements. This completes the proof. □

In the proof we used the subset $S_j := \{1, \ldots n\} \backslash \{j\} \subset \{1, \ldots, n\}$. Here we have that G is a subdirect product of $G_j \times G_{S_j}$, so by Goursat's lemma there is a group Q_j and a pair of homomorphisms $\psi_j := (\psi_j^{(1)}, \psi_j^{(2)})$ such that $G = G_j \times_{\psi_j} G_{S_j}$. The following corollary tells us that these are all the partitions we need to consider in order to determine whether or not G has abelian entanglements.

Corollary 2.2.3. *With the notation above, G has abelian entanglements if and only if Q_j is abelian for every $j \in \{1, \ldots, n\}$.*

Proof. One implication is trivial. Suppose that Q_j is abelian for every $j \in \{1, \ldots, n\}$. Then by the proof of Proposition 2.2.1, G is a normal subgroup of $G_1 \times \cdots \times G_n$, and again using Proposition 2.2.1, G has abelian entanglements, as claimed. □

Proposition 2.2.4. *Suppose that G is a normal subgroup of $G_1 \times \cdots \times G_n$ such that the projection maps $\pi_i : G \to G_i$ are surjective for all i. Then the quotient $(G_1 \times \cdots \times G_n)/G$ is abelian.*

Proof. We will proceed by showing that $(G_1 \times \cdots \times G_n)' \leqslant G$. Let $x := (x_1, \ldots x_n) \in (G_1 \times \cdots \times G_n)'$. By Proposition 2.2.1 G has abelian entanglements, so for each j, to the inclusion $G \hookrightarrow G_j \times G_{S_j}$ there corresponds an abelian quotient $G_j/\pi_j(N_j)$, where $N_j = (G_j \times \{1\}) \cap G$. The composition

$$G_1 \times \cdots \times G_n \xrightarrow{\pi_j} G_j \longrightarrow G_j/\pi_j(N_j)$$

gives an abelian quotient of $G_1 \times \cdots \times G_n$, hence $x_j = \pi_j(x_1, \ldots, x_n)$ is contained in $\pi_j(N_j)$. It follows that $(1, \ldots, 1, x_j, 1 \ldots, 1) \in G$. As j was arbitrary, and $\prod_j (1, \ldots, 1, x_j, 1 \ldots, 1) = x$, we conclude $x \in G$. □

Proposition 2.2.5. *Suppose G has abelian entanglements with respect to $G_1 \times \cdots \times G_n$ and let $S \subseteq \{1, \ldots, n\}$. Then G_S has abelian entanglements with respect to $\prod_{i \in S} G_i$.*

Proof. We will show that G_S is normal in $\prod_{i \in S} G_i$. Note that

$$G \leqslant \pi_S^{-1}(G_S) \leqslant G_1 \times \cdots \times G_n$$

and by Proposition 2.2.4 the quotient $(G_1 \times \cdots \times G_n)/G$ is abelian. It follows then that $\pi_S^{-1}(G_S)$ is normal in $G_1 \times \cdots \times G_n$, and denote the quotient by Φ_S. Now $\ker \pi_S \subset \pi_S^{-1}(G_S)$ so the map $G_1 \times \cdots \times G_n \to \Phi_S$ factors via $\prod_{i \in S} G_i$. Let ψ_S be such that the following diagram commutes

$$\begin{array}{ccc} G_1 \times \cdots \times G_n & & \\ \downarrow{\scriptstyle \pi_S} & \searrow & \\ \prod_{i \in S} G_i & \xrightarrow{\psi_S} & \Phi_S \end{array} .$$

It is easy to see that the kernel of ψ_S is precisely G_S, hence G_S is normal in $\prod_{i \in S} G_i$ and by Proposition 2.2.1 G_S has abelian entanglements with respect to $\prod_{i \in S} G_i$, as claimed. □

2.3 Elliptic curves with abelian entanglements

We consider here a family of elliptic curves with the property that the intersections of the different torsion fields of each curve in this family are abelian extensions.

We say that an elliptic curve E has *abelian entanglements* if the corresponding group $G(m_E) \leqslant G(\ell_1^{\alpha_1}) \times \cdots \times G(\ell_n^{\alpha_n})$ has abelian entanglements in the sense of section 2.2, where m_E as usual denotes the smallest split and stable integer for E, and has prime factorisation $m_E = \ell_1^{\alpha_1} \dots \ell_n^{\alpha_n}$.

Lemma 2.3.1. *The following two conditions are equivalent:*

(i) *E has abelian entanglements.*

(ii) *For each $m_1, m_2 \in \mathbb{N}$ which are relatively prime, the intersection*

$$\mathbb{Q}([m_1]) \cap \mathbb{Q}([m_2])$$

is an abelian extension of $\mathbb{Q}$.

Proof. Suppose E has abelian entanglements, and let m_1, m_2 be relatively prime. If m_1 and m_2 both divide m_E, then by Proposition 2.2.5 $G(m_1 m_2)$ has abelian entanglements with respect to $G(m_1) \times G(m_2)$. This implies the Goursat quotient $Q_{m_1 m_2}$ is abelian, and by Lemma 1.2.8 $\mathbb{Q}([m_1]) \cap \mathbb{Q}([m_2])$ is an abelian extension of $\mathbb{Q}$. For general m_1, m_2, let

$$m_1' = (m_1, m_E), \quad m_2' = (m_2, m_E).$$

Then m_1' and m_2' are relatively prime integers dividing m_E so be the same argument $\mathbb{Q}([m_1']) \cap \mathbb{Q}([m_2'])$ is an abelian extension of $\mathbb{Q}$. From Serre's open image Theorem if n is any integer and d is coprime to nm_E then

$$G(nd) = G(n) \times \mathrm{GL}_2(\mathbb{Z}/d\mathbb{Z}).$$

It follows that $Q_{m_1 m_2}$ is isomorphic to $Q_{m_1' m_2'}$, hence the claim. □

Corollary 2.3.2. *If E has abelian entanglements, then for any $m := \prod_i q_i^{s_i}$ we have that $G(m) \leqslant \prod_i G(q_i^{s_i})$ has abelian entanglements.*

Proof. This follows immediately from Corollary 2.2.3 and Lemma 2.3.1. □

Assume now that E is an elliptic curve over $\mathbb{Q}$ with abelian entanglements, and let m be a positive integer with prime factorisation $m = \prod_\ell \ell^{\alpha_\ell}$. Since E has abelian entanglements, by Corollary 2.3.2 and Proposition 2.2.4 there are a map ψ_m and a finite abelian group Φ_m that fit into the exact sequence

$$1 \longrightarrow G(m) \longrightarrow \prod_{\ell|m} G(\ell^{\alpha_\ell}) \xrightarrow{\psi_m} \Phi_m \longrightarrow 1. \tag{2.3.1}$$

Note that the group Φ_m measures the extent to which there are entanglements between the various ℓ^{α_ℓ}-torsion fields. For instance Φ_m is trivial if and only if for any two coprime integers m_1, m_2 dividing m one has $\mathbb{Q}(E[m_1]) \cap \mathbb{Q}(E[m_2]) = \mathbb{Q}$. The following lemma tells us that Φ_{m_E} measures the full extent to which the distinct torsion fields of E have any entanglements.

Lemma 2.3.3. *Let m be a positive integer and d be a positive integer coprime to m_E. Then $\Phi_{md} \simeq \Phi_m$.*

Proof. Again there is a map ψ_{md} and an abelian group Φ_{md} which fit into the short exact sequence

$$1 \longrightarrow G(md) \longrightarrow \prod_{\ell^{\alpha_\ell}||md} G(\ell^{\alpha_\ell}) \xrightarrow{\psi_{md}} \Phi_{md} \longrightarrow 1.$$

As d is coprime to m_E, by Serre's open image Theorem we have that

$$G(md) = G(m) \times \prod_{\ell^{\alpha_\ell}||d} G(\ell^{\alpha_\ell}) \tag{2.3.2}$$

It follows that $G(\ell^{\alpha_\ell})$ is contained in the kernel of ψ_{md} for any $\ell \mid d$, hence $\Phi_{md} \simeq \Phi_m$. □

For each prime $\ell \mid m$, let $S(\ell)$ be a subset of $G(\ell^{\alpha_\ell})$, and define

$$\mathcal{S}_m := \prod_{\ell|m} S(\ell), \quad \mathcal{G}_m := \prod_{\ell|m} G(\ell^{\alpha_\ell}).$$

so that $\mathcal{S}_m \subset \mathcal{G}_m$. The following theorem allows us to compute the fraction of elements in $G(m)$ that belong to $\prod_{\ell|m} S(\ell)$. It will play a key role in the method we will develop for computing entanglement correction factors as character sums. If A is an abelian group, then $\widehat{A}$ denotes the group of characters $\chi : A \to \mathbb{C}^\times$.

Theorem 2.3.4. *Assume $E/\mathbb{Q}$ has abelian entanglements, and let Φ_m be as in (2.3.1). For each $\tilde{\chi} \in \widehat{\Phi}_m$ a character of Φ_m, let χ be the character of $\mathcal{G}_m$ obtained by composing $\tilde{\chi}$ with ψ_m, and let χ_ℓ the restriction of χ to the component $G(\ell^{\alpha_\ell})$. Then*

$$\frac{|\mathcal{S}_m \cap G(m)|}{|G(m)|} = \left(1 + \sum_{\tilde{\chi} \in \widehat{\Phi}_m - \{1\}} \prod_{\ell|m} E_{\chi,\ell}\right) \frac{|\mathcal{S}_m|}{|\mathcal{G}_m|},$$

where

$$E_{\chi,\ell} = \sum_{x \in S(\ell)} \frac{\chi_\ell(x)}{|S(\ell)|}.$$

Proof. Let $\mathbb{1}_{\mathcal{S}_m}$ be the indicator function of $\mathcal{S}_m$ in $\mathcal{G}_m$, and $\mathbb{1}_{G(m)}$ that of $G(m)$. Also, to simplify notation we will use Φ in place of Φ_m. Then we have that

$$\frac{|\mathcal{S}_m \cap G(m)|}{|G(m)|} = \frac{1}{|G(m)|} \sum_{x \in \mathcal{G}_m} \mathbb{1}_{\mathcal{S}_m}(x) \mathbb{1}_{G(m)}(x).$$

By the orthogonality relations of characters (see for instance §VI.1 of [Ser73]) we have that if $x \in \mathcal{G}_m$, then

$$\sum_{\tilde{\chi} \in \widehat{\Phi}} \chi(x) = \begin{cases} [\mathcal{G}_m : G(m)] & \text{if } x \in G(m) \\ 0 & \text{if } x \notin G(m). \end{cases}$$

This implies that

$$\mathbb{1}_{G(m)} = \frac{1}{[\mathcal{G}_m : G(m)]} \sum_{\tilde{\chi} \in \widehat{\Phi}} \chi,$$

so it follows that

$$\begin{aligned}
\frac{|\mathcal{S}_m \cap G(m)|}{|G(m)|} &= \frac{1}{|\mathcal{G}_m|} \Big(\sum_{x \in \mathcal{G}_m} \mathbb{1}_{\mathcal{S}_m}(x) + \sum_{x \in \mathcal{G}_m} \sum_{\tilde{\chi} \in \widehat{\Phi} \setminus \{1\}} \mathbb{1}_{\mathcal{S}_m}(x) \chi(x) \Big) \\
&= \frac{|\mathcal{S}_m|}{|\mathcal{G}_m|} \Big(1 + \sum_{x \in \mathcal{G}_m} \sum_{\tilde{\chi} \in \widehat{\Phi} \setminus \{1\}} \frac{\mathbb{1}_{\mathcal{S}_m}(x) \chi(x)}{|\mathcal{S}_m|} \Big) \\
&= \frac{|\mathcal{S}_m|}{|\mathcal{G}_m|} \Bigg(1 + \sum_{\tilde{\chi} \in \widehat{\Phi} \setminus \{1\}} \Big(\prod_{\ell \mid m} \sum_{x \in G(\ell)} \frac{\mathbb{1}_{S(\ell)}(x) \chi_\ell(x)}{|S(\ell)|} \Big) \Bigg) \\
&= \frac{|\mathcal{S}_m|}{|\mathcal{G}_m|} \Bigg(1 + \sum_{\tilde{\chi} \in \widehat{\Phi} \setminus \{1\}} \Big(\prod_{\ell \mid m} \sum_{x \in S(\ell)} \frac{\chi_\ell(x)}{|S(\ell)|} \Big) \Bigg)
\end{aligned}$$

where the third equality follows from the fact that $\mathbb{1}_{\mathcal{S}_m}$ and χ are products of functions $\mathbb{1}_{S(\ell)}$ and χ_ℓ defined on the components $G(\ell^{\alpha_\ell})$. The result now follows from letting $E_{\chi,\ell}$ be the average value of χ_ℓ on $S(\ell)$, that is

$$E_{\chi,\ell} = \sum_{x \in S(\ell)} \frac{\chi_\ell(x)}{|S(\ell)|}.$$

□

2.4 Cyclic reduction of elliptic curves

In this section we consider an elliptic curve analogue of Artin's classical conjecture on primitive roots. Recall that this conjecture predicts the density of primes p such that a given rational number is a primitive root modulo p. In [LT77], Lang and Trotter formulated an analogous conjecture for elliptic curves over $\mathbb{Q}$. Namely, if P is a point of $E(\mathbb{Q})$ of infinite order, then the

problem is to determine the density of primes p for which $\tilde{E}(\mathbb{F}_p)$ is generated by $\tilde{P}$, the reduction of P modulo p.

Note that for there to exist even one prime p of good reduction with this property, a necessary condition is that the group $\tilde{E}(\mathbb{F}_p)$ be cyclic, and that is the question we consider here. In [Ser86], Serre showed assuming the Generalized Riemann Hypothesis that the set of primes p such that $\tilde{E}(\mathbb{F}_p)$ is cyclic has a density. He did this by adapting Hooley's argument of conditionally proving Artin's conjecture on primitive roots. Namely, we have the following:

Theorem 2.4.1 (Serre, 1976)**.** *Let E be an elliptic curve defined over $\mathbb{Q}$ with conductor N_E. Assuming GRH we have that*

$$|\{p \leqslant x \text{ prime} : p \nmid N_E,\ \tilde{E}(\mathbb{F}_p) \text{ is cyclic}\}| \sim C_E \frac{x}{\log x}$$

as $x \to \infty$, where $C_E := \sum_{n \geqslant 1} \frac{\mu(n)}{[\mathbb{Q}(E[n]) : \mathbb{Q}]}$.

We explicitly evaluate this density C_E as an Euler product. Note that the condition of $\tilde{E}(\mathbb{F}_p)$ being cyclic is completely determined by $\rho_E(G_{\mathbb{Q}})$. Indeed, $\tilde{E}(\mathbb{F}_p)$ is cyclic if and only if p does not split completely in the field $\mathbb{Q}(E[\ell])$ for any $\ell \neq p$. Note that this condition is automatically satisfied when $\ell > p$, since p splitting completely in $\mathbb{Q}(E[\ell])$ implies $p \equiv 1 \pmod{\ell}$. In other words, if for each prime ℓ we define the set $S(\ell) := G(\ell) - \{1\}$, then for all $p \nmid N_E$ the group $\tilde{E}(\mathbb{F}_p)$ is cyclic if and only if $\rho_\ell(\mathrm{Frob}_p) \in S(\ell)$ for any $\ell < p$, i.e. if p does not split completely in $\mathbb{Q}(E[\ell])$.

By the Chebotarev density theorem, the set of primes p that do not split completely in $\mathbb{Q}(E[\ell])$ has density equal to

$$\delta_\ell := \frac{|S(\ell)|}{|G(\ell)|} = 1 - \frac{1}{[\mathbb{Q}(E[\ell]) : \mathbb{Q}]}.$$

If we assume that the various splitting conditions at each prime ℓ are in-

dependent, then it is reasonable to expect that the density of primes p for which $\tilde{E}(\mathbb{F}_p)$ is cyclic is equal to $\prod_\ell \delta_\ell$. However as we know, this assumption of independence is not correct, as different torsion fields may have non-trivial intersection. To be precise, for each square-free integer d let

$$\mathcal{S}_d := \prod_{\ell \mid d} S(\ell), \quad \mathcal{G}_d := \prod_{\ell \mid d} G(\ell).$$

By Chebotarev, the density of primes p such that $p \nmid N_E$ and $\rho_\ell(\mathrm{Frob}_p) \in S(\ell)$ for all $\ell \mid d$ and $\ell \neq p$ is equal to $|\mathcal{S}_d \cap G(d)|/|G(d)|$. If we let d increase to infinity ranging over square-free integers, then Serre's above result implies that, assuming GRH,

$$C_E = \lim_{d \to \infty} \frac{|\mathcal{S}_d \cap G(d)|}{|G(d)|} \tag{2.4.1}$$

where the limit will be seen to exist.

Now let $m = \prod_{\ell \mid m_E} \ell$ be the square-free part of m_E, and let d be a square-free integer coprime to m. By (2.3.2) we have

$$\frac{|\mathcal{S}_{md} \cap G(md)|}{|G(md)|} = \frac{|\mathcal{S}_m \cap G(m)|}{|G(m)|} \prod_{\ell \mid d} \frac{|S(\ell)|}{|G(\ell)|}.$$

For ℓ coprime to m_E, we have that $|S(\ell)|/|G(\ell)|$ is $1 + \mathcal{O}(1/\ell^4)$ so the limit in (2.4.1) does indeed exist. Letting d tend to infinity over the square-free numbers then gives

$$C_E = \frac{|\mathcal{S}_m \cap G(m)|}{|G(m)|} \prod_{\ell \nmid m} \frac{|S(\ell)|}{|G(\ell)|}.$$

The above discussion implies that if we do take into account entanglements,

then assuming GRH we have

$$C_E = \mathfrak{C}_E \prod_\ell \delta_\ell \tag{2.4.2}$$

where $\mathfrak{C}_E$ is an *entanglement correction factor*, and explicitly evaluating such densities amounts to computing the correction factors $\mathfrak{C}_E$. The entanglement correction factor $\mathfrak{C}_E$ arises as the factor by which C_E differs from the uncorrected value $\lim_{d\to\infty} |\mathcal{S}_d|/|\mathcal{G}_d| = \prod_\ell \delta_\ell$. We will use Theorem 2.3.4 for evaluating $\mathfrak{C}_E$ as a character sum for elliptic curves with abelian entanglements.

Theorem 2.4.2. *Assume $E/\mathbb{Q}$ has abelian entanglements, and let Φ_m be as in (2.3.1). Let $\tilde{\chi} \in \widehat{\Phi}_m$ be a character of Φ_m and let χ be the character of $\mathcal{G}_m$ obtained by composing $\tilde{\chi}$ with ψ_m. Define $E_{\chi,\ell}$ by*

$$E_{\chi,\ell} = \begin{cases} 1 & \text{if } \chi \text{ is trivial on } G(\ell), \\ \frac{-1}{[\mathbb{Q}(E[\ell]):\mathbb{Q}]-1} & \text{otherwise.} \end{cases}$$

Then

$$C_E = \mathfrak{C}_E \prod_\ell \delta_\ell$$

where the entanglement correction factor $\mathfrak{C}_E$ is given by

$$\mathfrak{C}_E = 1 + \sum_{\tilde{\chi} \in \widehat{\Phi} - \{1\}} \prod_{\ell | m} E_{\chi,\ell}.$$

Proof. By Theorem 2.3.4 we have that

$$\frac{|\mathcal{S}_m \cap G(m)|}{|G(m)|} = \frac{|\mathcal{S}_m|}{|\mathcal{G}_m|} \left(1 + \sum_{\tilde{\chi} \in \widehat{\Phi} \setminus \{1\}} \prod_{\ell | m} E_{\chi,\ell} \right),$$

where $E_{\tilde{\chi},\ell}$ is the average value of χ_ℓ on $S(\ell)$. By (2.4.2), we know that

$$\begin{aligned}\mathfrak{C}_E &= \frac{C_E}{\prod_\ell \delta_\ell} \\ &= \frac{|\mathcal{S}_m \cap G(m)|/|G(m)|}{|\mathcal{S}_m|/|\mathcal{G}_m|}.\end{aligned}$$

Finally, notice that if χ is non-trivial on $G(\ell)$ then χ_ℓ is non-trivial, hence

$$\sum_{x \in S(\ell)} \chi_\ell(x) = \Big(\sum_{x \in G(\ell)} \chi_\ell(x) \Big) - \chi_\ell(1) = -1.$$

This completes the proof. □

Remark 2.4.3. Note that in the above theorem we may replace m by any square-free multiple of it. Indeed, for any $\tilde{\chi}$, it follows from Lemma 2.3.3 that $E_{\chi,\ell} = 1$ for any $\ell \nmid m$, hence the product $\prod_{\ell|m} E_{\chi,\ell}$ does not change, and the quotient of $|\mathcal{S}_{md} \cap G(md)|/|G(md)|$ and $|\mathcal{S}_{md}|/|\mathcal{G}_{md}|$ is constant as d tends to infinity.

In what follows we will use Theorem 2.4.2 to compute $\mathfrak{C}_E$ for various elliptic curves over $\mathbb{Q}$.

2.4.1 Serre curves

Consider the representation $\rho_E : G_\mathbb{Q} \to \mathrm{GL}_2(\widehat{\mathbb{Z}})$ given by the action of $G_\mathbb{Q}$ on $E(\overline{\mathbb{Q}})_{\mathrm{tors}}$. Serre has shown in [Ser72] that the image of ρ_E is always contained in a specific index 2 subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ and thus ρ_E is *never* surjective. Following Lang and Trotter, we define an elliptic curve E over $\mathbb{Q}$ to be a Serre curve if $[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : G] = 2$.

It follows from the result of Serre that Serre curves are elliptic curves over $\mathbb{Q}$ whose Galois action on their torsion points is as large as possible. Jones has shown in [Jon10] that "most" elliptic curves over $\mathbb{Q}$ are Serre curves (see Section 3.1 for the more precise statement) . Thus they are prevalent over

$\mathbb{Q}$ and we also have complete understanding of their Galois theory, and this makes their entanglement factors particularly easy to handle in conjunction with Theorem 2.4.2.

First we briefly describe the index 2 subgroup H_E of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ (see [Ser72], page 311 for more details). To this end let $\chi_\Delta : G_\mathbb{Q} \to \{\pm 1\}$ be the character associated to $K := \mathbb{Q}(\sqrt{\Delta})$, where Δ is the discriminant of any Weierstrass model of E over $\mathbb{Q}$, and note that χ_Δ does not depend on the choice of model. Let

$$\varepsilon : \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \longrightarrow \{\pm 1\}$$

be the signature map under any isomorphism $GL_2(\mathbb{Z}/2\mathbb{Z}) \simeq S_3$. Then as $K \subset \mathbb{Q}(E[2])$, one can check that $\chi_\Delta = \varepsilon \circ \rho_{E,2}$.

Note that $K \subset \mathbb{Q}(\zeta_{|D|})$, where D is the discriminant of $\mathbb{Q}(\sqrt{\Delta})$. Then there exists a unique quadratic character $\alpha : (\mathbb{Z}/|D|\mathbb{Z})^\times \to \{\pm 1\}$ such that $\chi_\Delta = \alpha \circ \det \rho_{E,|D|}$. From this it follows that $\varepsilon \circ \rho_{E,2} = \alpha \circ \rho_{E,|D|}$. If we then define $M_E = \mathrm{lcm}(|D|, 2)$ and

$$H_{M_E} := \{A \in \mathrm{GL}_2(\mathbb{Z}/M_E\mathbb{Z}) \ : \ \varepsilon(A \bmod 2) = \alpha(\det(A \bmod |D|))\},$$

then it follows from the above discussion that H_{M_E} contains $G(M_E)$. If we let H_E be the inverse image of H_{M_E} in $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ under the reduction map, then H_E is clearly an index 2 subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ which contains G. We have then that G is a Serre curve if and only if $\rho_E(G_\mathbb{Q}) = H_E$. It follows from the above discussion that all Serre curves have abelian entanglements.

Proposition 2.4.4. *Let $E/\mathbb{Q}$ be a Serre curve. Let D be the discriminant of $\mathbb{Q}(\sqrt{\Delta})$ where Δ is the discriminant of any Weierstrass model of E over $\mathbb{Q}$. Then*

$$C_E = \mathfrak{C}_E \prod_\ell \left(1 - \frac{1}{(\ell^2 - 1)(\ell^2 - \ell)}\right)$$

where the entanglement correction factor $\mathfrak{C}_E$ *is given by*

$$\mathfrak{C}_E = \begin{cases} 1 & \text{if } D \equiv 0 \pmod 4 \\ 1 + \prod_{\ell \mid 2D} \frac{-1}{(\ell^2-1)(\ell^2-\ell)-1} & \text{if } D \equiv 1 \pmod 4 \end{cases}$$

Proof. Since E is a Serre curve, we have that $G(\ell) = \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ holds for all ℓ, hence $[\mathbb{Q}(E[\ell]) : \mathbb{Q}] = (\ell^2-1)(\ell^2-\ell)$.

Now suppose first that $D \equiv 0 \pmod 4$. Then $m_E = |D|$ is divisible by 4, hence we have that

$$G(m) = \prod_{\ell \mid m} G(\ell)$$

for all square-free m. It follows that $\Phi_m \simeq \{1\}$ hence its character group is trivial and $\mathfrak{C}_E = 1$.

Now suppose $D \equiv 1 \pmod 4$. In this case $m_E = 2|D|$ is square-free, hence $G(m_E)$ is an index 2 subgroup of $\prod_{\ell \mid m_E} G(\ell)$ and $\Phi \simeq \{\pm 1\}$. For each $\ell > 2$ dividing m_E, χ_ℓ is the character given by the composition $G(\ell) \xrightarrow{\det} (\mathbb{Z}/\ell\mathbb{Z})^* \to \{\pm 1\}$, that is $\chi_\ell = \left(\frac{\det}{\ell}\right)$, and $\chi_2 := \varepsilon$ is the signature map under an isomorphism $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \simeq S_3$. If we let $\chi := \prod_{\ell \mid m_E} \chi_\ell$ then we have an exact sequence

$$1 \longrightarrow G(m_E) \longrightarrow \prod_{\ell \mid m_E} G(\ell) \xrightarrow{\chi} \{\pm 1\} \longrightarrow 1.$$

Clearly each χ_ℓ is non-trivial on $G(\ell)$ for each ℓ dividing m_E so the result follows from Theorem 2.4.2 and using that $\Phi_{m_E} \simeq \{\pm 1\}$. □

2.4.2 Example: $Y^2 + Y = X^3 - X^2 - 10X - 20$

We now consider the elliptic curve over $\mathbb{Q}$ defined by the Weierstrass equation $Y^2 + Y = X^3 - X^2 - 10X - 20$. The Galois theory for this elliptic curve has been worked out by Lang and Trotter in [LT74], and in particular they have

shown that $m_E = 2 \cdot 5^2 \cdot 11$, and that the following properties hold:

- $G(2) = \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$.
- E has a rational 5-torsion point, and $\mathbb{Q}(E[5]) = \mathbb{Q}(\zeta_5)$.
- $[\mathbb{Q}(E[5^2]) : \mathbb{Q}(E[5])] = 5^4$, hence 5 is stable.
- $\mathbb{Q}(E[5^2]) \cap \mathbb{Q}(E[11]) = \mathbb{Q}(\zeta_{11})^+$, where $\mathbb{Q}(\zeta_{11})^+$ is the real quadratic subfield of $\mathbb{Q}(\zeta_{11})$. This implies there is a map

  $$\phi_5 : G(5^2) \longrightarrow (\mathbb{Z}/11\mathbb{Z})^\times/\{\pm 1\}.$$

  We make this map explicit. There is a basis for $E[5^2]$ over $\mathbb{Z}/25\mathbb{Z}$ under which we have

  $$G(5^2) = \left\{ \begin{pmatrix} 1+5a & 5b \\ 5c & u \end{pmatrix} : a, b, c, d \in \mathbb{Z}/25\mathbb{Z},\ u \in (\mathbb{Z}/25\mathbb{Z})^\times \right\}.$$

  Define the (surjective) homomorphism

  $$\begin{aligned} \psi : G(5^2) &\longrightarrow \mathbb{Z}/5\mathbb{Z} \\ \begin{pmatrix} 1+5a & 5b \\ 5c & u \end{pmatrix} &\longmapsto a \mod 5. \end{aligned}$$

  Then ϕ_5 is given by

  $$A \longmapsto (\pm 2)^{\psi(A)},$$

  where we note that ± 2 is a generator of $(\mathbb{Z}/11\mathbb{Z})^\times/\{\pm 1\}$.
- $\mathbb{Q}(E[2]) \cap \mathbb{Q}(E[11]) = \mathbb{Q}(\sqrt{-11})$.

From this we conclude that E has abelian entanglements and

$$G(2\cdot 5^2\cdot 11) = \Big\{(g_2, g_{25}, g_{11}) \in G(2)\times G(5^2)\times G(11) : \\ \varepsilon(g_2) = \left(\frac{\det(g_{11})}{11}\right),\ \phi_5(g_5) = \phi_{11}(g_{11})\Big\}.$$

Proposition 2.4.5. *Let $E/\mathbb{Q}$ be the elliptic curve given by Weierstrass equation $Y^2 + Y = X^3 - X^2 - 10X - 20$. Then we have*

$$C_E = \frac{3}{4}\mathfrak{C}_E \prod_{\ell\neq 5}\left(1 - \frac{1}{(\ell^2-\ell)(\ell^2-1)}\right) \\ \approx 0.611597,$$

where $\mathfrak{C}_E$ is given by

$$\mathfrak{C}_E = 1 + \frac{1}{65995}.$$

Proof. As before we take $m = 2\cdot 5\cdot 11$ to be the square-free part of m_E. Because E has abelian entanglements there is an exact sequence

$$1 \longrightarrow G(2\cdot 5\cdot 11) \longrightarrow G(2)\times G(5)\times G(11) \xrightarrow{\chi} \Phi_{110} \longrightarrow 1$$

From the description of $G(2\cdot 5^2\cdot 11)$ it follows that $G(2\cdot 5\cdot 11) = G(22)\times G(5)$, hence $\Phi_{110} \simeq \{\pm 1\}$. It follows that if we set χ_2 equal to the sign character ε, χ_{11} to $\left(\frac{\det(g_{11})}{11}\right)$ and χ_5 be trivial, then $\chi = \chi_2\chi_5\chi_{11}$.

By Theorem 2.4.2 we have

$$C_E = \mathfrak{C}_E \prod_{\ell} \delta_\ell.$$

where

$$\mathfrak{C}_E = 1 + E_{\chi_2}E_{\chi_5}E_{\chi_{11}}.$$

From the description of $G(\ell)$ it is then straightforward to compute δ_ℓ as well as E_{χ_ℓ} for every ℓ. □

Remark 2.4.6. Note that in this example, even though the Galois theory of E was considerably more complicated than that of a Serre curve, at the 'square-free' torsion level it was still very similar. Indeed, the subgroup $G(110) \leqslant G(2)\times G(5)\times G(11)$ was still cut out only by a quadratic character.

2.5 Cyclic reduction for primes in an arithmetic progression

We now consider a variant of the problem on cyclic reduction of elliptic curves. We have been looking at the density of primes p for which the reduction $\tilde{E}(\mathbb{F}_p)$ is cyclic. Here we impose the additional requirement that p lie in a prescribed residue class modulo some integer f. This is just one of many possible generalizations one could consider, and in many of them one should still obtain a density assuming GRH. One of the difficulties that arises however, is the explicit computation of the density as an Euler product. The character sum method we have given allows us to do this in a relatively simple manner.

If we keep the same setup as in Theorem 2.4.2, then note that the condition we are imposing on p being satisfied is again completely determined by $\rho_E(G_{\mathbb{Q}})$. In this case however, it is not necessarily enough to consider only the 'square-free' torsion fields $\mathbb{Q}(E[\ell])$. Suppose then that we are interested in primes p such that

(i) $\tilde{E}(\mathbb{F}_p)$ is cyclic,

(ii) $p \equiv a \pmod{f}$.

For each prime power ℓ^α, define

$$\mathcal{D}_a(\ell^\alpha) := \{A \in \mathrm{GL}_2(\mathbb{Z}/\ell^\alpha\mathbb{Z}) : \det A \equiv a \pmod{\ell^\alpha}\},$$
$$\left(I + \ell M_2(\mathbb{Z}/\ell^\alpha\mathbb{Z})\right)^c := \{A \in \mathrm{GL}_2(\mathbb{Z}/\ell^\alpha\mathbb{Z}) : A \not\equiv I \pmod{\ell}\}.$$

Let $f = \prod_\ell \ell^{e_\ell}$ be the prime factorisation of f, and for each $\ell \mid f$ set

$$\begin{aligned}\Psi_a(\ell^{e_\ell}) :&= \mathcal{D}_a(\ell^{e_\ell}) \cap \big(I + \ell M_2(\mathbb{Z}/\ell^{e_\ell}\mathbb{Z})\big)^c \\ &= \{A \in \mathrm{GL}_2(\mathbb{Z}/\ell^{e_\ell}\mathbb{Z}) : A \not\equiv I \pmod{\ell},\ \det A \equiv a \pmod{\ell^{e_\ell}}\}.\end{aligned}$$

Then set

$$S(\ell) := G(\ell^{e_\ell}) \cap \Psi_a(\ell^{e_\ell})$$

for those ℓ dividing f, and just as in the case of the previous subsection, set $S(\ell) := G(\ell) - \{1\}$ for all other ℓ. Then it follows that $p \nmid N_E$ satisfies conditions (i) and (ii) above if and only if for any $\ell \nmid p$ one has

(i) $\rho_\ell(\mathrm{Frob}_p) \in S(\ell)$ if $\ell \nmid f$,

(ii) $\rho_{\ell^{e_\ell}}(\mathrm{Frob}_p) \in S(\ell)$ if $\ell \mid f$.

Then the density of p having the 'right' local behaviour at ℓ equals

$$\delta_\ell = \begin{cases} |S(\ell)|/|G(\ell)| & \text{if } \ell \nmid f \\ |S(\ell)|/|G(\ell^{e_\ell})| & \text{if } \ell \mid f \end{cases}$$

and the naive density of primes satisfying conditions (i) and (ii) equals $\prod_\ell \delta_\ell$.

To account for entanglements, we proceed more or less along the same line as the case without the condition of p lying in a prescribed residue class, with some slight modifications. That is, let

$$m := \prod_{\ell \mid (f, m_E)} \ell^{e_\ell} \prod_{\substack{\ell \mid m_E \\ \ell \nmid f}} \ell$$

For any square-free d coprime to m, define

$$\mathcal{S}_{md} := \prod_{\ell \mid md} S(\ell), \quad \mathcal{G}_{md} := \prod_{\ell \mid (f,m)} G(\ell^{e_\ell}) \prod_{\substack{\ell \mid md \\ \ell \nmid f}} G(\ell).$$

By Corollary 2.3.2

$$G(md) \leqslant \mathcal{G}_{md}$$

has abelian entanglements, hence we have an exact sequence

$$1 \longrightarrow G(md) \longrightarrow \mathcal{G}_{md} \xrightarrow{\psi_{md}} \Phi_{md} \longrightarrow 1$$

for some abelian group Φ_{md}. We again have by (2.3.2) that $\Phi_{md} \simeq \Phi_m$ for any square-free d coprime to m, and the density we are looking for is then

$$C_E(a,f) = \lim_{d\to\infty} \frac{|\mathcal{S}_{md} \cap G(md)|}{|G(md)|} = \frac{|\mathcal{S}_m \cap G(m)|}{|G(m)|} \prod_{\ell\nmid m} \frac{|S(\ell)|}{|G(\ell)|}.$$

Theorem 2.5.1. *Let $\tilde{\chi} \in \widehat{\Phi}_m$ be a character of Φ_m and let χ be the character of $\mathcal{G}_m$ obtained by composing $\tilde{\chi}$ with ψ_m. Define $E_{\chi,\ell}$ by*

$$E_{\tilde{\chi},\ell} = \sum_{x\in S(\ell)} \frac{\chi_\ell(x)}{|S(\ell)|}.$$

Then

$$C_E(a,f) = \mathfrak{C}_E(a,f) \prod_\ell \delta_\ell$$

where the entanglement correction factor $\mathfrak{C}_E(a,f)$ is given by

$$\mathfrak{C}_E(a,f) = 1 + \sum_{\tilde{\chi}\in\widehat{\Phi}_m - \{1\}} \prod_{\ell|m} E_{\chi,\ell}.$$

Proof. The proof is exactly as that of Theorem 2.3.4 with the obvious modifications. □

It follows from the previous theorem that in order to evaluate the correction factors $\mathfrak{C}_E(a,f)$ it suffices to compute the order of $S(\ell)$ as well as the average value of the χ_ℓ on $S(\ell)$.

2.5.1 Serre curves

In what follows we again consider the example of Serre curves. To simplify the following proofs we will henceforth assume a and f are coprime integers. If not, then for a prime ℓ dividing (a, f) we obtain $|\Psi_a(\ell^{e_\ell})| = 0$ hence $|S(\ell)| = 0$ and $C_E(a, f) = 0$, which we take to mean the conditions imposed are satisfied for only finitely many p.

Lemma 2.5.2. *Let $E/\mathbb{Q}$ be a Serre curve, and let a and f be coprime positive integers. Let D be the discriminant of $\mathbb{Q}(\sqrt{\Delta})$ where Δ is the discriminant of any Weierstrass model of E over $\mathbb{Q}$. Suppose that $|D| \neq 4, 8$. Then*

$$\delta_\ell = \begin{cases} \frac{1}{\phi(\ell^{e_\ell})} & \text{if } a \not\equiv 1 \pmod{\ell} \text{ and } \ell \mid f \\ \frac{1}{\phi(\ell^{e_\ell})}\left(1 - \frac{1}{\ell(\ell-1)(\ell+1)}\right) & \text{if } a \equiv 1 \pmod{\ell} \text{ and } \ell \mid f \\ 1 - \frac{1}{(\ell^2-1)(\ell^2-\ell)} & \text{if } \ell \nmid f. \end{cases}$$

Proof. If $\ell \nmid f$ then as before we obtain the local density $\delta_\ell = 1 - 1/(\ell^2 - 1)(\ell^2 - \ell)$. At $\ell \mid f$ we consider the two cases. If $a \not\equiv 1 \pmod{\ell}$ then

$$S(\ell) = \mathcal{D}_a(\ell^{e_\ell})$$

since any element with determinant $a \not\equiv 1$ cannot be trivial mod ℓ. It follows that for such ℓ one has $\delta_\ell = 1/\phi(\ell^{e_\ell})$. If $a \equiv 1 \pmod{\ell}$ then we need to count the fraction of elements of $\mathcal{D}_a(\ell^{e_\ell})$ which are non-trivial mod ℓ. There is a surjective map $G(\ell) \to (\mathbb{Z}/\ell\mathbb{Z})^*$ of degree $\ell(\ell-1)(\ell+1)$, and $\mathbb{Q}(E[\ell]) \cap \mathbb{Q}(\zeta_{\ell^{e_\ell}}) = \mathbb{Q}(\zeta_\ell)$ (since $|D| \neq 4, 8$) so it follows that this fraction is precisely $1 - 1/\ell(\ell-1)(\ell+1)$, as desired. □

Lemma 2.5.3. *Let E, a and f be as in Lemma 2.5.2. Suppose further that*

$|D| = 4$. *Then*

$$\delta_2 = \begin{cases} \frac{1}{\phi(2^{e_2})} & \text{if } a \equiv 3 \pmod 4 \text{ and } 4 \mid f \\ \frac{1}{\phi(2^{e_2})}(1 - \frac{1}{3}) & \text{if } a \equiv 1 \pmod 4 \text{ and } 4 \mid f \\ \frac{5}{6} & \text{if } 4 \nmid f. \end{cases}$$

Proof. The assumption on D implies that $\mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(i)$ and $m_E = 4$. Recall that $2^{e_2} || f$ is the highest power of 2 dividing f. If $e_2 \geqslant 2$ then a is odd, hence is 1 or 3 mod 4. Note that $\mathbb{Q}(\zeta_{2^{e_2}}) \cap \mathbb{Q}(E[2]) = \mathbb{Q}(i)$. Now the fraction of elements $A \in G(2^{e_2})$ such that $A \in \mathcal{D}_a(2^{e_2})$ equals $1/\phi(2^{e_2})$. If $a \equiv 3 \pmod 4$ then any such $A \in \mathcal{D}_a(2^{e_2})$ acts non-trivially on $\mathbb{Q}(i)$, hence is non-trivial mod 2. It follows that $S(2) = \mathcal{D}_a(2^{e_2})$ and $\delta_2 = 1/\phi(2^{e_2})$. If $a \equiv 1 \pmod 4$, then because $[\mathbb{Q}(E[2]) : \mathbb{Q}(i)] = 3$ exactly $1 - 1/3$ of the elements in $A \in \mathcal{D}_a(2^{e_2})$ are in $S(2)$. Finally suppose $e_2 < 2$. Then the only condition at 2 is being non-trivial mod 2, and the conclusion follows. □

Lemma 2.5.4. *Let E, a and f be as in Lemma 2.5.2. Suppose further that $|D| = 8$. Then*

(i) If $\mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(\sqrt{2})$ then

$$\delta_2 = \begin{cases} \frac{1}{\phi(2^{e_2})} & \text{if } a \equiv 3 \text{ or } 5 \pmod 8 \text{ and } 8 \mid f \\ \frac{1}{\phi(2^{e_2})}(1 - \frac{1}{3}) & \text{if } a \equiv 1 \text{ or } 7 \pmod 8 \text{ and } 8 \mid f \\ \frac{5}{6} & \text{if } 8 \nmid f. \end{cases}$$

(ii) $\mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(\sqrt{-2})$ then

$$\delta_2 = \begin{cases} \frac{1}{\phi(2^{e_2})} & \text{if } a \equiv 5 \text{ or } 7 \pmod 8 \text{ and } 8 \mid f \\ \frac{1}{\phi(2^{e_2})}(1 - \frac{1}{3}) & \text{if } a \equiv 1 \text{ or } 3 \pmod 8 \text{ and } 8 \mid f \\ \frac{5}{6} & \text{if } 8 \nmid f. \end{cases}$$

Proof. We proceed similarly to Lemma 2.5.3. The assumption on D implies that $\mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(\sqrt{\pm 2})$. If $e_2 \geqslant 3$ then in this case $\mathbb{Q}(\zeta_{2^{e_2}}) \cap \mathbb{Q}(E[2]) = \mathbb{Q}(\sqrt{\pm 2})$. In case (i), elements in $\mathcal{D}_a(2^{e_2})$ act non-trivially on $\mathbb{Q}(\sqrt{2})$ if and only if $a \equiv 3$ or $5 \pmod 8$, hence the conclusion. Case (ii) follows from the same argument. □

In what remains of this section we will deduce the correction factor $\mathfrak{C}_E(a, f)$. In the following lemmas we compute the local factors E_ℓ for the different primes ℓ dividing m_E. As is often the case, the prime 2 requires special consideration and we split the computation of the local correction factor E_2 into various cases. Keep the same notation for E, a, f and D, and suppose further that $|D| \neq 4, 8$. Then m_E contains at least one odd prime factor and we have an exact sequence

$$1 \longrightarrow G(m) \longrightarrow \prod_{\ell | (f, m_E)} G(\ell^{e_\ell}) \prod_{\substack{\ell | m_E \\ \ell \nmid f}} G(\ell) \xrightarrow{\chi} \{\pm 1\} \longrightarrow 1$$

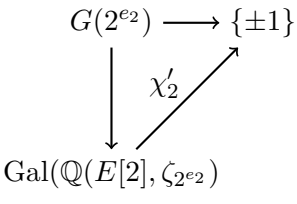
where $\chi = \prod_\ell \chi_\ell$ is a product of characters χ_ℓ. Here χ_ℓ is given by the composition $G(\ell^{e_\ell}) \to G(\ell) \xrightarrow{\det} (\mathbb{Z}/\ell\mathbb{Z})^* \to \{\pm 1\}$ and χ_2 is the character corresponding to the quadratic extension $\mathbb{Q}(E[2^{\alpha_2}]) \cap \mathbb{Q}(E[m/2^{\alpha_2}])$, where $2^{\alpha_2} || m$. When $e_2 = 1$ for instance, χ_2 is the signature map $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \to \{\pm 1\}$, corresponding to the quadratic extension $\mathbb{Q}(\sqrt{\Delta})$.

Lemma 2.5.5. *Suppose* $\mathrm{ord}_2(D) = 0$. *Then* $E_2 = -1/5$.

Proof. Since $D \equiv 1 \pmod 4$ it follows that $m_E = 2|D|$ and χ_2 is the signature map. Let $2^{e_2} || f$ be the largest power of 2 dividing f. If $e_2 \leqslant 0$ then $E_2 = -1/5$ by the same argument as in Proposition 2.4.1. If $e_2 > 1$, then $S(2) \subset G(e^{e_2})$ consists of the elements of $\mathcal{D}_a(2^{e_2})$ which are non-trivial mod 2.

Because $m_E = 2|D|$ with D odd, χ_2 is the signature map, hence it factors through the surjection $G(2^{e_2}) \to \mathrm{Gal}(\mathbb{Q}(E[2]), \zeta_{2^{e_2}})$, so we have a

commutative diagram



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Let $S'(2)$ be the image of $S(2)$ under the surjection $G(2^{e_2}) \to \mathrm{Gal}(\mathbb{Q}(E[2]), \zeta_{2^{e_2}})$. Then note that because $\mathbb{Q}(\zeta_{2^{e_2}}) \cap \mathbb{Q}(E[2]) = \mathbb{Q}$, for each $\sigma \in G(2)$ there is a unique $\sigma' \in \mathrm{Gal}(\mathbb{Q}(E[2]), \zeta_{2^{e_2}})$ such that $\sigma(\zeta_{2^{e_2}}) = \zeta_{2^{e_2}}^a$ and $\sigma' \equiv \sigma \pmod 2$. It follows that

$$\sum_{x \in S'(2)} \chi'(x) = -1$$

and the conclusion follows. □

Lemma 2.5.6. *Suppose* $\mathrm{ord}_2(D) = 2$. *We have*

(i) If $|D| \neq 4$ *and* $4 \mid f$ *then*

$$E_2 = -\left(\frac{a}{4}\right)\frac{1}{5}.$$

(ii) If $|D| = 4$ *or* $4 \nmid f$ *then*

$$E_2 = 0.$$

Proof. If $4 \nmid f$ then because $m_E = |D|$ it follows that $m_E \nmid m$, hence

$$G(m) = \prod_{\ell \mid (f, m_E)} G(\ell^{e_\ell}) \prod_{\substack{\ell \mid m_E \\ \ell \nmid f}} G(\ell)$$

and $\Phi_m \simeq \{1\}$, so $E_2 = 0$. Similarly if $|D| = 4$ then m_E has no odd prime factors and we again conclude $E_2 = 0$.

Now suppose $|D| \neq 4$ and $4 \mid f$. If we let Δ_{sf} denote the square-free part of Δ, then the assumption on $\mathrm{ord}_2(D)$ implies that $\Delta_{\mathrm{sf}} \equiv 3 \pmod 4$. Also, because $4 \mid f$, we have that $\mathbb{Q}(i) \subset \mathbb{Q}(E[2^{e_2})$, hence

$$\mathbb{Q}(\sqrt{i\Delta_{\mathrm{sf}}}) = \mathbb{Q}(E[2^{e_2}]) \cap \mathbb{Q}(E[m/2^{e_2}])$$

and χ_2 is the character corresponding to this quadratic extension. If we define

$$\chi_i : G(2^{e_2}) \to \{\pm 1\}, \quad \chi_\Delta : G(2^{e_2}) \to \{\pm 1\}$$

to be the characters corresponding to the quadratic extensions $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{\Delta})$, respectively, then $\chi_2 = \chi_i\chi_\Delta$. Now χ_i has constant value equal to $(\frac{a}{4})$ on $S(2)$, and by the same argument as in Lemma 2.5.5 χ_Δ has average value $-1/5$ on $S(2)$. It follows then that

$$\begin{aligned} E_2 &= \frac{1}{S(2)} \sum_{x \in S(2)} \chi_2(x) \\ &= \frac{1}{S(2)} \sum_{x \in S(2)} \chi_i(x)\chi_\Delta(x) \\ &= -\left(\frac{a}{4}\right)\frac{1}{5}. \end{aligned}$$

□

To deal with the case of $\mathrm{ord}_2(D) = 3$, we establish the following notation. Note that if $\mathrm{ord}_2(D) = 3$ then we must have that $2 \mid \Delta_{\mathrm{sf}}$. Let Δ' be such that $\Delta_{\mathrm{sf}} = 2\Delta'$.

Lemma 2.5.7. *Suppose* $\mathrm{ord}_2(D) = 3$*, and keep the notation above. We have*

(i) If $|D| \neq 8,\ 8 \mid f$ *and* $\Delta' \equiv 1 \pmod 4$ *then*

$$E_2 = \begin{cases} 1/5 & \text{if } a \equiv 1 \text{ or } 7 \pmod 8 \\ -1/5 & \text{if } a \equiv 3 \text{ or } 5 \pmod 8 \end{cases}.$$

(ii) If $|D| \neq 8,\ 8 \mid f$ *and* $\Delta' \equiv 3 \pmod 4$ *then*

$$E_2 = \begin{cases} 1/5 & \text{if } a \equiv 1 \text{ or } 3 \pmod 8 \\ -1/5 & \text{if } a \equiv 5 \text{ or } 7 \pmod 8 \end{cases}.$$

(iii) If $|D| = 8$ *or* $8 \nmid f$ *then*

$$E_2 = 0.$$

Proof. If $|D| = 8$ or $8 \nmid f$ then by the same reasoning as in Lemma 2.5.6 we conclude $E_2 = 0$. Assume then that $|D| \neq 8$ and $8 \mid f$. Because $8 \mid f$, we have that $\mathbb{Q}(\sqrt{\pm 2}) \subset \mathbb{Q}(E[2^{e_2}])$. Let

$$\chi_{\sqrt{2}} : G(2^{e_2}) \to \{\pm 1\}, \quad \chi_{\sqrt{-2}} : G(2^{e_2}) \to \{\pm 1\}, \quad \chi_\Delta : G(2^{e_2}) \to \{\pm 1\}$$

to be the characters corresponding to the quadratic extensions $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{\Delta})$, respectively. If $\Delta' \equiv 1 \pmod 4$ then

$$\mathbb{Q}(\sqrt{\Delta'}) = \mathbb{Q}(E[2^{e_2}]) \cap \mathbb{Q}(E[m/2^{e_2}])$$

and χ_2 is the quadratic character corresponding to this extension, with $\chi_2 = \chi_{\sqrt{2}}\chi_\Delta$. If $\Delta' \equiv 3 \pmod 4$ then

$$\mathbb{Q}(\sqrt{-\Delta'}) = \mathbb{Q}(E[2^{e_2}]) \cap \mathbb{Q}(E[m/2^{e_2}])$$

and χ_2 is the quadratic character corresponding to this extension, with $\chi_2 = \chi_{\sqrt{-2}}\chi_\Delta$. Now note that $\chi_{\sqrt{2}}$ has constant value on $S(2)$ equal to 1 if $a \equiv 1$ or $7 \pmod 8$, and -1 if $a \equiv 3$ or $5 \pmod 8$, and $\chi_{\sqrt{-2}}$ has constant value on $S(2)$ equal to 1 if $a \equiv 1$ or $3 \pmod 8$, and -1 if $a \equiv 5$ or $7 \pmod 8$ We conclude exactly as in Lemma 2.5.6. □

Proposition 2.5.8. *Let* $E/\mathbb{Q}$ *be a Serre curve, and let* a *and* f *be coprime positive integers. Let* D *be the discriminant of* $\mathbb{Q}(\sqrt{\Delta})$ *where* Δ *is the dis-*

criminant of any Weierstrass model of E over $\mathbb{Q}$. Suppose that $|D| \neq 4, 8$. Then

$$C_E(a,f) = \mathfrak{C}_E(a,f)\frac{1}{\phi(f)}\prod_{\ell\mid(a-1,f)}\left(1-\frac{1}{\ell(\ell-1)(\ell+1)}\right)\prod_{\ell\nmid f}\left(1-\frac{1}{(\ell^2-1)(\ell^2-\ell)}\right)$$

where the entanglement correction factor $\mathfrak{C}_E(a,f)$ is given by

$$\mathfrak{C}_E(a,f) = 1 + E_2\prod_{\substack{\ell\mid(D,f)\\ \ell\neq 2}}\left(\frac{a}{\ell}\right)\prod_{\substack{\ell\mid D\\ \ell\nmid 2f}}\frac{-1}{(\ell^2-1)(\ell^2-\ell)-1}.$$

Here E_2 is given by Lemmas 2.5.5, 2.5.6 and 2.5.7,

Proof. Since $|D| \neq 4, 8$, the equality involving $C_E(a,f)$ follows from using Lemma 2.5.2 for all ℓ. The form of the entanglement correction factor at 2 follows from Lemmas 2.5.5, 2.5.6 and 2.5.7. It remains to consider $\ell \neq 2$. By Theorem 2.5.1 if $\ell \nmid f$ and $\ell \mid D$ then $S(\ell) = G(\ell) - \{1\}$ and so

$$E_\ell = \frac{-1}{(\ell^2-1)(\ell^2-\ell)-1}.$$

Ir $\ell \mid (D,f)$ then because $\mathbb{Q}(E[\ell]) \cap \mathbb{Q}(\zeta_{\ell^{e_\ell}}) = \mathbb{Q}(\zeta_\ell)$ we have that χ_ℓ has constant value $(\frac{a}{\ell})$ on $S(\ell)$ and the result follows. □

Corollary 2.5.9. *For any (a,f) coprime integers, we have $C_E(a,f) > 0$.*

Proof. It is clear that the naive density $\prod_\ell \delta_\ell$ does not vanish, hence in order for $C_E(a,f)$ to be zero, we would need the correction factor $\mathfrak{C}_E(a,f)$ to be zero, which happens if and only if $\prod_\ell E_\ell = -1$. This is impossible as E_2 is always $\pm 1/5$ or 0. □

Corollary 2.5.10. *The correction factor $\mathfrak{C}_E(a,f)$ equals 1 if and only if* $\mathrm{ord}_2(D) > \mathrm{ord}_2(f)$.

Proof. From the form of the correction factor it follows that $\mathfrak{C}_E(a,f) = 1$ if and only if $E_2 = 0$, and the result follows. □

2.5.2 Example: $Y^2 = X^3 + X^2 + 4X + 4$

We look now at an example of a non-Serre curve where the constant $C_E(a, f)$ can vanish. This implies that conjecturally, there should only exist finitely many primes p such that $\tilde{E}(\mathbb{F}_p)$ is cyclic and $p \equiv a \pmod{f}$. Let E be the elliptic curve over $\mathbb{Q}$ given by the Weierstrass equation $Y^2 = X^3 + X^2 + 4X + 4$. In [Bra09], a description of the Galois theory of E is worked out. In particular, for this curve we have that $m_E = 120$, and the following properties hold:

- E has a rational 3-torsion point, and $G(3) \simeq S_3$.
- E has a rational two-torsion point, and $\mathbb{Q}(E[2]) = \mathbb{Q}(i)$.
- $G(4)$ has order 16, and $\mathbb{Q}(E[4]) \cap \mathbb{Q}(E[5]) = \mathbb{Q}(\sqrt{5})$.
- $G(8)$ has order 128, and $\mathbb{Q}(E[8]) \cap \mathbb{Q}(E[5]) = \mathbb{Q}(\zeta_5)$.
- $G(5) = \mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$
- $\mathbb{Q}(E[3]) \cap \mathbb{Q}(E[40]) = \mathbb{Q}$, hence $G(120) = G(3) \times G(40)$.

From all of this we conclude that

$$G(120) = \{(g_8, g_3, g_5) \in G(8) \times G(3) \times G(5) : g_8(\zeta_5) = \zeta_5^{\det g_5}\}$$

hence E has abelian entanglements and $G(120)$ fits into the exact sequence

$$1 \longrightarrow G(120) \longrightarrow G(8) \times G(3) \times G(5) \longrightarrow \Phi_{120} \longrightarrow 1,$$

where $\Phi_{120} \simeq (\mathbb{Z}/5\mathbb{Z})^\times$. Also, given coprime integers a and $f = \prod_\ell \ell^{e_\ell}$ we again set

$$m := \prod_{\ell \mid (f, 120)} \ell^{e_\ell} \prod_{\substack{\ell \mid 120 \\ \ell \nmid f}} \ell.$$

Lemma 2.5.11. *For any $\tilde{\chi} \in \widehat{\Phi}_m - \{1\}$ we have $E_{\chi,2} = 0$.*

Proof. Suppose first that $4 \nmid f$. Then m is square-free, and because

$$G(30) = G(2) \times G(3) \times G(5)$$

it follows that $\Phi_m \simeq \{1\}$, hence $E_{\chi,2} = 0$. Suppose now that $4 \mid f$, and let $\tilde{\eta}$ be a generator of $\tilde{\Phi}_{120}$. If $8 \mid f$, then $120 \mid m$, hence $\Phi_m \simeq \Phi_{120} \simeq (\mathbb{Z}/5\mathbb{Z})^\times$. Any $\tilde{\chi} \in \widehat{\Phi}_m - \{1\}$ is equal to $\tilde{\eta}^j$ for some $j \in \{1, 2, 3\}$ and χ_2 is equal to η_2^j, where

$$\eta_2 : G(2^{e_2}) \longrightarrow (\mathbb{Z}/5\mathbb{Z})^\times$$

is the character corresponding to the subfield $\mathbb{Q}(\zeta_5) \subset \mathbb{Q}(E[2^{e_2}])$. Now because $\mathbb{Q}(E[2]) = \mathbb{Q}(i) \subset \mathbb{Q}(\zeta_{2^{e_2}})$ it follows that $\mathbb{Q}(E[2], \zeta_{2^{e_2}}) \cap \mathbb{Q}(\zeta_5) = \mathbb{Q}$, hence

$$\begin{aligned} \sum_{g \in S(2)} \eta_2^j(g) &= \sum_{x \in (\mathbb{Z}/5\mathbb{Z})^\times} x \\ &= 0. \end{aligned}$$

We conclude that $E_{\chi,2} = 0$. If $4 || f$, then $\Phi_m \simeq \{\pm 1\}$ and we can use the same argument given that $\mathbb{Q}(i) \cap \mathbb{Q}(\zeta_5) = \mathbb{Q}$. This proves the claim. □

Proposition 2.5.12. *For any coprime (a, f) we have that $\mathfrak{C}_E(a, f) = 1$. Further,*

$$C_E(a, f) = 0 \iff 4 \mid f \text{ and } a \equiv 1 \pmod 4.$$

Proof. That $\mathfrak{C}_E(a, f) = 1$ follows directly from Theorem 2.5.1 and Lemma 2.5.11. It follows from this that

$$C_E(a, f) = \prod_\ell \delta_\ell.$$

For $\ell \neq 2$ we have that $\delta_\ell \neq 0$. Indeed,

$$\delta_3 = \begin{cases} \frac{1}{\phi(3^{e_3})} & \text{if } a \equiv 2 \pmod 3 \text{ and } 3 \mid f \\ \frac{1}{\phi(3^{e_3})}(1 - \frac{1}{3}) & \text{if } a \equiv 1 \pmod 3 \text{ and } 3 \mid f \\ \frac{5}{6} & \text{if } 3 \nmid f \end{cases},$$

and

$$\delta_\ell = \begin{cases} \frac{1}{\phi(\ell^{e_\ell})} & \text{if } a \not\equiv 1 \pmod \ell \text{ and } \ell \mid f \\ \frac{1}{\phi(\ell^{e_\ell})}\left(1 - \frac{1}{\ell(\ell-1)(\ell+1)}\right) & \text{if } a \equiv 1 \pmod \ell \text{ and } \ell \mid f \\ 1 - \frac{1}{(\ell^2-1)(\ell^2-\ell)} & \text{if } \ell \nmid f. \end{cases}$$

Finally, given that $\mathbb{Q}(E[2]) = \mathbb{Q}(i)$, it follows that $\delta_2 = 0$ if and only if $4 \mid f$ and $a \equiv 1 \pmod 4$, and the conclusion follows. □

Remark 2.5.13. Suppose a and f are coprime integers such that $a \equiv 1 \pmod 4$. The above proposition is saying that the only obstruction to the existence of infinitely many primes p such that $\tilde{E}(\mathbb{F}_p)$ is cyclic and $p \equiv a \pmod f$ is a local one at the prime 2. Meaning, for any prime p it is impossible for it to satisfy the required condition at the prime 2, that is, for Frob_p to lie in the set $S(2)$, which is the empty set. Note also that even when f is divisible by 4, we still have $E_{\chi,2} = 0$ and hence $\mathfrak{C}_E(a, f) = 1$. What this is encoding is the fact that $\mathbb{Q}(\zeta_{2^{e_2}}) \cap \mathbb{Q}(\zeta_5) = \mathbb{Q}$ for any e_2. The only entanglement of E occurs in the subfield $\mathbb{Q}(\zeta_5)$, and this field is disjoint from $\mathbb{Q}(\zeta_{2^\infty})$.

2.5.3 Example: $Y^2 + XY + Y = X^3 - X^2 - 91X - 310$

So far we have only considered examples where the constant $C_E(a, f)$ either does not vanish, or vanishes because there is a condition at some prime ℓ which cannot be satisfied. Another interesting possibility is when all δ_ℓ are

non-zero, yet the constant $C_E(a,f)$ still vanishes. This occurs if and only if the entanglement correction factor $\mathfrak{C}_E(a,f)$ vanishes and its expression as a product of local correction factors makes it easy to determine when this happens. The entanglement correction factor being zero means there is an obstruction coming from the entanglement fields which prevent there being infinitely many primes p satisfying the imposed conditions. We will now analyse an example when this occurs.

Consider the elliptic curve E over $\mathbb{Q}$ given by Weierstrass equation $Y^2 + XY + Y = X^3 - X^2 - 91X - 310$. The discriminant of our Weierstrass model is $\Delta = 17$. This curve has one rational torsion point of order 2 and $\mathbb{Q}(E[2]) = \mathbb{Q}(\sqrt{17})$. In fact, machine computation shows that $m = 34$, where m is the square-free part of m_E, and

$$G(34) = \{(g_2, g_{17}) \in G(2) \times \mathrm{GL}_2(\mathbb{Z}/17\mathbb{Z}) : \varepsilon(g_2) = \theta_{17} \circ \det(g_{17})\}$$

where as usual ε denotes the signature map and $\theta_{17} : (\mathbb{Z}/17\mathbb{Z})^* \to \{\pm 1\}$ denotes the unique quadratic character of $(\mathbb{Z}/17\mathbb{Z})^*$.

If we let D denote the discriminant of $\mathbb{Q}(\sqrt{\Delta})$, then $D = 17 \equiv 1 \pmod 4$, hence by a similar argument to Lemma 2.5.2 we obtain that

$$\prod_{\ell} \delta_\ell = \frac{1}{2}\frac{1}{\phi(f)} \prod_{\substack{\ell \mid (a-1,f) \\ \ell \neq 2}} \left(1 - \frac{1}{\ell(\ell-1)(\ell+1)}\right) \prod_{\substack{\ell \nmid f \\ \ell \neq 2}} \left(1 - \frac{1}{(\ell^2-\ell)(\ell^2-1)}\right)$$

which is non-zero for all a and f. By Theorem 2.5.1 we have that

$$C_E(a,f) = \mathfrak{C}_E(a,f)\frac{1}{2}\frac{1}{\phi(f)} \prod_{\substack{\ell \mid (a-1,f) \\ \ell \neq 2}} \left(1 - \frac{1}{\ell(\ell-1)(\ell+1)}\right) \prod_{\substack{\ell \nmid f \\ \ell \neq 2}} \left(1 - \frac{1}{(\ell^2-\ell)(\ell^2-1)}\right)$$

with

$$\mathfrak{E}_E(a,f) = 1 + \prod_{\ell \mid 34} E_\ell.$$

We conclude then the following.

Proposition 2.5.14. *For the above elliptic curve we have that $C_E(a, f) = 0$ if and only if $17 \mid f$ and a is a quadratic residue modulo 17.*

Proof. The naive density $\prod_\ell \delta_\ell$ is non-vanishing, hence $C_E(a, f) = 0$ if and only if $\mathfrak{C}_E(a, f) = 0$. Using the same argument as in Lemma 2.5.5, we deduce $E_2 = -1$ for all a, f. We have then that

$$\mathfrak{C}_E(a, f) = 0 \iff E_{17} = 1.$$

If $17 \nmid f$ then $E_{17} = -1/78335$. If $17 \mid f$ then $E_{17} = \left(\frac{a}{17}\right)$ and the conclusion follows. □

Remark 2.5.15. Note that if $17 \mid f$ and a is a quadratic residue mod 17, then for any prime $p \equiv a \pmod f$ we have that p splits in $\mathbb{Q}(\sqrt{17}) = \mathbb{Q}(E[2])$, so Frob_p would not satisfy the condition at the prime 2. The obstruction to the existence of infinitely many primes p such that $\tilde{E}(\mathbb{F}_p)$ is cyclic and $p \equiv a \pmod f$ is precisely the entanglement between the 2 and 17 torsion fields. The above proposition is saying that this the only obstruction that exists.

2.6 Koblitz's conjecture

In [Kob88], N. Koblitz made a conjecture on the asymptotic behaviour of the number of primes p for which the cardinality of the group $\tilde{E}(\mathbb{F}_p)$ is prime. In this section we use our character sum method to give a description of the constants appearing in this asymptotic.

Conjecture 2.6.1 (Koblitz)**.** *Let $E/\mathbb{Q}$ be a non-CM curve and let Δ be the discriminant of any Weierstrass model of E over $\mathbb{Q}$. Suppose that E is not $\mathbb{Q}$-isogenous to a curve with non-trivial $\mathbb{Q}$-torsion. Then*

$$|\{\text{primes } p \leqslant x : p \nmid \Delta, |\tilde{E}(\mathbb{F}_p)| \text{ is prime}\}| \sim C_E \frac{x}{(\log x)^2}$$

as $x \to \infty$ where C_E is an explicit positive constant.

In [Zyw11c], Zywina shows that the description of the constant C_E given by Koblitz is not always correct, and he gives a corrected description of the constant along with providing several interesting examples and numerical evidence for the refined conjecture. In particular the constant described by Zywina is not necessarily positive. The reason the original constant is not always correct is that it does not take into account that divisibility conditions modulo distinct primes need not be independent. Put another way, it could occur that there are non-trivial intersections between distinct ℓ-power torsion fields of E. The following is the refined Koblitz conjecture given by Zywina, which here we state restricted to non-CM curves over $\mathbb{Q}$.

Conjecture 2.6.2. *Let $E/\mathbb{Q}$ be a non-CM elliptic curve of discriminant Δ, and let t be a positive integer. Then there is an explicit constant $C_{E,t} \geqslant 0$ such that*

$$|\{\textit{primes } p \leqslant x : p \nmid \Delta, |\tilde{E}(\mathbb{F}_p)|/t \textit{ is prime}\}| \sim C_{E,t} \frac{x}{(\log x)^2}$$

as $x \to \infty$.

The condition on p that $|\tilde{E}(\mathbb{F}_p)|/t$ be prime can be given as a splitting condition in the various ℓ-torsion fields, so the character sum method we have developed again seems well suited to compute $C_{E,t}$. In his paper Zywina computes the constants $C_{E,t}$ via a different method than the one we use here, both in the CM and non-CM cases. Here we will restrict ourselves to non-CM curves with abelian entanglements over the rationals.

For each prime power ℓ^α, define

$$\Psi_t(\ell^\alpha) := \left\{ A \in \mathrm{GL}_2(\mathbb{Z}/\ell^\alpha\mathbb{Z}) : \ \det(I - A) \in t \cdot (\mathbb{Z}/\ell^\alpha\mathbb{Z})^\times \right\}.$$

For a prime $p \nmid N_E \ell$ note that $\tilde{E}(\mathbb{F}_p)/t$ is invertible modulo $\ell^\alpha/(\ell^\alpha, t)$ if and only if $\rho_{\ell^\alpha}(\mathrm{Frob}_p) \in G(\ell^\alpha) \cap \Psi_t(\ell^\alpha)$. Suppose that t has prime factorisation

$t = \prod_\ell \ell^{e_\ell}$. With this in mind, define the set of 'good' Frobenius elements to be

$$S_t(\ell) = \begin{cases} G(\ell^{e_\ell+1}) \cap \Psi_t(\ell^{e_\ell+1}) & \text{if } \ell \mid t \\ G(\ell) \cap \Psi_t(\ell) & \text{if } \ell \nmid t \ . \end{cases}$$

We now give a description of the constant $C_{E,t}$ in terms of our sets $S_t(\ell)$ as well as a crude heuristic of justifying it. This heuristic follows the same lines as that of Koblitz and Zywina. The key argument relies on the Cramer's model which asserts that, roughly speaking, the primes behave as if every random integer n is prime with probability $1/\log n$. If the sequence $\{|\tilde{E}(\mathbb{F}_p)|/t\}_{p \nmid N_E}$ were assumed to behave like random integers, then the proability that $|\tilde{E}(\mathbb{F}_p)|/t$ is prime would be

$$\frac{1}{\log\left(|\tilde{E}(\mathbb{F}_p)|/t\right)} \approx \frac{1}{\log(p+1) - \log t}.$$

The last approximation uses the fact that by Hasse's bound, $\tilde{E}(\mathbb{F}_p)$ is close to $p+1$.

It is not true however, that the $|\tilde{E}(\mathbb{F}_p)|/t$ behave like random integers with respect to congruences, and in order to get a better approximation we need to take these congruences into account. If we fix a prime ℓ, then for all but finitely many p. if $|\tilde{E}(\mathbb{F}_p)|/t$ is prime then it is invertible modulo ℓ. If ℓ does not divide t, then by Chebotarev, the density of primes $p \nmid N_E$ such that $\tilde{E}(\mathbb{F}_p)/t$ is invertible modulo ℓ is $|S_t(\ell)|/|G(\ell)|$. If $\ell \mid t$, then similarly the density of primes $p \nmid N_E$ such that $\tilde{E}(\mathbb{F}_p)$ is divisible by ℓ^{e_ℓ} *and* $\tilde{E}(\mathbb{F}_p)/t$ is invertible modulo ℓ equals $|S_t(\ell)|/|G(\ell^{e_\ell+1})|$. Meanwhile the density of natural numbers that are invertible mod ℓ is $(1 - 1/\ell)$. If we let d be a square-free integer coprime to t, then

$$\prod_{\ell \mid td} \frac{1}{1-1/\ell} \prod_{\ell \mid t} \frac{|S_t(\ell)|}{|G(\ell^{e_\ell+1})|} \prod_{\ell \mid d} \frac{|S_t(\ell)|}{|G(\ell)|} \cdot \frac{1}{\log(p+1) - \log t}$$

should constitute a better approximation to the probability that $|\tilde{E}(\mathbb{F}_p)|/t$ is prime, as it takes into account the congruences modulo all primes $\ell \mid td$. Taking into account all congruences amounts to letting d tend to infinity, hence this model suggests that for a randomly chosen p, $|\tilde{E}(\mathbb{F}_p)|/t$ is prime with probability

$$\prod_{\ell} \frac{\delta_\ell}{1-1/\ell} \cdot \frac{1}{\log(p+1) - \log t}$$

where

$$\delta_\ell = \begin{cases} |S_t(\ell)|/|G(\ell)| & \text{if } \ell \nmid t \\ |S_t(\ell)|/|G(\ell^{e_\ell+1})| & \text{if } \ell \mid t \ . \end{cases}$$

This is the constant that was given by Koblitz with $t = 1$ and later refined by Zywina. The problem that still remained with the approximation given by Koblitz, is that while it does take into account congruences modulo ℓ, is assumes that divisibility conditions modulo distinct primes are independent. In order to deal with this we take a similar approach as in the previous sections. That is, we let

$$m := \prod_{\ell \mid t} \ell^{e_\ell+1} \prod_{\substack{\ell \mid m_E \\ \ell \nmid t}} \ell$$

and for each square-free d coprime to m, let

$$\mathcal{S}_{md} := \prod_{\ell \mid md} S_t(\ell), \quad \mathcal{G}_{md} := \prod_{\ell \mid t} G(\ell^{e_\ell+1}) \prod_{\substack{\ell \mid md \\ \ell \nmid t}} G(\ell).$$

By Corollary 2.3.2

$$G(md) \leqslant \mathcal{G}_{md}$$

has abelian entanglements, hence we have an exact sequence

$$1 \longrightarrow G(md) \longrightarrow \mathcal{G}_{md} \xrightarrow{\psi_{md}} \Phi_{md} \longrightarrow 1$$

for some abelian group Φ_{md}. By (2.3.2) we have that $\Phi_{md} \simeq \Phi_m$ for any square-free d coprime to m. Note now that $|\mathcal{S}_{md} \cap G(md)|/|G(md)|$ is the density of p for which $|\tilde{E}(\mathbb{F}_p)|/t$ is an integer and invertible modulo md, hence by letting d tend to infinity over the square free integers coprime to m, the refined constant is

$$\begin{aligned} C_{E,t} &= \lim_{d\to\infty} \frac{|\mathcal{S}_{md} \cap G(md)|/|G(md)|}{1-1/\ell} \\ &= \left(\prod_{\ell|m} \frac{1}{1-1/\ell}\right) \cdot \frac{|\mathcal{S}_m \cap G(m)|}{|G(m)|} \prod_{\ell\nmid m} \frac{\delta_\ell}{1-1/\ell}. \end{aligned}$$

It follows by the prime number theorem that the expected number of primes p such that $|\tilde{E}(\mathbb{F}_p)|/t$ is prime is asymptotic to $C_{E,t} \cdot x/(\log x)^2$.

Applying Theorem 2.3.4 with m defined as above we obtain

$$C_{E,t} = \mathfrak{C}_{E,t} \prod_{\ell} \frac{\delta_\ell}{1-1/\ell} \tag{2.6.1}$$

where the entanglement correction factor $\mathfrak{C}_{E,t}$ is given by

$$\mathfrak{C}_{E,t} = 1 + \sum_{\tilde{\chi}\in\widehat{\Phi}_m - \{1\}} \prod_{\ell|m} E_{\chi,\ell}.$$

2.6.1 Serre curves

In this section we compute the constants $C_{E,1}$ in Conjecture 2.6.2 for Serre curves. This will amount to finding the average value of various quadratic characters on $S(\ell)$. In the case of Serre curves, the sets $S(\ell)$ are particularly easy to treat.

Proposition 2.6.3. *Let $E/\mathbb{Q}$ be a Serre curve. Let D be the discriminant of $\mathbb{Q}(\sqrt{\Delta})$ where Δ is the discriminant of any Weierstrass model of E over*

$\mathbb{Q}$. *Then*

$$C_{E,1} = \mathfrak{C}_{E,1} \prod_{\ell} \left(1 - \frac{\ell^2 - \ell - 1}{(\ell-1)^3(\ell+1)}\right)$$

where the entanglement correction factor $\mathfrak{C}_{E,1}$ *is given by*

$$\mathfrak{C}_{E,1} = \begin{cases} 1 & \text{if } D \equiv 0 \pmod 4 \\ 1 + \prod_{\ell \mid D} \dfrac{1}{\ell^3 - 2\ell^2 - \ell + 3} & \text{if } D \equiv 1 \pmod 4 \end{cases}$$

Proof. We begin by noting that, for Serre curves,

$$S_1(\ell) = \left\{ A \in \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : \ \det(I - A) \in (\mathbb{Z}/\ell\mathbb{Z})^{\times} \right\}.$$

We have then that

$$\begin{aligned} \delta_\ell &= \frac{|S_1(\ell)|}{|G(\ell)|} \\ &= 1 - \frac{|S_1(\ell)^c|}{|\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})|} \end{aligned}$$

where $S_1(\ell)^c = \{A \in \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : \ \det(I - A) = 0\}$. Thus $S_1(\ell)^c$ consists of those matrices whose eigenvalues are 1 and λ for some $\lambda \in (\mathbb{Z}/\ell\mathbb{Z})^{\times}$. It follows from Table 12.4 in §12, Chapter XVIII of [Lan02], that there are ℓ^2 elements of $\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$ with both eigenvalues equal to 1, and $\ell^2 + \ell$ elements with eigenvalues 1 and $\lambda \neq 1$. We obtain then that $|S_1(\ell)^c| = \ell^2 + (\ell-2)(\ell^2+\ell)$, hence we have that

$$\delta_\ell = 1 - \frac{\ell^2 + (\ell-2)(\ell^2+\ell)}{(\ell^2-\ell)(\ell^2-1)}$$

and a calculation yields that

$$\frac{\delta_\ell}{1 - 1/\ell} = 1 - \frac{\ell^2 - \ell - 1}{(\ell-1)^3(\ell+1)}.$$

From (2.6.1) it rests only to compute $\mathfrak{C}_{E,1}$. Because $t = 1$, m equals the square-free part of m_E, and we may proceed just as in the proof of Proposition 2.4.4. That is, when $D \equiv 0 \pmod 4$ then $\mathfrak{C}_{E,1} = 1$. If $D \equiv 1 \pmod 4$, then for each $\ell \mid 2D$ it suffices to compute the average value of χ_ℓ on $S_1(\ell)$.

Note that since the χ_ℓ are non-trivial, then $\sum_{x \in G(\ell)} \chi_\ell(x) = 0$. For $\ell > 2$ recall that $\chi_\ell = \left(\frac{\det}{\ell}\right)$, hence given an element $x \in S_1(\ell)^c$ with eigenvalues 1 and λ, we have that $\chi_\ell(x) = \left(\frac{\lambda}{\ell}\right)$. There are an equal number of squares and non-squares in $(\mathbb{Z}/\ell\mathbb{Z})^\times$, so we conclude then

$$\begin{aligned}
\sum_{x \in S_1(\ell)} \chi_\ell(x) &= -\sum_{x \in S_1(\ell)^c} \chi_\ell(x) \\
&= -\left(\ell^2 \left(\frac{1}{\ell}\right) + (\ell^2 + \ell) \sum_{\substack{\lambda \in (\mathbb{Z}/\ell\mathbb{Z})^\times \\ \ell \neq 1}} \left(\frac{\lambda}{\ell}\right)\right) \\
&= -(\ell^2 - (\ell^2 + \ell)) \\
&= \ell.
\end{aligned}$$

From this we obtain

$$\begin{aligned}
E_\ell &= \frac{\ell}{|G(\ell)| - |S_1(\ell)|} \\
&= \frac{\ell}{(\ell^2 - \ell)(\ell^2 - 1) - (\ell^2 + \ell)(\ell - 2) - \ell^2} \\
&= \frac{1}{\ell^3 - 2\ell^2 - \ell + 3}.
\end{aligned}$$

For $\ell = 2$ one can directly compute $S_1(2)$. It consists of the 2 matrices $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ both of which have order 3 and hence are even permutations. Since χ_2 is the signature character we conclude $E_2 = 1$, and this completes the proof. □