Seismology of magnetars

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Chapter 2

Excitation of f-modes and torsional modes by giant flares

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On the excitation of f-modes and torsional modes by giant flares
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Abstract

Magnetar giant flares may excite vibrational modes of neutron stars. Here we compute an estimate of initial post-flare amplitudes of both the torsional modes in the magnetar's crust and of the global f-modes. We show that while the torsional crustal modes can be strongly excited, only a small fraction of the flare's energy is converted directly into the lowest-order f-modes. For a conventional model of a magnetar, with the external magnetic field of $\sim 10^{15}$ G, the gravitational-wave detection of these f-modes with advanced LIGO is unlikely.

2.1 Introduction

The gamma- and x-ray flares from Soft Gamma Repeaters (SGRs; Mazetz et al. 1979, Hurley et al. 1998, 2004) are believed to be powered by a sudden release of magnetic energy stored in their host magnetars (Thompson & Duncan 1995). An SGR flare may excite vibrational modes of a magnetar (Duncan 1998). Indeed, torsional oscillations of a magnetar provide an attractive explanation for some of the quasi-periodic oscillations (QPOs) observed in the tails of giant flares (Barat et al. 1983, Israel et al. 2005, Strohmayer & Watts 2005, van Hoven & Levin 2011 (see chapter 3), Gabler et al. 2011, Colaiuda & Kokkotas 2011).

Excitation of low-order f-modes is also of considerable interest, because of the f-modes' strong coupling to potentially detectable gravitational radiation. The sensitivity of the ground-based gravitational-wave interferometers has dramatically improved over the last 5 years (Abott et al. 2009a, Acernese et al. 2008) and interesting upper limits on the f-mode gravitational- wave emission from the 2004 SGR 1806-20 giant flare, a possible 2009 SGR 1550–5418 giant flare and several less energetic bursts have recently been obtained (Abott et al. 2008, Abott et al. 2009b, Abadie et al. 2010, see also Kalmus et al. 2009). Advanced LIGO and VIRGO are expected to become operational in the next 5-7 years and it is of interest to predict the strength of expected gravitational-wave signal from future giant flares.

In this chapter, we compute a theoretical estimate for the amplitude of the torsional and f-modes expected to be excited in a giant flare. We show that only a small fraction of the flare energy is expected to be pumped into the low-order f-modes and estimate the signal-to-noise ratio for the future giant flare detection with advanced LIGO. By contrast, the torsional modes can be strongly excited and may well be responsible for some of the observed QPO's in magnetar flares.

2.2 The general formalism

The giant flares release a significant fraction of the free magnetic energy stored in their host magnetars. Two distinct mechanisms for this have been proposed: (1) Large-scale rearrangement of the internal field, facilitated by a major rupture of the crust (Thompson & Duncan 1995, 2001; we shall refer to it as the internal mechanism, IM) and (2) a large-scale rearrangement of the magnetospheric field, facilitated by fast reconnection (Lyutikov 2006, Gill & Heyl 2010; we shall refer to it as the external mechanism, EM). Both processes may well be at play: the IM would likely serve as a trigger for the EM (however, as was argued in Lyutikov 2003, EM may also be triggered by slow motion of the footpoints of a magnetospheric flux tube, leading to a sudden loss of magnetostatic equilibrium). Observationally, the extremely short, a few microseconds rise time of the 2004 giant flare in SGR 1806-20 (Hurley et al. 2004) gives reason to believe that EM was at play in that source: the IM operates on a much longer Alfvén crossing timescale of $0.05 - 0.1$ seconds. The long timescale for the IM implies that it would not be efficient in exciting the f-modes which have frequencies of over a kHz; this was recently independently emphasized by Kashiyama & Ioka (2011).

2.2.1 Excitation by the EM

During the large-scale EM event, the magnetic stresses at the stellar surface change rapidly by, at most[1], order 1. The magnetosphere comes to a new equilibrium, on the very short timescale of several Alfvén (light)-crossing times and the stresses change to new constant values. We shall characterize the change of the magnetic stress by the 3 components

$$\Delta T_{rr} = \frac{B^2}{4\pi} f_r(\theta, \phi),$$

[1] There is some observational evidence for the substantial magnetic-field reconfiguration in the magnetosphere, as seen from the difference between the persistent pre-flare and post-flare pulse profiles (Palmer et al. 2005). The global change of the magnetospheric twist would result in the comparable change in the tangential magnetic field at the surface, as is evident from e.g. the twisted-magnetosphere solution by Thompson, Lyutikov, & Kulkarni 2002.

$$\Delta T_{r\theta} = \frac{B^2}{4\pi} f_\theta(\theta, \phi), \tag{2.1}$$
$$\Delta T_{r\phi} = \frac{B^2}{4\pi} f_\phi(\theta, \phi),$$

where B is some characteristic value of the surface magnetic field and f_r, f_θ and f_ϕ are functions are of order 1 in the strongest possible flares and are smaller for the weaker flares. Consider now a normal mode of the star with an eigenfrequency ω_n and a displacement wavefunction $\boldsymbol{\xi}_n(r, \theta, \phi)$. We treat the changing surface magnetic stress as an external perturbation acting on the mode. We derive the mode excitation using the Lagrangian formalism; in Appendix 2.A we sketch the derivation directly from the equations of motion. The Lagrangian of the free (pre-perturbation) mode is given by

$$L_{\rm free}(a_n, \dot{a}_n) = \frac{1}{2} m_n \dot{a}_n^2 - \frac{1}{2} m_n \omega_n^2 a_n^2, \tag{2.2}$$

where a_n is the generalized coordinate corresponding to the normal mode, m_n is the effective mass given by

$$m_n = \int d^3 r \rho(\boldsymbol{r}) \boldsymbol{\xi}_n^2(\boldsymbol{r}), \tag{2.3}$$

and $\rho(\boldsymbol{r})$ is the density. The Lagrangian term characterising the mode's interaction with external stress is given by (cf. section 2 of Levin 1998)

$$L_{\rm int} = a_n \int R^2 \boldsymbol{\xi}_n \cdot \boldsymbol{F} \sin\theta d\theta d\phi, \tag{2.4}$$

where

$$\boldsymbol{F} = \Delta T_{rr} \boldsymbol{e}_r + \Delta T_{r\theta} \boldsymbol{e}_\theta + \Delta T_{r\phi} \boldsymbol{e}_\phi, \tag{2.5}$$

and the displacement $\boldsymbol{\xi}$ is evaluated at the radius of the star R. The full Lagrangian for the nth mode is given by[1]

$$L(a_n, \dot{a}_n) = L_{\rm free} + E_{\rm mag} \alpha_n \frac{a_n}{R}, \tag{2.6}$$

[1]We work in the linear regime and don't take into account the non-linear coupling between the modes. The mode amplitudes $\ll 1$ found at the end of our calculation indicate that this is a good approximation.

where

$$E_{\rm mag} = \frac{B^2 R^3}{4\pi} \tag{2.7}$$

is the characteristic energy stored in the star's magnetic field and and α_n is the coupling coefficient given by

$$\alpha_n = \int \boldsymbol{\xi}_n(R,\theta,\phi) \cdot \boldsymbol{f}(\theta,\phi) \sin\theta d\theta d\phi, \tag{2.8}$$

where

$$\boldsymbol{f} = f_r(\theta,\phi)\boldsymbol{e}_r + f_\theta(\theta,\phi)\boldsymbol{e}_\theta + f_\phi(\theta,\phi)\boldsymbol{e}_\phi \tag{2.9}$$

It is now trivial to find the motion resulting from the sudden introduction of the external stress at moment $t = 0$. The coordinate a_n oscillates as follows:

$$a_n(t) = \bar{a}_n \left[1 - \cos\left(\omega_n t\right)\right], \tag{2.10}$$

where the amplitude is given by

$$\bar{a}_n = \frac{\alpha_n E_{\rm mag}}{m_n \omega_n^2 R}. \tag{2.11}$$

The energy in the excited mode is given by

$$E_n = \frac{\alpha_n^2 E_{\rm mag}^2}{2 m_n \omega_n^2 R^2} \tag{2.12}$$

We now briefly revisit the mode excitation by the IM. In this case, the interaction Lagrangian of a mode with the magnetic field is described by the following volume integral:

$$L_{\rm int} = a_n \int d^3r \boldsymbol{f}_L(\boldsymbol{r}) \cdot \boldsymbol{\xi}_n(\boldsymbol{r}), \tag{2.13}$$

where $\boldsymbol{f}_L = [\nabla \times \boldsymbol{B}] \times \boldsymbol{B}$ is the lorentz force per unit volume. Since $f_L \sim B^2/R$, one can see that the coupling of the internal field variation to the mode is of the same order of magnitude as that of the external field variation, provided that the external and internal fields are of the same order of magnitude.

However, the IM mechanism acts on a much longer timescale[1] $\tau_{\rm Alfven} \sim 0.1$s than the typical f-mode period of $\tau_f \sim 0.0005$ s, so the f-mode oscillator would be adiabatically displaced without excitation of the periodic oscillations. One can show that the typical suppression factor of the IM relative to the EM excitation is *at least* of order $2\pi\tau_{\rm Alfven}/\tau_f$ in the mode amplitude[2]. This factor is so large that even if internal field was stronger than the external field by an order of magnitude, the IM excitation would still be suppresed relative to the EM one.

Is there a way around this suppression factor? Potentially, IM could feature a collection of many localized MHD excitations, with the timescale for each one being determined by the Alfvén-crossing time of each of the excitation domain. If the domains were small enough, their timescales could be more closely matched with the f-mode period (Melatos, private communications). However, in this case the magnitude of the overlap integral in Eq. (2.13) would be reduced by a factor $\sim (R/\Delta R)^3$, where ΔR is the characteristic size of the excited domain. The domains would contribute incoherently to the amplitude of the excited mode, thus the contribution of an individual domain would have to be multiplied by $(R/\Delta R)^{3/2}$ in the (somewhat unlikely) limit where the active domains occupy the whole star. Thus, while the timescale of the mini-flares could be well-matched with the f-mode period, their overall contribution to the overlap integral in Eq. (2.13) would be suppressed by $\sim (R/\Delta R)^{3/2}$. In the optimal case that the mini-flares have the same timescale as the f-mode period, $R/\Delta R \sim \tau_{\rm Alfven}/\tau_f$. Therefore, the collection of mini-flares would not give us any gain in the mode excitation amplitude, as compared to the IM estimate given in the previous paragraph. Two applications of the formalism

[1] This timescale could be shorter by a factor of $\sqrt{x_p} \sim 0.2$ (where x_p is the proton fraction) if the superfluid neutrons are decoupled from the MHD (Easson & Pethick 1979, van Hoven & Levin 2008 (see also chapter 1), Andersson, Glampedakis, & Samuelsson 2009). However, even in this case the timescale $\tau_{\rm Alfven}$ on which the IM acts is still a factor of $\sim$40 larger than the f-mode period τ_f.

[2] This can be formalized by the following argument: consider a harmonic oscillator of proper frequency ω_0, initially at rest, which is externally driven by force $f(t)$. The amplitude of the induced oscillation at the proper frequency is proportional to $\tilde{f}(\omega_0)$, the Fourier transform of $f(t)$ evaluated at ω_0. For a step function, representing the rapid transition (several light crossing times) to the new magnetospheric equillibrium in the EM, $\tilde{f}(\omega) \propto 1/\omega$. On the other hand, for a smooth pulse of duration τ, as expected in IM, the Fourier transform is suppressed and scales at most as $\tilde{f}(\omega) \propto (\omega\tau)^{-1} 1/\omega$ when $\omega\tau \gg 1$.

for the mode excitation by the EM mechanism developed above are presented in the next two sections.

2.3 f-modes and gravitational waves

In order to estimate an effective f-mode mass, we have computed the $l = 2$ f-mode displacement functions for a neutron star[1] in the Cowling approximation (see Appendix 2.B). Convenient scalings are

$$\begin{aligned} m_n &= q_M M, \\ \omega_n^2 &= q_\omega \frac{GM}{R^3}, \\ \xi_r(R,\theta,\phi) &= a_n Y_{2m}(\theta,\phi). \end{aligned} \tag{2.14}$$

In our fiducial model $q_M = 0.046$, where we have normalised the mode wave-function so that $\boldsymbol{e}_r \cdot \boldsymbol{\xi}_{2m}(R,\theta,\phi) = Y_{2m}(\theta,\phi)$. Our reference number $q_\omega = 1.35$ was obtained using a fitting formula for fully relativistic f-mode frequencies[2] from Andersson & Kokkotas (1996). The amplitude of the f-mode is given by

$$\frac{\bar{a}_{2m}}{R} = \frac{\alpha_{2m}}{q_m q_\omega} \frac{E_{\text{mag}}}{E_{\text{grav}}}, \tag{2.15}$$

where

$$E_{\text{grav}} = \frac{GM^2}{R} \tag{2.16}$$

[1] We constructed our neutron star model using the equation of state from Douchin & Haensel (2001) and Haensel & Pichon (1994). In calculating the f-mode we treated the whole star as a fluid, neglecting the effects of bulk- and shear moduli.

[2] We are not being consistent in, on the one hand, using the Cowling approximation for a Newtonian star to determine the effective mode mass, but on the other hand using the published relativistic calculations for the mode frequencies. Normally, Newtonian calculations would be sufficient, given the many unknown details of the flare and the many poorly constrained parameters we'd already introduced into the model and the formalism we developed in the previous section is manifestly Newtonian (but can be generalized to relativistic regime if the need arises). However, as we show below, the signal-to-noise ratio for the gravitational-wave detection is very sensitive to the mode frequency and therefore we try to be accurate in characterizing these frequencies.

is of the same order as the gravitational binding energy of the neutron star. We get

$$\frac{\bar{a}_{2m}}{R} \sim 3 \times 10^{-6} \alpha_{2m} \left(\frac{B}{10^{15}\mathrm{G}}\right)^2 \left(\frac{R}{10\mathrm{km}}\right)^4 \left(\frac{1.4M_\odot}{M}\right)^2. \tag{2.17}$$

The energy in the f-mode is

$$E_f = \frac{\alpha_{2m}^2}{2q_m q_\omega} \frac{E_{\mathrm{mag}}^2}{E_{\mathrm{grav}}} \sim 1.5 \times 10^{-6} \alpha_{2m}^2 E_{\mathrm{mag}} \tag{2.18}$$

for our fiducial parameters. This energy is drained from the star primarily through emission of gravitational waves. The total amount of energy carried by gravitational waves is therefore

$$E_{\mathrm{GW}} = E_f = \frac{2\pi^2 d^2 f^2 c^3}{G} \int_{-\infty}^{\infty} \langle h^2 \rangle dt \tag{2.19}$$

where $f = \omega_n / 2\pi$ is the f-mode frequency in Hz, $\langle h^2 \rangle$ is the direction and polarisation averaged value of the square of the gravitational-wave strain h as measured by observers at distance d from the source. This expression allows us to estimate the expected signal-to-noise ratio for ground based gravitational wave interferometers (cf. Abadie et al 2010). One can use the fact that nearly all the gravitational-wave signal is expected to arrive in a narrow-band around the f-mode frequency and that the signal form (the exponentially-decaying sinusoid) is known. The Wiener-filter expression for the signal-to-noise can be written as

$$\frac{S}{N} \approx \left[\frac{1}{S_h(f)} \int_{-\infty}^{\infty} |\tilde{h}^2(f')| df'\right]^{1/2} \tag{2.20}$$

$$\sim \left[\frac{G}{2\pi^2 c^3} \frac{E_f}{S_h(f) f^2 d^2}\right]^{1/2} \tag{2.21}$$

where $\tilde{h}(f)$ is the Fourier transform of the time-dependent gravitational wave strain $h(t)$. As is standard for narrow-band signal, we have used Parseval's theorem to convert the integral over f to the integral over t from Eq. (2.19), and, following Abadie et al. 2010, we have approiximated h^2 with the average $\langle h^2 \rangle$. At frequencies of a few kHz the spectral density, $S_h(f)$, of the ground based detectors like Advanced LIGO and Virgo is dominated by shot-noise and is proportional to f^2. This makes the signal-to-noise ratio for observations of magnetar f-modes excited in giant flares particularly sensitive to frequency ($\propto f^{-3}$). For Advanced LIGO we find

$$\frac{S}{N} \approx 0.07\ \alpha_{2m} \left(\frac{2\ \mathrm{kHz}}{f}\right)^3 \left(\frac{B}{10^{15}\ \mathrm{G}}\right)^2 \times \left(\frac{1\ \mathrm{kpc}}{d}\right) \left(\frac{R}{10\ \mathrm{km}}\right)^2 \left(\frac{0.07\ M_\odot}{m_n}\right)^{1/2} \tag{2.22}$$

Here we used tabulated[1] $S_h(f)$ from the LIGO document LIGO-T0900288, which gives $S_h(f) = 8.4 \cdot 10^{-47}\mathrm{Hz}^{-1}\,(f/2000\ \mathrm{Hz})^2$ for the shot-noise dominated part of the curve.

2.4 Torsional modes

Intuitively, one expects torsional modes to be strongly excited during the magnetar flares (Duncan 1998), since it is the free energy of the twisted magnetic field that is being released. These have much lower proper frequencies than the f-modes (with the fundamental believed to be in the range $10 - 40$ Hz, see Steiner & Watts 2009 and references therein), which can be well-matched to the Alfvén frequencies inside the star. Thus both EM and IM are likely to play a role in the torsional mode excitation. Here, we consider the EM explicitly but keep in mind that IM would give a similar answer.

For the torsional modes in the crust, the displacement is given by

$$\boldsymbol{\xi}_{nlm}(r,\theta,\phi) = g_n(r)\boldsymbol{r} \times \boldsymbol{\nabla} Y_{lm}(\theta,\phi), \tag{2.23}$$

and it is convenient to normalize the wavefunctions so that $g_n(R) = 1$. Here $n = 0, 1, ...$ is the number of radial nodes. With this normalization, the effective mode mass $m_{nlm} \sim m_{\mathrm{crust}} \sim 0.01M$ and from Eq. (2.11) one gets for the mode amplitude normalized by the star radius:

$$\frac{a_{nlm}}{R} \sim 0.01\alpha_{nlm} \left(\frac{B}{10^{15}\mathrm{G}}\right)^2 \frac{R}{10\mathrm{km}} \frac{0.014 M_\odot}{m_{nlm}} \left(\frac{100\mathrm{Hz}}{f}\right)^2. \tag{2.24}$$

[1] These sensitivity curves represent the incoherent sum of principal sources of noise as they are currently understood.

Thus we see that for a reasonable range of parameters it is feasible that the crustal torsional modes would be strongly excited by a giant flare.

2.4.1 Magnetic modes

Recently, Kashiyama & Ioka (KI, 2011) suggested that certain types of MHD modes that may be strongly excited during a giant flare, are coupled to gravitational radiation and may therefore become an interesting source for advanced LIGO. KI focus on the polar modes of Sotani & Kokkotas (2009); the MHD modes found by Lander & Jones (2011a,b) also satisfy some of the KI's criteria. While interesting, this idea has potential caveats that need further investigation. KI assume that the oscillations are long-lived, $\sim 10^7$ oscillation periods. However, MHD modes are notoriously capricious. While the idealized modes of Sotani & Kokkotas (2009) and Lander & Jones (2011a,b) are protected by symmetry, the global magnetic modes in more realistic configurations may couple to a variety of localized Alfvén-type modes and may thus be quickly damped via phase mixing and resonant absorption (Goedbloed & Poedts 2004, van Hoven & Levin 2011 (see chapter 3)). Thus, in our view, there is currently no compelling reason to believe that the magnetic modes can be substantially longer lived than the observed magnetar QPOs.

2.5 Discussion

In this chapter, we have computed the excitation of the f-modes and crustal tosional neutron-star modes by a giant flare. Corsi & Owen (2011) recently computed the magnetic energy that can be released during the flare[1] and found values comparable to $E_{\rm mag}$. However, in this work we showed that only a small fraction of the released flare energy is converted into the f-modes and that the associated gravitational-wave emission is correspondingly weaker

[1]These analytical calculations necessarily make simplifying assumptions about the structure of an equilibrium magnetic field inside the magnetar, but they are likely to give correct order-of-magnitude values.

than has been previously hoped (cf. Abadie et al. 2010 and Corsi & Owen 2011). From Eq. (2.22), our fiducial model does not look promising for future advanced LIGO detection of a giant flare, even if $\alpha_{2m} \sim 1$, i.e. if the flare comprises a global reconfiguration of the magnetospheric field so that the released electro-magnetic energy is of order of the total magnetic energy of the star, $\sim 10^{47}$ erg (the most energetic of the 3 observed giant flares released few$\times 10^{46}$ erg). However, if the surface field is significantly larger than 10^{15} G and/or the star is greater than 10 km in radius (which would reduce the f-mode frequency and increase the contact surface area), then one can become more hopeful about the potential detection. On the other hand, we have seen that there is no difficulty in exciting the crustal torsional modes to a large amplitude. Whether or not this leads to the observed quasi-periodic oscillations in the flare's tail (Israel et al. 2005, Strohmayer & Watts 2005, Watts & Strohmayer 2006) depends crucially on the dynamics of hydromagnetic coupling between the crustal modes and the Alfvén modes of the magnetar core (Levin 2006, 2007, van Hoven & Levin 2011 (see chapter 3), Gabler et al. 2011, Colaiuda & Kokkotas 2011).

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Appendix 2.A: Alternative derivation of the mode excitation

In this Appendix we derive, for completeness, the formalism for mode excitation directly from the equations of motion; cf. Unno et al (1989). Let $\xi(\boldsymbol{r}, t)$ be the small displacement of the star from its equilibrium position. The equations of motion are given by

$$\rho\ddot{\boldsymbol{\xi}} = \boldsymbol{F}(\boldsymbol{\xi}) + \boldsymbol{f}_{\mathrm{ext}}(\boldsymbol{r}, t), \tag{2.25}$$

where ρ is the density, $\boldsymbol{F}(\boldsymbol{\xi})$ is the restoring force linear in $\boldsymbol{\xi}$ and $\boldsymbol{f}_{\mathrm{ext}}$ is the external force per unit volume. For a normal-mode eigenfunction $\boldsymbol{\xi}_n$ with the angular frequency ω_n, one has $\boldsymbol{F}_n = -\rho\omega_n^2\xi_n$. We now decompose the star displacement into its eigenmodes

$$\boldsymbol{\xi}(\boldsymbol{r}, t) = \Sigma_n a_n(t)\boldsymbol{\xi}_n(\boldsymbol{r}) \tag{2.26}$$

and suibstitute this series into Eq. (2.25) to obtain

$$\Sigma_n[\ddot{a}_n + \omega_n^2 a_n]\boldsymbol{\xi}_n(\boldsymbol{r}) = \boldsymbol{f}_{\mathrm{ext}}(\boldsymbol{r}, t). \tag{2.27}$$

Taking a dot product of the above equation with $\boldsymbol{\xi}_k(\boldsymbol{r})$, integrating over the volume of the star and using the orthogonality relation

$$\int d^3r\rho\boldsymbol{\xi}_n \cdot \boldsymbol{\xi}_k \propto \delta_{mk}, \tag{2.28}$$

we obtain the equation of motion for a_k:

$$\ddot{a}_k + \omega_k^2 a_k = \frac{\alpha_k(t)}{m_k}, \tag{2.29}$$

where

$$\alpha_k = \int d^3r\boldsymbol{f}_{\mathrm{ext}} \cdot \boldsymbol{\xi}_k(\boldsymbol{r}) \tag{2.30}$$

and

$$m_k = \int d^3r\rho(\boldsymbol{r})\boldsymbol{\xi}_k^2(\boldsymbol{r}). \tag{2.31}$$

These equations of motion are identical to those derived from the Lagrangian in Eqs (2.2) and (2.13). For the case when the external force is applied at the surface, one recovers equations of motion derived from Eqs (2.2) and (2.4).

Appendix 2.B: Stellar oscillations

In this appendix we give the non-relativistic equations that govern adiabatic fluid motion in non-rotating, spherical stars. We derive linearized equations of motion for small fluid displacements $\boldsymbol{\xi}$ in the Cowling approximation. That is, we ignore perturbations of the gravitational potential resulting from the small fluid displacement $\boldsymbol{\xi}$. In non-rotating stars, the fluid flow obeys the continuity- and Euler equations,

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) \tag{2.32}$$

$$\rho \frac{d\boldsymbol{v}}{dt} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi, \tag{2.33}$$

where ρ is the mass-density, $\boldsymbol{v}$ is the velocity vector and P is the pressure. The gravitational potential Φ satisfies Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho. \tag{2.34}$$

Together with an equation for adiabatic motion,

$$\frac{dP}{dt} = \frac{\Gamma_1 P}{\rho} \frac{d\rho}{dt}, \tag{2.35}$$

where Γ_1 is the adiabatic constant, the above equations provide a complete dynamical description of the star. In order to find eigenmodes of the star, we proceed as follows:

(1) We construct an equillibrium stellar model. We assume that our star is non-rotating and neglect deformations due to magnetic pressure, which are expected to be small. Therefore, we adopt a spherically symmetric background stellar model that is a solution of the Tolman-Oppenheimer-Volkoff equation (TOV equation). We calculate the hydrostatic equillibrium using a SLy equation of state (Douchin & Haensel, 2001; Haensel & Potekhin, 2004; Haensel, Potekhin & Yakovlev, 2007). The model that we use here has a mass of $M_* = 1.4\ M_\odot$, a radius $R_* = 1.16 \cdot 10^6$ cm, a central density $\rho_c = 9.83 \cdot 10^{14}$ g cm^{-3} and cental pressure $P_c = 1.36 \cdot 10^{35}$ dyn cm^{-2}. As a further simplification, we treat the whole star as a fluid, neglecting effects due to non-zero bulk- and shear moduli in the crust.

(2) We introduce a small fluid displacement $\boldsymbol{\xi}(\boldsymbol{x}, t)$ and assume for this displacement a harmonic time dependence, i.e. $\boldsymbol{\xi}(\boldsymbol{x}, t) \propto \boldsymbol{\xi}(\boldsymbol{x})e^{i\omega t}$. Using the perturbation $\boldsymbol{\xi}$, we linearize the fluid equations (2.33) - (2.35) around the static equillibrium model. This yields the following pair of ordinary differential equations (see e.g. Cox, 1980; Unno et al., 1989; Christensen-Dalsgaard, 2003):

$$\frac{d\xi_r}{dr} = -\left[\frac{2}{r} + \frac{1}{\Gamma_1 P}\frac{dP}{dr}\right]\xi_r + \frac{r\omega^2}{c^2}\left[\frac{l(l+1)c^2}{r^2\omega^2} - 1\right]\xi_h \tag{2.36}$$

$$\frac{d\xi_h}{dr} = \frac{1}{r\omega^2}\left[\omega^2 - N^2\right]\xi_r + \left[\frac{N^2}{g} - \frac{1}{r}\right]\xi_h \tag{2.37}$$

where ξ_r and ξ_h are radial- and horizontal components of the fluid displacement. The integer l enters the equation due to an expansion of the perturbed quantities into spherical harmonics $Y_l^m(\theta, \phi)$. In terms of ξ_r and ξ_h, the displacement field $\boldsymbol{\xi}_{lm}(\boldsymbol{x})$ corresponding to a spherical harmonic degree l and order m, can be expressed as

$$\boldsymbol{\xi}_{lm}(\boldsymbol{x}) = \mathrm{Re}[\xi_r(r)\, Y_l^m \hat{\boldsymbol{r}} + \xi_h(r) r \boldsymbol{\nabla}\, Y_l^m], \tag{2.38}$$

where $\hat{\boldsymbol{r}}$ is the unit vector in the radial direction. Further, in Eq. (2.37), g is the gravitational acceleration and N^2 is the square of the buoyancy frequency (Brunt-Väisälä frequency) is given by:

$$N^2 = g\left(\frac{1}{\Gamma_1 P}\frac{dP}{dr} - \frac{1}{\rho}\frac{d\rho}{dr}\right) \tag{2.39}$$

(3) We supplement the equations (2.36) and (2.37) with boundary conditions at $r = 0$ and $r = R_*$. The boundary condition in the center of the star is obtained by requiring the solutions to be regular functions at $r = 0$. One may show (see e.g. Unno, 1989) that this leads to the condition

$$\xi_r = l\xi_h \qquad \text{at} \quad r = 0. \tag{2.40}$$

At the stellar surface, we enforce a zero-stress boundary condition, i.e. the Lagrangian perturbation of the pressure $\delta P = 0$. This gives

$$\xi_h = -\frac{1}{r\rho\omega^2}\frac{dP}{dr}\xi_r \qquad \text{at} \quad r = R_*. \tag{2.41}$$

Equations (2.36) and (2.37) augmented with the boundary conditions of equations (2.40) and (2.41), constitutes a boundary value problem, which yields, for each index l, a unique series of solutions (eigenmodes) $\xi_{r,ln}(r)$ and $\xi_{h,ln}(r)$ corresponding to eigenfrequencies $\omega = \omega_{ln}$. Here the index n denotes the number nodes along the radial axis. Since there are two separate classes of solutions for a given number of radial nodes, i.e. the low frequency g-modes and the high frequency p-modes, we label the g-modes with negative integer n and the p-modes with positive integer n. These two branches of modes are separated by the nodeless ($n = 0$) f-mode.

(4) We obtain solutions of the above boundary value problem by means of a shooting method. For a fixed value of ω, we integrate Eq's (2.36) and (2.37) from $r = 0$, where Eq. (2.40) is satisfied, to the surface at $r = R_*$, using the 4-th order Runge-Kutta scheme. Eigenvalues ω_{ln} and eigenfunctions $\xi_{r,ln}, \xi_{h,ln}$ are obtained by repeating this procedure for different values of ω, until the boundary condition Eq. (2.41) is satisfied at R_*.