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A flavour of family symmetries in a family of flavour models

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Citation

Adelhart Toorop, R. de. (2012, February 21). *A flavour of family symmetries in a family of flavour models*. Retrieved from <https://hdl.handle.net/1887/18506>

Version: Corrected Publisher's Version

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Note: To cite this publication please use the final published version (if applicable).

Chapter 3

Mixing patterns of finite modular groups

The most amazing thing happened to me to me tonight. I was coming here, on the way to the lecture, and I came in through the parking lot. And you won't believe what happened. I saw a car with the licence plate ARW 357. Can you imagine? Of all the millions of licence plates in the state, what was the chance that I would see that particular one tonight? Amazing!

(attributed to) Richard Feynman

3.1 Introduction

From bimaximal to tribimaximal to which neutrino mixing?

Slightly over a decade ago neutrino oscillations were discovered. Shortly after this observation, it was found that the PMNS neutrino mixing matrix can be characterized by two large mixing angles and one significantly smaller, possibly vanishing one. It was soon speculated that the neutrino mixing might correspond to a special pattern. Early data were in accordance with so-called bimaximal (BM) mixing [59–61] that is characterized by $\sin^2 \theta_{23}^l = \sin^2 \theta_{12}^l = 1/2$ and $\sin^2 \theta_{13}^l = 0$.

More precise measurements of the solar angle θ_{12}^l showed that the corresponding mixing is in fact not maximal; it is currently off by approximately 6σ . Instead, for a long time, the data were compatible with the tribimaximal (TBM) mixing pattern [8] of figure 1.18 and table 2.4. TBM mixing has $\sin^2 \theta_{12}^l = 1/3$ and the other two angles identical to the bimaximal values: the mixing corresponding to the atmospheric angle θ_{23}^l is still maximal and the reactor angle θ_{13}^l is still vanishing. Both the bimaximal mixing and the tribimaximal mixing can emerge when horizontal symmetry groups are assumed along the lines of section 2.4. In particular the small discrete groups A_4 and S_4 are often used. See e.g. [47,62] for reviews.

As mentioned in the previous chapter, recent data by the T2K collaboration gives evidence for non-zero reactor mixing angle. This signal is in accordance with earlier hints that a small but non-zero reactor angle can alleviate tensions between oscillation data extracted from KamLAND and from solar neutrinos and also in accordance with the first data of Double Chooz [63]. In table 2.4 we presented two recent independent fits of the neutrino oscillation data. In both fits, the hypothesis of zero θ_{13} is excluded by more than 3σ . The new neutrino oscillation data do not match the tribimaximal pattern anymore, at least not at the three sigma level. The detection of non-zero θ_{13} can lead to four directions in flavour-symmetry model-building space.

A first conclusion can be that the most important prediction of tribimaximal mixing, a vanishing reactor mixing angle, is not observed. The falsification of this most popular implementation of

flavour symmetries can be seen as a general argument against the use of flavour symmetries in general and a sign that one should give up on the approach. Secondly – and less negatively – one can keep the idea of flavour symmetries, but use them to build models that are less constraining in their predictions. An example is a model that provides tribimaximal mixing with a single finetuning instead of three as in the Standard Model. If this parameter is not exactly tuned this way, near-tribimaximal mixing results. The models of chapter 5 – motivated in a different way – are examples of such models.

Thirdly, one can try to reconcile the predictions with the data by allowing next-to-leading order (NLO) effects to alter the precise prediction. These corrections then modify the prediction of zero $\sin^2 \theta_{13}$ and, most likely, also modify the two other mixing angles. In models where quark and neutrino mixing have a common origin, NLO corrections in the quark sector also follow. At the end of section 2.4 we found that the requirement that the solar mixing angle should have only minor corrections in order to keep fitting the data, is quite restricting. Possibly, the old bimaximal mixing pattern is a better starting point: both the solar and the reactor angle are far away from the data at first order and they require comparable corrections in order to fit the data. This was done in [64] and is further explored in chapter 4.

A fourth possibility is to assume a different mixing pattern at leading order, preferably one that has a non-zero prediction for θ_{13} already at the lowest level. In this chapter we pursue this last option. We systematically consider all mixing patterns that are generated by a class of discrete groups of which A_4 and S_4 are the first members. See e.g. for [65, 66] for other scans of candidate groups and [67] for an analysis of how groups like these can naturally occur when an $SU(3)_F$ symmetry gets broken.

The key to generate this class of groups is in (2.73) and its analogue for S_4 . Both A_4 and S_4 are generated by two elements, generally called S and T that satisfy two relations that make them subgroups of the modular group Γ

$$S^2 = (ST)^3 = 1 . \quad (3.1)$$

The groups A_4 and S_4 are selected by additionally requiring a third relation between the generators. Both are of the form

$$T^N = 1 . \quad (3.2)$$

$N = 3$ corresponds to A_4 and $N = 4$ to S_4 . In section 3.2 we show that the list of groups generated by generalizing this relation is infinite, but that only a finite number of these contains three dimensional irreducible representations. For all these groups G_f we investigate in section 3.3 to which lepton mixing patterns it can naturally give rise when we assume that the flavour dynamics breaks the group, but ensures residual symmetries G_e and G_ν in respectively the charged lepton and the neutrino sectors.

As expected, tribimaximal mixing and bimaximal mixing occur in this list as well as several other candidate mixing-patterns. Some of these are closer to the data than others. Four mixing patterns are particularly interesting. These are related to two groups of the type $\Delta(6n^2)$ with respectively $n = 4$ and $n = 8$ (where $n = 2$ corresponds to S_4) that give rise to four mixing patterns that are quite close to the data as shown in table 3.1. In particular the patterns M3 and M4 are compatible with the present data at the 2σ level. Note that this is possible only because of the prediction of non-zero θ_{13}^l . We discuss the patterns M1 to M4 in more detail in section 3.4. In section 3.5 we present the conclusions of the chapter.

3.2 Finite modular groups and their representations

As mentioned in the introduction, the groups A_4 and S_4 have a common presentation in terms of two generators S and T satisfying for $N = 3$ or 4

$$S^2 = (ST)^3 = 1 \quad , \quad T^N = 1 . \quad (3.3)$$

For $N = 5$ the group A_5 follows. With this group added, the list contains the proper symmetry groups of the five platonic solids, that can be grouped into three dual pairs: cube/octahedron,

group	pattern	$\sin^2 \theta_{12}$	$\sin^2 \theta_{23}$	$\sin^2 \theta_{13}$
$\Delta(96)$	M1	$\frac{8-2\sqrt{3}}{13} \approx 0.349$	$\frac{5+2\sqrt{3}}{13} \approx 0.651$	$\frac{2-\sqrt{3}}{6} \approx 0.045$
	M2	$\frac{8-2\sqrt{3}}{13} \approx 0.349$	$\frac{8-2\sqrt{3}}{13} \approx 0.349$	$\frac{2-\sqrt{3}}{6} \approx 0.045$
$\Delta(384)$	M3	$\frac{4}{8+\sqrt{2}+\sqrt{6}} \approx 0.337$	$\frac{4-\sqrt{2}+\sqrt{6}}{8+\sqrt{2}+\sqrt{6}} \approx 0.424$	$\frac{4-\sqrt{2}-\sqrt{6}}{12} \approx 0.011$
	M4	$\frac{4}{8+\sqrt{2}+\sqrt{6}} \approx 0.337$	$\frac{4+2\sqrt{2}}{8+\sqrt{2}+\sqrt{6}} \approx 0.576$	$\frac{4-\sqrt{2}-\sqrt{6}}{12} \approx 0.011$
Central value [24]		0.312	0.420	0.025
Central value [27,28]		0.312	0.520	0.014

Table 3.1: Four of the mixing patterns produced by the groups we consider. In particular patterns M3 and M4 can be very close to the experimental data.

dodecahedron/icosahedron and the self-dual tetrahedron. Also the group A_5 has been used in flavour symmetric model building, see for instance [58], where the Golden Ratio mixing pattern is derived. It fits the data with a precision comparable to the tribimaximal pattern and has the same $\theta_{13} = 0$ prediction.

A natural question is whether these presentations extend to other finite groups for $N > 5$. It turns out that this is possible, although for $N \geq 7$ a fourth relation will be required next to the three already given to ensure that the groups considered are finite [68]. Though not excluded as candidates for a flavour symmetry, infinite discrete groups have the disadvantage that they can possess infinitely many irreducible representations of a given dimensionality, which makes them less appealing for model building as they eventually allows to reproduce any mixing pattern. In particular, we will be interested in the groups in this series that have three dimensional representations [69], as these can then be used to build models of lepton mixing.

We will embed the previous groups into an infinite set of finite groups, related to the modular group. To be able to study this set, we first study the modular group (over \mathbf{Z}) itself, which is an discrete *infinite* group.

The (inhomogeneous) modular group Γ is the group of linear fractional transformations acting on a variable z :

$$z \rightarrow \frac{az + b}{cz + d} . \quad (3.4)$$

The parameters a, b, c and d are integers and $ad - bc = 1$. Obviously, a transformation described by parameters $\{a, b, c, d\}$ is identical to a transformation defined by $\{-a, -b, -c, -d\}$.

As is clear from the definition, Γ is isomorphic to the group $PSL(2, \mathbf{Z}) = SL(2, \mathbf{Z})/\{\pm I\}$. Here $SL(2, \mathbf{Z})$ (the *homogeneous* modular group) is the group of 2×2 matrices with integer entries and determinant equal to one and to get $PSL(2, \mathbf{Z})$ (the *inhomogeneous* modular group), two matrices that only differ by an overall sign are identified as M and $-M$ determine the same transformation.

The modular group Γ is generated by two elements S and T satisfying [70]:

$$S^2 = (ST)^3 = 1 . \quad (3.5)$$

Note again that these relations are also satisfied by the generators of A_4 , S_4 and A_5 , although they are not sufficient to define these groups uniquely. With respect to the behaviour of the parameter z , S and T can be represented by the transformations

$$S : z \rightarrow \frac{-1}{z} . \quad T : z \rightarrow z + 1. \quad (3.6)$$

This corresponds to the two following matrices of $SL(2, Z)$:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (3.7)$$

We now generalize this construction by replacing integers by integers modulo N . Given a natural number $N > 1$, the homogeneous finite modular group $SL(2, Z_N)$ is defined as the group of 2×2 matrices with entries that are integers modulo N and determinant equal to one modulo N . Again, the inhomogeneous groups are defined by identifying in $SL(2, Z_N)$ two opposite matrices

$$\Gamma_N \sim PSL(2, Z_N) \equiv SL(2, Z_N)/\{\pm I\}.$$

For each N these groups are finite.

The order of the homogeneous finite modular groups $SL(2, Z_N)$ is [70, 71]

$$\left| SL(2, Z_N) \right| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2} \right) \quad (3.8)$$

The product extends to the prime p divisors of N . For $N = 2$, the identity I and its opposite $-I$ are indistinguishable and therefore $\Gamma_2 \equiv SL(2, Z_2)$. For $N > 2$ they are distinguishable and the order of the inhomogeneous groups is half of that of the homogeneous ones

$$\left| \Gamma_N \right| = \frac{1}{2} \left| SL(2, Z_N) \right|. \quad (3.9)$$

In table 3.2 we list the order of $SL(2, Z_N)$ and Γ_N , as well as the groups Γ_N is isomorphic to, for $2 \leq N \leq 11$

N	2	3	4	5	6	7	8	9	10	11
$SL(2, Z_N)$	6	24	48	120	144	336	384	648	720	1320
Γ_N	6	12	24	60	72	168	192	324	360	660
$\Gamma_N \sim$	S_3	A_4	S_4	A_5	$(S_3 \times A_4)$	$PSL(2, 7)$			$(S_3 \times A_5)$	$PSL(2, 11)$

Table 3.2: Properties of $SL(2, Z_N)$ and Γ_N for $2 \leq N \leq 11$.

The group Γ_2 has six elements and coincide with the permutation group S_3 . For $N = 3, 4, 5$, the groups Γ_N coincide with the proper symmetry groups of the Platonic solids and we have [71]: $\Gamma_3 \sim A_4$, $\Gamma_4 \sim S_4$ and $\Gamma_5 \sim A_5$. Our proposal is to investigate the whole series Γ_N . Notice that, if we regard the matrices S and T of equation (3.7) as representative of elements of Γ_N , we find that, besides the relations in equation (3.5), they also satisfy $T^N = 1$. However, in general, further relations are required to define the complete presentation of the corresponding group. For instance, Γ_7 is characterized by

$$S^2 = (ST)^3 = 1, \quad T^7 = 1, \quad (ST^{-1}ST)^4 = 1. \quad (3.10)$$

The group Γ_7 is isomorphic to the group¹ $PSL(2, 7)$. Note that without including the last relation in (3.10), the matrices S and T can generate a group of infinite order.

The group $SL(2, Z_N)$ is a double covering of the group Γ_N , for $N > 2$. There is a homomorphism between these two groups and the inhomogeneous group, Γ_N , can be regarded as an unfaithful copy of its homogeneous counterpart $SL(2, Z_N)$ (except for $N = 2$ where the relation is an isomorphism). As a consequence, all irreducible representations of Γ_N are also irreps of $SL(2, Z_N)$.

¹The 7 in $PSL(2, 7)$ stands for the Galois field of order 7, so a better notation would be $PSL(2, GF_7)$. It is non-trivial that $PSL(2, Z_7)$ is isomorphic to $PSL(2, GF_7)$. Actually, only for prime N the numbers 0 to $(N - 1)$ with addition and multiplication modulo N form a finite field.

In the following we will recall the complete classification of the irreducible representations of $SL(2, Z_N)$. In this way we will obtain all the representations of the group we are interested in, Γ_N , plus additional representations that we will discard.

We now like to find all three-dimensional representations of $SL(2, Z_N)$ (and thus of Γ_N). To study the irreps of $SL(2, Z_N)$, we consider the three possible situations for N

- 1) N is prime.
- 2) N is a power of a prime.
- 3) N is the product $\prod_p p^{\lambda_p}$ of primes and powers of primes.

We start with the case where N is a prime p . As remarked before, if $p = 2$, we have $\Gamma_2 = S_3$, which has two one-dimensional and one two-dimensional representations, but no three-dimensional ones. The dimensions d and multiplicities μ of the irreducible representations of $SL(2, Z_p)$ (p an odd prime) as a function of p are given in table 3.3.

We find that $SL(2, Z_N)$ has three dimensional irreps only for the primes 3, 5 and 7. The only related Γ_N groups that can have three dimensional irreps are thus A_4 , A_5 and $PSL(2, 7)$. We indeed find that A_4 has a three dimensional irrep, while A_5 and $PSL(2, 7)$ have two, in the latter case, a complex-conjugate pair. These representations are explicitly given in table 3.4 in a basis where T is diagonal.

d	μ	$d = 3$ if	μ if $d = 3$	related Γ_n .
1	1*	-	-	-
$p + 1$	$\frac{1}{2}(p - 3)$	$p \neq 2$	-	-
p	1	$p = 3$	1	A_4
$p - 1$	$\frac{1}{2}(p - 1)$	$p \neq 4$	-	-
$\frac{1}{2}(p + 1)$	2	$p = 5$	2	A_5
$\frac{1}{2}(p - 1)$	2	$p = 7$	2	$PSL(2, 7)$

Table 3.3: Dimensions d and multiplicities μ of the irreducible representations of $SL(2, Z_p)$, p being an odd prime and the possibility to have three dimensional representations. If the candidate three-dimensional representation is related to even p in the third column, it is crossed. *In the case $p = 3$, there are two extra 1d representations from the last row.

Next, we consider the case where N is a power p^λ of a prime. We separately discuss the cases where p is an odd prime and where $p = 2$. In table 3.5 we list the irreducible representations d of $SL(2, Z_{p^\lambda})$ with $p > 2$ and $\lambda > 1$, and the multiplicities μ of these representations. This table should be understood as follows. Given an integer $\bar{\lambda} > 1$, all groups $SL(2, Z_{p^\lambda})$ with $\lambda < \bar{\lambda}$ are unfaithful copies of the group $SL(2, Z_{p^{\bar{\lambda}}})$. It follows that the representations of $SL(2, Z_{p^\lambda})$ with $1 \leq \lambda < \bar{\lambda}$ are also representations of the group $SL(2, Z_{p^{\bar{\lambda}}})$. The irreducible representations of $SL(2, Z_{p^{\bar{\lambda}}})$ are given by those of table 3.3 ($\lambda = 1$) and by those listed in table 3.5, with $\lambda = 2, \dots, \bar{\lambda}$. For instance, if $\lambda = 3$ we should include both $\lambda = 2$ and $\lambda = 3$ from table 3.5. As a check we can compute the order of $SL(2, Z_{p^{\bar{\lambda}}})$ from tables 3.3 and 3.5 and we find $p^{3\bar{\lambda}}(1 - 1/p^2)$ in agreement with eq. (3.8). From table 3.5 it is easy to prove that for $p > 2$ and $\lambda > 1$ there are no other irreducible three-dimensional representations, apart from those already given in table 3.3. This concludes the discussion for $p > 2$.

The case $p = 2^\lambda$ is more complicated and a separate discussion for each λ is needed. Again the representations of $SL(2, Z_{2^\lambda})$ are also representations of the group $SL(2, Z_{2^{\bar{\lambda}}})$ for $1 \leq \lambda < \bar{\lambda}$. For $\lambda > 4$ there are no three-dimensional irreducible representations, different from those already induced by $\lambda = 2, 3, 4$ [72–74]. In table 3.6 we summarize the irreducible representations of $SL(2, Z_{2^\lambda})$ ($\lambda = 1, 2, 3, 4$). We conclude that Γ_4 , Γ_8 and Γ_{16} can give “new” three dimensional irreducible representations and are interesting from a model building point of view.

N	S	$\frac{1}{2\pi i} \log(T)$
3	$\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\text{diag}(0, \frac{1}{3}, \frac{2}{3})$
5	$\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\phi & \frac{1}{\phi} \\ \sqrt{2} & \frac{1}{\phi} & -\phi \end{pmatrix}$	$\text{diag}(0, \frac{1}{5}, \frac{4}{5})$
	$-\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \frac{1}{\phi} & -\phi \\ \sqrt{2} & -\phi & \frac{1}{\phi} \end{pmatrix}$	$\text{diag}(0, \frac{2}{5}, \frac{3}{5})$
7	$\frac{2}{\sqrt{7}} \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & -s_3 & s_1 \\ s_3 & s_1 & -s_2 \end{pmatrix}$	$\text{diag}(\frac{2}{7}, \frac{1}{7}, \frac{4}{7})$
	$\frac{2}{\sqrt{7}} \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & -s_3 & s_1 \\ s_3 & s_1 & -s_2 \end{pmatrix}$	$\text{diag}(\frac{5}{7}, \frac{6}{7}, \frac{3}{7})$

Table 3.4: Three dimensional irreducible representations of Γ_N , $N = 3, 5, 7$. We have defined $\phi \equiv \frac{1+\sqrt{5}}{2}$ and $s_j \equiv \sin(\frac{j\pi}{7})$.

d	$p^{\lambda-1}(p+1)$	$p^{\lambda-1}(p+1)$	$\frac{1}{2}p^{\lambda-2}(p^2-1)$
μ	$\frac{1}{2}p^{\lambda-2}(p-1)^2$	$\frac{1}{2}p^{\lambda-2}(p^2-1)$	$4p^{\lambda-1}$

Table 3.5: Dimensions d and multiplicities μ of the "new" irreducible representations of $SL(2, Z_{p^\lambda})$, p being an odd prime and $\lambda > 1$. See the text for explanations.

Lastly, we consider the case where N is a product of primes and powers of primes

$$N = \prod_p p^{\lambda_p}. \quad (3.11)$$

Now the group $SL(2, Z_N)$ factorizes as

$$SL(2, Z_N) = \prod_p SL(2, Z_{p^{\lambda_p}}) \quad (3.12)$$

Two examples can be checked on the second line of table 3.2: $SL(2, Z_6) = SL(2, Z_2) \times SL(2, Z_3)$ (the number of elements satisfies $144 = 6 \times 24$) and $SL(2, Z_{10}) = SL(2, Z_2) \times SL(2, Z_5)$ (where the number of elements satisfies $720 = 6 \times 120$). Due to the special position of Γ_2 , in these two cases, also the relations $\Gamma_6 = \Gamma_2 \times \Gamma_3 = S_3 \times A_4$ and $\Gamma_{10} = \Gamma_2 \times \Gamma_5 = S_3 \times A_5$ hold, as given on the lowest line.

Clearly, three dimensional representations of these product groups can be constructed using the three-dimensional representation of one of the groups and one-dimensional representations of all the others. Therefore, the cases where N is a product do not give new patterns.

In conclusion all independent three-dimensional representations of the finite modular groups can be studied by considering the six groups $SL(2, Z_N)$ ($N = 3, 4, 5, 7, 8, 16$). We have 33 distinct irreducible triplets. From table 3.3 we see that one is associated to $N = 3$, two are related to $N = 5$ and two to $N = 7$. Moreover, from table 3.6, we see that 4 irreducible triplets corresponds to $N = 4$, while $N = 8$ introduces 8 new irreducible triplets and finally 16 other independent irreducible triplets are associated to $N = 16$.

The full list of these 33 triplets is explicitly given in Appendix A of ref. [74], in terms of the S and T elements, in the basis where the T generator is diagonal. Of these 33 $SL(2, Z_N)$ representations

d	1	2	3	4	6	8	12	24	Order	Note
$SL(2, Z_2)$	2	1							6	Isomorphic to S_3
$SL(2, Z_4)$	4	2	4						48	Double cover of S_4
$SL(2, Z_8)$	4	6	12	2	6				384	
$SL(2, Z_{16})$	4	6	28	2	26	6	2	2	3072	

Table 3.6: Dimensions d and multiplicities of the irreducible representations of $SL(2, Z_{2^\lambda})$, for $\lambda < 5$. For each group all the irreducible representations are listed [72–74].

only a smaller subset are also representations of the corresponding inhomogeneous group Γ_N : those satisfying the relations in eq. (3.5). They are 19 and we collect them in table 3.4 and 3.7. In the latter table, the elements S are given in terms of a matrix

$$S \equiv \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix} \quad (3.13)$$

N	S	$\frac{1}{2\pi i} \log(T)$
4	S	$\text{diag}(0, \frac{1}{4}, \frac{3}{4})$
	$-S$	$\text{diag}(\frac{2}{4}, \frac{3}{4}, \frac{1}{4})$
8	S	$\text{diag}(\frac{6}{8}, \frac{7}{8}, \frac{3}{8})$
	S	$\text{diag}(\frac{2}{8}, \frac{5}{8}, \frac{1}{8})$
	$-S$	$\text{diag}(\frac{6}{8}, \frac{1}{8}, \frac{5}{8})$
	$-S$	$\text{diag}(\frac{2}{8}, \frac{3}{8}, \frac{7}{8})$
16	S	$\text{diag}(\frac{14}{16}, \frac{5}{16}, \frac{13}{16})$
	S	$\text{diag}(\frac{2}{16}, \frac{11}{16}, \frac{3}{16})$
	$-S$	$\text{diag}(\frac{6}{16}, \frac{13}{16}, \frac{5}{16})$
	$-S$	$\text{diag}(\frac{10}{16}, \frac{3}{16}, \frac{11}{16})$
	S	$\text{diag}(\frac{10}{16}, \frac{15}{16}, \frac{7}{16})$
	S	$\text{diag}(\frac{6}{16}, \frac{1}{16}, \frac{9}{16})$
	$-S$	$\text{diag}(\frac{2}{16}, \frac{7}{16}, \frac{15}{16})$
	$-S$	$\text{diag}(\frac{14}{16}, \frac{9}{16}, \frac{1}{16})$

Table 3.7: Three dimensional irreducible representations of Γ_N , $N = 4, 8, 16$. The matrix S is defined in the text.

In the next section we will study the application of Γ_N with $N = 3, 4, 5, 7, 8$ or 16 to the lepton sector. We suppose that the group functions as a horizontal symmetry group, that after it gets broken leaves a residual G_e symmetry in the charged lepton sector and a residual G_ν in the neutrino sector. Both G_e and G_ν are expressed in the generators S and T .

In the cases $N = 3, 4, 5$ and 7 , this is straightforward, but in the cases $N = 8$ and 16 an extra complication occurs. In these cases, the representations given by S and T are not faithful and generate subgroups of Γ_8 and Γ_{16} of order 96 and 384 respectively. These are the groups $\Delta(96)$ and $\Delta(384)$ that we study in sections 3.3.5 and 3.3.6. They belong to the series $\Delta(n^2)$ [75–77] that also $\Delta(24) \sim S_4$ is part of and are isomorphic to the semi-direct products of $Z_2 \times Z_2$ and S_3 .

3.3 Lepton mixing patterns from Γ_N

In this section we classify the lepton mixing patterns arising from the candidate flavour symmetry groups mentioned in the previous section. We have $G_f = \Gamma_N$ for $N = 3, 4, 5$ and 7 , while for $N = 8$ and 16 G_f is the subgroup of Γ_N that can be constructed from the S and T generators mentioned in table 3.7.

We aim at a complete classification under the following rules. We work in a certain leading order approximation, where the neutrino mass matrix and the charged lepton mass matrix are separately invariant under the subgroups G_ν and G_e of G_f , respectively. As has been discussed in detail in the literature [78, 79], this framework in which the misalignment between neutrino and charged lepton mass matrices is associated with the non-trivial breaking of a flavour symmetry is particularly interesting and predictive. Given G_f , we will scan all possible subgroups G_ν and G_e with the following restrictions.

Firstly we assume that neutrinos are Majorana particles as strongly hinted to by their light masses. The Majorana character of neutrinos shows up in the choice of G_ν . With a single generation, the only transformation of a Majorana neutrino leaving invariant its mass term is a change of sign. If there are three generations, it can be shown [78, 80] that the appropriate invariance group of the neutrino sector generalizes to the product of two commuting parities, the Klein group $Z_2 \times Z_2$, allowing for an independent relative change of sign of any neutrino.

Secondly, we discard non-Abelian residual symmetries for the charged leptons since the non-Abelian character of the subgroup would result in a complete or partial degeneracy of the mass spectrum, which is not in accordance with the hierarchy among the charged lepton masses. Hence, we choose G_e to be a cyclic group Z_n or a subgroup of these groups, such as $Z_2 \times Z_2$.

Thirdly, to minimize double counting and to select only those mixing patterns that reflect the properties of the full group G_f , we will ask that the subgroups $G_\nu = Z_2 \times Z_2$ and G_e generate the full group G_f .

Once we specify a three-dimensional representation ρ of G_f for the lepton doublets, the elements $g_{\nu i}$ of the subgroup G_ν and g_{ei} of the subgroup G_e are given by matrices $\rho(g_{\nu i})$ and $\rho(g_{ei})$. These matrices leave the neutrino mass matrix m_ν and the combination $(m_e^\dagger m_e)$ invariant².

$$\rho(g_{\nu i})^T m_\nu \rho(g_{\nu i}) = m_\nu, \quad \rho(g_{ei})^\dagger (m_e^\dagger m_e) \rho(g_{ei}) = (m_e^\dagger m_e) . \quad (3.14)$$

The matrices $\rho(g_{\nu i})$ and $\rho(g_{ei})$ can be diagonalized by two unitary transformations Ω_ν and Ω_e . This follows from the facts that ρ is a unitary representation and that G_ν and G_e are Abelian

$$\rho(g_{\nu i})_{\text{diag}} = \Omega_\nu^\dagger \rho(g_{\nu i}) \Omega_\nu, \quad \rho(g_{ei})_{\text{diag}} = \Omega_e^\dagger \rho(g_{ei}) \Omega_e . \quad (3.15)$$

By the above invariance requirements Ω_ν and Ω_e are also the transformations that diagonalize m_ν and $(m_e^\dagger m_e)$, respectively.

$$(m_\nu)_{\text{diag}} = \Omega_\nu^T m_\nu \Omega_\nu, \quad (m_e^\dagger m_e)_{\text{diag}} = \Omega_e^\dagger (m_e^\dagger m_e) \Omega_e . \quad (3.16)$$

The lepton mixing matrix is, up to phase redefinitions, given by

$$U_{\text{PMNS}} = \Omega_e^\dagger \Omega_\nu . \quad (3.17)$$

Phase redefinitions $\Omega_e \rightarrow \Omega_e K_e$ and $\Omega_\nu \rightarrow \Omega_\nu K_\nu$, with K_e and K_ν diagonal matrices of phases, can be used to make the eigenvalues of m_ν real and positive and to eliminate all but three phases in U_{PMNS} . One of these is the Dirac CP phase δ_{CP}^l that can be measured in neutrino oscillations (at least if $\theta_{13}^l \neq 0$); two others are Majorana phases. These cannot be predicted in our approach as

²In our convention, $SU(2)_L$ doublets are on the right of m_e

the neutrino masses remain unconstrained by the above requirements. When relevant, we report $|\sin \delta_{\text{CP}}^l|$ and the Jarlskog CP-invariant of equation (2.34).

The fact that the actual neutrino and charged lepton masses are not fixed by the requirements (3.14) and (3.15), means that these relations only fix the lepton mixing matrix up to interchange of rows and columns. We can use this to our advantage to obtain a U_{PMNS} that is in closest agreement with the data. Even if there is evidence of non-zero θ_{13}^l now, this angle is very small. We recall from (the lepton analogue of) equation (2.32) that the sine of this angle is given by the absolute value of the (1 3) element of the mixing matrix and therefore, we choose to represent the smallest element of the mixing matrices at the (1 3) position. Now we are just allowed an extra interchange of the first and second column and of the second and third row.

We choose to order the first and second columns such that the (1 1) element is larger in absolute value than the (1 2) element. Equation (2.32) tells us that $\tan \theta_{12}^l$ is then smaller than 1 and this implies $\sin^2 \theta_{12}^l < 1/2$ in accordance with the data in table 2.4. Next, we mention a slight tension between the two global neutrino oscillation fits reported in table 2.4. The fit [24] seems to point at a value of $\sin^2 \theta_{23}$ smaller than 1/2, implying that the (2 3) element of the PMNS matrix is larger than the (3 3) element. On the other hand the fit [27, 28] very slightly favours $\sin^2 \theta_{23} > 1/2$, which can be reproduced if the two above mentioned elements are ordered the other way around.

Equations (3.14) and (3.15) also show that the PMNS mixing matrix is invariant, when the matrices representing the elements of G_ν and/or G_e change overall sign. Furthermore, when these matrices are complex conjugated, the lepton mixing matrix becomes conjugated as well. We do not discuss these cases separately.

In the next six subsections, we systematically consider the cases $N = 3, 4, 5, 7, 8$ and 16. In each of the cases we present the conjugacy classes of the group with their order and a list of the Abelian subgroups that the group possesses. We consider all possible Abelian subgroups G_e that the charged lepton sector can be invariant under (as explained above, G_ν of the neutrino sector is fixed to be one of the Klein groups). For all these cases, we find the corresponding lepton mixing matrix and we comment on the compatibility with the data of these patterns.

3.3.1 The group $\Gamma_3 \sim A_4$

The group A_4 is the group of even permutations of four elements. Directly related, A_4 is also the symmetry group of the regular tetrahedron, the simplest of the Platonic solids. The group has three inequivalent one-dimensional irreducible representations 1, 1' and 1'' and one three dimensional irrep. As explained in section 3.2, it can be generated by elements S and T that satisfy

$$S^2 = (ST)^3 = T^3 = 1. \quad (3.18)$$

In the T -diagonal (Altarelli–Feruglio) basis of the three dimensional representation, S and T are given in table 3.4. The twelve elements of the group A_4 are members of four classes, with order 1, 2, 3 and 3 respectively as shown in table 3.8. We name these classes $a\mathcal{C}_b$, where a refers to the number of elements and b the order.

Class	Order	# Elements	Elements
$1\mathcal{C}_1$	1	1	E
$3\mathcal{C}_2$	2	3	S, T^2ST, ST^2ST
$4\mathcal{C}_3^{(1)}$	3	4	T, ST, TS, STS
$4\mathcal{C}_3^{(2)}$	3	4	$T^2, (ST)^2, (TS)^2, (STS)^2$

Table 3.8: Conjugacy classes of A_4

There is a unique $Z_2 \times Z_2$ Klein group, that is equal to the class of order 2 and there are four Z_3 s. We

can assume that each of these is generated by an element of the first class of order 3 as shown in table 3.9.

Subgroup		Generators
$Z_2 \times Z_2$	K	S, T^2ST
Z_3	C_1	T
	C_2	ST
	C_3	TS
	C_4	STS

Table 3.9: Abelian subgroups of A_4 and their generators.

The fact that there is only a single Klein group, automatically fixes G_ν . It also forces G_e to be one of the Z_3 groups, as it cannot also be the Klein group. The choices of G_e are all equivalent. In all cases the Klein group and the Z_3 -group together generate the full group A_4 and we obtain the so-called magic mixing matrix as the lepton mixing matrix [81,82].

$$U_{\text{PMNS}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \quad (3.19)$$

In this mixing pattern, both the solar and atmospheric mixing are maximal and θ_{13} fulfills $\sin^2 \theta_{13} = 1/3$. This pattern also leads to a Dirac CP phase $|\delta_{CP}| = \pi/2$ and $|J_{CP}| = 1/(6\sqrt{3}) \approx 0.096$.

We comment about the fact that in this scheme A_4 cannot reproduce the tribimaximal mixing. However there are many models in the literature that obtain TBM mixing from A_4 , including the model of section 2.4. The key is that in these models, the neutrinos are invariant under a Klein group generated not only by S , but also by the so-called $(\mu \tau)$ -invariant matrix U

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3.20)$$

This matrix is not an element of A_4 . Invariance under U can therefore not be forced by A_4 . It can however appear as an accidental symmetry because of the matter content of a specific model and/or auxiliary symmetries that are present next to A_4 . The matrix U is actually an element of S_4 and since this group also has a Z_3 subgroup, we will find tribimaximal mixing in the next section. The combination of S_4 and tribimaximal mixing is indeed also often seen in the literature.

The group A_4 appears in many places in this thesis and details of its group theory, such as Clebsch Gordan rules are thus often needed. We derive those properties in appendix 3.A that follows this chapter.

3.3.2 The group $\Gamma_4 \sim S_4$

The next group we consider is S_4 . It is the symmetric group of (even and odd) permutations of four elements. As such, it is twice as large as A_4 , having twenty-four elements. S_4 is also the symmetry group of two of the Platonic solids, the cube and the octahedron. It has five irreps: two one-dimensional, one two-dimensional and two inequivalent three-dimensional. The group can be generated by two elements S and T that satisfy

$$S^2 = (ST)^3 = T^4 = 1. \quad (3.21)$$

Specific realizations of S and T in the three-dimensional representation are given in table 3.7. We see that the two triplets are related by an overall change of sign of both $\rho(S)$ and $\rho(T)$. As explained in the introduction of this section these two cases lead to equivalent lepton mixing matrices and do not need to be discussed separately.

The group S_4 has five conjugacy classes, whose elements are listed in table 3.10. The Abelian subgroups are Klein groups as well as groups of order 3 and 4, as given in table 3.11. One of the Klein groups K is a conjugacy class on itself, making it a normal subgroup.

Class	Elements
$1\mathcal{C}_1$	E
$6\mathcal{C}_2$	$S, ST^2ST, T^3ST, T^2ST^2, ST^3ST^2, ST^2ST^3$
$3\mathcal{C}_2$	T^2, ST^2ST^2, ST^2S
$8\mathcal{C}_3$	$ST, ST^3, TS, T^3S, T^2ST, ST^3ST, T^3ST^2, T^2ST^3$
$6\mathcal{C}_4$	$T, ST^2, T^3, STS, T^2S, ST^3S$

Table 3.10: Conjugacy classes of S_4

Subgroup		Generators
$Z_2 \times Z_2$	K	T^2, ST^2S
	K_1	S, ST^2ST^2
	K_2	T^2, ST^2ST^3
	K_3	ST^2S, ST^3ST^2
Z_3	C_1	ST
	C_2	TS
	C_3	T^3ST^2
	C_4	T^2ST^3
Z_4	Q_1	T
	Q_2	ST^2
	Q_3	STS

Table 3.11: Abelian subgroups $Z_2 \times Z_2$, Z_3 and Z_4 of S_4 and their generators. K is a normal subgroup.

The possible mixing patterns are listed below. From table 3.11 we see that we have several different possible choices for G_ν and G_e . Some of them do not generate the entire group S_4 and can be disregarded. For instance K together with any other subgroup in table 3.11 always generate a group smaller than S_4 .

In the next three paragraphs, we consider the three choices of G_e : it can be a Z_3 , Z_4 or Klein subgroup. We focus on choices of the G_e and G_ν where it is possible to generate the full group S_4 . Because of the potential presence of many phases in the lepton mixing matrix, we report the absolute values of its elements. This is enough to calculate all three mixing angles.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_3$

When the lepton sector is invariant under an element of order 3, the full group S_4 can be generated if neutrino sector is invariant under a Klein group K_i , but not K . We then obtain the tribimaximal mixing pattern, as anticipated upon in the discussion of A_4 .

$$\|U_{\text{PMNS}}\| = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (3.22)$$

When we pick the invariant Klein group K for the neutrino sector, it is not possible to generate the full group S_4 , but a subgroup is generated instead. In this case, the subgroup is A_4 and the corresponding mixing pattern is the magic matrix of (3.19).

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_4$

If we assume the lepton sector to be invariant under an element of order 4, the mixing pattern obtained can be

$$||U_{\text{PMNS}}|| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (3.23)$$

An example is when $G_\nu = K_1$ and $G_e = Q_1$. This mixing pattern is bimaximal. As mentioned in the introduction, the mixing pattern itself does not describe the data well, as the solar angle is maximal and thus too large. However, with next-to-leading order corrections included, it may be very relevant, in particular since the order of magnitude of the correction to the solar and reactor angle are similar. Furthermore, the corrections in the lepton sector might be related to the generation of the Cabibbo angle, which is also approximately of the same size via quark-lepton complementarity; see e.g. [83–87] and the discussion in chapter 4.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_2 \times Z_2$

When both the neutrino sector and the charged lepton sector are assumed to be invariant under (different) Klein groups, the lepton mixing matrix can be the same as in the previous case. When the full group S_4 is generated, for instance by K_1 and K_2 , the mixing pattern is the bimaximal mixing of equation (3.23).

This concludes the discussion for S_4 . The group can be used to reproduce two of the most popular mixing patterns in the literature, tribimaximal and bimaximal mixing. In the case of bimaximal mixing, this is possible in two ways, with very different residual symmetries in the charged lepton sector. As for A_4 , the group theory of S_4 will be important in many places of this thesis, in particular in chapter 4. We derive the Clebsch-Gordan rules and transformation rules of different bases into each other in appendix 3.B.

3.3.3 The group $\Gamma_5 \sim A_5$

The third subgroup of the modular group is A_5 . It can be defined as the group of even permutations of 5 elements and has 60 elements and five conjugacy classes. Relevant Abelian subgroups are $Z_2 \times Z_2$, Z_3 and Z_5 . In total, there are 21 of them. Due to the growing number of elements, we no longer list all classes and generators of Abelian subgroups. Instead, these are collected in appendix 3.C at the end of this chapter.

The group has two inequivalent irreducible triplets. Two candidates for $\rho(S)$ and $\rho(T)$ are mentioned in table 3.4. We checked that if the set $\{S, T\}$ is given by one pair of matrices given in the table, that then the others can be written as $\{S' = T^2 S T^3 S T^2, T' = T^2\}$ and that this is a second independent representation that satisfies the algebra

$$S^2 = (ST)^3 = T^5 = 1. \quad (3.24)$$

In the following three subsections, we describe the lepton mixing when the neutrino residual symmetry is a Klein groups and the one of the charged leptons is either Z_3 , Z_5 or $Z_2 \times Z_2$.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_3$

When we take G_ν a Klein group and we take G_e generated by an element of order 3, we get the mixing pattern

$$\|U_{\text{PMNS}}\| = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2}\phi & \frac{\sqrt{2}}{\phi} & 0 \\ \frac{1}{\phi} & \phi & \sqrt{3} \\ \frac{1}{\phi} & \phi & \sqrt{3} \end{pmatrix} \approx \begin{pmatrix} 0.934 & 0.357 & 0 \\ 0.252 & 0.661 & 0.707 \\ 0.252 & 0.661 & 0.707 \end{pmatrix}. \quad (3.25)$$

This mixing pattern is for instance generated by $G_\nu = K_1$ and $G_e = C_1$. The mixing angles are vanishing θ_{13} and maximal θ_{23} together with $\sin^2 \theta_{12} = \frac{1}{3}(2 - \phi) = \frac{1}{6}(3 - \sqrt{5}) \approx 0.127$. Obviously, $J_{CP} = 0$. This is the pattern mentioned by Lam [80]. It would need large corrections to the solar mixing angle in order to match the current data.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_5$

When the lepton sector is invariant under an element of order 5, the mixing pattern becomes

$$\|U_{\text{PMNS}}\| = \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} & 0 \\ \sin \theta_{12}/\sqrt{2} & \cos \theta_{12}/\sqrt{2} & 1/\sqrt{2} \\ \sin \theta_{12}/\sqrt{2} & \cos \theta_{12}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \approx \begin{pmatrix} 0.851 & 0.526 & 0 \\ 0.372 & 0.602 & 0.707 \\ 0.372 & 0.602 & 0.707 \end{pmatrix}. \quad (3.26)$$

with $\tan \theta_{12} = 1/\phi$. This pattern is generated for any choice of G_ν and G_e . Again, we find vanishing θ_{13} and maximal atmospheric mixing θ_{23} , this time together with $\sin^2 \theta_{12} \approx 0.276$. Obviously $J_{CP} = 0$. This mixing pattern was discussed before e.g. in [58,88]. As mentioned at the beginning of section 3.2, its agreement with the data is of the same order as the tribimaximal mixing pattern.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_2 \times Z_2$

In case both the neutrino sector and the charged lepton sector are invariant under a Klein group, the mixing reads

$$\|U_{\text{PMNS}}\| = \frac{1}{2} \begin{pmatrix} \phi & 1 & \phi^{-1} \\ \phi^{-1} & \phi & 1 \\ 1 & \phi^{-1} & \phi \end{pmatrix} \approx \begin{pmatrix} 0.809 & 0.5 & 0.309 \\ 0.309 & 0.809 & 0.5 \\ 0.5 & 0.309 & 0.809 \end{pmatrix}. \quad (3.27)$$

Excluding the case in which G_ν and G_e are the same group, we always find the pattern in equation(3.27) to be generated. The mixing angles which can be extracted from equation(3.27) are: $\sin^2 \theta_{13} \approx 0.095$ and $\sin^2 \theta_{12} = \sin^2 \theta_{23} \approx 0.276$. Interchange of the second and third row gives $\sin^2 \theta_{23} \approx 0.724$. Also in this case there is no non-trivial Dirac CP phase, i.e. $J_{CP} = 0$. We mention that although the (1 2)-angle is in good agreement with the data, $\sin^2 \theta_{13}$ is too large even now it is clearly established to be non-zero and also the atmospheric mixing angle is too far away from maximum in order to fit the data.

3.3.4 The group $\Gamma_7 \sim PSL(2, 7)$

For N equal to 7, the modular subgroup Γ_N is isomorphic to $PSL(2, 7)$ [89, 90]. We recall from the previous section that the group can be generated from two generators S and T only if a fourth non-trivial relation is satisfied. A presentation of the group is then

$$S^2 = (ST)^3 = T^7 = (ST^{-1}ST)^4 = 1. \quad (3.28)$$

Possible matrix representations for S and T are given in table 3.4. The two representations there are each others complex conjugates and thus trivially give rise to the same mixing patterns. The group

$PSL(2, 7)$ has six conjugacy classes mentioned in table 3.C.3 in the appendix after this chapter and a total of 71 relevant Abelian subgroups of the type Z_3, Z_4, Z_7 and $Z_2 \times Z_2$. These are collected in table 3.C.4. In the following four paragraphs, we discuss the lepton mixing when the charged leptons are invariant under each of these groups, while the neutrinos are fixed to be invariant under the Klein group. None of these mixing patterns are very close to the neutrino data.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_3$

When the neutrino sector is invariant under a Klein group and the charged lepton sector under an element of order 3, it is possible to generate the whole group $PSL(2, 7)$ with the elements of G_e and G_ν and we find the following mixing pattern

$$\begin{aligned} \|U_{\text{PMNS}}\| &= \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{\frac{1}{2}(5+\sqrt{21})} & 1 & \frac{1}{2}(5-\sqrt{21})\sqrt{\frac{1}{2}(5+\sqrt{21})} \\ 1 & 2 & 1 \\ \frac{1}{2}(5-\sqrt{21})\sqrt{\frac{1}{2}(5+\sqrt{21})} & 1 & \sqrt{\frac{1}{2}(5+\sqrt{21})} \end{pmatrix} \\ &\approx \begin{pmatrix} 0.894 & 0.408 & 0.187 \\ 0.408 & 0.816 & 0.408 \\ 0.187 & 0.408 & 0.894 \end{pmatrix}. \end{aligned}$$

The mixing angles are $\sin^2 \theta_{13} = \frac{1}{12}(5-\sqrt{21}) \approx 0.035$ and $\sin^2 \theta_{12} = \sin^2 \theta_{23} = \frac{1}{14}(7-\sqrt{21}) \approx 0.173$. If the second and third row are interchanged, the atmospheric mixing is given by $\sin^2 \theta_{23} = \frac{1}{14}(7+\sqrt{21}) \approx 0.827$. The CP violating phase fulfills $|\sin \delta_{CP}| = \sqrt{7/8} \approx 0.935$ and thus $|J_{CP}| = 1/(24\sqrt{3}) \approx 0.024$. One possible choice of G_ν and G_e is: $G_\nu = K_1$ and $G_e = C_1$.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_4$

If we take the lepton sector invariant under an element of order 4, we can generate the following mixing pattern

$$\begin{aligned} \|U_{\text{PMNS}}\| &= \frac{1}{2} \begin{pmatrix} \sqrt{\frac{1}{2}(3+\sqrt{7})} & 1 & \frac{1}{2}(3-\sqrt{7})\sqrt{3+\sqrt{7}} \\ 1 & \sqrt{2} & 1 \\ \frac{1}{2}(3-\sqrt{7})\sqrt{3+\sqrt{7}} & 1 & \sqrt{\frac{1}{2}(3+\sqrt{7})} \end{pmatrix} \\ &\approx \begin{pmatrix} 0.840 & 0.5 & 0.210 \\ 0.5 & 0.707 & 0.5 \\ 0.210 & 0.5 & 0.840 \end{pmatrix}. \end{aligned} \quad (3.29)$$

The mixing angles are given by $\sin^2 \theta_{13} = \frac{1}{8}(3-\sqrt{7}) \approx 0.044$, $\sin^2 \theta_{12} = \sin^2 \theta_{23} = \frac{1}{9}(5-\sqrt{7}) \approx 0.262$. After interchange of the second and third rows, $\sin^2 \theta_{23}$ is equal to $\frac{1}{9}(4+\sqrt{7}) \approx 0.738$. The Jarlskog invariant fulfills $|J_{CP}| = 1/32 \approx 0.031$ and $|\sin \delta_{CP}| = \frac{1}{4}\sqrt{13-\sqrt{7}} \approx 0.804$. This pattern is produced for example for $G_\nu = K_1$ and $G_e = Q_1$.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_7$

If the charged lepton sector is invariant under an element of order 7, the mixing takes the form

$$\|U_{\text{PMNS}}\| = 2\sqrt{\frac{2}{7}} \begin{pmatrix} s_2 s_3 & s_1 s_3 & s_1 s_2 \\ s_1 s_2 & s_2 s_3 & s_1 s_3 \\ s_1 s_3 & s_1 s_2 & s_2 s_3 \end{pmatrix} \approx \begin{pmatrix} 0.815 & 0.452 & 0.363 \\ 0.363 & 0.815 & 0.452 \\ 0.452 & 0.363 & 0.815 \end{pmatrix}. \quad (3.30)$$

For the mixing angles we find $\sin^2 \theta_{13} \approx 0.132$ and $\sin^2 \theta_{12} \approx 0.235$; $\sin^2 \theta_{23}$ is approximately equal to 0.235 or, after interchange of rows, to 0.765. CP violation is characterized by $|J_{CP}| = 1/(8\sqrt{7}) \approx$

0.047. This pattern arises from any possible combination of a Klein group and an element of order seven.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_2 \times Z_2$

If both the neutrino sector and the charged lepton sector are invariant under a Klein group, the unique mixing pattern, compatible with our requirements, is

$$||U_{\text{PMNS}}|| = \frac{1}{2} \begin{pmatrix} \sqrt{2} & 1 & 1 \\ 1 & \sqrt{2} & 1 \\ 1 & 1 & \sqrt{2} \end{pmatrix}. \quad (3.31)$$

We find for the mixing angles $\sin^2 \theta_{13} = 1/4$, $\sin^2 \theta_{12} = 1/3$ and $\sin^2 \theta_{23} = 1/3$ or $2/3$. The quantity $|\sin \delta_{CP}|$ is $3\sqrt{7}/8 \approx 0.992$ and $|J_{CP}| = \sqrt{7}/32 \approx 0.083$. One choice of G_ν and G_e leading to this particular pattern is $G_\nu = K_1$ and $G_e = K_3$.

3.3.5 The subgroup $\Delta(96)$ of Γ_8

The next group to consider is Γ_8 . As explained at the end of section 3.2, it does not have three-dimensional faithful representations. Instead the three-dimensional representations given in table 3.7 can generate at most a subgroup of order 96. This group $\Delta(96)$ is also known as the tetrakisoctahedral group 4O .

The group $\Delta(96)$ has ten conjugacy classes as given in table 3.C.5; a list of generating elements for the Abelian subgroups $Z_2 \times Z_2$, Z_3 , Z_4 and Z_8 can be found in table 3.C.6 and that of the Z_2 symmetries in table 3.C.7. The reason for mentioning the latter is explained shortly. There are ten irreducible representations: two singlets, one doublet, six triplets (two of which unfaithful) and one sextet. The character table of $\Delta(96)$ can be found in [75, 76]. The elements S and T of the four faithful three-dimensional representations are given in table 3.7. They are related to each other by an overall change of sign and/or complex conjugation and thus give rise to the same mixing patterns. S and T fulfill the relations

$$S^2 = (ST)^3 = T^8 = (ST^{-1}ST)^3 = 1. \quad (3.32)$$

In the following we discuss all possible combinations of G_e and $G_\nu = Z_2 \times Z_2$ case by case. We encounter two new instances: firstly, we come across situations in which a certain combination of types of subgroups employed for G_e and $G_\nu = Z_2 \times Z_2$ does not allow to generate the original group $\Delta(96)$.

Secondly, it can be checked that the generating elements of the groups Q_1 , Q_2 and Q_3 for the faithful irreducible triplets (to which we assign the left-handed lepton generations) are represented by matrices which have two degenerate eigenvalues. As a consequence, it is not possible to distinguish among the three generations of leptons with these groups. On themselves they cannot be used to generate G_e . However, a combination of two of them or a combination with an additional Z_2 -generating element, resolves the degeneracy. For this reason we also include the cases $G_e = Z_2 \times Z_4$ and $G_e = Z_4 \times Z_4$ in our discussion.³

³We do not discuss all possible types of $Z_2 \times Z_4$ and $Z_4 \times Z_4$ subgroups, but only those in which the Z_4 group alone is not sufficient to distinguish the three generations of leptons.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_3$

We first discuss the case where the neutrino sector is invariant under a Klein group and the charged lepton sector under an element of order 3. The generated mixing pattern reads

$$\|U_{\text{PMNS}}\| = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}(\sqrt{3}+1) & 1 & \frac{1}{2}(\sqrt{3}-1) \\ 1 & 1 & 1 \\ \frac{1}{2}(\sqrt{3}-1) & 1 & \frac{1}{2}(\sqrt{3}+1) \end{pmatrix} \approx \begin{pmatrix} 0.789 & 0.577 & 0.211 \\ 0.577 & 0.577 & 0.577 \\ 0.211 & 0.577 & 0.789 \end{pmatrix}. \quad (3.33)$$

This leads to the following mixing angles: $\sin^2 \theta_{13} = \frac{2-\sqrt{3}}{6} \approx 0.045$ and $\sin^2 \theta_{12} = \sin^2 \theta_{23} = \frac{8-2\sqrt{3}}{13} \approx 0.349$. Alternatively, the second and third row can be exchanged, giving $\sin^2 \theta_{23} = \frac{5+2\sqrt{3}}{13} \approx 0.651$ instead. The fact that this pattern is close to the data, including a prediction for non-zero reactor mixing angle, makes the pattern very interesting. It is discussed in more detail in section 3.4. We refer to the two sets of mixing angles here as M2 and M1 respectively. A viable choice of G_e and G_ν leading to this pattern is $G_e = C_1$ and $G_\nu = K_1$.

It is interesting to note that this mixing pattern can also be realized when the flavour group is S_4 , although this does not happen when neutrinos are Majorana particles and thus required to satisfy $G_\nu = Z_2 \times Z_2$. If neutrinos are instead Dirac particles, $G_\nu = Z_4$ may be chosen. Together with any choice for a Z_3 subgroup as residual symmetry in the lepton sector, the pattern M1 or M2 follows.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_8$

If the original group $\Delta(96)$ is generated through the elements of G_e and G_ν , the resulting mixing pattern is bimaximal, see equation (3.23). One possible choice of G_e and G_ν is $G_e = O_1$ and $G_\nu = K_1$.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_2 \times Z_4$

As discussed above, we also consider the Z_4 subgroup contained in G_e to be one of the subgroups which are represented by matrices with degenerate eigenvalues (Q_1 , Q_2 or Q_3) for the irreducible faithful triplet. These are only sufficient to describe the charged leptons when combined with an additional Z_2 -group. The mixing pattern that arises if all our requirements are passed, is again the bi-maximal one of equation (3.23). An example of G_e and G_ν is $G_e = V_1 \times Q_1$ and $G_\nu = K_1$.

Three cases that do not generate the full group $\Delta(96)$

Lastly, we study the cases where G_ν is given by a Klein group and G_e is given by either

- a Z_4 group, but not Q_1 , Q_2 or Q_3 (that have degenerate eigenvalues),
- a Klein group $Z_2 \times Z_2$ or
- a product $Z_4 \times Z_4$, where each of the Z_4 groups is given by one of Q_1 , Q_2 or Q_3 .

None of the possible choices allows us to generate the original group $\Delta(96)$ using the elements of the subgroups G_e and G_ν .

3.3.6 The subgroup $\Delta(384)$ of Γ_{16}

The last group we consider in this chapter is Γ_{16} . Just like Γ_8 , none of the three-dimensional representations is faithful. The eight triplets in table 3.7 instead generate a subgroup of order 384. Like the group of the previous section, it is part of the series $\Delta(6n^2)$, this time with $n = 8$.

The group $\Delta(384)$ has 24 conjugacy classes and a total of 145 Z_n subgroups as given in the appendix. As not only the Z_2 elements, but also some of the Z_4 and Z_8 elements have degenerate eigenvalues, candidates for G_e also include $Z_4 \times Z_2$, $Z_4 \times Z_4$, $Z_4 \times Z_8$, $Z_8 \times Z_2$ and $Z_8 \times Z_8$. In these cases, the order 4 and 8 elements should come from Q_{1-3} and O_{1-3} .

The matrix representations for the elements S and T that can generate the group $\Delta(384)$ are given in table 3.7. The elements S and T satisfy four non-trivial relations

$$S^2 = (ST)^3 = T^{16} = (ST^{-1}ST)^3 = 1. \quad (3.34)$$

We observe that the eight faithful triplets of table 3.7 come in two groups of four, where the matrices in each group are related by an overall sign change and/or complex conjugation. The two groups are related by observing that if one set of S and T matrices satisfies the presentation (3.34), then so does any odd power of it, smaller than 16.

In the following, we calculate the mixing patterns that follow when G_ν is taken to be a Klein group and G_e is a Z_n -group, a Klein group or any of the product groups mentioned above.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_3$

The case we discuss is the one where the neutrino sector is invariant under a Klein group and the charged lepton sector under an element of order 3. The generated mixing pattern reads

$$\begin{aligned} \|U_{\text{PMNS}}\| &= \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{2}\sqrt{4+\sqrt{2}+\sqrt{6}} & 1 & \frac{1}{2}\sqrt{4-\sqrt{2}-\sqrt{6}} \\ \frac{1}{2}\sqrt{4+\sqrt{2}-\sqrt{6}} & 1 & \frac{1}{2}\sqrt{4-\sqrt{2}+\sqrt{6}} \\ \sqrt{1-\frac{1}{\sqrt{2}}} & 1 & \sqrt{1+\frac{1}{\sqrt{2}}} \end{pmatrix} \\ &\approx \begin{pmatrix} 0.810 & 0.577 & 0.107 \\ 0.497 & 0.577 & 0.648 \\ 0.312 & 0.577 & 0.754 \end{pmatrix}. \end{aligned} \quad (3.35)$$

With the ordering chosen in eq. (3.35), the mixing angles are

$$\begin{aligned} \sin^2 \theta_{13} &= \frac{4 - \sqrt{2} - \sqrt{6}}{12} \approx 0.011, \\ \sin^2 \theta_{12} &= \frac{4}{8 + \sqrt{2} + \sqrt{6}} \approx 0.337, \\ \sin^2 \theta_{23} &= \frac{4 - \sqrt{2} + \sqrt{6}}{8 + \sqrt{2} + \sqrt{6}} \approx 0.424. \end{aligned} \quad (3.36)$$

If we exchange the second and third rows in U_{PMNS} , the atmospheric mixing changes to

$$\sin^2 \theta_{23} = \frac{4 + 2\sqrt{2}}{8 + \sqrt{2} + \sqrt{6}} \approx 0.576. \quad (3.37)$$

We observe that CP is conserved in both cases. We refer to these two patterns as M3 and M4. Both patterns are very close to the data, with interestingly, M3 favoured by the global neutrino fit [24] and M4 by the fit [27,28]. Similar to the case where the mixing patterns M1 and M2 could be generated by S_4 instead of $\Delta(96)$ if neutrinos are Dirac particles, we find that these patterns can also be generated by the ‘simpler’ group $\Delta(96)$ if the neutrino are allowed to be Dirac particles invariant under a Z_3 symmetry and charged leptons under a Z_8 symmetry.

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_{16}$

We now study mixing patterns where the neutrinos are invariant under a Klein group and the charged leptons under one of the Z_{16} groups. It is possible to generate the full group $\Delta(384)$ for instance with K_2 and Y_1 . The mixing pattern that follows is the bimaximal mixing pattern (3.23)

Mixing patterns with $G_\nu = Z_2 \times Z_2$ and $G_e = Z_8 \times Z_2$

When the charged lepton sector is invariant under a $Z_8 \times Z_2$ group, with the Z_8 either O_1 , O_2 or O_3 , the full group $\Delta(384)$ is generated for instance if $G_e = O_1 \times V_2$ and $G_\nu = K_1$. The resulting mixing pattern is the familiar bimaximal one of equation (3.23).

Cases that do not generate the full group $\Delta(384)$

There are seven additional cases where G_e is in accordance with the requirements set in this section. When combined with any G_ν that is a Klein group, it is not possible to recover the complete group $\Delta(384)$. These cases are

- G_e is a Z_4 subgroup, unequal to Q_1 , Q_2 or Q_3 ,
- G_e is a Z_8 subgroup, unequal to O_1 , O_2 or O_3 ,
- G_e is a Klein group,
- G_e is a product group $Z_4 \times Z_2$, with the Z_4 factor given by either Q_1 , Q_2 or Q_3 ,
- G_e is a product group $Z_4 \times Z_4$, with both Z_4 factors from Q_{1-3} ,
- G_e is a product group $Z_4 \times Z_8$, with the Z_4 factor from Q_{1-3} and the Z_8 factor from O_{1-3} or
- G_e is a product group $Z_8 \times Z_8$ with both Z_8 factors from O_{1-3} .

This concludes all cases where the residual symmetry in the neutrino sector is a Klein group. There is one pattern that does not respect this rule, but that might be worth mentioning.

$$V = \begin{pmatrix} \cos \pi/16 & \sin \pi/16 & 0 \\ -\sin \pi/16 & \cos \pi/16 & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 0.981 & 0.195 & 0 \\ -0.195 & 0.981 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.38)$$

This can be produced by a number of residual subgroups, including Z_4 and Z_4 (e.g. Q_1 and Q_{18}); Z_4 and Z_8 (e.g. Q_1 and O_{24}); Z_4 and Z_{16} (e.g. Q_4 and Y_{12}); Z_8 and Z_8 (e.g. O_1 and O_{24}) and, lastly, Z_8 and Z_{16} (e.g. O_{24} and Y_{12}). Obviously, this is not a valuable pattern for neutrinos, but it might be relevant for quarks. If the two residual subgroups are those of the up- and down-quark sectors, the result is quite close to the CKM matrix. In a first approximation, one angle slightly smaller than the Cabibbo angle is generated. Subleading corrections may lead to the two other angles and possibly to very small effects on the neutrino mixing matrix. In the spirit of this chapter, we refrain from giving a more concrete model building realization. This is likely to be quite complicated, as now there are different residual symmetries in all four sectors (charged leptons, neutrinos, up and down quarks). In a realization with flavons, preventing them to couple to the wrong sectors is likely to be non-trivial.

3.4 Four interesting mixing patterns

In this section we comment on the four mixing patterns M1 to M4. All of these points are close to the current data on the neutrino mixing angles as shown already in table 3.1 at the beginning of this chapter and as graphically illustrated in figure 3.1.

Both (pairs of) mixing patterns appeared when $G_\nu = Z_2 \times Z_2$ and $G_e = Z_3$ was chosen for a group of the type $\Delta(6n^2)$. When these two subgroups are chosen in $\Delta(24) \sim S_4$ one gets the tribimaximal

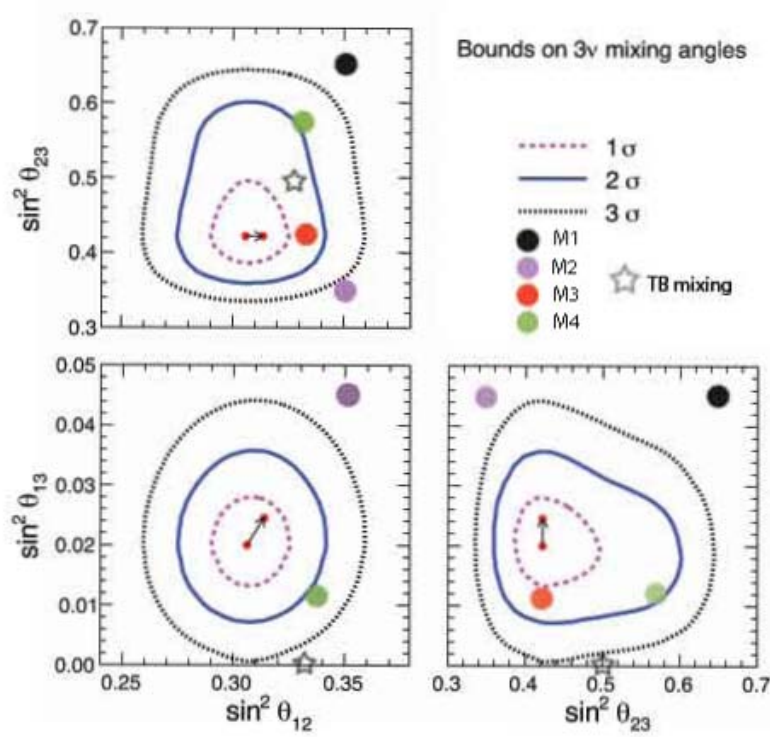


Figure 3.1: Values of $\sin^2 \theta_{ij}$ for the four different mixing patterns M1 (black), M2 (violet), M3 (red) and M4 (green) as well as the tribimaximal mixing pattern (star). The contours show the 1σ (pink dashed line), 2σ (blue solid line) and 3σ (black dotted line) levels and are taken from [24]. The small dots indicate the best fit values of the mixing angles and the arrows the effect of the new estimates of the reactor antineutrino flux. Note that in the $\sin^2 \theta_{12}$ - $\sin^2 \theta_{13}$ plane the points of M1 and M2 as well as of M3 and M4 lie on top of each other, since they only differ in the value of $\sin^2 \theta_{23}$.

mixing pattern given by equation (3.22), or, with the phases reintroduced in the standard way by

$$U_{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad (3.39)$$

The two new patterns are quite similar to tribimaximal mixing. This can be seen from the fact that also these patterns can be brought into a form in which the second column has three entries equal to $1/\sqrt{3}$. Indeed, the TBM mixing matrix can be modified by a rotation in the (1 3) plane acting from the right

$$U_{\text{PMNS}} = U_{\text{TBM}} U_{13}(\alpha) \quad \text{with} \quad U_{13}(\alpha) = \begin{pmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix}. \quad (3.40)$$

It is immediate to show that, by taking $\alpha = -\pi/12$ and $\alpha = \pi/24$, the resulting mixing matrices are identical, in absolute value, to the matrices in equations (refeq:mixD96) and (3.35), respectively. Such perturbations from TB mixing with α arbitrary have been already discussed in the literature [91–96]. For generic α , the mixing angles read

$$\sin^2 \theta_{12} = \frac{1}{2 + \cos 2\alpha}, \quad \sin^2 \theta_{23} = \frac{1}{2} - \frac{\sqrt{3} \sin 2\alpha}{4 + 2 \cos 2\alpha}, \quad \sin^2 \theta_{13} = \frac{2}{3} \sin^2 \alpha. \quad (3.41)$$

For small α , we can expand the results

$$\sin^2 \theta_{12} \approx \frac{1}{3} + \frac{2\alpha^2}{9}, \quad \sin^2 \theta_{23} \approx \frac{1}{2} - \frac{\alpha}{\sqrt{3}}, \quad \sin^2 \theta_{13} \approx \frac{2\alpha^2}{3} \quad (3.42)$$

showing that the deviation from the value of TB mixing of $\sin^2 \theta_{12}$, the best measured quantity among the three mixing angles, is quadratic in α , whereas the leading correction to $\sin^2 \theta_{23} = 1/2$ is linear in α , allowing reasonably large deviations from its maximal value as indeed still allowed by the data.

3.5 Conclusions of the chapter

Recent results of the T2K experiment and of a global fit of the neutrino oscillation data point to non-vanishing θ_{13} at the 3σ level. The best fit value of θ_{13} is around $0.15 \div 0.16$, smaller than the ones of the other angles, but much larger than 0.02 , the 1σ experimental error on the solar angle θ_{12} . If future data confirm this result, many models giving rise at LO to mixing patterns with vanishing θ_{13} , such as TB mixing, become disfavoured, because corrections, expected in these models, generically lead to too small θ_{13} .

A particular elegant mechanism to produce simple mixing patterns is based on discrete flavour symmetries. The latter are broken in a non-trivial way and as a consequence give rise to mixing angles whose values only depend on the properties of the flavour symmetry, but not on lepton masses. After the T2K data the natural question is whether such symmetries still remain a valuable tool to describe flavour mixing. In the beginning of the chapter we have given four possible answers to this question, ranging from ‘probably not’ to ‘possible with the right types of flavour symmetries’.

In this chapter we have scanned a large category of candidate flavour-symmetries. The discrete flavour group G_f is chosen to be a modular subgroup Γ_N or, in specific cases a subgroup hereof. G_f is supposed to be broken in such a way that the relevant mass matrices m_ν and $m_e^\dagger m_e$ have a residual invariance under the subgroups G_ν and G_e , respectively. The lepton mixing matrix originates from the mismatch of these two subgroups and from their specific embedding into G_f . We rediscovered the lepton mixing matrices that correspond to tribimaximal mixing, bimaximal mixing and the golden ratio mixing from the ‘smaller’ groups A_4 , S_4 and A_5 in the list of G_f . The group $PSL(2, 7)$ generates several mixing patterns, but none of them is particularly close to the data given by neutrino oscillations. The groups $\Delta(96)$ and $\Delta(384)$ provide us with two (or four if a degeneracy is taken into account) very interesting mixing patterns. In particular the pattern reproduced by $\Delta(384)$ is remarkably close to their experimental best fit values. More precise measurements are expected to show if the agreement between the patterns and the data are just coincidental or if they might be related by a deeper mechanism. If the latter is the case, the first thing on a theorist’s to-do list should obviously be a concrete model that derives these mixing angles and generates some additional predictions.

Appendices to chapter 3

In the previous chapter, we studied six groups of the type Γ_N . The groups A_4 and S_4 , corresponding respectively to $N = 4$ and $N = 5$, occur at many other places in this thesis. In the first two appendices here, we carefully derive their group theoretical properties. In the third appendix, we list some properties of the other finite modular groups under discussion, A_5 , $PSL(2, 7)$, $\Delta(96)$ and $\Delta(384)$.

3.A The alternating group A_4

The group A_4 is the group of even permutations of four elements. It thus has $4!/2 = 12$ elements. A general permutation of them can be seen as a mapping $p : i \mapsto p(i)$, with $i = 1 \cdots 4$. The mappings can be represented as

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ p(1) & p(2) & p(3) & p(4) \end{pmatrix}. \quad (3.A.1)$$

This is often simply represented as $p = (p(1), p(2), p(3), p(4))$. With only four elements, only a limited number of structures is available for permutations. Apart from the identity, these are transpositions, double transpositions, three-cycles and four-cycles.

A transposition is the interchange of two elements, while leaving the other two elements inert. The interchange of i_1 and i_2 is represented as $i_1 \mapsto i_2 \mapsto i_1$ or as (i_1, i_2) in the so-called cycle notation. A single transposition is obviously an odd permutation, so it is not an element of A_4 , but it is an important building block for permutations that are in A_4 .

A double transpositions, the combined and unrelated interchange of two pairs of elements is an even permutation and thus an element of A_4 . They can be represented as $i_1 \mapsto i_2 \mapsto i_1$ and $i_3 \mapsto i_4 \mapsto i_3$ or simply as $(i_1, i_2) (i_3, i_4)$. With four elements present, there are three double transpositions, one for each of the three ways to pair the numbers 1 to 4. Obviously (double) transpositions square to the identity, giving an important property of these transformations.

The third type are three-cycles $i_1 \mapsto i_2 \mapsto i_3 \mapsto i_1$, while the fourth element i_4 is inert. This is also written as (i_1, i_2, i_3) . A three-cycle can be seen as a combination of two transpositions, for instance as $(i_1, i_3) \circ (i_1, i_2)$. There are eight three-cycles in A_4 : there are four ways to pick the three elements (there are four choices for the single element that has to be left out) and each of these gives rise to two three-cycles, one being the square of the other. The third power of any element gives the identity. Four-cycles lastly, can be written as the product of three transpositions and as such are no elements of A_4 .

We can call one of the double transpositions S and one of the three-cycles T . It follows that all elements of A_4 can be written as simple combinations of S and T . It is in particular relevant to mention that the element ST is a three-cycle and as such satisfies $(ST)^3 = 1$. In table 3.A.1 we mention the 12 elements of A_4 in permutation notation, in cycle notation and as a product of S and T .

Permutation	Cycle	$f[S, T]$	Order	Class
(1, 2, 3, 4)	-	$\mathbb{1}$	1	E
(1, 3, 4, 2)	(2, 3, 4)	$(ST)^2$	3	$4C_3^{(2)}$
(1, 4, 2, 3)	(2, 4, 3)	ST	3	$4C_3^{(1)}$
(2, 1, 4, 3)	(1, 2) (3, 4)	S	2	$3C_2$
(2, 3, 1, 4)	(1, 2, 3)	T	3	$4C_3^{(1)}$
(2, 4, 3, 1)	(1, 2, 4)	$(STS)^2$	3	$4C_3^{(2)}$
(3, 1, 2, 4)	(1, 3, 2)	T^2	3	$4C_3^{(2)}$
(3, 2, 4, 1)	(1, 3, 4)	TS	3	$4C_3^{(1)}$
(3, 4, 1, 2)	(1, 3) (2, 4)	T^2ST	2	$3C_2$
(4, 1, 3, 2)	(1, 4, 2)	STS	3	$4C_3^{(1)}$
(4, 2, 1, 3)	(1, 4, 3)	$(TS)^2$	3	$4C_3^{(2)}$
(4, 3, 1, 2)	(1, 4) (2, 3)	ST^2ST	2	$3C_2$

Table 3.A.1: The 12 elements of A_4 with their permutation and cycle notation; the way to write them in S and T ; the order and the class they belong to.

In the discussion above, the group A_4 works on four abstract points. A very natural interpretation is possible, where the four points (1, 2, 3, 4) are the four vertices of a tetrahedron. An operation $i_1 \mapsto i_2 \dots$ represents the corresponding interchange of the vertices. Even permutations of the four vertices are exactly the rotations that leave the tetrahedron invariant, while odd permutations represent reflections.

We define S as the double transposition (1, 2) (3, 4) that interchanges vertices 1 and 2 as well as 3 and 4. It can be represented by a rotation over 180° over the axis connecting the middles of the edges 1-2 and 3-4. The three-cycle T is defined as the anti-clockwise rotation over 120° in the 1-2-3 plane, leaving vertex 4 invariant. This is shown in figure 3.A.1

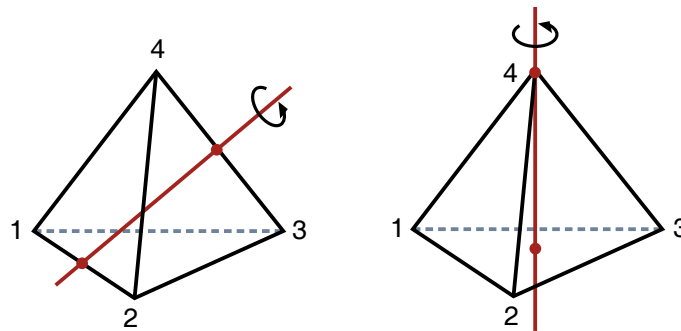


Figure 3.A.1: The generating elements S and T of A_4 .

In section 3.3.1 we calculated the conjugacy classes of A_4 . The group has four classes: E containing the identity, $3C_2$ containing the three elements of order 2 (including S) and two classes $4C_3$ and $4C_3'$ with order 3 elements (one containing T and the other T^2). The order and the conjugacy class any element belongs to are also given in table 3.A.1. For finite groups, the number of irreducible representations (irreps) up to equivalence is equal to the number of classes. Their dimensions are

related to the order of the group according to

$$\sum_{a \in I} d_a^2 = \text{order}(G) \quad (3.A.2)$$

Because A_4 has four classes, it should also have four irreps. The only way to satisfy equation (3.A.2) is if three of them have dimension 1 and one has dimension 3. The irreps of A_4 can thus be characterized as $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ and $\mathbf{3}$.

Next we construct the character table. We use the following requirements

- In the trivial representation, all elements are represented as 1.
- The identity operation, contained in the class E is represented by the number 1 for all three one-dimensional representations and by the (3×3) identity matrix (with trace 3) for the three-dimensional representation
- The elements of $3C_2$ should be represented by $+1$ or -1 in the irreps $\mathbf{1}'$ and $\mathbf{1}''$ since they square to the identity.
- The elements of $4C_3$ and $4C_3'$ should be represented by 1 , $\omega \equiv e^{2\pi i/3}$ or ω^2 in the irreps $\mathbf{1}'$ and $\mathbf{1}''$ as the identity is their third power.
- The irreps satisfy an orthonormality relation:

$$\frac{1}{\text{order}(G)} \sum_{i \in G} \chi^I(i) (\chi^J(i))^* = \delta^{IJ} .$$

Here $\chi^I(i)$ stands for the character of the element i in the irreducible representation I .

When we apply these conditions, there are just two character tables possible. One of them is given in table 3.A.2; the other simply has the irreps $\mathbf{1}'$ and $\mathbf{1}''$ interchanged⁴.

	E	$3C_2$ (containing S)	$4C_3$ (containing T)	$4C_3'$ (containing T^2)
$\mathbf{1}$	1	1	1	1
$\mathbf{1}'$	1	1	ω	ω^2
$\mathbf{1}''$	1	1	ω^2	ω
$\mathbf{3}$	3	-1	0	0

Table 3.A.2: The character table of A_4 .

Two explicit representations are popular in the literature. In the representation of Ma and Rajasekaran (MR) [97], the element S is diagonal in the three-dimensional representation. It has the further advantage that all elements are represented by real matrices in the triplet representation. In the basis of Altarelli and Feruglio (AF) [10, 11] the element T is chosen diagonal (and complex). The representations of S and T in the various irreducible representations and in the bases of MR and AF are given in table 3.A.3. Both representations are used in this thesis: the AF representation in chapters 2 and 3 and the MR representation in chapter 5.

The bases of Ma and Rajasekaran and of Altarelli and Feruglio are related by a basis transformation

$$M_{\text{MR}} = \Omega^\dagger M_{\text{AF}} \Omega, \quad \Omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} . \quad (3.A.3)$$

⁴In the original definition of Altarelli and Feruglio [11] the opposite definition for $\mathbf{1}'$ and $\mathbf{1}''$ is chosen. The 'Altarelli-Feruglio basis' given below is therefore not the original one, but it shares the most important property, namely that the T matrix is diagonal in the triplet representation.

Irrep	Ma and Rajasekaran		Altarelli and Feruglio	
1	$S = 1$	$T = 1$	$S = 1$	$T = 1$
1'	$S = 1$	$T = \omega$	$S = 1$	$T = \omega$
1''	$S = 1$	$T = \omega^2$	$S = 1$	$T = \omega^2$
3	$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$S = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$

Table 3.A.3: The generating elements S and T of A_4 in the bases of Ma and Rajasekaran and of Altarelli and Feruglio.

The multiplication rules for A_4 can be constructed from the character table by demanding that the characters satisfy the multiplication rules. We find

$$\begin{aligned}
1 \times r &= r \text{ for all representations } r, \\
1' \times 1' &= 1'', \quad 1'' \times 1'' = 1, \quad 1' \times 1'' = 1, \\
1' \times 3 &= 3, \quad 1'' \times 3 = 3, \\
3 \times 3 &= 1 + 1' + 1'' + 3 + 3.
\end{aligned} \tag{3.A.4}$$

In the last product, one of the resulting triplets can be chosen to be symmetric and the other to be antisymmetric in the elements of the original triplets.

Lastly, we construct the Clebsch-Gordan (CG) coefficients of the products above. We use α_i to indicate the elements of the first representation of the product and β_j to indicate those of the second representation. The products are then linear in α_i and β_j . The coefficients follow from the requirement that the products (3.A.4) 'commute' with S and T transformations. For instance if $(\alpha \times \beta)_{1'}$ indicates the $1'$ in the product 3×3 , we should have $((T_3 \alpha) \times (T_3 \beta))_{1'} = T_{1'}(\alpha \times \beta)_{1'}$. All results are given in table 3.A.4.

The calculation of the Clebsch-Gordan coefficients concludes this appendix. The multiplication rules and the CG coefficients of A_4 are the starting point for the analyses in the main chapters of this thesis.

	Ma-Rajasekaran	Altarelli-Feruglio
$1 \times r = r$	r	r
$1' \times 1' = 1''$	$\alpha\beta$	$\alpha\beta$
$1' \times 1'' = 1$	$\alpha\beta$	$\alpha\beta$
$1'' \times 1'' = 1'$	$\alpha\beta$	$\alpha\beta$
$1' \times 3 = 3$	$\alpha(\beta_1, \omega\beta_2, \omega^2\beta_3)$	$\alpha(\beta_3, \beta_1, \beta_2)$
$1'' \times 3 = 3$	$\alpha(\beta_1, \omega^2\beta_2, \omega\beta_3)$	$\alpha(\beta_2, \beta_3, \beta_1)$
$3 \times 3 \ni 1$	$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$	$\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2$
$3 \times 3 \ni 1'$	$\alpha_1\beta_1 + \omega^2\alpha_2\beta_2 + \omega\alpha_3\beta_3$	$\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_3$
$3 \times 3 \ni 1''$	$\alpha_1\beta_1 + \omega\alpha_2\beta_2 + \omega^2\alpha_3\beta_3$	$\alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1$
$3 \times 3 \ni 3_S$	$\frac{1}{2} \begin{pmatrix} \alpha_2\beta_3 + \alpha_3\beta_2 \\ \alpha_1\beta_3 + \alpha_3\beta_1 \\ \alpha_1\beta_2 + \alpha_2\beta_1 \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ -\alpha_1\beta_2 - \alpha_2\beta_1 + 2\alpha_3\beta_3 \\ -\alpha_1\beta_3 + 2\alpha_2\beta_2 - \alpha_3\beta_1 \end{pmatrix}$
$3 \times 3 \ni 3_{AS}$	$\frac{1}{2} \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ -\alpha_1\beta_3 + \alpha_3\beta_1 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ -\alpha_1\beta_3 + \alpha_3\beta_1 \end{pmatrix}$

Table 3.A.4: The Clebsch-Gordan coefficients for A_4 in the two bases used in this thesis, when α_i and β_j represent the elements of the terms of the product.

3.B The symmetric group S_4

The group S_4 is the group of all permutations of 4 elements. It has 24 elements that are of order 1, 2, 3 and 4. The irreps are of one-dimensional (twice), two-dimensional (once) and three-dimensional (twice). We denote the two one-dimensional irreps as 1_1 and 1_2 instead as 1 and $1'$ to use the common notation in S_4 models. The details of the classes and irreps can be calculated in the same way as in the previous section. This is summarized in the extended character table 3.B.1.

Class	χ_{1_1}	χ_{1_2}	χ_2	χ_{3_1}	χ_{3_2}	Elements
E	1	1	2	3	3	$\mathbb{1}$
$3C_2$	1	1	2	-1	-1	T^2, ST^2S, ST^2ST^2
$6C_2$	1	-1	0	1	-1	$S, T^3ST, TST^3, T^2ST^2, ST^2ST, TST^2S$
$8C_3$	1	1	-1	0	0	$TS, ST, (TS)^2, (ST)^2, T^2ST, TST^2, T^3ST^2, T^2ST^3$
$6C_4$	1	-1	0	-1	1	$T, T^3, ST^2, T^2S, STS, TST$

Table 3.B.1: Character table of the S_4 discrete group. In the last column we have reported the elements for each class in terms of the generators S and T of the group.

As for A_4 , all elements can be generated as by two elements, S of order 2 and T of order 4 that satisfy $(ST)^3 = \mathbb{1}$. S_4 is also the symmetry group of the octahedron. S , T and ST can be represented respectively by a rotation over 180° on an axis through two opposing edges; a rotation of 90° on an axis through the centres of two opposing faces and a rotation over 180° on an axis through two opposite vertices as given in figure 3.B.1

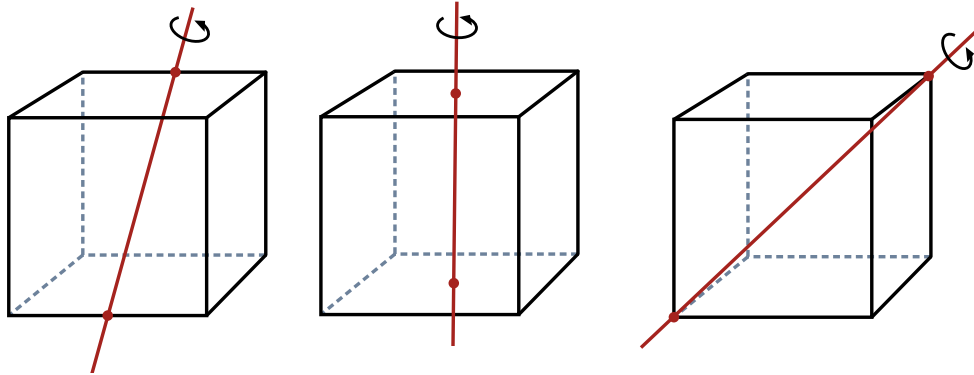


Figure 3.B.1: Graphical representation of the elements S , T and ST of S_4 .

Many matrix representations can be used for S_4 . In chapter 4 we choose a representation in which neither S nor T is diagonal in the three dimensional representations. We present this basis as well as S and T diagonal bases in table 3.B.2. There we also report the matrices that diagonalize our S and T elements according to $M_{\text{chap4}} = \Omega_{2,S,T}^\dagger M_{2,S,T-\text{diag}} \Omega_{2,S,T}$.

The product rules can be constructed from the character table. They read

$$\begin{aligned}
 1 \times r &= r \text{ for all representations } r, \\
 1_2 \times 1_2 &= 1_1, \quad 1_2 \times 2 = 2, \quad 1_2 \times 3_1 = 3_2, \quad 1_2 \times 3_2 = 3_1, \\
 2 \times 2 &= 1_1 + 1_2 + 2, \quad 2 \times 3_1 = 3_1 + 3_2, \quad 2 \times 3_2 = 3_1 + 3_2, \\
 3_1 \times 3_1 &= 3_2 \times 3_2 = 1_1 + 2 + 3_1 + 3_2, \quad 3_1 \times 3_2 = 1_2 + 2 + 3_1 + 3_2.
 \end{aligned} \tag{3.B.1}$$

For each of the bases the Clebsch-Gordan rules can be calculated. We report those for the basis used in chapter 4 in table 3.B.3.

Irrep	Basis of chapter 4		
$\mathbf{1}_1$	$S = 1$	$T = 1$	
$\mathbf{1}_2$	$S = -1$	$T = -1$	
$\mathbf{2}$	$S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$	$T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	
$\mathbf{3}_1$	$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$	
$\mathbf{3}_2$	$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	
Irrep	S-diagonal basis		Diagonalizing matrix
$\mathbf{1}_1$	$S = 1$	$T = 1$	
$\mathbf{1}_2$	$S = -1$	$T = -1$	
$\mathbf{2}$	$S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$T = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$	$\Omega_2 = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$
$\mathbf{3}_1$	$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$T = \frac{1}{2} \begin{pmatrix} 0 & -\sqrt{2} & -\sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix}$	$\Omega_S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 1 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$
$\mathbf{3}_2$	$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$T = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 1 & -1 \\ -\sqrt{2} & -1 & 1 \end{pmatrix}$	"
Irrep	T-diagonal basis		Diagonalizing matrix
$\mathbf{1}_1$	$S = 1$	$T = 1$	
$\mathbf{1}_2$	$S = -1$	$T = -1$	
$\mathbf{2}$	$S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$	$T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	
$\mathbf{3}_1$	$S = \frac{1}{2} \begin{pmatrix} 0 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 1 & -1 \\ -\sqrt{2} & -1 & 1 \end{pmatrix}$	$T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}$	$\Omega_T = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & -i & i \\ 0 & 1 & 1 \end{pmatrix}$
$\mathbf{3}_2$	$S = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix}$	$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$	"

Table 3.B.2: The generating elements S and T of S_4 in the basis of chapter 4, and in S - and T -diagonal ones.

Product rule	Clebsch-Gordan coefficients
$\mathbf{1}_1 \times \mathbf{r} = \mathbf{r}$	r
$\mathbf{1}_2 \times \mathbf{1}_2 = \mathbf{1}_1$	$\alpha\beta$
$\mathbf{1}_2 \times \mathbf{2} = \mathbf{2}$	$\begin{pmatrix} -\alpha\beta_2 \\ \alpha\beta_1 \end{pmatrix}$
$\mathbf{1}_2 \times \mathbf{3}_2 = \mathbf{3}_1$ and $\mathbf{1}_2 \times \mathbf{3}_1 = \mathbf{3}_2$	$\begin{pmatrix} \alpha\beta_1 \\ \alpha\beta_2 \\ \alpha\beta_3 \end{pmatrix}$
$\mathbf{2} \times \mathbf{2} = \mathbf{1}_1$	$\alpha_1\beta_1 + \alpha_2\beta_2$
$\mathbf{2} \times \mathbf{2} = \mathbf{1}_2$	$\alpha_1\beta_2 - \alpha_2\beta_1$
$\mathbf{2} \times \mathbf{2} = \mathbf{2}$	$\begin{pmatrix} \alpha_1\beta_2 + \beta_1\alpha_2 \\ \alpha_1\beta_1 - \alpha_2\beta_2 \end{pmatrix}$
$\mathbf{2} \times \mathbf{3}_1 = \mathbf{3}_1$ and $\mathbf{2} \times \mathbf{3}_3 = \mathbf{3}_2$	$\begin{pmatrix} \alpha_1\beta_1 \\ -\frac{1}{2}(\sqrt{3}\alpha_1 + \alpha_2)\beta_2 \\ \frac{1}{2}(\sqrt{3}\alpha_1 - \alpha_2)\beta_3 \end{pmatrix}$
$\mathbf{2} \times \mathbf{3}_1 = \mathbf{3}_2$ and $\mathbf{2} \times \mathbf{3}_2 = \mathbf{3}_1$	$\begin{pmatrix} \alpha_1\beta_1 \\ -\frac{1}{2}(\alpha_1\sqrt{3}\alpha_2)\beta_2 \\ -\frac{1}{2}(\alpha_1 + \sqrt{3}\alpha_2)\beta_3 \end{pmatrix}$
$\mathbf{3}_1 \times \mathbf{3}_1 = \mathbf{1}_1$, $\mathbf{3}_1 \times \mathbf{3}_1 = \mathbf{1}_1$ and $\mathbf{3}_1 \times \mathbf{3}_2 = \mathbf{1}_2$	$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$
$\mathbf{3}_1 \times \mathbf{3}_1 = \mathbf{2}$ and $\mathbf{3}_2 \times \mathbf{3}_2 = \mathbf{2}$	$\begin{pmatrix} \sqrt{3}(-\alpha_2\beta_2 + \alpha_3\beta_3) \\ 2\alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3 \end{pmatrix}$
$\mathbf{3}_1 \times \mathbf{3}_2 = \mathbf{2}$	$\begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3 \\ \sqrt{3}(\alpha_2\beta_2 - \alpha_3\beta_3) \end{pmatrix}$
$\mathbf{3}_1 \times \mathbf{3}_1 = \mathbf{3}_1$, $\mathbf{3}_1 \times \mathbf{3}_2 = \mathbf{3}_1$ and $\mathbf{3}_2 \times \mathbf{3}_2 = \mathbf{3}_2$	$\begin{pmatrix} \alpha_2\beta_3 + \alpha_3\beta_2 \\ \alpha_1\beta_3 + \beta_3\alpha_1 \\ \alpha_1\beta_2 + \alpha_2\beta_1 \end{pmatrix}$
$\mathbf{3}_1 \times \mathbf{3}_1 = \mathbf{3}_2$, $\mathbf{3}_1 \times \mathbf{3}_2 = \mathbf{3}_2$ and $\mathbf{3}_2 \times \mathbf{3}_2 = \mathbf{3}_1$	$\begin{pmatrix} -\alpha_2\beta_3 + \alpha_3\beta_2 \\ \alpha_1\beta_3 - \beta_3\alpha_1 \\ -\alpha_1\beta_2 + \alpha_2\beta_1 \end{pmatrix}$

Table 3.B.3: The Clebsch-Gordan coefficients for S_4 in the bases used in chapter 4. Again α_i and β_j represent the elements of the terms of the product.

3.C Tables of Abelian subgroups for A_5 , $PSL(2, 7)$, $\Delta(96)$ and $\Delta(384)$

The groups A_5 , $PSL(2, 7)$, $\Delta(96)$ and $\Delta(384)$ were very important in chapter 3, but they do not reappear in this thesis. Hence we refrain from giving their full group theoretical properties, such as character tables. These can be found in the mathematical and/or physical literature, e.g. [58,75,76,89]. In this appendix we just report the relevant Abelian subgroups of the groups, as these can be used as candidates for G_e and G_ν as described in the chapter.

3.C.1 Abelian subgroups of A_5

The group A_5 has five irreducible representations, namely the trivial singlet, two triplets and one four- and one five-dimensional representation. In the tables below the five conjugacy classes and all relevant subgroups of A_5 are given, together with a typical element or generator of them expressed in the elements S and T . We mention that all subgroups of a certain type are similar to each other.

Class	Typical element
$1C_1$	E
$15C_2$	S
$20C_3$	ST
$12C_5^{(1)}$	T, T^4
$12C_5^{(2)}$	T^2, T^3

Table 3.C.1: Conjugacy classes in A_5

Subgroup		Generators	Subgroup		Generators
$Z_2 \times Z_2$	K_1	$S, T^2ST^3ST^2$	Z_3	C_1	ST
	K_2	T^4ST, ST^3ST^2S		C_2	TS
	K_3	TST^4, ST^2ST^3S		C_3	TST^3
	K_4	T^2ST^3, ST^2ST		C_4	T^2ST^2
	K_5	T^3ST^2, TST^2S		C_5	T^3ST
Z_5	R_1	T		C_6	ST^3ST
	R_2	ST^2		C_7	ST^2ST^3
	R_3	T^2S		C_8	ST^3ST^2
	R_4	TST		C_9	ST^2ST^4
	R_5	TST^2		C_{10}	ST^2ST^2S
	R_6	T^2ST			

Table 3.C.2: Generators of Abelian subgroups of A_5 .

3.C.2 Abelian subgroups of $PSL(2, 7)$

Apart from the two complex conjugated three-dimensional representations $PSL(2, 7)$ has one singlet, one six-, one seven- and one eight-dimensional representation. All classes and relevant subgroups of $PSL(2, 7)$ are collected in the tables below. The cyclic subgroups are similar to each other, while the Klein groups can be divided into two categories, where each order-two element is part of both.

Class	Typical element	Class	Typical element
$1C_1$	E	$21C_2$	S
$56C_3$	ST	$42C_4$	ST^3
$24C_7^{(1)}$	T, T^2, T^4	$24C_7^{(1)}$	T^3, T^5, T^6

Table 3.C.3: Conjugacy classes in $PSL(2, 7)$

Subgroup		Generators	Subgroup		Generators
$Z_2 \times Z_2$	K_1	S, T^2ST^3ST	Z_3	C_1	ST
	K_2	S, TST^3ST^2		C_2	TS
	K_3	$T^4ST^3, T^2ST^4ST^2$		C_3	TST^5
	K_4	T^4ST^3, ST^4ST^3S		C_4	T^2ST^4
	K_5	T^5ST^2, ST^4ST^3S		C_5	T^3ST^3
	K_6	T^2ST^5, ST^3ST^4S		C_6	T^4ST^2
	K_7	T^3ST^4, ST^3ST^4S		C_7	T^5ST
	K_8	$T^3ST^4, T^2ST^4ST^2$		C_8	TST^3S
	K_9	T^6ST, ST^5ST^6		C_9	T^2ST^4S
	K_{10}	TST^6, ST^4ST^4		C_{10}	ST^2ST^5
	K_{11}	T^2ST^5, ST^4ST^4		C_{11}	ST^2ST^4
	K_{12}	T^6ST, ST^3ST^3		C_{12}	T^4ST^2S
	K_{13}	TST^6, ST^2ST		C_{13}	ST^5ST^2
	K_{14}	T^5ST^2, ST^2ST		C_{14}	ST^4ST^2
Z_4	Q_1	T^3S	C_{15}	ST^3ST	
	Q_2	ST^3	C_{16}	ST^2ST^4S	
	Q_3	TST^3	C_{17}	ST^4ST^2S	
	Q_4	T^2ST^2	C_{18}	TST^4ST^5	
	Q_5	TST^2	C_{19}	TST^4ST	
	Q_6	T^3ST	C_{20}	TST^3ST^4	
	Q_7	T^2ST	C_{21}	TST^5ST^2	
	Q_8	TST^5S	C_{22}	T^2ST^5ST	
	Q_9	T^2ST^3S	C_{23}	TST^5ST	
	Q_{10}	T^3ST^2S	C_{24}	ST^3ST^5ST	
	Q_{11}	ST^3ST^4	C_{25}	$ST^2ST^4ST^6$	
	Q_{12}	ST^4ST^3	C_{26}	$ST^2ST^4ST^2$	
	Q_{13}	ST^2ST^3	C_{27}	$ST^2ST^4ST^5$	
	Q_{14}	ST^3ST^2	C_{28}	ST^2ST^4ST	
	Q_{15}	ST^5ST	Z_7	P_1	T
	Q_{16}	ST^2ST^2S		P_2	STS
	Q_{17}	TST^4ST^2		P_3	T^2S
	Q_{18}	T^2ST^4ST		P_4	TST^4
	Q_{19}	TST^5ST^3		P_5	T^4ST
	Q_{20}	$T^2ST^5ST^2$		P_6	ST^2
		P_7	T^2ST^3		

Table 3.C.4: Generators of Abelian subgroups of $PSL(2, 7)$.

3.C.3 Abelian subgroups of $\Delta(96)$

The group $\Delta(96)$ has ten irreducible representations: two singlets, one doublet, six triplets and one sextet. In this appendix we list the conjugacy classes and the generators for the relevant Abelian subgroups of $\Delta(96)$. As explained in the text of section 3.3.5, these are of the type $Z_2 \times Z_2$, Z_3 , Z_4 and Z_8 . Also $Z_4 \times Z_2$ and $Z_4 \times Z_4$ can be used for specific choices of the Z_4 generator, namely Q_1 , Q_2 or Q_3 .

All Klein groups except K are similar to each other and so are the sixteen Z_3 subgroups C_i and all six Z_8 subgroups O_i . The twelve Z_4 subgroups Q_i fall into three categories applying similarity transformations belonging to $\Delta(96)$: the first contains Q_1 , Q_2 and Q_3 , the second one Q_4 , Q_5 and Q_6 and the third the others Q_7, \dots, Q_{12} . The generators of Q_{1-6} all commute. The product group $Q_4 \times Q_5 \times Q_6$ is a normal subgroup of 4O , since the elements of order two fill the class $3C_2$ and those of order 4 fill the class $6C_4$.

Class	Typical element	Class	Typical element
$1C_1$	E	$3C_2$	T^4
$12C_2$	S	$32C_3$	ST
$3C_4^{(1)}$	T^2	$3C_4^{(2)}$	T^6
$6C_4$	ST^2ST^4	$12C_4$	ST^4
$12C_8^{(1)}$	T, T^5	$12C_8^{(2)}$	T^3, T^7

Table 3.C.5: Conjugacy classes in $\Delta(96)$

Subgroup		Generators	Subgroup		Generators
$Z_2 \times Z_2$	K	T^4, ST^4S, ST^4ST^4	Z_8	O_1	T
	K_1	$ST^4\bar{S}T^4, S$		O_2	ST^2
	K_2	T^4, ST^2ST		O_3	T^2S
	K_3	ST^4S, T^7ST		O_4	STS
	K_4	ST^4ST^4, T^6ST^2		O_5	ST^4ST
	K_5	ST^4S, TST^7		O_6	T^5ST
	K_6	T^4, ST^6ST^3			
Z_3	C_1	ST	Z_4	Q_1	T^2
	C_2	ST^3		Q_2	ST^2S
	C_3	ST^5		Q_3	ST^2ST^2
	C_4	ST^7		Q_4	$\bar{S}T^4\bar{S}T^2$
	C_5	T^2ST		Q_5	ST^6ST^2
	C_6	T^2ST^3		Q_6	ST^2ST^4
	C_7	T^4ST		Q_7	$\bar{S}T^4$
	C_8	T^3ST^4		Q_8	T^3ST
	C_9	T^6ST		Q_9	ST^6ST
	C_{10}	TST^2		Q_{10}	T^2ST^2
	C_{11}	T^3ST^2		Q_{11}	TST^3
	C_{12}	T^5ST^2		Q_{12}	ST^2ST^3
	C_{13}	T^4ST^3			
	C_{14}	T^2ST^5			
	C_{15}	TST^4			
	C_{16}	TST^6			

Table 3.C.6: List of elements generating the subgroups $Z_2 \times Z_2$, Z_3 , Z_4 and Z_8 of $\Delta(96)$.

Z_2 groups									
V_1	ST^6ST^3	V_2	ST^2ST	V_3	TST^2S	V_4	T^4ST^2ST	V_5	TST^7
V_6	S	V_7	T^2ST^6	V_8	T^5ST^3	V_9	T^4ST^4	V_{10}	T^7ST
V_{11}	T^3ST^5	V_{12}	T^6ST^2	V_{13}	ST^4S	V_{14}	ST^4ST^4	V_{15}	T^4

Table 3.C.7: List of generating elements of the Z_2 subgroups of $\Delta(96)$.

3.C.4 Abelian subgroups of $\Delta(384)$

The group $\Delta(384)$ has 24 irreducible representations: two singlets, one doublet, 14 triplets (six of which unfaithful) and seven sextets. In this appendix, we list its relevant Abelian subgroups. There are 27 Z_2 subgroups, from which 13 Klein groups can be constructed. Furthermore, there are 64 Z_3 subgroups, 18 Z_4 subgroups, 24 Z_8 subgroups and 12 Z_{16} subgroups. As mentioned in the text about mixing patterns from $\Delta(384)$, apart from G_e being a Klein group or a Z_n group, also $Z_4 \times Z_2$, $Z_4 \times Z_4$, $Z_4 \times Z_8$, $Z_8 \times Z_2$ and $Z_8 \times Z_8$ can be considered.

All Klein groups, except K are similar to each other. The same holds for all Z_3 and Z_{16} subgroups. The Z_4 and Z_8 subgroups fall into three and five categories respectively. Of the Z_4 elements, all of Q_{1-6} commute. The first category contains the elements of Q_1 , Q_2 and Q_3 with only two unique eigenvalues; the second category contains the elements of Q_4 , Q_5 and Q_6 and the third one contains all other elements. A similar structure can be found in the Z_8 subgroups, where all of O_{1-12} commute and five categories are formed by respectively O_{1-3} ; O_{4-6} ; O_{7-9} ; O_{10-12} and all others.

Class	Typical element	Class	Typical element
$1\mathcal{C}_1$	E	$3\mathcal{C}_2$	T^8
$24\mathcal{C}_2$	S	$128\mathcal{C}_3$	ST
$3\mathcal{C}_4^{(1,2)}$	T^4 / T^{12}	$6\mathcal{C}_4$	$ST^4 ST^8$
$24\mathcal{C}_4$	ST^8	$3\mathcal{C}_8^{(1-4)}$	$T^2 / T^6 / T^{10} / T^{14}$
$6\mathcal{C}_8^{(1-6)}$	$ST^2 ST^4 / ST^2 ST^6 / ST^2 ST^8$	$24\mathcal{C}_8^{(1,2)}$	ST^4 / ST^{12}
	$ST^2 ST^{10} / ST^2 ST^{12} / ST^4 ST^{10}$	$24\mathcal{C}_{16}^{(1-4)}$	$T / T^3 / T^5 / T^7$

Table 3.C.8: Conjugacy classes in $\Delta(384)$

Z_2 groups							
V_1	TST^2S	V_2	$ST^{14}ST^7$	V_3	T^3ST^6S	V_4	$T^5ST^{10}S$
V_5	ST^2ST^9	V_6	ST^6ST^3	V_7	$ST^{10}ST^5$	V_8	ST^2ST
V_9	TST^{15}	V_{10}	$ST^{13}ST^{10}$	V_{11}	T^5ST^{11}	V_{12}	TST^3ST^5
V_{13}	$T^{15}ST$	V_{14}	T^4ST^{12}	V_{15}	T^6ST^{10}	V_{16}	T^8ST^8
V_{17}	$ST^6ST^{10}S$	V_{18}	ST^3ST^6	V_{19}	$ST^8ST^{12}ST^{12}$	V_{20}	T^2ST^{14}
V_{21}	$T^{11}ST^5$	V_{22}	S	V_{23}	ST^9ST^2	V_{24}	T^9ST^7
V_{25}	ST^8S	V_{26}	ST^8ST^8	V_{27}	T^8		

Table 3.C.9: List of generating elements of the Z_2 subgroups of $\Delta(384)$.

Klein groups			
K	T^8, ST^8S, ST^8ST^8	K_1	$ST^8S, T^{11}ST^5$
K_2	ST^8S, TST^{15}	K_3	$ST^8S, T^{15}ST$
K_4	ST^8S, T^5ST^{11}	K_5	ST^8ST^8, T^6ST^{10}
K_6	ST^8ST^8, T^4ST^{12}	K_7	ST^8ST^8, T^2ST^{14}
K_8	ST^8ST^8, S	K_9	$T^8, ST^{14}ST^7$
K_{10}	$T^8, ST^{10}ST^5$	K_{11}	T^8, ST^6ST^3
K_{12}	T^8, ST^2ST		

Table 3.C.10: List of generating elements of the Klein groups contained in $\Delta(384)$.

Z_3 groups							
C_1	ST	C_2	TS	C_3	ST^3	C_4	T^3S
C_5	T^5S	C_6	T^7S	C_7	T^9S	C_8	$T^{11}S$
C_9	TST^{14}	C_{10}	T^2ST^{13}	C_{11}	T^3ST^{12}	C_{12}	T^4ST^{11}
C_{13}	T^5ST^{10}	C_{14}	T^6ST^9	C_{15}	T^7ST^8	C_{16}	T^8ST^7
C_{17}	T^9ST^6	C_{18}	$T^{10}ST^5$	C_{19}	$T^{11}ST^4$	C_{20}	$T^{12}ST^3$
C_{21}	$T^{13}ST^2$	C_{22}	$T^{14}ST$	C_{23}	TST^{12}	C_{24}	T^2ST^{11}
C_{25}	T^3ST^{10}	C_{26}	T^4ST^9	C_{27}	T^5ST^8	C_{28}	T^6ST^7
C_{29}	T^7ST^6	C_{30}	T^8ST^5	C_{31}	T^9ST^4	C_{32}	$T^{10}ST^3$
C_{33}	$T^{11}ST^2$	C_{34}	$T^{12}ST$	C_{35}	TST^{10}	C_{36}	T^2ST^9
C_{37}	T^3ST^8	C_{38}	T^4ST^7	C_{39}	T^5ST^6	C_{40}	T^6ST^5
C_{41}	T^7ST^4	C_{42}	T^8ST^3	C_{43}	T^9ST^2	C_{44}	$T^{10}ST$
C_{45}	TST^8	C_{46}	T^2ST^7	C_{47}	T^3ST^6	C_{48}	T^4ST^5
C_{49}	T^5ST^4	C_{50}	T^6ST^3	C_{51}	T^7ST^2	C_{52}	T^8ST
C_{53}	TST^6	C_{54}	T^2ST^5	C_{55}	T^3ST^4	C_{56}	T^4ST^3
C_{57}	T^5ST^2	C_{58}	T^6ST	C_{59}	TST^4	C_{60}	T^2ST^3
C_{61}	T^3ST^2	C_{62}	T^4ST	C_{63}	TST^2	C_{64}	T^2ST

Table 3.C.11: List of generating elements of the Z_3 subgroups of $\Delta(384)$.

Z_4 groups							
Q_1	T^4	Q_2	ST^4S	Q_3	ST^4ST^4		
Q_4	ST^4ST^8	Q_5	ST^8ST^4	Q_6	ST^4ST^{12}		
Q_7	ST^8	Q_8	TST^7	Q_9	T^2ST^6	Q_{10}	T^3ST^5
Q_{11}	T^4ST^4	Q_{12}	T^5ST^3	Q_{13}	T^6ST^2	Q_{14}	T^7ST
Q_{15}	ST^2ST^5	Q_{16}	ST^6ST^7	Q_{17}	$ST^{10}ST^9$	Q_{18}	$ST^{14}ST^{11}$

Table 3.C.12: List of generating elements of the Z_4 subgroups of $\Delta(384)$.

Z_8 groups					
O_1	T^2	O_2	ST^2S	O_3	$ST^{14}ST^{14}$
O_4	ST^6ST^4	O_5	ST^4ST^6	O_6	$ST^{10}ST^{14}$
O_7	ST^2ST^4	O_8	ST^2ST^{14}	O_9	ST^4ST^2
O_{10}	ST^2ST^8	O_{11}	ST^6ST^{14}	O_{12}	ST^8ST^2
O_{13}	ST^4	O_{14}	T^4S	O_{15}	TST^{11}
O_{16}	TST^3	O_{17}	T^2ST^{10}	O_{18}	T^2ST^2
O_{19}	T^3ST^9	O_{20}	T^3ST	O_{21}	ST^2ST^3
O_{22}	ST^6ST	O_{23}	$ST^{10}ST^3$	O_{24}	$ST^{14}ST$

Table 3.C.13: List of generating elements of the Z_8 subgroups of $\Delta(384)$.

Z_{16} groups							
Y_1	T	Y_2	ST^2	Y_3	STS	Y_4	ST^6
Y_5	ST^{10}	Y_6	ST^{14}	Y_7	TST^5	Y_8	TST^9
Y_9	TST^{13}	Y_{10}	ST^4ST	Y_{11}	ST^8ST	Y_{12}	$ST^{12}ST$

Table 3.C.14: List of generating elements of the Z_{16} subgroups of $\Delta(384)$.