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Piecewise deterministic Markov processes : an analytic approach

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Piecewise Deterministic Markov Processes:
An Analytic Approach

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Piecewise Deterministic Markov Processes:
An Analytic Approach

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Printed in the Netherlands

To my mother Karemah
To my wife Maram

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Notation

Unless otherwise mentioned, (S, d) will denote a complete separable metric space, viewed as a measurable space with respect to its Borel σ -algebra $\mathcal{B}(S)$. We write $\mathcal{M}(S)$ to denote the real vector space of all signed finite Borel measures on S , containing $\mathcal{M}^+(S)$, the cone of positive measures. $\mathcal{P}(S)$ is the set of probability measures in $\mathcal{M}^+(S)$. We denote the total variation norm on $\mathcal{M}(S)$ by $\|\cdot\|_{TV}$ and write $\mathcal{M}(S)_{TV}$ for the Banach space consisting of $\mathcal{M}(S)$ endowed with this norm. We denote by $\text{BM}(S)$ the real vector space of all bounded Borel measurable functions from S to $\mathbb{R}$ (equipped with the supremum norm) and by $\text{BL}(S)$ the Banach space of bounded Lipschitz functions from S to $\mathbb{R}$, with the norm $\|f\|_{\text{BL}} = |f|_{\text{Lip}} + \|f\|_{\infty}$, where $|f|_{\text{Lip}}$ denotes the Lipschitz constant of f . We write 1_E for the indicator function of $E \subset S$. For $f : S \rightarrow \mathbb{R}$ measurable and $\mu \in \mathcal{M}(S)$, we write $\langle f, \mu \rangle$ for $\int_S f d\mu$. $C_b(S)$ denotes the Banach space of bounded continuous functions from S to $\mathbb{R}$, endowed with the supremum norm $\|\cdot\|_{\infty}$. For $x \in S$ and $r > 0$, $B(x, r)$ denotes the open ball centered at x with radius r , and $\overline{B}(x, r)$ the closed ball, in the metric space S under consideration. The support of a measure $\mu \in \mathcal{M}(S)$ is denoted by $\text{supp}(\mu)$ and is given by

$$\text{supp}(\mu) = \{x \in X : \mu(B(x, r)) > 0 \text{ for every } r > 0\}.$$

δ_x denotes the Dirac measure (point measure) at x . Let d_H denote the Hausdorff metric on the non empty closed and bounded subsets of X . That is, for $E, F \subset X$ bounded, define

$$\delta(E, F) := \sup_{x \in E} d(x, F) = \sup_{x \in E} \inf_{y \in F} \|x - y\|.$$

Then $d_H(E, F) := \max(\delta(E, F), \delta(F, E))$. Also, define

$$\delta(A, y) := \sup_{x \in A} \|x - y\|.$$

For general properties of the Hausdorff metric we refer to [49]. According to [49, Theorem 10.1.6, p.297], the hyperspace $\mathcal{H}(S)$ is complete (for the Hausdorff metric) when the underlying metric space S is complete.

Chapter 1

Introduction

The subject of this thesis, *piecewise deterministic Markov processes, an analytic approach*, is on the border between analysis and probability theory. Such processes can either be viewed as random perturbations of deterministic dynamical systems in an impulsive fashion, or as a particular kind of stochastic processes in continuous time in which parts of the sample trajectories are deterministic. Accordingly, questions concerning these processes may be approached starting from either side. The approach we used is an analytic one flavoured by probability, as the title says. The topic we study in this thesis is the long-term behaviour of a system that results from a particular type of stochastic perturbation of a deterministic dynamical system, namely by means of random jumps at fixed equidistant time points, where the distribution for the jumps is allowed to depend on the state of the system just before the jump.

1.1 Real world problems

Before giving a precise meaning to the notion of piecewise deterministic Markov processes (PDMP) we shall first show examples of such systems as they appear naturally in the modelling of particular real world phenomena.

1.1.1 Experimental sampling and sustainable harvesting

Here one can think of harvesting from a growing bacterial colony, or a marine-ecosystem from which fish are harvested with a net. In experimental set-ups where one grows bacteria or fungi like baker's yeast in a fermenter, or plant cells in a cell suspension culture, one regularly draws small samples for inspection as

part of the experimental protocol. These interventions will interfere with further growth of the population. The question is to quantify how small a sample should be or how ‘much’ this sampling process will disturb the dynamics of the population.

Intensive fishery has reduced the world ocean fish population drastically. A crucial question is to determine fishing quota that ascertain that a fish population will not go extinct. Such Knowledge may lead to environmentally sustainable ‘harvesting’. In the fishing industry fixed sized nets are thrown out regularly into the sea. The catch size may be considered random, with an obvious maximum. Although much is controlled in modern fishing industry, the fish population is certainly not homogeneously distributed in the waters, leading to the assertion of random catch size.

1.1.2 Treatment of bacterial infections

A common method to treat bacterial infections in animals and humans is by applying antibiotics, orally or by injection. When the antibiotic arrives at the location of infection it will either kill the bacteria present there or prevent them from reproducing. If the local dose is too low some of the bacteria may survive and may even become resistant to the antibiotic given. Antibiotic resistance is a serious phenomenon that is occurring more–and–more often, unfortunately. It can lead to multidrug resistant (MDR) bacteria, which are sometimes called ‘superbugs’. These form a great threat to public health and consequently are of big concern in medicine.

Antibiotics are given regularly in many treatments. The local dose will reduce the number of bacteria in the infected part of the body. The size of reduction will depend on the number of bacteria present before the dose arrives. Moreover, randomness in this reduction is expected, because for example of changes in the (complex) physiology of the patient, resulting in changes in the antibiotics level at the location of infection. This randomness may be captured by a stochastic description. The size of the infectious bacterial colony may have an effect on the effectiveness of the antibiotic (e.g. a bio-film structure). Therefore, it is natural to assume that the reduction is random, with a distribution that is biased by the number of bacteria present.

During the time between the doses the bacteria will grow again. The change in bacterial numbers may be considered deterministic.

If the given dose of antibiotics is too low compared to colony size and time

between application of doses, a persistent bacterial infection may be the result, with unwanted medical consequences.

The problem is to provide conditions on dosage (size and timing) that almost surely or with sufficiently high confidence will drive the infectious population to extinction within the frame of treatment. Note that a very high antibiotics dose must be avoided, because this will also kill beneficial bacteria in the intestines. This will cause other serious health problems.

1.1.3 Pest control

The example of treatment of a bacterial infection provided in the previous section is actually an example of *pest control*: a population of ‘unwanted’ organisms is growing amidst and feeding from a ‘wanted’ population of organisms. One may consider as another example the Colorado potato beetle (*Leptinotarsa decemlineata*) whose larvae in particular feed on potato plant leaves. This beetle is known to be able to quickly devastate potato crop fields. Larvae will form pupae in the soil that grow out to adult beetles during the growth season. Female beetles lay eggs on the plant leaves that yield new larvae. At the end of the growth season some soil-based pupae will enter diapause, survive winter and start a new population in the new growth season [42]. While the development during the growth season may be viewed varying deterministically, with spatial effects, the survival in winter can better be viewed as a random variable.

One may then look at the problem, how insecticides, or biologic counter measures like the introduction of a natural predator, will affect the dynamics of the population, taken the uncontrollable and random weather effects into account.

1.2 Analytic model description

The mathematical interpretation of the above discussed models is that, the dynamics of the system is described by a deterministic dynamical system, while at discrete points in time the dynamics of the latter is interrupted by quick interventions that change the state of the system randomly to a new position in state space according to a law that depends on the state just before this intervention. The times of intervention may either be equally spaced and predetermined as we consider in Chapters 3 and 4 or be random themselves. We study intensively the first case, where intervention times are known and fixed while the jumps are random with state-dependent distribution. This type of dynamics occurs naturally

in the modelling and analysis of the biological systems described in Section 1.1. Intervention times may also be random with the distribution for the duration of the time intervals between interventions dependent on the state at the start. More precisely, the law depends on the state just after the last intervention, but the change in state is deterministic. An example of this type is a model for cell growth and division given in [55]. A more complicated version is explored in [45]. The precise mathematical formulation of the problems could be summarized as follows: The evolution of the system between the interventions is given by the deterministic system $(\phi_t)_{t \geq 0}$ on a subset S of a vector space X endowed with a metric. Assume that x^* is an equilibrium. The population just before the next intervention will be $x' = \phi_{\Delta t}(x)$ where $x \in S$ is the state just after the previous intervention and Δt is the time between two interventions. Then the population y just after the intervention will be in a Borel set $A \subseteq X$ with a certain probability. If the probability of a jump of which the size is in B and which starts at x is given by $Q_x(B)$, then the probability that the population just after the intervention is in A equals $Q_{x'}(A - x')$. In fact, we end in A if and only if the deviation is $y - x'$ with $y \in A$. This happens with probability $Q_{x'}(A - x')$. Thus, the distribution of the population size just after the next intervention is $Q_{\phi_{\Delta t}(x)}(\cdot - \phi_{\Delta t}(x))$. The intervention may move the state out of the set S , on which $(\phi_t)_{t \geq 0}$ is defined, *a priori*. This point will get particular attention later on.

Now, if the population size just after the n -th intervention would have distribution μ , concentrated on S , then the distribution just after the $(n + 1)$ - intervention equals $P\mu$ given by

$$(P\mu)(A) := \int_S Q_{\phi(x)}(A - \phi(x)) d\mu(x), \quad (1.1)$$

for any μ in the set $\mathcal{P}(S)$ of Borel probability measures on S and $A \subseteq X$ Borel where $\phi = \phi_{\Delta t}$. Note that $P\mu$ is defined as a measure on X . The operator P is a Markov operator.

We shall prove the existence of a region B^* within X containing the equilibrium x^* such that P maps $\mathcal{P}(B^*)$ into itself. Hence, P yields a Markov operator on B^* . We provide conditions under which P has a unique ergodic invariant measure μ^* , which is asymptotically stable, that is $P^n \delta_x \rightarrow \mu^*$ as $n \rightarrow \infty$ for all $x \in B$.

1.3 Piecewise deterministic Markov processes

As an alternative to our approach presented in Section 1.2, one could describe the model by a Markov chain defined (informally) by

$$X_{n+1} := \phi(X_n) + Y_n, \quad Y_n \sim Q_{\phi(X_n)}, \quad (1.2)$$

when X_0 is taken in a neighbourhood of a stable fixed point x^* of ϕ .

Piecewise deterministic Markov processes (PDMPs) are a family of Markov processes involving deterministic motion (evolution) perturbed by random jumps. Davis [20] was the first to introduce the PDMPs in the literature as a general class of stochastic models complementing the well-studied class of diffusion processes. Our presentation of the PDMP follows [16], except for excluding the case that the deterministic flow can hit the boundary.

Let $S \subset \mathbb{R}^n$ be a closed subset with non-empty interior. Assume that a semi-group $(\phi_t)_{t \geq 0}$ of locally Lipschitz maps $\phi_t : S \rightarrow S$ is given, i.e. a dynamical system in S in continuous time. A jump rate $\lambda : S \rightarrow \mathbb{R}_+$ is given, measurable such that for each $x \in S$, there exists $\tau_x > 0$ such that

$$\int_0^{\tau_x} \lambda(\phi_s(x)) ds < \infty. \quad (1.3)$$

Moreover, a map $x \mapsto Q'_x : S \rightarrow \mathcal{P}(S)$ is assumed such that $x \mapsto Q'_x(E)$ is measurable for all Borel sets $E \subset S$. This is equivalent to $x \mapsto Q'_x$ being Bochner measurable as map $S \rightarrow \mathcal{S}_{BL}$ ([90, Proposition 3.2.4]). Put

$$\Delta(x, t) := \int_0^t \lambda(\phi_s(x)) ds, \quad x \in S, \quad t \geq 0.$$

The above mentioned data yield a continuous time process X_t on S starting from $x \in S$ as follows (see for details [16]): Take a random variable T_1 such that it has cumulative distribution function corresponding to $P_x(T_1 > t) = e^{-\Delta(x, t)}$.

If $\lim_{t \rightarrow \infty} \Delta(x, t)$ exists, then T_1 may be $+\infty$ with positive probability. Then put $X_t = \phi_t(x)$ for all t .

Otherwise, select independently X_1 having distribution $Q'_{\phi_{T_1}(x)}$. Now, starting afresh from X_1 we select the next interjump time $T_2 - T_1$ and new location X_2 similarly. Thus, we arrive at a sequence $\{T_n\}_{n \in \mathbb{N}}$ of intervention times and states

$\{X_n\}_{n \in \mathbb{N}}$ of the system just after intervention.

The continuous-time process is defined as

$$X_t := \phi_{t-T_n}(X_n) \quad \text{if } T_n < t \leq T_{n+1}$$

with the convention that $T_0 = 0$ and $X_0 = x$.

Note that in our model as explained in Section 1.2 Q_x represents the additive jump instead of the relocation Q'_x used above. Note also that the presented framework for PDMPs, does not support deterministic times of interventions. Moreover, the assumptions exclude that the interjump time is bounded below, away from 0. The set-up does not seem limited to domains in $\mathbb{R}^n$. However, it is not clear to what extent the results obtained in [16] extend to general Polish spaces, non-locally compact ones in particular.

According to [16], the results developed in [62, 87] and related works would be hard to apply –from practical point of view– for PDMPs, because the transition semigroup of the PDMP as well as its associated resolvent kernel cannot be explicitly calculated from its local characteristics: the flow, the jump rate and the transition measure.

Related concepts of PDMP appear e.g. in [50] and [75].

1.4 Further mathematical motivation

The applications in population biology and related problems such as mentioned in Section 1.1 form one motivation to study the type of stochastic processes that are the subject of this thesis. Purely mathematical there are further motivating questions.

First, there is the question of ‘*persistence of stability*’. It may be –as is often the case– that the deterministic system without the random interventions has asymptotically stable steady states. Addition of random interventions may have a destabilizing effect. An important mathematical problem is to provide conditions (as weak as possible) that allow to conclude that the resulting stochastic process is still (asymptotically) stable in a suitable probabilistic sense. It may be, e.g. that a Markov operator associated to the stochastic system has an invariant measure, connected to the deterministic steady state, such that any measure ‘close to it’ will be ‘attracted’ by this invariant measure. This problem set in the generality of possibly non-locally compact Polish state space will be the main focus of Chapter 4.

Secondly, it may also be that the random perturbations have a destabilizing effect in points of the domain of attraction of the deterministic stable steady state. The stochastic process starting in this part may move away from the steady state with some positive probability. In Chapter 3 this is shown in the setting of a sampling or harvesting model, where for some initial states (forming an open set, so ‘quite large’), the population may become extinct with positive probability. The work is further motivated by the fact that fixed time interventions do not fit readily in the PDMP–framework. Also non–locally compact Polish spaces, e.g. separable subsets of Banach spaces, do not fit in the latter mentioned framework as it has been developed up till now.

Also in the theory of stochastic differential equations the problem of persistence of stability has been studied extensively, with the generality of the random perturbations gradually increasing from additive white noise to noise generated by Lévy process with state dependent noise coefficient, see, e.g. [73] and references therein.

The investigations in this thesis may be seen as a next step, in which also the shape of the noise distribution may be state dependent. We restrict ourselves here to impulsive perturbations.

1.5 The analytic approach

The analytic approach as mentioned in Section 1.2 is essentially to study the Markov chain through analytic properties of the associated Markov operator, viewed as an operator on the cone of positive finite Borel measures, equipped with an appropriate topology.

Examples of such properties that we employ here are: continuity of particular associated maps, non-expansiveness for natural metrics on measures, or the equicontinuity of the family of iterates of the operator. Moreover, we use Banach lattice techniques where appropriate. This approach follows the Polish school on iterated function systems founded by Lasota and further developed with his coworkers, among others Myjak and Szarek [56, 57, 58, 82, 83, 84]. It is further influenced by Lasota’s view on dynamical systems and probability theory [54] and applications [55].

1.6 Outline of thesis

In **Chapter 2** we provide a concise summary of concepts and results on Markov operators. We start by defining the dual bounded Lipschitz norm and the Fortet–Mourier norm of a signed finite measure. We discuss the Banach lattice structure of the vector space of finite measures. Then we thoroughly discuss some fundamentals of Markov operators, like regularity concepts, in which we present and connect most of the known concepts like regular Markov operator, Markov–Feller, strong Feller, ultra Feller, regular Markov operator with the e-property, nonexpansive Markov operator and the relation between these concepts. We also present the definitions of invariant and ergodic measures. Concepts like overlapping supports and strictly overlapping supports are also presented. The relation between the support of $P\delta_x$ and the support of $P\mu$ is also discussed. Moreover, we obtain the equivalence of uniform and global stability of ergodic measures for Markov-Feller operators on compact spaces that satisfy the e-property to a novel condition involving the successive iterates of these operators. It is shown that tracing the change of supports of measures under application of an associated Markov operator suffices to establish a lower bound condition. A separate section of this chapter is devoted to emphasize the implications of lower bound estimates, such as existence and asymptotic stability of invariant measures.

Chapter 3 is devoted to the study of a Markov operator corresponding to a deterministic system with impulsive random intervention. The system is rather natural - it is composed of a logistic map disturbed by interventions that occur regularly in time. The interventions may be described as adding a random variable with distribution depending on the state just before intervention. The main assumption crucial in our analysis is that the random variables are independent and distributed according to some density with respect to Lebesgue measure that is allowed to vary with the state of intervention in a continuous manner. Thanks to this assumption, to obtain ergodicity, it is possible to make use of the classical theory due to Meyn and Tweedie. The studied system is inspired by the sampling or harvesting model that we explained in Section 1.1. The main result says the system possesses two invariant ergodic measures. We also discuss in detail the extinction problem.

In **Chapter 4** we study map iterations in subsets of Banach spaces that are randomly perturbed. The law of the latter change in state is allowed to depend on state. It is shown that in the vicinity of a deterministic stable fixed point of the map iteration, asymptotic stability of the ergodic measure of the Markov

chain defined by the perturbed system holds, which is regarded as ‘*persistence of stability*’.

Chapter 5 develops tools rather than applications. We investigate how a set of continuous functions that are defined on different intervals, with values in a fixed metric space may be metrized. We present metrics that are inspired by the Skorohod metric. In particular, we discuss what conditions must be satisfied in these constructions of such metrics such that the resulting metric space is Polish, i.e. separable and complete. We consider such metric spaces because non-equidistant intervention times do not yield –as in Chapters 3 and 4– one fixed Markov operator. We use state spaces consisting of functions on different intervals to tackle this problem. In this chapter we also define a metric space where elements are sequences of functions defined on different intervals.

In the last chapter, **Chapter 6**, we give an outlook on further research which extends the investigations of Chapters 3 and 4 to cover the case of non-equidistant intervention times using as state space the newly constructed metric space from Chapter 5.

Three of the chapters are based on papers:

- Chapter 3 is based on the paper ‘*Ergodicity and stability of a dynamical system perturbed by impulsive random interventions*’ [3] (with minor modifications), which is joint work with Sander Hille and Onno van Gaans, and has been published in Journal of Mathematical Analysis and Applications.
- Chapter 4 is based on the paper ‘*Persistence of stability for equilibria of map iterations in Banach spaces under small random perturbations*’ [5] (with minor modifications), which is joint work with Sander Hille and Onno van Gaans, and which has been submitted.
- Chapter 5 is based on the paper ‘*On Metrization of unions of function spaces on different intervals*’ [4] (with minor modifications), which is joint work with Sander Hille and Onno van Gaans, and which has been published in Journal of the Australian Mathematical Society .

Chapter 2

Results on Markov operators on measures

In this chapter we collect basic definitions and results on spaces of measures and Markov operators on measures. Markov operators on spaces of L^1 functions have been studied extensively, in particular from the functional analytic view point of strongly continuous, positive linear semigroups in ordered Banach spaces, see e.g. [29, 54]. Here we focus on Markov operators defined on (probability) measures, as arising naturally in the context of probability theory, the theory of Markov processes in particular.

2.1 Norms and topologies on spaces of measures

Let S be a Polish space and fix a metric d on S that metrizes the topology of S and turns S into a complete separable metric space. On $\text{BL}(S)$ we can introduce two norms, $\|\cdot\|_{\text{BL}}$ and $\|\cdot\|_{\max}$ that make it a Banach space. $\|\cdot\|_{\text{BL}}$ was defined in the notation section, while

$$\|f\|_{\max} := \max\{|f|_{\text{Lip}}, \|f\|_{\infty}\}.$$

$\mathcal{M}(S)$ embeds into $\text{BL}(S)^*$ [26, Lemma 6]). The restriction of the dual norm in $\text{BL}(S)^*$ to $\mathcal{M}(S)$, for each of the norms on $\text{BL}(S)$, yields two norms on $\mathcal{M}(S)$:

$$\|\mu\|_{BL}^* := \sup_{f \in \text{BL}(S), \|f\|_{BL} \leq 1} \left| \int f d(\mu) \right| = \sup_{f \in \text{BL}(S)} \frac{|\langle f, \mu \rangle|}{\|f\|_{BL}},$$

and the *Fortet–Mourier norm* given by the formula

$$\|\mu\|_{FM} = \sup_{f \in \text{BL}(S), \|f\|_{\max} \leq 1} |\langle f, \mu \rangle|.$$

One has $|\langle f, \mu \rangle| \leq \|\mu\|_{BL}^* \cdot \|f\|_{BL}$ and similar for $\|\cdot\|_{FM}$ and $\|\cdot\|_{\max}$. If no topology on $\mathcal{M}^+(S)$ or $\mathcal{P}(S)$ is explicitly specified, we will consider the topology induced by $\|\cdot\|_{BL}^*$.

Remark 2.1.1. Note the difference between the $\|\cdot\|_{BL}^*$ norm and $\|\cdot\|_{FM}$ norm where in the first we take the sup $|\langle f, \mu \rangle|$ over all $f \in \text{BL}(S)$ such that $\{|f|_{\text{Lip}} + \|f\|_{\infty}\} \leq 1$ where in the latter the supremum is taken over the larger set of all $f \in \text{BL}(S)$ such that $\max\{|f|_{\text{Lip}}, \|f\|_{\infty}\} \leq 1$. Note also that $\|\cdot\|_{BL}^*$ and $\|\cdot\|_{FM}$ are equivalent norms. More specifically, for any $\mu \in \mathcal{M}^+(S)$,

$$\|\mu\|_{BL}^* \leq \|\mu\|_{FM} \leq 2 \|\mu\|_{BL}^*.$$

We denote by $\mathcal{S}_{BL}$ the closure of D in $\text{BL}(S)^*$, where $D := \text{span}\{\delta_x : x \in S\}$. $\mathcal{S}_{BL}$ equals the closure of $\mathcal{M}(S)$ in $\text{BL}(S)^*$ [46]. $\mathcal{S}_{BL}^+$ denotes the the closure of the positive span of $\{\delta_x : x \in S\}$ in $\text{BL}(S)^*$, thus, $\mathcal{S}_{BL}^+ = \mathcal{M}^+(S)_{BL}$. One says that $\mu_n \in \mathcal{M}^+(S)$ converges weakly to $\mu \in \mathcal{M}^+(S)$ if $\langle f, \mu_n \rangle \rightarrow \langle f, \mu \rangle$ for all $f \in C_b(S)$. The $\sigma(\mathcal{M}(S), C_b(S))$ -weak topology restricted to $\mathcal{M}^+(S)$ coincides with the restriction of the $\|\cdot\|_{BL}^*$ or $\|\cdot\|_{FM}$ -norm topology to $\mathcal{M}^+(S)$ [26, 27].

2.2 The Banach lattice structure of finite measures

Let S be a set with a σ -algebra Σ . $\mathcal{M}(S)$ is an ordered vector space: $\mu \leq \nu$ if and only if $\mu(A) \leq \nu(A)$ for all $A \in \Sigma$. Moreover, $\mathcal{M}(S)$ is a vector lattice, $\|\cdot\|_{TV}$ is vector lattice norm on it, and $(\mathcal{M}(S), \|\cdot\|_{TV})$ is a Banach lattice. If $\mu_1, \mu_2 \in \mathcal{M}(S)$, then (cf. Bogachev, [11, p. 176 and p. 279])

$$(\mu_1 \wedge \mu_2)(A) = \inf \{\mu_1(B) + \mu_2(A \setminus B) : B \subset A \text{ measurable}\}. \quad (2.1)$$

$$(\mu_1 \vee \mu_2)(A) = \sup \{\mu_1(B) + \mu_2(A \setminus B) : B \subset A \text{ measurable}\}.$$

The Jordan decomposition $\mu = \mu^+ - \mu^-$ relates to the lattice structure: $\mu^+ = \mu \vee 0$ and $\mu^- = -(\mu \wedge 0)$. Moreover, $|\mu| = \mu^+ + \mu^-$. The total variation norm has

different expressions. We shall use

$$\|\mu\|_{TV} = |\mu|(S) = \sup_{B \in \mathcal{B}(S)} \mu(B) - \inf_{B \in \mathcal{B}(S)} \mu(B) = \sup_{B \in \mathcal{B}(S)} |\mu(B)|. \quad (2.2)$$

Also, if $\mu_1, \mu_2 \in \mathcal{M}^+(S)$, then $\|\mu_1 + \mu_2\|_{TV} = \|\mu_1\|_{TV} + \|\mu_2\|_{TV}$, moreover, μ_1 and μ_2 are mutually singular if and only if $\mu_1 \wedge \mu_2 = 0$.

Lemma 2.2.1. *For any $\mu, \nu \in \mathcal{M}^+(S)$ with $\mu(S) = \nu(S)$ there exist mutually singular measures $\mu', \nu' \in \mathcal{M}^+(S)$ such that $\mu - \nu = \mu' - \nu'$, and*

$$\|\mu'\|_{TV} = \|\nu'\|_{TV} = \frac{\|\mu - \nu\|_{TV}}{2}.$$

Proof. Take $\mu' = (\mu - \nu)^+$ and $\nu' = (\mu - \nu)^-$. Then $\mu' - \nu' = \mu - \nu$ and $\mu' + \nu' = |\mu - \nu|$. Also, $\mu' \wedge \nu' = 0$. Now $\mu'(S) - \nu'(S) = \mu(S) - \nu(S) = 0$, so $\mu'(S) = \nu'(S)$, that is $\|\mu'\|_{TV} = \|\nu'\|_{TV}$. Moreover, $\|\mu' + \nu'\|_{TV} = \|\mu - \nu\|_{TV}$ and $\|\mu' + \nu'\|_{TV} = \|\mu'\|_{TV} + \|\nu'\|_{TV}$, so $\|\mu'\|_{TV} + \|\nu'\|_{TV} = \|\mu - \nu\|_{TV}$. Hence, $\|\mu'\|_{TV} = \|\nu'\|_{TV} = \|\mu - \nu\|_{TV} / 2$. $\square$

2.3 Fundamentals of Markov operators

Let us firstly recall some facts on Markov operators. We use the notation that appear on page viii. An operator $P : \mathcal{M}^+(S) \rightarrow \mathcal{M}^+(S)$ is called a *Markov operator on S* if $P\mu(S) = \mu(S)$, for all $\mu \in \mathcal{M}^+(S)$, and

$$P(\lambda_1 \mu_1 + \lambda_2 \mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$$

for $\lambda_1, \lambda_2 \geq 0$ and $\mu_1, \mu_2 \in \mathcal{M}^+(S)$ (cf. [83, 84, 90]).

2.3.1 Regularity concepts

A Markov operator P is said to be *regular* if there exists an operator U which maps $\text{BM}(S)$ into itself such that $\langle Uf, \mu \rangle = \langle f, P\mu \rangle$, for all $f \in \text{BM}(S)$ and $\mu \in \mathcal{M}^+(S)$. In that case U is unique and is called the *dual* of P . Note that $U : \text{BM}(S) \rightarrow \text{BM}(S)$ is a positive bounded linear operator when $\text{BM}(S)$ is equipped with $\|\cdot\|_\infty$ -norm, $\|U\| \leq 1$ and $U\mathbf{1} = \mathbf{1}$. A regular Markov operator on S is *Markov-Feller* if its dual maps $C_b(S)$ into itself. It is *strong Feller* if its dual maps $\text{BM}(S)$ into $C_b(S)$. Finally, a Markov operator P is *ultra-Feller* if the map $x \rightarrow P\delta_x$ from S to $\mathcal{M}(S)_{TV}$ is continuous (c.f. [79, 90]).

Remark 2.3.1. If P and Q are two Markov operators on $\mathcal{M}^+(S)$ that are both strong Feller, then the product PQ is ultra-Feller. In particular, if P is strong Feller, then P^2 is ultra-Feller ([22, Theorem IX.18] or [79]). There exist strong Feller operators which are not ultra-Feller, even if S is compact.

Let X and Y be two metric spaces, and F a family of functions from X to Y . The family F is equicontinuous at a point $x_0 \in X$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_0), f(x)) < \epsilon$ for all $f \in F$ and all x such that $d(x_0, x) < \delta$. The family is *equicontinuous* if it is equicontinuous at each point of X . Let $P : \mathcal{M}^+(S) \rightarrow \mathcal{M}^+(S)$ be a regular Markov operator. Following Lasota, Szarek and others (cf. [86, 90]) we say that P has the *e-property* if $(U^n f)_{n \in \mathbb{N}_0}$ is equicontinuous for every $f \in \text{BL}(S)$. Note that having the e-property depends on the choice of metric; the same P may have the e-property for one metric, but not for another that yields the same metric topology. Moreover, P has the e-property if and only if the family $(P^n)_{n \in \mathbb{N}}$ is an equicontinuous family in $C_b(\mathcal{S}_{\text{BL}}^+, \mathcal{S}_{\text{BL}}^+)$. (cf. [90, Theorem 7.2.2., p. 139]).

We follow Lasota and Szarek and call a Markov operator P *nonexpansive* if

$$\|P\mu - P\nu\|_{FM} \leq \|\mu - \nu\|_{FM},$$

for all $\mu, \nu \in \mathcal{M}^+(S)$.

Remark 2.3.2. Every regular Markov operator with the e-property is Markov-Feller. This follows directly from ([90, Lemma 7.2.1., p. 138]), which says: If P is a regular Markov operator with dual U such that $Uf \in C_b(S)$ for every $f \in \text{BL}(S)$, then P is Markov-Feller.

The ultra-Feller property implies the strong Feller property which implies the e-property which in turn implies the Markov-Feller property. It is not difficult to check that a nonexpansive Markov operator has the e-property (hence, it is Markov-Feller, see Remark 2.3.2). The following is a slight reformulation of [90, Proposition 3.3.2.], together with a characterization of the Feller property in terms of lower semicontinuity (see also [63, Chapter 6], where the latter is taken as definition). Recall that a function $f : S \rightarrow \mathbb{R}$ is *lower semicontinuous* if $\{x : f(x) > a\}$ is open for each $a \in \mathbb{R}$.

Proposition 2.3.3. *Let P be a Markov operator on a complete separable metric space (S, d) . Then the following are equivalent:*

- i) P is Markov-Feller.
- ii) $P : \mathcal{M}^+(S)_{\text{BL}} \rightarrow \mathcal{M}^+(S)_{\text{BL}}$ is continuous.

iii) The map $x \mapsto P\delta_x : S \rightarrow \mathcal{M}^+(S)_{BL}$ is continuous.

iv) For each $O \subset S$ open, $x \mapsto P\delta_x(O) : S \rightarrow \mathbb{R}_+$ is lower semicontinuous.

Proof. i) $\Rightarrow$ ii): Let $\mu_n, \mu \in \mathcal{M}^+(S)_{BL}$ be such that $\|\mu_n - \mu\|_{BL}^* \rightarrow 0$. This is equivalent to $\langle f, \mu_n \rangle \rightarrow \langle f, \mu \rangle$ for every $f \in C_b(S)$. Thus,

$$\langle f, P\mu_n \rangle = \langle Uf, \mu_n \rangle \rightarrow \langle Uf, \mu \rangle = \langle f, P\mu \rangle,$$

for every $f \in C_b(S)$. Thus, by the equivalence of the convergence in the weak topology and convergence in $\|\cdot\|_{BL}^*$ -norm on $\mathcal{M}^+(S)$, $\|P\mu_n - P\mu\|_{BL}^* \rightarrow 0$.

ii) $\Rightarrow$ iii): Since $x \mapsto \delta_x : S \rightarrow \mathcal{M}^+(S)_{BL}$ is Lipschitz, and the map $P : \mathcal{M}^+(S)_{BL} \rightarrow \mathcal{M}^+(S)_{BL}$ is continuous, the composition map

$$x \mapsto \delta_x \mapsto P\delta_x : S \rightarrow \mathcal{M}^+(S)_{BL}$$

is continuous.

iii) $\Rightarrow$ i): Let $f \in C_b(S)$, then $x \mapsto Uf(x) = \langle f, P\delta_x \rangle$ is continuous.

iii) $\Rightarrow$ iv): This is a consequence of the Portmanteau or Alexandrov's Theorem.

iv) $\Rightarrow$ iii) Let $x_0 \in S$, $(x_n) \subset S$ be such that $x_n \rightarrow x_0$. Let $\epsilon > 0$, and $U \subset S$ open. Since $x \mapsto P\delta_x(U)$ is lower semicontinuous at x_0 , there exists an open neighbourhood V_{x_0} of x_0 such that $P\delta_x(U) \geq P\delta_{x_0}(U) - \epsilon$ for all $x \in V_{x_0}$. So there exists N such that $x_n \in V_{x_0}$ for all $n \geq N$. Consequently,

$$\liminf_{n \rightarrow \infty} P\delta_{x_n}(U) \geq P\delta_{x_0}(U) - \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\liminf_{n \rightarrow \infty} P\delta_{x_n}(U) \geq P\delta_{x_0}(U).$$

The Portmanteau Theorem yields that $P\delta_{x_n} \rightarrow P\delta_{x_0}$ weakly, hence $x \mapsto P\delta_x : S \rightarrow \mathcal{M}^+(S)_{BL}$ is continuous. $\square$

Note also that P is strong Feller if and only if $x \mapsto P\delta_x(U)$ is continuous for every measurable set U . Also, we will use the following results, which are Corollary 3.2.8 on page 48 and Corollary 3.2.6 on page 47 in [90],

Remark 2.3.4. For any $\mu \in \mathcal{M}^+(S)$, $\mu = \int_S \delta_x d\mu(x)$, as a Bochner integral in $\mathcal{S}_{BL}$.

It follows that, for a regular Markov operator P ,

$$P\mu = P \left(\int_S \delta_x d\mu(x) \right) = \int_S P\delta_x d\mu(x). \quad (2.3)$$

Remark 2.3.5. Assume that $p : \Omega \rightarrow \mathcal{M}^+(S)$ is strongly measurable as a map into $\mathcal{S}_{BL}$, such that $v := \int_\Omega p(\omega) d\mu(\omega)$ exists in $\mathcal{S}_{BL}$ for $f \in \text{BM}(S)$. Then

$$\int_S f dv = \int_\Omega \langle f, p(\omega) \rangle d\mu(\omega). \quad (2.4)$$

2.3.2 Invariant measures

A measure $\mu \in \mathcal{M}^+(S)$ is called *invariant* under P if $P\mu = \mu$. We will require that an invariant measure is always a probability measure. A Borel set E is *P-invariant* if for all $x \in E$, the measure $P\delta_x$ is concentrated on E (i.e. $P\delta_x(E) = 1$). In the literature many different definitions of the concept of ergodic measure exist, see e.g. [76]. We follow [44] and [95, p. 18] and say that: $\mu \in \mathcal{P}(S)$ is *ergodic* with respect to P if $\mu(E) = 0$ or $\mu(E) = 1$ for every P -invariant Borel set E . The ergodic measures (in this sense) are the extreme points of the convex set of invariant probability measures. In case that the set of ergodic measures is finite, a measure is invariant if and only if it is a convex combination of ergodic measures (cf. [2], Theorem 19.25).

The following notion is defined by Szarek in [84]: A Markov operator P *overlaps supports* if for every $x, y \in S$ there exists $n_0 \in \mathbb{N}$, such that

$$\text{supp}(P^{n_0}\delta_x) \cap \text{supp}(P^{n_0}\delta_y) \neq \emptyset.$$

We say that P *strictly overlaps supports* if for every $x, y \in S$ there exists $n_0 \in \mathbb{N}$, such that the intersection contains a non-empty open set, say U , i.e.

$$\text{supp}(P^{n_0}\delta_x) \cap \text{supp}(P^{n_0}\delta_y) \supseteq U.$$

For a particular class of Markov operators, uniqueness of invariant probability measures may be established by employing the following result, initially proved in [84] for P with the e-property that overlaps support and extended in [90] to P having the weaker Cesàro e-property and a different overlap condition. ([90, Theorem 7.4.6. p. 156]):

Theorem 2.3.6. *Let P be a regular Markov operator on (S, d) with the (Cesàro) e-property. Suppose that for every $x, y \in S$ there exists a $z \in S$, such that for every $r > 0$ there are $n_1, n_2 \in \mathbb{N}$ such that*

$$P^{n_1} \delta_x (B(z, r)) > 0 \text{ and } P^{n_2} \delta_y (B(z, r)) > 0.$$

Then P has at most one invariant probability measure on S .

2.3.3 Dynamics of supports

Define the *support map* associated to P by

$$\psi_p(E) := \overline{\bigcup_{x \in E} \text{supp}(P\delta_x)}, \quad E \subseteq S. \quad (2.5)$$

Lasota and Myjak [56] discuss properties of ψ_p in detail, in particular when P is Markov–Feller. One has the following

Proposition 2.3.7. *Let P be a regular Markov operator on S . Then*

- i) For all $\mu \in \mathcal{M}^+(S)$, $\text{supp}(P\mu) \subseteq \psi_p(\text{supp}(\mu))$.*
- ii) If P is Feller, then $\text{supp}(P\mu) = \psi_p(\text{supp}(\mu))$.*
- iii) If P is Feller and μ^* is an invariant measure for P , then $\text{supp}(\mu^*)$ is a fixed point of the map ψ_p , i.e. $\psi_p(\text{supp}(\mu^*)) = \text{supp}(\mu^*)$.*

Proof. i) Let $z \notin \psi_p(\text{supp}(\mu)) = \overline{\bigcup_{x \in \text{supp}(\mu)} \text{supp}(P\delta_x)} =: D$. Then there exists $r > 0$ such that $B := B(z, r)$ and D are disjoint. We use that

$$P\mu(B) = \int_S P\delta_y(B) \mu(dy) = \int_{\text{supp}(\mu)} P\delta_y(B) \mu(dy).$$

Since $y \in \text{supp}(\mu)$ implies $B \cap \text{supp}(P\delta_y) = \emptyset$, $P\delta_y(B) = 0$. Thus, $P\mu(B) = 0$. So, $z \notin \text{supp}(P\mu)$.

For ii) see [56, Corollary 5.1].

iii) follows immediately from ii). □

2.4 Stability of invariant measures

As Markov operators often describe transition probabilities of stochastic systems, it is essential to understand their iterates in order to understand the dynamics of the described system. A very important property is stability of invariant

measures. Our discussion includes a *new* result (Theorem 2.4.3 (iii)), which provides a novel equivalence of uniform and global stability of unique ergodic measures for regular Markov operators on compact metric spaces that satisfy the *e-property* to a condition involving the successive iterates of these operators.

It is worthy to mention that there exist various notions of asymptotic stability in the theory of Markov operators and semigroups originating from a probabilistic point of view. We advocate a view from dynamical systems theory: an invariant measure μ^* for a Markov operator P is called *locally asymptotically stable*, if there exists an open neighborhood U of μ^* in $\mathcal{P}(S)$ such that $P^n\nu \rightarrow \mu^*$ for all $\nu \in U$ as $n \rightarrow \infty$. If U can be taken equal to $\mathcal{P}(S)$ we call μ^* *globally asymptotically stable*. Recall that an invariant measure μ^* is called *asymptotically stable* on a set $B \subset S$ with $B \supset \text{supp}(\mu^*)$, if $P^n\mu \rightarrow \mu^*$ as $n \rightarrow \infty$, for all $\mu \in \mathcal{P}(S)$ with $\text{supp}(\mu) \subset B$. The latter is equivalent to $P^n\delta_x \rightarrow \mu^*$ as $n \rightarrow \infty$ for all $x \in B$. A Markov operator is called *asymptotically stable* if there exists an invariant measure $\mu^* \in \mathcal{P}(S)$ such that

$$w - \lim_{n \rightarrow \infty} P^n\mu = \mu^* \text{ for every } \mu \in \mathcal{P}(S). \quad (2.6)$$

where $w - \lim$ denotes the limit for the weak topology. Obviously, a measure μ^* satisfying (2.6) is unique. Note that P is asymptotically stable if and only if it has an invariant measure which is asymptotically stable on S . However, in general condition (2.6) doesnot imply that μ^* is invariant. Yet, this implication holds for Markov-Feller operators (see [56, p. 314]). Note that (2.6) is equivalent to $\lim_{n \rightarrow \infty} \|P^n\mu - \mu^*\|_{BL}^* = 0$, provided that S is separable and complete.

Throughout the remainder of this section let (S, d) be a complete separable metric space. The results here are of general interest for applications in the theory of Markov-Feller operators. We start with two results on Markov operators on compact metric spaces.

Proposition 2.4.1. *Let (S, d) be a compact metric space and P a Markov operator on S with the e-property. Then $(P^n)_n$ is precompact in $(C_b(\mathcal{P}(S), \mathcal{S}_{BL}), d_\infty)$.*

Proof. Let $\mathcal{F} = \{P^n : n \in \mathbb{N}\} \subset C_b(\mathcal{P}(S), \mathcal{S}_{BL})$. Since P has the e-property, $\mathcal{F}$ is equicontinuous in $C_b(\mathcal{P}(S), \mathcal{S}_{BL})$ (cf. [90, Theorem 7.2.2]). Denote $\mathcal{F}(\mu) = \{P^n\mu : \mu \in \mathcal{P}(S), n \in \mathbb{N}\} \subset \mathcal{P}(S)$, $\mu \in \mathcal{P}(S)$. As $\|P^n\mu\|_{BL}^* = \|P^n\mu\|_{TV} = 1$, $\mathcal{F}(\mu)$ is bounded in $\mathcal{S}_{BL}$. Let κ stand for the Kuratowski measure of non-

compactness. According to Ambrosetti ([8], or [23, Proposition 1.2.10, p17]),

$$\kappa(\mathcal{F}) = \sup_{\mu \in \mathcal{P}(S)} \kappa_{\mathcal{S}_{BL}}(\mathcal{F}(\mu)).$$

Since $\mathcal{F}(\mu) \subset \mathcal{P}(S)$ and $\mathcal{P}(S)$ is compact, $\kappa_{\mathcal{S}_{BL}}(\mathcal{F}(\mu)) \leq \kappa_{\mathcal{S}_{BL}}(\mathcal{P}(S)) = 0$. Hence, $\kappa(\mathcal{F}) = 0$. Thus, $\mathcal{F}$ is precompact in $(C_b(\mathcal{P}(S), \mathcal{S}_{BL}), d_\infty)$. $\square$

Lemma 2.4.2. *If P is a Markov-Feller operator on S with the e-property and Q is a Markov-Feller operator on S such that there exists an open set $U \subset \mathcal{P}(S)$ such that $P^n \mu \rightarrow Q\mu$ for each $\mu \in U$, then $P^n \mu \rightarrow Q\mu$ uniformly on each compact subset of U .*

Proof. Let $K \subset \mathcal{P}(S)$ be compact and contained in U and let $\epsilon > 0$. First, observe that Q is a Markov-Feller operator if and only if $Q : \mathcal{S}_{BL}^+ \rightarrow \mathcal{S}_{BL}^+$ is continuous (Proposition 2.3.3). Thus, for each $\mu \in K$ there exists V_μ open, contained in U , such that $\|Q\mu - Q\nu\|_{BL}^* < \epsilon$ for all $\nu \in V_\mu$. Since P has the e-property, for all $\mu \in K$, there exists U_μ open in $\mathcal{S}_{BL}^+$, with $\mu \in U_\mu$ such that $\|P^n \nu - P^n \mu\|_{BL}^* < \epsilon$ for all $\nu \in U_\mu$ and for all $n \in \mathbb{N}$. Since K is compact, there exist $\mu_1, \mu_2, \dots, \mu_k$ such that $K \subset \bigcup_{i=1}^k U_{\mu_i} \cap V_{\mu_i}$. For each j there exists N_j such that $\|P^n \mu_j - Q\mu_j\|_{BL}^* < \epsilon$ for all $n \geq N_j$. Let $N := \max\{N_1, \dots, N_k\}$. For $\nu \in K$, there exists j such that $\nu \in U_{\mu_j} \cap V_{\mu_j}$ and for all $n \geq N$

$$\|P^n \nu - Q\nu\|_{BL}^* \leq \|P^n \nu - P^n \mu_j\|_{BL}^* + \|P^n \mu_j - Q\mu_j\|_{BL}^* + \|Q\mu_j - Q\nu\|_{BL}^* < 3\epsilon.$$

$\square$

Theorem 2.4.3 below asserts that the concepts of global and uniform asymptotic stability coincide on compact spaces for Markov operators having the e-property. It also provides a remarkable technical condition that we discovered to be equivalent to global and uniform stability in the compact case. The proof of Theorem 2.4.3 relies on Lemma 2.4.2 and the auxiliary Proposition 2.4.1, which we believe to be of independent interest.

Theorem 2.4.3. *Let (S, d) be a compact metric space and let P be a Markov operator on S with the e-property. If P has a unique ergodic measure μ^* , then the following are equivalent:*

- i) μ^* is globally asymptotically stable.
- ii) μ^* is uniformly stable, i.e. $\lim_{n \rightarrow \infty} P^n \mu = \mu^*$ uniformly for all $\mu \in \mathcal{P}(S)$.

iii) For all $\mu \in \mathcal{P}(S)$, $\|P^{n+1}\mu - P^n\mu\|_{BL}^* \rightarrow 0$ as $n \rightarrow \infty$.

Proof. For the implication iii) implies i), we have, since S is compact, that Proposition 2.4.1 yields that $(P^n) \subset C_b(\mathcal{P}(S), \mathcal{S}_{BL})$ is precompact. Thus, there exists a subsequence (n_k) and an operator $Q \in C_b(\mathcal{P}(S), \mathcal{S}_{BL})$ such that $P^{n_k} \rightarrow Q$ with respect to d_∞ . Hence, by Lemma 2.4.2, $P^{n_k}\mu \rightarrow Q\mu$ uniformly for all $\mu \in \mathcal{P}(S)$. Let $\mu \in \mathcal{P}(S)$. Because P is Markov-Feller, hence continuous on $\mathcal{S}_{BL}^+$, condition iii) implies

$$\|(P - I)Q\mu\|_{BL}^* = \lim_{k \rightarrow \infty} \|(P - I)P^{n_k}\mu\|_{BL}^* = 0.$$

Hence, $Q\mu$ is an invariant measure for P . Since μ^* is the unique ergodic measure for P , $Q\mu = \mu^*$. So, $P^{n_k}\mu \rightarrow \mu^*$ uniformly. Since (P^n) is equicontinuous at μ^* , for all $\epsilon > 0$, there exists U_{μ^*} open with $\mu^* \in U_{\mu^*}$ such that $\|P^n\nu - \mu^*\|_{BL}^* = \|P^n\nu - P^n\mu^*\|_{BL}^* < \epsilon$ for all n and for all $\nu \in U_{\mu^*}$. Thus, by uniform convergence of $P^{n_k}\mu$ to μ^* , there exists $N \in \mathbb{N}$ such that

$$\|P^n P^{n_k}\mu - P^n\mu^*\|_{BL}^* < \epsilon \text{ for all } n, \text{ for all } k \geq N, \text{ and for all } \mu \in U_{\mu^*}. \quad (2.7)$$

Also observe that iii) implies that, for all m , $\|P^n P^m\mu - P^n\mu\|_{BL}^* \rightarrow 0$ as $n \rightarrow \infty$. Thus, there exists $N' \in \mathbb{N}$ such that

$$\begin{aligned} \|P^n\mu - \mu^*\|_{BL}^* &\leq \|P^n\mu - P^n P^{N'}\mu\|_{BL}^* + \|P^n P^{N'}\mu - \mu^*\|_{BL}^* \\ &< 2\epsilon \text{ for all } n \geq N'. \end{aligned}$$

Hence, $P^n\mu \rightarrow \mu^*$ as $n \rightarrow \infty$ for all $\mu \in \mathcal{P}(S)$ which establishes i). For the implication i) implies ii) we have $P^n\mu \rightarrow \mu^*$ for every $\mu^* \in \mathcal{P}(S)$. Let Q be the Markov-Feller operator on S defined by $Q\mu = \mu^*$, $\mu \in \mathcal{P}(S)$. Then by Lemma 2.4.2, $P^n\mu \rightarrow \mu^*$ uniformly.

For the implication ii) implies iii), we have that ii) implies for any $\mu \in \mathcal{P}(S)$ that $(P^n\mu)_n$ is a Cauchy sequence, so there exists N such that $\|P^n P^m\mu - P^n\mu\|_{BL}^* < \epsilon$ for all $n \geq N$ and $m \geq 0$. $\square$

2.5 Characterizations through lower bound estimates

In this section we discuss relations between so-called lower bound type estimates and existence and asymptotic stability of invariant measures of Markov operators.

In particular, one can employ equicontinuity conditions on the Markov operator (the ‘e-property’) together with a form of the general lower bound technique to establish asymptotic stability of an ergodic measure associated to a Markov-Feller operator linked to the model. It fits into a series of approaches to studying ergodic properties of Markov operators, semigroups and the associated Markov processes. Doeblin, [25] employed a *uniform* lower-bound condition for transition probabilities. According to Lasota and Szarek [58], Markov essentially used a version of the technique already in the years 1906-1908. Harris, [41] established a generalization of Doeblin’s condition. A similar approach was used recently in Hairer and Mattingly [39]. Szarek and co-authors substantially extended the applicability of the technique to Markov operators and semigroups with the e-property on Polish spaces, (c.f. [85]). Worm [90], subsequently weakened the conditions in the results for Markov semigroups of the latter: e.g. the e-property could be replaced by eventual e-property, while the lower-bound condition itself could be simplified. The results that will be presented in the following sections are discrete-time versions of results for continuous time Markov semigroups presented in [90], Chapter 7. The latter results have not (yet) been published unfortunately. The proofs of the discrete-time versions follow closely those of the continuous time results. We give a complete proof of the discrete-time version of the local asymptotic stability result, Theorem 2.5.4, only. The existence result, Theorem 2.5.1, can be checked to hold, the asymptotic stability result Theorem 2.5.11 for the Markov operator is much more involved. The purpose of bringing these three theorems together in this section is to raise awareness for the implications of lower-bound estimates of the form appearing in the mentioned theorems. They are characterizations of existence and local stability of invariant measures and asymptotic stability of the Markov operator both in discrete and continuous time. Therefore, they deserve more attention. They are not used however in the remainder of the thesis, as we employ different, particular results implying asymptotic stability of Markov operators— for $\|\cdot\|_{TV}$ and $\|\cdot\|_{FM}$ in Chapters 3 and 4. Namely, Theorem 3.3.4, which is a version of Orey’s Theorem, and Theorem 4.6.2, which is due to Szarek [82]. Because of this independence, we felt free to provide a detailed proof of the discrete-time local asymptotic stability result, Theorem 2.5.4, only, because Markov operators (with the e-property) generally will have more than one ergodic measure.

2.5.1 Existence of invariant measures

Define the Cesaro averages $P^{(n)}$ of P^n as $P^{(n)} := \frac{1}{n} \sum_{k=0}^{n-1} P^k$. The following theorem is an adaptation to a discrete time setting of [90, Theorem 7.4.4.]. Its proof mimics that of the continuous time version.

Theorem 2.5.1. *Let P be a Markov operator on a complete separable metric space (S, d) with the e-property. Then there exist invariant measures for P if and only if there exist $x, z \in S$ such that for every $r > 0$,*

$$\liminf_{n \rightarrow \infty} P^{(n)} \delta_x (B(z, r)) > 0.$$

Note that the lower-bound condition mentioned in the above theorem is necessary and sufficient for existence.

2.5.2 Asymptotic Stability of ergodic measures

We singled out particular partial results in the proof of the main theorem (Theorem 2.5.4) that deserve separate attention.

Define the following sets:

$$\begin{aligned} S_1 &= \{x \in S : (P^{(n)} \delta_x) \text{ is tight}\}, \\ S_2 &= \{x \in S_1 : (P^{(n)} \delta_x) \text{ converges in } \mathcal{S}_{BL}\}, \\ S_3 &= \left\{x \in S_2 : \epsilon_x = \lim_{n \rightarrow \infty} P^{(n)} \delta_x \text{ is invariant}\right\}, \end{aligned}$$

and for any P -invariant measure μ^* , let

$$S_{\mu^*} := \{x \in S : P^{(n)} \delta_x \rightarrow \mu^* \text{ as } n \rightarrow \infty\}. \quad (2.8)$$

The following result is taken from Worm [90, Lemma 5.3.2.]:

Lemma 2.5.2. *If P is a Markov-Feller operator and P has a unique invariant probability measure μ^* , then $S_1 = S_2 = S_3 = S_{\mu^*}$.*

Remark 2.5.3. If S is compact, and P is a Markov-Feller operator with a unique invariant probability measure μ^* , then $S_{\mu^*} = S$. Indeed, since $\mathcal{P}(S)$ is compact in this case, $S_1 = S$.

The lower bound result for local asymptotic stability alluded to in the introduction of this section is given next. It is the discrete time version of [90,

Theorem 7.4.11], which generalized [85, Theorem 2]. Although the idea of proof follows closely the mentioned continuous time versions, we include a full but ‘stream lined’ presentation of the proof here, in order to have a thoroughly verified discrete-time version of the result available for reference.

Theorem 2.5.4. *Let P be a Markov operator with the e-property. Assume that P has an ergodic measure μ^* and let S_{μ^*} be defined by (2.8). If there exist $x \in \text{supp}(\mu^*)$ and $z \in S$, such that*

$$\liminf_{n \rightarrow \infty} P^n \delta_x (B(z, r)) > 0 \quad (2.9)$$

for every $r > 0$, then for every $\nu_1, \nu_2 \in \mathcal{P}(S)$ with $\nu_1(S_{\mu^*}) = \nu_2(S_{\mu^*}) = 1$,

$$\lim_{n \rightarrow \infty} \|P^n \nu_1 - P^n \nu_2\|_{BL}^* = 0.$$

In particular, $P^n \nu \rightarrow \mu^*$ for every $\nu \in \mathcal{P}(S)$ such that $\nu(S_{\mu^*}) = 1$. Moreover, $z \in \text{supp}(\mu^*)$ necessarily.

Before proving Theorem 2.5.4 we derive

Corollary 2.5.5. *Let (S, d) be a compact metric space and let P be a Markov operator on S with the e-property. If P has a unique ergodic measure μ^* , then the following are equivalent:*

i) There exist $x \in \text{supp}(\mu^*)$ and $z \in S$, such that

$$\liminf_{n \rightarrow \infty} P^n \delta_x (B(z, r)) > 0$$

for every $r > 0$.

ii) μ^* is uniformly stable, i.e. $\lim_{n \rightarrow \infty} P^n \mu = \mu^*$ uniformly for all $\mu \in \mathcal{P}(S)$.

Moreover, $z \in \text{supp}(\mu^*)$ necessarily.

Proof. If i) holds, then global stability follows from Remark 2.5.3 and Theorem 2.5.4 and uniform stability follows from Theorem 2.4.3. If ii) holds, then i) follows from the Portmanteau Theorem and the fact that $z \in \text{supp}(\mu^*)$. In particular, we get for any $x \in S$ and $r > 0$, $\liminf_{n \rightarrow \infty} P^n \delta_x (B(z, r)) \geq \mu^*(B(z, r)) > 0$. $\square$

The proof of Theorem 2.5.4 relies on several lemmas.

Lemma 2.5.6. *Let P be a Markov operator that has the e-property. Assume that P has an ergodic measure μ^* . If there exist $x \in \text{supp}(\mu^*)$ and $z \in S$, such that $\liminf_{n \rightarrow \infty} P^n \delta_x(B(z, r)) > 0$ for every $r > 0$, then, for each $r > 0$ there exists $\alpha = \alpha(r) > 0$, such that*

$$\liminf_{n \rightarrow \infty} P^n \nu(B(z, r)) > \alpha$$

for every $\nu \in \mathcal{P}(S)$ satisfying $\nu(S_{\mu^*}) = 1$. (Note that α is independent of ν).

Proof. Let $r > 0$. Define $\gamma := \frac{1}{2} \liminf_{n \rightarrow \infty} P^n \delta_x(B(z, \frac{r}{2}))$. By assumption $\gamma > 0$. Let $f := [1 - \frac{2}{r} d(x, B(z, \frac{r}{2}))]^+$. Then $1_{B(z, \frac{r}{2})} \leq f \leq 1_{B(z, r)}$ and $f \in \text{BL}(S)$. By the e-property, there exists an $s > 0$ such that $|\langle P^n \delta_y - P^n \delta_x, f \rangle| < \gamma$ for every $y \in B(x, s)$ and n . Thus, for all $y \in B(x, s)$ and n ,

$$\begin{aligned} P^n \delta_y(B(z, r)) &\geq \langle P^n \delta_y, f \rangle \geq \langle P^n \delta_x, f \rangle - |\langle P^n \delta_y - P^n \delta_x, f \rangle| \\ &> P^n \delta_x(B(z, r/2)) - \gamma. \end{aligned}$$

Hence,

$$\liminf_{n \rightarrow \infty} P^n \delta_y(B(z, r)) \geq \liminf_{n \rightarrow \infty} P^n \delta_x(B(z, r/2)) - \gamma = \gamma \quad (2.10)$$

for every $y \in B(x, s)$. Set $\theta := \frac{1}{2} \mu^*(B(x, s))$. Then $\theta > 0$, since $x \in \text{supp}(\mu^*)$. Fix $\nu \in \mathcal{P}(S)$ such that $\nu(S_{\mu^*}) = 1$. Thus, $P^{(n)}\nu \rightarrow \mu^*$. By the Portmanteau Theorem,

$$\liminf_{n \rightarrow \infty} P^{(n)}\nu(B(x, s)) \geq \mu^*(B(x, s)) \geq 2\theta > 0.$$

Then there exists $n_0 > 0$ such that $P^{n_0}\nu(B(x, s)) > \theta$. Thus, by Fatou's Lemma

$$\begin{aligned} \liminf_{n \rightarrow \infty} P^n \nu(B(z, r)) &= \liminf_{n \rightarrow \infty} P^{n+n_0} \nu(B(z, r)) \\ &\geq \int_S \liminf_{n \rightarrow \infty} P^n \delta_y(B(z, r)) d[P^{n_0}\nu](y) \\ &\geq \int_{B(x, s)} \liminf_{n \rightarrow \infty} P^n \delta_y(B(z, r)) d[P^{n_0}\nu](y) > \gamma\theta, \end{aligned}$$

using (2.10). Taking $\alpha = \gamma\theta$ completes the proof. $\square$

Lemma 2.5.7. *Let P be a Markov operator and E a P -invariant Borel set in S . If $\mu_0 \in \mathcal{P}(S)$ is such that $\mu_0(E) = 1$ and there exist $n \in \mathbb{N}$, $\alpha > 0$ and $B \subset S$ measurable set in S such that $P^n \mu_0(B) > \alpha$, then there exist $\nu_1, \mu_1 \in \mathcal{P}(S)$ with*

$\text{supp}(\nu_1) \subset \overline{B}$, $\mu_1(E) = 1$ and

$$P^n \mu_0 = \alpha \nu_1 + (1 - \alpha) \mu_1.$$

Proof. Define $\alpha_1 := (P^n \mu_0)(B) > \alpha$. Denote by $(P^n \mu_0)_B$ the restriction of $P^n \mu_0$ to B , i.e. $(P^n \mu_0)_B(F) := P^n \mu_0(F \cap B)$. Clearly

$$P^n \mu_0 = \alpha \left(\frac{1}{\alpha_1} (P^n \mu_0)_B \right) + \left(P^n \mu_0 - \frac{\alpha}{\alpha_1} (P^n \mu_0)_B \right).$$

Define $\nu_1 := \frac{1}{\alpha_1} (P^n \mu_0)_B$ and $\mu_1 := \frac{1}{1-\alpha} \left(P^n \mu_0 - \frac{\alpha}{\alpha_1} (P^n \mu_0)_B \right)$. Then $\nu_1, \mu_1 \in \mathcal{P}(S)$ and $\text{supp}(\nu_1) \subset \overline{B}$, while $P^n \mu_0 = \alpha \nu_1 + (1 - \alpha) \mu_1$. Note that $\mu_1 \leq \frac{1}{1-\alpha_1} (P^n \mu_0)$. Since E is P -invariant and $\mu_0(E) = 1$, one has $P^n \mu_0(E) = 1$, or $P^n \mu_0(S \setminus E) = 0$. Then $\mu_1(S \setminus E) = 0$, hence $\mu_1(E) = 1$. $\square$

Note that one may take for P the identity map in Lemma 2.5.7.

Lemma 2.5.8. *If $\nu_1, \nu_2 \in \mathcal{P}(S)$ have $\text{supp}(\nu_i) \subset B(z, r)$, then $\|\nu_1 - \nu_2\|_{BL}^* \leq 2r$. Consequently, if P satisfies the e-property, then for every $\epsilon > 0$, there exists $r > 0$ such that for all $\nu_i \in \mathcal{P}(S)$ with $\text{supp}(\nu_i) \subset B(z, r)$, for $i \in \{1, 2\}$, one has $\|P^n \nu_1 - P^n \nu_2\|_{BL}^* < \epsilon$ for all n .*

Proof. Fix $\epsilon > 0$. If $f \in BL(S)$ with $\|f\|_{Lip} \leq 1$ and $\|f\|_{BL} \leq 1$, define

$$f_{\inf} := \inf_{x \in B(z, r)} f(x) \quad \text{and} \quad f_{\sup} := \sup_{x \in B(z, r)} f(x).$$

Then $f_{\sup} - f_{\inf} \leq 2r$, since by $\|f\|_{Lip} \leq 1$, we have $f_{\sup} \leq f(z) + r$ and $f_{\inf} \leq f(z) - r$. We have $f_{\inf} \leq \langle \nu_i, f \rangle \leq f_{\sup}$, for $i \in \{1, 2\}$, thus, $|\langle \nu_1 - \nu_2, f \rangle| \leq 2r$. So, $\|\nu_1 - \nu_2\|_{BL}^* \leq 2r$. Now, let $\epsilon > 0$. Since P satisfies the e-property, there exists $\delta > 0$ such that $\|P^n \nu_1 - P^n \delta_z\|_{BL}^* < \epsilon/2$ for all ν with $\|\nu - \delta_z\| < \delta$ and all $n \in \mathbb{N}$. Put $r = \frac{1}{2}\delta$. If ν_1, ν_2 are such that $\text{supp}(\nu_i) \subset B(z, r)$ and $n \in \mathbb{N}$ then,

$$\|P^n \nu_1 - P^n \nu_2\|_{BL}^* \leq \|P^n \nu_1 - P^n \delta_z\|_{BL}^* + \|P^n \nu_2 - P^n \delta_z\|_{BL}^* < \epsilon.$$

$\square$

Lemma 2.5.9. *Let Q be a Markov-Feller operator on S . Suppose that there exist $x, z \in S$, $m \in \mathbb{N}$ and $r > 0$ such that $Q^m \delta_x(B(z, r)) \geq \beta > 0$. Then for each*

$R > r$ and $0 < \beta' < \beta$ there exists an open neighborhood U_x of x in S , such that

$$Q\delta_y(B(z, R)) \geq \beta' \text{ for all } y \in U_x.$$

Proof. Let $R > r$, $0 < \beta' < \beta$. Put $\epsilon := \beta - \beta'$. There exists $f \in BL(S)$ such that $1_{B(z, r)} \leq f \leq 1_{B(z, R)}$ (see the proof of Lemma 2.5.6). Since Q is continuous, there exists an open neighborhood V_x of δ_x in $\mathcal{P}(S)$ such that $\|Q\mu - Q\delta_x\|_{BL}^* < \frac{\epsilon}{\|f\|_{BL}}$ for all $\mu \in V_x$. Let $U_x := \{y \in S : \delta_y \in V_x\}$. Since $y \mapsto \delta_y : S \rightarrow \mathcal{P}(S)$ is continuous (see Proposition 2.3.3), U_x is open and $x \in U_x$. Moreover,

$$|\langle Q\delta_y, f \rangle - \langle Q\delta_x, f \rangle| \leq \|Q\delta_y - Q\delta_x\|_{BL}^* \cdot \|f\|_{BL} < \epsilon$$

for all $y \in U_x$. Since $\langle Q\delta_y, f \rangle \leq Q\delta_y(B(z, R))$ and $\langle Q\delta_x, f \rangle \geq Q\delta_x(B(z, r))$, we find for $y \in U_x$ that

$$\begin{aligned} Q\delta_y(B(z, R)) &\geq \langle Q\delta_y, f \rangle \geq \langle Q\delta_x, f \rangle - |\langle Q\delta_y, f \rangle - \langle Q\delta_x, f \rangle| \\ &> Q\delta_x(B(z, r)) - \epsilon \geq \beta - \epsilon = \beta'. \end{aligned}$$

□

Lemma 2.5.10. *Let P be a Markov-Feller operator on S with invariant measure μ^* . If $y \in S$ is such that for each $r > 0$, there exist $n \in \mathbb{N}$ and $x \in \text{supp}(\mu^*)$ satisfying $P^n\delta_x(B(y, \frac{1}{2}r)) > 0$, then $y \in \text{supp}(\mu^*)$.*

Proof. We have to show $\mu^*(B(y, r)) > 0$ for each $r > 0$. Fix $r > 0$. Take $n \in \mathbb{N}$ and $x \in \text{supp}(\mu^*)$ such that $\beta := P^n\delta_x(B(y, \frac{1}{2}r)) > 0$. According to Lemma 2.5.9 with $\beta' = \frac{1}{2}\beta$ and $R = r > \frac{1}{2}r$, and $Q = P^n$, there exists U_x , an open neighborhood of x , such that $P^n\delta_z(B(y, r)) \geq \beta' > 0$ for all $z \in U_x$. Thus, using $\mu^* = P^n\mu^* = \int_S P^n\delta_z d\mu^*(z)$,

$$\begin{aligned} \mu^*(B(y, r)) &= \int_S P^n\delta_z(B(y, r)) d\mu^*(z) \\ &\geq \int_{U_x} P^n\delta_z(B(y, r)) d\mu^*(z) \geq \beta'\mu^*(U_x) > 0. \end{aligned}$$

□

We now use Lemmas 2.5.6-2.5.10 to present the proof of Theorem 2.5.4.

Proof. (Proof of Theorem 2.5.4) Let $x \in \text{supp}(\mu^*)$ and $z \in S$ be such that (2.9) holds for every $r > 0$. Let $\nu_i \in \mathcal{P}(S)$ be such that $\nu_i(S_{\mu^*}) = 1$, $i \in \{1, 2\}$. Let $\epsilon > 0$ and let r be as in the second part of Lemma 2.5.8 with ϵ replaced by $\frac{\epsilon}{2}$. Put $B = B(z, \frac{r}{2})$. Using Lemma 2.5.6, there exist α such that $\liminf_{n \rightarrow \infty} P^n \nu(B) > \alpha$ for all $\nu \in \mathcal{P}(S)$ satisfying $\nu(S_{\mu^*}) = 1$. Thus, there exists a sequence (n_k) in $\mathbb{N}$ such that $P^{n_k} \nu_1(B) > \alpha$ for all k . Since

$$\liminf_{k \rightarrow \infty} P^{n_k} \nu_2(B) \geq \liminf_{n \rightarrow \infty} P^n \nu_2(B) > \alpha,$$

we obtain a subsequence (n_{k_m}) such that $P^{n_{k_m}} \nu_2(B) > \alpha$ for all m . Thus, there exists $n'_1 \in \mathbb{N}$ with $P^{n'_1} \nu_i(B) > \alpha$ ($i = 1, 2$). Since P has the e -property, S_{μ^*} is P -invariant (c.f. [90, Corollary 7.3.12]). Lemma 2.5.7 yields existence of $\nu_i^1, \mu_i^1 \in \mathcal{P}(S)$ with $\text{supp}(\nu_i^1) \subset \overline{B}$, $\mu_i^1(S_{\mu^*}) = 1$ and $P^{n'_1} \nu_i = \alpha \nu_i^1 + (1 - \alpha) \mu_i^1$. Replacing ν_i by μ_i^1 and proceeding inductively, we obtain a sequence (n'_j) such that $P^{n'_{j+1}} \mu_i^j = \alpha \nu_i^{j+1} + (1 - \alpha) \mu_i^{j+1}$ with $\nu_i^{j+1}, \mu_i^{j+1} \in \mathcal{P}(S)$, $\mu_i^{j+1}(S_{\mu^*}) = 1$, while $\text{supp}(\nu_i^{j+1}) \subset \overline{B}$. Fix k such that $(1 - \alpha)^k < \frac{\epsilon}{4}$. Write $m_l = \sum_{j=l}^k n'_j$, $l \geq 1$. Then

$$P^{m_1+n} \nu_i = \alpha \sum_{j=1}^k (1 - \alpha)^{j-1} P^{m_{j+1}+n} \nu_i^j + (1 - \alpha)^k P^n \mu_i^k.$$

Consequently, using Lemma 2.5.8,

$$\begin{aligned} \|P^{m_1+n} \nu_1 - P^{m_1+n} \nu_2\|_{BL}^* &\leq \alpha \sum_{j=1}^k (1 - \alpha)^{j-1} \|P^{m_{j+1}+n} \nu_1^j - P^{m_{j+1}+n} \nu_2^j\|_{BL}^* \\ &\quad + (1 - \alpha)^k \|P^n \mu_1^k - P^n \mu_2^k\|_{BL}^* \\ &\leq \alpha \sum_{j=1}^k (1 - \alpha)^{j-1} \frac{\epsilon}{2} + \frac{\epsilon}{4} \cdot 2 < \epsilon, \end{aligned}$$

for all n . Lemma 2.5.10 yields $z \in \text{supp}(\mu^*)$. □

Verifying the lower-bound estimate (2.9) in Theorem 2.5.4 can be a tedious ‘exercise’. In Chapter 4, Section 4.6 we provide an example of an approach to establishing the estimate using ‘support tracing’ for the iterates $P^n \delta_x$. The result stated there is even stronger than 2.9: it is a *uniform* lower-bound estimate.

2.5.3 Asymptotic stability of the Markov operator

A uniform lower-bound estimate yields stronger results concerning the Markov operator. The following Theorem is an adaptation to the discrete-time setting of [86, Corollary 5.4. p.1134].

Theorem 2.5.11. *Let P be a Markov operator on a complete separable metric space (S, d) with the e-property. Then P is asymptotically stable if and only if there exists a $z \in S$ such that for all $r > 0$*

$$\inf_{x \in S} \liminf_{n \rightarrow \infty} P^n \delta_x (B(z, r)) > 0. \quad (2.11)$$

The proof for the continuous-time case is tedious and long. We believe that the proof can be modified for the discrete-time case to hold. Theorem 2.5.11 is not needed in the end in Chapter 4, since an existing result on nonexpansiveness could be used. Therefore, we did not do the detailed verification of this assertion.

Chapter 3

Ergodicity and extinction for a One-dimensional case of sampling

In this chapter we determine all ergodic measures and their stability properties of a Markov operator that is associated to a Markov chain which ensues from impulsive random interventions in a one-dimensional deterministic dynamical system at equally spaced time points. This setting is inspired by a biological application in population dynamics, where samples (‘catches’) are drawn regularly from a growing population or part of a bacterial population is eradicated, e.g. through antibiotics. On the way, we formulate and prove a version of Orey’s convergence Theorem and exponential ergodicity using essentially Banach lattice arguments and Banach’s Fixed Point Theorem, valid in the generality of a Polish state space. We use the Krylov-Bogoliubov-Beboutov-Yosida decomposition to show that we found all ergodic measures. Finally, we prove that the extinction probability is a continuous function of the initial population size that is strictly positive on part of the state space.

This chapter considers the long-term behavior of a system that results from a particular type of stochastic perturbation of a deterministic dynamical system. That is, the dynamics of the latter is interrupted by a fast intervention through which the state of the system is changed randomly to a new position in state space according to a law that depends on the state just before this intervention. The times of intervention may either be equally spaced and predetermined – as we consider here – or be random themselves, with the distribution for the duration of the time intervals between interventions dependent on the state at the start

too.

This type of dynamics occurs naturally for example in the modeling of particular biological systems. One can think of a growing bacterial colony, modeled deterministically, from which samples of random size are drawn regularly in an experiment, a marine ecosystem from which fish are harvested with a net or a simple model for the development of a bacterial infection that is treated with antibiotics that do not eradicate all (at once). One may further consider the modeling and analysis of stochasticity in gene regulatory networks with feedback through a cell's signaling system [45, 60], or a more quantitative study of developmental heterogeneity in populations of genetically identical bacteria when exposed to a randomly changing environment [21]. See [55] for application to cell growth and division.

The associated mathematical models deserve further attention than they obtained over the past years, in particular in view of the applications. The vast amount of results on stochastic differential equations (SDEs) with either a Wiener or a jump process as random perturbation (cf. [67, 74]) seems inadequate for these settings. In the current SDE approach only the amplitude of the perturbations can depend on the state of the system, not the *shape* of its distribution, nor can the random time between interventions in a jump process. This is an essential modeling ingredient in these applications. Ji, *et. al.* [51] considers an SDE version of the system that we study here.

In this chapter we consider as deterministic system the solution to the logistic or Verhulst equation

$$\frac{dv}{dt} = rv\left(1 - \frac{v}{K}\right), \quad (3.1)$$

which may model autonomous growth of bacteria in a colony or a fish population. We take a one-dimensional system in order to focus on mathematical issues in proving existence, (non)uniqueness and stability of invariant measures for the Markov chain associated to (3.1) that is obtained by diminishing the solution at times $t_n := n\Delta t$ ($\Delta t > 0$), with a random amount, the *catch size*, that is drawn from the fixed interval $[0, m_c]$ according to a law $Q_{v'}$. Essential for the applications is the dependence of this law on the state v' just before intervention. The *maximal catch size* m_c is fixed. Any negative outcome as a result of the random perturbation is reduced directly to the value $v = 0$ (*extinction*). After intervention the deterministic system (3.1) continues with the new state as initial condition.

We explore an analytic approach towards existence and stability of ergodic

measures, motivated by our background in functional analysis and operator theory. That is, we associate a Markov operator P on finite Borel measures on $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ to the mentioned Markov chain (see Section 3.2) and subsequently study and employ the functional analytic properties of this operator, e.g. that P is ultra-Feller. Essentially, we use Banach lattice techniques to show that P is a contraction on measures supported on a suitable invariant interval in $\mathbb{R}_+$, for the total variation norm that is naturally associated to the lattice structure. In this manner exponential ergodicity (cf. [63]) follows from Banach's Fixed Point Theorem. Our proof is inspired by that of Orey's Convergence Theorem (cf. [70], or [63], Theorem 18.1.2), but takes full advantage of the Banach lattice formulation. Results on exponential ergodicity with a similar spirit of reducing the problem to a fixed point argument can be found in [40].

In preparation of these existence and stability results we 'trace' the supports of iterates of the transition kernel associated to P , i.e. $P^n \delta_x$, where δ_x is the Dirac measure at x (Section 3.3). It allows us to compute the support of the ergodic measure that was shown to exist (Section 3.5). This is particularly interesting for the applications, as it provides at least a range in which the population size will be on the long term. Determining the precise density of this invariant measure was not feasible, given the generality in which we studied the problem.

Finally, we show that the system has *precisely two ergodic measures*, namely the trivial measure δ_0 ('population extinct') and the non-trivial ergodic measure μ^* mentioned above. We do so (in Section 3.6) by employing the so-called Krylov-Bogoliubov-Beboutov-Yosida (KBBY) ergodic decomposition of the state space (cf. [92] Chapter XIII, [96] Chapter 2 or [91]). It provides *a priori* a description of all ergodic measures. By means of the results obtained so far, we can then show that this description can yield only the two measures δ_0 and μ^* . As a further result of the KBBY decomposition we obtain that the *extinction probability* is continuous as a function of the population size $v(0)$ at the start and is zero if $v(0)$ is beyond a computable threshold v_-^* , given the parameters r, K and m_c .

We are aware that the part on existence and stability of the ergodic measure μ^* can also be obtained through the theory on T -chains (and e-chains) and results on exponential ergodicity as presented in for example [63] (e.g. Theorem 16.0.2 or 18.0.2). The main difficulty in application of the characterizations in [63] is of course in verifying suitable conditions, e.g. aperiodicity. We discuss in Section 3.7 how this approach could proceed based upon the wealth of (sometimes technical) probabilistic results for this purpose in [63]. In these arguments we need to use,

too, the information on supports of iterates of the transition kernel of Section 3.3 that we used in our analytic approach. It may depend on background and scope of the reader which approach he enjoys most. It is worth mentioning here that similar problems were studied by Young and coauthors, for instance [93, 94]. In the latter paper the state space is a Hilbert space, but the jumps are assumed independent and identically distributed. Moreover, such questions have been addressed also in papers by S. Kuksin and A. Shirikyan where the authors studied dissipative systems perturbed by random kick-forces (see [53] and the references therein).

3.1 Model description

Fix $\Delta t > 0$ and put $t_n := n\Delta t$, $n \in \mathbb{N}_0$. The sequence $(t_n)_{n=1}^\infty$ should be viewed as the times of random interventions in the deterministic system that we shall describe now.

3.1.1 The deterministic population model

In this chapter we consider the *logistic equation*, which is the simplest, though realistic, model for a developing population, given by (3.1), where $v(t)$ is the expected number of individuals in the population at time t (often called ‘victims’ in a predator prey model), $r > 0$ is its maximum per-capita rate of change, and $K > 0$ is the so-called *carrying capacity* of the environment. The proof of the following lemma is straightforward.

Lemma 3.1.1. *The unique solution to (3.1) with $v(0) = v_0$ is explicitly given by*

$$v(t) = \phi_t(v_0) := \left(\frac{1}{K} + \left(\frac{1}{v_0} - \frac{1}{K} \right) e^{-rt} \right)^{-1}. \quad (3.2)$$

The steady states are $v = 0$ and $v = K$, where K is stable, while 0 is unstable. Moreover, the intervals $[0, K]$ and $[K, \infty)$ are positively invariant.

3.1.2 Stochastic interventions

We will consider stochastic interventions at times t_n , $n \in \mathbb{N}$. The distribution for the jump to the new state after intervention will be state dependent. We

suppose that at the intervention times the number of individuals is diminished by a random amount, the *catch size*, which is between 0 and a fixed maximum m_c . We need the following crucial assumptions on the law Q_x for the catch size, where x is the population size just before the intervention. We call Q_x the *catch size distribution*. We assume that for every $x \in [0, \infty)$, Q_x is a probability measure on $[0, m_c]$ (viewed as measure on $\mathbb{R}$) such that

A1) $\text{supp}(Q_x) = [0, m_c]$ for $x > 0$.

A2) Q_x has density function $y \mapsto q(x, y)$ with respect to Lebesgue measure on $\mathbb{R}$ for each $x > 0$.

A3) The map $x \mapsto q(x, \cdot) : (0, \infty) \rightarrow L^1(\mathbb{R})$ is continuous.

A4) $Q_x([0, x]) \rightarrow 0$ as $x \downarrow 0$.

Assumption A4) expresses, essentially, that the fewer individuals there are, the more likely it will be to catch all.

Lemma 3.1.2. *If $x \mapsto q(x, y)$ is continuous on $[0, \infty)$ for almost every $y \in \mathbb{R}$, then q satisfies A3).*

Proof. Since $q(x, y)$ and $q(x_0, y)$ are densities, due to the Vitali-Scheff Theorem (Scheff [78], Bogachev [11]) a sufficient condition that

$$\lim_{x \rightarrow x_0} \int_{\mathbb{R}} |q(x, y) - q(x_0, y)| dy = 0$$

in $[0, \infty)$ is that $\lim_{x \rightarrow x_0} q(x, y) = q(x_0, y)$ for almost every y , which we have by assumption. $\square$

3.1.3 Introduction of the Markov operator P

If the population size just before an intervention equals x , then the population y just after the intervention will be in a Borel set $A \subseteq [0, \infty)$ with probability $Q_x(x - A)$. So, if the population size at time 0 equals x , then the population size just before the first intervention at time Δt equals $\phi_{\Delta t}(x)$, thus, the distribution of the population size just after the first intervention is $Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - \cdot)$. Furthermore, if the population size just after the n -th intervention would have distribution μ , then the distribution just after the $(n + 1)$ - intervention equals $P\mu$ where P is defined as follows. First put

$$(\tilde{P}\mu)(A) = \int_{[0,\infty)} Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - A) d\mu(x)$$

for any $\mu \in \mathcal{M}^+([0, \infty))$. Note that $\tilde{P} : \mathcal{M}^+([0, \infty)) \rightarrow \mathcal{M}^+(\mathbb{R})$ is a total mass conserving positively linear map. It has kernel¹ $\tilde{p}(x, \cdot) := \tilde{P}\delta_x = Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - \cdot)$. Finally, define P as the push-forward of $\tilde{P}$ under $F(x) = x^+ := \max(x, 0)$:

$$P\mu := F^*\tilde{P}\mu = (\tilde{P}\mu) \circ F^{-1}. \quad (3.3)$$

Note that $P\delta_0(\{0\}) = \tilde{P}\delta_0((-\infty, 0]) = Q_0([0, \infty)) = 1$, irrespective of the choice of Q_0 . Thus, $P\delta_0 = \delta_0$. We call δ_0 the trivial invariant measure of P .

The population size just after the n -th intervention is a random variable X_n . Then $(X_n)_n$ is a Markov chain and P is its corresponding Markov operator.

The following property of our model is crucial in our analysis:

Theorem 3.1.3. *The Markov operator P defined by (3.3) with Q_x satisfying A1)-A4) is ultra-Feller on $[0, \infty)$.*

Proof. We need to show that the map $x \mapsto P\delta_x : [0, \infty) \rightarrow \mathcal{M}(\mathbb{R})_{TV}$ is continuous. First we show right continuity at 0. For $x > 0$, such that $\phi_{\Delta t}(x) < m_c$, one has:

$$d_x := P\delta_x(\{0\}) = Q_{\phi_{\Delta t}(x)}([\phi_{\Delta t}(x), m_c]) = 1 - Q_{\phi_{\Delta t}(x)}([0, \phi_{\Delta t}(x)]) > 0.$$

Then $v_x := P\delta_x - d_x\delta_0$ is a positive measure on $\mathbb{R}_+$ with $\|v_x\| = 1 - d_x$. Consequently, $\|P\delta_0 - P\delta_x\| = \|(1 - d_x)\delta_0 - v_x\|_{TV} \leq 2(1 - d_x)$. The latter converges to 0 as $x \downarrow 0$, according to A4). Now consider continuity on $(0, \infty)$. Fix x_0 in $(0, \infty)$. We may write $\tilde{P}\delta_x$ as $T_{\hat{x}}[q(\hat{x}, \cdot)] dm$ where T_z denotes the translation-reflection map $T_z g(y) := g(z - y)$ on $L^1(\mathbb{R})$, $\hat{x} := \phi_{\Delta t}(x)$, and m is Lebesgue measure. Consider

$$f : (x, z) \mapsto q(\hat{x}, \hat{z} - \cdot) = T_{\hat{z}}[q(\hat{x}, \cdot)] : (0, \infty) \times (0, \infty) \rightarrow L^1(\mathbb{R}).$$

Since $(T_z)_{z \geq 0}$ is strongly continuous semigroup on $L^1(\mathbb{R})$, the map $(z, g) \mapsto T_z g$ is jointly continuous on $[0, \infty) \times K$, for any $K \subset L^1(\mathbb{R})$ compact (e.g. [31], Lemma I.5.2). Now let $(x_n)_n$ be a sequence in $(0, \infty)$ such that $x_n \rightarrow x_0$, then

¹A function $p : S \times \mathcal{B}(S) \rightarrow [0, 1]$, defined as $p(x, E) = P\delta_x(E)$ for $x \in S$ and $E \in \mathcal{B}(S)$ is called the *transition function* (transition kernel) (c.f. Ethier and Kurtz, [32]).

$(q(\hat{x}_n, \cdot))_{n=0}^\infty$ is contained in a compact subset of $L^1(\mathbb{R})$, according to assumption A3). So, $f(x_n, x_n) \rightarrow f(x_0, x_0)$, i.e. $x \mapsto \tilde{P}\delta_x$ is continuous for $\|\cdot\|_{TV}$ at x_0 . Note that the push-forward under a continuous map is continuous for $\|\cdot\|_{TV}$. Hence, $x \mapsto P\delta_x = F^*\tilde{P}\delta_x$ is continuous for $\|\cdot\|_{TV}$. $\square$

3.1.4 An Example of a distribution that satisfies A1)-A4).

If there are many individuals a catch may be expected to be large, if there are few, the catch will be small. A distribution which captures these properties and satisfies assumptions A1)-A4) is a suitably scaled Beta distribution;

$$Q_x \sim \text{Beta}\left(0, m_c, \beta, \beta \frac{x}{x^*}\right), \quad (3.4)$$

for some positive parameters β and x^* . The meaning of x^* is that it is the population size just before intervention at which the distribution Q_{x^*} has expectation $\frac{1}{2}m_c$.

Recall that the Beta distribution $\text{Beta}(a, b, \alpha, \beta)$ is defined on $[a, b]$ and given by

$$\frac{1}{B(\alpha, \beta)} (x - a)^{\alpha-1} \frac{(b - x)^{\beta-1}}{(b - a)^{\alpha+\beta-1}}, \quad a < x < b,$$

where $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the Beta function.

Note that the density function q associated to Q_x is continuous on $[0, \infty) \times \mathbb{R} \setminus \{0, m_c\}$, but may have singularities for y at 0 and m_c . Thus, q satisfies Lemma 3.1.2, hence condition A3). Clearly Q_x satisfies A1) and A2). Note that Q_x also satisfies A4).

In this example $P\delta_x$ is given by

$$P\delta_x = \begin{cases} F_x(y) dy & \text{for } x : \phi_{\Delta t}(x) \geq m_c \\ F_x(y) dy + d_x \delta_0 & \text{for } x : \phi_{\Delta t}(x) < m_c \end{cases},$$

where

$$F_x(y) = \begin{cases} \text{Beta}\left(\phi_{\Delta t}(x) - m_c, \phi_{\Delta t}(x), \beta \frac{\phi_{\Delta t}(x)}{x^*}, \beta\right)(y) & \text{for } 0 \leq y \leq \phi_{\Delta t}(x) \\ 0 & \text{elsewhere} \end{cases},$$

and

$$d_x = 1 - \int_0^{\phi_{\Delta t}(x)} F_x(y) \, dy.$$

3.2 Tracing supports

We shall determine all ergodic measures for the Markov operator P on $[0, \infty)$ in Sections 3.5 and 3.6 that follow. For applications it is interesting to know the supports of these measures. Zaharopol [95] provides expressions for their supports, in terms of the sets $\text{supp}(P^n \delta_x)$, $x \in [0, \infty)$, $n \in \mathbb{N}$. In this section we determine the latter sets explicitly. This will prove useful both in establishing existence, uniqueness and stability of the non-trivial ergodic measure and the computations of its support.

At this point we first would like to analyze what happens in the extreme deterministic cases: nothing is caught at times t_n or more interestingly, if it is ‘always’ the maximum catch size m_c . In the first case, if the initial population size is x , then the population size at $t = t_n$ equals $a_n^+(x) := \phi_{n\Delta t}(x)$. If at each time the maximum size m_c is caught, then the population size at time t_n will be $a_n^-(x) := \psi^n(x)$, where

$$\psi : [0, \infty) \rightarrow [0, \infty) : x \mapsto (\phi_{\Delta t}(x) - m_c)^+.$$

It will turn out that $\text{supp}(P^n \delta_x) = [a_n^-(x), a_n^+(x)]$, see Proposition 3.2.3.

For the latter case we have to analyze the iterates of the map ψ .

In order to formulate our results conveniently, define

$$m_c^* := K \cdot \frac{\left(e^{\frac{1}{2}r\Delta t} - 1\right)}{\left(e^{\frac{1}{2}r\Delta t} + 1\right)},$$

the *critical catch size*, and for m_c satisfying $(K - m_c)^2 \geq \frac{4Km_c}{e^{r\Delta t} - 1}$,

$$v_{\pm}^* := \frac{1}{2}(K - m_c) \pm \frac{1}{2}\sqrt{(K - m_c)^2 - \frac{4Km_c}{e^{r\Delta t} - 1}}, \quad (3.5)$$

which is a real number. Note that $m_c^* < K$. Moreover, if $m_c \leq K$, then

$$(K - m_c)^2 \geq \frac{4Km_c}{e^{r\Delta t} - 1} \quad \text{if and only if} \quad m_c \leq m_c^*.$$

Proposition 3.2.1. (*Fixed points of ψ*) We distinguish the following cases:

1. If $m_c^* < m_c$, then ψ has one fixed point, which is 0.
2. If $0 < m_c < m_c^*$, then ψ has three distinct fixed points in $[0, \infty)$ which are $0, v_-^*, v_+^*$ given by equation (3.5).
3. If $m_c = m_c^*$, then v_-^* and v_+^* coincide and ψ has two fixed points in $[0, \infty)$; 0 and $v_\pm^*$.

Proof. One can solve the fixed point equation $\phi_{\Delta t}(v^*) - m_c = v^*$ explicitly for v^* using expression (3.2) and obtain the stated result by straightforward calculation. $\square$

Proposition 3.2.2. (*Stability of fixed points of ψ*) Suppose that $0 < m_c < m_c^*$. Then one has the following cases:

1. If $0 < x < v_-^*$, then $a_n^-(x) \downarrow 0$ as $n \rightarrow \infty$ and there exists n_0 such that $a_n^-(x) = 0$ for all $n \geq n_0$.
2. If $v_-^* < x < v_+^*$, then $a_n^-(x) \uparrow v_+^*$ as $n \rightarrow \infty$.
3. If $x \geq v_+^*$, then $a_n^-(x) \downarrow v_+^*$.
4. If $x \in (0, \infty)$, then $a_n^+(x) \rightarrow K$.

Proof. The results follow easily from the interpretation of the graph of ψ . $\square$

We have the following result on the support of $P^n \delta_x$:

Proposition 3.2.3. For each $x \in [0, \infty)$, and $n \in \mathbb{N}$,

$$\text{supp}(P^n \delta_x) = [a_n^-(x), a_n^+(x)].$$

Proof. According to assumption A1), we see that $P\delta_x$ has as support the interval

$$[\psi(x), \phi_{\Delta t}(x)] = [a_1^-(x), a_1^+(x)].$$

If $P^n \delta_x$ is supported on the interval $I_n = [\alpha, \beta]$, then

$$P^{n+1} \delta_x = P(P^n \delta_x) = P \left(\int_{[0, \infty)} \delta_y [P^n \delta_x](dy) \right) = \int_{I_n} P \delta_y [P^n \delta_x](dy),$$

which, by monotonicity of ψ and $\phi_{\Delta t}$, is supported on

$$\overline{\bigcup_{y \in I_n} [\psi(y), \phi_{\Delta t}(y)]} = [\psi(\alpha), \phi_{\Delta t}(\beta)].$$

Induction yields the result. $\square$

Corollary 3.2.4. *If $0 < m_c < m_c^*$, then the sets $[v_-^*, K]$, $[v_+^*, K]$, $[v_-^*, \infty)$ and $[v_+^*, \infty)$ are P -invariant. In particular, P leaves $\mathcal{M}^+([v_+^*, K])$, $\mathcal{M}^+([v_-^*, K])$, $\mathcal{M}^+([v_+^*, \infty))$ and $\mathcal{M}^+([v_-^*, \infty))$ invariant.*

Proof. The first part follows from Propositions 3.2.2 and 3.2.3. For the second part, take a measure μ such that $\text{supp} \mu \subset [v_-^*, \infty)$. If $y < v_-^*$ and $r > 0$ such that $y + r < v_-^*$, then, with $E = (y - r, y + r)$, $P\mu(E) = \int_{[v_-^*, \infty)} P\delta_x(E) d\mu(x) = 0$, according to Propositions 3.2.2 and 3.2.3. Hence, $y \notin \text{supp}(P\mu)$ and we conclude $\text{supp}(P\mu) \subset [v_-^*, \infty)$. The other cases are shown by similar reasoning. $\square$

3.3 A Banach lattice approach to exponential ergodicity

We now consider the case with random catch size. We begin by an abstract result, Theorem 3.3.4, which holds for an arbitrary Polish space S . It yields convergence of the n -step transition probabilities, as time goes to ∞ , to a unique invariant probability measure in the total variation topology. The proof of Theorem 3.3.4 is partially inspired by Orey, [70], but takes a Banach lattice formulation. Note that P is a positive contraction in $(\mathcal{M}(S), \|\cdot\|_{TV})$. We present conditions under which it is also a *strict* contraction on the probability measures, so that, for each probability measure μ , $P^n \mu$ converges to a unique fixed point within the probability measures. We use essentially Banach lattice arguments. The stability results in total variation norm and exponential ergodicity as found in [63] e.g. Theorem 18.0.2 thus boil down to a Banach fixed point argument. In this sense our approach is similar to [40]. We first discuss the measurability of the map $(x, y) \mapsto \|P^n \delta_x - P^n \delta_y\|_{TV}$.

Lemma 3.3.1. *Let $n \in \mathbb{N}$, and let P be a regular Markov operator.*

1. *For every $A \in \mathcal{B}(S)$ the map $(x, y) \mapsto (P^n \delta_x - P^n \delta_y)(A) : S \mapsto \mathbb{R}$ is measurable.*

2. The map $(x, y) \mapsto \|P^n \delta_x - P^n \delta_y\|_{TV} : S \mapsto [0, \infty)$ is measurable.

Proof. (1) As P is regular, there exists a bounded linear map $U : BM(S) \rightarrow BM(S)$ which is dual to P . Since $x \mapsto 1_A(x)$ is measurable, the map $x \mapsto (U1_A)(x)$ is measurable and therefore

$$x \mapsto P\delta_x(A) = \langle P\delta_x, 1_A \rangle = \langle \delta_x, U1_A \rangle = U1_A(x)$$

is measurable.

(2) Since (S, d) is separable, $\mathcal{B} := \mathcal{B}(S)$ is countably generated (separable). Thus, there exists a countable subset $\mathcal{A}$ of $\mathcal{B}$ such that $\mathcal{A}$ is an algebra and a generator of $\mathcal{B}$ and such that for each μ and each $B \in \mathcal{B}$, and for any $\epsilon > 0$, there exists $A \in \mathcal{A}$, such that $|\mu(A) - \mu(B)| < \epsilon$ (see Bogachev [11]). Hence, $\sup_{B \in \mathcal{B}} \mu(B) = \sup_{A \in \mathcal{A}} \mu(A)$ and $\inf_{B \in \mathcal{B}} \mu(B) = \inf_{A \in \mathcal{A}} \mu(A)$. So, for $x, y \in S$ and $n \in \mathbb{N}$, we have by (2.2)

$$\|P^n \delta_x - P^n \delta_y\|_{TV} = \sup_{A \in \mathcal{A}} (P^n \delta_x - P^n \delta_y)(A) - \inf_{A \in \mathcal{A}} (P^n \delta_x - P^n \delta_y)(A).$$

For each $A \in \mathcal{A}$ the map $(x, y) \mapsto (P^n \delta_x - P^n \delta_y)(A) : S \mapsto \mathbb{R}$ is measurable. Because $\mathcal{A}$ is countable, it follows that $(x, y) \mapsto \|P^n \delta_x - P^n \delta_y\|_{TV}$ is measurable. $\square$

Lemma 3.3.2. *Let P be a regular Markov operator on S . For every $\mu, \nu \in \mathcal{M}^+(S)$ and $n \in \mathbb{N}$,*

$$\left\| \iint (P^n(\delta_x - \delta_y)) \mu(dx) \nu(dy) \right\|_{TV} \leq \iint \|P^n(\delta_x - \delta_y)\|_{TV} \mu(dx) \nu(dy). \quad (3.6)$$

Proof. The difficulty is that the integral at the left hand side is not a Bochner integral in $\mathcal{M}(S)_{TV}$. Recall that according to Lemma 3.3.1 (1), the map $(x, y) \mapsto (P^n \delta_x - P^n \delta_y)(E) : S \mapsto \mathbb{R}$ is measurable for all $n \geq 1$ and for all $E \in \mathcal{B}(S)$. So, the integral at the left hand side is well-defined when viewed as a set-wise integral. According to Lemma 3.3.1 (2), the map $(x, y) \mapsto \|P^n \delta_x - P^n \delta_y\|_{TV} : S \mapsto [0, \infty)$ is measurable for all $n \geq 1$, and therefore, the integral at right hand side is

well-defined. For $B \in \mathcal{B}(S)$,

$$\begin{aligned} \left| \iint (P^n(\delta_x - \delta_y))(B) \mu(dx) \nu(dy) \right| &\leq \iint |(P^n(\delta_x - \delta_y))(B)| \mu(dx) \nu(dy) \\ &\leq \iint \|P^n(\delta_x - \delta_y)\|_{TV} \mu(dx) \nu(dy) \end{aligned}$$

and (3.6) follows. $\square$

Remark 3.3.3. The Markov operator P extends to a positive linear operator on $\mathcal{M}(S)_{TV}$ given by $P\mu := P\mu^+ - P\mu^-$. Since $\|P\mu\|_{TV} = \|\mu\|_{TV}$ for all $\mu \in \mathcal{M}^+(S)$, this extension is bounded with operator norm equal 1. Hence, for $\mu, \nu \in \mathcal{M}^+(S)$, the map $n \mapsto \|P^n(\mu - \nu)\|_{TV}$ is non-increasing.

Theorem 3.3.4. *Let P be a regular Markov operator on S such that there exists $n_1 \in \mathbb{N}$ for which $\alpha := \inf_{x,y \in S} \|P^{n_1}\delta_x \wedge P^{n_1}\delta_y\|_{TV} > 0$. Then for all $n \geq n_1$ one has*

$$\|P^n(\mu - \nu)\|_{TV} \leq \theta^n \|\mu - \nu\|_{TV},$$

where $\theta = (1 - \alpha)^{\frac{1}{n_1}} < 1$. Consequently, there exists a unique ergodic measure μ^* and

$$\sup_{\mu \in \mathcal{P}(S)} \|P^n\mu - \mu^*\|_{TV} \leq 2\theta^n \rightarrow 0$$

as $n \rightarrow \infty$.

Proof. Let $\mu, \nu \in \mathcal{P}(S)$ be two probability measures. Recall that $\|P^n(\mu - \nu)\|_{TV}$ is non-increasing as a function of n (see Remark 3.3.3). We should show that there exists an n_1 and $\rho_1 < 1$, independent of μ and ν such that

$$\|P^{n_1}(\mu - \nu)\|_{TV} \leq \rho_1 \|\mu - \nu\|_{TV}, \quad (3.7)$$

then iteration gives $\|P^{kn_1}(\mu - \nu)\|_{TV} < \rho_1^k \|\mu - \nu\|_{TV}$, $k \in \mathbb{N}$ and this combined with the monotonicity of $\|P^n(\mu - \nu)\|_{TV}$ implies the ergodicity of P . Note that inequality (3.7) indicates that P^{n_1} is a strict contraction with norm $\|P^{n_1}\|_{TV} = \rho < 1$. Due to Lemma 2.2.1 there exist $\mu', \nu' \in \mathcal{M}^+(S)$, such that $\mu - \nu = \mu' - \nu'$, $\|\mu'\|_{TV} = \|\nu'\|_{TV} = \|\mu - \nu\|_{TV}/2$, and $\mu' \wedge \nu' = 0$. The following argument will show that it suffices to prove (3.7) for the case $\mu = \delta_x$, $\nu = \delta_y$, $x \neq y$. Observe

that according to Lemma 3.3.2

$$\begin{aligned} \|P^n(\mu - \nu)\|_{TV} &= \|P^n(\mu' - \nu')\|_{TV} \\ &= (\|\mu'\|_{TV})^{-1} \left\| \iint (P^n(\delta_x - \delta_y)) \mu'(dx) \nu'(dy) \right\| \\ &\leq (\|\mu'\|_{TV})^{-1} \iint \|P^n(\delta_x - \delta_y)\| \mu'(dx) \nu'(dy). \end{aligned}$$

Consequently, once we prove that $\|P^n(\delta_x - \delta_y)\|_{TV} \leq 2\rho$ for any $x, y \in S$, we will get for any two probability measures μ, ν

$$\begin{aligned} \|P^n(\mu - \nu)\|_{TV} &\leq 2\rho (\|\mu'\|_{TV})^{-1} \|\mu'\|_{TV} \|\nu'\|_{TV} \\ &\leq 2\rho \frac{1}{2} \|\mu - \nu\|_{TV} = \rho \|\mu - \nu\|_{TV}. \end{aligned}$$

Now using the vector lattice identity $|\mu - \nu| = \mu + \nu - 2(\mu \wedge \nu)$ we obtain

$$\begin{aligned} \|P^n\delta_x - P^n\delta_y\|_{TV} &= \|P^n\delta_x\|_{TV} + \|P^n\delta_y\|_{TV} - 2\|P^n\delta_x \wedge P^n\delta_y\|_{TV} \\ &\leq 2(1 - \|P^n\delta_x \wedge P^n\delta_y\|_{TV}) = 2(1 - \alpha). \end{aligned}$$

By taking $\rho := 1 - \alpha < 1$, the result follows. $\square$

3.4 Unique ergodicity and total variation stability on $[v_-^*, \infty)$

Consider again the Markov operator P given by (3.3). Theorem 3.3.4 yields:

Theorem 3.4.1. *If $0 < m_c < m_c^*$, and A1)-A4) hold, then the restriction of the Markov operator P defined by (3.3) is uniformly exponentially ergodic on $\mathcal{P}([r, R])$, for any $v_-^* \leq r \leq v_+^*$ and any $R \geq K$. That is,*

$$\sup_{\mu \in \mathcal{P}([r, R])} \|P^n\mu - \mu^*\|_{TV} \rightarrow 0$$

at an exponential rate as $n \rightarrow \infty$.

Proof. We need to show that $\|P^n\delta_x \wedge P^n\delta_y\|_{TV} > 0$. For this purpose we first show that P strictly overlaps supports. Recall that for $x \in [r, R]$, $\text{supp}(P^n\delta_x) = [a_n^-(x), a_n^+(x)]$, $a_n^-(x) \rightarrow v_+^*$ and $a_n^+(x) \rightarrow K$. So, one can take n_0 large such that for all $x, y \in [r, R]$, $(a_n^-(x), a_n^+(x)) \cap (a_n^-(y), a_n^+(y)) \neq \emptyset$, because $a_n^+(x) >$

$a_n^-(y)$ for all $n \geq n_0$. Then the intersection of the supports of $P^n \delta_x$ and $P^n \delta_y$ contains a non-empty open interval. This yields that $P^n \delta_x \wedge P^n \delta_y \neq 0$. Hence, $\|P^n \delta_x \wedge P^n \delta_y\|_{TV} > 0$. Next, since $x \mapsto P^n \delta_x : [r, R] \rightarrow \mathcal{M}(S)_{TV}$ is continuous and $(\mathcal{M}(S), \|\cdot\|_{TV})$ is a Banach lattice, the map $(x, y) \mapsto P^n \delta_x \wedge P^n \delta_y$ is continuous. Since the norm is a continuous mapping, $(x, y) \mapsto \|P^n \delta_x \wedge P^n \delta_y\|_{TV}$ is continuous. Since $[r, R]$ is compact, there exists $\alpha > 0$ such that for all $x, y \in [r, R]$ one has

$$\|P^n \delta_x \wedge P^n \delta_y\|_{TV} \geq \alpha > 0.$$

The result now follows due to Theorem 3.3.4. $\square$

The non-compact case $[v_-^*, \infty)$ is captured in the following:

Theorem 3.4.2. *For every $\mu \in \mathcal{P}([v_-^*, \infty))$ one has $\|P^n \mu - \mu^*\|_{TV} \rightarrow 0$ as $n \rightarrow \infty$. Hence, P is uniquely ergodic² on $[v_-^*, \infty)$. The convergence is not uniform.*

Proof. Let $\mu \in \mathcal{P}([v_-^*, \infty))$ be such that $\text{supp} \mu \subseteq [v_-^*, \infty)$. Fix $\epsilon > 0$. Take $\mu_\epsilon \in \mathcal{P}([v_-^*, \infty))$ such that $\|\mu - \mu_\epsilon\|_{TV} < \epsilon/2$ and such that $\text{supp} \mu_\epsilon$ is compact. Then $\|P^n \mu_\epsilon - \mu^*\|_{TV} \rightarrow 0$ as $n \rightarrow \infty$ by Corollary 3.4.1. Take n_0 such that

$$\|P^n \mu_\epsilon - \mu^*\|_{TV} < \epsilon/2$$

for all $n \geq n_0$. Then

$$\begin{aligned} \|P^n \mu - \mu^*\|_{TV} &\leq \|P^n \mu_\epsilon - \mu^*\|_{TV} + \|P^n (\mu_\epsilon - \mu)\|_{TV} \\ &\leq \|P^n \mu_\epsilon - \mu^*\|_{TV} + \|\mu_\epsilon - \mu\|_{TV} \leq \epsilon \end{aligned}$$

for $n \geq n_0$. Thus, $P^n \mu \rightarrow \mu^*$ in $\|\cdot\|_{TV}$. To show that the convergence is not uniform in δ_x for $x \in [v_-^*, \infty)$, take $y \in [v_-^*, \infty)$, $y > x$ such that $a_{n_1}^-(y) > a_{n_1}^+(x)$ for some n_1 (see Proposition 3.2.2). Then there is no overlapping in the supports (Proposition 3.2.3), i.e. $P^n \delta_x \wedge P^n \delta_y = 0$, hence $\|P^n \delta_x - P^n \delta_y\|_{TV} = 2$. Suppose the convergence is uniform, then $\sup_{x \in [v_-^*, \infty)} \|P^n \delta_x - \mu^*\|_{TV} \rightarrow 0$. So, there exists n_1 such that $\|P^n \delta_x - \mu^*\|_{TV} < 1$ for all $x \in [v_-^*, \infty)$. Hence,

$$\|P^n \delta_x - P^n \delta_y\|_{TV} \leq \|P^n \delta_x - \mu^*\|_{TV} + \|\mu^* - P^n \delta_y\|_{TV} < 2,$$

which is a contradiction. $\square$

²We call P *uniquely ergodic* if P has exactly one invariant probability measure, which is then an ergodic measure (cf. Zaharopol [95], p18).

Moreover, one has

Theorem 3.4.3. *If $0 < m_c < m_c^*$, and A1)-A4) hold, then the ergodic measure μ^* for the Markov operator P defined by (3.3) has support $[v_+, K]$. Hence, P is strictly ergodic³ on $[v_+, K]$.*

Proof. In order to prove that $\text{supp}(\mu^*) = [v_+, K]$, first note that for any $x \in [v_-, K]$, $a_n(x)$ converges to v_+ (see Proposition 3.2.2) and $\phi_{n\Delta t}(x)$ to K . For any probability measure μ on $[v_-, K]$ and $0 \leq k \leq n$ one has

$$(P^n \mu)(A) = \int_{[v_-, K]} (P^{n-k} \delta_x)(A) d[P^k \mu](x),$$

for any $A \subseteq [v_-, K]$ Borel. In particular, $P^n \mu^* = \int_{[v_-, K]} P^n \delta_x d\mu^*(x)$. Hence,

$$\text{supp}(P^n \mu^*) \subseteq \overline{\bigcup_{x \in [v_-, K]} \text{supp}(P^n \delta_x)}.$$

Since

$$\text{supp}(P^n \delta_x) \subseteq [a_n^-(x), \phi_{n\Delta t}(x)] \subseteq [a_n^-(v_-^*), \phi_{n\Delta t}(K)] = [a_n^-(v_-^*), K]$$

for all $x \in [v_-, K]$, $\text{supp}(P^n \mu^*) \subseteq [a_n^-(v_-^*), K]$. Since $P^n \mu^* = \mu^*$ for all n , we obtain

$$\text{supp}(\mu^*) = \text{supp}(P^n \mu^*) \subseteq \bigcap_{n=1}^{\infty} [a_n^-(v_-^*), K] = [v_+, K].$$

For the other inclusion, let $x \in (v_+, K)$ and $U \subseteq [v_-, K]$ open with $x \in U$. Take $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq U$ and its closure $[x - \delta, x + \delta] \subseteq (v_+, K)$. Take n so large that $a_n^-(K) < x - \delta$ and $\phi_{n\Delta t}(v_+^*) > x + \delta$. Then for every $y \in [v_+, K]$,

$$\text{supp}(P^n \delta_y) = [a_n^-(y), \phi_{n\Delta t}(y)] \supseteq [a_n^-(K), \phi_{n\Delta t}(v_+^*)] \supseteq [x - \delta, x + \delta],$$

³If P is uniquely ergodic and the support of the unique invariant probability is the entire space S , then P is called *strictly ergodic* (cf. Zaharopol [95], p18).

so $P^n \delta_y((x - \delta, x + \delta)) > 0$. Hence, using invariance of μ^* ,

$$\begin{aligned} \mu^*((x - \delta, x + \delta)) &= \int_{[v_+^*, K]} P^n \delta_y((x - \delta, x + \delta)) d\mu^*(y) \\ &\geq \int_{[v_+^*, K]} P^n \delta_y((x - \delta, x + \delta)) d\mu^*(y) > 0, \end{aligned}$$

so that $\mu^*(U) > 0$. Therefore, $x \in \text{supp}(\mu^*)$. It follows that $[v_+^*, K] \subseteq \text{supp}(\mu^*)$. Hence, $\text{supp}(\mu^*) = [v_+^*, K]$. $\square$

3.5 Dynamics on the full state space $[0, \infty)$

If we consider the perturbed dynamical system of Section 3.2 on the entire state space $[0, \infty)$, there will be more than one invariant measure. Since $\{0\}$ is an invariant set, the point measure δ_0 will be an invariant measure of P , in addition to μ^* of Section 3.5. The main objective of this section is to prove that the ergodic measures μ^* and δ_0 are the only ergodic measures of P on $[0, \infty)$. Moreover, we examine the dynamics of P on this space. In particular, we investigate the *extinction probability* of the population, which is $\lim_{n \rightarrow \infty} P^n \delta_x(\{0\})$. We shall show that this limit exists and that it is a continuous function of x .

Based on [90] we define on $\mathcal{M}(S)$ the dual bounded Lipschitz norm, $\|\cdot\|_{BL}^*$, as follows:

$$\|\mu\|_{BL}^* := \sup_{f \in BL, \|f\|_{BL} \leq 1} \left| \int f d\mu \right| = \sup_{f \in BL \setminus \{0\}} \frac{|\langle f, \mu \rangle|}{\|f\|_{BL}}.$$

Thus, $|\langle f, \mu \rangle| \leq \|\mu\|_{BL}^* \cdot \|f\|_{BL}$. One says that $(\mu_n), \mu_n \in \mathcal{M}(S)$, converges weakly to $\mu \in \mathcal{M}(S)$ if $\lim_{n \rightarrow \infty} \langle f, \mu_n \rangle = \langle f, \mu \rangle$ for every continuous bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$. The convergence $\lim_{n \rightarrow \infty} \|\mu_n - \mu\|_{BL}^* = 0$ for $\mu_n, \mu \in \mathcal{P}(S)$ is equivalent to the weak convergence of $(\mu_n)_{n \geq 1}$ to μ (see Dudley, [26, 27]). We denote by $\mathcal{S}_{BL}$ the closure of D in $BL(S)^*$, where $D := \text{span}\{\delta_x : x \in S\}$.

In order to show that there cannot be ergodic measures other than μ^* and δ_0 , we employ the so-called Krylov-Bogoliubov-Beboutov-Yosida (KBBY) ergodic decomposition of the state space (cf. Ch.2 in [95] or Ch XIII in Yoshida [92] and the references found there for the case of locally compact state spaces and Markov-Feller operators. For more general Markov operators see Zaharopol [96]

and Worm [91] for the general case of a Polish state space). Central in the KBBY decomposition is the consideration of Cesaro averages of a Markov operator P and associated limits in $\mathcal{S}_{BL}$:

$$P^{(n)} = \frac{1}{n} \sum_{k=0}^{n-1} P^k, \quad \epsilon_x := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k \delta_x,$$

whenever the limit exists. In our setting, this is the case for any $x \in [0, \infty)$, ϵ_x is an invariant measure necessarily, since P is Markov-Feller. This follows from the Krylov-Bogoliubov Theorem for x in the P -invariant compact set $[0, K]$, and from Theorem 3.4.2 for $x > K$. The KBBY decomposition yields a subset $\Gamma_{\text{cpie}} \subset [0, \infty)$ such that ϵ_x is an ergodic measure for each $x \in \Gamma_{\text{cpie}}$. Moreover, each ergodic measure equals ϵ_x for some $x \in \Gamma_{\text{cpie}}$ (cf. [95], Theorem 2.1.3). Note that if a sequence converges, then its Cesaro average converges, with the same limit. This yields that if $x \in [v_-^*, \infty)$, then $\epsilon_x = \mu^*$ since $P^k \delta_x \rightarrow \mu^*$.

Corollary 3.4.1 and Theorem 3.4.2 yield that $\epsilon_x = \mu^*$ for all $x \in [v_-^*, \infty)$. Clearly, $\epsilon_0 = \delta_0$.

Proposition 3.5.1. *If $x \in (0, v_-^*)$, then ϵ_x is not ergodic.*

Proof. According to Proposition 3.2.3,

$$\text{supp}(P^{n_0} \delta_x) = [a_n^-(x), a_n^+(x)]$$

and, according to Proposition 3.2.2 (1), there exists n_0 such that $a_n^-(x) = 0$ for all $n \geq n_0$. Take $w^* \in (0, v_-^*)$, such that $\phi_{\Delta t}(w^*) < m_c$. Then, for $n \geq n_0$,

$$(0, w^*) \cap (0, a_n^+(x))$$

is a non-empty open set in the support of $P^n \delta_x$, so $P^n \delta_x((0, w^*)) > 0$. For $y \in (0, w^*)$ one has

$$P \delta_y(\{0\}) = F^* \tilde{P} \delta_y(\{0\}) = \tilde{P} \delta_y(F^{-1}\{0\}) = \tilde{P} \delta_y((-\infty, 0]),$$

because $F(x) = x^+$. Hence

$$\begin{aligned} P \delta_y(\{0\}) &= \int_{[0, \infty)} Q_{\phi_{\Delta t}(u)}(\phi_{\Delta t}(u) - (-\infty, 0]) d\delta_y(u) \\ &= Q_{\phi_{\Delta t}(y)}([\phi_{\Delta t}(y), \infty)) > 0, \end{aligned}$$

since $\phi_{\Delta t}(y) < \phi_{\Delta t}(w^*) < m_c$ and $\text{supp}(Q_{\phi_{\Delta t}(y)}) = [0, m_c]$. Therefore,

$$\begin{aligned} c &:= (P^{n_0+1}\delta_x)(\{0\}) = \int_{[0, \infty)} P\delta_y(\{0\}) dP^{n_0}\delta_x(y) \\ &\geq \int_{[0, w^*)} P\delta_y(\{0\}) dP^{n_0}\delta_x(y) > 0. \end{aligned}$$

Then $c \in (0, 1]$, and $P^{n_0+1}\delta_x \geq c\delta_0$. Hence, for all $n \geq n_0 + 1$ one has $P^n\delta_x \geq c\delta_0$. Recall that

$$\begin{aligned} P^{(n)}\delta_x &= \frac{1}{n} \sum_{k=0}^{n-1} P^k\delta_x = \frac{1}{n} \sum_{k=0}^{n_0} P^k\delta_x + \frac{1}{n} \sum_{k=n_0+1}^{n-1} P^k\delta_x \\ &= \frac{1}{n} \left(\sum_{k=0}^{n_0} P^k\delta_x \right) + \frac{n - n_0 - 1}{n} \cdot \frac{1}{n - n_0 - 1} \sum_{k=n_0+1}^{n-1} P^k\delta_x \\ &\geq \frac{1}{n} \alpha + \left(1 - \frac{n_0 + 1}{n} \right) c\delta_0 \\ &\geq \left(1 - \frac{n_0 + 1}{n} \right) c\delta_0, \quad n \geq n_0 + 1. \end{aligned}$$

So, $P^{(n)}\delta_x - \left(1 - \frac{n_0+1}{n} \right) c\delta_0 \geq 0$. Since $\mathcal{M}^+([0, \infty))$ is closed in $\mathcal{S}_{BL}$, the limit is positive as well. That is, $\epsilon_x - c\delta_0 \geq 0$. Hence, $\epsilon_x \geq c\delta_0$. Similarly, for any $z^* \in (v_-^*, K)$, there exists $m_0 > 0$ such that $z^* \in \text{supp}(P^{m_0}\delta_x)$. Thus,

$$d := P^{m_0}\delta_x([v_-^*, K]) > 0.$$

Since $[v_-^*, K]$ is P -invariant, $P^{k+1}\delta_x([v_-^*, K]) \geq d$ for all $k \geq m_0$. Then for $n \geq m_0$,

$$\begin{aligned} P^{(n)}\delta_x([v_-^*, K]) &= \frac{1}{n} \sum_{k=0}^{m_0} P^k\delta_x([v_-^*, K]) + \frac{1}{n} \sum_{k=m_0+1}^{n-1} P^k\delta_x([v_-^*, K]) \\ &\geq \frac{n - m_0 - 1}{n(n - m_0 - 1)} \cdot \sum_{k=m_0+1}^{n-1} P^k\delta_x([v_-^*, K]) \\ &\geq \frac{1}{n} (n - m_0 - 1) d \\ &= \left(1 - \frac{m_0 + 1}{n} \right) d, \quad n \geq m_0 + 1. \end{aligned}$$

The Portmanteau Theorem then yields that

$$\epsilon_x([v_-^*, K]) \geq \limsup_{n \rightarrow \infty} P^{(n)} \delta_x([v_-^*, K]) \geq d.$$

So, one has $\epsilon_x \neq c\delta_0$. Also, $\epsilon_x \neq \mu^*$ because $\epsilon_x(\{0\}) \geq c\delta_0(\{0\}) = c$ and $\mu^*(\{0\}) = 0$ because $0 \notin \text{supp}\mu^*$. Thus,

$$\epsilon_x = (\epsilon_x - c\delta_0) + c\delta_0 = (1 - c) \left(\frac{1}{1 - c} (\epsilon_x - c\delta_0) \right) + c\delta_0, \quad c \in (0, 1).$$

That is, ϵ_x is a non-trivial convex combination of two invariant probability measures and therefore ϵ_x is not an extreme point of the set of invariant probability measures. Thus, ϵ_x is not ergodic. $\square$

Thus, from the KBBY decomposition we can deduce that δ_0 and μ^* are all (and the only) ergodic measures.

Theorem 3.5.2. *Under the conditions as in Theorem 3.4.3, the Markov operator P defined by (3.3) on $[0, \infty)$ has only two ergodic invariant measures: δ_0 and μ^* .*

Proof. As we discussed above, the set of ergodic measures equals $\{\epsilon_x : x \in [0, \infty)\}$. Clearly $\epsilon_0 = \delta_0$ is ergodic. For $x \in (0, v_-^*)$, Proposition 3.5.1 yields that ϵ_x is not ergodic. Also, $\epsilon_x = \mu^*$ for $x \in [v_-^*, \infty)$. Hence, the only ergodic measures are δ_0 and μ^* . $\square$

Corollary 3.5.3. *If μ is invariant, then there exists $p \in [0, 1]$ such that*

$$\mu = p\delta_0 + (1 - p)\mu^*.$$

The following Lemma could be found (in a more general situation) in [90], Proposition 7.3.3. p. 148.

Proposition 3.5.4. *The map $x \mapsto \epsilon_x : S \rightarrow \mathcal{S}_{BL}$ is continuous.*

Proof. The map $x \mapsto \epsilon_x : S \rightarrow \mathcal{S}_{BL}$ is continuous if and only if the map

$$x \mapsto \langle \epsilon_x, f \rangle : S \rightarrow \mathbb{R}$$

is continuous for all $f \in C_b(S)$. One has

$$\begin{aligned} |\langle \epsilon_x - \epsilon_y, f \rangle| &= \lim_{n \rightarrow \infty} |\langle \frac{1}{n} \sum_{k=0}^{n-1} P^k \delta_x - P^k \delta_y, f \rangle| \\ &\leq \sup_n \frac{1}{n} \sum_{k=0}^{n-1} \|P^k \delta_x - P^k \delta_y\|_{TV} \|f\|_{\infty}. \end{aligned}$$

Since P is ultra Feller, there exists an open neighborhood U_x of x such that for $y \in U_x$ and for all k ,

$$\|P^k \delta_x - P^k \delta_y\|_{TV} \leq \|P \delta_x - P \delta_y\|_{TV} < \frac{\epsilon}{1 + \|f\|_{\infty}}.$$

Hence, $|\langle \epsilon_x - \epsilon_y, f \rangle| < \epsilon$ for $y \in U_x$. $\square$

Proposition 3.5.5. *There exists a continuous function $p : [0, \infty) \rightarrow [0, 1]$ such that for every $x \in [0, \infty)$, $\lim_{n \rightarrow \infty} P^{(n)} \delta_x = p(x) \delta_0 + (1 - p(x)) \mu^*$ in $\mathcal{M}([0, \infty))_{BL}^+$. Moreover, $p(x) = 0$ for $x \in [v_-^*, \infty)$, $p(0) = 1$, and $0 < p(x) < 1$ for $x \in (0, v_-^*)$.*

Proof. We argued above, that according to the KBBY decomposition, ϵ_x is an invariant probability measure for each $x \in [0, \infty)$. Thus, ϵ_x is a unique convex combination of δ_0 and μ^* , because of Corollary 3.5.3. Let $p(x) \in [0, 1]$ be the unique number such that $\epsilon_x = p(x) \delta_0 + (1 - p(x)) \mu^*$. Now let $0 < \delta < v_-^*$, and define $f(x) = [1 - \frac{1}{\delta} x]^+$. Then $f \in BL([0, \infty))$. Since $x \mapsto \epsilon_x : [0, \infty) \rightarrow \mathcal{M}([0, \infty))_{BL}^+$ is continuous, $x \mapsto \langle \epsilon_x, f \rangle = p(x)$ is continuous. Since $f = 0$ on $\text{supp}(\mu^*)$, we have $\langle \mu^*, f \rangle = 0$, so $\langle \epsilon_x, f \rangle = p(x) < \delta_0, f \rangle = p(x) f(0) = p(x)$. Clearly, $p(0) = 1$. For $x \in [v_-^*, \infty)$, $p(x) = 0$ follows from Proposition 3.4.1 and Theorem 3.4.2. For $x \in (0, v_-^*)$, ϵ_x is not ergodic (Proposition 3.5.1), so $0 < p(x) < 1$ for $x \in (0, v_-^*)$. $\square$

Recall that for $x \in [v_-^*, \infty)$, we have a stronger convergence than stated in Proposition 3.5.5: according to Theorem 3.4.2, $P^n \delta_x \rightarrow \mu^*$ for $\|\cdot\|_{TV}$.

In the remainder of this section we show that $P^n \delta_x(\{0\}) \rightarrow p(x)$ and

$$P^n \delta_x([v_-^*, \infty)) \rightarrow 1 - p(x)$$

as $n \rightarrow \infty$. Thus, $p(x)$ may be interpreted as extinction probability. This proceeds through the following series of lemmas.

Lemma 3.5.6. *For every $\gamma \in (0, 1)$, there exists $\delta > 0$ such that for all $y \in [0, \delta)$ one has $P\delta_y(\{0\}) \geq 1 - \gamma$.*

Proof. Due to Theorem 3.1.3, for every Borel set A the evaluation map $x \mapsto P\delta_x(A) : [0, \infty) \rightarrow [0, 1]$ is continuous. Hence, $x \mapsto P\delta_x(\{0\})$ is continuous. At $x = 0$ one has $P\delta_0(\{0\}) = 1$. Thus, by the continuity at 0, there exists $\delta > 0$ such that $P\delta_x(\{0\}) \geq 1 - \gamma$ for all $y \in [0, \delta)$. $\square$

Lemma 3.5.7. *Let $\gamma \in (0, 1)$ and $\epsilon_n \geq 0$ with $\epsilon_n \rightarrow 0$. If $\alpha_n \geq 0$, $n \in \mathbb{N}$, satisfy $\alpha_{n+1} \leq \gamma\alpha_n + \epsilon_n$, for every $n \in \mathbb{N}$, then $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Define $\beta_n := \frac{1}{\gamma^n}\alpha_n$. Then $\beta_{n+1} = \frac{1}{\gamma^{n+1}}\alpha_{n+1} \leq \frac{1}{\gamma^{n+1}}\gamma\alpha_n + \frac{1}{\gamma^{n+1}}\epsilon_n = \beta_n + \frac{1}{\gamma^{n+1}}\epsilon_n$. By induction, $\beta_{n+1} = \beta_0 + \sum_{k=0}^n \frac{1}{\gamma^{k+1}}\epsilon_k$. So, $\alpha_{n+1} = \gamma^{n+1}\alpha_0 + \sum_{k=0}^n \gamma^{n-k}\epsilon_k$. For $n \geq m$ we have

$$\begin{aligned} \sum_{k=0}^n \gamma^{n-k}\epsilon_k &= \sum_{k=0}^{m-1} \gamma^{n-k}\epsilon_k + \sum_{k=m}^n \gamma^{n-k}\epsilon_k \\ &\leq \sum_{k=0}^{m-1} \gamma^{n-k} \sup_{j \geq 0} \epsilon_j + \sum_{k=m}^n \gamma^{n-k} \sup_{j \geq m} \epsilon_j \\ &= \gamma^n \sum_{k=0}^{m-1} \gamma^{-k} \sup_{j \geq 0} \epsilon_j + \frac{1}{1-\gamma} \sup_{j \geq m} \epsilon_j. \end{aligned}$$

So, $\limsup_{n \rightarrow \infty} \sum_{k=0}^n \gamma^{n-k}\epsilon_k \leq \frac{1}{1-\gamma} \sup_{j \geq m} \epsilon_j$ for all m . Thus, $\alpha_{n+1} \rightarrow 0$. $\square$

Lemma 3.5.8. *For every $x \in [0, v_-^*)$ and every closed $K \subset (0, v_-^*)$, $P^{(n)}\delta_x(K) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Note that by Portmanteau, since $P^{(n)}\delta_x \rightarrow \epsilon_x$, for the closed set K one has

$$\limsup_{n \rightarrow \infty} P^{(n)}\delta_x(K) \leq \epsilon_x(K).$$

Note that (from Proposition 3.5.5) $\epsilon_x = p(x)\delta_0 + (1-p(x))\mu^*$. Since $K \subset (0, v_-^*)$, $\delta_0(0, v_-^*) = 0$ and $\mu^*(0, v_-^*) = 0$, we obtain $\epsilon_x(K) = 0$. It follows that $\limsup_{n \rightarrow \infty} P^{(n)}\delta_x(K) \leq 0$, hence $\lim_{n \rightarrow \infty} P^{(n)}\delta_x(K) = 0$. $\square$

Lemma 3.5.9. *For every $x \in [0, \infty)$, $P^{(n)}\delta_x((0, v_-^*)) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Let $\gamma \in (0, 1)$ and take $\delta > 0$ as in Lemma 3.5.6. For $y \in [0, \delta)$,

$$P\delta_y((0, v_-^*]) \leq P\delta_y((0, \infty)) \leq P\delta_y([0, \infty)) - P\delta_y(\{0\}) \leq 1 + \gamma - 1 = \gamma.$$

Note also that $P\delta_y((0, v_-^*]) \leq 1$. Hence,

$$\begin{aligned}
 P^{n+1}\delta_x((0, v_-^*]) &= \int_{[0, \infty)} P\delta_y((0, v_-^*]) dP^n\delta_x(y) \\
 &= \int_{(0, v_-^*]} P\delta_y((0, v_-^*]) dP^n\delta_x(y) \\
 &= \int_{(0, \delta)} P\delta_y((0, v_-^*]) dP^n\delta_x(y) + \int_{[\delta, v_-^*]} P\delta_y((0, v_-^*]) dP^n\delta_x(y) \\
 &\leq \gamma P^n\delta_x((0, \delta)) + P^n\delta_x([\delta, v_-^*]).
 \end{aligned}$$

Hence,

$$\sum_{k=1}^n P^k\delta_x((0, v_-^*]) \leq \gamma \sum_{k=0}^{n-1} P^k\delta_x((0, \delta)) + \sum_{k=0}^{n-1} P^k\delta_x([\delta, v_-^*]).$$

Thus,

$$\begin{aligned}
 P^{(n+1)}\delta_x((0, v_-^*]) &= \frac{1}{n+1} \sum_{k=0}^n P^k\delta_x((0, v_-^*]) \\
 &= \frac{1}{n+1} \delta_x((0, v_-^*]) + \frac{1}{n+1} \sum_{k=1}^n P^k\delta_x((0, v_-^*]) \\
 &\leq \frac{1}{n+1} + \frac{1}{n+1} \gamma \sum_{k=0}^{n-1} P^k\delta_x((0, \delta)) + \frac{1}{n+1} \sum_{k=0}^{n-1} P^k\delta_x([\delta, v_-^*]) \\
 &= \frac{1}{n+1} + \gamma \frac{n}{n+1} P^{(n)}\delta_x((0, \delta)) + \frac{n}{n+1} P^{(n)}\delta_x([\delta, v_-^*]) \\
 &= \frac{1}{n+1} + \gamma \left(1 - \frac{1}{n+1}\right) P^{(n)}\delta_x((0, \delta)) \\
 &\quad + \left(1 - \frac{1}{n+1}\right) P^{(n)}\delta_x([\delta, v_-^*]).
 \end{aligned}$$

Hence, $P^{(n+1)}\delta_x((0, \delta)) \leq \gamma P^{(n)}\delta_x((0, \delta)) + \epsilon_n$, where

$$\begin{aligned}
 \epsilon_n &= \frac{1}{n+1} - \frac{\gamma}{n+1} P^{(n)}\delta_x((0, \delta)) + \left(1 - \frac{1}{n+1}\right) P^{(n)}\delta_x([\delta, v_-^*]) \\
 &\quad - P^{(n+1)}\delta_x([\delta, v_-^*]).
 \end{aligned}$$

Put $\alpha_n := P^{(n)}\delta_x((0, \delta))$. Then, one has $\alpha_{n+1} \leq \gamma\alpha_n + \epsilon_n$. Also, by Lemma 3.5.8 $\epsilon_n \rightarrow 0$. Hence, using Lemma 3.5.7, $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. $\square$

Lemma 3.5.10. *For all $x \in [0, \infty)$, $P^n\delta_x((0, v_-^*)) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Note that for all $x \in [0, \infty)$, the map $n \mapsto P^n\delta_x((0, v_-^*))$ is decreasing. As $(P^n\delta_x((0, v_-^*)))_n$ is a decreasing sequence bounded from below by 0, it is convergent and

$$\lim_{n \rightarrow \infty} P^n\delta_x((0, v_-^*)) = \lim_{n \rightarrow \infty} P^{(n)}\delta_x((0, v_-^*)) = 0.$$

$\square$

The extinction probability of the population starting at x equals $\lim_{n \rightarrow \infty} P^n\delta_x(\{0\})$ and is given precisely by the following proposition.

Proposition 3.5.11. *One has:*

$$\lim_{n \rightarrow \infty} P^n\delta_x(\{0\}) = p(x), \quad \lim_{n \rightarrow \infty} P^n\delta_x([v_-^*, \infty)) = 1 - p(x).$$

Proof. Recall that, since $P^n\delta_x$ is a probability measure on $[0, \infty)$ for all n , one has $P^n\delta_x([0, \infty)) = 1$ for all n . So,

$$\lim_{n \rightarrow \infty} \{P^n\delta_x(\{0\}) + P^n\delta_x((0, v_-^*)) + P^n\delta_x([v_-^*, \infty))\} = 1.$$

Lemma 3.5.10 says that $P^n\delta_x((0, v_-^*)) \rightarrow 0$. By the Portmanteau Theorem, one has $\limsup_{n \rightarrow \infty} P^{(n)}\delta_x(\{0\}) \leq \epsilon_x(\{0\}) \leq p(x)$. If we put

$$L_0 = \lim_{n \rightarrow \infty} P^n\delta_x(\{0\}) = \limsup_{n \rightarrow \infty} P^{(n)}\delta_x(\{0\}),$$

then $L_0 \leq p(x)$. Also,

$$\limsup_{n \rightarrow \infty} P^{(n)}\delta_x([v_-^*, \infty)) \leq \epsilon_x([v_-^*, \infty)) \leq 1 - p(x).$$

If we put $L_K = \lim_{n \rightarrow \infty} P^n\delta_x([v_-^*, \infty)) = \limsup_{n \rightarrow \infty} P^{(n)}\delta_x([v_-^*, \infty))$, then $L_K \leq 1 - p(x)$. Thus, $1 = L_0 + L_K \leq p(x) + L_K \leq p(x) + 1 - p(x) = 1$. Hence, $\lim_{n \rightarrow \infty} P^n\delta_x(\{0\}) = p(x)$ and $\lim_{n \rightarrow \infty} P^n\delta_x([v_-^*, \infty)) = 1 - p(x)$. $\square$

3.6 An alternative approach to exponential ergodicity

One could roughly say that a version of the Feller Property combined with some type of irreducibility yields some kind of ergodicity. Many results of this type are known and go back to a classical result often derived from theorems of Khas'minski and Doob, which states that topological irreducibility together with the strong Feller property implies uniqueness of the invariant measure. For many purposes the strong Feller property is too restrictive, and there are different ways known in which it can be relaxed to a weaker property that is still stronger than the Feller property. Meyn and Tweedie [63] use T -chains, combined with (ψ) -irreducibility and aperiodicity to get exponential ergodicity (Theorem 16.0.2). More recently, Hairer and Mattingly [38], have devised an *asymptotic strong Feller* property, which is weaker than the strong Feller property and which in combination with a weak type of irreducibility yields ergodicity.

In our situation the Markov operator P given by (3.3) has the ultra Feller property, which is stronger than the strong Feller property. Therefore, we obtained ergodicity on $[v_-^*, K]$ by only using a fairly weak irreducibility condition, which we established by tracing supports (see Theorems 3.3.4 and 3.4.1).

Instead, the theory of Meyn and Tweedie could have been applied. In the remaining part we shall sketch how this could proceed. It seems that proving aperiodicity also requires the result on supports of iterates of the kernel that are essential in our approach (Section 3.3). We use terminology from [63].

Proposition 3.6.1. *The Markov operator P defined by (3.3), and restricted to $[v_-^*, K]$ yields a ψ -irreducible T -chain, Φ , with $\psi = P\delta_K$.*

Proof. That P induces a T -chain follows from Theorem 6.2.9 of [63] and the support of $P\delta_K$ having non-empty interior (Proposition 3.2.3). For ψ -irreducibility use [63] Theorem 6.1.5, which requires Φ to be a strong Feller chain. This follows from Theorem 3.1.3. It also requires the state space S to have a reachable point. According to Proposition 3.2.3, the point $x^* = K$ is reachable, since for every open set U containing K , and for every $x \in [v_-^*, K]$, there exists n such that $\text{supp}(P^n\delta_x) \cap U \neq \emptyset$. $\square$

In order to apply the Meyn and Tweedie $\|\cdot\|_{TV}$ -stability results (for instance, Theorem 16.0.2.) we need to prove aperiodicity.

Proposition 3.6.2. *The ψ -irreducible T -chain induced by P is aperiodic.*

Proof. If the chain were periodic, there would be a d -cycle with $d \geq 2$. That is, there exist $D_i \subset [v_-^*, K]$, $i = 1, 2, \dots, d$ pairwise disjoint such that for each $x \in D_i$, $P\delta_x(D_{i+1(\bmod d)}) = 1$ and $(\cup_{i=1}^d D_i)^c$ is ψ -null (see [63], Theorem 5.4.4). Let $x \in D_1$, then $P^n\delta_x(D_{n+1(\bmod d)}) = 1$ for all n . According to proposition 3.2.3, $P^n\delta_x([a_n^-(x), a_n^+(x)] \setminus D_{n+1(\bmod d)}) = 0$. Since $P^n\delta_x$ has strictly positive density (a.e.) with respect to Lebesgue measure λ , we find that $\lambda([a_n^-(x), a_n^+(x)] \setminus D_{n+1}) = 0$ for all n . Now take n so large that $(a_n^-(x), a_n^+(x)) \cap (a_{n+1}^-(x), a_{n+1}^+(x)) \neq \emptyset$. Then, taking indices of D modulo d ,

$$\begin{aligned} 0 &\neq \lambda([a_n^-(x), a_n^+(x)] \cap [a_{n+1}^-(x), a_{n+1}^+(x)]) \\ &\leq \lambda((D_{n+1} \cap D_{n+2}) \cup ([a_n^-(x), a_n^+(x)] \setminus D_{n+1}) \\ &\quad \cup ([a_{n+1}^-(x), a_{n+1}^+(x)] \setminus D_{n+2})) \\ &= 0 \end{aligned}$$

because $D_{n+1} \cap D_{n+2} = \emptyset$, and we arrive at a contradiction. $\square$

Theorem 6.0.1. (ii) of [63] yields that any compact set is petite, so $[v_-^*, K]$ is petite. Theorem 5.5.7. of [63] yields that $[v_-^*, K]$ is small. Finally, Theorem 16.0.2 of [63] yields uniform exponential ergodicity from part (v). So we have

Corollary 3.6.3. *The Markov operator P defined by (3.3), when restricted to $[v_-^*, K]$, is uniformly exponentially ergodic.*

A similar argument applies to restriction of P to $[r, R]$, as in Theorem 3.4.1.

Chapter 4

Persistence of stability for a general class of infinite dimensional cases

This chapter addresses the long-term behaviour –in a suitable probabilistic sense– of map iteration in subsets of Banach spaces that are randomly perturbed. The law of the latter change in state is allowed to depend on state. We provide quite general conditions under which a stable fixed point of the deterministic map iteration induces an asymptotically stable ergodic measure of the Markov chain defined by the perturbed system, which is regarded as ‘persistence of stability’. The support of this invariant measure is characterized. The applicability of the framework is illustrated for deterministic dynamical systems that are subject to random interventions at fixed equidistant time points. In particular, we consider systems motivated by population dynamics: a model in ordinary differential equations, a model derived from a reaction–diffusion system and a class of delay equations.

4.1 Introduction

Consider a closed subset S of a separable Banach space $(X, \|\cdot\|)$ and a map $\phi : S \rightarrow S$ with suitable regularity as specified in A3) and A5) below. Iteration of ϕ yields a dynamical system in S in discrete time. It is randomly perturbed in the following way. Assume that for each $x \in S$, a Borel probability measure Q_x on X is given, which is the law of a random additive perturbation Y that can

take place at state x . We consider the long-term behaviour of the Markov chain defined (informally) by

$$X_{n+1} := \phi(X_n) + Y_n, \quad Y_n \sim Q_{\phi(X_n)}, \quad (4.1)$$

when X_0 is taken in a neighbourhood of a stable fixed point x^* of ϕ . Of course, conditions need to be imposed on the support of Q_x such that there exists a closed set $B^* \subset S$, containing x^* , such that X_n remains in B^* for all n (almost surely). Our analysis approach centers around the associated Markov operator P given by

$$(P\mu)(A) := \int_S Q_{\phi(x)}(A - \phi(x)) d\mu(x), \quad (4.2)$$

on Borel probability measures μ , concentrated on B^* . We provide quite general conditions on ϕ and Q_x (‘smallness’) that yield that such a set B^* exists and that P is a *nonexpansive* Markov operator on B^* [56, 82]. The latter leads to asymptotic stability: there is a unique invariant probability measure μ^* on B^* such that $P^n\mu$ converges to μ^* weakly for any probability measure μ on B^* as $n \rightarrow \infty$, even uniformly in μ .

The Markov chain (4.1) on B^* relates to so-called *nonlinear state space* (NSS) models (cf. [63], see also [47]) and more closely to *piecewise deterministic Markov processes* (PDMPs), (cf. [16, 19, 50, 75]). The NSS models are tailored towards random switching among a family of dynamical systems, driven by a hidden noise variable, rather than the ‘impulsive’ change in state described in (4.1). The latter abstracts a fast, short-lasting intervention process into an immediate jump in state space. The probabilistic framework of PDMPs fits better to (4.1), but still not completely: typically the state space is assumed locally compact [16, 19] with notable exceptions [50, 75], while also the times of intervention are random, with (essential) exponential distribution for the length of the intervention-free time period. Thus, fixed equidistant times of intervention do not fit readily into this framework.

However, modeling population dynamics in biology leads naturally to systems of the form (4.1). For example in cases when the population is confronted occasionally (but regularly) with external interference, like experimental sampling or commercial harvesting. It can even be ‘essential biology’: when the organism’s life cycle is such that there are non-overlapping generations. This occurs when the next generation is ‘born’ from eggs or seeds deposited by the previous gener-

ation when the latter has died completely, e.g. in winter. A common theme for insects and annual plants.

In [3] we investigated the long-term dynamics of the logistic growth model perturbed by random interventions (sampling, harvesting) to investigate what is technically needed in its mathematical analysis. In this chapter we focus on the part in [3] concerned with stability properties of an ergodic measure of the system. We now look at perturbing *infinite dimensional systems* mainly, that arise e.g. from models for spatially diffusing and interacting populations [52, 66, 69] or type structured population models with delay [43]. See Section 4.7 for examples and further details on possible applications.

Examples of models with random (but discrete) times of intervention can be found in the PDMP–literature mentioned above. [55] provides a model for cell division. See also [45] for an application to gene regulation, similar in spirit to our approach. Young *et al.* [94] studied the dynamics of the 2D Navier–Stokes equations. The study of PDMPs in infinite dimensional state space, motivated by applications such as in [45, 75, 94] deserve further mathematical attention.

In the field of stochastic differential equations (SDEs) the formulation of conditions that ensure the persistence of stability of equilibria of the deterministic system to an invariant measure for the process defined by the SDE has been studied extensively, see e.g. [17, 37, 48]. This well-studied framework for ‘continuous random perturbation’ cannot accommodate currently for models in which also the *shape* of the distribution of jumps can depend on state, though.

Our approach employs the non-expansiveness of the Markov operator P on B^* , defined in (4.2). Convergence of the supports of the measures $P^n \delta_x$ to a unique set $E^* \subset B^*$ in Hausdorff distance allows us to derive a fundamental lower bound estimate (Theorem 4.6.3) which yields that P is locally concentrating. A result due to Szarek [82] then gives the asymptotic stability of P , on B^* , which is interpreted as a ‘persistence of stability’–result for (4.1).

Our analysis framework and approach applies to deterministic systems defined by a well-posed system of ordinary differential equations, too. This setting is well-covered by the techniques from [63], yielding stability in total variation norm even. The infinite dimensional case does not properly fit into [63] however, because of the assumption of local compactness of state space in [63].

4.2 Abstract model description

let $(X, \|\cdot\|)$ be a separable Banach space and let S be a closed subset of X , viewed as a complete separable metric space for the metric induced by the norm in X . We assume that a Lipschitz map $\phi : S \rightarrow S$ is given with a steady state $x^* \in S$ which is exponentially attracting, i.e. there exists a ball B centered at x^* and $0 \leq \theta < 1$ such that

$$\|\phi(x) - x^*\| \leq \theta \|x - x^*\| \quad (4.3)$$

for all $x \in B$.

We will consider stochastic interventions at fixed, equally spaced discrete times $0 < t_1 < t_2 \dots < t_n < \dots$ at which the state is changed to a new state randomly. Put $\Delta t := t_{i+1} - t_i$. The state x just before an intervention is changed to a new state y according to some probability distribution Q_x for the jump in the state, i.e. $\text{Prob}(y \in A) = Q_x(A - x)$. Note that we allow Q_x to change with x . We will discuss the following questions:

Is there an invariant set in S for the perturbed system? Does there exist an invariant probability measure concentrated on this set? Is it unique? Is it stable?

We provide conditions on ϕ and on Q_x in relation to Δt , M and ω , such that there exists a unique stable invariant probability measure. The conditions are motivated by general classes of examples which we present in Section 4.7. Moreover, these conditions are actually applicable as also discussed in Section 4.7. By use of the *hyperspace* associated to a metric space (e.g. see [49]) we can draw further conclusions on the dynamics of the supports of states of P . The hyperspace of B^* is the collection $\mathcal{H}(B^*)$ of all closed and bounded subsets of B^* endowed with the Hausdorff metric d_H .

The conditions are as follows:

A1 There exists $L \geq 0$ such that $\|Q_x - Q_y\|_{FM} \leq L \|x - y\|$ for all $x, y \in S$.

A2 $C_x := \text{supp}(Q_x)$ is bounded for each $x \in S$ and $d_H(C_x, C_y) \leq L \|x - y\|$ for all $x, y \in S$.

The conditions on ϕ come down to the following; with a slight strengthening of (4.3):

A3 There exists $R^* > 0$ such that the closed $\|\cdot\|$ -ball $B^* = \overline{B}(x^*, R^*)$ contained in S and centered at the fixed point x^* of ϕ , is invariant under ϕ , and there

exists $\theta \in [0, 1)$ such that

$$\| \phi(x) - \phi(y) \| \leq \theta \| x - y \| \quad \text{for all } x, y \in B^*.$$

A4 C_x satisfies $\sup_{y \in C_x} \| y \| < (1 - \theta(1 + L)) R^*$ for all $x \in B^*$.

A5 $\phi : S \rightarrow S$ is a compact map: $\phi(B)$ has compact closure for every closed ball $B \subseteq S$.

Remark 4.2.1. Condition A1 does not impose constraints on the closeness of the supports Q_x . We also need to control this, leading to A2. Concerning A2, for example one may check that $d_H(\overline{B}(x, r), \overline{B}(y, r)) \leq \| x - y \|$. Moreover, If $\| \cdot \|$ does not satisfy A3, then in general classes of examples (see Section 4.7) one can construct an equivalent norm which does. Note that the topology in $\mathcal{M}(X)_{TV}$ depends on the topology of X , not on the norm. Note also that $V(x) = 1 + \| x - x^* \|$ yields a bounded Lyapunov function on B^* .

Assumption A4 expresses the ‘smallness’ of the stochastic interventions, while A2 limits the change in size of these interventions when x is varying.

Moreover we have

Remark 4.2.2. In particular ecological models for the dispersal of species with synchronous, non-overlapping generations that have a sedentary stage and a dispersal stage, $x \in S$ represents the population distribution over a spatial domain Ω at the end of the life cycle of the current generation. A function $f : S \rightarrow S$ models the transition of the population to the next generation and start of their dispersal stage. Then

$$\phi(x)(\omega) := \int_{\Omega} K(\omega - \omega') f(x)(\omega') d\omega', \quad (4.4)$$

where K is *redistribution kernel*. The latter has been estimated from experimental observations for numerous species (see [52, 66] for further details and references). If f is locally Lipschitz and the redistribution kernel K is sufficiently nice, then ϕ is expected to be a (non-linear) compact map, as composition of a local Lipschitz map and compact linear integral operator. So A5 appears natural.

Our approach is that we consider iterates of a Markov operator P induced by the perturbed system that we describe below. In Section 4.7 we explain how these conditions are naturally satisfied in several cases of semi-linear evolution equations.

The evolution of the system between the interventions is given by the deterministic system $(\phi_t)_{t \geq 0}$. That is, the population just before the next intervention will be $x' = \phi_{\Delta t}(x)$ where $x \in S$ is the state just after the previous intervention. Then the population y just after the intervention will be in a Borel set $A \subseteq X$ with probability $Q_{x'}(A - x')$. In fact, we end in A if and only if the deviation is $y - x'$ with $y \in A$. This happens with probability $Q_{x'}(A - x')$. Thus, the distribution of the population size just after the next intervention is $Q_{\phi_{\Delta t}(x)}(\cdot - \phi_{\Delta t}(x))$. The intervention may move the state out of the set S , on which ϕ is defined, a priori. This point will get particular attention later on. Furthermore, if the population size just after the n -th intervention would have distribution μ , concentrated on S , then the distribution just after the $(n + 1)$ - intervention equals $P\mu$ given by (4.2) for any $\mu \in \mathcal{P}(S)$ and $A \subseteq X$ Borel. Note that $P\mu$ is defined as a measure on X . Observe that P is a mass preserving operator $P : \mathcal{M}^+(S) \rightarrow \mathcal{M}^+(X)$ with transition kernel¹ $p(x, \cdot) = P\delta_x = Q_{\phi_{\Delta t}(x)}(\cdot - \phi_{\Delta t}(x))$. So,

$$(P\mu)(A) = \int_S (P\delta_x)(A) d\mu(x). \quad (4.5)$$

We shall prove the existence of a region B^* within X containing the equilibrium x^* such that P maps $\mathcal{P}(B^*)$ into itself. Hence, P yields a Markov operator on B^* . Due to (4.5), it suffices to understand the iterates $(P^n\delta_x)_n$ in $\mathcal{P}(B^*)$ in order to determine the dynamics of P .

4.3 Existence of a P -invariant neighbourhood

In this section assume that the conditions A1, A4 and A5 are satisfied. We show that if the perturbations are sufficiently small, as make precise by A5, the set B^* in assumption A4 is invariant under the operator P of the perturbed system defined in (4.2). That is, for every $x \in B^*$, $\phi(x) + C_{\phi(x)} \subseteq B^*$. First define

$$(P_0\mu)(A) = \int_S Q_x(A - x) d\mu(x). \quad (4.6)$$

Observe that $P_0\delta_x = Q_x(\cdot - x)$ and thus $P\delta_x = P_0\delta_{\phi(x)}$.

¹A function $p : S \times \mathcal{B}(S) \rightarrow [0, 1]$, defined as $p(x, E) = P\delta_x(E)$ for $x \in S$ and $E \in \mathcal{B}(S)$ is called the *transition function* (transition kernel). (c.f. Ethier and Kurtz, [32]).

Lemma 4.3.1. *For all $x \in S$, $\text{supp}(P_0\delta_x) = x + C_x$. Hence,*

$$\text{supp}(P\delta_x) = \phi(x) + C_{\phi(x)}.$$

Proof. By definition of P_0 , properties of the support and A2,

$$\text{supp}(P_0\delta_x) = \text{supp}(Q_x(\cdot - x)) = \text{supp}(Q_x) + x = x + C_x.$$

Now by definition of P and the above equality,

$$\text{supp}(P\delta_x) = \text{supp}(P_0\delta_{\phi(x)}) = \phi(x) + C_{\phi(x)}.$$

□

Lemma 4.3.2. *Assume A2 holds true. Then for $x, y \in S$ one has*

$$d_H(\text{supp}(P\delta_x), \text{supp}(P\delta_y)) \leq \theta(1 + L) \|x - y\|.$$

Proposition 4.3.3. *Assume that A2, A3 and A4 hold and let ψ_p be given by (2.5). Then $\psi_p(E) \subset B^*$ for any subset $E \subset B^*$. Consequently, the operator P given by (4.2) leaves invariant the set*

$$\mathcal{P}^* := \{\mu \in \mathcal{P}(S) : \text{supp}(\mu) \subset B^*\}.$$

Proof. Let $x \in B^*$. As $\phi(x^*) = x^*$, using Lemma 4.3.2 and Assumption A4,

$$\begin{aligned} \delta(\phi(x) + C_{\phi(x)}, x^*) &\leq \delta(\phi(x) + C_{\phi(x)}, \phi(x^*) + C_{x^*}) + \delta(\phi(x^*) + C_{x^*}, x^*) \\ &\leq d_H(\text{supp}(P\delta_x), \text{supp}(P\delta_{x^*})) + \delta(x^* + C_{x^*}, x^*) \\ &\leq \theta(1 + L) \|x - x^*\| + \sup_{y \in C_{x^*}} \|x^* + y - x^*\| \\ &\leq \theta(1 + L) R + \sup_{y \in C_{x^*}} \|y\| \\ &\leq \theta(1 + L) R + (1 - \theta(1 + L)) R = R. \end{aligned}$$

Thus, for all $x \in B^*$, $\delta(\phi(x) + C_{\phi(x)}, x^*) < R$ and consequently, $\phi(x) + C_{\phi(x)} \subset B^*$, as desired. According to the first part of the proof, $\psi_p(\{x\}) \subseteq B^*$ for every $x \in B^*$. Since B^* is closed, application of Proposition 2.3.7(i) and Lemma 4.3.1 completes the proof. □

4.4 Non-expansiveness of P

In Section 4.3 we have established that P as defined by (4.2) is a Markov operator on B^* , if the perturbations are sufficiently small –as made precise by A4. We consider the problem of existence of invariant measures for P on B^* .

Lemma 4.4.1. *Assume A1–A4 holds. If $f \in BL(B^*)$, then $Uf \in BL(B^*)$ with $|Uf|_{Lip} \leq \theta(1+L) \|f\|_{\max}$. Thus, $\|Uf\|_{\max} \leq \max(1, \theta(1+L)) \|f\|_{\max}$.*

Proof. Let $f \in BL(B^*)$ and $x, y \in B^*$. Then $\phi(x) + C_{\phi(x)} \subset B^*$, similarly for y . Moreover, there exists $\tilde{f} \in BL(X)$, an extension of f , such that $\|\tilde{f}\|_{\max} = \|f\|_{\max}$ (cf. [26, Lemma 5]). Consequently,

$$\begin{aligned}
 |Uf(x) - Uf(y)| &= |\langle f, P\delta_x \rangle - \langle f, P\delta_y \rangle| \\
 &= \left| \int_X \tilde{f}(z + \phi(x)) Q_{\phi(x)}(dz) - \int_X \tilde{f}(z + \phi(y)) Q_{\phi(y)}(dz) \right| \\
 &\leq \left| \int_X \left\{ \tilde{f}(z + \phi(x)) - \tilde{f}(z + \phi(y)) \right\} Q_{\phi(x)}(dz) \right| \\
 &\quad + \left| \int_X \tilde{f}(z + \phi(y)) \{Q_{\phi(x)} - Q_{\phi(y)}\}(dz) \right| \\
 &\leq \|\tilde{f}\|_{Lip} \|\phi(x) - \phi(y)\| \int_{B^*} Q_{\phi(x)} dz \\
 &\quad + \|\tilde{f}(\cdot + \phi(y))\|_{\max} \cdot \|Q_{\phi(x)} - Q_{\phi(y)}\|_{FM} \\
 &\leq \|\tilde{f}\|_{\max} \theta \|x - y\| + \|\tilde{f}\|_{\max} L \|\phi(x) - \phi(y)\| \\
 &= \theta(1+L) \|f\|_{\max} \|x - y\|.
 \end{aligned}$$

Thus, $|Uf|_{Lip} \leq \theta(1+L) \|f\|_{\max}$. Also, $\|Uf\|_{\infty} \leq \|f\|_{\infty}$. Consequently, $\|Uf\|_{\max} \leq \max(1, \theta(1+L)) \|f\|_{\max}$. $\square$

Proposition 4.4.2. *P satisfies $\|P\mu - P\nu\|_{FM} \leq \max(1, \theta(1+L)) \|\mu - \nu\|_{FM}$. Hence, if $\theta(1+L) \leq 1$, then P is nonexpansive. In particular, P satisfies the e -property.*

Proof. By Lemma 4.4.1, for $\mu, \nu \in \mathcal{M}^+(S)$

$$\begin{aligned}
 |\langle f, P\mu - P\nu \rangle| &= |\langle Uf, (\mu - \nu) \rangle| \leq \|Uf\|_{\max} \|\mu - \nu\|_{FM} \\
 &\leq \max(1, \theta(1+L)) \|f\|_{\max} \|\mu - \nu\|_{FM}.
 \end{aligned}$$

Hence, by the assumption $\theta(1+L) \leq 1$,

$$\begin{aligned} \|P\mu - P\nu\|_{FM} &= \sup_{f \in BL(B^*), \|f\|_{\max} \leq 1} |\langle f, P\mu - P\nu \rangle| \\ &\leq \|\mu - \nu\|_{FM}. \end{aligned}$$

Thus, P is nonexpansive. Consequently, P^n is nonexpansive. It follows that $\{U^n f : n \geq 0\}$ is equicontinuous. That is, P satisfies the e -property. $\square$

4.5 Dynamics of the supports

Recall ψ_p defined by (2.5). Under the assumptions A2, A3, A4, ψ_p maps closed subsets of B^* to closed subsets of B^* . ψ_p is monotonic, that is, if $A \subset B$, then $\psi_p(A) \subset \psi_p(B)$. Note that $\psi_p(E)$ is the closure of the set of possible states after the first intervention when starting with states in E , according to Lemma 4.3.1. Based on this observation, starting at x_0 , after n -interventions, the possible set of states is $\psi_p^n(\{x_0\})$.

The following is a simple consequence of Proposition 2.3.7(ii) concerning the supports of $P^n \delta_{x_0}$.

Corollary 4.5.1. *For each $x_0 \in B^*$ and $n \in \mathbb{N}$, one has the following:*

1. $\text{supp}(P^n \delta_{x_0}) = \psi_p^n(\{x_0\})$.
2. If $0 \in C_x$ for all $x \in B^*$, then $x_n = \phi_{\Delta t}^n(x_0) \in \text{supp}(P^n \delta_{x_0})$.

Proposition 4.5.2. *Assume that A2, A3 and A4 hold. Then the map ψ_p is a contraction on the hyperspace $\mathcal{H}(B^*)$.*

Proof. Note that ψ_p leaves B^* invariant due to Proposition 4.3.3. A straightforward computation shows that $d_H(\psi_p(E), \psi_p(F)) \leq \theta(1+L)d_H(E, F)$ for $E, F \in \mathcal{H}(B^*)$. So we get a contraction if $\theta(1+L) < 1$, which follows from A4. $\square$

A direct consequence of Proposition 4.5.2, the completeness of B^* , and the Banach Fixed Point Theorem is

Corollary 4.5.3. *Assume that A2, A3 and A4 hold. Then ψ_p has a unique fixed point E^* in $\mathcal{H}(B^*)$ and $\psi_p^n(E) \rightarrow E^*$ as $n \rightarrow \infty$ for all $E \in \mathcal{H}(B^*)$.*

The following result is a direct consequence of Proposition 2.3.7 and Corollary 4.5.3.

Corollary 4.5.4. *Assume that A2, A3 and A4 hold. If P has an invariant measure μ^* , then $\text{supp}(\mu^*) = E^*$.*

We like to discuss some more tools that may be used for showing convergence of sequences of sets. One such a tool is the topological lower limit. Let us first recall its definition (see [95, p.24],[60]).

Definition 4.5.5. The (Kuratowski) topological lower limit of a sequence of subsets $(A_n)_{n \in \mathbb{N}}$ of a separable metric space X is

$$\text{Li}_{n \rightarrow \infty} A_n = \{x \in X : \exists (x_n) \in X, N \in \mathbb{N} : x_n \in A_n \forall n \geq N \text{ and } x_n \rightarrow x\},$$

where the convergence $x_n \rightarrow x$ is in the metric topology of X . Moreover, the topological upper limit of $(A_n)_{n \in \mathbb{N}}$ is

$$\text{Ls}_{n \rightarrow \infty} A_n = \{x \in X : \exists (x_{n_k}) \in X : x_{n_k} \in A_{n_k} \forall k \in \mathbb{N} \text{ and } x_{n_k} \rightarrow x\},$$

If $\text{Li}_{n \rightarrow \infty} A_n = \text{Ls}_{n \rightarrow \infty} A_n$, the sequence $(A_n)_{n \in \mathbb{N}}$ is called topologically convergent with topological limit denoted by $\text{Lt}_{n \rightarrow \infty} A_n$.

It should be noted that in the case when the underlying metric space X is compact, $\text{Li}_{n \rightarrow \infty} A_n = A$ if and only if $\{A_n\}$ converges to A in the Hausdorff distance (cf. [65, p. 382]). In an arbitrary metric space $\lim_{n \rightarrow \infty} d_H(A_n, A) = 0$ implies $\text{Li}_{n \rightarrow \infty} A_n = A$, but the reverse implication is not true (cf. [56, p. 309]). In [56, Theorem 5.1] it is shown that if P is Markov–Feller such that it is asymptotically stable with (unique) invariant probability measure μ^* , then $\text{Lt}(\psi_p^n(A)) = \text{supp}(\mu^*)$, for any $A \subset \text{supp}(\mu^*)$, closed, non-empty.

4.6 Existence, uniqueness and stability of an invariant measure

In this section we show existence, uniqueness and stability of an invariant measure μ^* for P as defined by (4.2). First we recall concepts defined in [56].

Definition 4.6.1. A Markov operator P on $\mathcal{M}(S)$ is called *globally concentrating* if for every $\epsilon > 0$ and every bounded Borel set $A \subset S$ there exists a bounded

Borel set $B \subset S$ and an integer n_0 such that $P^n \mu(B) \geq 1 - \epsilon$ for $n \geq n_0$ and $\mu \in \mathcal{P}(A)$. It is called *locally concentrating* if for every $\epsilon > 0$ there exists a number $\alpha > 0$ such that for every bounded Borel set $A \subset S$ there exists a Borel set $C \subset S$ with diameter $\text{diam } C \leq \epsilon$ and an integer n_* such that $P^n \mu(C) \geq \alpha$ for all $n \geq n_*$ and $\mu \in \mathcal{P}(A)$.

The following result is due to Szarek [56, Theorem 5.4.], for a complete proof see [82].

Theorem 4.6.2. *Let (S, d) be a separable, complete metric space. Then a non-expansive, locally and globally concentrating Markov operator is asymptotically stable.*

Note that globally concentrating is immediate in our case as S is bounded. The following theorem shows that P is locally concentrating.

Theorem 4.6.3. *Assume A1–A5 to hold. Let $E^* \in \mathcal{H}(B^*)$ be the unique fixed point of ψ_p and let $z \in E^*$. Then for all $r > 0$, there exists $N_r \in \mathbb{N}$ and $\alpha > 0$, such that*

$$P^n \delta_x(B(z, r)) \geq \alpha$$

for all $x \in B^*$ and $n \geq N_r$. In particular, P is locally concentrating on B^* .

Proof. Recall that E^* exists due to Corollary 4.5.3. Let $r > 0$ and $x_0 \in B^*$. Then $\psi_p^n(\{x_0\}) \rightarrow E^*$ in $\mathcal{H}(B^*)$ by Corollary 4.5.3. So, there exists $N = N_{r, x_0}$ such that

$$d_H(\psi_p^n(\{x_0\}), E^*) < \frac{1}{4}r$$

for all $n \geq N_{r, x_0}$. Let $x \in B^*$. Then, with $\theta_p := \theta(1 + L) < 1$,

$$d_H(\psi_p^n(\{x\}), \psi_p^n(\{x_0\})) \leq \theta_p^n d(x, x_0) \leq 2R^* \theta_p^n,$$

where we use Proposition 2.3.7 ii) and Lemma 4.3.2. Since $\theta_p < 1$ (Assumption A5), there is N_0 , independent of x , such that $2R^* \theta_p^n < \frac{1}{4}r$ for all $n \geq N_0$. Then

$$d_H(\psi_p^n(\{x\}), E^*) < \frac{1}{4}r + \frac{1}{4}r = \frac{1}{2}r$$

for all $n \geq N_r := \max(N_0, N_{r,x_0})$ and all $x \in B^*$. Hence,

$$\begin{aligned} \frac{1}{2}r &> d_H(E^*, \psi_p^n(\{x\})) = d_H(E^*, \text{supp } P^n \delta_x) \\ &\geq \sup_{e \in E^*} d(e, \text{supp } P^n \delta_x) \\ &\geq d(z, \text{supp } P^n \delta_x) \\ &= \inf_{y \in \text{supp } P^n \delta_x} d(z, y) \quad \text{for all } n \geq N_r, x \in B^*. \end{aligned}$$

Thus, for all $x \in B^*$ and all $n \in \mathbb{N}$ such that $n \geq N_r$, there exists $x_n \in \text{supp } P^n \delta_x$ such that $d(z, x_n) < r$, i.e. $x_n \in B(z, r)$. So, $P^n \delta_x(B(z, r)) > 0$ for all $n \geq N_r$ and all $x \in B^*$. Moreover, for $n \geq N_r$,

$$P^n P_0 \delta_x(B(z, r)) = \int_{\text{supp}(P_0 \delta_x)} P^n \delta_y(B(z, r)) dP_0 \delta_x(y) > 0.$$

Hence, for every $x \in B^*$, $P^n P_0 \delta_x(B(z, r)) > 0$ for all $n \geq N_r$. Next we use a compactness argument to obtain uniformity in x in the lower bound.

Let D be the closure in X of $\phi(B^*)$. Then by Assumption A5, D is compact in X . Observe that

$$P^{n+1} \delta_x = P^n (P_0 \delta_{\phi(x)}).$$

Since P is Feller, P^n is Feller. Also, according to Lemma 4.4.1, $x \mapsto P_0 \delta_x : S \rightarrow \mathcal{M}^+(S)_{BL}$ is continuous. Proposition 2.3.3 yields that P_0 is Feller. Thus, $P^n P_0$ is Feller. Hence $x \mapsto P^n P_0 \delta_x(B(z, r)) : S \rightarrow \mathbb{R}_+$ is lower semicontinuous. Moreover, a strictly positive lower semicontinuous function on a compact set is bounded away from zero on this set.

So, by compactness of D , for each $n \geq N_r$, there exists $\beta_n > 0$ such that

$$\min_{x \in D} P^n P_0 \delta_x(B(z, r)) \geq \beta_n,$$

and for all $x \in B^*$ and $n \geq N_r$.

Finally, we show that $P^m \delta_x(B(z, r)) \geq \beta_n$ for all $x \in B^*$ and $m \geq N_r + 1$. Indeed, for $m \geq N_r + 1$ and $x \in B^*$, and using the invariance of B^* under P :

$$\begin{aligned} P^m \delta_x(B(z, r)) &= \int_{B^*} (P^n \delta_y)(B(z, r)) dP^{m-n} \delta_x(y) \\ &\geq \beta_n \cdot P^{m-n} \delta_x(B^*) = \beta_n > 0. \end{aligned}$$

□

Corollary 4.6.4. *Assume A1–A5 to hold. Let $z \in E^*$. Then for all $r > 0$,*

$$\inf_{x \in B^*} \liminf_{n \rightarrow \infty} P^n \delta_x (B(z, r)) > 0. \quad (4.7)$$

Theorem 4.6.5. *(Existence, uniqueness and stability) Assume A1–A5 to hold. Then there exists a unique ergodic measure μ^* of P restricted to B^* , which is asymptotically stable on B^* .*

Proof. By Theorem 4.6.3, P is locally concentrating on B^* . Since B^* is bounded, P is globally concentrating on B^* . Due to Proposition 4.4.2, P is nonexpansive. Theorem 4.6.2 now gives the result. □

4.7 General classes of examples: semi-linear equations

In this section we present three general classes of abstract examples that satisfy the conditions A3–A5, with specific examples derived from population dynamics. Thus, our set-up covers a range of interesting examples, while we also show how to establish technically the fundamental assumptions A3 and A5 in concrete cases. A5 will be discussed in generality, A3 on a mere case-by-case basis. The remaining assumptions A1–A2 and A4 concern the type of random perturbation. A1 and A2 are quite flexible. For example, one may take the support of each Q_x the same (as in [3]), A4 is a constraint on the ‘speed’ at which Q_x is allowed to change in relation to the contractiveness of the map ϕ determined by A3. Thus we focus in this section on A3 and A5. The random interventions can then be adjusted to satisfy the remaining assumptions.

Consider the following abstract Cauchy problem in a Banach space $(X, \|\cdot\|)$:

$$x'(t) = Ax(t) + F(x(t)), \quad x(0) = x_0 \in X. \quad (4.8)$$

where $A : \mathcal{D}(A) \subset X \rightarrow X$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on X and $F : X \rightarrow X$ is a local Lipschitz map. There exist $M \geq 1, \omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. The classical results on C_0 -semigroups and semilinear equations (e.g. [30, 59, 72]) yield the existence of a unique mild solution to (4.8) defined on a maximal interval of existence $[0, \tau_{x_0})$,

i.e. a continuous map $x : [0, \tau_{x_0}) \rightarrow X$ such that

$$x(t) = T(t)x_0 + \int_0^t T(t-s)F(x(s))ds \quad (4.9)$$

holds for all $t \in [0, \tau_{x_0})$. The solution depends continuously on x_0 .

We need to assume that $(T(t))_{t \geq 0}$ is either immediately compact or eventually compact, i.e., respectively, $T(t)$ is a compact operator for $t > 0$ or there exists $t_0 > 0$ such that $T(t_0)$ is compact, hence $T(t)$ is compact for all $t \geq t_0$.

Below we consider the following particular situations:

1. A nonlinear system of ODEs: X is finite dimensional and A is a square matrix.
2. $(T(t))_{t \geq 0}$ is an analytic, immediately compact, uniformly exponentially stable semigroup.
3. $(T(t))_{t \geq 0}$ is an eventually compact, uniformly exponentially stable semigroup.

Assume now that $(X, \|\cdot\|)$ is separable and that $(T(t))_{t \geq 0}$ is immediately compact as in cases (1) and (2) above. E.g. $(T(t))_{t \geq 0}$ may be an analytic semigroup with compact resolvent [30, Theorem II.4.29]. For each $B \subset X$ bounded there exists $\tau_B > 0$ such that all mild solutions $x(\cdot, x_0)$ to (4.8), with $x_0 \in B$, are at least defined on $[0, \tau_B]$. (This follows from the standard existence proof using Picard iteration, [72]). Put $\phi_t(x_0) := x(t; x_0)$, the solution map.

The following is a slightly restrictive reformulation of [64, Proposition 1.6], which is needed below:

Lemma 4.7.1. *Let X be a separable Banach space. If (f_n) is a sequence in $C([a, b], X)$ and $h(t) := \beta(\{f_n(t) | n \in \mathbb{N}\})$, where β is the Hausdorff measure of non-compactness in X , then h is a measurable function on $[a, b]$ and*

$$\beta \left(\left\{ \int_a^b f_n(t) dt | n \in \mathbb{N} \right\} \right) \leq \int_a^b h(t) dt. \quad (4.10)$$

Now we have

Proposition 4.7.2. *If $B \subset X$ is bounded, then $E_t^B := \{x(t; x_0) | x_0 \in B\}$ is relatively compact in X for each $0 < t \leq \tau_B$.*

Proof. Fix $0 < t \leq \tau_B$. Let (x_n) be a sequence in E_t^B . There exist $x_0^{(n)} \in B$ such that $x_n = x(t; x_0^{(n)})$. Put $y_n(s) := T(t-s)F(x(s; x_0^{(n)}))$. The sequence $x(s; x_0^{(n)}) \subset X$ is bounded for each $0 \leq s \leq t$; F is locally Lipschitz, hence, $\{F(x(s; x_0^{(n)})) | n \in \mathbb{N}\}$ is bounded too. $T(t-s)$ is compact for $0 \leq s < t$, so $\{y_n(s) | n \in \mathbb{N}\}$ is relatively compact. Now apply Lemma 4.7.1 by putting $f_n = y_n \in C([0, t], X)$ and $h(s) = 0$ for $0 \leq s < t$. Thus,

$$\left\{ \int_0^t T(t-s)F(x(s; x_0^{(n)})) ds | n \in \mathbb{N} \right\}$$

is relatively compact. Moreover, $\{T(t)x_0^{(n)} | n \in \mathbb{N}\}$ is relatively compact. The sum of two compact sets is compact, hence $\{x_n | n \in \mathbb{N}\}$ is relatively compact and (x_n) has a convergent subsequence. Thus, E_t^B is relatively sequentially compact, hence relatively compact. $\square$

If $B \subset X$ is a bounded set such that the solution maps $\phi_t : B \rightarrow X$ leave B invariant for $t \in [0, \tau_B]$, then solutions starting in B exist for all time and remain in B . The solution operator $\phi_t : B \rightarrow B$ is defined for all $t > 0$. We obtain from Proposition 4.7.2,

Corollary 4.7.3. *The operator ϕ_t satisfies Assumption A5 with $S = B$ for all $t > 0$.*

Assume that a steady state $x^* \in X$ exists for (4.8) (or (4.9)). We need the following condition on F near x^* :

C1 For every $\epsilon > 0$ there exists R such that for any $x_i \in \overline{B}(x^*, R)$ one has $\|F(x_1) - F(x_2)\| \leq \epsilon \|x_1 - x_2\|$.

For example, if U is an open neighbourhood of x^* in X and $F : U \rightarrow X$ is C^1 , and $DF(x^*) = 0$, then C1 is satisfied. Note that if C1 holds for the norm $\|\cdot\|$, then it holds for any norm equivalent to $\|\cdot\|$. Under Condition C1, x^* will be exponentially attracting on a neighbourhood of x^* . However, the dynamics of (4.8) need not be contractive near x^* as required in A3. For instance, it could be that a solution $t \mapsto x(t, x_0)$ follows an elliptic spiral orbit so that its distance to x^* may increase before converging to zero. We will show that the dynamics is contracting for the norm $\|\cdot\|_A$ defined by

$$\|x\|_A := \sup_{t \geq 0} e^{-\omega t} \|T(t)x\|, \quad x \in X, \quad (4.11)$$

which is equivalent to $\|x\|: \|x\| \leq \|x\|_A \leq M \|x\|$. Moreover,

$$\|e^{At}x\|_A \leq e^{\omega t} \|x\|_A.$$

Denote by $B_A(x, r)$ and $\overline{B}_A(x, r)$ the open and closed ball for $\|\cdot\|_A$ centered at x of radius r .

Lemma 4.7.4. *Let U be an open neighbourhood of x^* in X , let $F : U \rightarrow X$ be such that C1 holds. Let $u_1, u_2 \in U$ and let $x_1(t), x_2(t)$ be the solutions of (4.8) with $x_i(0) = u_i$, $i = 1, 2$, for all t for which both solutions exist. Then for every $\epsilon > 0$ there exists an $R > 0$ and $\tau > 0$ such that for all $u_i \in \overline{B}_A(x^*, R)$, $x_i(t)$ is defined for all $0 \leq t \leq \tau$ and*

$$\|x_1(t) - x_2(t)\|_A \leq \|u_1 - u_2\|_A e^{(\epsilon + \omega)t}.$$

Proof. Let $\epsilon > 0$ and take, with the aid of C1, $R_0 > 0$ such that $\|F(x) - F(y)\| \leq \frac{\epsilon}{M} \|x - y\|$ for all $x, y \in \overline{B}(x^*, R_0)$. Then $\|F(x) - F(y)\|_A \leq \epsilon \|x - y\|_A$. Choose $R \in (0, \frac{1}{2}R_0]$ such that $\overline{B}_A(x^*, R) \subseteq \overline{B}(x^*, \frac{1}{2}R_0)$. There exists $L > 0$ such that for $u_i \in \overline{B}(x^*, \frac{1}{2}R_0)$, one has $x_i(t) \in \overline{B}(x^*, R_0)$ for all $t \in [0, L]$. In particular, for $u_1, u_2 \in \overline{B}_A(x^*, R)$ and $t \in [0, L]$ we have

$$\begin{aligned} \|x_1(t) - x_2(t)\|_A &\leq \|T(t)(u_1 - u_2)\|_A \\ &\quad + \int_0^t \|T(t-s)\|_A \|F(x_1(s)) - F(x_2(s))\|_A ds \\ &\leq e^{\omega t} \|u_1 - u_2\|_A + \int_0^t e^{\omega(t-s)} \epsilon \|x_1(s) - x_2(s)\|_A ds. \end{aligned}$$

Thus, by Gronwall's Lemma (see e.g. [88]), for $t \in [0, L]$,

$$\|x_1(t) - x_2(t)\|_A \leq \|u_1 - u_2\|_A e^{(\epsilon + \omega)t}. \quad (4.12)$$

□

The following is an immediate consequence of Lemma 4.7.4.

Corollary 4.7.5. *Suppose all conditions of Lemma 4.7.4 are satisfied. Assume that $\epsilon + \omega < 0$. Then*

(i) Solutions with initial condition $x_0 \in \overline{B}_A(x^*, R)$ exist globally and are contained in $\overline{B}_A(x^*, R)$.

Let $(\phi_t)_{t \geq 0}$ be the associated dynamical system in $\overline{B}_A(x^*, R)$.

(ii) For each $t > 0$ there exist a $\theta \in [0, 1)$ such that for all $x, y \in \overline{B}_A(x^*, R)$

$$\|\phi_t(x) - \phi_t(y)\|_A \leq \theta \|x - y\|_A.$$

That is, Assumption A3 holds for ϕ_t on $\overline{B}_A(x^*, R)$.

4.7.1 Finite dimensional state space

Let X be finite dimensional and assume that F is such that the system of ODEs (4.8) is well-posed. The solution map $\phi_t(x_0) := x(t; x_0)$ is then defined for all $t \geq 0$ and depends continuously on x_0 . In a finite dimensional Banach space, any continuous map is compact, so A5 follows immediately.

For discussion of the contractiveness condition A3), consider the following example from population dynamics.

Specific example: The Rosenzweig-MacArthur Model (RMM)

The Rosenzweig-MacArthur Model (RMM), sometimes called a Kolmogorov type predator-prey model (c.f. [36, 77]), combines logistic growth of the prey population (v —for victims) with satiation of predation by the predator population (p), modeled with a Holling type II functional response:

$$\dot{v} = rv \left(1 - \frac{v}{K}\right) - \frac{av}{b + v} \cdot p, \quad (4.13)$$

$$\dot{p} = -dp + h \cdot \frac{av}{b + v} \cdot p. \quad (4.14)$$

It can be considered as the simplest biologically realistic predator-prey model beyond the classical Lotka–Volterra type models. The biologically relevant parameter settings are $a, b, d, h, r, K > 0$, which are assumed to hold. Moreover, we focus our attention on solutions in $\mathbb{R}_+^2$, since only these are biologically interpretable. We write (4.13)–(4.14) simply as an equation: $\dot{x}(t) = f(x(t))$, with $x = (v, p)^T \in \mathbb{R}^2$.

Remark 4.7.6. Since the right-hand side of (4.13)–(4.14) is locally Lipschitz on the set where $|v - b| \geq \delta > 0$, local existence and uniqueness of a solution to the RMM with the initial value (v_0, p_0) at $t = 0$ is clear, using standard arguments (see e.g. [88]). Positive solutions exist globally.

Assume $ha \neq d$ and put $\eta := \frac{d}{ha-d}$. The RMM has a unique steady state $x^* := (b\eta, \frac{rb}{a}(1+\eta)(1-\frac{b}{K}\eta))$ in $\text{Int}(\mathbb{R}_+^2)$ if and only if $a > (1 + \frac{b}{K}) \cdot \frac{d}{h}$. The linearisation of f around x^* is

$$A := Df(x^*) = \begin{bmatrix} \frac{rd}{ha}(1 - \frac{b}{K}(1+2\eta)) & \frac{-d}{h} \\ r(h - \frac{d}{a})(1 - \frac{b\eta}{K}) & 0 \end{bmatrix}. \quad (4.15)$$

The matrix A has eigenvalues given explicitly by

$$\lambda_{\pm} = \frac{(c - r + \frac{hac}{d}) \pm \sqrt{D}}{2},$$

where $c = \frac{rd}{ha}(1 - \frac{b\eta}{K})$, $D = (c - r + \frac{hac}{d})^2 - 4c(ah - d)$. So x^* is linearly stable when $\eta < \frac{K}{b} < 1 + 2\eta$. Let $g(x) := f(x) - Ax$. Then $g \in C^1(\mathbb{R}_+^2, \mathbb{R}^2)$ and $Dg(x^*) = 0$. So the RMM satisfies the assumptions of Lemma 4.7.4.

Moreover, for the uniformly continuous semigroup e^{tA} generated by A , the growth bound ω equals its spectral bound $s(A) = \max(\lambda_+, \lambda_-) = \lambda_+$ (cf. [30, Corollary IV.2.4.]). Thus, $\omega < 0$. As a consequence of Lemma 4.7.4 or Corollary 4.7.5, we obtain that there is an invariant ball $\overline{B}_A(x^*, R)$ on which the solution operator is a strict contraction. That is, Assumption A3 holds.

Let S be the normalized matrix of eigenvectors of A . Then $A = S \cdot \Lambda \cdot S^{-1}$, where Λ is the diagonal matrix with λ_+ and λ_- on its diagonal. So, $e^{At} = S \cdot e^{\Lambda t} \cdot S^{-1}$ and $\|e^{At}\| \leq \|S\| \|e^{\Lambda t}\| \|S^{-1}\|$. Hence,

$$\sup_{t \geq 0} e^{-\omega t} \|e^{At}\| \leq \|S\| \|S^{-1}\|. \quad (4.16)$$

Now let $C_x = C = [-c_{\max}, 0] \times [-c_{\max}, 0]$, i.e. one removes individuals from the population at intervention time ('harvesting', 'sampling'), with a maximal 'catch size' $c_{\max}$. Using (4.16) we obtain an effective condition for A4 in terms of the original Euclidean metric $\|\cdot\|_E$ on $\mathbb{R}_+^2$:

$$\text{diam}_E(C) \leq \frac{1}{\|S\| \|S^{-1}\|} (1 - \theta(1 + L)) R^*.$$

Consequently, for concrete parameter values, a condition on the size of C can be computed explicitly. If one defines $Q_{(v,p)} = Q_v^1 \otimes Q_p^2$, where Q_x^i are Beta distributions on $[-c_{\max}, 0]$ with parameters $-c_{\max}, 0, \beta_i, \alpha_i^* \frac{v}{v+p}$, then $x \mapsto Q_x^i$ is Lipschitz continuous for the total variation norm (cf. Example 2.4. in [3]). Hence A1 is satisfied. A2 is trivial.

4.7.2 Infinite dimensional state space: immediately compact semigroups

We shall now continue with a specific example in population dynamics derived from the RMM to illustrate the verification and validity of assumptions A3 and A5 in applications.

Specific example 1: RMM with diffusion.

One may add diffusion of species to the RMM model. So consider an open bounded set Ω in $\mathbb{R}^2$, with uniformly C^2 - boundary and dynamics of the prey and predator species governed by

$$\frac{\partial v}{\partial t} = D_v \Delta v + rv \left(1 - \frac{v}{K}\right) - \frac{av}{b+v} \cdot p, \quad (4.17)$$

$$\frac{\partial p}{\partial t} = D_p \Delta p - dp + h \cdot \frac{av}{b+v} \cdot p, \quad (4.18)$$

complemented by Neumann boundary conditions for both v and p on $\partial\Omega$. All parameters are assumed to be strictly positive and such that the associated RMM has a unique stable steady state (v^*, p^*) in the interior of $\mathbb{R}_+^2$, see Section 4.7.1.

The Laplacian Δ defines a sectorial operator $(A, \mathcal{D}(A))$ in $C(\overline{\Omega})$ with

$$\mathcal{D}(A) = \left\{ f \in \cap_{p \geq 1} W^{2,p}(\Omega) \mid \Delta f \in C(\overline{\Omega}), \frac{\partial f}{\partial \nu} = 0 \text{ on } \partial\Omega \right\},$$

$Af = \Delta f$. Moreover, $\mathcal{D}(A) \subset C^1(\overline{\Omega})$ (cf. [59, Corollary 3.1.24, p. 100]). Because Ω is bounded, $\mathcal{D}(A)$ is dense in $C(\overline{\Omega})$ and $(A, \mathcal{D}(A))$ generates a strongly continuous analytic semigroup $(T(t))_{t \geq 0}$ in $C(\overline{\Omega})$ (cf. [30, 59]). $\mathcal{D}(A)$, equipped with the graph norm, embeds continuously into $C(\overline{\Omega})$. Since $\mathcal{D}(A) \subset C^1(\overline{\Omega})$, by the Closed Graph Theorem $\mathcal{D}(A)$ embeds continuously into $C^1(\overline{\Omega})$, too. $\overline{\Omega}$ is compact. The Arzela–Ascoli Theorem yields that $C^1(\overline{\Omega})$ embeds into $C(\overline{\Omega})$ compactly. One concludes that $(A, \mathcal{D}(A))$ has compact resolvent (cf. [30, Proposition II.4.25]). Thus, $(T(t))_{t \geq 0}$ is an immediately compact semigroup in $C(\overline{\Omega})$ [30, Theorem II.4.29].

Note that the reaction terms in (4.17)–(4.18) contain the factor $\frac{v}{b+v}$ that may cause discontinuities when v is allowed to be negative at places. However, only non-negative solutions are biologically relevant. For these $\frac{v}{b+v}$ is bounded. Thus, let $C(\overline{\Omega})_+$ be the closed convex cone in $C(\overline{\Omega})$ consisting of everywhere non-negative functions: the positive cone.

Proposition 4.7.7. *For each initial condition $(v^0, p^0) \in C(\overline{\Omega})_+^2$, there exists a unique non-negative mild solution $(v(t), p(t))$ to (4.17)–(4.18) that exists for all time $t \geq 0$. The corresponding non-linear solution operator*

$$\phi_t : C(\overline{\Omega})_+^2 \rightarrow C(\overline{\Omega})_+^2 : \phi_t(v^0, p^0) := (v(t), p(t))$$

is a locally Lipschitz, compact map for $t > 0$. In particular, ϕ_t satisfies Assumption A5.

Proof. The reaction terms in (4.17)–(4.18) define a local Lipschitz perturbation $C(\overline{\Omega})_+^2 \rightarrow C(\overline{\Omega})_+^2$ of the positive C_0 -semigroup $T_v(t) \times T_p(t)$ in $C(\overline{\Omega})^2$ defined by the diffusion semigroups $(T_v(t))_{t \geq 0}$ and $(T_p(t))_{t \geq 0}$ in $C(\overline{\Omega})$ associated to $D_v \Delta$ and $D_p \Delta$ respectively. The proof of local existence and positivity follows the standard approach for semilinear equations by using Picard iteration. To deduce positivity it is handier to consider a modified version of (4.17)–(4.18). Let $B \subset C(\overline{\Omega})_+^2$ be $\|\cdot\|_\infty$ -bounded. There exists $\delta > 0$ such that $\|v\|_\infty + \|p\|_\infty \leq \delta$ for all $(v, p) \in B$. Put

$$B' := \left\{ (v, p) \in C(\overline{\Omega})_+^2 \mid \|v\|_\infty + \|p\|_\infty \leq 2\delta \right\}.$$

Rewrite (4.17)–(4.18) into

$$\frac{\partial v}{\partial t} = D_v \Delta v + \left(r - \frac{2r\delta}{K} - \frac{2a}{b}\delta \right) v + \frac{r}{K} v (2\delta - v) + av \left(\frac{2\delta}{b} - \frac{p}{b+v} \right) \quad (4.19)$$

$$\frac{\partial p}{\partial t} = D_p \Delta p - dp + ha \cdot \frac{v}{b+v} \cdot p \quad (4.20)$$

The operator $v \mapsto D_v \Delta v + \left(r - \frac{2r\delta}{K} - \frac{2a}{b}\delta \right) v$ is the generator of the positive C_0 -semigroup $\left(e^{\left(r - \frac{2r\delta}{K} - \frac{2a}{b}\delta \right)t} T_v(t) \right)_{t \geq 0}$ in $C(\overline{\Omega})$, while $v \mapsto D_p \Delta p - dp$ generates the positive C_0 -semigroup $\left(e^{-dt} T_p(t) \right)_{t \geq 0}$. Denote these two semigroups by $(T_v^\delta(t))_{t \geq 0}$ and $(T_p^\delta(t))_{t \geq 0}$ respectively. Put $T^\delta(t) := T_v^\delta(t) \times T_p^\delta(t)$ and

$$F_\delta(v, p) := \begin{bmatrix} \frac{r}{K} v (2\delta - v) + av \left(\frac{2\delta}{b} - \frac{p}{b+v} \right) \\ ha \cdot \frac{v}{b+v} \cdot p \end{bmatrix}$$

Then F_δ is a *positive* Lipschitz map from B' into $C(\overline{\Omega})^2$. There exists $\tau = \tau_{B, \delta} > 0$ such that the mild solution to (4.19)–(4.20) with initial condition $X_0 = (v_0, p_0) \in B$ is obtained as limit in $B'_\phi := \{u \in C([0, \tau], B') \mid u(0) = X_0\}$ of the

sequence u_n defined by $u_0(t) := \hat{T}(t)\phi$ and

$$u_{n+1}(t) := \hat{T}(t)\phi + \int_0^t \hat{T}(t-s) F_\delta(u_n(s)) ds.$$

Note that $u_0(t) \geq 0$ for all t . Inductively, $u_n(t) \geq 0$ too. τ is chosen such that $u_n \in B'_\phi$ for all n and the series (u_n) converges in $C(\bar{\Omega})_+^2$. Since $C(\bar{\Omega})_+$ is closed in $C(\bar{\Omega})$, the limit of (u_n) is positive. Application of Gronwall's Lemma yields local Lipschitz continuity of the solution map ϕ_t for sufficiently small t .

The semigroup property yields that ϕ_t is locally Lipschitz for any $t \geq 0$. Moreover, because $\bar{\Omega}$ is compact, $C(\bar{\Omega})$ is separable and Proposition 4.7.2 gives the compactness of ϕ_t for $t > 0$ small, hence for all $t > 0$.

Now for the global existence, let $x_0 \in B$ and u the unique maximally defined positive solution to (4.17)–(4.18) or (4.19)–(4.20) with $u(0) = x_0$. Let $[0, \tau_{x_0})$ be the maximal time interval on which u is defined. Then either $\tau_{x_0} = \infty$ or $\tau_{x_0} < \infty$ and $\lim_{t \uparrow \tau_{x_0}} \|x(t)\| = \infty$. We now show that blow-up in finite time is impossible. Put $x(t) = (v(t), p(t))$. Then by positivity of v, p and $T_v(t)$,

$$\begin{aligned} 0 \leq v(t) &= T_v(t)v(0) + \int_0^t T_v(t-s) \left[rv(s) - \frac{r}{K}v(s)^2 - \frac{av(s)}{b+v(s)} \cdot p(s) \right] ds \\ &\leq T_v(t)v(0) + \int_0^t T_v(t-s) [rv(s)] ds \end{aligned}$$

as ordering in the Banach lattice $C(\bar{\Omega})$. Consequently,

$$\begin{aligned} \|v(t)\|_\infty &\leq \|T_v(t)v(0)\|_\infty + \int_0^t \|T_v(t-s)[rv(s)]\|_\infty ds \\ &\leq M_v e^{\omega_v t} \|v(0)\|_\infty + r M_v \int_0^t e^{\omega_v(t-s)} \|v(s)\|_\infty ds. \end{aligned}$$

Gronwall's Lemma yields

$$\|v(t)\|_\infty \leq M_v \|v(0)\|_\infty e^{(\omega_v + r M_v)t} \quad (4.21)$$

for all $t \in [0, \tau_\phi)$. Similarly,

$$\begin{aligned} 0 \leq p(t) &= T_p(t) p(0) + \int_0^t T_p(t-s) \left[-dp(s) + h \cdot \frac{av(s)}{b+v(s)} \cdot p(s) \right] ds \\ &\leq T_p(t) p(0) + \int_0^t T_p(t-s) [ah \cdot p(s)] ds \end{aligned}$$

in $C(\overline{\Omega})$, so

$$\|p(t)\|_\infty \leq M_p \|p(0)\|_\infty e^{(w_p+ah)t} \quad (4.22)$$

for all $t \in [0, \tau_{x_0})$. Estimates (4.21) and (4.22) yield that $\|x(t)\|_\infty$ remains bounded on $[0, \tau_{x_0})$ where τ_{x_0} were finite. So $\tau_{x_0} = \infty$: positive solutions exist globally. $\square$

Establishing the validity of Assumption A3) in a neighbourhood of the steady state (v^*, p^*) of constant functions in $C(\overline{\Omega})_+^2$ is more involved. There are results in the literature that allow to deduce the persistence of stability of the steady state of the ODE as equilibrium of the PDE, see e.g. [14]. The latter requires the existence of a positively invariant region, which (4.17)–(4.18) does not have, unfortunately. Moreover, our framework requires *contractiveness* of the solution operator ϕ_t associated to (4.17)–(4.18) on a neighbourhood, which is stronger than convergence towards a steady state. Our approach to establishing Assumption A3) for ϕ_t centers on Condition C1, Lemma 4.7.4 and Corollary 4.7.5. However, the diffusion semigroup $T_d(t) = T_v(t) \times T_p(t)$ is contractive in $C(\overline{\Omega})^2$ (and $L^p(\Omega)$, $1 \leq p \leq \infty$), but its growth bound ω_0 equals 0.

We proceed as follows. Consider the matrix A defined by (4.15). It defines a bounded linear operator $\mathcal{A}$ in $C(\overline{\Omega})^2$ by means of

$$\mathcal{A}(v, p)(x) := A \begin{bmatrix} v(x) \\ p(x) \end{bmatrix}.$$

Note that the conditions for existence and (linear) stability of (v^*, p^*) for RMM are such that

$$A = \begin{bmatrix} -\alpha & -\beta \\ \gamma & 0 \end{bmatrix}, \quad \alpha, \beta, \gamma > 0.$$

Now first consider the linear system of PDEs describing the perturbation of $T_d(t)$

by $\mathcal{A}$ in $H^1(\Omega)^2 = W^{1,2}(\Omega)^2$:

$$\partial_t v(t) = D_v \Delta v(t) - \alpha v(t) - \beta p(t), \quad (4.23)$$

$$\partial_t p(t) = D_p \Delta p(t) + \gamma v(t). \quad (4.24)$$

Multiplication of (4.23) by $v(t)$, (4.24) by $p(t)$, integrating over Ω and applying the Divergence Theorem and the Neumann boundary conditions yields

$$\frac{d}{dt} \|v(t)\|_2^2 = -D_v \|\nabla v(t)\|_2^2 - \alpha \|v(t)\|_2^2 - \beta \int_{\Omega} v(t) p(t) dx, \quad (4.25)$$

$$\frac{d}{dt} \|p(t)\|_2^2 = -D_p \|\nabla p(t)\|_2^2 + \gamma \int_{\Omega} v(t) p(t) dx. \quad (4.26)$$

Put $\nu(t) := \gamma \|v(t)\|_2^2 + \beta \|p(t)\|_2^2$. Then (4.25)–(4.26) yields

$$\frac{d}{dt} \nu(t) \leq -\alpha \gamma \|v(t)\|_2^2 \leq 0$$

and consequently $\nu(t) \leq \nu(0)$ for all t . Even more precise,

$$0 \leq \nu(t) \leq \nu(0) - \alpha \gamma \int_0^t \|v(s)\|_2^2 ds.$$

Therefore,

$$\int_0^t \|v(s)\|_2^2 ds \leq \frac{\nu(0)}{\alpha \gamma}. \quad (4.27)$$

Integration of (4.25) and (4.26) leads to

$$\begin{aligned} \alpha \int_0^t \|v(s)\|_2^2 ds + D_v \int_0^t \|\nabla v(s)\|_2^2 ds &\leq \|v(0)\|_2^2 - \beta \int_0^t \int_{\Omega} v(s) p(s) dx ds, \\ D_p \int_0^t \|\nabla p(s)\|_2^2 ds &\leq \|p(0)\|_2^2 + \gamma \int_0^t \int_{\Omega} v(s) p(s) dx ds. \end{aligned}$$

Consequently, for all $t \geq 0$,

$$-\frac{1}{\gamma} \|p(0)\|_2^2 \leq \int_0^t \int_{\Omega} v(s) p(s) dx ds \leq \frac{1}{\beta} \|v(0)\|_2^2 \quad (4.28)$$

and we obtain

$$\int_0^t \|\nabla p(s)\|_2^2 ds \leq \frac{\nu(0)}{\beta D_p}, \quad \int_0^t \|\nabla v(s)\|_2^2 ds \leq \frac{\nu(0)}{\gamma D_v}. \quad (4.29)$$

For a bound on $p(t)$ we return to (4.23)–(4.24), multiplying the first by $p(t)$, the second by $v(t)$, add the resulting equations and integrate over space and time. It yields

$$\begin{aligned} \beta \int_0^t \|p(s)\|_2^2 ds &= \int_{\Omega} v(0) p(0) dx - \int_{\Omega} v(t) p(t) dx + \int_0^t \|v(s)\|_2^2 ds \\ &\quad - (D_p + D_v) \int_0^t \int_{\Omega} \nabla p(s) \cdot \nabla v(s) dx ds \\ &\quad - \alpha \int_0^t \int_{\Omega} v(s) p(s) dx ds. \end{aligned}$$

Using $\nu(t) \leq \nu(0)$, the Cauchy–Schwarz inequality and (4.27), (4.28) and (4.29), we arrive at

$$\begin{aligned} \int_0^t \|p(s)\|_2^2 ds &\leq \|v(0)\|_2 \|p(0)\|_2 + \nu(0) \left[\frac{1}{\sqrt{\beta\gamma}} \left(1 + \frac{D_v + D_p}{\sqrt{D_v D_p}} \right) + \frac{1}{\alpha} \right] \\ &\quad + \frac{\alpha}{\gamma} \|p(0)\|_2^2. \end{aligned}$$

Thus, the perturbed semigroup $S(t)$ in $H^1(\Omega)^2$, defined by (4.23)–(4.24) satisfies

$$\int_0^{\infty} \|S(t)x\|_{H^1 \times H^1}^2 ds < \infty, \quad (4.30)$$

for all $x \in H^1(\Omega)^2$. According to [30, Theorem V.1.8], $(S(t))_{t \geq 0}$ is uniformly

exponentially stable. This is equivalent to the growth bound of $(S(t))_{t \geq 0}$ in $H^1(\Omega)^2$ being strictly negative.

The regularizing effect of diffusion allows to obtain a similar result in $C(\overline{\Omega})^2$:

Theorem 4.7.8. *The perturbed linear semigroup $(S(t))_{t \geq 0}$ in $C(\overline{\Omega})^2$ defined by (4.23)–(4.24) is uniformly exponentially stable. In particular, its growth bound ω_s is strictly negative.*

Proof. Let $t_0 > 0$ and take $t > 2t_0$. The L^p -to- L^q regularity effect of diffusion (with $1 \leq p \leq q \leq \infty$, see e.g. [7, Proposition 12.5]) yields that $S(t_0)$ maps $W^{k,p}(\Omega)^2$ into $W^{k,q}(\Omega)^2$, $k \in \mathbb{N}_0$, provided that $t \mapsto t^{-(\frac{1}{p}-\frac{1}{q})}$ is integrable over $[0, 1]$, because then the Bochner integral in the variations of constants formula (4.9) converges actually in $W^{k,q}(\Omega)$. As Ω is bounded, $W^{k,q}$ embeds continuously into $W^{k,p}(\Omega)$. By means of the Closed Graph Theorem we conclude that $S(t_0)$ is a bounded linear map from $W^{k,p}(\Omega)^2$ into $W^{k,q}(\Omega)^2$, for $k \in \mathbb{N}_0$, $\frac{1}{p} - \frac{1}{q} < 1$. Similarly, it can be shown that $S(t_0)x \in \mathcal{D}(\mathcal{A}_{\text{diff}}^\alpha)$ for all $0 \leq \alpha < 1$, where $\mathcal{A}_{\text{diff}}^\alpha$ is the α -fractional power of the generator of the diffusion semigroup $T_v \times T_p$, $\mathcal{A}_{\text{diff}}$. We conclude that $S(t_0)x \in H^1(\Omega)^2$ when $x \in L^2(\Omega)^2$. Now, let $x \in C(\overline{\Omega})^2$ and let $t > 2t_0$. Then, with $y = S(t_0)x \in H^1(\Omega)^2$,

$$\begin{aligned} \int_{2t_0}^t \|S(s)x\|_{L^\infty \times L^\infty}^2 ds &= \int_{2t_0}^t \|S(t_0)S(s-2t_0)S(t_0)x\|_{L^\infty \times L^\infty}^2 ds \\ &\leq c_1 \int_{2t_0}^t \|S(s-2t_0)y\|_{L^2 \times L^2}^2 ds \\ &\leq c_1 \cdot c_2 \int_{2t_0}^t \|S(s-2t_0)y\|_{H^1 \times H^1}^2 ds. \end{aligned}$$

According to (4.30), $\int_0^\infty \|S(s)x\|_{L^\infty \times L^\infty}^2 ds < \infty$ for all $x \in C(\overline{\Omega})^2$. $(S(t))_{t \geq 0}$ is strongly continuous in $C(\overline{\Omega})^2$, so we can conclude by the Datko–Pazy Theorem [30, Theorem V.1.8.] that $(S(t))_{t \geq 0}$ is uniformly exponentially stable in $C(\overline{\Omega})^2$ too. Thus, its growth bound is strictly negative, according to [30, Proposition V.1.7.]. $\square$

Now we observe that the map $f(v, p)$ defining the right-hand side of (4.13)–(4.14) in the RMM is real analytic on a neighbourhood of (v^*, p^*) . Define

$$G(v, p) := F(v, p) - \mathcal{A}(v, p).$$

Then there exists a neighbourhood U of the constant function (v^*, p^*) in $C(\overline{\Omega})_+^2$ such that $G \in C^1(U, C(\overline{\Omega})^2)$ and $DG(v^*, p^*) = 0$. Thus, Condition C1 holds for G . As observed, then C1 holds for the modified norm $\|\cdot\|_{A_S}$ for $(S(t))_{t \geq 0}$ as defined in (4.11) too. Now we can apply Lemma 4.7.4 and Corollary 4.7.5 and obtain

Proposition 4.7.9. *There exists $R > 0$ such that the closed $\|\cdot\|_{A_S}$ -ball B^* of radius R around (v^*, p^*) in $C(\overline{\Omega})_+^2$ is invariant under ϕ_t while ϕ_t is a strict contraction on B^* , for $t > 0$. That is, Assumption A3 holds for ϕ_t on B^* for $t > 0$.*

Any random perturbation that meets conditions A1), A2), and A4) will make our framework apply to the RMM with diffusion, with $\phi = \phi_t$, $t > 0$. Thus, small random perturbations at fixed equidistant points will result in a Markov chain on $B^* \subset C(\overline{\Omega})_+^2$ that is asymptotically stable with invariant measure supported in B^* .

Specific example 2: Dispersion of organisms with non-overlapping generations.

As a concrete example of the application of our framework in an infinite dimensional setting, we consider a modification of models in ecology for the dispersal of organisms with synchronous, non-overlapping generations as discussed in [52, 66]. One may think of particular insect species or annual plants, which have two distinct phases in their life cycle: a *dispersal phase* in which the organism (or its seeds) spread in a spatial region, and a *sedentary phase* in which it is immobile (e.g. deposited eggs, larvae stage, settled seed or full-grown plant). The old generation of mature individuals will die, e.g. in winter, while the individuals in the sedentary phase will partially survive into spring and hatch or germinate to become the next generation.

To formulate the model mathematically, in the spirit of [66], suppose that dispersal takes place in a bounded open set Ω in $\mathbb{R}^2$, with uniformly C^2 -boundary, by means of diffusion. Let us consider a system of two insect species, both with synchronous, non-overlapping generations, and both with a dispersal and sedentary phase, but one being prey and the other predator. Let $v_n(t)$ and $p_n(t)$ denote the spatial distribution of each at time t , of the n -th generations dispersal stage. We model dispersal by diffusion, while the prey-predator interaction is local, given by dynamics of the RMM, i.e. v_n and p_n satisfy (4.17)–(4.18), with Neumann boundary conditions. Thus, individuals cannot leave Ω . We consider

mild solutions in $C(\overline{\Omega})_+^2$. (See Proposition 4.7.7). We assume that the (positive) parameters are such that the RMM, hence (4.17)–(4.18), has a unique stable steady state (v^*, p^*) , as before.

Both prey and predator species lay eggs during their dispersal stage. The spatial distribution of eggs during the life-time of the n -th generation is represented by $w_n(t)$ for prey and $q_n(t)$ for predator:

$$w_n(t) = \int_0^t R_v(s, v_n(s)) ds =: \rho_v(v_n), \quad (4.31)$$

$$q_n(t) = \int_0^t R_p(s, p_n(s)) ds =: \rho_p(p_n), \quad (4.32)$$

where $R_i : \mathbb{R}_+ \times C(\overline{\Omega}) \rightarrow C(\overline{\Omega})$ represent the rate at which species $i \in \{v, p\}$ lay eggs. The time-dependence in (4.31) and (4.32) allows to model seasonal effects. We assume that R_i are such that $w_n(t), q_n(t) \geq 0$ when $v_n(t), p_n(t) \geq 0$.

The distribution of eggs at the end $t = t_e$ of the n -th generation determines the population distributions $v_{n+1}(0)$ and $p_{n+1}(0)$ at the start of the dispersal stage of the next generation. That is:

$$v_{n+1}(0) = S_v(w_n(t_e)), \quad p_{n+1}(0) = S_p(q_n(t_e)).$$

The maps S_v and $S_p : C(\overline{\Omega})_+ \rightarrow C(\overline{\Omega})_+$ model deterministically the survival of offspring (eggs) during the winter season and their successful hatching and growth during the sedentary stage, yielding next generation individuals that will disperse and reproduce.

In its simplest form, R_i and S_i , $i \in \{v, p\}$ will be local operators, i.e. superposition or Nemytskii operators associated to functions $\sigma_i : \mathbb{R}_+ \rightarrow \mathbb{R}$, $r_i : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$, where $0 \leq \sigma_i(x) \leq x$.

For the purpose of this elaborate example we take the simplest expression for σ_i and r_i to which our framework applies:

$$\sigma_i(x) = \sigma_i x, \quad 0 < \sigma_i \leq 1, \quad \text{and} \quad r_i(t, x) = r_i(t) \geq 0,$$

for all $x \in \Omega$, with $r_i(\cdot) \in L^1([0, t_e])$. It is crucial to assume that there exists $0 < t_r < t_e$ such that $r_i(t) = 0$ for $0 \leq t \leq t_r$. That is, only after time t_r reproduction can start. Thus, this condition is not too restrictive from the point of view of applications.

Let $\Phi : \mathbb{R}_+ \times C(\overline{\Omega})_+^2 \rightarrow C(\overline{\Omega})_+^2$ be the semiflow associated to ϕ_t : $\Phi(t, x) := \phi_t(x)$. Denote the v and p -component of $\Phi(t, x)$ by the corresponding subscript.

Now define the operator ϕ on $C(\overline{\Omega})_+^2$ as

$$\phi(x) := \left(S_v \circ \rho_v(\Phi(\cdot, x)_v), S_p \circ \rho_p(\Phi(\cdot, x)_p) \right). \quad (4.33)$$

Let $x_n^0 := (v_n(0), p_n(0)) \in C(\overline{\Omega})_+^2$ denote the population composition at the start of the n -th generation's dispersal stage (and $\hat{x}_n^0$ similarly, for a different initial condition $(v_1(0), p_1(0))$). Let B^* be the closed $\|\cdot\|_{A_S}$ -ball of radius R around the constant function $x^* = (v^*, p^*)$ in $C(\overline{\Omega})_+^2$, as in Proposition 4.7.9. Assume that x^* is a fixed point of ϕ , i.e.

$$v^* = S_v(\rho_v(v^*)), \quad p^* = S_p(\rho_p(p^*)).$$

Then one must have

$$\sigma_i \int_0^{t_e} r_i(t) dt = 1, \quad i = v \text{ or } p.$$

Let $\mathcal{D}_t$ be the matrix operator on $C(\overline{\Omega})^2$ defined by the 2×2 diagonal matrix $D_t = \text{diag}(\sigma_v r_v(t), \sigma_p r_p(t))$.

Lemma 4.7.10. *For all $x, y \in B^*$, one has*

$$\|\phi(x) - \phi(y)\|_{A_S} \leq \|x - y\|_{A_S} \int_0^{t_e} \|\mathcal{D}_t\|_{A_S} e^{(\epsilon + \omega_s)t} dt$$

where ϵ and ω_s are as in Specific example 1.

Proof. Let $x, y \in B^*$. Then using

$$\begin{aligned} \|\phi(x) - \phi(y)\|_{A_S} &\leq \left\| \int_0^{t_e} \mathcal{D}_s [\phi_s(x) - \phi_s(y)] ds \right\|_{A_S} \\ &\leq \int_0^{t_e} \|\mathcal{D}_s\|_{A_S} \|\phi_s(x) - \phi_s(y)\|_{A_S} ds \\ &\leq \int_0^{t_e} \|\mathcal{D}_s\|_{A_S} e^{(\epsilon + \omega_s)s} ds \cdot \|x - y\|_{A_S} \end{aligned}$$

□

Now

$$\begin{aligned} \|\mathcal{D}_t x\|_{A_S} &\leq M \|\mathcal{D}_t x\|_{C(\overline{\Omega})^2} \leq M \cdot \max_{i \in \{v, p\}} (\sigma_i r_i(t)) \|x\| \\ &\leq M \cdot \max_{i \in \{v, p\}} (\sigma_i r_i(t)) \|x\|_{A_S}. \end{aligned}$$

Thus, using conditions (4.30) and $r_i(t) = 0$ for $t \in [0, t_r]$, we obtain

$$\int_0^{t_e} \|\mathcal{D}_t\|_{A_S} e^{(\epsilon + \omega_s)t} dt \leq 2M e^{(\epsilon + \omega_s)t_r},$$

which can be made smaller than 1, when $t_e \geq |\epsilon + \omega_s| \log(2M)$ and t_r is sufficiently large. Here $M \geq 1$ is the constant in the growth bound for $S(t)$ in $C(\overline{\Omega})^2$: $\|S(t)\| \leq M e^{(\omega_s t)}$. If $r_i \in L^\infty([0, t_e])$, $i \in \{v, p\}$, then $\|\mathcal{D}_t\|_{A_S}$ is bounded, say by M' , and

$$\int_0^{t_e} \|\mathcal{D}_t\|_{A_S} e^{(\epsilon + \omega_s)t} dt \leq M' (t_e - t_r) e^{(\epsilon + \omega_s)t_r},$$

which can also be made arbitrary small when t_r is sufficiently close to t_e .

Summarizing, we have

Proposition 4.7.11. *If either $r_i \in L^1([0, t_e])$ and $t_e \geq |\epsilon + \omega_s| \log(2M)$ or $r_i \in L^\infty([0, t_e])$, then for sufficiently large t_r in $[0, t_e]$, ϕ is a strict contraction on B^* . That is, Assumption A3) is satisfied.*

Now consider Assumption A5, i.e. the compactness of ϕ .

Lemma 4.7.12. *Assume that $R_i : [0, t_e] \times C(\overline{\Omega}) \rightarrow C(\overline{\Omega})$, $i \in \{v, p\}$, are jointly continuous. Then the map $\Phi : S \rightarrow S$ given by*

$$\Phi(v^0, p^0)(t) := (w(t_e), q(t_e))$$

is compact for all $t_e > 0$.

Proof. By compactness of the solution map ϕ_t , $t > 0$ (Proposition 4.7.7), if $B \subset S$ is bounded, then $\phi_t(B) \subset S$ is precompact for all $t > 0$. We need to show that $\Phi(B) \subset S$ is precompact. To that end we use Lemma 4.7.1 with $X = C(\overline{\Omega})^2$, which is a separable Banach space, because Ω is bounded. Now, let $(\hat{w}_n, \hat{q}_n)$ be a

sequence in $\Phi(B)$. Then there exist $(v_n^0, p_n^0) \in B$ such that

$$\hat{w}_n = \int_0^{t_e} R_v(t, v_n(t)) dt, \quad \hat{q}_n = \int_0^{t_e} R_\omega(t, p_n(t)) dt,$$

where $(v_n(t), p_n(t)) = \phi_t(v_n^0, p_n^0)$. Put first $x_n(t) := R_v(t, v_n(t))$. The assumptions on R_v yield that $x_n \in C([0, t_e], C(\bar{\Omega})_+)$. Moreover, for each $t \in (0, t_e]$, $\{v_n(t) | n \in \mathbb{N}\} \subset C(\bar{\Omega})_+$ is precompact. Therefore, by continuity of $R_v(t, \cdot)$, $\{x_n(t) | n \in \mathbb{N}\}$ is precompact. Thus, according to (4.10), $\{\hat{w}_n : n \in \mathbb{N}\} \subset C(\bar{\Omega})_+$ is precompact. Similarly, we obtain $\{\hat{q}_n : n \in \mathbb{N}\} \subset L^1(\Omega)_+$ is precompact. Hence, $(\hat{w}_n, \hat{q}_n) \subset S$ has a convergent subsequence and we conclude that $\Phi(B)$ is precompact. $\square$

Corollary 4.7.13. *If $S_i : C(\bar{\Omega})_+ \rightarrow C(\bar{\Omega})_+$, $i = \{v, p\}$, are continuous, then the deterministic next generation map ϕ defined by (4.33) is compact. That is, ϕ satisfies Assumption A5.*

4.7.3 Delay equations

In various population dynamical examples a model in terms of delay equations appears naturally. For example, one can consider a population of bacteria growing in a chemostat that are ‘fed’ with a random amount of nutrients at fixed times. When starving, some bacteria will form spores– with a time delay. Existing spores will germinate when there is a sufficient supply of nutrients– again with a delay. In [43], one considers ODE models for populations extended by adding delay. These examples have the form

$$\begin{cases} x'(t) = Cx(t) + Lx_t + f(x(t)), & t \geq 0 \\ x(t) = u(t), & -r \leq t \leq 0 \end{cases} \quad (4.34)$$

where $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$, C is a $d \times d$ matrix, $L : L^p([-r, 0]; \mathbb{R}^d) \rightarrow \mathbb{R}^d$ is a bounded linear map for some $1 \leq p < \infty$, and $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Lipschitz, differentiable at 0, $f(0) = 0$ and $Df(0) = 0$. Due to random interventions, the trajectories will typically be discontinuous, so that the formulation using the Delfour–Mitter space $X = \mathbb{R}^d \times L^p([-r, 0]; \mathbb{R}^d)$ is more suitable than the space $C([-r, 0]; \mathbb{R}^d)$. For terminology and results concerning delay equations we refer to [12, 24].

There exists an $\eta \in L^q([-r, 0]; \mathbb{R}^{d \times d})$ such that $Lu = \int_{-r}^0 \eta(\theta) u(\theta) d(\theta)$, $u \in L^p([-r, 0]; \mathbb{R}^d)$, where $\frac{1}{p} + \frac{1}{q} = 1$. For every $u \in L^p([-r, 0]; \mathbb{R}^d)$ and $\alpha \in \mathbb{R}^d$ there is a unique $x : [-r, \infty) \rightarrow \mathbb{R}^d$ with $x|_{[0, \infty)}$ continuous, $x_0 = u$, and

$$x(t) = \alpha + \int_0^t x(s) ds + \int_0^t \int_{-r}^0 \eta(\theta) x(s + \theta) d\theta ds + \int_0^t f(x(s)) ds, \quad t \geq 0.$$

Moreover, $v(t) = (x(t), x_t)$ satisfies $v'(t) = Av(t) + F(v(t))$, where $F(\alpha, u) = (f(\alpha), 0)$, and A is the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on $\mathbb{R}^d \times L^p[-r, 0]$ and $T(t)$ is compact for $t > r$. Hence, in the linear case, when $f \equiv 0$, $\phi_{\Delta t}$ satisfies A5. We show next that the same is true in the nonlinear case. Clearly, $0 \in \mathbb{R}^d \times L^p([-r, 0]; \mathbb{R}^d)$ is a steady state. If $\varphi : [-r, \infty) \mapsto \mathbb{R}^{d \times d}$ is the fundamental solution of $x'(t) = Cx(t) + Lx_t$ and $v = (\alpha, u) \in \mathbb{R}^d \times L^p([-r, 0]; \mathbb{R}^d)$, then

$$\begin{aligned} x(t) = & \varphi(t) \alpha + \int_0^{\min\{t, r\}} \int_{-r}^{-s} \eta(\theta) \varphi(t-s) u(s+\theta) d\theta ds \\ & + \int_0^t \int_{-r}^s \eta(\theta) \varphi(t-s) x(s+\theta) d\theta ds + \int_0^t \varphi(t-s) f(x(s)) ds. \end{aligned}$$

Moreover, if $ML_f < \alpha$, then there exist $M_1 > 0$ and $\alpha_1 > 0$ such that $|x(t)| \leq M_1 e^{-\alpha_1 t} \|v\|$ for all $t \geq 0$ where $\|v\| = |\alpha| + \|u\|_{L^p}$. Also, it can be shown that on

$$\{v = (\alpha, u) \in \mathbb{R}^d \times L^p([-r, 0]; \mathbb{R}^d) : \|v\| = |\alpha| + \|u\|_p \leq R, \phi_{\Delta t}(u)\},$$

the function $\theta \mapsto x(\Delta t + \theta)$ is Lipschitz with Lipschitz constant at most

$$(L_\varphi + 2\Delta t L_\varphi \|\eta\|_{TV} + 2\|\varphi\|_\infty \|\eta\|_{TV} + \Delta t L_\varphi L_f M_1 + \|\varphi\|_\infty L_f M_1) R,$$

provided that $\Delta t > r$. Thus, if B is a ball in $\mathbb{R}^d \times L^p([-r, 0]; \mathbb{R}^d)$, then there is an $R > 0$, such that $\|v\| \leq R$ for all $v \in B$ and then

$$\begin{aligned} \{\phi_{\Delta t}(v) : v \in B\} \subseteq \\ \{(\alpha, u) : u : [-r, 0] \rightarrow \mathbb{R}^d \text{ Lipschitz with } |\cdot|_{\text{Lip}} \leq DR \text{ and } |\alpha| \leq M_1 e^{-\alpha_1 \Delta t} R\}. \end{aligned}$$

Hence, $\phi_{\Delta t}(B)$ has compact closure in $\mathbb{R}^d \times L^p([-r, 0]; \mathbb{R}^d)$. Define for $v \in \mathbb{R}^d \times L^p([-r, 0]; \mathbb{R}^d)$:

$$|||v||| := \sup_{t \in [0, \Delta t]} \|T(t)v\|.$$

Then ([30]): $|||T(t)v||| \leq e^{-\alpha t} |||u|||$. Moreover, for $t > \Delta t - r$, it can be shown that

$$|||(x_1)_t - (x_2)_t||| \leq e^{(DL_f - \alpha)t} |||u_1 - u_2|||,$$

with $L_f < \frac{\alpha}{D}$ where x_1 and x_2 are two solutions of (4.34) with initial conditions v_1 and v_2 respectively. Having in mind that $\phi_{\Delta t}(v_1) = ((x_1)_{(\Delta t)}, (x_1)_{\Delta t})$, we observe that $\phi_{\Delta t}$ is a contraction with respect to $|||\cdot|||$. In conclusion, A3 and A5 are satisfied.

If $d = 2$ and Q_x for $x \in \mathbb{R}^2$ is a measure on $\mathbb{R}^2$ as in RMM example, then for each $x \in \mathbb{R}^2$ and $u \in L^p[-r, 0]$, $\tilde{Q}_{(x,u)}(A \times B) = Q_x(A) \delta_0(B)$, $A \subseteq \mathbb{R}^2$, $B \subseteq L^p[-r, 0]$ Borel sets, extends to a measure on $\mathbb{R}^2 \times L^p[-r, 0]$ satisfying A1–A2.

Chapter 5

On metrization of unions of function spaces on different intervals

This chapter investigates a class of metrics that can be introduced on the set consisting of the union of continuous functions defined on different intervals with values in a fixed metric space, where the union ranges over a family of intervals. Its definition is motivated by the Skorohod metric(s) on càdlàg functions. We show what is essential in transferring the ideas employed in the latter metric to our setting and obtain a general construction for metrics in our case. Next, we define the metric space where elements are sequences of functions from the above mentioned set. We provide conditions that ensure separability and completeness of the constructed metric spaces.

5.1 Introduction

The space $C([a, b])$ of continuous functions defined on an interval $[a, b]$ is used in many branches of analysis. Apart from the uniform topology, there are various other useful topologies, metrics and norms that have been considered on $C([a, b])$ for various purposes. The definition of a topology, in particular a metric, on a set consisting of continuous functions defined on *different* intervals attracted less attention, if at all so far. Such sets - and the need of a suitable metric - appear naturally when studying stochastic perturbations of deterministic dynamical systems in metric spaces. They arise when one considers a deterministic system

with continuous trajectories, that is subject to interventions at discrete points in time randomly, in which the deterministic system jumps instantaneously to a new position, also randomly, see for instance, Lasota and Mackey [54]. We have been motivated by the modeling in population biology of a fish population with occasional fishing in such a fashion. Other examples are in financial mathematics [10, 34], or impulsive systems [81].

The sample trajectories of the resulting stochastic process in the metric state space S of the unperturbed dynamical system may be viewed as a *concatenation* of the continuous trajectories of the latter in between the times of interventions at which the jump take place. Thus, these continuous paths are taken from a union over $\tau > 0$ of the collection $C_s([0, \tau])$ of all continuous S -valued maps defined on the interval $[0, \tau]$. We denote this union by C_s^τ .

For application of numerous results from probability theory it is advantageous to have a metric on C_s^τ which makes it a complete and separable metric space. The main objective of this chapter is to exhibit a general construction for metrics on C_s^τ that yield completeness and separability based on collections of homeomorphisms between intervals $[0, \tau]$ and $[0, \tau']$, and a *penalty function* that measures the amount of deformation that these homeomorphisms cause to the intervals. We believe that the results are of general mathematical interest.

The construction is motivated by the metric on càdlàg¹ functions $D_s([0, 1])$ or $D_s([0, \infty))$ as exhibited in [9, 33, 80]. In fact, it turns out that the penalty function used for càdlàg functions may not be the only possible penalty function that yields a complete (separable) metric space, but it is one that has highly convenient additional properties. Our approach to the stochastic process contrasts with the common approach in probability theory of viewing the trajectories directly in the space of càdlàg functions $D_s([0, \infty))$, see Skorohod [80], Ethier and Kurtz [33] and Billingsley [9].

The structure of the chapter is as follows. In Section 5.2 we present properties on the sets of homeomorphisms between the intervals (D1), (D2) and on the penalty function (P1)-(P4) that allow us to define a metric on C_s^τ . We show that the penalty function for [9, 33, 80] satisfies these properties (and more). In Section 5.3 we provide additional conditions under which the metric yields a separable or a complete metric space. It turns out that one needs a slightly stronger condition

¹A càdlàg (French "continue à droite, limite à gauche" means "continuous on (the) right, limit on (the) left"), function is a function defined on the real numbers (or a subset of them) that is everywhere right-continuous and has left limits everywhere. The collection of càdlàg functions on a given domain is known as Skorohod space.

(P3') instead of (P3) for completeness (Theorem 5.3.5). For separability, one needs separability of the spaces $C_s([0, \tau])$. The subsequent Section 5.4 is devoted to sequences with values in C_s^τ , which relates to coding of the sample trajectories of the stochastic process that results from the perturbation and a brief discussion of the relation to càdlàg functions. The latter is subject of further study. Let us conclude the introduction by an example where a space of functions on different intervals is natural.

Motivating Example Consider a time homogeneous dynamical system on a metric space S given by a continuous map $\Phi : [0, \infty) \times S \rightarrow S$. That means a state $x \in S$ at time $t_0 \geq 0$ evolves to the state $\Phi(t - t_0, x)$ at time t . For instance, S could be R^n or an infinite dimensional Banach space and $\Phi(t, x)$ could be the solution operator of a well-posed system of evolutionary ordinary or partial differential equations.

Suppose that the system is subject to random interventions at times $0 < t_1 < t_2 \dots$, in the sense that the continuous evolution given by Φ stops at time t_k , and the system jumps instantaneously and randomly to a new position after which it starts its evolution from there again according to Φ . The position of the system in S at time t , represented by $Y(t)$, can be described in the case of jump sizes being independent random variables, $J_1, J_2, \dots$, as:

$$Y(t) = Y_k(t - t_k) \text{ if } t \in [t_k, t_{k+1}),$$

where

$$\begin{aligned} Y_0(t) &= \Phi(t, x_0) \text{ if } t \in [0, t_1], \\ Y_k(t) &= \Phi(t, Y_{k-1}(t_k - t_{k-1}) + J_k) \text{ if } t \in [0, t_{k+1} - t_k], \quad k \geq 1, \end{aligned}$$

for some initial value $x_0 \in S$. For convenience we have put $t_0 = 0$. The full trajectories Y can thus be encoded as a sequence (Y_k) that takes values in $\bigcup_{k=0}^{\infty} C_s([0, t_{k+1} - t_k])$. In the case that the interventions are equidistant, that is $t_{k+1} - t_k = \tau$ for all k , each Y_k is a random element of $C_s([0, \tau])$, and $(Y_k)_{k=0}^{\infty}$ is a Markov chain in the space $C_s([0, \tau])$. In the case that the intervention times are not equidistant, we propose to view each Y_k as an element of the space $\bigcup_{k=0}^{\infty} C_s([0, t_{k+1} - t_k])$. Thus, some complications of continuous time Markov processes can be avoided. Such non-equidistant intervention times appear in particular when stochastic times are considered.

Recall that, for application of the theory of Markov chains it is a great advan-

tage if the state space is a separable complete metric space. In the subsequent sections we will consider various metrics on $\bigcup_{k=0}^{\infty} C_s([0, t_{k+1} - t_k])$ and study separability and completeness.

5.2 Metrics on the space C_s^τ

Let (S, d_s) be a metric space and let $\tau = (\tau_\alpha)_{\alpha \in A}$ be a family of distinct elements τ_α of $[0, \infty)$; that is $\tau_\alpha \neq \tau_\beta$ when $\alpha \neq \beta$ and $(\tau_\alpha) \subset [0, \infty)$. Put $C_s^\tau := \bigcup_\alpha C_s([0, \tau_\alpha])$ where

$$C_s([0, \tau_\alpha]) := \{f : [0, \tau_\alpha] \rightarrow S : f \text{ continuous}\}.$$

We will define a class of metrics on C_s^τ . Let for each pair (α, β) a nonempty collection, $\Delta_{\alpha, \beta}$, of maps λ be given:

$$\Delta_{\alpha, \beta} \subseteq \left\{ \lambda : [0, \tau_\alpha] \rightarrow [0, \tau_\beta], \lambda \text{ strictly increasing continuous}; \right. \\ \left. \lambda(0) = 0, \lambda(\tau_\alpha) = \tau_\beta \right\}.$$

Note that each λ is invertible. Assume that these collections satisfy:

$$(D1) \quad \Delta_{\beta, \alpha} = \Delta_{\alpha, \beta}^{-1} := \{\lambda^{-1} : \lambda \in \Delta_{\alpha, \beta}\}.$$

$$(D2) \quad \Delta_{\beta, \gamma} \circ \Delta_{\alpha, \beta} := \{\lambda_1 \circ \lambda_2 : \lambda_1 \in \Delta_{\beta, \gamma}, \lambda_2 \in \Delta_{\alpha, \beta}\} \subset \Delta_{\alpha, \gamma}.$$

Note that (D1) and (D2) imply $I \in \Delta_{\alpha, \alpha}$, where I is the identity map. Moreover, any $\lambda \in \Delta_{\alpha, \beta}$ must be a homeomorphism. Put $\Delta := \bigcup_{\alpha, \beta} \Delta_{\alpha, \beta}$.

Assume that a function $P : \Delta \rightarrow [0, \infty)$ is given, a *penalty function*, that satisfies the following properties:

$$(P1) \quad P(I) = 0.$$

$$(P2) \quad P(\lambda^{-1}) = P(\lambda).$$

$$(P3) \quad P(\lambda_1 \circ \lambda_2) \leq P(\lambda_1) + P(\lambda_2).$$

$$(P4) \quad \text{If } \lambda_n \in \Delta_{\alpha, \beta} \text{ and } P(\lambda_n) \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ then } \alpha = \beta \text{ and } \lambda_n \rightarrow I \text{ uniformly on } [0, \tau_\alpha].$$

The function $P(\lambda)$ ‘penalizes’ stretching of an interval by the map λ , as illustrated by the following good candidates for such Δ and P (as will be proved below).

Example 5.2.1. $\Delta_{\alpha,\beta}$ consists of all strictly increasing, continuous mappings of $[0, \tau_\alpha]$ onto $[0, \tau_\beta]$,

$$P(\lambda) = \|\lambda - I\| = \sup_{t \in [0, \tau_\alpha]} |\lambda(t) - t|, \quad \lambda \in \Delta_{\alpha,\beta}. \quad (5.1)$$

Example 5.2.2. $\Delta_{\alpha,\beta}$ consists of all increasing lipeomorphisms²,

$$P(\lambda) = \sup_{0 \leq t < s \leq \tau_\alpha} \left| \log \frac{\lambda(s) - \lambda(t)}{s - t} \right|, \quad \lambda \in \Delta_{\alpha,\beta}. \quad (5.2)$$

We proceed to define a metric on C_s^τ making use of the data (Δ, P) . Given a family $\Delta_{\alpha,\beta}$ and P as above satisfying (D1)-(D2) and (P1)-(P4) we define a function $\hat{d}$ on $C_s^\tau \times C_s^\tau$ as follows:

For $f, g \in C_s^\tau$ with $f \in C_s([0, \tau_\alpha])$, $g \in C_s([0, \tau_\beta])$, let

$$\begin{aligned} d_\lambda(f, g) &:= \sup_{t \in [0, \tau_\alpha]} d_s(f(t), g \circ \lambda(t)), \\ \hat{d}(f, g) &:= \inf_{\lambda \in \Delta_{\alpha,\beta}} \{d_\lambda(f, g) + P(\lambda)\}. \end{aligned}$$

Remark 5.2.3. If $f, g \in C_s([0, \tau_\alpha])$ then

$$\hat{d}(f, g) = \inf_{\lambda \in \Delta_{\alpha,\beta}} \{d_\lambda(f, g) + P(\lambda)\} \leq \{d_{\lambda=I}(f, g) + P(I)\} = \sup_{t \in [0, \tau_\alpha]} d_s(f(t), g(t)),$$

according to the assumptions $I \in \Delta_{\alpha,\alpha}$ and (P1).

Proposition 5.2.4. If $\Delta_{\alpha,\beta}$ and $P : \Delta \rightarrow [0, \infty)$ satisfy (D1), (D2) and (P1)-(P4) respectively, then $\hat{d}$ is a metric on C_s^τ .

Proof. For $f, g \in C_s^\tau$ with $f \in C_s([0, \tau_\alpha])$, $g \in C_s([0, \tau_\beta])$ let $\hat{d}(f, g) = 0$. Then there exist $\lambda_n : [0, \tau_\alpha] \rightarrow [0, \tau_\beta] \in \Delta_{\alpha,\beta}$ such that

$$\lim_{n \rightarrow \infty} \{d_{\lambda_n}(f, g) + P(\lambda_n)\} = 0.$$

Consequently,

$$P(\lambda_n) \rightarrow 0, \quad d_{\lambda_n}(f, g) \rightarrow 0.$$

Now, $P(\lambda_n) \rightarrow 0$ gives $\tau_\alpha = \tau_\beta$ and $\lambda_n \rightarrow I$ uniformly on $[0, \tau_\alpha]$ according to (P4).

²A function is called a *lipeomorphism* if it is Lipschitz, bijective and has a Lipschitz inverse. Also, an increasing lipeomorphism should be strictly increasing.

Therefore,

$$\lim_{n \rightarrow \infty} d_{\lambda_n}(f, g) = 0$$

gives

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, \tau_\alpha]} d_s(f(t), g \circ \lambda_n(t)) = 0$$

and this yields

$$\sup_{t \in [0, \tau_\alpha]} d_s(f(t), g \circ I(t)) = 0.$$

Thus, $f = g$. For the other way around, if $f = g$ and $f, g : [0, \tau_\alpha] \rightarrow S$, then, as mentioned above,

$$\hat{d}(f, g) \leq \sup_{t \in [0, \tau_\alpha]} d_s(f(t), g(t))$$

and having d_s a metric gives $\hat{d}(f, g) = 0$. For symmetry we have:

$$\begin{aligned} d_\lambda(f, g) &= \sup_{t \in [0, \tau_\alpha]} d_s(f(t), g \circ \lambda(t)) \\ &= \sup_{t \in [0, \tau_\beta]} d_s(f(\lambda^{-1}(t)), g(t)) = d_{\lambda^{-1}}(g, f). \end{aligned}$$

By assumption (D1) and (P2),

$$\begin{aligned} \hat{d}(f, g) &:= \inf_{\lambda \in \Delta_{\alpha, \beta}} \{d_\lambda(f, g) + P(\lambda)\} \\ &= \inf_{\lambda^{-1} \in \Delta_{\alpha, \beta}^{-1} = \Delta_{\beta, \alpha}} \{d_{\lambda^{-1}}(g, f) + P(\lambda^{-1})\} \\ &= \hat{d}(g, f). \end{aligned}$$

For the triangle inequality, let f, g be as stated above and let $h \in C_s([0, \tau_\gamma])$. Then one has:

$$\begin{aligned} d_{\lambda_1}(f, g) + d_{\lambda_2}(g, h) &= \sup_{t \in [0, \tau_\alpha]} d_s(f(t), g(\lambda_1(t))) + \sup_{t \in [0, \tau_\beta]} d_s(g(t), h(\lambda_2(t))) \\ &\geq \sup_{t \in [0, \tau_\alpha]} d_s(f(t), g(\lambda_1(t))) + \sup_{s \in [0, \tau_\alpha]} d_s(g(\lambda_1(t)), h(\lambda_2\lambda_1(t))) \\ &\geq \sup_{t \in [0, \tau_\alpha]} d_s(f(t), h(\lambda_2\lambda_1(t))) \\ &= d_{\lambda_2\lambda_1}(f, h). \end{aligned}$$

Note that $\lambda_2\lambda_1 \in \Delta_{\beta,\gamma}\Delta_{\alpha,\beta}$ and by assumption (D2), $\lambda_2\lambda_1 \in \Delta_{\alpha,\gamma}$. Therefore,

$$\begin{aligned}
 \hat{d}(f, g) + \hat{d}(g, h) &= \inf_{\lambda_1 \in \Delta_{\alpha,\beta}} \{d_{\lambda_1}(f, g) + P(\lambda_1)\} + \inf_{\lambda_2 \in \Delta_{\beta,\gamma}} \{d_{\lambda_2}(g, h) + P(\lambda_2)\} \\
 &= \inf_{\{\lambda_1 \in \Delta_{\alpha,\beta}, \lambda_2 \in \Delta_{\beta,\gamma}\}} \{d_{\lambda_1}(f, g) + d_{\lambda_2}(g, h) + P(\lambda_1) + P(\lambda_2)\} \\
 &\geq \inf_{\{\lambda_1 \in \Delta_{\alpha,\beta}, \lambda_2 \in \Delta_{\beta,\gamma}\}} \{d_{\lambda_2\lambda_1}(f, h) + P(\lambda_2\lambda_1)\} \\
 &\geq \inf_{\lambda_2\lambda_1 \in \Delta_{\alpha,\gamma}} \{d_{\lambda_2\lambda_1}(f, h) + P(\lambda_2\lambda_1)\} \\
 &\geq \inf_{\lambda \in \Delta_{\alpha,\gamma}} \{d_\lambda(f, h) + P(\lambda)\} = \hat{d}(f, h).
 \end{aligned}$$

□

Let us call the following condition (P4'):

(P4') If $\lambda_n \in \Delta_{\alpha,\beta_n}$ and $P(\lambda_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\tau_{\beta_n} \rightarrow \tau_\alpha$ as $n \rightarrow \infty$ and $\lambda_n \rightarrow I$ uniformly on $[0, \tau_\alpha]$.

Note that (P4') implies (P4). With (P4') one can clarify the concept of convergence under the metric $\hat{d}$. Indeed, if it is given that $(f_n) \in C_s([0, \tau_{\alpha_n}])$ and $f \in C_s([0, \tau_\alpha])$ such that $f_n \rightarrow f$ under the metric $\hat{d}$ then we conclude that there exist $\lambda_n \in \Delta_{\alpha_n,\alpha}$ such that

$$d_{\lambda_n}(f_n, f) = \sup_{t \in [0, \tau_\alpha]} d_s(f_n(\lambda_n^{-1}(t)), f(t)) \rightarrow 0,$$

that is, $f_n \circ \lambda_n^{-1} \rightarrow f$ uniformly on the interval on which f is defined, $[0, \tau_\alpha]$, under the metric d_s . Moreover, $P(\lambda_n) \rightarrow 0$ which, by (P2) and (P4'), implies that

$$\tau_{\alpha_n} \rightarrow \tau_\alpha. \quad (5.3)$$

Conversely, if we have $(f_n) \in C_s([0, \tau_{\alpha_n}])$ and we would prove $\hat{d}(f_n, f) \rightarrow 0$ then we should find $f \in C_s([0, \tau_\alpha])$ and functions $\lambda_n \in \Delta_{\alpha_n,\alpha}$ (we should only consider $\lambda_n : [0, \tau_{\alpha_n}] \rightarrow [0, \tau_\alpha]$, or $\lambda_n^{-1} : [0, \tau_\alpha] \rightarrow [0, \tau_{\alpha_n}]$, that is, either the domain or the range of λ_n is independent of n) such that $P(\lambda_n) \rightarrow 0$ and

$$\sup_{t \in [0, \tau_{\alpha_n}]} d_s(f_n(t), f \circ \lambda_n(t)) = \sup_{t \in [0, \tau_\alpha]} d_s(f_n \circ \lambda_n^{-1}(t), f(t)) \rightarrow 0.$$

We will establish more properties of the penalty function (5.2):

Proposition 5.2.5. $P(\lambda)$ defined by (5.2) satisfies properties (P1)-(P3) mentioned above. Moreover, if $\lambda_n \in \Delta_{\alpha_n, \beta_n}$ and $P(\lambda_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\frac{\tau_{\beta_n}}{\tau_{\alpha_n}} \rightarrow 1$ as $n \rightarrow \infty$ and

$$\frac{1}{\tau_{\alpha_n}} \sup_{s \in [0, \tau_{\alpha_n}]} |\lambda_n(s) - s| \rightarrow 0.$$

As a special case, if (τ_{α_n}) is bounded, then $\tau_{\beta_n} - \tau_{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$. Consequently, $P(\lambda)$ satisfies (P4') and, therefore, (P4).

Proof. For (P1) and (P2) the proof is obvious. For (P3), let $\lambda_2 : [0, \tau_\alpha] \rightarrow [0, \tau_\beta]$ and $\lambda_1 : [0, \tau_\beta] \rightarrow [0, \tau_\gamma]$ then we have (we will abbreviate $\lambda_1 \circ \lambda_2$ to $\lambda_1 \lambda_2$)

$$\begin{aligned} \left| \log \frac{\lambda_1 \lambda_2(s) - \lambda_1 \lambda_2(t)}{s - t} \right| &= \left| \log \frac{\lambda_1 \lambda_2(s) - \lambda_1 \lambda_2(t)}{\lambda_2(s) - \lambda_2(t)} \cdot \frac{\lambda_2(s) - \lambda_2(t)}{s - t} \right| \\ &\leq \left| \log \frac{\lambda_1 \lambda_2(s) - \lambda_1 \lambda_2(t)}{\lambda_2(s) - \lambda_2(t)} \right| + \left| \log \frac{\lambda_2(s) - \lambda_2(t)}{s - t} \right| \\ &= \left| \log \frac{\lambda_1(w) - \lambda_1(v)}{w - v} \right| + \left| \log \frac{\lambda_2(s) - \lambda_2(t)}{s - t} \right|. \end{aligned}$$

Taking the supremum on the interval $[0, \tau_\alpha]$ for both sides of the inequality we get (P3). Furthermore, let $\lambda_n \in \Delta_{\alpha_n, \beta_n}$ be such that $P(\lambda_n) \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\sup_{0 \leq t < s \leq \tau_{\alpha_n}} \left| \log \frac{\lambda_n(s) - \lambda_n(t)}{s - t} \right| \rightarrow 0,$$

with $t = 0$ and $s = \tau_{\alpha_n}$ we obtain $\left| \log \frac{\tau_{\beta_n}}{\tau_{\alpha_n}} \right| \rightarrow 0$. Hence, $\frac{\tau_{\beta_n}}{\tau_{\alpha_n}} \rightarrow 1$ as $n \rightarrow \infty$. Given $\epsilon > 0$, choose $\epsilon_1 > 0$ such that

$$\max \{ (e^{\epsilon_1} - 1), (1 - e^{-\epsilon_1}) \} < \epsilon.$$

There exists N such that $\left| \log \frac{\lambda_n(s) - \lambda_n(t)}{s - t} \right| < \epsilon_1$ for all $\tau_{\alpha_n} \geq s > t \geq 0$ and for all $n \geq N$. Thus, at $t = 0$, $e^{-\epsilon_1}s < \lambda_n(s) < e^{\epsilon_1}s$ for all $s > 0$ and for all $n \geq N$. Hence,

$$-(1 - e^{-\epsilon_1})s < \lambda_n(s) - s < (e^{\epsilon_1} - 1)s$$

for all $s > 0$ and for all $n \geq N$. Hence,

$$\frac{1}{\tau_{\alpha_n}} \sup_{s \in [0, \tau_{\alpha_n}]} |\lambda_n(s) - s| \leq \max \{ (e^{\epsilon_1} - 1), (1 - e^{-\epsilon_1}) \} < \epsilon$$

for all $n \geq N$. Thus, if (τ_{α_n}) is bounded, then

$$\tau_{\beta_n} - \tau_{\alpha_n} = \left(\frac{\tau_{\beta_n}}{\tau_{\alpha_n}} - 1 \right) \tau_{\alpha_n} \rightarrow 0$$

as $n \rightarrow \infty$. The proof that $P(\lambda)$ satisfies (P4') follows immediately. $\square$

Remark 5.2.6. The definitions (5.2.1) and (5.2) for a penalty function may be used for nondecreasing functions λ , not necessarily continuous, when allowing $+\infty$ as value of the penalty function. In case (2), let λ be a nondecreasing function that maps $[0, \tau_\alpha]$ to $[0, \tau_\beta]$ satisfying $\lambda(0) = 0$ and $\lambda(\tau_\alpha) = \tau_\beta$. If $P(\lambda) < \infty$, then λ is strictly increasing and Lipschitz continuous. The inverse satisfies $P(\lambda^{-1}) = P(\lambda) < \infty$. Thus, also λ^{-1} is Lipschitz. Moreover, $P(\lambda) < \infty$ if and only if λ is an increasing lipeomorphism. On the other way around, there exist strictly increasing continuous functions λ for which $P(\lambda) = \infty$.

Proof. If $P(\lambda) < \infty$, then the slopes $\frac{\lambda(s) - \lambda(t)}{s - t}$ should satisfy $0 < \frac{\lambda(s) - \lambda(t)}{s - t} < \infty$ for $s > t$. Therefore, λ is strictly increasing. Now to show that λ is a lipeomorphism when $P(\lambda)$ is finite, let $P(\lambda) = c < \infty$, then $\left| \log \frac{\lambda(s) - \lambda(t)}{s - t} \right| \leq c$ for all $\tau_\alpha \geq s > t \geq 0$, so, $e^{-c}(s - t) \leq \lambda(s) - \lambda(t) \leq e^c(s - t)$. Thus, $|\lambda(s) - \lambda(t)| \leq \max\{e^{-c}, e^c\} |s - t| = L |s - t|$, for all $\tau_\alpha \geq s > t \geq 0$, where $L = \max\{e^{-c}, e^c\}$. So, λ is Lipschitz. By (P2) and using the same argument as above, λ^{-1} is Lipschitz. Therefore, λ is a lipeomorphism and hence is a member of $\Delta_{\alpha, \beta}$. Conversely, if λ is a lipeomorphism, then $|\lambda(s) - \lambda(t)| \leq L |s - t|$, and $|\lambda^{-1}(s) - \lambda^{-1}(t)| \leq \ell |s - t|$ with $\ell, L \geq 1$. Consequently, $0 < \frac{1}{\ell} \leq \left| \frac{\lambda(s) - \lambda(t)}{s - t} \right| \leq L$ for $s \neq t$. Hence, $\left| \log \frac{\lambda(s) - \lambda(t)}{s - t} \right| \leq c$ for all $\tau_\alpha \geq s > t \geq 0$, where $c = \max(-\log \ell, \log L)$. Thus, $P(\lambda) \leq c < \infty$. $\square$

Remark 5.2.7. The penalty function introduced in (5.2) satisfies, for $\lambda \in \Delta_{\alpha, \beta}$,

$$P(\lambda) = \sup_{0 \leq t < s \leq \tau_\alpha} \left| \log \frac{\lambda(s) - \lambda(t)}{s - t} \right| = \text{ess sup}_{t \in [0, \tau_\alpha]} |\log \lambda'(t)|.$$

(See [33, p. 117]). This function is used in [33] in the definition of the Skorohod metric on càdlàg functions.

An example of a function P such that $\hat{d}$ is not a metric, where the definiteness property fails, but it is a semimetric is

Example 5.2.8. Let $P(\lambda) = \log \left(|\lambda|_{\text{Lip}} |\lambda^{-1}|_{\text{Lip}} \right)$, where $|\lambda|_{\text{Lip}}$ denotes the Lipschitz constant of a Lipschitz λ , with

$$\Delta_{\alpha,\beta} := \{\lambda: [0, \tau_\alpha] \rightarrow [0, \tau_\beta] : \lambda \text{ increasing lipeomorphism}\}.$$

Taking $f(t) = t$, $t \in [0, 1]$, $g(t) = 2t$, $t \in [0, \frac{1}{2}]$ gives that $\hat{d}(f, g) = 0$ but $f \neq g$.

In a special case one does obtain a metric space with this P :

Example 5.2.9. For a fixed τ the space $\left(C_s([0, \tau]), \hat{d} \right)$ with the penalty function $P(\lambda) = \log \left(|\lambda|_{\text{Lip}} |\lambda^{-1}|_{\text{Lip}} \right)$ and

$$\Delta_{\alpha,\beta} := \{\lambda: [0, \tau] \rightarrow [0, \tau] : \lambda \text{ increasing lipeomorphism}\}$$

is a metric space. Here we need to show that P satisfies (P4). For this purpose we need the following lemma.

Lemma 5.2.10. *Let $\lambda_n : [0, \tau] \rightarrow [0, \tau]$ be strictly increasing lipeomorphisms such that $|\lambda_n|_{\text{Lip}} |\lambda_n^{-1}|_{\text{Lip}} \rightarrow 1$. Then we have:*

1. $|\lambda_n|_{\text{Lip}} \rightarrow 1$ and $|\lambda_n^{-1}|_{\text{Lip}} \rightarrow 1$.
2. $\lambda_n(t) \rightarrow t$, for all t (pointwise convergence).
3. $\lambda_n \rightarrow I$ uniformly ($\|\lambda_n - I\| \rightarrow 0$).

Proof. We have $|\lambda_n(\tau) - \lambda_n(0)| \leq |\lambda_n|_{\text{Lip}} |\tau - 0|$ but having $\lambda_n(0) = 0$, $\lambda_n(\tau) = \tau$ gives $|\lambda_n(\tau) - \lambda_n(0)| = |\tau - 0|$. So, $|\lambda_n|_{\text{Lip}} \geq 1$. λ_n is increasing, therefore, λ_n^{-1} is increasing and hence $|\lambda_n^{-1}|_{\text{Lip}} \geq 1$ as well. Together with $|\lambda_n|_{\text{Lip}} |\lambda_n^{-1}|_{\text{Lip}} \rightarrow 1$, we get $|\lambda_n|_{\text{Lip}} \rightarrow 1$ and $|\lambda_n^{-1}|_{\text{Lip}} \rightarrow 1$. For the second point of the lemma we need to show that $\lim_{n \rightarrow \infty} \lambda_n(t) = t$ for all t . We will do this by contradiction. So, assume there exists t_0 such that $\lambda_n(t_0)$ does not converge to t_0 . Then $t_0 \neq 0$ and $t_0 \neq \tau$. Moreover, there exist $\epsilon > 0$ and a subsequence (n_k) such that either $\lambda_{n_k}(t_0) \leq t_0 - \epsilon$ or $\lambda_{n_k}(t_0) \geq t_0 + \epsilon$ for all k . In the first case, $|\lambda_n(t_0) - \lambda_n(\tau)| \leq |\lambda_n|_{\text{Lip}} |t_0 - \tau|$ or $|\lambda_n|_{\text{Lip}} \geq \frac{|\lambda_n(t_0) - \lambda_n(\tau)|}{|t_0 - \tau|}$:

$$|\lambda_{n_k}|_{\text{Lip}} \geq \frac{|\lambda_{n_k}(t_0) - \lambda_{n_k}(\tau)|}{|t_0 - \tau|} = \frac{\tau - \lambda_{n_k}(t_0)}{\tau - t_0} \geq \frac{\tau - t_0 + \epsilon}{\tau - t_0} = 1 + \frac{\epsilon}{\tau - t_0} > 1.$$

In the second case,

$$|\lambda_{n_k}|_{\text{Lip}} \geq \frac{|\lambda_{n_k}(t_0) - \lambda_{n_k}(0)|}{|t_0 - 0|} = \frac{\lambda_{n_k}(t_0)}{t_0} \geq \frac{t_0 + \epsilon}{t_0} = 1 + \frac{\epsilon}{t_0} > 1.$$

This contradicts $|\lambda_n|_{\text{Lip}} \rightarrow 1$ as $n \rightarrow \infty$. For the last part of the lemma, let $\epsilon > 0$. From the pointwise convergence we have: there exist N_x such that: $|\lambda_n(x) - x| < \frac{\epsilon}{3}$ for all $n \geq N_x$. Let $M = \sup_n |\lambda_n|_{\text{Lip}}$. Then, $M \geq 1$. Now, if $|x - y| < \frac{\epsilon}{3M}$ then we have:

$$\begin{aligned} |\lambda_n(y) - y| &\leq |\lambda_n(y) - \lambda_n(x)| + |\lambda_n(x) - x| + |x - y| \\ &\leq |\lambda_n|_{\text{Lip}} |y - x| + \frac{\epsilon}{3} + \frac{\epsilon}{3M} \\ &< \frac{\epsilon |\lambda_n|_{\text{Lip}}}{3M} + \frac{\epsilon}{3} + \frac{\epsilon}{3M} \quad (\text{note } \frac{|\lambda_n|_{\text{Lip}}}{M} \leq 1, \text{ also } \frac{1}{M} \leq 1) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Since $[0, \tau]$ is compact, there exist finitely many $x_i \in [0, \tau]$ such that $[0, \tau]$ is covered by open intervals $(x_i - \frac{\epsilon}{3M}, x_i + \frac{\epsilon}{3M})$. Put $N = \max_i (N_{x_i})$ then

$$\sup_{y \in [0, \tau]} |\lambda_n(y) - y| < \epsilon$$

for all $n \geq N$. That is, $\|\lambda_n - I\| \rightarrow 0$. □

5.3 Separability and completeness of the space

$$C_s^\tau$$

The main result on separability is given by the following Proposition.

Proposition 5.3.1. *Suppose that there exists a countable subset*

$$Q \subseteq \{\tau_\alpha : \tau_\alpha \in \mathbb{R}_+, \alpha \in \Lambda\}$$

such that for every τ_α there exist τ_n in Q and there exist $\lambda_n \in \Delta_{\alpha, n}$, such that $P(\lambda_n) \rightarrow 0$. If $C_s([0, \tau_\alpha])$ is separable for all α , then the space $C_s^\tau := \bigcup_\alpha C_s([0, \tau_\alpha])$ equipped with the metric $\hat{d}$ is separable.

Proof. For each α , let $M[0, \tau_\alpha]$ be a countable dense subset of $C_s([0, \tau_\alpha])$. We claim that the set $M = \bigcup_{\tau_\alpha \in Q} M[0, \tau_\alpha]$, is a countable dense subset in C_s^τ .

Obviously, taking the disjoint union over $\tau_\alpha \in Q$ makes M countable. Now, to show that the defined set M is dense we only need to show that if $f \in C_s[0, \tau_\alpha]; \tau_\alpha \notin Q$ then f can be approximated by functions from M (the other case, $f \in C_s[0, \tau_\alpha]; \tau_\alpha \in Q$ is obvious). So, let $f \in C_s([0, \tau_\alpha]); \tau_\alpha \notin Q$ and take $\tau_n \in Q$ and $\lambda_n : [0, \tau_\alpha] \rightarrow [0, \tau_n]$ such that $P(\lambda_n) \rightarrow 0$. Define $f_n \in C_s([0, \tau_n])$ as $f_n = f \circ \lambda_n^{-1}$. Then we have:

$$f_n(\lambda_n(t)) = f(\lambda_n^{-1}(\lambda_n(t))) = f(t).$$

Therefore, $d_{\lambda_n^{-1}}(f_n, f) = 0$. Thus,

$$\hat{d}(f_n, f) \leq d_{\lambda_n^{-1}}(f_n, f) + P(\lambda_n) = P(\lambda_n) \rightarrow 0$$

as $n \rightarrow \infty$. Let $g_n \in M[0, \tau_n]$ such that $\hat{d}(g_n, f_n) < \frac{1}{n}$. Then,

$$\hat{d}(g_n, f) \leq \hat{d}(g_n, f_n) + \hat{d}(f_n, f) \leq \frac{1}{n} + P(\lambda_n) \rightarrow 0.$$

□

Remark 5.3.2. The penalty functions of Example 5.2.1 and 5.2.2 satisfy the required condition in the previous proposition when $\{\tau_\alpha : \alpha \in \Lambda\} = (0, \infty)$.

Proof. Choose $Q = \mathbb{Q} \cap (0, \infty)$. For $\tau_\alpha \in (0, \infty)$ choose τ_n in Q such that $\tau_n \rightarrow \tau_\alpha$. Define $\lambda_n(t) := \frac{\tau_n}{\tau_\alpha}t$, clearly, $\lambda_n \in \Delta_{\alpha, n}$. Moreover, in the first case,

$$P(\lambda_n) = \|\lambda_n - I\| = \sup_{t \in [0, \tau_\alpha]} |\lambda_n(t) - t| = \sup_{t \in [0, \tau_\alpha]} \left| \frac{\tau_n}{\tau_\alpha} - 1 \right| t \leq \left| \frac{\tau_n}{\tau_\alpha} - 1 \right| \tau_\alpha \rightarrow 0.$$

In the second case,

$$P(\lambda_n) = \sup_{\tau_\alpha \geq s > t \geq 0} \left| \log \frac{\lambda_n(s) - \lambda_n(t)}{s - t} \right| = \sup_{\tau_\alpha \geq s > t \geq 0} \left| \log \frac{\tau_n}{\tau_\alpha} \right| \rightarrow 0.$$

□

Whether the spaces $C_s([0, \tau_\alpha])$ are separable depends on S . Separability of S is not sufficient, pathwise connectedness conditions seem to be required. We have the following particular result that suffices in most practical settings:

Proposition 5.3.3. *Let X be a normed space. Let S be a convex subset of X which is separable in the relative topology, then the space $C_s([0, \tau_\alpha])$ equipped with*

the uniform metric is separable. In particular, $(C_s([0, \tau_\alpha]), \hat{d})$ is separable for any α .

Proof. Let E be a countable dense subset of S and let $\xi = \{f : [0, \tau_\alpha] \rightarrow S\}$ be such that there exist $0 = t_1 < t_2 < \dots < t_n = \tau_\alpha$, $t_i \in \mathbb{Q}$, $f(t_1), f(t_2), \dots, f(t_n) \in E$ satisfying

$$f(\lambda t_i + (1 - \lambda)t_{i+1}) = \lambda f(t_i) + (1 - \lambda)f(t_{i+1})$$

for all $1 \leq i < n$ and $\lambda \in [0, 1]$. Countability of ξ is obvious. We need to show that ξ is dense in $C_s([0, \tau_\alpha])$. For this purpose let $f \in C_s([0, \tau_\alpha])$. Given $\epsilon > 0$, choose $\delta > 0$ such that $d_s(f(s), f(t)) < \epsilon$ whenever $|s - t| < \delta$. Choose $0 = t_1 < t_2 < \dots < t_n = \tau_\alpha$ in $[0, \tau_\alpha] \cap \mathbb{Q}$ such that $t_{i+1} - t_i < \delta$. Choose $e_1, e_2, \dots, e_n \in E$ such that $d_s(f(t_i), e_i) < \epsilon$. Construct $g \in \xi$ such that $g(t_i) = e_i$. One could check that $\sup_{t \in [0, \tau_\alpha]} d_s(f(t), g(t)) \leq 5\epsilon$, then Remark 5.2.3 yields $\hat{d}(f, g) < 5\epsilon$, and this ends the proof. $\square$

Establishing the property of completeness requires more elaboration. In particular we need a condition stronger than (P3): For the completeness of the general metric space $(C_s^\tau, \hat{d})$ we will need the following condition (P3') that will replace (P3).

(P3') If $\mu_m \in \Delta_{m, m+1}$ for $m = 1, 2, \dots$ are such that $\sum_{m=1}^\infty P(\mu_m) < \infty$, then the sequence $(\mu^{(m)})_{m=1}^\infty$ defined by $\mu^{(m)} := \mu_m \circ \mu_{m-1} \circ \dots \circ \mu_1$ converges uniformly to some $\lambda \in \Delta$ and $P(\lambda) \leq \sum_{m=1}^\infty P(\mu_m)$.

Note that (P3') implies (P3).

Let $(\mu_n^{(m)})_{m=1}^\infty$ denote the sequence in Δ defined by $\mu_n^{(m)} := \mu_{n+m} \circ \dots \circ \mu_{n+1} \circ \mu_n$.

Remark 5.3.4. Suppose (Δ, P) satisfy (P3') and (μ_n) is a sequence satisfying the conditions of (P3'). Define

$$\lambda_n := \lim_{n \rightarrow \infty} \mu_n^{(m)} = \lim_{m \rightarrow \infty} \mu_{n+m} \dots \mu_{n+1} \mu_n.$$

Then $P(\lambda_n) \rightarrow 0$ as $n \rightarrow \infty$. Moreover, there exists $\tau_\beta; 0 < \tau_\beta < \infty$ such that $\lambda_n \in \Delta_{n, \beta}$ and $\tau_n \rightarrow \tau_\beta$. Hence,

$$\lambda_n : [0, \tau_n] \rightarrow [0, \lim_{m \rightarrow \infty} \tau_{n+m+1}] = [0, \tau_\beta]$$

and $\tau_{n+m+1} \leq M$ for some $M \in \mathbb{R}$.

Proof. By (P3'), $P(\lambda_n) \leq \sum_{m=n}^{\infty} P(\mu_m)$. Thus, $P(\lambda_n) \rightarrow 0$ as $n \rightarrow \infty$. Also the other part follows from (μ_n) satisfy (P3'), which in turn yields, λ_n is the uniform limit of $(\mu_n^{(m)})$ and $\lambda_n \in \Delta$. That is, there exist $\tau_{\beta_n}; 0 < \tau_{\beta_n} < \infty$ such that $\lambda_n \in \Delta_{n,\beta_n}$. Then, Since $\tau_n = \mu_{n-1} \dots \mu_1(\tau_1)$,

$$\lambda_n(\tau_n) = \lim_{m \rightarrow \infty} \mu_{n+m} \dots \mu_{n+1} \mu_n(\tau_n) = \lim_{m \rightarrow \infty} \mu_{n+m} \dots \mu_1(\tau_1) = \lambda_1(\tau_1),$$

for each n , so $\tau_{\beta_n} = \tau_{\beta} := \lambda_1(\tau_1)$ for all n . Further, $\tau_n \rightarrow \tau_{\beta}$. Also,

$$\lim_{m \rightarrow \infty} \tau_{n+m+1} = \lambda_1(\tau_1) = \tau_{\beta}.$$

Thus, for fixed n , $(\tau_{n+m+1})_m$ converges, so it is bounded. $\square$

Theorem 5.3.5. *Let (Δ, P) satisfy conditions (D1), (D2), (P1), (P2), (P3') and (P4). If (S, d_s) is a complete metric space, then the space $(C_s^{\tau}, \hat{d})$ is complete.*

Proof. The proof is inspired by Billingsley [9]. Let (f_k) be a Cauchy sequence in $C_s^{\tau} = \bigcup_{\alpha} C_s([0, \tau_{\alpha}])$ under the metric $\hat{d}$ associated to (Δ, P) . It contains a subsequence $(g_n) = (f_{k_n})$ such that $g_n \in C_s([0, \tau_n])$ and $\hat{d}(g_n, g_{n+1}) < \frac{1}{2^n}$. We shall prove that (g_n) is convergent. Now, by $\hat{d}(g_n, g_{n+1}) < \frac{1}{2^n}$, Δ contains $\mu_n : [0, \tau_n] \rightarrow [0, \tau_{n+1}]$ such that $d_{\mu_n}(g_n, g_{n+1}) + P(\mu_n) < \frac{1}{2^n}$. This implies

$$d_{\mu_n}(g_n, g_{n+1}) = \sup_{t \in [0, \tau_n]} d_s(g_n(t), g_{n+1}(\mu_n t)) < \frac{1}{2^n} \quad (5.4)$$

and

$$P(\mu_n) < \frac{1}{2^n}. \quad (5.5)$$

We need to find a function g in $C_s([0, \tau_{\beta}])$, $(\tau_{\beta} \neq \infty)$, and functions λ_n in $\Delta_{n,\beta}$ for which $P(\lambda_n) \rightarrow 0$ and $g_n(\lambda_n^{-1}(t)) \rightarrow g(t)$ uniformly in t with respect to the metric d_s . Since $P(\mu_n) < \frac{1}{2^n}$, $\sum_{n=1}^{\infty} P(\mu_n) < \infty$. Thus, by (P3'), $\lambda_n = \lim_{m \rightarrow \infty} \mu_n^{(m)}$ exists for each n , $\lambda_n \in \Delta$ and $P(\lambda_n) \leq \sum_{m=n}^{\infty} P(\mu_m) \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\lambda_n = \lim_{m \rightarrow \infty} \mu_{n+m} \dots \mu_{n+1} \mu_n,$$

$$\lambda_n = \lambda_{n+1} \circ \mu_n. \quad (5.6)$$

Therefore, λ_n and λ_{n+1} must have the same range. Since $\lambda_n, \lambda_{n+1} \in \Delta$, $\lambda_n \in \Delta_{n,\beta}$ for some fixed τ_{β} . This means that the range of $\lambda_n : [0, \tau_n] \rightarrow [0, \tau_{\beta}]$ is independent of n , and this will make it possible to evaluate the supremum below. Now (5.6)

is equivalent to $\lambda_{n+1}^{-1} = \mu_n \lambda_n^{-1}$. Therefore, by (5.4)

$$\sup_{t \in [0, \tau_\beta]} d_s(g_n(\lambda_n^{-1}t), g_{n+1}(\lambda_{n+1}^{-1}t)) = \sup_{\omega \in [0, \tau_n]} d_s(g_n(\omega), g_{n+1}(\mu_n \omega)) < \frac{1}{2^n}.$$

It follows that the functions $g_n(\lambda_n^{-1}t)$, which are elements of $C_s([0, \tau_\beta])$, are uniformly Cauchy and hence converge uniformly to a limit function $g(t)$. Finally, since

$$\sup_{t \in [0, \tau_n]} d_s(g_n(t), g(\lambda_n t)) \rightarrow 0$$

and $P(\lambda_n) \rightarrow 0$, we have $\hat{d}(g_n, g) \rightarrow 0$. $\square$

Note that the proof of Theorem 5.3.5 will not work for the penalty function (5.1). This is because $\sup_t |\lambda_n(t) - t| \leq \frac{1}{2^{n-1}}$ does not imply that λ_n is strictly increasing and therefore we cannot assume that the limit function $\lambda_n = \lim_{m \rightarrow \infty} \mu_{n+m} \dots \mu_{n+1} \mu_n$ will still be invertible. The penalty function given by (5.2), however, is particularly nice. It satisfies (P3'). Hence it yields a complete metric space when S is complete, among others. To prove this we need a lemma:

Lemma 5.3.6. *Let $P(\lambda)$ be the penalty function (5.2). If $\lambda \in \Delta$, then*

$$\sup_{t \in [0, \tau_\alpha]} |\lambda(t) - t| \leq \tau_\alpha (e^{P(\lambda)} - 1).$$

Proof. Since $|x - 1| \leq e^{|\log x|} - 1$ for $x > 0$,

$$\begin{aligned} |\lambda(t) - t| &= t \left| \frac{\lambda(t) - \lambda(0)}{t - 0} - 1 \right| \leq t \left(e^{|\log \frac{\lambda(t) - \lambda(0)}{t - 0}|} - 1 \right) \\ &\leq t \left(e^{\sup_{0 \leq s < r \leq \tau_\alpha} |\log \frac{\lambda(r) - \lambda(s)}{r - s}|} - 1 \right) \\ &= t (e^{P(\lambda)} - 1). \end{aligned}$$

Taking the supremum over the interval $[0, \tau_\alpha]$ we get the required result. $\square$

Proposition 5.3.7. *Let $P(\lambda)$ be the penalty function (5.2). Let*

$$\mu_m : [0, \tau_m] \rightarrow [0, \tau_{m+1}] \in \Delta_{m, m+1}$$

be such that $\sum_{m=1}^\infty P(\mu_m) < \infty$. Then the following holds:

1. *There exist m, M ; $0 < m < M$ such that $m \leq \tau_n \leq M$ for all n .*

2. For each n , the sequence $\left(\mu_n^{(m)}\right)_{m=1}^{\infty}$ in Δ defined by

$$\mu_n^{(m)} := \mu_{n+m} \circ \dots \circ \mu_{n+1} \circ \mu_n$$

is Cauchy in the uniform metric.

3. The uniform limit $\lambda_n := \lim_{m \rightarrow \infty} \mu_n^{(m)}$ is in Δ , $P(\lambda_n) \leq \sum_{m=0}^{\infty} P(\mu_{n+m})$ and consequently $P(\lambda_n) \rightarrow 0$ as $n \rightarrow \infty$.

In particular, (Δ, P) satisfies $(P\mathcal{J})$.

Proof. In order to prove that there exist m, M ; $0 < m < M$ such that $0 < m \leq \tau_n \leq M$, we consider the following: $P(\mu_n) = \sup_{0 \leq t < s \leq \tau_n} \left| \log \frac{\mu_n(s) - \mu_n(t)}{s - t} \right|$, and with $t = 0$, $s = \tau_n$, we have

$$\left| \log \frac{\mu_n(\tau_n) - \mu_n(0)}{\tau_n - 0} \right| = |\log(\tau_{n+1}) - \log(\tau_n)| \leq P(\mu_n).$$

Since $P(\mu_n)$ is summable, $(\log(\tau_n))$ is Cauchy. Hence, $(\log(\tau_n))$ converges and it is bounded. That is, $|\log \tau_n| \leq L$, for some $L \in \mathbb{R}$. Thus, $e^{-L} \leq \tau_n \leq e^L$. Or, $0 < m \leq \tau_n \leq M$, where $m = e^{-L}$ and $M = e^L$. Also, $\log(\tau_n) \rightarrow \zeta$ implies $\tau_n = e^{\log \tau_n} \rightarrow e^{\zeta} := \tau_{\beta}$. That is, $\lim_{n \rightarrow \infty} \tau_n = \tau_{\beta}$ where $0 < \tau_{\beta} < \infty$. Now to prove $\mu_n^{(m)}$ is Cauchy in the uniform metric we consider the following, by Lemma 5.3.6,

$$\begin{aligned} \sup_{t \in [0, \tau_n]} |\mu_n^{(m+1)}(t) - \mu_n^{(m)}(t)| &= \sup_{t \in [0, \tau_n]} |\mu_{n+m+1} \dots \mu_{n+1} \mu_n(t) - \mu_{n+m} \dots \mu_{n+1} \mu_n(t)| \\ &= \sup_{s \in [0, \tau_{n+m+1}]} |\mu_{n+m+1}(s) - s| \\ &\leq \tau_{n+m+1} (e^{P(\mu_{n+m+1})} - 1). \end{aligned}$$

Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\lim_{m \rightarrow \infty} P(\mu_m) = 0$. Let N be such that $P(\mu_m) \leq \frac{1}{2}$ for all $m \geq N$. Since $e^x - 1 \leq 2x$ for $0 \leq x \leq \frac{1}{2}$, then for all $m \geq N$ we have

$$\begin{aligned} \sup_{t \in [0, \tau_n]} |\mu_n^{(m+1)}(t) - \mu_n^{(m)}(t)| &\leq \tau_{n+m+1} 2P(\mu_{n+m+1}) \\ &\leq 2MP(\mu_{n+m+1}). \end{aligned}$$

Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\sum_{m=1}^{\infty} 2MP(\mu_{n+m+1}) < \infty$. Thus, $\left(\mu_n^{(m)}\right)_{m=1}^{\infty}$ is uniformly Cauchy and there exists $\lambda_n = \lim_{m \rightarrow \infty} \mu_n^{(m)}$. Moreover, for the third part,

for fixed n , the function λ_n is a uniform limit of continuous functions and thus continuous. Also, it is nondecreasing and satisfies $\lambda_n(0) = 0$,

$$\lambda_n(\tau_n) = \lim_{m \rightarrow \infty} \mu_{n+m} \circ \dots \circ \mu_{n+1} \circ \mu_n(\tau_n) = \lim_{m \rightarrow \infty} \tau_{n+m+1} = \tau_\beta.$$

Since

$$\begin{aligned} \left| \log \frac{\mu_{n+m} \dots \mu_{n+1} \mu_n(t) - \mu_{n+m} \dots \mu_{n+1} \mu_n(s)}{t - s} \right| &\leq P(\mu_{n+m} \dots \mu_{n+1} \mu_n) \\ &\leq P(\mu_{n+m}) + \dots + P(\mu_{n+1}) + P(\mu_n) = \sum_{i=n}^{n+m} P(\mu_i), \end{aligned}$$

taking the limit as $m \rightarrow \infty$ gives

$$\left| \log \frac{\lambda_n(t) - \lambda_n(s)}{t - s} \right| \leq \lim_{m \rightarrow \infty} \sum_{i=n}^{n+m} P(\mu_i) = \sum_{i=n}^{\infty} P(\mu_i).$$

Thus, $P(\lambda_n) \leq \sum_{i=n}^{\infty} P(\mu_i) < \infty$. Hence, by Remark 5.2.6, $\lambda_n \in \Delta$. Also, $P(\lambda_n) \rightarrow 0$ as $n \rightarrow \infty$. $\square$

Corollary 5.3.8. *If (S, d_s) is a complete metric space and $\hat{d}$ is considered with the penalty function (5.2), then the space $(C_s^\tau, \hat{d})$ is complete.*

Part of the nice properties established in Proposition 5.3.7 may already be derived from the following general property which we call (P5):

(P5) There exists a continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{x} = L$$

(exists) and such that for $\lambda \in \Delta_{\alpha, \beta}$

$$\sup_{t \in [0, \tau_\alpha]} |\lambda(t) - t| \leq \tau_\alpha f(P(\lambda)).$$

According to Lemma 5.3.6, the function $P(\lambda)$ given by (5.2) satisfies (P5).

Note that (P5) implies (P4). Moreover, we have the following partial result towards establishing (P3').

Lemma 5.3.9. *Assume (Δ, P) satisfies the condition (P5). Let $\mu_m : [0, \tau_m] \rightarrow [0, \tau_{m+1}] \in \Delta_{m, m+1}$ be such that $\sum_{m=1}^{\infty} P(\mu_m) < \infty$. Then the following holds:*

1. There exist $m, M; 0 < m < M$ such that $m \leq \tau_n \leq M$ for all n .

2. For each n , the sequence $\left(\mu_n^{(m)}\right)_{m=1}^{\infty}$ in Δ defined by

$$\mu_n^{(m)} := \mu_{n+m} \circ \dots \circ \mu_{n+1} \circ \mu_n$$

is Cauchy in the uniform metric.

3. If $\lambda_n := \lim_{m \rightarrow \infty} \mu_n^{(m)}$ is in Δ for some $n = n_0 \in \mathbb{N}$, then $\lambda_n \in \Delta$ for all $n \geq n_0$.

Proof. Since $\lim_{x \rightarrow 0^+} \frac{f(x)}{x} = L$, $f(x) \leq (L + \epsilon)x$ for sufficiently small x and $f(x) \rightarrow 0$ as $x \rightarrow 0$. Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $P(\mu_m) \rightarrow 0$ as $m \rightarrow \infty$. Thus, $f(P(\mu_m)) \rightarrow 0$ as $m \rightarrow \infty$. Moreover, since $f(P(\mu_m)) \leq (L + \epsilon)P(\mu_m)$ and $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\sum_{m=1}^{\infty} f(P(\mu_m)) < \infty$. By (P5),

$$\begin{aligned} |\tau_{m+1} - \tau_m| &= |\mu_m(\tau_m) - \tau_m| \\ &\leq \sup_{0 \leq t \leq \tau_m} |\mu_m(t) - t| \leq \tau_m f(P(\mu_m)). \end{aligned}$$

Thus, $\left| \frac{\tau_{m+1}}{\tau_m} - 1 \right| \leq f(P(\mu_m)) \rightarrow 0$ as $m \rightarrow \infty$. Or, $\frac{\tau_{m+1}}{\tau_m} \rightarrow 1$ as $m \rightarrow \infty$. Since $\log x \leq x - 1$ for all $x > 0$,

$$\begin{aligned} |\log(\tau_{m+1}) - \log(\tau_m)| &\leq \max \left(\frac{\tau_{m+1}}{\tau_m} - 1, \frac{\tau_m}{\tau_{m+1}} \left(1 - \frac{\tau_{m+1}}{\tau_m} \right) \right) \\ &\leq \left(1 + \frac{\tau_m}{\tau_{m+1}} \right) \left| \frac{\tau_{m+1}}{\tau_m} - 1 \right|. \end{aligned}$$

Since $\frac{\tau_m}{\tau_{m+1}} \rightarrow 1$ as $m \rightarrow \infty$, $\left(1 + \frac{\tau_m}{\tau_{m+1}} \right) \leq 3$. Hence,

$$|\log(\tau_{m+1}) - \log(\tau_m)| \leq 3 \left| \frac{\tau_{m+1}}{\tau_m} - 1 \right| \leq 3f(P(\mu_m)).$$

Since $\sum_{m=1}^{\infty} f(P(\mu_m)) < \infty$, $(\log \tau_m)_{m=1}^{\infty}$ is Cauchy. Hence, it is convergent and bounded. That is to say, there exist $m, M; 0 < m < M$ such that $m \leq \tau_n \leq M$ for all n . For the second part,

$$\begin{aligned} \sup_{t \in [0, \tau_n]} |\mu_n^{(m+1)}(t) - \mu_n^{(m)}(t)| &= \sup_{s \in [0, \tau_{n+m+1}]} |\mu_{n+m+1}(s) - s| \\ &\leq \tau_{n+m+1} f(P(\mu_{n+m+1})). \end{aligned}$$

Since $\sum_{m=1}^\infty P(\mu_m) < \infty$, $\lim_{m \rightarrow \infty} P(\mu_m) = 0$. Let N be such that $P(\mu_m) \leq \delta$ for all $m \geq N$. Then for all $m \geq N$ we have

$$\begin{aligned} \sup_{t \in [0, \tau_n]} |\mu_n^{(m+1)}(t) - \mu_n^{(m)}(t)| &\leq \tau_{n+m+1} (L + \epsilon) P(\mu_{n+m+1}) \\ &\leq M (L + \epsilon) P(\mu_{n+m+1}). \end{aligned}$$

Since $\sum_{m=1}^\infty P(\mu_m) < \infty$, $\sum_{m=1}^\infty M (L + \epsilon) P(\mu_{n+m+1}) < \infty$. Thus, $(\mu_n^{(m)})_{m=1}^\infty$ is uniformly Cauchy. Moreover, there exists $\lambda_n = \lim_{m \rightarrow \infty} \mu_n^{(m)}$. For the third part, let $\lambda_n = \lim_{m \rightarrow \infty} \mu_n^{(m)}$ be in Δ for some $n \geq n_0$. Then $\lambda_n^{-1} \in \Delta$. By (5.6), $\lambda_{n+1}^{-1} = \mu_n \lambda_n^{-1}$. So by induction $\lambda_n^{-1} \in \Delta$ for all $n \geq n_0$. Hence, $\lambda_n \in \Delta$ for all $n \geq n_0$. $\square$

Remark 5.3.10. In the general case, (P5) does not seem to allow to draw any conclusion on the domination of $P(\lambda_n)$ by $\sum_{m=n}^\infty P(\mu_m)$ as required in (P3') unfortunately. Additional properties of (Δ, P) are required to establish that part.

5.4 Spaces of sequences in C_s^τ and càdlàg functions

For an introduction to càdlàg functions, see for instance, Billingsley [9]. As mentioned in the introduction, sequences of functions in C_s^τ are a means of coding the continuous deterministic parts of a sample trajectory of the stochastic process that results from random jump interventions on dynamical system in S . In this section we investigate the ‘classical’ two metric properties of sequence spaces, and the relation to càdlàg functions. We denote the space of all sequences of continuous functions from C_s^τ by $(C_s^\tau)^\mathbb{N}$ or briefly $(C_s^\mathbb{N})$, that is,

$$(C_s^\mathbb{N}) := \{(f_n) = (f_1, f_2, \dots) : f_i \in C_s([0, \tau_i])\}.$$

Now, we would like to define a metric on $(C_s^\mathbb{N})$. For this purpose we will make use of the metric $\hat{d}$. Let $\hat{d}_0(f, g) = \frac{\hat{d}(f, g)}{1 + \hat{d}(f, g)}$ then $\hat{d}_0$ is a metric on C_s^τ equivalent to $\hat{d}$, which gives C_s^τ finite diameter. Moreover, completeness of C_s^τ under $\hat{d}$ yields that C_s^τ is complete under $\hat{d}_0$. If $(f_n), (g_n) \in (C_s^\mathbb{N})$, then we define a metric d as

follows:

$$d((f_n), (g_n)) = \sum_{i=1}^{\infty} 2^{-i} \hat{d}_0(f_i, g_i). \quad (5.7)$$

Proposition 5.4.1. *The space $((C_s^{\mathbb{N}}), d)$ is complete, whenever $(C_s^{\tau}, \hat{d})$ is complete.*

Proof. The space $((C_s^{\mathbb{N}}), d)$ is the Cartesian product of the complete metric spaces $(C_s^{\tau}, \hat{d})$ and hence it is complete. $\square$

Proposition 5.4.2. *If $(C_s^{\tau}, \hat{d})$ is separable, then the space $((C_s^{\mathbb{N}}), d)$ is separable.*

Proof. The space $((C_s^{\mathbb{N}}), d)$ is the countable Cartesian product of the separable metric spaces $(C_s^{\tau}, \hat{d}_0)$ which are of finite diameter. Thus, the product $(C_s^{\mathbb{N}})$ with the metric (5.7) is separable. Moreover, if M is a countable dense subset of C_s^{τ} and we fix $f_0 \in M$, then a countable dense subset of the the space $((C_s^{\mathbb{N}}), d)$, $\mathfrak{M}$, could be defined as follows:

$$\mathfrak{M} = \{(f_1, f_2, f_3, \dots, f_m, f_0, f_0, f_0, \dots) : f_i \in M \ \forall i, m \in \mathbb{N}\}.$$

$\square$

The components of a sequence (f_k) in $(C_s^{\mathbb{N}})$ may be *concatenated* to yield an S -valued function on $[0, \tau_{\infty})$ where $\tau_{\infty} = \sum_n \tau_n$ whenever $f_n \in C_s([0, \tau_n])$. This function has jump discontinuities at times $t_k = \sum_{n=1}^k \tau_n$. At these points one needs to decide which value to attain, for instance, one may pick $f_{k+1}(0)$. Thus, one obtains a right-continuous functions on $[0, \tau_{\infty})$ with left-limits. That is, we may define a concatenation map as follows:

$$\gamma : (C_s^{\mathbb{N}}) \rightarrow \bigcup_{0 < \tau \leq \infty} S^{[0, \tau)},$$

$$\gamma(f_1, f_2, \dots)(t) := f_k(t), \quad \tau_1 + \dots + \tau_{k-1} \leq t < \tau_1 + \dots + \tau_k.$$

In order to get a map into the càdlàg functions on $[0, \infty)$ one therefore needs to restrict to a subcollection of sequences, $(C_s^{\mathbb{N}})_{\infty}$. It consists of all sequences $(f_n) \in (C_s^{\mathbb{N}})$ such that $\sum_{i=1}^{\infty} \tau_i = \infty$. It is a proper subspace of $(C_s^{\mathbb{N}})$. Moreover,

we may restrict the concatenation map γ , from $(C_s^\mathbb{N})_\infty$ into the space of all càdlàg functions $D_s([0, \infty))$:

$$\gamma : (C_s^\mathbb{N})_\infty \rightarrow D_s([0, \infty)),$$

with the same definition as above.

It is interesting to observe that this concatenation map is not injective. This could be easily seen from the following example:

Example 5.4.3. Put $f_n(t) = 7$, for $t \in [0, 2^n]$, $g_n(t) = 7$, for $t \in [0, 1]$. Then clearly $f_n \in C_s([0, 2^n])$, $g_n \in C_s([0, 1])$, $(f_n) \neq (g_n)$ and $\gamma((f_n)) = \gamma((g_n))$, namely the constant function 7 in $D_s([0, \infty))$.

Continuity or measurability properties of the concatenation map are not yet completely clear. It will be the topic of further investigation in subsequent work.

We continue by providing some properties of $(C_s^\mathbb{N})_\infty$.

Lemma 5.4.4. *The space $(C_s^\mathbb{N})_\infty$ is not a closed subspace of $(C_s^\mathbb{N})$. Hence, $((C_s^\mathbb{N})_\infty, d)$ is not complete.*

Proof. We give an example of a sequence $(f_k^{(n)})_{k=1}^\infty \in (C_s^\mathbb{N})_\infty$ such that $(f_k^{(n)}) \rightarrow (f_k)$ and $(f_k) \notin (C_s^\mathbb{N})_\infty$. Let $(f_k^{(n)}) \in C_s([0, \frac{1}{n^2} + \frac{1}{2^k}])$ be such that $(f_k^{(n)}) \rightarrow (f_k)$ for some $(f_k) \in (C_s^\mathbb{N})$. Then $(f_k^{(n)}) \in (C_s^\mathbb{N})_\infty$ since $\sum_k \sum_n \frac{1}{n^2} + \frac{1}{2^k} = \infty$. But $(f_k) \notin (C_s^\mathbb{N})_\infty$ since $\sum_k \tau_k = \sum_k \frac{1}{2^k} \neq \infty$. $\square$

Proposition 5.4.5. *If $(C_s^\tau, \hat{d})$ is separable, then the space $((C_s^\mathbb{N})_\infty, d)$ is separable.*

Proof. The proof is similar to the one for the space $((C_s^\mathbb{N}), d)$, Proposition 5.4.2, with slight differences. As f_0 is fixed, so its interval of definition, therefore, $\sum_{i=1}^\infty \tau_i = \infty$. $\square$

Lemma 5.4.6. *$(C_s^\mathbb{N})_\infty$ is dense in $((C_s^\mathbb{N}), d)$. In particular, when $(C_s^\tau, \hat{d})$ is complete, then $((C_s^\mathbb{N}), d)$ is a completion of the space $((C_s^\mathbb{N})_\infty, d)$.*

Proof. From the proof of Proposition 5.4.2, $\mathfrak{M}$ is dense in $((C_s^\mathbb{N}), d)$ and

$$\mathfrak{M} \subseteq (C_s^\mathbb{N})_\infty \subseteq (C_s^\mathbb{N}),$$

so $(C_s^\mathbb{N})_\infty$ is dense in $(C_s^\mathbb{N})$. Thus, when $(C_s^\tau, \hat{d})$ is complete, it follows from Lemma 5.4.1 that $((C_s^\mathbb{N}), d)$ is a completion of $(C_s^\mathbb{N})_\infty$. $\square$

Chapter 6

Continuous time process: an outlook

This chapter gives an outlook on extensions of the investigations in Chapters 3 and 4 to include explicitly the time intervals between interventions in the description of the dynamics, which allows for replacement of the deterministic system by a stochastic process with continuous trajectories and to extend the scope to interventions taking place at non-equidistant time points. It will be explained how the metric space of continuous functions on different intervals that has been introduced in Chapter 5, may serve as a natural state space. In particular, it allows to use a similar (analytic) approach even in the case when the deterministic process between interventions is replaced by a continuous time stochastic process with continuous trajectories. We like to point out again, that the question of asymptotic stability of the associated Markov operator for the PDMPs is still challenging when the state space is a non-locally compact Polish space. The locally compact case is to a large extent covered by [15, 16].

6.1 A description on partial trajectories of constant duration

The Markov operators studied in Chapters 3 and 4 only describe the dynamics of the considered models at the time points just after intervention. Since the dynamics between two interventions is purely deterministic, that information is sufficient to recover the continuous time process.

In this section we present how the Markov chain on states at equidistant times

of interventions can be related to a Markov operator on the space of partial trajectories between intervention times. This view will then be extended to cover the case of random time interventions in the following sections.

Let S be a complete separable subset of a normed vector space X and let $\phi : [0, \infty) \times S \rightarrow S$ be a dynamical system such that $t \mapsto \phi(t, x) = \phi_t(x)$ is continuous for all $x \in S$ and $x \mapsto \phi(t, x)$ is Lipschitz for all $t \in [0, \infty)$ and such that for each compact set B in S ,

$$\sup_{t \in [0, t^*]} |\phi_t|_B|_{\text{Lip}} < \infty$$

for each $t^* > 0$. Let for each $x \in S$ the ‘jump distribution’ Q_x be a Borel probability measure on X such that the measure $A \mapsto Q_x(A - x)$ is concentrated on S , for all $x \in S$. Let $\Delta t > 0$ and let $t_n = n\Delta t$, $n \in \mathbb{N}$ be the intervention times. As in Chapters 3 and 4 we suppose that the dynamics follow the deterministic dynamical system $(\phi_t)_t$ between interventions and that at each t_n the system jumps from the present state x to a new state y , where the jump $y - x$ is random with distribution Q_x . Assume that Q_x satisfies A1)–A5) of Chapter 4. Now define the jump operator $P_J : \mathcal{M}^+(S) \rightarrow \mathcal{M}^+(S)$ as

$$P_J \mu := \int_S Q_y(\cdot - y) d\mu(y). \quad (6.1)$$

If μ is the distribution of the position just before the jump, then $P_J \mu$ is the distribution of the position just after the jump. If μ is the distribution just after intervention at t_n and $t \in (t_n, t_{n+1})$, then the probability distribution for the state of the system at time t is given by $(\phi_{t-t_n})_{\#} \mu$. Define $\Phi : S \rightarrow C_S([0, \Delta t])$ by

$$\Phi(x)(t) := \phi_t(x), \quad t \in [0, \Delta t], \quad x \in S,$$

and for $\mu \in \mathcal{M}^+(S)$ define $P_{\Phi} \mu := \Phi_{\#} \mu$. Note that the map Φ defined above is continuous (this is an easy special case of Lemma 6.2.4 below). Hence, P_{Φ} maps $\mathcal{M}^+(S)$ into $\mathcal{M}^+(C_S([0, \Delta t]))$. Also, define $\text{end}(f) := f(\Delta t)$, $f \in C_S([0, \Delta t])$ and $P_{\text{end}} \mu := \text{end}_{\#} \mu$, $\mu \in \mathcal{M}^+(C_S[0, \Delta t])$. We can now construct a Markov operator describing the dynamics of the trajectories. Indeed, define

$$\mathbb{P} : \mathcal{M}^+(C_S([0, \Delta t])) \rightarrow \mathcal{M}^+(C_S([0, \Delta t]))$$

by

$$\mathbb{P}\mu := P_\Phi P_J P_{\text{end}}\mu.$$

Then $\mathbb{P}$ is a Markov operator on $C_S([0, \Delta t])$ and $\mathbb{P}\mu$ is the distribution of the partial trajectories after a jump, if μ is the distribution of the partial trajectories before that jump.

6.2 Non-equidistant state dependent random intervention times

If the intervention times $0 = t_0 < t_1 < t_2 < \dots$ are not equidistant, then it is still possible to describe the dynamics of the states just after intervention by means of iterations of one fixed Markov operator. This is even possible if the lengths of the intervals are random with state dependent distribution. We now present two approaches to this setting.

6.2.1 A model on $S \times (0, \infty)$

We follow the approach of [45, 55]. The idea there is to consider as a state the pair (x, τ) rather than only the position x just after an intervention, where τ is the duration of time before the next intervention. Suppose that for any state $x \in S$ just after intervention the length of the time interval until the next interval has law π_x on $(0, \infty)$. That is, almost surely an immediate new jump will not occur, while there will be a next jump. For a Borel measure μ on $S \times (0, \infty)$ define

$$P_1\mu(A) = \int_{S \times (0, \infty)} 1_A(\phi_\tau(x)) d\mu(x, \tau), \quad A \subseteq S \text{ Borel}.$$

Then $P_1\mu$ is the distribution of the position just before a jump if μ is the distribution of the position just after the previous jump. The distribution of position just after the jump is then given by $P_2\mu := P_J P_1\mu$, where P_J is given by (6.1). Finally, define for $A \subseteq S$, $B \subseteq (0, \infty)$ Borel,

$$P_{\text{prod}}\mu(A \times B) := \int_A \pi_x(B) d\mu(x).$$

P_{prod} extends to a measure on the Borel sets of $S \times (0, \infty)$ in standard way. Now the state of the system just after intervention, with random jumps and random times of intervention, both taken state dependent, is given by the Markov operator P on $S \times (0, \infty)$ defined by

$$P\mu := P_{\text{prod}}P_2\mu = P_{\text{prod}}P_JP_1\mu. \quad (6.2)$$

In view of the analysis in Section 4.4, it is an interesting problem to determine under which conditions on ϕ , Q and π the Markov operator P is nonexpansive or satisfies the e-property. An initial result in this direction is provided by the following:

Lemma 6.2.1. *If $f \in BL(S \times (0, \infty))$ and $x \mapsto \pi_x$ is Lipschitz with Lipschitz constant L for the $\|\cdot\|_{FM}$, then the map $\psi_f : x \mapsto \int_{(0, \infty)} f(x, t) \pi_x(dt)$ is Lipschitz with Lipschitz constant less than or equal $\|f\|_{\max} (1 + L) \|x - x'\|$.*

Proof. One has

$$\begin{aligned} |\psi_f(x) - \psi_f(x')| &\leq \left| \int_{(0, \infty)} f(x, t) \pi_x(dt) - \int_{(0, \infty)} f(x', t) \pi_{x'}(dt) \right| \\ &\leq \left| \int_{(0, \infty)} f(x, t) \pi_x(dt) - \int_{(0, \infty)} f(x, t) \pi_{x'}(dt) \right| \\ &\quad + \int_{(0, \infty)} |f(x, t) - f(x', t)| \pi_{x'}(dt) \\ &\leq \|f(x, \cdot)\|_{\max} \|\pi_x - \pi_{x'}\|_{FM} \\ &\quad + \|f\|_{\text{Lip}} \|x - x'\| \int_{(0, \infty)} \pi_{x'}(dt) \\ &\leq \|f\|_{\max} \cdot L \|x - x'\| + \|f\|_{\max} \|x - x'\| \\ &\leq (1 + L) \|f\|_{\max} \|x - x'\|. \end{aligned}$$

□

Corollary 6.2.2. *If the maps $x \mapsto \pi_x$ and $x \mapsto Q_x$ are both Lipschitz for $\|\cdot\|_{FM}$ with Lipschitz constants L and L' respectively and P_1 is strict contractive with Lipschitz constant satisfying $\|P_1\|_{\text{Lip}, FM} (1 + L) (1 + L') \leq 1$, then P is nonexpansive.*

Proof. For $\mu, \nu \in \mathcal{M}^+(S)$, and $f \in BL(S \times (0, \infty))$,

$$\begin{aligned} |\langle f, P_{\text{prod}}\mu - P_{\text{prod}}\nu \rangle| &= \left| \int_S \int_{(0, \infty)} f(x, t) \pi_x(dt) d\mu(dx) \right. \\ &\quad \left. - \int_S \int_{(0, \infty)} f(x, t) \pi_x(dt) d\nu(dx) \right| \\ &= |\langle \psi_f, \mu - \nu \rangle| \\ &\leq \| \psi_f \| \| \mu - \nu \|_{FM} . \end{aligned}$$

Hence,

$$\| P_{\text{prod}}\mu - P_{\text{prod}}\nu \|_{FM} \leq (1 + L) \| \mu - \nu \|_{FM} .$$

Thus, by the assumption $|P_1|_{Lip, FM} (1 + L) (1 + L') \leq 1$,

$$\begin{aligned} \| P\mu - P\nu \|_{FM} &= \sup_{f \in BL} |\langle f, P_{\text{prod}}P_J P_1\mu - P_{\text{prod}}P_J P_1\nu \rangle| \\ &\leq |P_1|_{Lip, FM} |P_J|_{Lip, FM} |P_{\text{prod}}|_{Lip, FM} \| \mu - \nu \|_{FM} \\ &\leq |P_1|_{Lip, FM} (1 + L) (1 + L') \| \mu - \nu \|_{FM} \\ &\leq \| \mu - \nu \|_{FM} . \end{aligned} \quad \square$$

6.2.2 A model on the space of partial trajectories

In the case of equidistant times, the Markov operator describing the dynamics of the states just after intervention can easily be lifted to a Markov operator giving the dynamics of the trajectories between interventions, as discussed in Section 6.1. The latter is a Markov operator acting on $\mathcal{M}^+(C_S[0, \Delta t])$ in that case. By means of the space C_S^τ presented in Chapter 5, a similar procedure works for the case of non-equidistant intervention times. We start with a convenient, technical, additional result on C_S^τ .

Let $\text{dur}(f)$ be the duration of $f \in C_S^\tau$, i.e. $f : [0, \text{dur}(f)] \rightarrow S$. Put $\text{end}_\theta(f) := f(\theta \text{dur}(f))$, for every $0 \leq \theta \leq 1$. We simply write $\text{end}(f)$ instead of $\text{end}_1(f)$, i.e. the end point of f .

Lemma 6.2.3. *Assume that (P4') of Section 5.2 holds. Then*

1. $\text{dur} : C_S^\tau \rightarrow (0, \infty)$ is continuous.
2. $\text{end}_\theta : C_S^\tau \rightarrow S$ is continuous for every $0 \leq \theta \leq 1$.

Proof. Note that if $f_n \rightarrow f$ in C_S^τ , and $\text{dur}(f_n) = \tau_n$, $\text{dur}(f) = \tau$, then $\tau_n \rightarrow \tau$ if (P4') holds. For part (1): let $f_n \rightarrow f$ in C_S^τ . Then the claim is to show that $\text{dur}(f_n) \rightarrow \text{dur}(f)$ in $(0, \infty)$. This follows from (P4'), since $\tau_{\alpha_n} \rightarrow \tau_\alpha$. For part (2): let again $f_n \rightarrow f$ in C_S^τ . We have that for each $n \in \mathbb{N}$, there exists $\lambda_n : [0, \tau] \rightarrow [0, \tau_n]$ such that $\sup_{s \in [0, \tau]} |\lambda_n(s) - s| \rightarrow 0$ as $n \rightarrow \infty$ and $\sup_{s \in [0, \tau]} |f_n(\lambda_n(s)) - f(s)| \rightarrow 0$ as $n \rightarrow \infty$. Then

$$\begin{aligned} |f_n(\theta\tau_n) - f(\theta\tau)| &= |f_n(\lambda_n(\lambda_n^{-1}(\theta\tau_n))) - f(\theta\tau)| \\ &\leq |f_n(\lambda_n(\lambda_n^{-1}(\theta\tau_n))) - f(\lambda_n^{-1}(\theta\tau_n))| \\ &\quad + |f(\lambda_n^{-1}(\theta\tau_n)) - f(\theta\tau)| \\ &\leq \sup_{s \in [0, \tau]} |f_n(\lambda_n(s)) - f(s)| + |f(\lambda_n^{-1}(\theta\tau_n)) - f(\theta\tau)|. \end{aligned}$$

Also, with $s_n = \lambda_n^{-1}(\theta\tau_n)$,

$$\begin{aligned} |\lambda_n^{-1}(\theta\tau_n) - \theta\tau| &\leq |\lambda_n^{-1}(\theta\tau_n) - \theta\tau_n| + |\theta\tau_n - \theta\tau| \\ &= |s_n - \lambda_n(s_n)| + |\theta\tau_n - \theta\tau| \\ &\leq \sup_{s \in [0, \tau]} |s - \lambda_n(s)| + |\theta\tau_n - \theta\tau|, \end{aligned}$$

which tends to zero as $n \rightarrow \infty$. Since f is continuous, we obtain that

$$f(\lambda_n^{-1}(\theta\tau_n)) - f(\theta\tau) \rightarrow 0$$

as $n \rightarrow \infty$. Hence,

$$\text{end}_\theta(f_n) = f_n(\theta\tau_n) \rightarrow f(\theta\tau) = \text{end}_\theta(f). \quad \square$$

We shall assume that (P4') of Section 5.2 is satisfied. Recall that if $P(\lambda)$ is as in Example 5.2.2, then it holds.

Let $\mu_0 \in \mathcal{P}(C_S^\tau)$ be the distribution for the first partial trajectory. Let $\nu'_0 \in \mathcal{P}(S)$ be the distribution for the end points of the partial trajectories associated to μ_0 . That is,

$$\nu'_0 = \text{end}_\# \mu_0 =: P_{\text{end}} \mu_0.$$

Then $\nu_1 := P_J \nu'_0$ is the probability distribution of the starting points in S of the next partial trajectory. Further, $\nu_2 := P_{\text{prod}} \nu_1$ is the joint distribution on $S \times (0, \infty)$ of state and end time of the next partial trajectory. Now define the semiflow map $\Phi : S \times (0, \infty) \rightarrow C_S^\tau$ by $\Phi(x, \tau)(t) = \phi_t(x)$, $t \in [0, \tau]$, $x \in S$.

Lemma 6.2.4. *Assume that the metric on C_S^τ is defined by $P(\lambda)$ given by Example 5.2.2. and that for each compact set B in S and compact $K \subset \mathbb{R}_+$, there exists a $C_{B,K} \geq 0$ such that*

$$d_S(\phi_t(x), \phi_{t'}(x)) \leq C_{B,K} |t - t'|$$

for all $x \in B$ and $t, t' \in K$. Then $\Phi : S \times (0, \infty) \rightarrow C_S^\tau$ is continuous.

Proof. Let $(x_n, t_n) \in S \times (0, \infty)$ converge to (x^*, t^*) . The claim is to show

$$\Phi(x_n, t_n) \rightarrow \Phi(x^*, t^*)$$

in C_S^τ . Let $B = \{x_n : n \in \mathbb{N}\} \cup \{x^*\}$. Then B is a compact set in S . Similarly, the convergent sequence (t_n) is contained in a compact $K \subset \mathbb{R}_+$. We may assume that K contains $\cup_n [0, t_n]$. Then for any $\lambda_n \in \Delta_{t_n, t^*}$

$$\begin{aligned} d(\Phi(x_n, t_n) \circ \lambda_n^{-1}, \Phi(x^*, t^*)) &= \sup_{t \in [0, t^*]} d_S(\Phi(x_n, t_n)(\lambda_n^{-1}(t)), \Phi(x^*, t^*)(t)) \\ &\leq \sup_{t \in [0, t^*]} d_S(\phi_{\lambda_n^{-1}(t)}(x_n), \phi_t(x_n)) \\ &\quad + d_S(\phi_t(x_n), \phi_t(x^*)) \\ &\leq \sup_{t \in [0, t^*]} d_S(\phi_{\lambda_n^{-1}(t)}(x_n), \phi_t(x_n)) \\ &\quad + |\phi_t \upharpoonright_B|_{\text{Lip}} d(x_n, x^*) \\ &\leq C_{B,K} \sup_{t \in [0, t^*]} |\lambda_n^{-1}(t) - t| + |\phi_t \upharpoonright_B|_{\text{Lip}} d(x_n, x^*). \end{aligned}$$

Hence,

$$\begin{aligned} \hat{d}(\Phi(x_n, t_n), \Phi(x^*, t^*)) &= \inf_{\lambda \in \Delta_{t_n, t^*}} \{d(\Phi(x_n, t_n) \circ \lambda^{-1}, \Phi(x^*, t^*)) + P(\lambda)\} \\ &\leq \inf_{\lambda \in \Delta_{t_n, t^*}} \sup_{t \in [0, t^*]} \{C_{B,K} |\lambda^{-1}(t) - t| \\ &\quad + |\phi_t \upharpoonright_B|_{\text{Lip}} d(x_n, x^*) + P(\lambda)\} \\ &\leq \{C_{B,K} \sup_{t \in [0, t^*]} \left| \frac{t_n}{t^*} t - t \right| + \sup_{t \in [0, t^*]} |\phi_t \upharpoonright_B|_{\text{Lip}} d(x_n, x^*) \\ &\quad + \sup_{t \in [0, t^*]} \log \frac{t^*}{t_n}\}, \end{aligned}$$

which tends to zero as $n \rightarrow \infty$. □

Then the Markov operator $\mathbb{P}$ on the partial trajectory space C_S^τ associated to our piecewise deterministic Markov process (PDMP) with random interventions and random times between interventions is

$$\mathbb{P}\mu_0 := \Phi_{\#}\nu_2 = P_{\Phi}P_{\text{prod}}P_J(P_{\text{end}}\mu_0). \quad (6.3)$$

Define for $\mu \in \mathcal{M}^+(C_S^\tau)$, $P_2\mu := P_JP_{\text{end}}\mu$. Then for any $A \subseteq C_S^\tau$ Borel,

$$\mathbb{P}\mu(A) = \int_S \int_0^\infty 1_A(\Phi(x, t)) d\pi_x(t) dP_2\mu(x).$$

6.2.3 Equidistant times as a special case

If the duration of the partial trajectories are all equal to Δt with probability 1, then the models of 2.1 and 2.2 should reduce to the model studied in Chapter 4. Let us check that that is indeed true.

Take for the distribution of τ the point measure at Δt , i.e. $\pi_x = \delta_{\Delta t}$. Take $\mu_0 = \nu_0 \otimes \delta_{\Delta t}$, where ν_0 is a distribution of positions just after a jump. Then

$$\begin{aligned} P_1\mu_0(A) &= \int_S 1_A(\phi_{\Delta t}(x)) d\nu_0(x) \\ &= \nu_0 \circ \phi_{\Delta t}^{-1}(A) = (\phi_{\Delta t})_{\#} \nu_0(A). \end{aligned}$$

Also,

$$\begin{aligned} P_2\mu_0(A) &= P_JP_1\mu(A) = \int_S Q_y(A - y) dP_1\mu(y) \\ &= \int_S Q_{\phi_{\Delta t}(y)}(A - \phi_{\Delta t}(y)) d\nu_0(y). \end{aligned}$$

Hence,

$$P\mu_0 = P_{\text{prod}}P_JP_1\mu = P_{\text{chap4}}\nu_0 \otimes \delta_{\Delta t},$$

where P_{chap4} is the Markov operator of the equidistant time case discussed in Chapter 4.

6.2.4 Approaches to asymptotic stability

In order to prove existence and stability of an invariant measure for either P on $S \times (0, \infty)$ defined by (6.2) or $\mathbb{P}$ on C_S^τ , one could use the lower bound technique as in Section 4.7. The difficulty will be to find suitable conditions on ϕ , Q and π under which the desired lower bound holds true. Another difficulty is establishing nonexpansiveness of P or $\mathbb{P}$, or more general, the e-property. Concerning the e-property, and the applicability of the lower-bound technique, it is desirable to have tractable conditions to establish the e-property. Unfortunately, such conditions—apart from nonexpansiveness—are lacking (still). Further research in this direction is needed, preferably based on concrete examples from applications.

Then $\mathbb{P}$ is a Markov operator describing the transition from a partial trajectory between two interventions to the partial trajectory between the next two interventions.

6.3 Comments and further extensions

The description on $\mathcal{M}^+(C_S^\tau)$ is more involved than that on $S \times (0, \infty)$. Nevertheless we think that the latter is useful in studying extensions of the set up discussed in this thesis. One should note that one could start working with cadlag functions with Skorohod metric, $\mathbb{D}_s$, if it is known that the process/deterministic part has continuous trajectories (almost surely) instead of the space C_S^τ constructed in Chapter 5. However, cadlag functions can be unnecessarily complicated since they may have countably many jumps on a finite interval, while our C_S^τ is more convenient since there are, of course, no jumps at all on finite intervals. Moreover, the partial trajectories – viewed as cadlag functions – are not in one specific space $\mathbb{D}_s([0, \tau])$, nor $\mathbb{D}_s([0, \infty))$: the duration is varying, or one has to truncate the function in the latter setting. That is, consider $\mathbb{D}_s([0, \infty)) \times [0, \infty)$. However, (f, τ) and (g, τ) define the same partial trajectory when $f(t) = g(t)$ for $t \in [0, \tau]$. thus, one needs to take a quotient space of $\mathbb{D}_s([0, \infty)) \times [0, \infty)$ as state space. Quotient metric spaces require special care and are not as easy to handle as e.g. quotient Banach spaces. Therefore we prefer to use the space C_S^τ from Chapter 5 when applicable.

6.3.1 Markov process instead of deterministic ϕ_t

The scope of the framework presented in the previous sections can be extended. For instance, the deterministic dynamical system given by ϕ may be replaced by a continuous time stochastic process with continuous trajectories. Such processes could be given by particular stochastic differential equations. In this section we present possible applications from SDE with state dependent jump distribution. One may replace the operator composition $P_\Phi \circ P_{\text{prod}}$ in equation (6.3) of Section 2.2 by a more general Markov operator $P_{\text{gen}} : \mathcal{M}^+(S \times (0, \infty)) \rightarrow \mathcal{M}^+(C_S^\tau)$ mapping a joint distribution of position and duration time to a distribution of partial trajectories. This operator could for instance be given by a stochastic process with continuous trajectories and a stopping time.

6.3.2 Applications to financial mathematics

A combined model of a continuous time stochastic process with continuous trajectories and random jumps at discrete, but random time points is quite common in, for instance, financial mathematics [13, 35, 68]. The setting with state dependent jump distribution allows a more general dependence of jumps on the state than can be covered by the standard theory of stochastic differential equations with semimartingales [74], since the shape of the jump distributions can changes not only the size of jumps.

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Samenvatting

Het onderzoek beschreven in dit proefschrift gaat over een analytische aanpak voor de bestudering van stuksgewijs deterministische Markov-processen. Dit is een onderwerp dat zich bevindt op het grensvlak van analyse en kansrekening. Stuksgewijs deterministische Markov-processen kunnen enerzijds gezien worden als deterministische systemen met stochastische instantane sprongen en anderzijds als een speciaal soort stochastische processen waarbij delen van de paden deterministisch zijn. De bestudering van zulke processen kan zowel vanuit de analyse als vanuit de kansrekening benaderd worden. Het centrale thema van het onderzoek is de bestudering van het lange termijngedrag van Markov processen die voortkomen uit een speciek soort verstoring van deterministische dynamische systemen.

Deze stuksgewijs deterministische Markov-processen ontstaan op een natuurlijke wijze bij het modelleren van diverse verschijnselen. Men kan bijvoorbeeld denken aan het ‘oogsten’ uit een groeiende bacteriekolonie of aan een marien ecosysteem waaruit vis wordt weggevangen met een net. In een experimentele opzet waar men bacteriën kweekt, of schimmels zoals bakkersgist in een fermenteervat, of plantencellen in een celsuspensie, neemt men regelmatig monsters in het kader van het experimentele protocol. Deze interventies beïnvloeden de groei van de populatie. De vraag rijst hoe men kan kwantificeren hoe erg het nemen van monsters de dynamica van de populatie verstoort afhankelijk van de grootte van de monsters.

De precieze wiskundige formulering van zulke problemen kan als volgt worden samengevat. De evolutie van het systeem tussen de interventies wordt gegeven door een deterministisch dynamisch systeem $(\phi_t)_{t \geq 0}$ op een deelverzameling S van een vectorruimte X voorzien van een metriek. Neem aan dat x^* een evenwicht is van het systeem. De populatie net voor de volgende interventie zal gelijk zijn aan $x' = \phi_{\Delta t}(x)$, waar $x \in S$ de toestand is net na de vorige interventie en Δt de tijd tussen twee interventies. De populatie y net na de interventie is dan in een Borel-verzameling $A \subseteq X$ met een bepaalde kans. Als de kans op een sprong die begint in x en die een grootte heeft in B gelijk is aan $Q_x(B)$, dan is de kans dat de populatie net na de interventie een element van A is gelijk aan $Q_{x'}(A - x)$. Immers, we eindigen in A als en alleen als de verandering $y - x'$ is met $y \in A$. Dit

gebeurt met kans $Q_{x'}(A - x')$. De verdeling van de populatie net na de volgende interventie is dus $Q_{\phi_{\Delta t}(x)}(\cdot - \phi_{\Delta t}(x))$. De interventie kan a priori de toestand uit de verzameling S schuiven. Dit punt krijgt speciale aandacht.

Als de grootte van de populatie net na de n -de interventie verdeling μ heeft, waar μ een kansmaat is die geconcentreerd is op S , dan is de verdeling net na de $(n + 1)$ -ste interventie gelijk aan $P\mu$, gegeven door

$$(P\mu)(A) := \int_S Q_{\phi(x)}(A - \phi(x))d\mu(x),$$

met μ in de verzameling $\mathcal{P}(S)$ van Borel-kansmaten op S , $A \subseteq X$ een Borelverzameling en $\phi = \phi_{\Delta t}$.

We bewijzen de existentie van een gebied B^* in X dat het evenwicht x^* bevat en zo is dat P de verzameling $\mathcal{P}(B^*)$ in zichzelf afbeeldt. Dit betekent dat P een Markov operator op B^* induceert. We geven voorwaarden waaronder P een unieke ergodische invariante maat μ^* heeft, die asymptotisch stabiel is in de zin dat $P^n\delta_x \rightarrow \mu^*$ voor iedere $x \in B$.

In Hoofdstuk 2 geven wij een beknopte samenvatting van concepten en resultaten over Markov-operatoren. We beginnen met de definitie van de duale begrensde Lipschitz-norm en de Fortet-Mourier-norm van een getekende maat. We bespreken de Banach-roosterstructuur van de vectorruimte van eindige maten. Daarna bespreken we grondig enkele fundamentele aspecten van de theorie van Markov-operatoren, zoals regulariteitsbegrippen, waarin we begrippen als Markov-operator, Markov-Feller-operator, sterk Feller, ultra-Feller, reguliere Markov-operator met de equicontinuiteitseigenschap (e-eigenschap) en niet-expanderende Markov-operator en de verbanden ertussen uiteenzetten. We geven ook de definities van invariante en ergodische maten. Concepten als overlappende dragers en strikt overlappende dragers worden eveneens besproken. Verder bespreken we het verband tussen de dragers van $P\delta_x$ en $P\mu$, waar P een Markov operator is, δ_x de Dirac-maat in het punt x en μ een kansmaat. Bovendien verkrijgen we de equivalentie van uniforme en globale stabiliteit van ergodische maten voor Markov-Feller-operatoren op compacte ruimten die de e-eigenschap hebben met een nieuwe voorwaarde in termen van de opeenvolgende iteraties van deze operatoren. We laten zien dat het analyseren van de veranderingen van de dragers van maten door toepassing van een geassocieerde Markov-operator voldoende is om de gewenste ondergrensvoorwaarde af te leiden. Een aparte paragraaf belicht de implicaties van ondergrensschattingen voor onder andere existentie en

asymptotische stabiliteit van invariante maten.

Hoofdstuk 3 bestudeert Markov-operatoren die corresponderen met een deterministisch systeem met instantane stochastische sprongen. Het systeem is heel natuurlijk. Het is opgebouwd uit een logistische afbeelding verstoord door regelmatig in de tijd voorkomende ingrepen. Een ingreep kan worden beschreven als het bij de toestand optellen van een toevalsgrootte met een verdeling die afhankelijk is van de toestand vlak voor de ingreep. De belangrijkste aanname die cruciaal is voor onze analyse is dat de toevalsgrootheden onafhankelijk zijn en verdeeld volgens een dichtheid ten opzichte van de Lebesgue-maat die op een continue manier afhangt van de toestand. Door deze aanname is het mogelijk om ergodiciteit af te leiden met behulp van de klassieke theorie van Meyn en Tweedie. Het bestudeerde dynamische systeem is geïnspireerd door een model voor het nemen van monsters of oogsten. We bestuderen in detail de mogelijkheid dat de populatie uitsterft.

In Hoofdstuk 4 bestuderen we iteraties van afbeeldingen op deelverzamelingen van Banach-ruimten met stochastische verstoringen. De verdeling van de instantane verandering van de toestand mag van de toestand afhangen. Er wordt bewezen dat in de omgeving van een stabiel vast punt van de iteratieafbeelding de Markov-keten gedefinieerd door het verstoorde systeem asymptotisch stabiel is. Dit verschijnsel duiden we aan met *‘volharding van stabiliteit’*.

Hoofdstuk 5 gaat meer over het ontwikkelen van gereedschap dan over toepassingen. We onderzoeken hoe een verzameling van continue functies met waarden in een metrische ruimte die gedefinieerd zijn op verschillende intervallen voorzien kan worden van een geschikte metriek. In het bijzonder gaan we na onder welke voorwaarden onze constructie een metrische ruimte oplevert die separabel en volledig is, met andere woorden, die Pools is. De ruimten die we construeren zijn van belang omdat niet-equidistante interventietijden geen Markov-operator opleveren zoals in de Hoofdstukken 3 en 4 het geval is maar dat wel zo is met de ruimten uit dit hoofdstuk. In dit hoofdstuk definiëren we ook een metrische ruimte waarvan de elementen rijen van functies zijn die gedefinieerd zijn op verschillende intervallen.

In het laatste hoofdstuk, Hoofdstuk 6, blikken we vooruit hoe vervolgonderzoek het werk uit de Hoofdstukken 3 en 4 uit kan breiden naar het geval van niet-equidistante interventietijden door als toestandsruimte één van de metrische ruimten uit Hoofdstuk 5 te gebruiken.

Curriculum vitae

Taleb Alkurdi was born in Jerash–Jordan on December 15, 1981. In 1999 he got his second–secondary diploma (*Tawjihi*) from Jerash high school. Then he studied Mathematics at The University of Jordan and he obtained his bachelor’s degree in 2003. In September 2005 he started his Master program also at The University of Jordan. Under the supervision of Prof. dr. Roshdi Khalil he wrote his master thesis *Remotal sets in some function spaces*, which he defended in 2008 resulting in his Master degree. He moved to the Netherlands in December 2009 where he has been a PhD student at Leiden University until August 2013. His PhD research, leading to this thesis, was supervised by dr. Sander Hille, dr. Onno van Gaans, and prof. dr. Sjoerd Verduyn Lunel.