Chapter 1

Hydromagnetic waves in a superfluid neutron star with strong vortex pinning

Based on:
Hydromagnetic waves in a superfluid neutron star with strong vortex pinning
Maarten van Hoven & Yuri Levin, 2008, published in MNRAS

Abstract

Neutron-star cores may be hosts of a unique mixture of a neutron superfluid and a proton superconductor. Compelling theoretical arguments have been presented over the years that if the proton superconductor is of type II, than the superconductor fluxtubes and superfluid vortices should be strongly coupled and hence the vortices should be pinned to the proton-electron plasma in the core. We explore the effect of this pinning on the hydromagnetic waves in the core and discuss two astrophysical applications of our results: (1) We show that even in the case of strong pinning, the core Alfvén waves thought to be responsible for the low-frequency magnetar quasi-periodic oscillations (QPO) are not significantly mass-loaded by the neutrons. The decoupling of ~ 0.95 of the core mass from the Alfvén waves is in fact required in order to explain the QPO frequencies, for simple magnetic geometries and for magnetic fields not greater than 10^{15} Gauss. (2) We show that in the case of strong vortex pinning, hydromagnetic stresses exert stabilizing influence on the Glaberson instability, which has recently been proposed as a potential source of superfluid turbulence in neutron stars.

1.1 Introduction

Since the late 1950's, it has been realized that neutron-star interior may consist of a number of quantum fluids (see Shapiro & Teukolsky 1983 for a review). Currently, it is thought that both neutron superfluid and proton superconductor are likely to coexist in the neutron-star cores (see, e.g., Link 2007 for a discussion). Several researchers have argued that if the proton superconductivity were of the type II, then the superconductor fluxtubes would couple strongly to the neutron superfluid vortices. This line of reasoning is based on the fact that nuclear forces contain velocity-dependent terms, which results in the entrainment of protons in the neutron super current (Alpar, Langer & Sauls, 1984). Therefore, the vortices are sheathed by charged currents entrained in the superfluid flow and are strongly magnetized. Magnetic fluxtubes interact strongly with the magnetized vortices, similar to the way in which the fluxtubes interact between each other (Ruderman, Zhu, & Chen 1998 and references therein). As a result of this coupling, the vortices get strongly pinned to the proton-electron plasma in the core. Such pinning would have important implications for the neutron-star phenomenology. Ruderman, Zhu, & Chen (1998) have argued that the vortex-pinning in the core may be responsible for the observed glitches in the pulsar rotation rates. Link (2003) has considered the effect of the vortex-fluxtube interaction on the dynamics of the precessing pulsar PSR 1828-11 (observed by Stairs, Lyne, & Shemar 2000). Building on the theoretical work by Shaham (1977) and Sedrakian, Wasserman, & Cordes (1999), he has concluded that the interaction, if present, would ultimately lead to the fast precession. Since PSR 1828-11 is precessing slowly and persistently, Link (2003) has argued that the core vortex pinning is excluded by the observations and hence that either the proton superconductor might be of type I, or that both proton and neutron condensates do not coexist inside that pulsar. While Link's argument is suggestive, we believe it is premature to rule out strong vortex pinning in the cores of all neutron stars.

In this chapter we consider hydromagnetic waves in the case when the neutron vortices are strongly pinned to the proton-electron plasma in the core. We have two main astrophysical motivations for studying this problem. The first one is due to the fairly recent observations of the quasi-periodic oscillations (QPOs) of the x-ray luminosity in the tails of giant magnetar flares (Israel et al. 2005, Strohmayer & Watts 2005, 2006, see also earlier but lower signal-to-noise measurements of Barat et al. 1983). Israel et al. 2005 has argued that the lowest-frequency and the longest-lived QPO of ~ 18 Hz is likely to represent an Alfvén-type oscillation in the magnetar core (this frequency is too low to be associated with the torsional modes of the crust). Levin (2006) has shown that for a magnetar-strength field the crustal motion [which is thought to be either powering the flare (Thompson & Duncan 1995) or responding to a global reconnection event in the magnetosphere (Lyutikov 2003)] would excite the core Alfvén waves on the timescale of several oscillation periods. Since then, a significant body of theoretical work has been devoted to a study of global magnetar vibrations, which would involve both hydromagnetic waves in the core and elasto-magnetic shear waves in the crust [Glampedakis, Samuelsson, & Andersson 2006, Levin 2007 (from here on L07), Sotani, Kokkotas, & Stergioulas 2007, Lee 2007]. In particular, L07 has argued that the long-lived low-frequency QPOs are associated with the special spectral points of the Afven continuum in the magnetar core. For simple B-field configurations these special points can be found analytically, and do not depend on the details of the crust. For example, for a uniform internal B-field, the lowest QPO is expected at the frequency

$$\nu_{\rm Alfven} \sim \frac{B_{\rm eff}}{2R\sqrt{4\pi\rho_c}} \tag{1.1}$$

where R is the radius of the fluid part of the star, ρ_c is the density of the the part of the fluid which is coupled to the Alfvén waves and $B_{\rm eff}$ is the effective magnetic field which is given by[1]

$$B_{\rm eff} = \sqrt{BB_{\rm cr}}. \tag{1.2}$$

[1]The occurence of $B_{\rm eff}$ in Eq. (1.1) and (1.2) can be understood as follows: the magnetic tension force acting on surface Σ perpendicular to the fluxtubes containing N fluxtubes is given by $N\Delta\Sigma\cdot B_{\rm cr}^2/4\pi = \Sigma\cdot T_{\rm eff}$, where $\Delta\Sigma$ is the cross section of a single fluxtube and $T_{\rm eff}$ is the effective tensile stress. The magnetic flux through Σ is $\Phi = \Sigma\cdot B = N\Delta\Sigma\cdot B_{\rm cr}$ and we find $T_{\rm eff} = BB_{\rm cr}/4\pi = B_{\rm eff}^2/4\pi$ (as opposed to $T = B^2/4\pi$, where T is the corresponding part of the Maxwell stress tensor). (A detailed derivation is given in Easson & Pethick, (1977)).

Here B is the average magnetic field, $B_{\rm cr} \simeq 10^{15}$ G is the value of the critical magnetic field confined to the fluxtubes. From Eq. (1.2) we see that the magnetar QPO frequencies could provide an interesting constraint on the magnetic-field strength and geometry. However, interaction between neutron and proton superfluids could affect core Alfvén waves, by effectively mass-loading them with neutrons. We shall consider the extreme case of such interaction—the strong vortex pinning on the fluxtubes, and show that it does not significantly alter the Alfvén-wave propagation in slowly-spinning magnetars (but is important for the Alfvén waves in the fast-spinning radio-pulsars). This simplifies the interpretation of the QPO frequencies and shows that it is valid to assume that ρ_c is just the density of protons, about 5% of the total fluid density.

Our second motivation is the recent theoretical discussion of the superfluid turbulence in the neutron-star cores. Superfluid turbulence has been discussed in the context of the laboratory Helium fluid for the last 30 years (see, e.g., Donnelly 1991 for a review). In a ground-breaking series Peralta, Melatos, Giacobello, & Ooi at the University of Melbourne (2005, 2006; hereafter PMGO5 and PMGO6) have applied the superfluid-helium ideas to neutron-star interiors. PMGO have developed from scratch a code which studies numerically the 2-component superfluid dynamics. The two components in PMGO are the neutron superfluid and the normal neutron fluid which are coupled via the mutual friction force at the superfluid vortices; this mixture is expected if the core temperature is a significant fraction of the critical temperature of the superfluid. The equations of motions used by PMGO were derived by Hall and Vinen (1956) and Bekharevich and Khalatnikov (1961). PMGO5 have studied, for the first time, the superfluid spherical Taylor-Couette flow and find that it becomes turbulent in 2 steps: (1) The normal component develops meridional circulation due to the Eckman pumping (see, e.g., Pedlosky 1987), and (2) the component of the meridional flow of the normal fluid which is directed *along* the superfluid vortices drives the vortex Kelvin waves unstable; this is known as the Glaberson (or sometimes Donnelly-Glaberson) instability (Glaberson, Johnson, Ostermeier 1974, Donnelly 1991). In PMGO5 simulations, the Glaberson instability leads to turbulence. Interestingly, PMGO6 and Melatos and Peralta (2007) demonstrate that the superfluid turbulence could affect the pulsar spin and could be behind the well-known pulsar timing noise and spin glitches.

More recently, Sidery, Andersson and Gomer (2007, hereafter SAG) and Glampedakis, Andersson and Jones (2007, hereafter GAJ1 and 2008, hereafter GAJ2) developed an analytical theory of the Glaberson instability in neutron stars. Their 2-component fluid consists of the neutron superfluid and the proton superconductor, which are, like in PMGO, coupled via the mutual-friction force. SAG have considered the limit of the weak mutual friction (see below) and infinitely heavy proton superfluid and found that it was the inertial waves in the neutron superfluid which were subject to the Glaberson instability. GAJ1 and GAJ2 have extended this analysis to include the regime of realistic proton-to-neutron mass ratio and of arbitrarily strong mutual friction. Notably, they found that the Glaberson instability operated even in the regime of strong pinning. But where would the initial relative flow between the protons and neutrons come from? GAJ have argued that if the pinned neutron vortices were misaligned with the angular velocity of the protons, then this would naturally lead to the relative proton-neutron flow which would potentially be strong enough to drive the turbulence in some parts of the star. Without turbulence, the star with misaligned pinned vortices would undergo fast precession (Shaham 1977) which, although probably hard to detect, has yet not been observed in any of the currently-timed radio pulsars or magnetars. In GAJ2 the authors argue that this Glaberson-instability-induced turbulence may generically prevent the star getting into a configuration with the fast precession. However, this conclusion is premature. One important piece of physics which is not considered in GAJ is the strong hydromagnetic stress inside the proton superfluid, which, as we show below, has a suppressing effect on the Glaberson instability and hence on the development of the superfluid turbulence. We will derive, however, a robust upper limit on the angle of fast precession, which is determined by the maximum possible mutual torque between the neutron superfluid and the proton superconductor. The maximal precession angle turns out to be much smaller than 1 degree and thus it is not surprising that the fast precession has never been observed in real neutron stars.

The plan of the chapter is as follows. In the next section we derive the dispersion relation for the hydromagnetic waves when the superfluid vortices are rigidly attached to the core plasma. In sections 1.3 and 1.4, we consider applications to oscillating magnetars and precessing pulsars, respectively. We end with the general discussion in section 1.5.

1.2 Dispersion relations

As a starting point, we utilize the plane-wave analysis outlined in GAJ. We follow closely the notation of and reasoning behind GAJ1's basic equations (1)–(7) and our derived dispersion relation is identical to their Eq. (10) in the limit of zero hydromagnetic stress, but has important extra terms when the hydromagnetic stress is included. We begin with the two-fluid dynamical equations, cf. Eqs. (1) and (2) in GAJ1:

$$D_t^n \boldsymbol{v}_n + \boldsymbol{\nabla}\psi_n = 2\boldsymbol{v}_n \times \boldsymbol{\Omega} + \boldsymbol{f}_{\rm mf} \tag{1.3}$$

$$D_t^p \boldsymbol{v}_p + \boldsymbol{\nabla}\psi_p = 2\boldsymbol{v}_p \times \boldsymbol{\Omega} - \boldsymbol{f}_{\rm mf}/x_p + \nu_{\rm ee}\nabla^2 \boldsymbol{v}_p + \boldsymbol{f}_{\rm hm} \tag{1.4}$$

Here $\boldsymbol{v}_n$ and $\boldsymbol{v}_p$ are the velocity vectors of the neutron and proton condensates respectively (throughout this thesis vectors are denoted by bold symbols), the full time derivatives D_t are defined in the usual way as $D_t^{n,p} = \partial/\partial t + \boldsymbol{v}_{n,p} \cdot \boldsymbol{\nabla}$, $\psi_{n,p}$ is the sum of specific chemical and gravitational potentials, $\boldsymbol{f}_{\rm mf}$ is the acceleration of the neutron superfluid due to the mutual friction between its vortices and the charged plasma, $x_p = \rho_p/\rho_n$ is the density ratio between the proton charged and neutral components of the interior ($\sim 5\%$), Ω is the angular velocity of the rotating frame in which all of the velocities are defined, $\nu_{\rm ee}$ is the kinetic viscosity of the plasma due to electron-electron scattering, and

$$\boldsymbol{f}_{\rm hm} = \boldsymbol{B}_{\rm eff} \cdot \boldsymbol{\nabla}\boldsymbol{B}_{\rm eff}/4\pi\rho_p \tag{1.5}$$

is the acceleration of the plasma due to the hydromagnetic stress. We note that because in a type-II superconductor the distance between the fluxtubes is much larger than the fluxtube diameter, we ignore magnetic pressure. In writing down Eq. (1.3), we have followed GAJ and neglected explicitly the effect of superfluid entrainment (which we expect will not qualitatively change our results) and the individual tension force for the vortices (which can be neglected if the wavelength of the waves is much greater than the inter-vortex distance). We shall use the following conventional form (Hall & Vinen 1956, Bekharevich & Khalatnikov 1961, PMGO, SAG and GAJ) for the mutual-friction force:

$$\boldsymbol{f}_{\rm mf} = \frac{R}{1+R^2}\hat{\boldsymbol{\omega}}_n \times (\boldsymbol{\omega}_n \times \boldsymbol{w}_{\rm np}) + \frac{R^2}{1+R^2}\boldsymbol{\omega}_n \times \boldsymbol{w}_{\rm np}, \tag{1.6}$$

where $\boldsymbol{\omega}_n = \boldsymbol{\nabla}\times\boldsymbol{V}_n$ is the vorticity of the neutron fluid in a non-rotating frame (here $\boldsymbol{V}_n$ is the neutron velocity in the non-rotating frame)[1], $\hat{\boldsymbol{\omega}}_n = \boldsymbol{\omega}_n/|\boldsymbol{\omega}_n|$ is the associated unit vector, $\boldsymbol{w}_{\rm np} = \boldsymbol{v}_n - \boldsymbol{v}_p$ and R is the dimensionless number measuring the strength of the drag between the neutron vortices and the plasma. When $R \ll 1$ (the weak-drag limit), the first term on the right-hand side dominates. This entails that the neutron vortices mostly follow the motion of the neutron superfluid in the direction perpendicular to $\boldsymbol{\omega}_n$. When $R \gg 1$ (the strong-drag limit), the second term on the right-hand side dominates. This entails that the neutron vortices mostly follow the plasma motion. When $R = \infty$, which is the case on which this chapter focuses, the vortices get pinned to the plasma. In this limit, the plasma and the neutron superfluid interact exclusively via the Magnus force arising from the relative motion between the neutron vortices and neutron superfluid.

We choose the background state as follows: 1. the z-axis is directed along Ω; 2. the neutron vortices are aligned with $\boldsymbol{\Omega} = \Omega\boldsymbol{e}_z$, and are at rest in the rotating frame; 3. in the same frame, the plasma has a background velocity $\boldsymbol{w}_0 = w_0\boldsymbol{e}_z$, which is directed along the vortices; 4. the mean magnetic field is directed along the vortices, $\boldsymbol{B} = B\boldsymbol{e}_z$. We consider waves which are propagating along the z-axis. We are interested in the waves for which the restoring force is the combination of hydromagnetic stress, the Coriolis force and the Magnus force. This means that the wave must be nearly incompressible, which

[1] In GAJ1, $\boldsymbol{\omega}_n$ is erroneously defined as $\boldsymbol{\nabla}\times\boldsymbol{v}_n$. However, in their subsequent calculations they, most likely, use the correct expression.

implies

$$\boldsymbol{k} \cdot \delta \boldsymbol{v}_{n,p} = 0. \tag{1.7}$$

Here $\boldsymbol{k}$ is the wavevector, $\delta \boldsymbol{v}_{n,p}$ is the neutron/proton velocity perturbation due to the wave. Incompressibility and assumed homogeneity of the background state imply $\delta\psi_{n,p} = 0$.

Let us introduce the Lagrangian displacement vectors $\boldsymbol{\xi}_{n,p}$ of the neutron and proton fluids from their background positions, with $\delta \boldsymbol{v}_{n,p} = D_t^{n,p} \boldsymbol{\xi}_{n,p}$. We are looking for the solutions of the form

$$\boldsymbol{\xi}_{n,p}(z,t) = \left[\xi_{x0}^{n,p} \boldsymbol{e}_x + \xi_{y0}^{n,p} \boldsymbol{e}_y\right] e^{i(\sigma t + kz)}, \tag{1.8}$$

where σ is the angular frequency of the wave. We now perturb Equations (1.3), (1.4) and (1.6); we set $R = \infty$ in the latter. To the linear order in the velocity perturbation, we have:

$$\begin{aligned} D_t^n \delta \boldsymbol{v}_n = \partial^2 \boldsymbol{\xi}_n / \partial t^2 = -\sigma^2 \boldsymbol{\xi}_n, \\ D_t^p \delta \boldsymbol{v}_p = -(\sigma + k w_0)^2 \boldsymbol{\xi}_p, \\ \delta \boldsymbol{f}_{\mathrm{mf}} = 2\boldsymbol{\Omega} \times (\delta \boldsymbol{v}_n - \delta \boldsymbol{v}_p) - (\boldsymbol{\nabla} \times \delta \boldsymbol{v}_n) \times \boldsymbol{w}_0, \\ \nu_{\mathrm{ee}} \nabla^2 \delta \boldsymbol{v}_p = -i \nu_{\mathrm{ee}} k^2 (\sigma + k w_0) \boldsymbol{\xi}_p, \\ \delta \boldsymbol{f}_{\mathrm{hm}} = c_A^2 \partial^2 \boldsymbol{\xi}_p / \partial z^2 = -c_A^2 k^2 \boldsymbol{\xi}_p. \end{aligned} \tag{1.9}$$

Here $c_A = \sqrt{B B_{\mathrm{cr}}/(4\pi\rho_p)}$ is the Alfvén velocity in the plasma. The expression for $\delta \boldsymbol{f}_{\mathrm{hm}}$ is obtained using the magnetic induction equation. Substituting these into Eqs. (1.3) and (1.4) and using $\boldsymbol{\nabla} \times \boldsymbol{\xi} = i\boldsymbol{k} \times \boldsymbol{\xi}$ together with $\delta \boldsymbol{v}_n = i\sigma \boldsymbol{\xi}_n$ and $\delta \boldsymbol{v}_p = i(\sigma + k w_0) \boldsymbol{\xi}_p$, we get two linear vector equations for $\boldsymbol{\xi}_n$ and $\boldsymbol{\xi}_p$. It is now convenient to proceed as follows: Let us represent a vector $\boldsymbol{\xi} = \xi_x \boldsymbol{e}_x + \xi_y \boldsymbol{e}_y$ by a complex number $\tilde{\xi} = \xi_x + i\xi_y$. Then a vector $\boldsymbol{e}_z \times \boldsymbol{\xi}$ is represented by $i\tilde{\xi}$. By using this, we can immediately rewrite the two real vector equations as two complex scalar equations:

$$\begin{aligned} \sigma \tilde{\xi}_n + 2\Omega \tilde{\xi}_p = 0, \\ \left[\bar{\sigma} + 2\Omega \left(1 - \frac{1}{x_p}\right) - \left(i\nu_{\mathrm{ee}} + c_A^2/\bar{\sigma}\right) k^2\right] \bar{\sigma} \tilde{\xi}_p + \frac{2\Omega - k w_0}{x_p} \sigma \tilde{\xi}_n = 0, \end{aligned} \tag{1.10}$$

where $\bar{\sigma} = \sigma + kw_0$. This pair of equation yields immediately the complex dispersion relation:

$$\bar{\sigma}^2 + \left[2\Omega\left(1 - \frac{1}{x_p}\right) - i\nu_{\rm ee}k^2\right]\bar{\sigma} - \frac{2\Omega(2\Omega - kw_0)}{x_p} - c_A^2 k^2 = 0. \tag{1.11}$$

The dispersion relation for arbitrary R is derived, for completeness, in Appendix 1.A. In the next two sections we consider two applications of the relation Eq. (1.11).

1.3 Hydromagnetic waves in magnetars

In this section we assume that there is no $\mathbf{\Omega}$-directed relative proton-neutron flow, i.e. we assume $w_0 = 0$. We also set $\nu_{\rm ee}$ to zero, since the ratio of the viscous to hydromagnetic stress is given by

$$\nu_{\rm ee}\sigma/c_A^2 \ll 1. \tag{1.12}$$

With these simplifications, the dispersion relation (1.11) gives

$$\sigma = -\Omega\left(1 - \frac{1}{x_p}\right) \pm \sqrt{\Omega^2\left(1 + \frac{1}{x_p}\right)^2 + c_A^2 k^2}. \tag{1.13}$$

It is important to note that in this expression c_A is a function of only the proton density ρ_p ($c_A^2 \equiv BB_{\rm cr}/4\pi\rho_p$). All observed magnetars are slowly rotating, with $\Omega \sim 1$ rad s^{-1}. The observed lowest angular frequency for a magnetar QPO is 18 Hz, thus $\sigma \sim 113$ rad s^{-1}. The sum of Magnus and Coriolis forces, represented by the terms with Ω, contribute only a fraction $\delta\sigma/\sigma$ to the wave frequency, given by

$$\delta\sigma/\sigma \simeq \frac{\Omega}{x_p\sigma} = 0.18\left(\frac{\Omega}{1\ \text{rad s}^{-1}}\right)\left(\frac{113\ \text{rad s}^{-1}}{\sigma}\right)\left(\frac{0.05}{x_p}\right). \tag{1.14}$$

We note that this does *not* depend on the assumption that σ represents some fundamental Alfvén mode. From the above equation, it is clear that for hydromagnetic waves associated with the magnetar QPO frequencies of 18Hz and higher, the magnus forces from neutron superfluid introduce only a small correction to their propagation. Thus we conclude that the magnetar oscillations (as seen in the giant-flare QPOs) even in the case of strong pinning, do not couple significantly to the neutron superfluid.[1] Therefore, given the knowledge of the internal magnetic field, one should use $\rho_c \simeq x_p \rho_n$ in Eq. (1.1) to determine the frequency of the lowest QPO which, according to Levin (2007), corresponds to the turning point of the Alfvén continuum in the core. For Levin's the simplest computable magnetar model (uniform internal magnetic field and density), with the typical magnetar parameters, $B = 10^{15}$ G, $R = 10$ km, $\rho = 10^{15}$ g cm^{-3} and $x_p = 0.05$, Eq. (1.1) gives $\nu_a \simeq 22$ Hz, which is in qualitative agreement with the observed 18 Hz. If the whole neutron superfluid would mass-load the wave, this frequency would be reduced by a factor of ~ 4. While suggestive, the numbers above certainly should not be taken as evidence of neutron superfluidity, since the strength and geometry of magnetic fields inside the magnetar are highly uncertain.

1.4 Precession of neutron stars

Consider now a precessing neutron star where the neutron angular velocity and the crust+proton angular velocity[2] are equal in magnitude Ω but are misaligned by an angle θ. Suppose that this angle is fixed due to the strong vortex pinning. The relative velocity of the proton superfluid along the vortices is given by

$$w_0 = \Omega \sin\theta x_2, \tag{1.15}$$

[1] We note that we have used the plane-wave analysis for what is likely a non-plane-wave oscillation. This is clearly a limitation of our formalism. However, the plane-wave analysis illustrates the physics which is also valid for oscillatory mode of any geometry, namely that for high-frequency waves Alfvén restoring forces are much greater than the Magnus forces. This occurs essentially because the Magnus force is proportional to the velocity and thus scales as σ, while the total restoring force scales as σ^2. Thus, we believe that our analysis is qualitatively correct in the high-frequency regime, for non-plane-wave Alfvén-type oscillations.

[2] The crust and the core protons are co-precessing; this is enforced by the hydro-magnetic stresses (Levin & D'Angelo, 2004).

where x_2 is the coordinate measured along $\mathbf{\Omega}_n \times \mathbf{\Omega}_p$. Note that this expression agrees with Eq. (18) in GAJ1 when one notes that for small θ their wobble angle θ_w equals $I_p\theta/I_n$, where I_p and I_n are the moments of inertia of the proton and neutron components, respectively.

1.4.1 Glaberson instability criterion

It is convenient to express the general solution of Eq. (1.11) as follows:

$$\sigma = -kw_0 - \Omega\left(1 - \frac{1}{x_p}\right) + \frac{i\nu_{\rm ee}k^2}{2} \pm \sqrt{D},$$

where

$$D = \Omega^2\left(1 + \frac{1}{x_p}\right)^2 + c_A^2k^2 - \frac{2kw_0\Omega}{x_p} - \frac{\nu_{\rm ee}^2k^4}{4} - i\nu_{\rm ee}k^2\Omega\left(1 - \frac{1}{x_p}\right) \quad (1.16)$$

This is essentially the same as Eq. (10) of GAJ1 when $c_A = 0$.

First, let us consider the non-viscous case with $\nu_{\rm ee} = 0$. Then the minimum of D is attained for $k = \Omega w_0/(x_p c_A^2)$ and equals

$$D_{\rm min} = \Omega^2\left(1 + \frac{1}{x_p}\right)^2 - \frac{\Omega^2 w_0^2}{x_p^2 c_A^2}. \quad (1.17)$$

Thus the Glaberson instability appears only for

$$w_0 > c_A(1 + x_p) \simeq c_A. \quad (1.18)$$

We now allow for the viscous term in Eqs. (1.16) and (1.16). We find that a weak, viscosity-driven instability appears at a smaller velocity

$$w_0 > 2\sqrt{x_p}c_A = \sqrt{\frac{BB_{\rm cr}}{\pi\rho_n}}, \quad (1.19)$$

for the wave-vector range

$$k_- < k < k_+ \quad (1.20)$$

where

$$k_{\pm} = \frac{\Omega}{x_p c_A^2}\left[w_0 \pm \sqrt{w_0^2 - 4c_A^2 x_p}\right] ; \tag{1.21}$$

see Appendix 1.B for the mathematical details.

Equation (1.19) expresses the lowest relative proton-neutron velocity which is required for the initiation of the Glaberson instability. Whether this velocity is achieved depends on the misalignment angle θ between the proton and neutron anglular velocity vectors. In the next subsection, we derive a simple but rigorous upper bound on θ.

1.4.2 The maximum misalignment angle for fast precession

The misalignment angle θ can be constrained, by requiring that the precessional torque τ_p of the proton component not exceed the maximum torque τ_m that the vortices can exert on the fluxtubes through magnetic stresses. The precessional torque is given by[1]

$$\tau_p = L_n \Omega_p \sin\theta \tag{1.22}$$

Here $L_n = I\Omega$ is the proton angular momentum, I is the total stellar moment of inertia and $\Omega_p = \Omega$. We find that for a typical neutron star with the mass of $M = 1.4\ M_\odot$ and radius of $R = 10^6$ cm, the precessional torque is given by

$$\tau_p \simeq 4 \cdot 10^{46} \sin\theta (P/1\ \mathrm{s})^{-2}\ \mathrm{g\ cm^2\ s^{-2}}. \tag{1.23}$$

Here P is the neutron-star rotational period. The maximal physically-admissible torque τ_m can be computed by assuming that the vortices have maximal contact with the fluxtubes, i.e. that the vortex is pushed/pulled with the stress

[1] Since the neutrons are pinned to the protons, the torque acting on the neutrons is given by the cross product of the instantaneous angular velocity of the protons and the neutrons angular momentum, and is therefore independent of x_p. In our derivation we assume that the angular velocities of the protons and the neutrons have the same magnitude Ω.

of $B_{\rm cr}^2/(8\pi)$ accross its whole side surface. The maximal torque exerted on a single vortex is given by

$$\tau_v = \frac{B_{\rm cr}^2}{8\pi} l^2 d, \tag{1.24}$$

where $d \sim 2 \cdot 10^{-12}$ cm is the vortex diameter and l is the vortex half-length. The total number of vortices is given by

$$N = \pi R^2 n_v \sim 3 \cdot 10^{16} (R/10^6 \text{ cm})^2 (P/1 \text{ s})^{-1}, \tag{1.25}$$

where n_v is the per-area vortex density (Shapiro and Teukolsky 1983, Link 2003). For a spherical star with a dense grid of the linear vortices, the average value of l^2 is $R^2/2$. Thus, by combining Eqs. (1.24) and (1.25), we arrive to the following expression:

$$\tau_m = \frac{B_{\rm cr}^2}{16} R^4 d n_v \simeq 1.3 \cdot 10^{45} (P/1 \text{ s})^{-1} \text{ g cm}^2 \text{ s}^{-2}. \tag{1.26}$$

From Eqs. (1.26) and (1.23), we see that our requirement $\tau_m > \tau_p$ implies that

$$\theta < 2^o (P/1 \text{ s}), \tag{1.27}$$

and therefore

$$w_0 \sim \Omega\theta R < 2 \cdot 10^5 \text{ cm s}^{-1}. \tag{1.28}$$

This upper limit on w_0 is spin-independent.

So, is this velocity sufficient to drive the Glaberson instability? Equation (1.19) tells us that for $x_p = 0.05$, $B = 10^{12}$ G and $\rho_n = 10^{15}$ g cm^{-3}, the critical relative velocity is $w_0 \sim 6 \cdot 10^5$ cm s^{-1}. Thus we conclude that in the presence of strong vortex pinning and magnetic fields $B > 10^{11}$ G the misalignment between the proton- and neutron angular velocities is unlikely to become large enough to provide wind velocities w_0 that are sufficient to drive the Glaberson instability.

1.5 Discussion

The calculations of this chapter have two main astrophysical implications. First, we have shown that the Alfvén waves which are associated with magnetar QPOs are not significantly mass-loaded by a neutron superfluid, even if the superfluid vortices are strongly pinned to the proton-electron plasma. For $B = 10^{15}$ G and the simplest B-field geometry, the expected frequency of lowest magnetar QPO is in remarkable agreement with observations, *if* only the protons, i.e. about 0.05 of the core mass, are mass-loading the Alfvén waves. Strong vortex pinning will, however, have a strong effect on the Alfvén waves in rapidly spinning and relatively non-magnetic neutron stars, i.e. those ones in the Low-Mass X-ray Binaries. In these stars the Alfvén waves may play an important role in damping of the r-mode instability, as discussed by Mendell (2001) and Kinney and Mendell (2003) for the cases of non-superfluid and superfluid core, respectively. Kinney and Mendell (2003) had not considered the vortex pinning (see also Mendell 1991); however from Eq. (1.14) and from the fact that $\sigma \sim \Omega$ for an r-mode, it is clear that the strong vortex pinning would have a huge (of order $1/x_p$) effect on the Alfvén waves with the r-mode frequency.

Second, we have shown that the hydromagnetic stresses generally suppress the Glaberson instability in the proton-neutron superfluid mixture, in the case of strongly pinned vortices[1]. We have also provided a robust upper bound Eq. (1.27) on the angle between proton and neutron angular velocities in the fast-precessing neutron stars. Even for slowly-spinning magnetars, the misalignment angle cannot exceed 20 degrees, which implies a wobble angle no greater than 1 degree. Thus a detection of neutron-star fast precession is difficult, if not impossible, observationally. An inspection of the XMM timing data on known anomalous x-ray pulsars produces no statistically-significant periodic signal which could be interpreted as fast precession [van Kerkwijk 2008, private communications].

[1] We have not considered here the PMGO case when some normal neutron component is present and is driving the instability.

Acknowledgements

We thank Andrew Melatos and Kostas Glampedakis for useful discussions, the anonymous referee for valuable suggestions. We thank Maarten van Kerkwijk for a search of fast magnetar precession in the XMM data.

Appendix 1.A: Dispersion relation for arbitrary drag

In this Appendix we perform a plane-wave analysis of the two-fluid dynamical equations defined in Eq's (1.3) and (1.4) and derive a dispersion relation for arbitrary drag strength R. Using the plane wave solutions from Eq. (1.8), we can perturb Eq's (1.3) and (1.4). Retaining linear terms we get:

$$D_t^n \delta \boldsymbol{v}_n = -\sigma^2 \boldsymbol{\xi}_n \tag{1.29}$$

$$D_t^p \delta \boldsymbol{v}_p = -\bar{\sigma}^2 \boldsymbol{\xi}_p \tag{1.30}$$

$$2\delta \boldsymbol{v}_n \times \boldsymbol{\Omega} = 2i\Omega\sigma \boldsymbol{\xi}_n \times \boldsymbol{e}_z \tag{1.31}$$

$$2\delta \boldsymbol{v}_p \times \boldsymbol{\Omega} = 2i\Omega\bar{\sigma} \boldsymbol{\xi}_p \times \boldsymbol{e}_z \tag{1.32}$$

$$\begin{aligned} \delta \boldsymbol{f}_{\rm mf} = \frac{R}{1+R^2} \big[& 2i\Omega \boldsymbol{e}_z \times \big(\boldsymbol{e}_z \times \big(\sigma \boldsymbol{\xi}_n - \bar{\sigma} \boldsymbol{\xi}_p \big) \big) \\ & + k w_0 \sigma \boldsymbol{e}_z \times ((\boldsymbol{e}_z \times \boldsymbol{\xi}_n) \times \boldsymbol{e}_z) \big] \\ + \frac{R^2}{1+R^2} \big[2i\Omega \boldsymbol{e}_z \times & \big(\sigma \boldsymbol{\xi}_n - \bar{\sigma} \boldsymbol{\xi}_p \big) - w_0 k \sigma \boldsymbol{e}_z \times (\boldsymbol{e}_z \times \boldsymbol{\xi}_n) \big] \end{aligned} \tag{1.33}$$

$$\nu_{\rm ee} \nabla^2 \delta \boldsymbol{v}_p = -\nu_{\rm ee} k^2 \bar{\sigma} \boldsymbol{\xi}_p \tag{1.34}$$

$$\delta \boldsymbol{f}_{\rm hm} = -c_A^2 k^2 \boldsymbol{\xi}_p \tag{1.35}$$

Here $\bar{\sigma} \equiv \sigma + w_0 k$ and $c_A = \sqrt{B B_{\rm cr}/4\pi\rho_c}$ is the Alfvén velocity in the plasma. We can simplify these expressions by using the same trick as in Section 1.2: Let us represent the vector $\boldsymbol{\xi} = \xi_x \boldsymbol{e}_x + \xi_y \boldsymbol{e}_y$ by a complex number $\tilde{\xi} = \xi_x + i\xi_y$. The cross product $\boldsymbol{e}_z \times \boldsymbol{\xi}$ corresponding to a simple rotation in the xy-plane, can then be represented by $i\tilde{\xi}$. Using this, we convert the two real vector equations (1.3) and (1.4) into two complex scalar equations:

$$-2\Omega C \bar{\sigma} \tilde{\xi}_p = \left[\bar{\sigma} + (C-1)(w_0 k - 2\Omega) \right] \sigma \tilde{\xi}_n \tag{1.36}$$

$$\left[\bar{\sigma} + \left(2\Omega\left(1 - \frac{C}{x_p}\right) - i\nu_{\rm ee}k^2\right) - \frac{c_A^2}{\bar{\sigma}}k^2\right]\bar{\sigma}\tilde{\xi}_p = \frac{C}{x_p}\left(w_0k - 2\Omega\right)\sigma\tilde{\xi}_n \quad (1.37)$$

where $C \equiv \frac{R(i+R)}{1+R^2}$. This pair of equations yields the following complex dispersion relation:

$$\begin{aligned} \bar{\sigma}^3 + \bar{\sigma}^2 &\left[(C-1)\left(w_0k - 2\Omega\right) + 2\Omega\left(1 - \frac{C}{x_p}\right) - i\nu_{\rm ee}k^2\right] \\ +\bar{\sigma}&\left[\left(w_0k - 2\Omega\right)\left((C-1)\left(2\Omega - i\nu_{\rm ee}k^2\right) + \frac{2\Omega C}{x_p}\right) - c_A^2k^2\right] \\ &-c_A^2k^2\left(C-1\right)\left(w_0k - 2\Omega\right) = 0 \end{aligned} \quad (1.38)$$

In the strong drag limit $R \to \infty$ ($C = 1$) this cubic equation simplifies significantly:

$$\bar{\sigma}^2 + \bar{\sigma}\left[2\Omega\left(1 - \frac{1}{x_p}\right) - i\nu_{\rm ee}k^2\right] - \left[\frac{2\Omega\left(2\Omega - w_0k\right)}{x_p} + c_A^2k^2\right] = 0, \quad (1.39)$$

which is Eq. (1.11) in the text.

Appendix 1.B: Instability criterion for non-zero viscosity

In this Appendix we derive a criterion for instability in the case of non-negligible viscosity, i.e. Eq. (1.19). We rewrite Eq. (1.16) as follows:

$$\sigma = A + iB \pm \sqrt{C + iD} \quad (1.40)$$

where

$$A = -w_0k - \Omega\left(1 - \frac{1}{x_p}\right)$$

$$B = \frac{\nu_{\rm ee}k^2}{2}$$

$$C = \Omega^2\left(1 + \frac{1}{x_p}\right)^2 + c_A^2k^2 - \frac{2w_0k\Omega}{x_p} - B^2 \quad (1.41)$$

$$D = -2\Omega\left(1 - \frac{1}{x_p}\right)B$$

The state of marginal stability is given by

$$Im\,(\sigma) = B \pm Im\left(\sqrt{C + iD}\right) = 0. \tag{1.42}$$

In order to find a convenient expression for $Im\left(\sqrt{C+iD}\right)$ we write $\sqrt{C+iD}$ in polar form. The imaginary part is then given by

$$Im\left(\sqrt{C + iD}\right) = \left(C^2 + D^2\right)^{\frac{1}{4}} \sin\left(\frac{1}{2}\arccos\left(\frac{C}{\sqrt{C^2+D^2}}\right)\right) \tag{1.43}$$

Combining this with Eq. 1.42), we find

$$B^2 = \frac{1}{2}\sqrt{C^2 + D^2} - \frac{1}{2}C \tag{1.44}$$

Taking the square of this expression and using Eqs (4.46) we arrive at

$$\frac{2\Omega\,(2\Omega - w_0 k)}{x_p} + c_A^2 k^2 = 0 \tag{1.45}$$

And therefore,

$$k_{\pm} = \frac{\Omega}{x_p c_A^2}\left[w_0 \pm \sqrt{w_0^2 - 4c_A^2 x_p}\right] \tag{1.46}$$

The unstable waves have wave-vectors in the interval $k_- < k < k_+$, provided that k_- and k_+ are real. Thus the criterion for the instability is

$$w_0 > 2c_A\sqrt{x_p}, \tag{1.47}$$

this is Eq. (1.19) of the main text. We now make an estimate of the growth rate in this instability window. For a realistic neutron star, we take $\nu_{\rm ee} \approx 10^9$ cm^2 s^{-1} at $T \approx 10^7$ K (Flowers & Itoh 1979, Cutler & Lindblom 1987, Andersson, Comer & Glampedakis, 2005), $\Omega \approx 2\pi$ rad/s, $c_A \approx 10^6$ cm s^{-1} and therefore for $w_0 \sim c_A$, the wave-vector of an unstable wave is $k \sim 10^{-4}$ cm^{-1}. We now note that $\nu_{\rm ee}k^2/2 << \Omega/x_p$. Therefore the terms B from Eq. (3.69) and B^2 from Eq. (4.46) have a negligible contribution to $Im\,(\sigma)$ and can be ignored.

Consider first the case where $C < 0$. We note that $Im\,(\sigma)$ is completely dominated by C so that we arrive at the criterion of Eq. (1.18) again. With a growth rate of

$$-\,Im\,(\sigma) = \sqrt{C} \tag{1.48}$$

Next consider $C > 0$, i.e. $w_0 < (1 + x_p)\, c_A$. By means of a simple analysis in the complex plane one can show that

$$\frac{\sqrt{D}}{2} < -Im\,(\sigma) < \sqrt{\frac{D}{2}} \tag{1.49}$$

For $w_0 > 2\sqrt{x_p} c_A$, there is a range of k where the instability occurs; see Eq. (1.20). Substituting $k = k_+$ into Eq. (1.49) for the maximum k in this range, we find the growth rate of the instability

$$-\,Im\,(\sigma) \approx \sqrt{\frac{\nu_{\mathrm{ee}}\Omega^3}{2x_p^3 c_A^4}} \left[w_0 + \sqrt{w_0^2 - 4c_A^2 x_p} \right] \tag{1.50}$$

We note that because the viscosity is small, this growth rate is much smaller than the one in Eq. (1.48).