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## SPECIAL VALUES OF CANONICAL GREEN'S FUNCTIONS

ROBIN DE JONG

ABSTRACT. We give a precise formula for the value of the canonical Green's function at a pair of Weierstrass points on a hyperelliptic Riemann surface. Further we express the 'energy' of the Weierstrass points in terms of a spectral invariant recently introduced by N. Kawazumi and S. Zhang. It follows that the energy is strictly larger than  $\log 2$ . Our results generalize known formulas for elliptic curves.

## 1. INTRODUCTION

Let  $M$  be a compact and connected Riemann surface of genus  $h \geq 1$ . On  $M$  one has a canonical Kähler form  $\mu$  given as follows. The space  $H$  of holomorphic differentials on  $M$  is a complex vector space of dimension  $h$ . It carries a natural hermitian inner product given by  $\langle \alpha, \beta \rangle = \frac{\sqrt{-1}}{2} \int_M \alpha \wedge \bar{\beta}$  for elements  $\alpha, \beta$  in  $H$ . Let  $(\omega_1, \dots, \omega_h)$  be an orthonormal basis of  $H$ . By the Riemann-Roch theorem, the  $\omega_i$  do not simultaneously vanish at any point of  $M$ , and we put

$$\mu = \frac{\sqrt{-1}}{2h} \sum_{i=1}^h \omega_i \wedge \bar{\omega}_i.$$

The associated metric has been studied in many contexts, ranging from arithmetic geometry [3] [9] [10] [19] [21] to perturbative string theory [1] [2] [5] [6] [13]. The metric is known to have everywhere non-positive curvature, and the curvature vanishes at  $x \in M$  if and only if  $x$  is a Weierstrass point on a hyperelliptic Riemann surface [20], Main Theorem.

In this paper we study the canonical Green's function  $g_\mu$  associated to  $\mu$  (see Section 2 for definitions), and in particular its special values  $g_\mu(w_i, w_j)$  at pairs  $(w_i, w_j)$  of distinct Weierstrass points on a hyperelliptic Riemann surface. The starting point of our discussion is the case  $h = 1$ , with  $M$  given as a 2-sheeted cover of the Riemann sphere  $\mathbb{CP}^1$  branched at four points, say  $\alpha_1, \alpha_2, \alpha_3$  and  $\infty$ . Let  $w_1, w_2, w_3, o$  be the four critical points of  $M$  lying over  $\alpha_1, \alpha_2, \alpha_3, \infty$ . It is then known [7] [15] [22] that the formula

$$(1.1) \quad g_\mu(w_i, w_j) = \frac{1}{3} \log 2 + \frac{1}{12} \log \left( \frac{|\alpha_i - \alpha_j|^2}{|\alpha_i - \alpha_k| \cdot |\alpha_j - \alpha_k|} \right)$$

holds. By the translation invariance of  $g_\mu$  we obtain the formula

$$(1.2) \quad \sum_{i=1}^3 g_\mu(w_i, o) = \log 2.$$

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We will return to both these formulas in the text below.

Our purpose is to generalize formulas (1.1) and (1.2) to the case of a hyperelliptic Riemann surface of genus  $h \geq 2$ . Let  $M$  be such a surface, given as the 2-sheeted cover of  $\mathbb{CP}^1$  branched at the  $2h + 2$  points  $\alpha_1, \dots, \alpha_{2h+2}$  (these may or may not include  $\infty$ ). We put, for each pair  $(i, j)$  of distinct indices,

$$(1.3) \quad \delta_{ij} = \frac{(\alpha_i - \alpha_j)^{2h(2h+1)} \prod_{r \neq s} (\alpha_r - \alpha_s)}{\prod_{r \neq i} (\alpha_i - \alpha_r)^{2h+1} \prod_{r \neq j} (\alpha_j - \alpha_r)^{2h+1}}.$$

In this formula, we disregard any difference of two roots when one of the roots is  $\infty$ . It is readily verified that  $\delta_{ij}$  is invariant under the action of  $\text{Aut}(\mathbb{CP}^1) = \text{PGL}_2(\mathbb{C})$  on the  $\alpha_i$ . In particular  $\delta_{ij}$  defines an analytic modular invariant of the pair  $(w_i, w_j)$ , where  $w_i, w_j$  are the critical points of  $M$  lying over  $\alpha_i, \alpha_j$ . More intrinsically  $\delta_{ij}$  is the discriminant of the branch set that one obtains by sending  $\alpha_i, \alpha_j$  to  $0, \infty$  using an element of  $\text{Aut}(\mathbb{CP}^1)$  and by normalizing the other  $2h$  roots such that their product is unity [12].

We note that there is a tight connection with the more familiar cross ratios on the  $\alpha_i$ . Put  $\eta_{ijk} = \delta_{ik} \delta_{jk}^{-1}$ , and let

$$\mu_{ijk} = (\alpha_i - \alpha_k)(\alpha_j - \alpha_r)(\alpha_j - \alpha_k)^{-1}(\alpha_i - \alpha_r)^{-1}$$

be the cross ratio of the 4-tuple  $(\alpha_i, \alpha_j, \alpha_k, \alpha_r)$ . Then the relation

$$\mu_{ijk}^{2h(2h+1)} = \eta_{ijk} \eta_{ijr}^{-1} = \delta_{ik} \delta_{jk}^{-1} \delta_{jr} \delta_{ir}^{-1}$$

holds. It is straightforward to verify that for each index  $i$ , the product  $\prod_{j \neq i} \delta_{ij}$  is equal to unity.

The *energy* of the Weierstrass points on  $M$  is defined to be the following invariant of  $M$ :

$$\psi(M) = \frac{1}{2h+2} \sum_{i \neq j} g_\mu(w_i, w_j),$$

where the summation runs over all pairs  $i, j$  of distinct indices. Our first result is the following.

**Theorem A.** *Let  $w_i, w_j$  be two distinct Weierstrass points of  $M$ . Then the formula*

$$g_\mu(w_i, w_j) = \frac{1}{4h(2h+1)} \log |\delta_{ij}| + \frac{1}{2h+1} \psi(M)$$

*holds.*

In our second result we give an expression for  $\psi(M)$ . Consider for the moment an arbitrary compact and connected Riemann surface  $M$  of genus  $h \geq 1$ . Let  $\Delta_\mu$  be the Laplacian on  $L^2(M, \mu)$  given by putting  $\partial \bar{\partial} f = \pi \sqrt{-1} \Delta_\mu(f) \mu$  for  $C^\infty$  functions in  $L^2(M, \mu)$ . Let  $(\phi_\ell)_{\ell=0}^\infty$  be an orthonormal basis of real eigenfunctions of  $\Delta_\mu$ , with corresponding eigenvalues  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ . We then let  $\varphi(M)$  denote the invariant

$$\varphi(M) = \sum_{\ell > 0} \frac{2}{\lambda_\ell} \sum_{m, n=1}^h \left| \int_M \phi_\ell \omega_m \wedge \bar{\omega}_n \right|^2$$

of  $(M, \mu)$ . This fundamental invariant was introduced and studied recently by N. Kawazumi [18] and S. Zhang [24]. Note that  $\varphi(M) \geq 0$ , and that  $\varphi(M) = 0$  if and only if  $M$  has genus one. In [17] the asymptotic behavior of the  $\varphi$ -invariant is determined for surfaces degenerating into a stable curve with a single node. The

result shows in particular that  $\varphi$  can be viewed as a Weil function, with respect to the boundary, on the stable (Deligne-Mumford) compactification of the moduli space of surfaces of a fixed genus  $h \geq 2$ .

**Theorem B.** *Let  $M$  be a hyperelliptic Riemann surface of genus  $h \geq 2$ , and let  $\varphi(M)$  be the Kawazumi-Zhang invariant of  $M$ . Then for the energy  $\psi(M)$  of the set of Weierstrass points of  $M$ , the formula*

$$\psi(M) = \frac{1}{2h} \varphi(M) + \log 2$$

holds.

In particular we find that  $\psi(M) > \log 2$ . With  $h = 1$ , the formulas in Theorems A and B specialize to equations (1.1) and (1.2), respectively.

In order to prove Theorem B we proceed as follows. First we prove that, when viewed as functions on the moduli space of hyperelliptic Riemann surfaces, both  $\psi$  and  $\frac{1}{2h}\varphi$  have the same image under  $\partial\bar{\partial}$ . Then we study the asymptotic behavior of the difference of  $\psi$  and  $\frac{1}{2h}\varphi$  on a family of hyperelliptic Riemann surfaces degenerating into a stable curve of compact type. By using induction on  $h$  we will then deduce that the difference is a constant on the moduli space, equal to  $\log 2$ . For the degeneration argument we will rely heavily on results obtained by R. Wentworth [23].

## 2. CANONICAL GREEN'S FUNCTIONS

In this section we collect some basic results on canonical Green's functions. The main sources are [3] [10]. Proposition 2.2, about a 'parallel' property of certain  $(1, 1)$ -forms related to the universal Riemann surface over the moduli space of Riemann surfaces, seems not to be well known, and is possibly of independent interest.

As before, let  $M$  be a compact and connected Riemann surface of genus  $h \geq 1$  and  $\mu$  its canonical Kähler form. Let  $\Delta$  be the diagonal on the complex manifold  $M \times M$ . The canonical Green's function  $g_\mu = g_\mu(x, y)$  is the  $C^\infty$ -function on  $M \times M \setminus \Delta$  uniquely characterized by the following conditions:

- (1)  $\partial\bar{\partial}g_\mu(x, y) = \pi\sqrt{-1}(\mu(y) - \delta_x)$  for all  $x \in M$ ;
- (2)  $\int_M g_\mu(x, y)\mu(y) = 0$  for all  $x \in M$ ;
- (3)  $g_\mu(x, y) = g_\mu(y, x)$  for all  $x \neq y \in M$ ;
- (4)  $g_\mu(x, y) - \log|z(x) - z(y)|$  is bounded for all  $x \neq y$  in a coordinate chart  $(U, z: U \xrightarrow{\sim} D)$  of  $M$ , where  $D$  is the open unit disk.

The canonical Green's function inverts the Laplacian  $\Delta_\mu$  in the sense that

$$f(x) = - \int_M g_\mu(x, y) \Delta_\mu(f)(y) \mu(y) + \int_M f \mu$$

holds for all  $x \in M$  and all  $f$  in  $C^\infty(M)$ . As a consequence we have a formal development

$$g_\mu(x, y) = \sum_{\ell > 0} \frac{\phi_\ell(x)\phi_\ell(y)}{\lambda_\ell}$$

for  $g_\mu$ , where  $\phi_\ell$  and  $\lambda_\ell$  are the eigenfunctions and eigenvalues of  $\Delta_\mu$  that we have introduced above.

In [10] Section 7 using standard harmonic analysis explicit formulas are derived for the  $\phi_\ell$  and  $\lambda_\ell$  in the case  $h = 1$ . As a corollary of these formulas [10] Section 7

contains a derivation of equation (1.2) from the resulting formal expression for  $g_\mu$ . We would like to present here an alternative argument.

**Proposition 2.1.** *Assume  $(M, o)$  is a complex torus, and let  $N$  be a positive integer. Let  $x_1, \dots, x_{N^2-1}$  be the non-trivial  $N$ -torsion points of  $M$ . Then the formula*

$$\sum_{i=1}^{N^2-1} g_\mu(x_i, o) = \log N$$

holds.

*Proof.* From properties (1) and (2) it follows that for every  $x, y$  on  $M$  with  $Nx \neq y$  the equality

$$g_\mu(Nx, y) = \sum_{w: Nw=y} g_\mu(x, w)$$

holds. Now choose  $x$  in a standard euclidean coordinate chart  $z: U \xrightarrow{\sim} D$  around  $o$ , where  $D$  is the open unit disk. Then we have

$$\begin{aligned} g_\mu(Nx, o) &= \sum_{w: Nw=o} g_\mu(x, w) \\ &= \sum_{i=1}^{N^2-1} g_\mu(x_i, x) + g_\mu(x, o) \\ &= \sum_{i=1}^{N^2-1} g_\mu(x_i, x) + \log |z(x)| + a + o(1) \end{aligned}$$

as  $x \rightarrow o$ , where  $a$  is some constant, by properties (3) and (4). On the other hand we have

$$\begin{aligned} g_\mu(Nx, o) &= \log |Nz(x)| + a + o(1) \\ &= \log N + \log |z(x)| + a + o(1) \end{aligned}$$

as  $x \rightarrow o$ , by property (4). The desired equality follows.  $\square$

One obtains (1.2) by taking  $N = 2$  in the above proposition.

Let  $\mathcal{M}$  denote the moduli space of compact and connected Riemann surfaces of genus  $h \geq 2$ , and denote by  $\pi: \mathcal{C} \rightarrow \mathcal{M}$  the universal surface over  $\mathcal{M}$ . Both are viewed as orbifolds. The canonical Green's functions on the fibers of  $\pi$  determine a generalized function  $g$  on the fiber product  $\mathcal{C} \times_{\mathcal{M}} \mathcal{C}$  with logarithmic singularities along the diagonal  $\Delta$ . Let  $\nu$  be the  $(1, 1)$ -form on  $\mathcal{C} \times_{\mathcal{M}} \mathcal{C}$  determined by the condition  $\partial\bar{\partial}g = \pi\sqrt{-1}(\nu - \delta_\Delta)$ . Let  $e^A$  denote the restriction of  $\nu$  to  $\Delta$ , which we view as another copy of  $\mathcal{C}$ . By [3] the form  $e^A$  restricts to  $2 - 2h$  times the canonical  $(1, 1)$ -form  $\mu$  in each fiber of  $\pi: \mathcal{C} \rightarrow \mathcal{M}$ .

The  $(1, 1)$ -forms  $e^A$  and  $\nu$  are parallel over  $\mathcal{M}$  in the following sense.

**Proposition 2.2.** *Let  $s, t$  be arbitrary holomorphic sections of  $\pi$  over an open subset  $U$  of  $\mathcal{M}$ . Then the equalities  $s^*e^A = t^*e^A = (s, t)^*\nu$  hold on  $U$ .*

*Proof.* Without loss of generality we may replace  $\mathcal{M}$  by a finite cover so that we may assume that  $\mathcal{M}$  is equipped with a universal theta characteristic  $\alpha$ , i.e. a consistent choice of a divisor class  $\alpha$  of degree  $h - 1$  on each surface  $M$  such that  $2\alpha$  is the canonical class. Let  $\mathcal{J} \rightarrow \mathcal{M}$  be the universal jacobian over  $\mathcal{M}$  and let  $\Phi: \mathcal{C} \times_{\mathcal{M}} \mathcal{C} \rightarrow \mathcal{J}$  be the map over  $\mathcal{M}$  given by sending a triple  $(M, x, y)$  to the pair

consisting of the jacobian  $J(M)$  of  $M$  and the class of the divisor  $h \cdot x - y - \alpha$  in  $J(M)$ . Let  $\bar{\pi}: \mathcal{C} \times_{\mathcal{M}} \mathcal{C} \rightarrow \mathcal{C}$  be the projection on the first factor and let  $\bar{s}, \bar{t}: \pi^{-1}U \rightarrow \mathcal{C} \times_{\mathcal{M}} \mathcal{C}$  be the local sections of  $\bar{\pi}$  obtained by pulling back the given local sections  $s, t$  of  $\pi$ . It follows from [14] Lemma 3.2 that there exists a  $(1, 1)$ -form  $\kappa$  on  $\mathcal{C}$  and a  $(1, 1)$ -form  $w$  on  $\mathcal{J}$  which is translation-invariant in the fibers of  $\mathcal{J} \rightarrow \mathcal{M}$ , such that  $\Phi^*w = h\nu + \bar{\pi}^*\kappa$  holds. We deduce from this that

$$h \bar{s}^*\nu = (\Phi\bar{s})^*w - \kappa \quad \text{and} \quad h \bar{t}^*\nu = (\Phi\bar{t})^*w - \kappa.$$

Let  $T_{x,y}$  denote translation by  $[x - y]$  over  $\mathcal{J}|_U$ . Then  $\Phi\bar{t} = T_{s,t}\Phi\bar{s}$  and since  $w$  is fiberwise translation-invariant we obtain  $\bar{s}^*\nu = \bar{t}^*\nu$  from the above equalities. We derive  $(s, t)^*\nu = s^*\bar{t}^*\nu = s^*\bar{s}^*\nu = (s, s)^*\nu = s^*e^A$ . By symmetry property (3) we have  $(s, t)^*\nu = (t, s)^*\nu = t^*e^A$  as well.  $\square$

### 3. PROOF OF THEOREM A

In this section we give a proof of Theorem A. Let  $x: M \rightarrow \mathbb{C}\mathbb{P}^1$  be the 2-sheeted cover with branch points  $\alpha_1, \dots, \alpha_{2h+2}$ . Let  $w_1, \dots, w_{2h+2}$  be the corresponding Weierstrass points on  $M$ . For the moment we fix two distinct indices  $i, j$ . Consider then the meromorphic function  $f_{ij} = (x - \alpha_i)(x - \alpha_j)^{-1}$  on  $M$ . We note that  $\text{div}(f_{ij}) = 2(w_i - w_j)$ . From properties (1) and (2) we therefore obtain an equality

$$g_\mu(w_i, z) - g_\mu(w_j, z) = \frac{1}{2} \log |f_{ij}(z)| - \frac{1}{2} \int_M \log |f_{ij}| \cdot \mu$$

of generalized functions on  $M$ . In particular we find for any pair of indices  $k, r$

$$\begin{aligned} g_\mu(w_i, w_k) - g_\mu(w_i, w_r) - g_\mu(w_j, w_k) + g_\mu(w_j, w_r) \\ &= \frac{1}{2} \log |f_{ij}(w_k)f_{ij}(w_r)^{-1}| \\ &= \frac{1}{2} \log |\mu_{ijk r}| \\ &= \frac{1}{4h(2h+1)} \log |\eta_{ijk}\eta_{ijr}^{-1}| \end{aligned}$$

where we recall that  $\eta_{ijk} = \delta_{ik}\delta_{jk}^{-1}$  with  $\delta_{ij}$  the expression from (1.3). This equality implies that there exists a constant  $c_{ij}$  such that

$$g_\mu(w_i, w_k) - g_\mu(w_j, w_k) = \frac{1}{4h(2h+1)} \log |\eta_{ijk}| + c_{ij}$$

for each index  $k$ . It follows from [16], Theorem 1.4 that

$$\sum_{k \neq i} g_\mu(w_i, w_k) = \sum_{k \neq j} g_\mu(w_j, w_k).$$

From the fact that  $\prod_{k \neq i, j} \delta_{ij}$  equals unity we obtain that  $\sum_{k \neq i, j} \log |\eta_{ijk}| = 0$ . It follows that

$$2h c_{ij} = \sum_{k \neq i, j} (g_\mu(w_i, w_k) - g_\mu(w_j, w_k)) = 0,$$

hence  $c_{ij} = 0$  and

$$g_\mu(w_i, w_k) - g_\mu(w_j, w_k) = \frac{1}{4h(2h+1)} \log |\eta_{ijk}| = \frac{1}{4h(2h+1)} \log |\delta_{ik}\delta_{jk}^{-1}|.$$

Now fix the index  $k$  and vary the indices  $i, j$ . We find that there exists a constant  $c_k$  such that

$$g_\mu(w_i, w_k) = \frac{1}{4h(2h+1)} \log |\delta_{ik}| + c_k$$

for all indices  $i \neq k$ . We obtain  $c_k = \frac{1}{2h+1} \psi(M)$  by summing over  $i \neq k$ . The formula in Theorem A follows.

Note that a similar argument in the case  $h = 1$  allows one to deduce (1.1) from (1.2).

#### 4. PROOF OF THEOREM B

As was announced in the Introduction, we proceed in several steps. Let  $\mathcal{H}$  be the moduli space of hyperelliptic Riemann surfaces of a fixed genus  $h \geq 2$ . Let  $\pi: \mathcal{C} \rightarrow \mathcal{H}$  be the universal surface over  $\mathcal{H}$ . Both are viewed as orbifolds. A first result is the following.

**Proposition 4.1.** *The equality of (1, 1)-forms*

$$\partial\bar{\partial}\psi = \frac{1}{2h}\partial\bar{\partial}\varphi$$

*holds on  $\mathcal{H}$ .*

*Proof.* Let  $\mathcal{U}$  be an open cover of  $\mathcal{H}$  such that for each  $U$  in  $\mathcal{U}$  there exist  $2h+2$  holomorphic sections  $w_1, \dots, w_{2h+2}: U \rightarrow \pi^{-1}U$  of  $\pi^{-1}U \rightarrow U$  such that each  $w_i$  is a Weierstrass point in each fiber of  $\pi^{-1}U \rightarrow U$ . Let  $U$  be an element of  $\mathcal{U}$ . It suffices to prove the required equality on  $U$ . Theorem 3.1 of [18] gives an expression for  $\partial\bar{\partial}\varphi$  over the moduli space of Riemann surfaces of genus  $h$  (note that [18] considers the invariant  $a$  with  $a = \frac{1}{2\pi}\varphi$ ). Since the ‘harmonic volume’ of a Weierstrass point on a hyperelliptic Riemann surface is trivial, this expression immediately implies that on  $U$  one has the equality

$$\partial\bar{\partial}\varphi = 2h(2h+1)\pi\sqrt{-1}w_i^*e^A,$$

for any choice of the index  $i$ . On the other hand by Proposition 2.2 we have

$$\partial\bar{\partial}\psi = \pi\sqrt{-1}\sum_{j \neq i} (w_i, w_j)^* \nu = (2h+1)\pi\sqrt{-1}w_i^*e^A$$

for each index  $i$ . The proposition follows.  $\square$

We next determine the limit behavior of the difference  $\psi - \frac{1}{2h}\varphi$  in a holomorphic family  $M_t$  of hyperelliptic Riemann surfaces of genus  $h$  degenerating into a stable curve  $M_0$  which is the union of two hyperelliptic Riemann surfaces  $M_1, M_2$  of genera  $h_1, h_2 \geq 1$ , respectively, with two points  $p_1 \in M_1$  and  $p_2 \in M_2$  identified, as in Chapter III of the book ‘Theta functions on Riemann surfaces’ by J. Fay [11]. Here  $t$  runs through a small punctured open disk around 0 in the complex plane. Note that  $h = h_1 + h_2$ . Let  $g_1, g_2$  be the canonical Green’s functions on  $M_1, M_2$ , respectively, and let  $z_1, z_2$  be the local coordinates around  $p_1, p_2$  on  $M_1, M_2$  that come with the degeneration model. We put, following [23], Section 6:

$$\log k_1 = \lim_{x \rightarrow p_1} [g_1(x, p_1) - \log |z_1(x)|], \quad \log k_2 = \lim_{x \rightarrow p_2} [g_2(x, p_2) - \log |z_2(x)|],$$

and then make the reparametrization  $\tau = k_1 k_2 t$ .

**Theorem 4.2.** *For the surfaces  $M_t$  in Fay's degeneration model, the limit formula*

$$\lim_{t \rightarrow 0} \left[ \frac{1}{2h} \varphi(M_t) + \frac{h_1 h_2}{h^2} \log |\tau| \right] = \frac{1}{2h} \varphi(M_1) + \frac{1}{2h} \varphi(M_2)$$

*holds.*

**Theorem 4.3.** *For the energy  $\psi(M_t)$  of the Weierstrass points of  $M_t$ , the limit formula*

$$\lim_{t \rightarrow 0} \left[ \psi(M_t) + \frac{h_1 h_2}{h^2} \log |\tau| \right] = \frac{h_1}{h} \psi(M_1) + \frac{h_2}{h} \psi(M_2)$$

*holds.*

We deduce Theorem B from Theorems 4.2 and 4.3 by induction on  $h$ . Note that the assertion in Theorem B is true for  $h = 1$  by (1.2) and the fact that  $\varphi$  vanishes for  $h = 1$ . Now assume that the assertion in Theorem B is true for all genera smaller than  $h$ . Consider a holomorphic family  $M_t$  of hyperelliptic Riemann surfaces of genus  $h$  as above. We find

$$\begin{aligned} \lim_{t \rightarrow 0} \left[ \psi(M_t) - \frac{1}{2h} \varphi(M_t) \right] &= \frac{h_1}{h} \psi(M_1) - \frac{1}{2h} \varphi(M_1) + \frac{h_2}{h} \psi(M_2) - \frac{1}{2h} \varphi(M_2) \\ &= \frac{h_1}{h} \left( \psi(M_1) - \frac{1}{2h_1} \varphi(M_1) \right) + \frac{h_2}{h} \left( \psi(M_2) - \frac{1}{2h_2} \varphi(M_2) \right) \\ &= \frac{h_1}{h} \log 2 + \frac{h_2}{h} \log 2 = \log 2. \end{aligned}$$

Let  $\overline{\mathcal{H}}$  be the closure of  $\mathcal{H}$  in the Deligne-Mumford compactification  $\overline{\mathcal{M}}$  of the moduli space  $\mathcal{M}$  of compact Riemann surfaces of genus  $h$ . Let  $\tilde{\mathcal{H}} \subset \overline{\mathcal{H}}$  be the partial compactification of  $\mathcal{H}$  given by stable curves of compact type (i.e. whose jacobians are principally polarized abelian varieties). Write  $\chi = \psi - \frac{1}{2h} \varphi$ . The above calculation implies that for hyperelliptic Riemann surfaces in  $\mathcal{H}$  tending to a point in the boundary of  $\mathcal{H}$  in  $\tilde{\mathcal{H}}$ , the function  $\chi$  tends to  $\log 2$ . In particular the function  $\chi$  extends as a continuous function over  $\tilde{\mathcal{H}}$ . By Proposition 4.1 we know that  $\chi$  is a pluriharmonic function on  $\mathcal{H}$ . By the Riemann extension theorem we find that  $\chi$  extends as a pluriharmonic function on  $\tilde{\mathcal{H}}$ , which is a constant function with value  $\log 2$  on  $\tilde{\mathcal{H}} \setminus \mathcal{H}$ .

Let  $\mathcal{A}^*$  be the Satake compactification of the moduli space  $\mathcal{A}$  of principally polarized abelian varieties of genus  $h$ . This is a normal projective variety [4]. There exists a holomorphic map  $t: \overline{\mathcal{H}} \rightarrow \mathcal{A}^*$  given by assigning to a stable curve  $M$  the jacobian of its normalization. The map  $t$  is injective on  $\mathcal{H}$ , and the function  $\chi$  descends as a pluriharmonic function on  $t(\tilde{\mathcal{H}})$ , which is a constant function with value  $\log 2$  on  $t(\tilde{\mathcal{H}} \setminus \mathcal{H})$ .

The boundary of  $\tilde{\mathcal{H}}$  in  $\overline{\mathcal{H}}$  is a finite union of divisors  $\Xi_i$ , see [8] for example. From the moduli theoretic description of the points in the divisors  $\Xi_i$  it follows that the map  $t$  restricted to the boundary has positive dimensional fibers. Hence the boundary of  $t(\tilde{\mathcal{H}})$  in the closure of  $t(\tilde{\mathcal{H}})$  in  $\mathcal{A}^*$  has codimension at least two. Let  $M$  be a point in  $t(\mathcal{H})$ . Using that  $\mathcal{A}^*$  is projective one sees from intersecting with suitable hyperplanes in a projective embedding that the variety  $t(\tilde{\mathcal{H}})$  contains a complete curve  $X$  passing through  $M$ . As  $t(\mathcal{H})$  is affine, the curve  $X$  necessarily contains a point of  $t(\tilde{\mathcal{H}} \setminus \mathcal{H})$ . Let  $Y \rightarrow X$  be the normalization of  $X$ . The function

$\chi$  pulls back to a pluriharmonic function on  $Y$ , which is then necessarily constant. It follows that the value of  $\chi$  at  $M$  is equal to  $\log 2$ , and Theorem B is proven.

For the proof of Theorem 4.2 we refer to [17]. It remains to prove Theorem 4.3. We need the next result, which follows from [23], Theorem 6.10.

**Theorem 4.4.** *Let  $g_t$  be the canonical Green's function on  $M_t$ . Denote by  $p$  both the point  $p_1$  on  $M_1$  and the point  $p_2$  on  $M_2$ . Then for local distinct holomorphic sections  $x, y$  of the family  $M_t$  specializing onto  $M_1$  we have*

$$\lim_{t \rightarrow 0} \left[ g_t(x, y) - \left( \frac{h_2}{h} \right)^2 \log |\tau| \right] = g_1(x, y) - \frac{h_2}{h} (g_1(x, p) + g_1(y, p)) .$$

For local distinct holomorphic sections  $x, y$  with  $x$  specializing onto  $M_1$  and  $y$  specializing onto  $M_2$  we have

$$\lim_{t \rightarrow 0} \left[ g_t(x, y) + \frac{h_1 h_2}{h^2} \log |\tau| \right] = \frac{h_1}{h} g_1(x, p) + \frac{h_2}{h} g_2(y, p) .$$

Now let  $w_1, \dots, w_{2h+2}$  be the Weierstrass points on  $M_t$ . As  $t \rightarrow 0$  a portion of  $2h_1 + 1$  of them degenerate onto  $M_1$ , and  $2h_2 + 1$  of them degenerate onto  $M_2$ . The point  $p$  is a Weierstrass point of both  $M_1, M_2$ . Let  $I$  be the set of indices such that  $i \in I$  if and only if  $w_i$  degenerates onto  $M_1$ .

**Lemma 4.5.** *The equality*

$$\sum_{\substack{i, j \in I \\ i \neq j}} g_1(w_i, w_j) = 2h_1 \psi(M_1)$$

holds.

*Proof.* We calculate

$$(2h_1 + 2) \psi(M_1) = \sum_{\substack{i, j \in I \cup \{p\} \\ i \neq j}} g_1(w_i, w_j) = \sum_{\substack{i, j \in I \\ i \neq j}} g_1(w_i, w_j) + 2 \psi(M_1) .$$

The required equality follows.  $\square$

Using Theorem 4.4 we find

$$\begin{aligned} (2h + 2) \psi(M_t) &= \sum_{i \neq j} g_t(w_i, w_j) \\ &= \sum_{\substack{i, j \in I \\ i \neq j}} g_t(w_i, w_j) + \sum_{\substack{i, j \in I^c \\ i \neq j}} g_t(w_i, w_j) + 2 \sum_{\substack{i \in I \\ j \in I^c}} g_t(w_i, w_j) \\ &= \sum_{\substack{i, j \in I \\ i \neq j}} \left( \left( \frac{h_2}{h} \right)^2 \log |\tau| + g_1(w_i, w_j) - \frac{h_2}{h} (g_1(w_i, p) + g_1(w_j, p)) \right) \\ &\quad + \sum_{\substack{i, j \in I^c \\ i \neq j}} \left( \left( \frac{h_1}{h} \right)^2 \log |\tau| + g_2(w_i, w_j) - \frac{h_1}{h} (g_2(w_i, p) + g_2(w_j, p)) \right) \\ &\quad + 2 \sum_{\substack{i \in I \\ j \in I^c}} \left( -\frac{h_1 h_2}{h^2} \log |\tau| + \frac{h_1}{h} g_1(w_i, p) + \frac{h_2}{h} g_2(w_j, p) \right) + o(1) \end{aligned}$$

as  $t \rightarrow 0$ . With the help of Lemma 4.5 this can be rewritten as

$$\begin{aligned}
(2h+2)\psi(M_t) &= \left( (2h_1+1)2h_1 \left(\frac{h_2}{h}\right)^2 + (2h_2+1)2h_2 \left(\frac{h_1}{h}\right)^2 \right) \log|\tau| \\
&\quad - 2(2h_1+1)(2h_2+1) \frac{h_1 h_2}{h^2} \log|\tau| \\
&\quad + \left( 2h_1 - \frac{4h_1 h_2}{h} + \frac{2h_1(2h_2+1)}{h} \right) \psi(M_1) \\
&\quad + \left( 2h_2 - \frac{4h_1 h_2}{h} + \frac{2h_2(2h_1+1)}{h} \right) \psi(M_2) + o(1) \\
&= (2h+2) \left( -\frac{h_1 h_2}{h^2} \log|\tau| + \frac{h_1}{h} \psi(M_1) + \frac{h_2}{h} \psi(M_2) \right) + o(1)
\end{aligned}$$

as  $t \rightarrow 0$ . Theorem 4.3 follows.

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