Anomalous diffusion of Dirac fermions
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Summary

Imagine an indecisive hiker standing somewhere along a path, constantly tossing a coin. Whenever the result is "heads", he walks one step further. Whenever the result is "tails", he takes a step back. How does this situation relate to someone heating (for a short while) a metal bar with a lighter? Both the "density of the hiker" (i.e. the probability to find him per piece of path), and the density of heat in the rod (determined by the temperature) spread with time according to the diffusion equation. This equation essentially states that the extent of the density profile grows as the square root of time. In the case of the hiker this means that if he has (averaged over many repetitions of this tiresome procedure) moved from his starting point by 20 meters after 15 minutes, we can expect him to have moved by 40 meters on average after one hour. Similar processes are abundant in nature and include the spreading of impurities in crystals as well as the sprawling of infected mosquitos.

The square-root-law of diffusion is remarkably universal, yet there exist cases in nature where the random spreading does not follow this law. Such diffusion is called *anomalous* – the width of the density profile spreads as a power of time which lies somewhere between 0 and 1, different from the power 1/2 of normal diffusion. For example, it has been found that contaminations in ground water spread slower than if they would be governed by normal diffusion. This is due to the pollutants getting trapped underground for long times too often to sustain normal diffusion. Slower-than-normal diffusion is called subdiffusion. Superdiffusion, faster than normal, is also possible, and has been observed for example in the transmission of skylight through cloudy atmospheres.

Anomalous diffusion can also occur in electrical conduction through disordered metals and semiconductors. At low temperatures quantum mechanics enters as another mechanism by which the diffusion of electrons can become anomalous. It may slow down to a complete halt due to destructive interference, while if the interference is constructive, the opposite effect can happen.

This thesis addresses a variety of systems in which the diffusion is anomalous, mainly motivated by recent experimental developments.

In Chapter 2 we look at the consequences that slower-than-normal diffusion on fractals has for shot noise, the time-dependent fluctuations of the electric current. Fractals are self-similar geometric objects characterized by a non-integer dimension. (Cloud patterns are a familiar example of fractals.) Our study is motivated by the fractals which form from "puddles" of electrons and holes in graphene, a one-atom thick sheet of carbon.

Faster-than-normal diffusion requires long steps to happen sufficiently more often than in the case of normal diffusion. Recently, superdiffusion of photons was reported in a medium consisting of glass spheres with a large range of diameters. Because the arrangement of the spheres is fixed in time, the steps which the photon makes in the medium are correlated: a long step in one direction is likely to be followed by another long step in the opposite direction. To assess the importance of the correlations we examine in Chapter 3 the effects correlations have on superdiffusion in one dimension. Working in one dimension enhances the correlations (thus providing a worst-case estimate) and allows an exact solution in closed form.

Electrons in graphene have the unusual property that they are massless, and their wave equation is the relativistic Dirac equation, rather than the nonrelativistic Schrödinger equation. (These massless electrons are the Dirac fermions from the title.) Quantum interference in graphene is constructive rather than destructive on average. As a consequence, while increasing the disorder strength in normal materials makes it more difficult for electrons to travel, in the case of graphene the opposite effect can occur. In Chapter 4 we develop and demonstrate the usefulness of a method to simulate the anomalous diffusion of Dirac fermions in a computer.

The last two chapters of this thesis address other properties of Dirac fermions, not directly related to anomalous diffusion. These appear in socalled topological insulators, a novel class of materials which are insulating in the bulk but can conduct via metallic edge states. These edge states are unique because spin-up and spin-down electrons move in opposite directions, regardless of any obstacles they might find along their way. Chapter 5 proposes and analyzes a spin precession experiment in topological insulators. In Chapter 6 we present the mechanism for the conversion of an ordinary insulator into a topological insulator by disorder. This conversion was reported in the literature, on the basis of computer simulations, but had remained unexplained.