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ADMISSIBLE CONSTANTS FOR GENUS 2 CURVES

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ABSTRACT. S.-W. Zhang recently introduced a new adelic invariant φ for curves of genus at least 2 over number fields and function fields. We calculate this invariant when the genus is equal to 2.

1. INTRODUCTION

Let X be a smooth projective geometrically connected curve of genus $g \geq 2$ over a field k which is either a number field or the function field of a curve over a field. Assume that X has semistable reduction over k . For each place v of k , let Nv be the usual local factor connected with the product formula for k .

In a recent paper [11] S.-W. Zhang proves the following theorem:

Theorem 1.1. *Let $(\omega, \omega)_a$ be the admissible self-intersection of the relative dualizing sheaf of X . Let $\langle \Delta_\xi, \Delta_\xi \rangle$ be the height of the canonical Gross-Schoen cycle on X^3 . Then the formula:*

$$(\omega, \omega)_a = \frac{2g-2}{2g+1} \left(\langle \Delta_\xi, \Delta_\xi \rangle + \sum_v \varphi(X_v) \log Nv \right)$$

holds, where the $\varphi(X_v)$ are local invariants associated to $X \otimes k_v$, defined as follows:

- if v is a non-archimedean place, then:

$$\varphi(X_v) = -\frac{1}{4}\delta(X_v) + \frac{1}{4} \int_{R(X_v)} g_v(x, x) ((10g+2)\mu_v - \delta_{K_{X_v}}),$$

where:

- $\delta(X_v)$ is the number of singular points on the special fiber of $X \otimes k_v$,
- $R(X_v)$ is the reduction graph of $X \otimes k_v$,
- g_v is the Green's function for the admissible metric μ_v on $R(X_v)$,
- K_{X_v} is the canonical divisor on $R(X_v)$.

In particular, $\varphi(X_v) = 0$ if X has good reduction at v ;

- if v is an archimedean place, then:

$$\varphi(X_v) = \sum_\ell \frac{2}{\lambda_\ell} \sum_{m,n=1}^g \left| \int_{X(\bar{k}_v)} \phi_\ell \omega_m \bar{\omega}_n \right|^2,$$

where ϕ_ℓ are the normalized real eigenforms of the Arakelov Laplacian on $X(\bar{k}_v)$ with eigenvalues $\lambda_\ell > 0$, and $(\omega_1, \dots, \omega_g)$ is an orthonormal basis for the hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2} \int_{X(\bar{k}_v)} \omega \bar{\eta}$ on the space of holomorphic differentials.

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Apart from giving an explicit connection between the two canonical invariants $(\omega, \omega)_a$ and $\langle \Delta_\xi, \Delta_\xi \rangle$, Zhang's theorem has a possible application to the effective Bogomolov conjecture, *i.e.*, the question of giving effective positive lower bounds for $(\omega, \omega)_a$. Indeed, the height of the canonical Gross-Schoen cycle $\langle \Delta_\xi, \Delta_\xi \rangle$ is known to be non-negative in the case of a function field in characteristic zero, and should be non-negative in general by a standard conjecture of Gillet-Soulé (*op. cit.*, Section 2.4). Further, the invariant φ should be non-negative, and Zhang proposes, in the non-archimedean case, an explicit lower bound for it which is positive in the case of non-smooth reduction (*op. cit.*, Conjecture 1.4.2). Note that it is clear from the definition that φ is non-negative in the archimedean case; in fact it is positive (*op. cit.*, Remark after Proposition 2.5.3).

Besides $\varphi(X_v)$, Zhang also considers the invariant $\lambda(X_v)$ defined by:

$$\lambda(X_v) = \frac{g-1}{6(2g+1)}\varphi(X_v) + \frac{1}{12}(\varepsilon(X_v) + \delta(X_v)),$$

where:

- if v is a non-archimedean place, the invariant $\delta(X_v)$ is as above, and:

$$\varepsilon(X_v) = \int_{R(X_v)} g_v(x, x)((2g-2)\mu_v + \delta_{K_{X_v}}),$$

- if v is an archimedean place, then:

$$\delta(X_v) = \delta_F(X_v) - 4g \log(2\pi)$$

with $\delta_F(X_v)$ the Faltings delta-invariant of the compact Riemann surface $X(\bar{k}_v)$, and $\varepsilon(X_v) = 0$.

The significance of this invariant is that if $\deg \det R\pi_*\omega$ denotes the (non-normalized) geometric or Faltings height of X one has a simple expression:

$$\deg \det R\pi_*\omega = \frac{g-1}{6(2g+1)}\langle \Delta_\xi, \Delta_\xi \rangle + \sum_v \lambda(X_v) \log Nv$$

for $\deg \det R\pi_*\omega$, as follows from the Noether formula:

$$12 \deg \det R\pi_*\omega = (\omega, \omega)_a + \sum_v (\varepsilon(X_v) + \delta(X_v)) \log Nv.$$

Now assume that X has genus $g = 2$. Our purpose is to calculate the invariants $\varphi(X_v)$ and $\lambda(X_v)$ explicitly. For the λ -invariant we obtain:

- if v is non-archimedean, then:

$$10\lambda(X_v) = \delta_0(X_v) + 2\delta_1(X_v),$$

where $\delta_0(X_v)$ is the number of non-separating nodes and $\delta_1(X_v)$ is the number of separating nodes in the special fiber of $X \otimes k_v$;

- if v is archimedean, then:

$$10\lambda(X_v) = -20 \log(2\pi) - \log \|\Delta_2\|(X_v),$$

where $\|\Delta_2\|(X_v)$ is the normalized modular discriminant of the compact Riemann surface $X(\bar{k}_v)$ (see below).

Thus, the $\lambda(X_v)$ are precisely the well-known local invariants corresponding to the discriminant modular form of weight 10 [6] [9] [10]. In particular we have:

$$\deg \det R\pi_*\omega = \sum_v \lambda(X_v) \log Nv$$

and we recover the fact that the height of the canonical Gross-Schoen cycle vanishes for X .

2. THE NON-ARCHIMEDEAN CASE

Let k be a complete discretely valued field. Let X be a smooth projective geometrically connected curve of genus 2 over k . Assume that X has semistable reduction over k . In this section we give the invariants $\varphi(X)$ and $\lambda(X)$ of X .

The proof of our result is based on the classification of the semistable fiber types in genus 2 and consists of a case-by-case analysis. The notation we employ for the various fiber types is as in [8]. We remark that there are no restrictions on the residue characteristic of k .

Theorem 2.1. *The invariant $\varphi(X)$ is given by the following table, depending on the type of the special fiber of the regular minimal model of X :*

Type	δ_0	δ_1	ε	φ
I	0	0	0	0
$II(a)$	0	a	a	a
$III(a)$	a	0	$\frac{1}{6}a$	$\frac{1}{12}a$
$IV(a, b)$	b	a	$a + \frac{1}{6}b$	$a + \frac{1}{12}b$
$V(a, b)$	$a + b$	0	$\frac{1}{6}(a + b)$	$\frac{1}{12}(a + b)$
$VI(a, b, c)$	$b + c$	a	$a + \frac{1}{6}(b + c)$	$a + \frac{1}{12}(b + c)$
$VII(a, b, c)$	$a + b + c$	0	$\frac{1}{6}(a + b + c) + \frac{1}{6}\frac{abc}{ab+bc+ca}$	$\frac{1}{12}(a + b + c) - \frac{5}{12}\frac{abc}{ab+bc+ca}$

For $\lambda(X)$ the formula:

$$10\lambda(X) = \delta_0(X) + 2\delta_1(X)$$

holds.

Let us indicate how the theorem is proved. Let r be the effective resistance function on the reduction graph $R(X)$ of X , extended bilinearly to a pairing on $\text{Div}(R(X))$. By Corollary 2.4 of [2] the formula:

$$\varphi(X) = -\frac{1}{4}(\delta_0(X) + \delta_1(X)) - \frac{3}{8}r(K, K) + 2\varepsilon(X)$$

holds, where K is the canonical divisor on $R(X)$. The invariant $r(K, K)$ is calculated by viewing $R(X)$ as an electrical circuit. The invariant ε is calculated on the basis of explicit expressions for the admissible measure and admissible Green's function; see [7] and [8] for such computations. The results we find are as follows:

Type	δ_0	δ_1	$r(K, K)$	ε
<i>I</i>	0	0	0	0
<i>II(a)</i>	0	a	$2a$	a
<i>III(a)</i>	a	0	0	$\frac{1}{6}a$
<i>IV(a, b)</i>	b	a	$2a$	$a + \frac{1}{6}b$
<i>V(a, b)</i>	$a + b$	0	0	$\frac{1}{6}(a + b)$
<i>VI(a, b, c)</i>	$b + c$	a	$2a$	$a + \frac{1}{6}(b + c)$
<i>VII(a, b, c)</i>	$a + b + c$	0	$2\frac{abc}{ab+bc+ca}$	$\frac{1}{6}(a + b + c) + \frac{1}{6}\frac{abc}{ab+bc+ca}$

The values of φ follow.

The formula for $\lambda(X)$ is verified for each case separately.

3. THE ARCHIMEDEAN CASE

Let X be a compact and connected Riemann surface of genus 2. In this section we calculate the invariants $\varphi(X)$ and $\lambda(X)$ of X . Let $\text{Pic}(X)$ be the Picard variety of X , and for each integer d denote by $\text{Pic}^d(X)$ the component of $\text{Pic}(X)$ of degree d . We have a canonical theta divisor Θ on $\text{Pic}^1(X)$, and a standard hermitian metric $\|\cdot\|$ on the line bundle $\mathcal{O}(\Theta)$ on $\text{Pic}^1(X)$. Let ν be its curvature form. We have:

$$\int_{\text{Pic}^1(X)} \nu^2 = \Theta \cdot \Theta = 2.$$

Let K be a canonical divisor on X , and let \mathbf{P} be the set of 10 points P of $\text{Pic}^1(X) - \Theta$ such that $2P \equiv K$. Denote by $\|\theta\|$ the norm of the canonical section θ of $\mathcal{O}(\Theta)$. We let:

$$\|\Delta_2\|(X) = 2^{-12} \prod_{P \in \mathbf{P}} \|\theta\|^2(P),$$

the normalized modular discriminant of X , and we let $\|H\|(X)$ be the invariant of X defined by:

$$\log \|H\|(X) = \frac{1}{2} \int_{\text{Pic}^1(X)} \log \|\theta\| \nu^2.$$

These two invariants were introduced in [1].

Theorem 3.1. *For the φ -invariant and the λ -invariant of X , the formulas:*

$$\varphi(X) = -\frac{1}{2} \log \|\Delta_2\|(X) + 10 \log \|H\|(X)$$

and

$$10\lambda(X) = -20 \log(2\pi) - \log \|\Delta_2\|(X)$$

hold.

The key to the proof is the following lemma. Let Φ be the map:

$$X^2 \rightarrow \text{Pic}^1(X), \quad (x, y) \mapsto [2x - y].$$

Lemma 3.2. *The map Φ is finite flat of degree 8.*

Proof. Let $y \mapsto y'$ be the hyperelliptic involution of X . We have a commutative diagram:

$$\begin{array}{ccc} X^2 & \xrightarrow{\Phi} & \text{Pic}^1(X) \\ \alpha \downarrow & & \uparrow \beta \\ X^2 & \xrightarrow{\Phi^\vee} & \text{Pic}^3(X) \end{array}$$

where α and β are isomorphisms, with:

$$\begin{aligned} \alpha: X^2 &\rightarrow X^2, & \Phi^\vee: X^2 &\rightarrow \text{Pic}^3(X), & \beta: \text{Pic}^3(X) &\rightarrow \text{Pic}^1(X), \\ (x, y) &\mapsto (x, y'), & (x, y) &\mapsto [2x + y], & [D] &\mapsto [D - K]. \end{aligned}$$

It suffices to prove that Φ^\vee is finite flat of degree 8. Let $p: X^{(3)} \rightarrow \text{Pic}^3(X)$ be the natural map; then p is a \mathbf{P}^1 -bundle over $\text{Pic}^3(X)$, and Φ^\vee has a natural injective lift to $X^{(3)}$. A point D on $X^{(3)}$ is in the image of this lift if and only if D , when seen as an effective divisor on X , contains a point which is ramified for the morphism $X \rightarrow \mathbf{P}^1$ determined by the fiber $|D|$ of p in which D lies. Since every morphism $X \rightarrow \mathbf{P}^1$ associated to a D on $X^{(3)}$ is ramified, the map Φ^\vee is surjective. As every morphism $X \rightarrow \mathbf{P}^1$ associated to a D on $X^{(3)}$ has only finitely many ramification points, the map Φ^\vee is quasi-finite, hence finite since Φ^\vee is proper. As X^2 and $\text{Pic}^3(X)$ are smooth and the fibers of Φ^\vee are equidimensional, the map Φ^\vee is flat. By Riemann-Hurwitz the generic $X \rightarrow \mathbf{P}^1$ associated to a D on $X^{(3)}$ has 8 simple ramification points. It follows that the degree of Φ^\vee is 8. \square

Let $G: X^2 \rightarrow \mathbf{R}$ be the Arakelov-Green's function of X , and let Δ be the diagonal divisor on X^2 . We have a canonical hermitian metric on the line bundle $\mathcal{O}(\Delta)$ on X^2 by putting $\|1\|(x, y) = G(x, y)$, where 1 is the canonical section of $\mathcal{O}(\Delta)$. Denote by h_Δ the curvature form of $\mathcal{O}(\Delta)$. We have:

$$\int_{X^2} h_\Delta^2 = \Delta \cdot \Delta = -2.$$

Restricting $\mathcal{O}(\Delta)$ to a fiber of any of the two natural projections of X^2 onto X and taking the curvature form we obtain the Arakelov (1,1)-form μ on X . We have $\int_X \mu = 1$ and:

$$\int_X \log G(x, y) \mu(x) = 0$$

for each y on X . Let (ω_1, ω_2) be an orthonormal basis of $H^0(X, \omega_X)$, the space of holomorphic differentials on X . We can write explicitly:

$$h_\Delta(x, y) = \mu(x) + \mu(y) - i \sum_{k=1}^2 (\omega_k(x) \bar{\omega}_k(y) + \omega_k(y) \bar{\omega}_k(x))$$

and:

$$\mu(x) = \frac{i}{4} \sum_{k=1}^2 \omega_k(x) \bar{\omega}_k(x).$$

By [11, Proposition 2.5.3] we have:

$$\varphi(X) = \int_{X^2} \log G h_\Delta^2.$$

We compute the integral using our results from [4] and [5]. Let W be the divisor of Weierstrass points on X , and let $p_1: X^2 \rightarrow X$ be the projection onto the first

coordinate. The divisor W is reduced effective of degree 6. According to [3, p. 31] there exists a canonical isomorphism:

$$\sigma: \Phi^* \mathcal{O}(\Theta) \xrightarrow{\cong} \mathcal{O}(2\Delta + p_1^* W)$$

of line bundles on X^2 , identifying the canonical sections on both sides. In [4, Proposition 2.1] we proved that this isomorphism has a constant norm over X^2 . Thus, the curvature forms on both sides are equal:

$$\Phi^* \nu = 2h_\Delta + 6\mu(x) \quad \text{on } X^2.$$

Squaring both sides of this identity we get:

$$h_\Delta^2 = \frac{1}{4} \Phi^*(\nu^2) - 6h_\Delta \mu(x),$$

since $\mu(x)^2 = 0$. Denote by $S(X)$ the norm of σ . Then we have:

$$2 \log G(x, y) + \sum_w \log G(x, w) = \log \|\theta\|(2x - y) + \log S(X)$$

for generic $(x, y) \in X^2$, where w runs through the Weierstrass points of X . By fixing y and integrating against $\mu(x)$ on X we find that:

$$\log S(X) = - \int_X \log \|\theta\|(2x - y) \mu(x).$$

By integrating against h_Δ^2 on X^2 we obtain:

$$2\varphi(X) + \sum_w \int_{X^2} \log G(x, w) h_\Delta^2 = -2 \log S(X) + \int_{X^2} \log \|\theta\|(2x - y) h_\Delta^2.$$

As we have:

$$h_\Delta^2 = 2\mu(x)\mu(y) - \sum_{k,l=1}^2 (\omega_k(x)\bar{\omega}_l(x)\bar{\omega}_k(y)\omega_l(y) + \bar{\omega}_k(x)\omega_l(x)\omega_k(y)\bar{\omega}_l(y))$$

it follows that:

$$\int_{X^2} \log G(x, w) h_\Delta^2 = 0$$

for each w in W and hence we simply have:

$$2\varphi(X) = -2 \log S(X) + \int_{X^2} \log \|\theta\|(2x - y) h_\Delta^2.$$

Using our earlier expression for h_Δ^2 this becomes:

$$2\varphi(X) = -2 \log S(X) + \int_{X^2} \log \|\theta\|(2x - y) \left(\frac{1}{4} \Phi^*(\nu^2) - 6h_\Delta \mu(x) \right).$$

It is easily verified that $h_\Delta \mu(x) = h_\Delta \mu(y) = \mu(x)\mu(y)$ and hence:

$$\int_{X^2} \log \|\theta\|(2x - y) h_\Delta \mu(x) = \int_{X^2} \log \|\theta\|(2x - y) \mu(x)\mu(y) = - \log S(X).$$

From Lemma 3.2 it follows that:

$$\int_{X^2} \log \|\theta\|(2x - y) \Phi^*(\nu^2) = 8 \int_{\text{Pic}^1(X)} \log \|\theta\| \nu^2 = 16 \log \|H\|(X).$$

All in all we find:

$$\varphi(X) = 2 \log S(X) + 2 \log \|H\|(X).$$

Let $\delta_F(X)$ be the Faltings delta-invariant of X . According to [5, Corollary 1.7] the formula:

$$\log S(X) = -16 \log(2\pi) - \frac{5}{4} \log \|\Delta_2\|(X) - \delta_F(X)$$

holds, and in turn, according to [1, Proposition 4] we have:

$$\delta_F(X) = -16 \log(2\pi) - \log \|\Delta_2\|(X) - 4 \log \|H\|(X).$$

The formula:

$$\varphi(X) = -\frac{1}{2} \log \|\Delta_2\|(X) + 10 \log \|H\|(X)$$

follows.

By definition we have:

$$\lambda(X) = \frac{1}{30} \varphi(X) + \frac{1}{12} \delta_F(X) - \frac{2}{3} \log(2\pi)$$

so we obtain:

$$10\lambda(X) = -20 \log(2\pi) - \log \|\Delta_2\|(X)$$

by using [1, Proposition 4] once more.

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