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The lead zeppelin : a force sensor without a handle

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Citation

Waarde, B. van. (2016, November 2). *The lead zeppelin : a force sensor without a handle*. *Casimir PhD Series*. Retrieved from <https://hdl.handle.net/1887/43816>

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Author: Waarde, Bob van

Title: The lead zeppelin : a force sensor without a handle

Issue Date: 2016-11-02

Chapter 2

Designing the Lead Zeppelin

2.1 Basic Idea

The Lead Zeppelin experiment consists of a (sub-)millimeter sized superconducting particle that is levitated in a potential well created by a suitable magnetic field profile. It was proposed theoretically by Romero-Isart et al.[20]. Due to the Meissner effect[24], the superconducting particle has a lower energy in low magnetic field than in higher magnetic fields. Therefore, a magnetic field that varies in space can serve as a trapping potential.

Our superconductor of choice is Lead. Because of Lead's relatively high critical temperature, measurements at 4 K are possible. Furthermore, Lead has quite a low melting point, which allows us to use a torch to fabricate (near) spherical particles.

Also, because Lead is a Type I superconductor, there is no pinned flux to worry about. Early experiments, that were performed by placing Lead and Niobium particles between two permanent magnets in a liquid Helium cryostat with optical access, showed that the Lead particles levitated at 4 K, whereas the Niobium particles did not. Niobium, being a Type II superconductor, likely trapped flux and subsequently did not levitate. (With the introduction of an adjustable levitating magnetic field by using coils of current-carrying wire, measurements with a Niobium Zeppelin could be feasible; however, we have not performed such measurements.)

In the following sections of this chapter, we describe the design of the experiment and requirements necessary for a successful measurement. We begin with the heart of the experiment: the various coils and their dimensions, as well as their fabrication and the fabrication of Lead Zeppelins. Then follows some theory: starting with a quick estimation of parameters to give us a feeling of what's going on, in the subsequent section we go into more detail. Next, we treat two notable sources for damping of the motion of the Lead Zeppelin. After addressing some relevant Fourier analysis, we then consider the feasibility of

measurements of the thermal motion of the Lead Zeppelin and the use of the Zeppelin as a probe for small gravitational forces. We discuss the external vibrations coupling into the experiment and end by reviewing the measures we took to fight off electromagnetic noise.

2.2 Experimental Setup

2.2.1 Ingredients

To do the Lead Zeppelin experiment, we need a number of building blocks. First of all, we need levitation coils that provide the confining magnetic potential for the levitating superconductor. Then, we need a way to read out the motion: we do this by measuring flux changes in a pickup coil connected to a SQUID. We also add an excitation coil with which to control the Zeppelin. Since we work with superconductors, we need to work at sufficiently low temperatures. We perform our experiments on a vacuum dipstick inserted in a Dewar of liquid Helium. A schematic overview of the experimental setup is given in figure 2.1.

To make the experiment work it is important to consider the sensitivity of the system to noise. The flux change in the detection coil of the experiment is

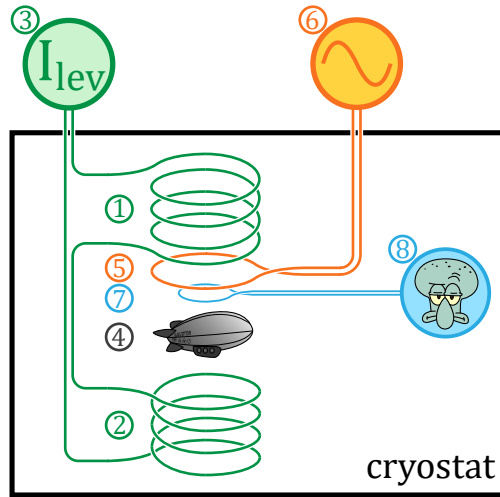


Figure 2.1: The basic Lead Zeppelin experimental setup. We see the **upper (1)** and **lower (2)** levitation coils, wound with opposite polarity, that provide the levitating magnetic field by running a current I_{lev} through them with a **current source (3)**. Placed between the levitation coils is the Lead Zeppelin (4). To control the motion of the Lead Zeppelin, we add an **excitation coil (5)** to the setup, connected to a **function generator (6)**. We read out the motion of the Lead Zeppelin with a **pickup coil (7)** and a **SQUID (8)**. The experiments are performed at low temperatures, typically at 4 K, indicated by the black box.

measured using a SQUID, one of the most sensitive flux sensors available. Because the SQUID is so sensitive it is necessary to shield it from external, unwanted magnetic fields. Very importantly, the current noise in the levitation coils should be very low, as described in section 2.9. We further use the Persistent Current Switch (PCS) described in chapter 3 to reduce the effect of the fluctuations in the current sent by our current source from the experiment.

It is equally important to pay attention to the vibrations in the system. The levitated Lead Zeppelin is a very soft mechanical resonator with a spring constant of $k \approx 10^{-2} \times I_{lev} \frac{\text{N}}{\text{m}}$, I_{lev} in Ampères (section 2.4.2). Its resonance frequencies typically lie between 0.5 Hz and 50 Hz. The measures taken to mitigate external vibrations are described in section 2.8.

In addition to placing a PCS between the levitation coils and the current source, we place line filters between the current source and the leads going to the experiment. We find that without filters, we cannot operate the SQUID while leaving the current source to the levitation coil connected to the experiment, even when the PCS is short circuited in the persistent mode. See also section 2.9. For the same reason, the function generator that we connect to the excitation coil is filtered by using both a line filter and a transformer.

2.2.2 Fabrication of Zeppelins

Early results were obtained with Zeppelins that were simply scraped off of a block of Lead with a knife. These Zeppelins were very chunky; an example can be seen in the bottom right inset in figure 2.2.

To incorporate more symmetry into the experiment, we started fabricating Zeppelins by torching the scraped-off pieces of Lead with a butane gas flame. When the temperature supersedes the melting point of Lead, the particles will form (almost-perfect) spheres, due to the effect of surface tension. For the sake of controlled asymmetry, if desired, the now spherical Zeppelins can be dented slightly. Figure 2.2 shows some of the Zeppelins that were fabricated.

2.2.3 Fabrication of Coils

Figure 2.4 shows the design of the coils that we use. We wind the levitation coils on spindles made from Macor, which is a non-magnetic, easily machinable, ceramic material. The pieces of Macor are held together by a brass clamp, and the whole is inserted into a superconducting Niobium tube.

The coils are made from 100 μm diameter unclad NbTi wire in a Formvar insulation. The levitation coils are each given 275 windings. The excitation coil consists of a single loop. In calculations, we take the center values of the coils as their dimension, as indicated in the figure, and as tabulated in table 2.1.

To read out the motion of the Lead Zeppelin, we use a superconducting pickup coil. Early results were obtained with the chip of figure 2.3 (left). We find (see section 4.1) that this pickup coil distorts the Zeppelin motion because of its

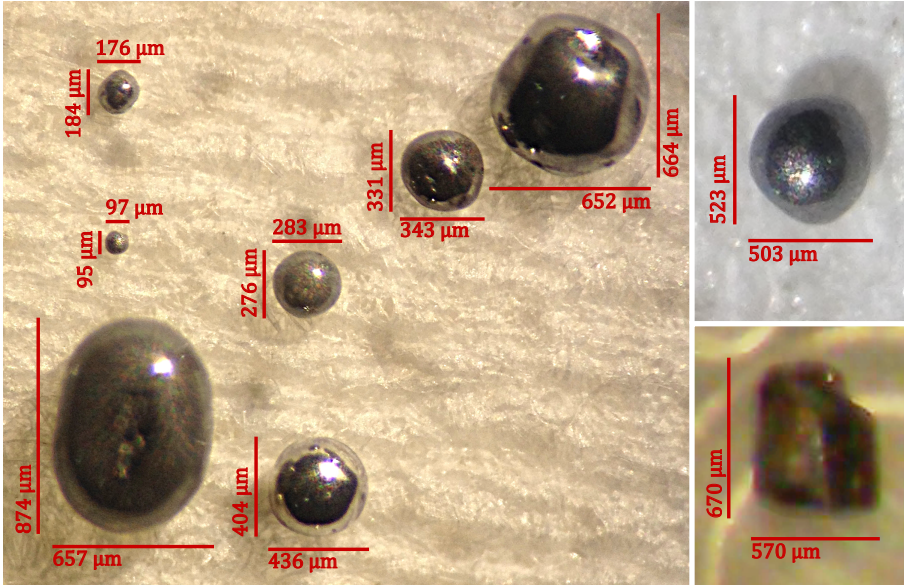


Figure 2.2: Some Zeppelins that were made by torching small chunks of Lead. Some are made slightly aspherical by denting them. The sizes that are indicated next to the Zeppelins are correct to $\sim 10\%$. The two Zeppelins in the top right and bottom left corners of the main panel, as well as the one in the top right inset, have been used in experiments described in this thesis. The bottom right panel shows a more crude Zeppelin that has not been torched, which was used in early experiments.

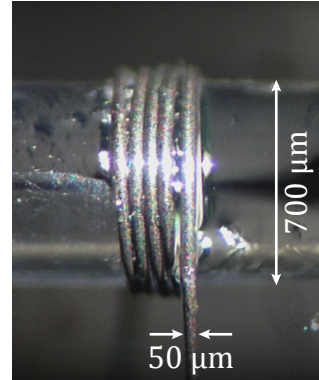
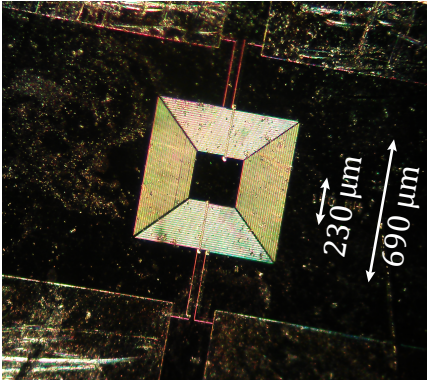


Figure 2.3: Left: an on-chip pickup coil used in early experiments. It is a transformer chip of 2×22 Niobium windings. The two coils of the transformer are wirebonded in series for use as a pickup coil. Right: a self-wound pickup coil during fabrication. It is wound from $50 \mu\text{m}$ diameter Niobium wire with Formvar insulation, around a nylon fishing wire of $700 \mu\text{m}$ diameter. We fixate the coil with some superglue.



Figure 2.4: (a) The design for the coils and the clamp that holds them. The dimensions in millimeters are indicated in the figure. In light blue the pieces of Macor onto which we wind the coils: the two deep, 2 mm wide slots carry the levitation coils; the middle, shallow groove carries the excitation coil. In gold the brass clamp that holds everything together. The dotted lines indicate the chamber in which the Zeppelin is free to vibrate, and where the pickup coil is placed. (b) Zoomed in to the coils. Indicated are the levitation coils, the pickup coil, the excitation coil and their dimensions, the $z = 0$ line and the Lead Zeppelin. (c) A photo of the actual clamp and coils. We wound the levitation coils using 100 μm diameter NbTi wire with a 13 μm Formvar insulating layer and were able to put $N_{lev} = 275$ windings on each Macor coil holder. The $\sim 650 \mu\text{m}$ diameter Zeppelin from figure 2.2 can be seen next to the coils. A green piece of printed circuit board can be seen near the upper levitation coil, behind which the excitation coil is situated. The excitation coil is a single loop made from the same wire as the levitation coils. The piece of PCB is a remnant of an older version of the experiment. Also visible is the Lead capillary that shields the pickup coil wires to the SQUID.

R_{lev}	R_{exc}	R_p
2.5 mm	3.05 mm	375 μm
z_{lev}	z_{exc}	z_p
2.25 mm	200 μm	500 μm
N_{lev}	N_{exc}	N_p
275	1	5

Table 2.1: The dimensions of important components of the experiment, see figure 2.4.

superconducting repulsion of magnetic fields. To make this effect smaller, we wound the pickup coil of figure 2.3 (right).

All cold connections are made by either spot welding or wirebonding. This ensures superconductivity to a very large extent: e.g. as we find out in chapter 3, spot welded connections have a resistance of less than 3.3 p Ω . This means that there is no heating at the cold parts of the experiment, and that there is no loss of signal between the pickup coil and the SQUID.

2.2.4 SQUID Read-out

The motion of the Lead Zeppelin is read-out by a Superconducting QUantum Interference Device (SQUID). A SQUID is a superconducting loop interrupted by two Josephson junctions. The flux in the SQUID is quantized in chunks of the magnetic flux quantum Φ_0 .

$$\Phi_0 = \frac{h}{2e} = 2.067833 \dots \times 10^{-15} \text{ Wb}. \quad (2.1)$$

The addition of the Josephson junctions yields a flux-dependent potential difference $V(\Phi)$ due to the Josephson effect[25].

We use a 2-stage SQUID readout, see figure 2.5. We have a ‘sensor SQUID’, to which we feed the flux from the Zeppelin by connecting the pickup coil to the input coil of the SQUID. Our sensor SQUID is a commercially available Quantum

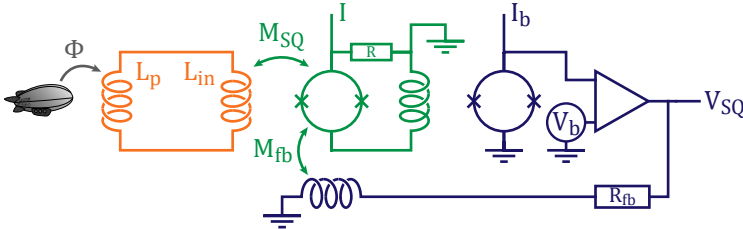


Figure 2.5: The SQUID read-out of the Zeppelin motion. The flux change caused by the Zeppelin in the pickup coil L_p is led to the input coil L_{in} of a sensor SQUID. The potential difference across this SQUID is then fed to an amplifier SQUID. In Flux Locked Loop mode, we keep the flux in the sensor SQUID constant with a feedback coil. We measure V_{SQ} , which is directly linked to the position of the Lead Zeppelin.

Design SQUID that can reach a noise level of $5 \frac{\mu\Phi_0}{\sqrt{\text{Hz}}}$ at 4.2 K[26]. Further, in order not to be limited by room-temperature pre-amplifier noise, we use an ‘amplifier SQUID’ as a low-noise pre-amplifier of $V(\Phi)$. With a resistor R we feed $V(\Phi)$ to the input coil of the amplifier SQUID.

The potential $V(\Phi)$ is dependent on the bias current I through the SQUID. We tune I such that the slope of the potential $\frac{dV(\Phi)}{d\Phi}$ is largest, which means that we are very sensitive to flux changes. We do feedback on the flux in the SQUID with a feedback coil: by keeping the flux in the SQUID loop constant, we stay at large $\frac{dV(\Phi)}{d\Phi}$ and stay very sensitive. Ultimately, V_{SQ} as indicated in figure 2.5 is the signal that we use in this thesis to characterize the position of the Lead Zeppelin.

2.3 The Principle of Levitation

The Lead Zeppelin levitates because of the Meissner effect. In a magnetic field $\mathbf{B}$, a small spherical superconducting particle will expel all magnetic field and will have a magnetisation $\mathbf{M} = -\mu_0\mathbf{B}$, with $\mu_0 = 4\pi \times 10^{-7} \frac{\text{N}}{\text{A}^2}$ the magnetic permeability of free space. It will have a magnetic moment $\mathbf{m} = V\mathbf{M}$, with V the volume of the particle. The energy of the system is $E = -\mathbf{m} \cdot \mathbf{B} = \frac{V}{\mu_0} |\mathbf{B}|^2$. We assume for now that the magnetic field is constant over the size of the particle. Even though this is not completely accurate in our case, it does quickly sketch a fairly decent picture of what’s going on. In section 2.4, we will dive deeper into the theory.

The magnetic field profile between the two levitation coils, which are put in an anti-Helmholtz configuration, is shown in figure 2.6. It shows a zero in magnetic field in the center of symmetry of the coils. Near this zero the magnetic field is well approximated by its first order Taylor expansion

$$\mathbf{B} = \frac{\partial B_r}{\partial r} r \hat{\mathbf{r}} + \frac{\partial B_z}{\partial z} z \hat{\mathbf{z}} \equiv G_r r \hat{\mathbf{r}} + G_z z \hat{\mathbf{z}}, \quad (2.2)$$

where the magnetic field gradients $G_{r,z}$ will depend on the actual configuration of the levitation coils. Substituting the magnetic field of the anti-Helmholtz configuration into the expression for the energy of the particle results in a harmonic potential given by

$$E = \frac{V}{\mu_0} (G_r^2 r^2 + G_z^2 z^2). \quad (2.3)$$

The force on the particle due to the anti-Helmholtz magnetic field is given by $\mathbf{F}_{\text{aHH}} = -\nabla E = -2 \frac{V}{\mu_0} (G_r^2 r \hat{\mathbf{r}} + G_z^2 z \hat{\mathbf{z}})$. Simultaneously, the gravitational force acting on it is given by $\mathbf{F}_g = -m_{zep} g \hat{\mathbf{z}}$. Since $\mathbf{F}_{\text{aHH}} + \mathbf{F}_g = \mathbf{0}$ in equilibrium, we can solve for the equilibrium height z_{eq} at which the particle levitates:

$$z_{eq} = -\frac{\mu_0 m_{zep} g}{2V G_z^2} = -\frac{\mu_0 \rho_{zep} g}{2G_z^2}. \quad (2.4)$$

When we fill in G_z in units of $\frac{\text{T}}{\text{m}}$, this reduces to

$$z_{eq} = -\frac{70}{G_z^2} \text{ mm}, \quad (2.5)$$

The force $\mathbf{F}_{aHH}$ also immediately yields the spring constants $k_{r,z} = 2 \frac{V}{\mu_0} G_{r,z}^2$. This in turn gives the resonance frequencies

$$f_{r,z} = \frac{1}{2\pi} \sqrt{\frac{k_{r,z}}{m_{zep}}} = \frac{1}{2\pi} \sqrt{\frac{2G_{r,z}^2}{\mu_0 \rho_{zep}}} = 1.9 \times |G_{r,z}| \text{ Hz}, \quad (2.6)$$

with $G_{r,z}$ in $\frac{\text{T}}{\text{m}}$. Note that to this order, $f_{r,z}$ and z_{eq} are not determined by the size of the particle, but only by the magnetic field gradients $G_{r,z}$ and the density of the Lead Zeppelin $\rho_{zep} = \rho_{Pb} = 11.34 \times 10^3 \frac{\text{kg}}{\text{m}^3}$.

One could envision using other superconductors than Lead. For instance Niobium, which is a Type II superconductor, but which has a higher critical temperature than Lead, or Aluminium, which has a lower critical field and temperature, but which is much lighter than Lead. The densities of Niobium and Aluminium are $\rho_{Nb} = 8.57 \times 10^3 \frac{\text{kg}}{\text{m}^3}$ and $\rho_{Al} = 2.70 \times 10^3 \frac{\text{kg}}{\text{m}^3}$. Therefore

$$f_{r,z} = 2.2 \times |G_{r,z}| \text{ Hz}, \quad z_{eq} = -\frac{53}{G_z^2} \text{ mm}, \quad \text{Niobium}, \quad (2.7)$$

and

$$f_{r,z} = 3.9 \times |G_{r,z}| \text{ Hz}, \quad z_{eq} = -\frac{17}{G_z^2} \text{ mm}, \quad \text{Aluminium}, \quad (2.8)$$

where again we fill in $G_{r,z}$ in units of $\frac{\text{T}}{\text{m}}$.

To get a feeling of what to expect experimentally, we can extract some numbers from the simple picture sketched above. We begin with the on-axis magnetic field produced by a loop of wire of radius R carrying a current I , which is given by

$$B = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}. \quad (2.9)$$

Its derivative to z is

$$G_z = -\frac{3\mu_0 I}{2} \frac{R^2 z}{(R^2 + z^2)^{5/2}}. \quad (2.10)$$

For two anti-Helmholtz coils of radius $R_{lev} = 2.5 \text{ mm}$, positions $z_{lev} = 2.25 \text{ mm}$, current $I_{lev} = 1.0 \text{ A}$ and number of windings $N = 275$ the combined gradient evaluated at $z = 0$ is

$$G_z = -3\mu_0 N I_{lev} \frac{R_{lev}^2 z_{lev}}{(R_{lev}^2 + z_{lev}^2)^{5/2}} = -34 \frac{\text{T}}{\text{m}}. \quad (2.11)$$

It then follows that

$$f_z = 65 \text{ Hz}, \quad z_{eq} = -61 \text{ } \mu\text{m}. \quad (2.12)$$

Likewise $f_z = 75 \text{ Hz}$, $z_{eq} = -46 \text{ } \mu\text{m}$ for Niobium and $f_z = 133 \text{ Hz}$, $z_{eq} = -15 \text{ } \mu\text{m}$ for Aluminium.

The levitating field $\mathbf{B}$ scales linearly with the current I_{lev} sent through the levitation coils. Consequently, the spring constants $k_{r,z} \propto I_{lev}^2$ and the resonance frequencies $f_{r,z} \propto I_{lev}$. This will serve as our ‘smoking gun’: as we change the levitation current I_{lev} , the resonances should change along with it.

For a Zeppelin of size $V = (500 \text{ } \mu\text{m})^3$ the assumption that the magnetic field is constant everywhere in the particle is not valid with coils the size of ours. In the next section, we present a more detailed analysis of the levitating Lead Zeppelin.

2.4 The Finite Sized Lead Zeppelin

In the previous section, we calculated the levitation parameters of the Lead Zeppelin experiment while assuming the Lead Zeppelin to be infinitesimally small. In this section, we find out what happens when we don’t neglect the finite size of the Lead Zeppelin.

We begin by calculating the flux in the pickup coil and SQUID as a consequence of there being a levitating superconductor between the levitation coils, as a function of position and levitation current. We then calculate the spring constants and resonance frequencies in the three spatial directions, taking the size of the Lead Zeppelin into account to 1st order. This calculation also yields the excitation force exerted by the excitation coil on the Lead Zeppelin, which means that we have all the relevant parameters in the equation of motion. After that, we calculate the equilibrium position due to the effect of gravity, and we find out the limits of levitation. The equilibrium position helps us understand that the Lead Zeppelin will behave as a Duffing oscillator at low levitation currents, and that if the Zeppelin is too large the magnetic field may not be smaller than the critical field everywhere inside the Zeppelin.

2.4.1 Coupling the Particle Motion to the SQUID

In order to calculate what the flux in the pickup coil is as a function of the position of the Lead Zeppelin in the levitating magnetic field, we approximate the Zeppelin as being cylindrically shaped. We use the equation derived in [27] that relates the surface current density $\mathbf{j}_S$ on the surface S of a superconductor of arbitrary shape in a magnetic field $\frac{1}{\mu_0} \mathbf{B}_{lev} = \mathbf{H}_{lev}$:

$$\mathbf{j}_S = 2 \hat{\mathbf{n}} \times \mathbf{H}_{lev}, \quad (2.13)$$

with $\hat{\mathbf{n}}$ the unit vector normal to S . We further use that the flux Φ in the pickup coil is $\Phi = MI$ and that the mutual inductance between two coaxial, circular loops of current with radii $R_{1,2}$ a distance l apart is

$$M(R_1, R_2, l) = \frac{\mu_0}{2} \int_0^{2\pi} \frac{R_1 R_2 \cos \xi}{(R_1^2 + R_2^2 + l^2 - 2R_1 R_2 \cos \xi)^{1/2}} d\xi, \quad (2.14)$$

which follows from the Neumann formula for the mutual inductance. The end result is a function $\Phi(z_{zep})$ that calculates the flux in the pickup coil from the position z_{zep} of the Zeppelin.

We shall only look at movement in the z -direction here, leaving movement in r as an exercise for the future. This would require using a more complete expression for the mutual inductance, where the two loops no longer remain on the same axis.

We model the Lead Zeppelin as a cylindrical superconductor with radius R_{zep} and thickness h_{zep} , that is placed longitudinally to the z -axis of a cylindrical coordinate system in which $z = z_{zep}$ is the location of the superconductor's center point. We place it in the experimental setup as shown in figure 2.4. There are two levitation coils on either side of the superconductor in an anti-Helmholtz configuration; one coil of N_{up} windings above the superconductor at $z = z_{up}$, given a current I_{up} , and another with N_{down} windings below the superconductor at $z = z_{down}$ given a current I_{down} . Both coils have a radius R_{lev} . At $z = z_p$ we place a pick-up coil of radius R_p , that is used to pick up the flux Φ induced in it by the surface currents on the Zeppelin.

According to the Biot-Savart law, a single infinitesimally thin circular loop of wire of radius R , carrying a current I , that is positioned coaxially to the z -axis of a cylindrical coordinate system (r, z, φ) at a height $z = z_{loop}$, induces a magnetic field of:

$$\begin{aligned} H_r(r, z) &= \frac{IR}{4\pi} \int_0^{2\pi} \frac{(z - z_{loop}) \sin \xi}{(R^2 + r^2 + (z - z_{loop})^2 - 2rR \sin \xi)^{3/2}} d\xi, \\ H_z(r, z) &= \frac{IR}{4\pi} \int_0^{2\pi} \frac{R - r \sin \xi}{(R^2 + r^2 + (z - z_{loop})^2 - 2rR \sin \xi)^{3/2}} d\xi, \\ H_\varphi &= 0. \end{aligned} \quad (2.15)$$

The total field produced by the two levitation coils will be taken to be the sum of many such thin loops: in the above we replace $R \rightarrow R_{up,down}$, $z_{loop} \rightarrow z_{up,down}$ and $I \rightarrow N_{up,down} I_{up,down}$ for the two coils and then we add their contributions together, $\mathbf{H}_{lev} = \mathbf{H}_{up} + \mathbf{H}_{down}$. Also, from here on we employ the symmetries $R_{up} = R_{down} \equiv R_{lev}$, $z_{up} = -z_{down} \equiv z_{lev}$, $I_{up} = -I_{down} \equiv I_{lev}$ and $N_{up} = N_{down} \equiv N_{lev}$ that reflect the current experimental setup of figure 2.4. For the record: we are taking identical R_{lev} and z_{lev} for each of the 275 windings that make up a levitation coil.

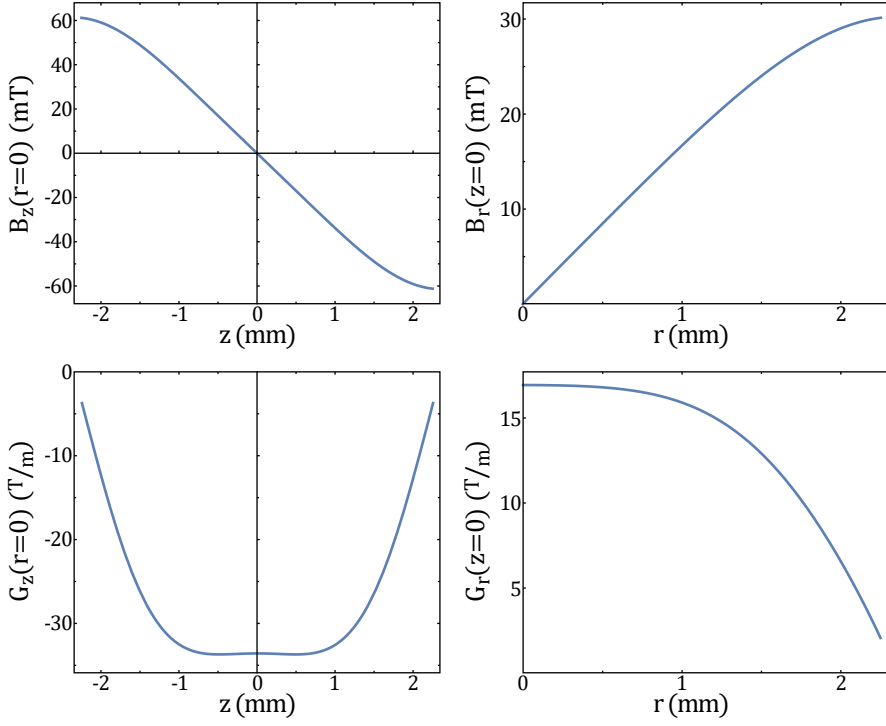


Figure 2.6: The levitating magnetic fields $B_{r,z}$ at $I_{lev} = 1$ A and the gradients $G_{r,z}$ ($B_{r,z} \propto I_{lev}$).

In figure 2.6 we plot the magnetic field components of $\mu_0 \mathbf{H}_{lev} = \mathbf{B}_{lev}$ and the gradients, at $I_{lev} = 1$ A for the standard experimental setup parameters as listed in table 2.1.

With these expressions for the levitating magnetic field, we calculate the surface current density and consequently the flux in the pickup coil. To this end, let us decompose the superconducting cylinder into its side, top and bottom surfaces. Beginning with the side surface, consider an infinitesimally thin circular loop of radius R_{zep} and height dz . For the side surface, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ so

$$\mathbf{j}_{S,side} = 2H_z \hat{\boldsymbol{\varphi}}, \quad (2.16)$$

and since $\Phi = MI$ the flux in the pick-up coil caused by the thin loop is

$$d\Phi_{side} = M j_{S,side} dz = 2M H_z dz. \quad (2.17)$$

The total contribution from the superconductor's side surface to the flux is the

sum of many of such thin loops:

$$\Phi_{side} = 2 \int_{z_{zep} - \frac{1}{2}h_{zep}}^{z_{zep} + \frac{1}{2}h_{zep}} M(R_p, R_{zep}, z_p - z) H_z(R_{zep}, z) dz. \quad (2.18)$$

For the top surface, consider an infinitely thin circular loop of inner radius r and outer radius $r + dr$. Here, $\hat{\mathbf{n}} \times \mathbf{H}_{lev} = -H_r \hat{\boldsymbol{\phi}}$ and the contribution from the thin loop to the flux is $d\Phi_{top} = M j_{S,top} dr$. Summing:

$$\Phi_{top} = -2 \int_0^{R_{zep}} M(R_p, r, z_p - (z_{zep} + \frac{1}{2}h_{zep})) H_r(r, z_{zep} + \frac{1}{2}h_{zep}) dr. \quad (2.19)$$

Similarly, for the bottom surface we calculate

$$\Phi_{bottom} = 2 \int_0^{R_{zep}} M(R_p, r, z_p - (z_{zep} - \frac{1}{2}h_{zep})) H_r(r, z_{zep} - \frac{1}{2}h_{zep}) dr. \quad (2.20)$$

The total flux through the pick-up coil due to the surface currents in the superconducting cylinder induced by the levitation coils is the sum of these three contributions:

$$\Phi = \Phi_{side} + \Phi_{top} + \Phi_{bottom}. \quad (2.21)$$

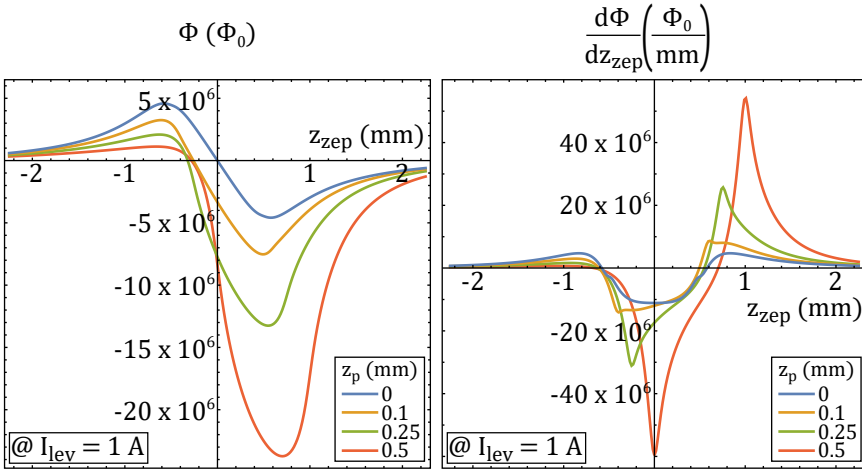


Figure 2.7: Left: the flux in the pickup coil Φ as a function of the position of the Zeppelin z_{zep} in units of Φ_0 ; right: $\frac{d\Phi}{dz_{zep}}$ as a function of z_{zep} . Calculated for some locations of the pickup coil z_p , $I_{lev} = 1$ A ($\Phi \propto I_{lev}$), the experimental parameters of table 2.1, and a cylindrical Zeppelin of dimensions $R_{zep} = 325$ μm , $h_{zep} = 1$ mm. We see that the flux change is largest when the Zeppelin enters and leaves the pickup coil.

Figure 2.7 shows the flux in the pickup coil and its derivative as a function of the position of the Zeppelin, for a few locations of the pickup coil z_p . It is calculated for the standard values of the setup of table 2.1, and $I_{lev} = 1$ A, $R_{zep} = 325$ μm and $h_{zep} = 1$ mm. We see that the flux change is biggest at the moments that the Zeppelin enters or leaves the pickup coil. The graphs of figure 2.7 have the symmetry $\Phi(z_{zep}, z_p) = \Phi(-z_{zep}, -z_p)$, i.e. when the pickup coil is moved from z_p to $-z_p$, the graphs are mirrored in the Φ -axis.

As depicted in figure 2.5, we connect the pickup coil to the input coil of a SQUID to measure this flux. The measured flux in the SQUID Φ_{SQ} follows from Kirchhoff's Laws:

$$\Phi_{SQ} = \Phi \frac{M_{SQ}}{L_{in} + L_p + \frac{R}{i\omega}} = \frac{\Phi}{220}. \quad (2.22)$$

With $L_p \approx 0.2$ μH the inductance of the pickup coil, $L_{in} = 2$ μH the inductance of the input coil of the SQUID, $M_{SQ} = 10$ nH the mutual inductance from L_{in} to the SQUID loop and R the resistance between L_p and L_{in} which is zero for a superconducting connection.

For future reference, we calculate Φ_{SQ} for some realistic values. In our case $z_p = 0.5$ mm, so at $z_{zep} = -0.7$ mm, figure 2.7 tells us that at $I_{lev} = 1$ A, the flux in the pickup coil is $\Phi = 1.1 \times 10^6 \Phi_0$ and $\frac{d\Phi}{dz_{zep}} = 1.9 \times 10^5 \frac{\Phi_0}{\text{mm}}$. At the more representative levitation current $I_{lev} = 0.35$ A, the flux in the SQUID Φ_{SQ} then is:

$$\Phi_{SQ} = 1750 \Phi_0, \quad \frac{d\Phi_{SQ}}{dz_{zep}} = 302 \frac{\Phi_0}{\text{mm}}. \quad (2.23)$$

For more signal, it would be better to replace the pickup coil to a more optimal position: it could increase $\frac{d\Phi}{dz_{zep}}$ greatly, especially for low equilibrium positions of the Zeppelin around $z_{eq} = -0.5$ mm.

2.4.2 Force Balance, Equation of Motion

The equation of motion of a damped, driven, harmonic oscillator is given by

$$m\ddot{x}(t) + \gamma\dot{x}(t) + kx(t) = F_{exc}(t). \quad (2.24)$$

In the case of the Lead Zeppelin, we can say to good approximation that the spring constant is provided by the levitation coils $k \propto I_{lev}^2$, and that the excitation coil provides the drive $F_{exc}(t) \propto I_{exc}(t)$.

For a driving force $F_{exc}(t) = F_0 \cos(\omega t)$ at frequency $\omega = 2\pi f$ (a phase could be added, but we will omit this for now) the solution to the equation of motion can be split in a steady state solution and a transient solution. For large enough t , the transient solution will have died out leaving only the steady state solution. The steady state solution is

$$x(t) = x_{\text{ampl}}(\omega) \cos(\omega t - \theta(\omega)), \quad (2.25)$$

with

$$x_{\text{ampl}}(\omega) = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{\omega\omega_0}{Q}\right)^2}}, \quad (2.26)$$

and

$$\theta(\omega) = \arctan\left(\frac{\omega\omega_0}{Q(\omega^2 - \omega_0^2)}\right). \quad (2.27)$$

The Q -factor is related to the damping coefficient as $Q = \frac{\omega_0 m}{\gamma}$ and $\omega_0 = 2\pi f_0 = \sqrt{\frac{k}{m}}$.

To calculate how exactly k and F_0 depend on the system parameters, we will begin by calculating the force on the Lead Zeppelin in a magnetic field $\mathbf{B}_{ext}$ using the formula derived in [27]

$$\mathbf{F} = \frac{2}{\mu_0} \oint dS [\mathbf{B}_{ext}(\hat{\mathbf{n}}_{zep} \cdot \mathbf{B}_{ext}) - \hat{\mathbf{n}}_{zep} |\mathbf{B}_{ext}|^2], \quad (2.28)$$

where the integral covers the whole of the Lead Zeppelin's surface S and where $\hat{\mathbf{n}}_{zep}$ is the unit vector normal to S . The magnetic field $\mathbf{B}_{ext}$ will be taken to be the sum of the magnetic fields provided by the two levitation coils and the excitation coil

$$\mathbf{B}_{ext} = \mathbf{B}_{lev} + \mathbf{B}_{exc}. \quad (2.29)$$

The magnetic fields will be assumed to be induced by sums of circular current loops, equation (2.15).

We go from the cylindrical coordinates of the previous section back to standard cartesian coordinates and give the Zeppelin dimensions $L_{x,y,z}$. We no longer assume the Zeppelin to be a small cylinder, as in the approximation we are about to make, this no longer makes any difference. What we gain is that we can include dynamics in the x - and y -directions.

In order for the integral to be calculable in a time span short enough to be practical, we shall restrict ourselves to three 1st order Taylor expansions: in the dimensions of the Lead Zeppelin $L_{x,y,z}$, the movement away from the origin $(x, y, z) = (\varepsilon_x, \varepsilon_y, \varepsilon_z)$ and the excitation current I_{exc} . The end result then looks like this:

$$\mathbf{F}_{tot} = k_x \varepsilon_x \hat{\mathbf{x}} + k_y \varepsilon_y \hat{\mathbf{y}} + (k_z \varepsilon_z + \kappa I_{exc}) \hat{\mathbf{z}}. \quad (2.30)$$

The parameter κ takes the excitation current to the excitation force $F_{exc} = \kappa I_{exc}$.

To first order, the excitation coil can be used only to excite the Lead Zeppelin in the z -direction, due to the symmetry of the setup. The excitation coil can in fact be used to excite the other directions as well, which is derived in appendix

A, but this is a higher order effect: the Lead Zeppelin has to be somewhere other than $x = 0$ or $y = 0$ for this to be possible.

It should be noted that the effect of flux reduction by the levitation coils in the persistent mode, as explained in appendix B, is not taken into account here; i.e. the flux produced by the excitation coil is counteracted, causing a less efficient excitation, and the flux produced by the motion of the Zeppelin is counteracted, causing a small reduction of the picked up flux in the pickup coil.

The spring constants are

$$k_x = k_y = K (5R_{lev}^2 - 3z_{lev}^2), \quad k_z = K (2R_{lev}^2 + 6z_{lev}^2), \quad (2.31)$$

with

$$K = \frac{3\mu_0}{2\pi} \frac{L_x L_y L_z R_{lev}^2 z_{lev}^2}{(R_{lev}^2 + z_{lev}^2)^5} I_{lev}^2. \quad (2.32)$$

The resonance frequencies are $f_{x,y,z} = \frac{1}{2\pi} \sqrt{\frac{k_{x,y,z}}{m_{zep}}}$. Like we found in section 2.3, here too we find the resonance frequencies to be proportional to I_{lev} and to be independent of the dimensions of the Zeppelin.

The coefficient κ is

$$\begin{aligned} \kappa = & -K_I \frac{1}{z_{exc}} (R_{exc}^2 + z_{exc}^2) (R_{lev}^2 + z_{lev}^2) \\ & \times (R_{lev} (R_{exc}^3 + (R_{exc} + R_{lev}) z_{exc}^2) + z_{lev}^2 z_{exc}^2), \end{aligned} \quad (2.33)$$

with

$$K_I = \frac{3\mu_0}{2\pi} \frac{L_x L_y L_z R_{exc} R_{lev} z_{exc} z_{lev}}{(R_{exc}^2 + z_{exc}^2)^{7/2} (R_{lev}^2 + z_{lev}^2)^{7/2}} I_{lev}. \quad (2.34)$$

We note that $\kappa \propto I_{lev}$; in other words, the amount by which we can excite the Lead Zeppelin depends not only on the strength of the excitation I_{exc} , but also on the strength of the magnetic levitation potential.

For a Zeppelin of dimensions $L_x = 300 \mu\text{m}$, $L_y = 400 \mu\text{m}$, $L_z = 500 \mu\text{m}$, therefore having a mass $m = 0.68 \text{ mg}$, and at $I_{lev} = 1 \text{ A}$, we find

$$k_x = k_y = 7.5 \times 10^{-3} \frac{\text{N}}{\text{m}}, \quad k_z = 20 \times 10^{-3} \frac{\text{N}}{\text{m}}, \quad (2.35)$$

leading to

$$f_x = f_y = 16.7 \text{ Hz}, \quad f_z = 27.2 \text{ Hz}, \quad (2.36)$$

and

$$\kappa = -1.06 \times 10^{-7} \frac{\text{N}}{\text{A}}. \quad (2.37)$$

2.4.3 Equilibrium Position and the Limits of Levitation

For the levitating force F on the Lead Zeppelin over the whole range of z , we favour the formula for spherical superconductors derived in [28] over equation (2.28), because it doesn't require Taylor expanding. (It is, however, harder to include an r -dependence, for which reason we did not use it earlier.)

Let the Zeppelin be a sphere of radius R_{zep} located at $z = 0$. When there is a circular loop of current I_{lev} with radius R_{lev} at a height z_{lev} , coaxial to the z -axis, the force on the Zeppelin due to the current loop is

$$\mathbf{F} = \alpha I_{lev}^2 \hat{\mathbf{z}}, \quad (2.38)$$

where

$$\begin{aligned} \alpha = & -\mu_0 R_{zep} S_{lev}^3 (S_{lev}^2 - R_{zep}^2) \cos \theta_{lev} \sin^2 \theta_{lev} \\ & \times \int_0^\pi \frac{d\xi \cos \xi}{(R_{zep}^4 + S_{lev}^4 - 2R_{zep}^2 S_{lev}^2 (\cos^2 \theta_{lev} + \sin^2 \theta_{lev} \cos \xi))^{3/2}}, \end{aligned} \quad (2.39)$$

in which

$$S_{lev} = \sqrt{R_{lev}^2 + z_{lev}^2}, \quad \theta_{lev} = \arctan \left(\frac{R_{lev}}{z_{lev}} \right). \quad (2.40)$$

In our case there are two such current loops at $\pm z_{lev}$ of N_{lev} windings. Including a z -dependence by doing $z_{lev} \rightarrow z_{lev} - z_{zep}$, filling in the experimental setup parameters of table 2.1, and using $R_{zep} = 325 \mu\text{m}$, yields the levitating force F on the Zeppelin for $-z_{lev} \leq z_{zep} \leq z_{lev}$; we show a plot of F in figure 2.8.

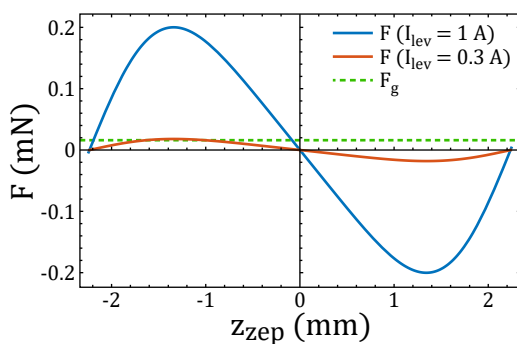


Figure 2.8: The levitating force F versus the position of the Lead Zeppelin z_{zep} , when it is a spherical particle of $650 \mu\text{m}$ diameter ($m = 1.6 \text{ mg}$). For comparison, also the gravitational force $F_g = mg = 16 \mu\text{N}$ is plotted. At $I_{lev} = 1 \text{ A}$, F_g pulls the Zeppelin away from $z = 0$ by only a small amount; we're still in the linear part of F . At $I_{lev} = 0.3 \text{ A}$, gravity pulls the Zeppelin off-center so much, that we are deep in the non-linear part of the levitating force.

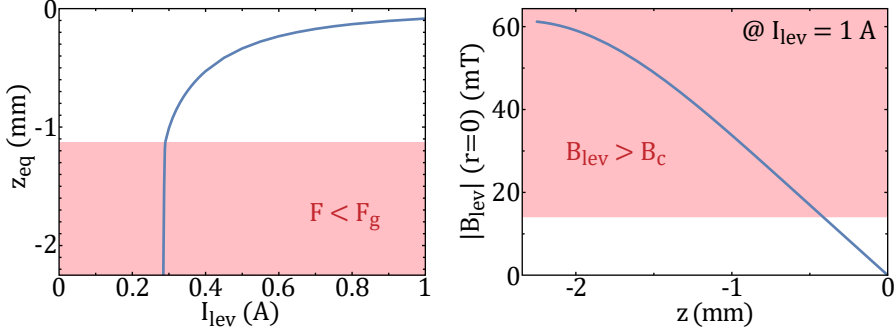


Figure 2.9: Left: the equilibrium position z_{eq} where the Zeppelin levitates as a function of levitation current I_{lev} , for a spherical particle of 650 μm diameter ($m = 1.6$ mg). Right: the levitating magnetic field B_{lev} at $I_{lev} = 1$ A as a function of Zeppelin position z . In red we indicate where the Zeppelin does not levitate: at too low levitation currents the force of gravity is stronger than the levitating force, and at too high levitation currents the levitating magnetic field is higher than the critical magnetic field of Lead at $T = 4.2$ K.

Due to gravity, the Zeppelin will be pulled downwards. For high enough levitation currents, e.g. 1 A, the Lead Zeppelin will hover in the linear regime of the force curve. However, at lower levitation currents the levitating force gets weaker and gravity will make the Zeppelin hover in the non-linear part.

To get the equilibrium position z_{eq} at a certain I_{lev} , we equate the levitation force to the gravitational force, $F(z_{eq}) = mg$, and solve for z_{eq} . The result of doing this can be seen in figure 2.9, where we plot z_{eq} as a function of I_{lev} . In this particular calculation, the Zeppelin doesn't levitate below $I_{lev} \approx 0.29$ A.

In figure 2.9 we also plot the strength of the levitating magnetic field. In red, we indicate in both plots the limit of levitation: the levitating force must be strong enough to overcome the force of gravity, but the levitating field B_{lev} must remain smaller than the critical field $B_{c,Pb}$ of Lead. We calculate $B_{c,Pb}$ according to

$$B_{c,Pb}(T) = \left(1 - \frac{T}{T_{c,Pb}}\right)^2 B_{c,Pb}(T = 0 \text{ K}) = 13.9 \text{ mT @ } 4.2 \text{ K}, \quad (2.41)$$

where $T_{c,Pb} = 7.2$ K and $B_{c,Pb}(T = 0 \text{ K}) = 80$ mT[29].

2.4.4 Duffing Oscillator

The force as plotted in figure 2.8 shows that for big deviations from $z = 0$, the curve departs from linear behaviour and acquires a z^3 component. We can therefore expect that when the Zeppelin moves in this non-linear part of the curve, it will behave as a Duffing oscillator.

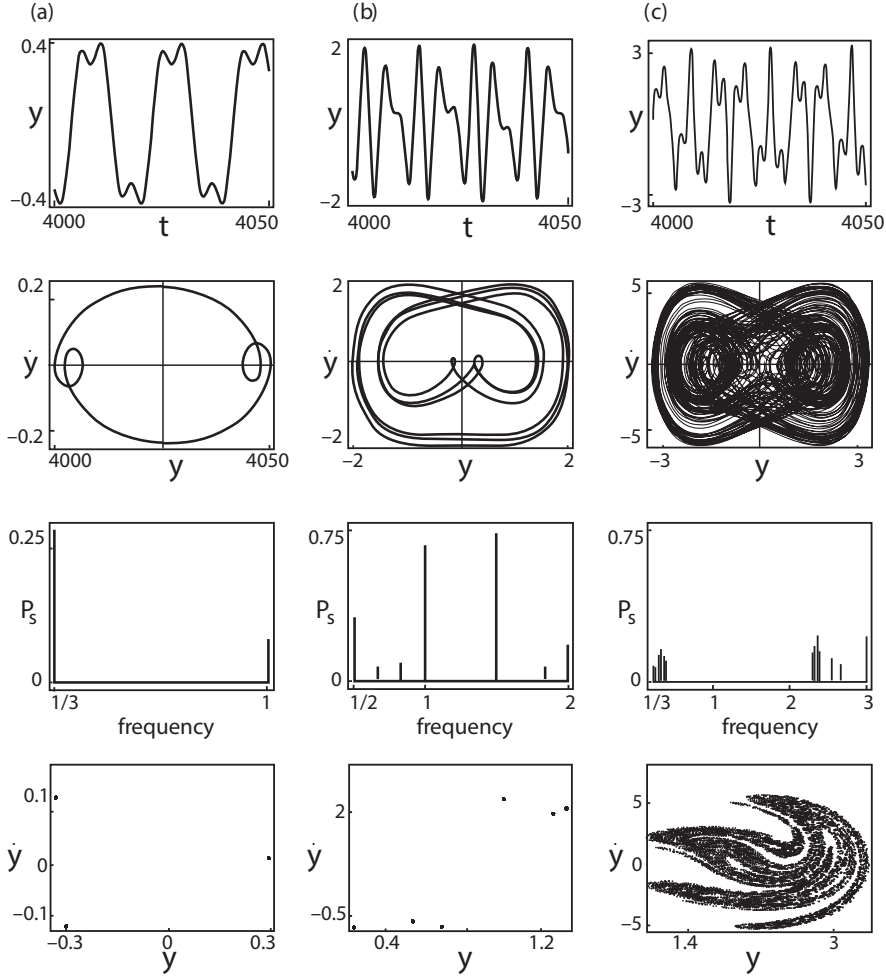


Figure 2.10: Reprinted with permission from [30, fig. 5.13], © John Wiley & Sons, Ltd. Some solutions to the Duffing equation (2.42), for $\omega_d = 1 \frac{\text{rad}}{\text{s}}$, $\frac{\delta}{m} = 1 \frac{\text{N}}{\text{m}^3}$, $\frac{\gamma}{m} = 0.02 \text{ s}^{-1}$ and zero spring constant. Left to right: $\frac{F_0}{m} = 0.1, 0.9, 0.7 \frac{\text{m}}{\text{s}^2}$. Top to bottom: time plot, phase plot, power spectrum, Poincaré section. The Poincaré sections are obtained by evaluating $(y(t), \dot{y}(t))$ at integer multiples of $T_d = \frac{2\pi}{\omega_d}$.

The equation of motion for a Duffing oscillator is equal to the one for a harmonic oscillator (eq. (2.24)) plus a cubic term:

$$m\ddot{z}(t) + \gamma\dot{z}(t) + k_z z(t) + \delta z^3(t) = F_{ext}(t). \quad (2.42)$$

The Duffing parameter δ determines the amount of non-linearity of the oscillator.

The Duffing equation (2.42) is notoriously hard to solve. In some limiting cases, however, it gets a little easier. In our case, for instance, the spring constant vanishes at the extrema of the force curve $k = \frac{dF}{dz} = 0$. At low levitation currents, gravity will pull the Zeppelin down towards the extremum below $z = 0$.

So suppose that gravity has pulled the Zeppelin all the way down to this extremum. Also, suppose that we drive the Zeppelin with $F_{ext} = F_0 \cos(\omega_d t)$ at $\omega_d = 1 \frac{\text{rad}}{\text{s}}$ and that $\frac{\delta}{m} = 1 \frac{\text{N}}{\text{m}^3}$. This particular form of the Duffing oscillator is also known as the ‘Ueda oscillator’[30]. Figure 2.10 shows some solutions to this equation.

There are many interesting features about these solutions, but here we only want to point out the form of the time signals and the emergence of sub-harmonics in the power spectrum. In our experiments, we have found that the Lead Zeppelin sometimes produces time-signals that greatly resemble the ones of figure 2.10. We will show some of those in section 4.3. Depending on the shape of the Zeppelin, the amplitude of its motion and the equilibrium position on the force curve, there will be a high degree of chaos (c), or less chaos, with an almost ‘square’ time trace (a). The fact that sub-harmonics show up, is interesting to keep in mind in future attempts to link peaks in the power spectrum to Zeppelin resonances.

2.5 Lockin Measurement

A typical measurement that we perform involves exciting the Lead Zeppelin with the excitation coil connected to a function generator. The function generator supplies an AC voltage at a frequency f of amplitude V_{fg} , which we can convert to an excitation force:

$$F_{ext}(t) = \kappa I_{exc}(t) = \kappa \frac{V_{fg}}{R_{fg}} \cos(2\pi f t), \quad (2.43)$$

where R_{fg} is the sum of the function generator’s impedance (50 Ω) and the resistance in the wiring to the excitation coil, which is usually $\sim 10 \Omega$, making it a total of $R_{fg} = 60 \Omega$. Via the pickup coil and the SQUID, the Lead Zeppelin’s response to this force is measured with a lockin amplifier.

When the Lead Zeppelin obeys the equation of motion (2.24), and armed with $\Phi_{SQ}(z)$ from section 2.4.1, we can model our SQUID lockin measurements. The signal from the levitating Lead Zeppelin causes a voltage across the SQUID, but we must be careful, because so does the signal coming from the excitation coil. Without this crosstalk, a measurement with a lockin amplifier yields the amplitude and phase of the motion of the Zeppelin for a given excitation frequency.

We will derive here what we expect to measure including crosstalk. Also, because the Lead Zeppelin has relatively low frequency and high Q , measurements tend to be rather time consuming. We shall discuss our lockin measurements quite detailed, because we often measure more quickly than what is necessary to allow the Zeppelin to equilibrate properly.

The typical measurement is to do a frequency sweep. We excite the Zeppelin at a time t_A with a certain frequency f_A such that the amplitude of oscillation is x_A . After having done this for a time Δt we move on to the next frequency f_B . The new steady state amplitude of oscillation x_B is found through (2.26), but it will take some time before it is reached. To avoid having to use the full transient solution, we instead simply state that the change to the new amplitude should be exponential and will depend on the Q -factor:

$$x_{A \rightarrow B}(t) = x_A + (x_B - x_A) \left(1 - e^{-t/\tau_{zep}}\right), \quad (2.44)$$

where $\tau_{zep} = \frac{Q}{\pi f_0}$.

Now, in reality, a frequency sweep is often done faster than what is dictated by τ_{zep} . The amplitude of oscillation will then not yet have reached x_B . To incorporate this into our model, we feed the motion of the Zeppelin through the steady state solution by setting up a frequency vector $\omega_i = 2\pi f_i$ with the frequencies at which we measure and by then calculating

$$x_i(t') = x_{i-1}(\Delta t) + (x_{\text{ampl}}(\omega_i) - x_{i-1}(\Delta t))(1 - e^{-t'/\tau_{zep}}), \quad (2.45)$$

where we let $t' = 0 \rightarrow \Delta t$ for each i , and where we take $x_{\text{ampl}}(\omega_i)$ from equation (2.26). Depending on the actual measurement being performed, we initialize $x_1(t' = 0) = 0$ or $x_1(t' = 0) = x_{\text{ampl}}(\omega_1)$ (although this detail usually doesn't really matter).

For each i , we describe the Zeppelin's motion by

$$x_{zep,i}(t') = x_i(t') \cos(2\pi f_i t' + \theta_i). \quad (2.46)$$

We assume that the phase changes instantaneously to its new value:

$$\theta_i = \theta(\omega_i), \quad (2.47)$$

with $\theta(\omega)$ as in equation (2.27). This will cause steps between consecutive $x_{zep,i}$, because the cosines don't match up. The real, physical Lead Zeppelin will not show this behaviour, because there cannot be a discontinuity in its position. However, in the current model, we will keep this as is; steps are a high frequency effect, and since the model involves low-pass filtering anyway, this doesn't make much of a difference to our result. Again, we are simply avoiding having to use the full solution to the equation of motion.

The flux in the pickup coil in time per frequency is

$$\Phi_i(t') = \Phi(x_{zep,i}(t') + z_{eq}). \quad (2.48)$$

We also insert the term z_{eq} to incorporate the fact that the Lead Zeppelin has an equilibrium position away from $z = 0$.

The flux $\Phi_i(t')$ can be converted to a voltage in the SQUID. Namely, we know how the flux changes from pickup coil to SQUID by equation (2.22), and we have a calibration of the voltage per flux in the SQUID, V_Φ . We can measure the current per flux for our SQUID I_Φ and multiply this by the feedback resistance R_{fb} and the gain G of our room-temperature pre-amplifier. Typical values are $I_\Phi = 2.83 \frac{\mu A}{\Phi_0}$, $R_{fb} = 100 \text{ k}\Omega$ and $G = 1$ so

$$V_\Phi = I_\Phi R_{fb} G = 0.283 \frac{V}{\Phi_0}. \quad (2.49)$$

Then the voltage in the SQUID due to the motion of the Lead Zeppelin is

$$V_{SQ,zep,i}(t') = \frac{\Phi_i(t')}{220} V_\Phi. \quad (2.50)$$

In our setup, there is also a lot of crosstalk: the flux in the pickup coil is not only given by the Zeppelin, but by the excitation coil as well.

$$V_{SQ,ct,i}(t') = V_{ct} \cos(2\pi f_i t' + \theta_{ct,i}). \quad (2.51)$$

The parameter V_{ct} depends, among other things, on the mutual inductance of the excitation coil to the pickup coil. We will keep it as a fit parameter, obtainable from the SQUID signal off Zeppelin resonance.

In total, we measure

$$V_{SQ,i}(t') = V_{SQ,zep,i}(t') + V_{SQ,ct,i}(t') - V_{\text{offset}}. \quad (2.52)$$

We account for the fact that the SQUID is insensitive to flux offsets prior to being flux-locked by subtracting a constant voltage V_{offset} . This $V_{SQ,i}(t')$ is fed to the lockin.

The lockin amplifier takes the SQUID voltage, and outputs its amplitude R and its phase Θ . It does so by multiplying the input signal with a reference signal, and feeding this through a low-pass filter.

$$\begin{aligned} \hat{X}_i(t') &= V_{SQ,i}(t') \times V_{ref} \sin(2\pi f_{ref} t' + \theta_{ref}), \\ \hat{Y}_i(t') &= V_{SQ,i}(t') \times V_{ref} \cos(2\pi f_{ref} t' + \theta_{ref}). \end{aligned} \quad (2.53)$$

If the input signal were a simple sine, then $\hat{X}$ and $\hat{Y}$ contain (co)sines at the sum and difference frequencies $f_{sig} \pm f_{ref}$. In particular, for $f_{ref} = f_{sig}$ there is a component at $2f_{sig}$, which is filtered away by the low-pass filter, and a DC component that scales with the (co)sine of the phase difference between signal and reference.

$$\begin{aligned} \hat{X}_i(t') &= \frac{V_{sig} V_{ref}}{2} \cos(\theta_{sig} - \theta_{ref}) + \dots \\ \hat{Y}_i(t') &= \frac{V_{sig} V_{ref}}{2} \sin(\theta_{sig} - \theta_{ref}) + \dots \end{aligned} \quad (2.54)$$

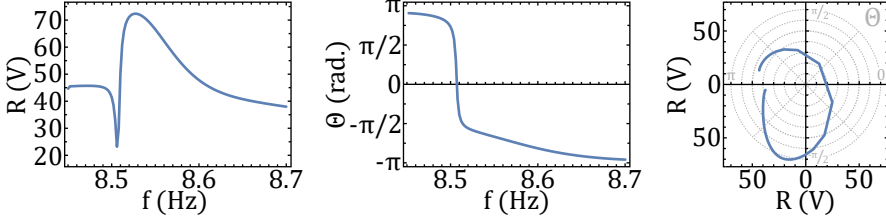


Figure 2.11: A simulation of a lockin measurement of the driven Lead Zeppelin. Left to right: the amplitude of the signal R , the phase of the signal Θ and a phase plot $R \cos \Theta$ vs. $R \sin \Theta$. The strength of the excitation has been taken as $F_0 = 1$ nN, which was chosen based on $I_{lev} = 0.3$ A, such that $\kappa \approx 10^{-8} \frac{\text{N}}{\text{A}}$ (eq. (2.37)), and on $I_{exc} = 0.1$ A. We also used the following values, which were chosen with $I_{lev} = 0.3$ A in mind: $k = 2 \times 10^{-3} \frac{\text{N}}{\text{m}}$, $m = 7 \times 10^{-7}$ kg, $\gamma = 3 \times 10^{-8} \frac{\text{N}}{\text{m/s}}$ and $z_{eq} = -700$ μm , such that $f_0 = 8.507$ Hz, $Q = 1247$ and $\tau_{zep} = 47$ s. Also $\tau_{LI} = 0.3$ s, $\Delta t = 2.0$ s, $\Delta f = 2$ mHz, $V_{SQ,ct} = 40$ V, $\theta_{ct} = \frac{\pi}{2}$, $\theta_{ref} = 0$.

In our case, these functions won't look as simple as this; the above is just an illustration of the working principles of a lockin amplifier.

Filtering is done by an RC -filter. For some input signal across a resistor and a capacitor in series, the voltage across the capacitor is the first order low-pass filtered version of the input signal. In the time domain, this can be expressed as a convolution with an exponential, which in the case of $\hat{X}$ and $\hat{Y}$ looks like:

$$\begin{aligned} X_i(t') &= \frac{1}{\tau_{LI}} \int_0^{t'} \hat{X}_i(s) e^{(s-t')/\tau_{LI}} ds + X_{0,i} e^{-t'/\tau_{LI}}, \\ Y_i(t') &= \frac{1}{\tau_{LI}} \int_0^{t'} \hat{Y}_i(s) e^{(s-t')/\tau_{LI}} ds + Y_{0,i} e^{-t'/\tau_{LI}}. \end{aligned} \quad (2.55)$$

The lockin time constant $\tau_{LI} = RC$ can be chosen by the user on the lockin front panel. The initial voltages $X_{0,i} = X_{i-1}(\Delta t)$ and $Y_{0,i} = Y_{i-1}(\Delta t)$ are exponentially repressed, but when Δt is similar to τ_{LI} , we can't forget about them.

Finally, the lockin output $R_i(t')$ measures the amplitude of the input signal and $\Theta_i(t')$ measures $\theta_{sig} - \theta_{ref}$:

$$\begin{aligned} R_i(t') &= \frac{2}{V_{ref}} \sqrt{X_i^2(t') + Y_i^2(t')}, \\ \Theta_i(t') &= \arctan \left(\frac{Y_i(t')}{X_i(t')} \right). \end{aligned} \quad (2.56)$$

We read out at $t' = \Delta t$: $R_i(\Delta t)$, $\Theta_i(\Delta t)$, at each frequency f_i for each i . This model captures the ringing up in time of the Zeppelin during the lockin measurement of R and Θ , as well as our impatience in waiting for the ring-up to be completed.

In figure 2.11 we show the results for R , Θ and the corresponding polar plot across the frequency sweep, of a simulation of a lockin measurement of the driven Zeppelin motion for some typical values. While for a high Q resonator without crosstalk the resonance is represented in a polar plot as a circle beginning and ending at the origin, its north pole coinciding with the origin, the presence of crosstalk moves the circle. For certain values of the crosstalk $V_{SQ,ct}$ with phase θ_{ct} , the circle encircles the origin causing R to have a dip before its peak and Θ to drop by 2π across the Zeppelin resonance. After the resonance, the signal relaxes to its off-resonant value as dictated by τ_{zep} .

2.6 Damping Mechanisms

The motion of the Lead Zeppelin will be damped by several factors. Here, we discuss two obvious causes of damping: the damping due to interaction with surrounding Helium gas and the damping due to eddy currents in normal metals in the vicinity of the Lead Zeppelin.

2.6.1 Background Gas

To have good thermalization, we leave the Helium background pressure in the vacuum container at around $P = 10^{-2}$ mbar. We cannot go much lower in pressure because the experiment in its current form depends on the cooling provided by the background gas: e.g. the Niobium shortcut wires of the PCS (chapter 3) heat up when injecting a current into the levitation coils, and need to be kept cold. The Helium will set a limit on the Q factor of the levitating Lead Zeppelin.

When the Lead Zeppelin is a sphere of radius R_{zep} , the damping due to collisions with the surrounding gaseous Helium is given by[31]:

$$\gamma = \frac{4\pi R_{zep}^2 P}{\bar{v}_{Th}} = 8.2 \times 10^{-9} \frac{\text{N}}{\text{m/s}}, \quad (2.57)$$

for $P = 1 \text{ Pa}$ ($= 10^{-2}$ mbar) and $R_{zep} = 325 \text{ }\mu\text{m}$. Also, $\bar{v}_{Th}$ denotes the thermal root mean square velocity of the gas molecules, given by $\bar{v}_{Th} = \sqrt{\frac{3k_B T}{m_{\text{He}}}} = 162 \frac{\text{m}}{\text{s}}$, when $T = 4.2 \text{ K}$ and $m_{\text{He}} = 4 \text{ amu} = 6.64 \times 10^{-27} \text{ kg}$.

This contribution to the damping could be lowered by using a smaller Zeppelin, and by decreasing the Helium background pressure, which requires better thermalization of the experiment.

2.6.2 Eddy Currents

The clamp that holds the levitation coils is made from brass, because it is stiff and easily machinable. However, being a normal, resistive metal also at low temperatures, the movement of the Lead Zeppelin will cause dissipation in it,

because the moving Zeppelin constitutes a time-varying flux: $P_{\text{diss}} = \frac{1}{R} \left(\frac{d\Phi}{dt} \right)^2$. When we call $v = \frac{dz}{dt}$, we can write the dissipation as $P_{\text{diss}} = \frac{1}{R} \left(\frac{d\Phi}{dz} \right)^2 v^2$. This can be compared to $P_{\text{diss}} = \mathbf{F} \cdot \mathbf{v} = \gamma v^2$, which follows from the damping force $\mathbf{F} = \gamma \mathbf{v}$. Therefore the damping coefficient is

$$\gamma = \frac{1}{R} \left(\frac{d\Phi}{dz} \right)^2. \quad (2.58)$$

In section 2.4.1 we calculated $\frac{d\Phi}{dz}$ for the pickup coil with which we read out the motion of the Lead Zeppelin. We recalculate it with R_p the radius of the part of the brass clamp of interest and z_p its distance from the Zeppelin.

The resistance R depends on the geometry of the clamp and the material from which it is made. The parts of the brass clamp that keep the Macor levitation coil holders in place, are in the current design (fig. 2.4) hollow cylinders of dimensions $r_{\text{in}} = 3.5$ mm, $r_{\text{out}} = 7$ mm, $h = 4$ mm. Let us assume that here eddy currents flow in the φ -direction. The resistance can then be calculated from $R = \rho \frac{l}{A}$, with ρ the specific resistance: dividing the cylinder up in infinitesimal volume elements with dimensions dr , ds and $d\varphi$, we write $R = \int \rho \frac{rd\varphi}{drds}$. Since we assume the current to flow in the φ -direction, resistivity is summed over $d\varphi$, but conductivity $\frac{1}{R}$ is summed over dr and ds . Therefore, the total resistance of our hollow cylinder is

$$R = \frac{2\pi\rho}{h \ln \frac{r_{\text{out}}}{r_{\text{in}}}}, \quad (2.59)$$

and the damping due to eddies in the surrounding normal metal is

$$\gamma = \frac{1}{2\pi\rho} \int_{r_{\text{in}}}^{r_{\text{out}}} dr \int_{s_{\text{min}}}^{s_{\text{max}}} ds \left[\frac{1}{r} \left(\frac{d\Phi}{dz}(R_p = r, z_p = s) \right)^2 \right]. \quad (2.60)$$

We calculate the flux derivative in a loop of radius $R_p = 5.25$ mm at location $z_p = 9.5$ mm to be

$$\frac{d\Phi}{dz} = -1.18 \times 10^8 \frac{\Phi_0}{\text{m}}, \quad (2.61)$$

and assume the flux derivative to have this value in the whole cylinder. The specific resistance of Copper at 4 K is [32] $\rho_{\text{Cu}}(4 \text{ K}) = 2 \times 10^{-11} \Omega \text{ m}$. This assumes ultra-pure, annealed Copper. Our clamps are in fact made of an (unannealed) Copper-alloy: brass. We estimate brass's specific resistivity at 4 K to be roughly 100 times higher than that of pure Copper, on the basis of its room temperature specific resistivity being about 5 times higher and its residual resistance ratio being much lower:

$$\rho_{\text{brass}}(4 \text{ K}) = 2 \times 10^{-9} \Omega \text{ m}. \quad (2.62)$$

We add the two contributions from the upper and lower cylindrical part of the clamp together, and get

$$\gamma = 2.6 \times 10^{-8} \frac{\text{N}}{\text{m/s}}. \quad (2.63)$$

If we were to make the coil clamps out a non-metallic material, this contribution to the damping will have gone away.

2.7 The Sensitivity of the Lead Zeppelin

In this section we will explore the sensitivity of the Lead Zeppelin and check the feasibility of a gravitational experiment with small source masses. First we will cover some of the basics of measurements on the Lead Zeppelin and their Fourier analysis. We will look at the Power Spectral Density (PSD) and root mean square (RMS) amplitude of signals, such that in the following subsections we can describe the thermal motion and force noise experienced by the Lead Zeppelin, as well as the influence of external vibrations, and compare it to the gravitational forces generated by a test mass oscillating with a given amplitude some distance away.

To do a (gravitational) measurement of driven Zeppelin motion, there are several important criteria:

$$\begin{aligned} S_F &> S_{F,Th}, \\ S_x &> S_{x,lim}, \\ S_x &> H^2(\omega) \times S_{x,E}. \end{aligned} \quad (2.64)$$

The force on the Zeppelin must be larger than the thermal force noise $S_{F,Th}$ acting on it. Also, the Zeppelin motion must be larger than the position detection limit $S_{x,lim}$ of our measurement. Finally, and sadly very importantly, the external vibrations $S_{x,E}$ reaching the Zeppelin via transfer function $H(\omega)$ must be sufficiently small.

2.7.1 Basics of Fourier Transform Measurements

Our definition of the continuous Fourier transform is

$$\mathcal{F}\{x(t)\}(\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt, \quad (2.65)$$

with $\omega = 2\pi f$. When we measure for a time T_m and sample the time signal in N chunks, we use the discrete Fourier transform:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i\omega_k t_n}, \quad (2.66)$$

with $X_k = X(\omega_k)$, $x_n = x(t_n)$ and $\omega_k = 2\pi f_k$, $f_k = k\Delta f = k\frac{1}{T_m}$, $t_n = n\Delta t = n\frac{T_m}{N}$. The discrete Fourier transform is (approximately) equal to the continuous Fourier transform on the interval $(0, T_m)$, up to a factor Δt , through a Riemann summation

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i\omega_k t_n} \approx \frac{1}{\Delta t} \int_0^{T_m} x(t) e^{-i\omega_k t} dt = \frac{1}{\Delta t} X(\omega_k). \quad (2.67)$$

Let us assume that we drive the Lead Zeppelin at exactly a frequency ω_d with an infinitely narrow linewidth, i.e. a δ -peak in frequency space:

$$\mathcal{F}\{F_0 \cos(\omega_d t)\}(\omega) = \pi F_0 \left(\delta(\omega - \omega_d) + \delta(\omega + \omega_d) \right). \quad (2.68)$$

Now, if the system behaves linearly (which we assume), the Zeppelin's response (eq. (2.26)) will also be a δ -peak in frequency space.

We will not be too concerned with phase information and negative frequency components, since the PSD that we will be calculating throws away phase information and is symmetric around $\omega = 0$.

The δ -peak will be smeared out over a wider range of frequencies, by the fact that we measure for a finite time T_m . For a rectangular window $w_R(t) = 1$ ($0 \leq t \leq T_m$ and zero otherwise), i.e. we do not modify the signal apart from breaking it in T_m -long chunks, the Zeppelin signal is convolved with a sinc function in frequency space. Namely, up to a complex phase factor, the Fourier transform of $w_R(t)$ is

$$W_R(\omega) = T_m \operatorname{sinc} \left(\frac{\omega T_m}{2} \right), \quad (2.69)$$

and $\mathcal{F}\{x(t)w_R(t)\}(\omega) = (X * W_R)(\omega)$.

The sinc function is defined as $\operatorname{sinc} x = \frac{\sin x}{x}$. The height of the main peak of the window is $W_R(0) = T_m$. We define the width δf of the sinc as the Full Width at Half Maximum (FWHM) of the main peak of the square of the sinc. It requires solving $x^2 = 2 \sin^2 x$ for x , and results in $\delta f = \frac{0.88}{T_m}$. The longer we measure, the higher and narrower the measured peak becomes.

The convolution of a function with a δ -function, yields the function itself. Therefore, at a δ -peak drive of the Zeppelin, we will measure a sinc function with finite height and linewidth whose peak coincides with the δ -peak.

Another widely used window is the Hann window. The Fourier transform of the Hann window $W_H(\omega)$, from the time-window $w_H(t) = \sin^2 \left(\frac{\pi t}{T_m} \right)$ ($0 \leq t \leq T_m$ and zero otherwise), is

$$W_H(\omega) = \left(1 + \frac{\omega}{2} \frac{1}{\omega - \frac{2\pi}{T_m}} + \frac{\omega}{2} \frac{1}{\omega + \frac{2\pi}{T_m}} \right) \frac{T_m}{2} \operatorname{sinc} \left(\frac{\omega T_m}{2} \right). \quad (2.70)$$

At $\omega = 0$, it is only half as high as the rectangular window. However, it has reduced spectral leakage as compared to the rectangular window, meaning that closely adjacent frequency components are better distinguishable.

The PSD is a handy tool in the frequency analysis of a signal, as it gives a handle on how strongly a certain frequency is present in the signal. In discrete time, it is calculated from the Fourier transform X_n as

$$S_n = \frac{1}{\text{NPBW}} \frac{1}{\Delta f} \frac{1}{N^2} |X_n|^2. \quad (2.71)$$

The factor NPBW is the Noise Power Bandwidth of the Window, a correction factor for the used window; e.g. for a rectangular window it is equal to 1, for a Hann window it equals 1.5. When we use a window other than the rectangular window, we should in the above replace X_n with its convolved counterpart.

When x_n is measured in meters, the units of the PSD are $\frac{\text{m}^2}{\text{Hz}}$. Often we will be talking of $S_n^{1/2}$, rather sloppily, as the power spectral density too, even though it really is an amplitude spectral density.

We can calculate what the value is of the peak in the PSD due to driving the Lead Zeppelin with an infinitesimally sharp excitation at a frequency ω_d . The Zeppelin will acquire an amplitude $x_{\text{ampl}}(\omega_d)$, according to equation (2.26), and move as a perfect cosine at ω_d (we ignore the phase). In frequency space, this becomes a δ -function, which is smeared out (depending on the chosen window, let us choose a simple rectangular window here) to a sinc function. Going to discrete time and frequency, we find that

$$X_n = \frac{1}{2} N x_{\text{ampl}}(\omega_d) \text{sinc} \frac{(\omega_n - \omega_d) T_m}{2}. \quad (2.72)$$

It should be noted that in going from the continuous to the discrete Fourier transform, strictly speaking the Dirac δ -function becomes a Kronecker δ , which, in our definitions, means that we have had a factor of 2π to correct for.

We can now calculate the PSD:

$$S_n = \frac{T_m}{4} x_{\text{ampl}}^2(\omega_d) \text{sinc}^2 \frac{(\omega_n - \omega_d) T_m}{2}, \quad (2.73)$$

or

$$S_n^{1/2} = \frac{1}{2} \sqrt{T_m} x_{\text{ampl}}(\omega_d) \left| \text{sinc} \frac{(\omega_n - \omega_d) T_m}{2} \right|. \quad (2.74)$$

If we were to choose T_m , N and ω_d such that there is a frequency $\omega_n = \omega_d$, at this frequency the value will be

$$S_{n @ \omega_d}^{1/2} = \frac{1}{2} \sqrt{T_m} x_{\text{ampl}}(\omega_d). \quad (2.75)$$

When we choose the Hann window instead of the rectangular window, the peak value is

$$S_{n@{\omega_d}, \text{Hann}}^{1/2} = \frac{1}{2\sqrt{6}} \sqrt{T_m} x_{\text{ampl}}(\omega_d). \quad (2.76)$$

It is not always the case that T_m , N and ω_d are precisely such that $\omega_n = \omega_d$ exists. For an ω_n slightly off ω_d , the measured peak value will be lower than it would otherwise be, resulting in a wrong estimation of $x_{\text{ampl}}(\omega_d)$. In this case, we need to do a little more work.

We begin by invoking Parseval's Theorem, which states that

$$\sum_{n=0}^{N-1} |x_n|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |X_n|^2. \quad (2.77)$$

When we take the sum over x_n to be a Riemann sum and convert it to an integral as in equation (2.67), and when we convert X_n to S_n with equation (2.71), we find that

$$\int_0^{T_m} |x(t)|^2 dt = \frac{1}{\text{NPBW}} \sum_{n=0}^{N-1} S_n. \quad (2.78)$$

The integral can be calculated easily, as we have $x(t) = x_{\text{ampl}}(\omega_d) \cos(\omega_d t)$ and therefore

$$\int_0^{T_m} |x(t)|^2 dt = x_{\text{ampl}}^2(\omega_d) \left(\frac{T_m}{2} + \frac{\sin(\omega_d T_m)}{4\omega_d} \right). \quad (2.79)$$

When $T_m \gg \frac{1}{2\omega_d}$, which is normally true in our measurements, the rightmost term can be ignored, and we are left with

$$\frac{1}{2} T_m x_{\text{ampl}}^2(\omega_d) = \frac{1}{\text{NPBW}} \sum_{n=0}^{N-1} S_n. \quad (2.80)$$

When we constrain the sum to PSD values in the very vicinity of the peak, which carry the most energy of the original signal and therefore contribute the most, throwing out the copy of the peak at $-\omega_d$ as we go, we write

$$x_{\text{ampl}}^2(\omega_d) = 4 \frac{1}{T_m} \frac{1}{\text{NPBW}} \sum_{n \sim \omega_d} S_n. \quad (2.81)$$

This result coincides with the earlier result where $\omega_n = \omega_d$ exists: in this case only the peak value of the sinc survives, whilst all the other frequency bins coincide with the zero-points of the sinc. This new result, however, can also deal with the more general case.

We want to remark here that the summing of PSD components in a peak allows us to be somewhat impatient, in the sense that we can get away with measurement times shorter than what is dictated by the quality factor of a resonance. If the Zeppelin motion is a (sharp) Lorentzian, whose width is given by $\delta f = \frac{f_0}{Q}$, then only for measurement times longer than $T_m = \frac{Q}{f_0}$ will there be more than one PSD data point in the peak. For typical values $Q = 3000$ and $f_0 = 10$ Hz, this means $T_m > 300$ s. Luckily, Parseval's Theorem means that the energy in the peak is conserved in the PSD components; it is just spread to adjacent frequency bins. Summing them yields a good estimate for the amount of energy in the actual Zeppelin motion. On the other hand, we cannot measure for too short either, because then the peak gets spread out so much that it may not be visible over the detection noise.

Finally, we discuss what happens when the Lead Zeppelin is driven by white noise, rather than an infinitely sharp drive at ω_d as in the above. When driven by white noise, it is hard to assign an amplitude x_{ampl} to the motion $x(t)$ of the Zeppelin. We instead use the Root Mean Square (RMS) amplitude x_{RMS} , which is defined on the time interval T_m as

$$x_{\text{RMS}} = \sqrt{\langle x^2(t) \rangle}, \quad \langle x^2(t) \rangle = \frac{1}{T_m} \int_0^{T_m} x^2(t) dt. \quad (2.82)$$

In the previous simple case that $x(t) = x_{\text{ampl}} \cos(\omega t)$ and we average over a single cycle, we find $x_{\text{RMS}} = \frac{x_{\text{ampl}}}{\sqrt{2}}$.

When we invoke the continuous version of Parseval's Theorem

$$\int_0^{T_m} |x(t)|^2 dt = \frac{1}{2\pi} \int_0^\infty |X(\omega)|^2 d\omega, \quad (2.83)$$

as well as the continuous version of the PSD

$$S(\omega) = \frac{1}{T_m} |X(\omega)|^2, \quad (2.84)$$

we may write

$$x_{\text{RMS}}^2 = \frac{1}{2\pi} \int_0^\infty S(\omega) d\omega. \quad (2.85)$$

When the Zeppelin is excited by a white noise source $\eta_x(\omega)$, it acquires a position noise $S_{x,\eta}$ through the equation of motion (2.26), with $k = m\omega_0^2$ and $S_F^{1/2} = kS_x^{1/2}$, of

$$S_{x,\eta}^{1/2}(\omega) = \frac{\omega_0^2}{\sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{\omega\omega_0}{Q}\right)^2}} \eta_x(\omega). \quad (2.86)$$

The integral of a Lorentzian evaluates to

$$\int_0^\infty \frac{d\omega}{(\omega^2 - \omega_0^2)^2 + \left(\frac{\omega\omega_0}{Q}\right)^2} = \frac{\pi}{2} \frac{Q}{\omega_0^3}, \quad (2.87)$$

such that the RMS amplitude of the Zeppelin motion becomes

$$x_{\text{RMS}} = \frac{1}{2} \sqrt{\omega_0 Q} \eta_x(\omega). \quad (2.88)$$

(As an example, when the Zeppelin is driven thermally, $\eta_x = \frac{1}{k} \sqrt{4k_B T \gamma}$ (section 2.7.3), we find that $k x_{\text{RMS}}^2 = k_B T$.)

2.7.2 Detection Noise

The Zeppelin signal $S_n^{1/2}$ should be higher than the detection noise. Our SQUIDs have a noise floor of typically $S_{\Phi, SQ}^{1/2} = 100 \frac{\mu\Phi_0}{\sqrt{\text{Hz}}}$. In equation (2.23) we calculated that the SQUID flux derivative to Zeppelin motion is $\frac{d\Phi_{SQ}}{dz} = 3.0 \times 10^5 \frac{\Phi_0}{\text{m}}$. This leads to a position detection limit of

$$S_{x, \text{lim}}^{1/2} = \frac{S_{\Phi, SQ}^{1/2}}{\left(\frac{d\Phi_{SQ}}{dz}\right)} = 3.3 \times 10^{-10} \frac{\text{m}}{\sqrt{\text{Hz}}}. \quad (2.89)$$

Note that this number can be improved upon in two ways: using a less noisy SQUID and placing the pickup coil at a better position. It is not uncommon for SQUIDs to have a noise floor of $\sim 1 \frac{\mu\Phi_0}{\sqrt{\text{Hz}}}$, even at a temperature of 4 K, which means two orders of magnitude improvement of $S_{x, \text{lim}}^{1/2}$. Moving the pickup coil can also be beneficial (figure 2.7), possibly gaining us about two orders of magnitude if we're optimistic.

Currently, at $T_m = 100$ s, the Lead Zeppelin needs to have an oscillation amplitude of at least (eq. (2.74))

$$x_{\text{ampl}}(\omega_d) > \frac{2}{\sqrt{T_m}} S_{x, \text{lim}}^{1/2} = 6.7 \times 10^{-11} \text{ m} \quad (2.90)$$

to be discernible above the SQUID noise. As we will see in the coming sections, the thermal motion of the Zeppelin and the motion of the Zeppelin because of external vibrations are both well above this value.

2.7.3 Thermal Motion

From the equipartition theorem, we know that the force noise on a damped harmonic oscillator due to thermal fluctuations is[3]

$$S_{F, Th} = 4k_B T \gamma, \quad (2.91)$$

with k_B Boltzmann's constant, T the temperature and γ the damping factor from the equation of motion (2.24). The levitating Lead Zeppelin, when driven only by this thermal noise, will have a thermal displacement noise of (eq. (2.26))

$$S_{x,Th}(\omega) = \frac{4k_B T \gamma / m^2}{(\omega^2 - \omega_0^2)^2 + \left(\frac{\omega \omega_0}{Q}\right)^2}. \quad (2.92)$$

The value at resonance $S_{x,Th}(\omega_0) = \frac{4k_B T Q}{m \omega_0^3}$, so

$$S_{x,Th}^{1/2}(\omega_0) = 1.2 \times 10^{-9} \frac{\text{m}}{\sqrt{\text{Hz}}}. \quad (2.93)$$

at $T = 4.2$ K, $m = 2$ mg, $\frac{\omega_0}{2\pi} = 10$ Hz and $Q = 3000$. This is well above the detection noise, equation (2.89) — we should already be able to measure the thermal motion of the Lead Zeppelin. However, we will see in section 2.8 that currently the external vibrations give larger motion than the thermal motion.

2.7.4 Gravitational Experiments

In this section we calculate what the requirements are to use a Lead Zeppelin to measure the gravitational force between it and a source mass, at frequencies between 10 Hz and 100 Hz. The smallest source mass used to date to measure gravitational forces is about 90 grams [33].

Suppose we have a block of mass $M = 1$ g at a distance $r = 3$ cm from the Lead Zeppelin, and we shake it around with an amplitude of $a = 1.5$ mm. The gravitational force between M at rest and the Zeppelin, of mass $m = 1$ mg, is

$$F_g = G \frac{mM}{r^2} = 74 \text{ aN}. \quad (2.94)$$

At the two end points of oscillation of the mass

$$F_g(r \pm a) = \frac{GmM}{(r \pm a)^2} \approx \frac{GmM}{r^2} \left(1 \mp 2\frac{a}{r}\right). \quad (2.95)$$

So the amplitude of the gravitational driving force on the Zeppelin is given by

$$F_0 = \frac{1}{2} \Delta F_g = 2a \frac{GmM}{r^3} = 7.4 \text{ aN}. \quad (2.96)$$

This gravitational driving force should be obtainable with a simple electromagnetic drive motor, as described in e.g. [34]. Of course, there is the subtlety that the drive motor and the Lead Zeppelin must be vibrationally decoupled from each other, such that the motor doesn't deliver its drive through the experimental setup rather than purely gravitationally — which may prove to be difficult at $r = 3$ cm in a cryostat. Alternatively, one can think of a block of mass $M = 1$

kg at a distance $r = 1$ m and amplitude $a = 5$ cm, which yields the same gravitational driving force. This corresponds roughly to a (heavy) cylinder piston in a car engine [35], which can be made to oscillate between 600 rpm (10 Hz) and 6000 rpm (100 Hz) easily.

If the mass M is made to oscillate at a resonance of the Zeppelin, the Zeppelin will acquire an amplitude (equation (2.26)) of

$$x_{\text{ampl}}(\omega_0) = \frac{F_0}{m} \frac{Q}{\omega_0^2} = 2a \frac{GM}{r^3} \frac{Q}{\omega_0^2} = 5.6 \text{ pm}, \quad (2.97)$$

for typical numbers $\frac{\omega_0}{2\pi} = 10$ Hz and $Q = 3000$. From equation (2.74), this leads to a PSD component with the value

$$S_{x,g}^{1/2}(\omega_0) = \frac{aGM}{r^3} \frac{Q}{\omega_0^2} \sqrt{T_m} = 2.8 \times 10^{-11} \frac{\text{m}}{\sqrt{\text{Hz}}}, \quad (2.98)$$

when $T_m = 100$ s and the window is rectangular. In our current experimental setup, this is below the noise floor by an order of magnitude, eq. (2.89). However, there are several factors that can pull this number towards measurability. As stated before, lowering the SQUID noise can gain us as much as two orders of magnitude, and placing the pickup coil in a more optimal position will also be beneficial. Then, the gravitational force between the oscillating mass M and the Lead Zeppelin will be visible over the detection noise. We can also play with the other parameters: e.g. when Q were $10\times$ bigger or when we were more patient $T_m = 10^4$ s ($= 2\frac{3}{4}$ hours), then too the Zeppelin begins to respond measurably to the gravitational force of M .

A big caveat comes from the criterion on the external vibrations. As we will see in the next section, the vibrations of the environment are, at present, much too large for the gravitational measurement to be feasible.

2.8 External Vibrations

In order to shield the experiment from vibrations, we hoist the Helium Dewar, in which the experiment is mounted, off the floor with a crane attached to the ceiling of our laboratory using bungee cords (figures 2.12, 2.16). The movement of the Dewar $x_d(t)$ caused by movement of the crane $x_c(t)$ is governed by the equation

$$m_d \ddot{x}_d(t) + \gamma_d \dot{x}_d(t) + k_d(x_d(t) - x_c(t)) = 0. \quad (2.99)$$

Here, m_d is the mass of the Helium Dewar, k_d is the spring constant of the bungee cords and γ_d the damping of the system. We assume that the major source of damping that the crane experiences does not come from the bungee cords. Going to frequency space, we find the transfer function

$$H_{cd}(\omega) = \left| \frac{X_d(\omega)}{X_c(\omega)} \right| = \left| \frac{\omega_d^2}{\omega_d^2 - \omega^2 + i \frac{\omega \omega_d}{Q_d}} \right|. \quad (2.100)$$

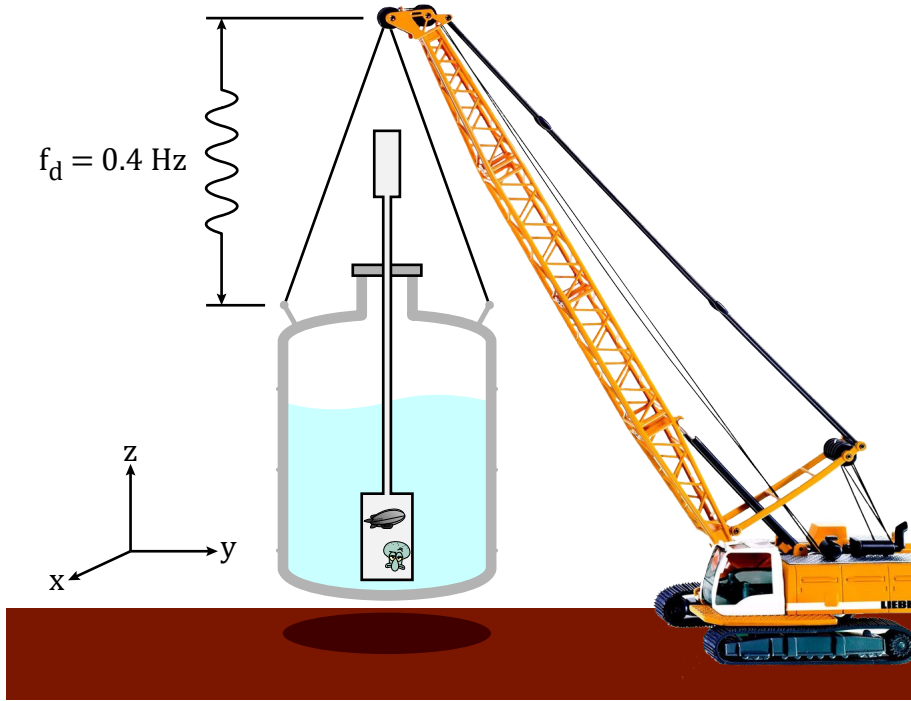


Figure 2.12: To keep external vibrations from reaching the experiment, we hoist the Helium Dewar off the floor using a crane and bungee cord. This acts as a 2nd order filter on movement of the crane beyond the resonance of the hanging Dewar $f_d \approx 0.4 \text{ Hz}$.

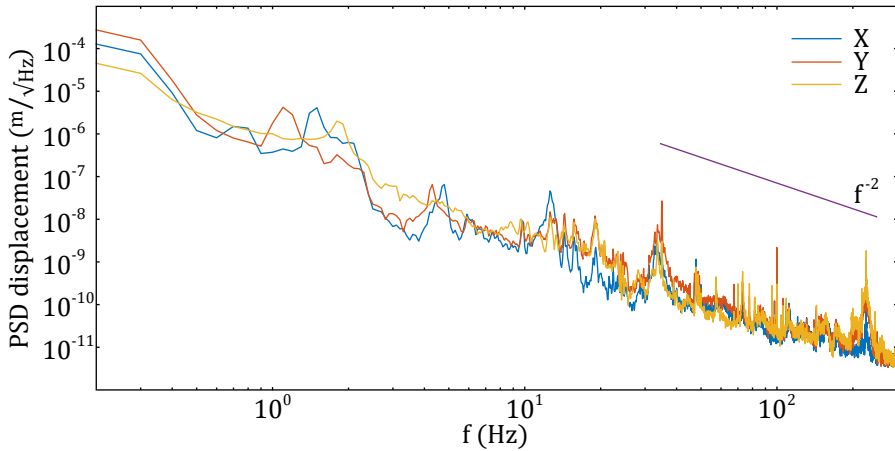


Figure 2.13: The movement of the hanging Helium Dewar, measured with geophones. The frequency resolution is $\Delta f = 0.1 \text{ Hz}$. Each PSD is an average of 10 PSDs.

We used $\gamma_d = \frac{m_d \omega_d}{Q_d}$ and $\omega_d = \sqrt{\frac{k_d}{m_d}}$.

We note that $H_{cd}(\omega \ll \omega_d) = 1$. At low frequencies, the Dewar follows the crane motion exactly. Also, $H_{cd}(\omega \gg \omega_d) = \left(\frac{\omega_d}{\omega}\right)^2$. Above the resonance of the Dewar, the Dewar-bungee cord (mass-spring) system serves as a second order filter on movement of the crane. The lower ω_d , therefore, the better vibrations from the crane are isolated from the Dewar.

The bungee cord stretches approximately $u = 1.5$ meters before the Helium Dewar comes off the floor, from which the resonance frequency follows:

$$f_d = \frac{1}{2\pi} \sqrt{\frac{g}{u}} \approx 0.4 \text{ Hz}, \quad (2.101)$$

with $g = 9.8 \frac{\text{m}}{\text{s}^2}$ the gravitational acceleration.

We measured the motion of the hanging Helium Dewar $S_{x,\text{Dewar}}$ in the three spatial directions using three geophones, see figure 2.13.

The experimental setup is mounted on a dipstick and stuck in the liquid Helium, so the movement of the Dewar will affect the movement of the Lead Zeppelin. Let us assume that the motion of the Dewar equals the motion of the experiment $S_{x,E} = S_{x,\text{Dewar}}$. Then, the displacement noise of the Zeppelin due to the motion of the Dewar is

$$S_{x,\text{zep}}^{1/2}(\omega) = S_{x,E}^{1/2}(\omega) H(\omega). \quad (2.102)$$

Here, $H(\omega)$ is the same as $H_{cd}(\omega)$ from equation (2.100), but now using the Zeppelin's resonance frequency ω_0 and quality factor Q , instead of the Dewar parameters.

In some limiting cases of interest, we rewrite $H(\omega)$ to

$$H(\omega) = \begin{cases} 1, & \omega \ll \omega_0, \\ Q, & \omega = \omega_0, \\ \left(\frac{\omega_0}{\omega}\right)^2, & \omega \gg \omega_0. \end{cases} \quad (2.103)$$

For frequencies much smaller than a Zeppelin resonance ω_0 , the Zeppelin will follow the motion of the Dewar exactly. However, since we are assuming that the Dewar and the experiment move together rigidly, there is no motion between the Zeppelin and the excitation coil and the levitation coils. Therefore, this causes no flux change in the pickup coil, and this is no problem. For frequencies much bigger than ω_0 the motion of the Dewar is damped to second order.

The big problem lies with the motion of the Dewar at ω_0 . Here, $H(\omega_0) = Q$: the Dewar motion is enhanced by Q . In order for this to be no problem, the motion of the Dewar has to be at least Q times lower than the Zeppelin motion of interest.

When we want to measure the thermal motion as described in section 2.7.3, which is $S_{x,Th}^{1/2}(\omega_0) = 1.2 \times 10^{-9} \frac{\text{m}}{\sqrt{\text{Hz}}}$ at $\frac{\omega_0}{2\pi} = 10$ Hz and $Q = 3000$ (equation (2.93)), the external vibrations must stay below

$$S_{x,E}^{1/2}(\omega_0) < \frac{1}{Q} S_{x,Th}^{1/2}(\omega_0) = 3.9 \times 10^{-13} \frac{\text{m}}{\sqrt{\text{Hz}}}. \quad (2.104)$$

When it is the gravitational interaction as described in section 2.7.4 that we want to measure, where we found that $S_{x,g}^{1/2}(\omega_0) = 2.8 \times 10^{-11} \frac{\text{m}}{\sqrt{\text{Hz}}}$ (equation (2.98)), the criterion gets even more stringent:

$$S_{x,E}^{1/2}(\omega_0) < \frac{1}{Q} S_{x,g}^{1/2}(\omega_0) = 9.4 \times 10^{-15} \frac{\text{m}}{\sqrt{\text{Hz}}}. \quad (2.105)$$

At present, we can see in figure 2.13 that the Dewar motion at 10 Hz is about $10^{-9} - 10^{-8} \frac{\text{m}}{\sqrt{\text{Hz}}}$.

The seismically isolated islands in the newly constructed measurement hall of Leiden University, which will be opened shortly, have been measured to have a displacement noise of $\sim 2 \times 10^{-10} \frac{\text{m}}{\sqrt{\text{Hz}}}$ at 10 Hz[36]. This would make our vibration sensitive measurements a lot closer to being doable, but it is not quite enough yet. What is more, the vibration isolation system of Advanced LIGO, most likely the best in the world, reaches an impressive displacement noise level of under $10^{-12} \frac{\text{m}}{\sqrt{\text{Hz}}}$ at 10 Hz[37]. It is clear that we need to revert also to other measures than just vibration isolation if we want the Lead Zeppelin to be a useful force sensor.

In the case of the thermal noise, we can go to a higher frequency: the thermal noise peak drops as $\omega_0^{-3/2}$, but simultaneously, the Dewar motion falls off as ω^{-2} , and life gets easier as $\sqrt{\omega_0}$. Similarly, lowering the Zeppelin's mass means increasing the thermal noise, but the Dewar motion is unaffected; it then gets easier as $\sqrt{m}$. A higher Q increases the thermal noise peak as $\sqrt{Q}$, but the Zeppelin motion due to the Dewar motion goes as Q ; it gets harder as $\sqrt{Q}$.

In the case of the gravitational measurement, tinkering with ω_0 , m or Q has no net effect. Here, the easiest solution is to use longer measurement times T_m , which scales as $\sqrt{T_m}$. But, in order for these not to become unrealistically long, we can additionally go for a larger source mass M (linear scaling), at a smaller distance r (scaling r^{-3}), with a larger amplitude a (linear scaling).

A final remark on the vibrations reaching the Lead Zeppelin is in order. In the above we have been assuming that the experiment moves rigidly with the Dewar. We can get away with this only to a certain extent. The levitation coils are tightly held in place by the brass clamp, which is in turn tightly screwed to the Niobium tube in which it is put, which is in turn tightly screwed to the dipstick (figure 2.16). The resonances of all of these stages are likely to be a lot higher in frequency than the Zeppelin resonances that we measure, so we can assume that the stages' transfer functions roughly equal unity (eq. (2.100)). The motion of the dipstick, then, is fed one-to-one to the levitation coils.

The dipstick itself, however, is essentially a long rod with a mass (the experiment) attached to its end, clamped on one end to the Dewar. It is possible for the boiling off of Helium, a process known as boiling noise[38], to set the dipstick in motion. At the moment, we do not know how serious this contribution to the displacement noise is, but we keep in mind that this process exists.

2.9 Measures Against Electromagnetic Noise

A SQUID is a notoriously sensitive magnetometer, with a sensitivity of typically $\sim 1 \frac{\mu\Phi_0}{\sqrt{\text{Hz}}}$. To give an idea of the smallness of this: $1 \mu\Phi_0$ is equal to the amount of flux through the bottom of a coffee cup caused by a refrigerator magnet 10 km away. It is clear that we must take special care to shield external (magnetic) noise.

A major cause of noise is the current source that we use to power the coils in the experiment. Any noise in the current creates flux noise in the levitation coils, which is picked up by the pickup coil and fed to the SQUID. In the ideal case that the pickup coil is located exactly in the middle between the levitation coils, both levitation coils are wound exactly identically, and there is no crosstalk to the leads between pickup coil and SQUID, this noise is not picked up. Real life, however, is of course not so ideal.

Say, for instance, that one of the levitation coils has one winding more than the other and that the only noise is the current noise specified by the manufacturer of the current source[39] $I_n = 0.1 \text{ mA}_{\text{RMS}}$. For typical numbers of the experiment $R_{\text{winding}} = 2.5 \text{ mm}$, $R_p = 375 \mu\text{m}$, and taking the distance between the extra winding and the pickup coil as the largest possible in our setup (i.e. the least-bad-case scenario) $l = 3.75 \text{ mm}$, we find that the mutual inductance $M = 10^{-11} \text{ H}$ (by equation (2.14)) and the noise in the SQUID (by equation (2.22)):

$$\Phi_{SQ,n} = 50 \text{ m}\Phi_{0,\text{RMS}}. \quad (2.106)$$

Clearly, this is bad — and this is the best case scenario!

Indeed, in our case we quickly found that the amount of noise coupling into the SQUID is many times worse. It is in fact so bad, that the SQUID modulation is completely drowned in the noise, which renders the SQUID unusable. We took several measures to overcome this problem, which can be seen in figure 2.15.

By putting the levitation coils in the Persistent Mode, a lot of the noise troubles are dealt with. This is explained in more detail in chapter 3.

We took care to make the experiment sit inside a closed cage of Faraday. This mitigates electric noise, like radios and cell phones. Inevitably, the wiring to the coils and the wiring to the SQUID punch a hole in the cage of Faraday, which we kept as small as possible. With an Aluminium clamp, the dipstick is securely connected to the Helium Dewar with as low an electrical impedance as possible, making the dewar part of the cage of Faraday.

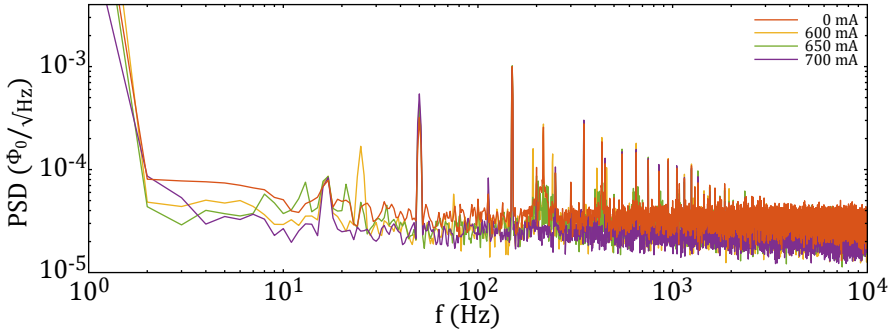


Figure 2.14: The PSDs of the SQUID signals with the levitation coils in the persistent mode, after installing the electromagnetic shielding, for some different levitation currents. Even though the noise floor is quite high for a SQUID at $\sim 30 - 40 \frac{\mu\Phi_0}{\sqrt{\text{Hz}}}$, it does not change with changing I_{lev} (nor does it change when the current source is completely detached).

The experiment and the SQUID are each put in a Niobium tube. The wires from the pickup coil to the SQUID are led between these Nb tubes through a thin Lead capillary. Additionally, the vacuum pot of the dipstick is covered in a layer of Niobium foil. All of these measures together mitigate magnetic noise.

The SQUID wires going from the SQUID in the vacuum pot to a connector on the warm side of the dipstick are comprised of 4 twisted pairs jointly shielded by a CuNi-braiding. The SQUID read-out box¹ is placed directly on the connector.

The wires going from the coils to the outside world, are comprised of twisted pairs jointly shielded by a CuNi-braiding from the vacuum pot to a connector on the warm end of the dipstick. From there on, they are shielded per twisted pair, and jointly shielded as well. The coil wiring ends in a specially designed breakout box. In this breakout box, each twisted pair is run through a line filter² and each separate wire is run through a π -filter³ as well. The current source is connected via shielded twisted pairs, which are wound around ferrite rings.

For the wires designated to the excitation coil, an extra filter measure is taken: in addition to the line filter and π -filters, a transformer is added that electrically separates the function generator from the excitation coil. This is just an extra safety feature, made possible by the fact that the excitation coil will carry AC signals. The transformer⁴ has a flat frequency response between 20 Hz and 20 kHz. The function generator is connected to the breakout box with a BNC cable wound around a ferrite ring.

Figure 2.14 shows the SQUID noise at a few different levitation currents. The noise level is the same for all currents. Completely detaching the current source from the experiment, also does not change the noise level.

Figure 2.16 shows the actual implementation of the experiment.

¹Magnicon GmbH, XXF-1-6/3 SQUID Electronics.

²Epcos, TDK Europe GmbH, B84111AB60 filter, 6A, chassis.

³Tusonix, Inc., 4209-053LF filter, pi-section, 5500PF, 350V, 10A.

⁴JensenTM Transformers, Inc., JT-MB-CPC microphone transformer.

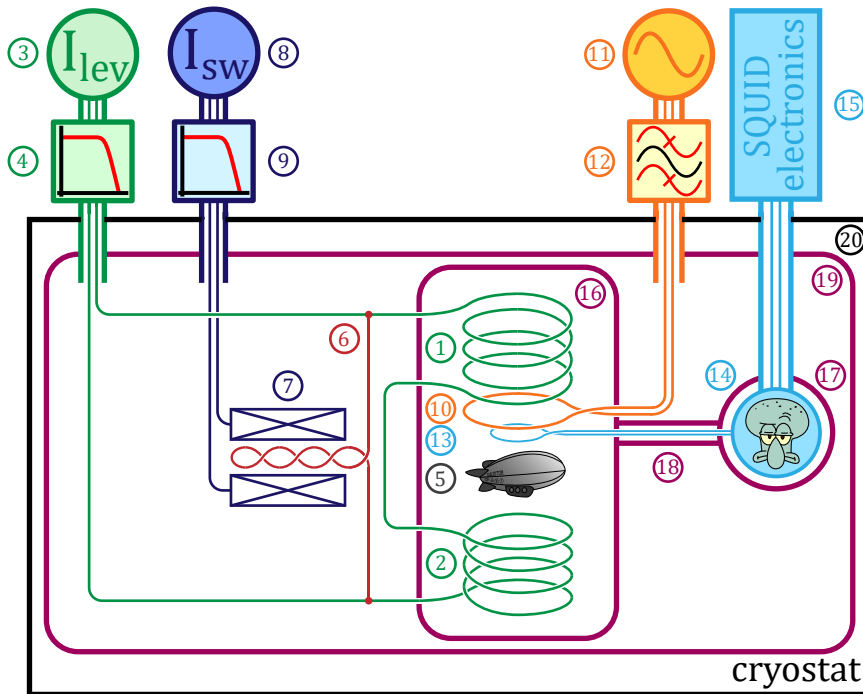


Figure 2.15: A more complete schematic of the experimental setup. We see the **upper** (1) and **lower** (2) **levitation coils**, connected to a **current source** (3). Added is **filtering** (4), which consists of a line filter and per wire a π -filter. Between the levitation coils we find the **Lead Zeppelin** (5). The levitation coils are shorted with a **Niobium wire** (6), which is twisted around itself and stuck in the **switch coil** (7) of the Persistent Current Switch (see chapter 3). By running a current I_{sw} through the switch coil with a **current source** (8), the Niobium shortcut wire is brought to its resistive state and a current can be injected into the levitation coils. Even though the Niobium shortcut wire is twisted around itself, thereby minimizing the mutual inductance between it and the switch coil, we still add **filtering** (9) to make sure there is as little noise as possible coupling into the levitation coils. This filtering consists of a line filter and per wire a π -filter. The **excitation coil** (10) is connected to a **function generator** (11) which is **filtered** (12) by a line filter, per wire a π -filter and additionally a transformer. The Zeppelin motion is picked up by a **pickup coil** (13) connected to a **SQUID** (14), which is read out with **SQUID electronics** (15). The levitation coils, excitation coil, pickup coil and Lead Zeppelin are placed inside **superconducting Niobium shielding** (16). The SQUID is also secured inside **Niobium shielding** (17). The wires from the pickup coil to the SQUID are led between the two shieldings through a **superconducting Lead capillary** (18). Around the whole cold part of the setup, we place an extra **Niobium shield** (19). We make sure that the Dewar in which we do our experiments (20) is securely attached to the shieldings of the various wires that leave the experiment to the outside world.

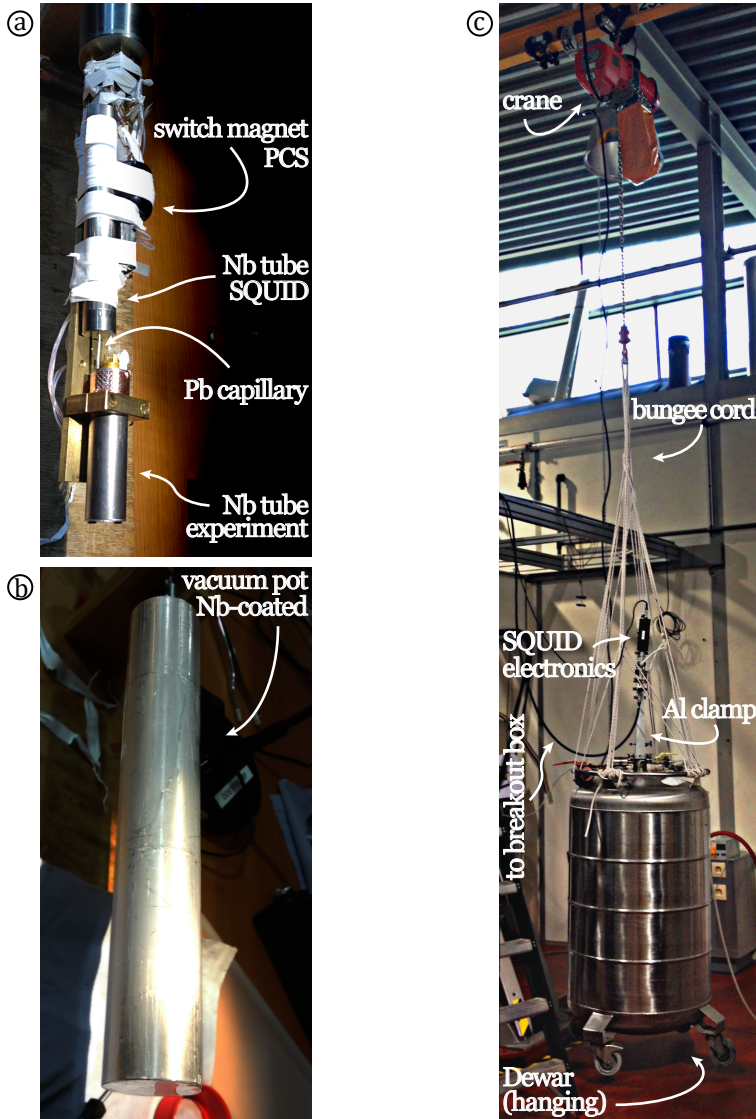


Figure 2.16: (a) The experiment as in figure 2.4 is placed in a Niobium tube and mounted on a dipstick. Via a Lead capillary, the wires of the pickup coil go to another Niobium tube in which the SQUID is placed. Also visible is the switch coil of the PCS. (b) We place a vacuum pot around the experiment, which is coated with Niobium foil. This way, the non-shielded parts of wiring are still shielded. (c) The dipstick is stuck in a Helium Dewar, which is hoisted off the floor with a crane and bungee cord. The SQUID electronics are placed directly on the dipstick. The dipstick and Dewar are connected to each other with an Aluminium clamp.

The following chapter has been published as: B. van Waarde, O. Benningshof and T. Oosterkamp, “A Magnetic Persistent Current Switch at milliKelvin Temperatures”, *Cryogenics*, vol. 78, p. 74, 2016.

In appendix B we explore whether we should worry that the levitation coils in the persistent mode, obeying flux conservation, cancel the excitation coil’s drive of the Zeppelin, and the flux in the pickup coil coming from the Zeppelin.