The Stochastic Geometry of non-Gaussian Fields

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4

EXTREMA STATISTICS FOR NONLOCAL PERTURBATIONS

In this chapter we focus on non-Gaussianities of nonlocal origin, so that the field cannot be expressed as $h(\boldsymbol{r}) = f(H(\boldsymbol{r}))$ as before. Instead, $h(\boldsymbol{r})$ may also depend on e.g. ∇H, thereby introducing a mixing between the field values at different points.

Such a nonlocal non-Gaussianity may arise for instance as the result of a nonlinear diffusion equation. The non-Gaussianities that develop may provide clues about the details of the underlying microscopic mechanisms. Consider for example a gas-liquid phase transition. In the early stages, there are many small volumes in which all the molecules are in the same phase, distributed randomly. Over time, these volumes will grow and merge, thereby gradually replacing the Gaussian disorder with structure [5].

As a concrete example, let us consider a field that is Gaussian at $t = 0$ and evolves according to the heat equation:

$$\frac{\partial h(\boldsymbol{r}, t)}{\partial t} = D\nabla^2 h(\boldsymbol{r}, t). \tag{4.1}$$

The solution is then given by

$$\begin{aligned} h(\boldsymbol{r}, t) &= \sum_{\boldsymbol{k}} A_0(\boldsymbol{k}) e^{-k^2 Dt} \cos(\boldsymbol{k} \cdot \boldsymbol{r} + \phi_{\boldsymbol{k}}) \\ &= \sum_{\boldsymbol{k}} \tilde{A}(\boldsymbol{k}, t) \cos(\boldsymbol{k} \cdot \boldsymbol{r} + \phi_{\boldsymbol{k}}), \end{aligned} \tag{4.2}$$

where $A_0(\boldsymbol{k})$ is the amplitude spectrum of the initial field at $t = 0$. We see that the field remains Gaussian for $t > 0$, but the amplitude spectrum changes with time.

Now suppose that we add an extra term:

$$\frac{\partial h(\boldsymbol{r}, t)}{\partial t} = D\nabla^2 h(\boldsymbol{r}, t) + f_{NL}(h, \nabla h), \tag{4.3}$$

where f_{NL} is any nonlinear function. Now, even when h is a Gaussian field at $t = 0$, non-Gaussianities will emerge as a consequence of the last term.

A variety of known diffusion equations has the general form above. For instance, when f_{NL} takes the form $-h^2$ we get Fisher's equation, which can be used as a model to describe the growth and saturation of a population. Another example is the Cahn-Hilliard equation for the development of order after a phase transition [5]. Several models of structure formation, in both condensed matter [7] and cosmology [13], also belong to this class.

We return our focus to the relative difference between maxima and minima, as in chapter 2. We derive a general formula for this imbalance in the case of a general perturbation, up to first order.

To illustrate this result, we apply it to the case of a field obeying the deterministic KPZ equation [32], for which $f_{NL} = \frac{\lambda}{2}(\nabla h)^2$. This equation is often used to model the height profile of a growing surface. A field that starts out as a Gaussian field will acquire non-Gaussian characteristics as time progresses. We use our formula to quantify the effect of the gradient-squared term on the relative difference in densities of maxima and minima. We verify the analytical predictions by comparing them with results from computer simulations.

The outline of this chapter is as follows. In section 4.1 we determine a general expression for the imbalance between maxima and minima for a non-Gaussian field. This is applied to the KPZ equation in section 4.2. Finally, section 4.3 summarizes our conclusions.

4.1 NON-GAUSSIAN FIELDS

In what follows, we concentrate on homogeneous and isotropic fields $h(\boldsymbol{r})$, which we assume to be in the form of a Gaussian $H(\boldsymbol{r})$ with the addition of a perturbation. Unlike chapters 2 and 3, we will not restrict ourselves to a perturbation of the local kind, i.e. where the perturbation at any point $\boldsymbol{r}$ is a function of $H(\boldsymbol{r})$ only. We will now also accommodate perturbations which depend on $\boldsymbol{\nabla} H$ for instance, or evolve over time. Such perturbations introduce a mixing between the values of the field at different points, which we will designate as *nonlocal* perturbations.

We will investigate the effect of a perturbation on the densities of maxima and minima. As mentioned before, for a Gaussian field these are the same due to symmetry. For a non-Gaussian field, they can differ. It is noteworthy that the density of saddle points, the other type of critical points, is always

equal to the density of maxima and minima combined as a consequence of Morse theory [39] (see also chapter 1).

A maximum (minimum) $\boldsymbol{r_0}$ of h is defined by the condition $h_x(\boldsymbol{r_0}) = h_y(\boldsymbol{r_0}) = 0$, along with the inequalities $h_{xx}(\boldsymbol{r_0})h_{yy}(\boldsymbol{r_0}) - h_{xy}(\boldsymbol{r_0})^2 > 0$ (if this were negative, $\boldsymbol{r_0}$ would be a saddle point) and $h_{xx}(\boldsymbol{r_0})$, $h_{yy}(\boldsymbol{r_0})$ negative (positive); note that the first condition implies that $h_{xx}(\boldsymbol{r_0})$ and $h_{yy}(\boldsymbol{r_0})$ have the same sign. The x and y subscripts indicate derivatives with respect to the coordinates of the two-dimensional plane.

The general procedure that we use is very similar to the one in chapter 3 and is as follows: We consider a fixed point $\boldsymbol{r_0}$; due to the homogeneity of h, the analysis will not depend on this choice. We determine the joint probability distribution of h_x, h_y, h_{xx}, h_{yy} and h_{xy}, since these stochastic variables are the ingredients from which maxima and minima are defined, as outlined above. This distribution can be determined via the generating function, which in turn can be constructed by determining the relevant cumulants involving the five stochastic variables. Once the probability distribution is obtained, we set $h_x = h_y = 0$ and integrate the second derivatives over the region defining a minimum (maximum) in order to get the density of minima (maxima).

As we did in section 3.2, we transform to another coordinate system, based on the complex coordinates $z = x + iy$ and z^*, which will allow us to make full use of the homogeneity and isotropy of h later on. In this new basis, we have (see eq. (3.9))

$$\frac{\partial}{\partial z} = \tfrac{1}{2}\frac{\partial}{\partial x} - \tfrac{1}{2}i\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z^*} = \tfrac{1}{2}\frac{\partial}{\partial x} + \tfrac{1}{2}i\frac{\partial}{\partial y}. \tag{4.4}$$

In this coordinate system, the definition of a maximum (minimum) becomes $h_z(\boldsymbol{r_0}) = 0$, $|h_{zz^*}(\boldsymbol{r_0})| > |h_{zz}(\boldsymbol{r_0})|$ and $h_{zz^*}(\boldsymbol{r_0})$ is negative (positive).[1]

We now need the cumulants. In the previous chapter we were able to argue that there is only a limited number of nonzero cumulants, based on the specific form of the variables we were dealing with, but this argument does not apply here. In principle, there are now infinitely many nonzero cumulants. However, a field that is generated by a nonlinear differential equation, like eq. (4.3), typically has small cumulants of high order. In particular, if f_{NL} is a quadratic function and the initial conditions are Gaussian, then the n-th order cumulants scale like f_{NL}^{n-2} (for $n > 2$) (see appendix D). Therefore we will only need to determine cumulants up to third order to get the correction to leading order.

1 Note that $h_{zz^*}(\boldsymbol{r_0})$ is real valued.

As in section 3.2, we can use symmetries to make our lives easier. From rotational symmetry it follows that each cumulant must have the same total number of z and z^* derivatives, otherwise it is zero. Translational invariance implies that any correlation should be constant with respect to $\boldsymbol{r}$, which leads to relations like

$$0 = \partial_{z^*}\langle h_z^2 h_{z^*}\rangle = \langle h_z^2 h_{z^*z^*}\rangle + 2\langle h_z h_{z^*} h_{zz^*}\rangle, \tag{4.5}$$

which gives us the relation present in eq. (4.6c).

Therefore, there are only a few independent cumulants that are (potentially) nonzero:

$$\sigma = \langle |h_z|^2\rangle, \tag{4.6a}$$
$$\alpha = \langle |h_{zz}|^2\rangle = \langle h_{zz^*}^2\rangle, \tag{4.6b}$$
$$\beta = \langle |h_z^2| h_{zz^*}\rangle = -\tfrac{1}{2}\langle h_z^2 h_{z^*z^*}\rangle = -\tfrac{1}{2}\langle h_{z^*}^2 h_{zz}\rangle, \tag{4.6c}$$
$$\gamma = \langle h_{zz^*}^3\rangle, \tag{4.6d}$$
$$\delta = \langle |h_{zz}|^2 h_{zz^*}\rangle. \tag{4.6e}$$

Here we introduced the shorthand notation $|h_z|^2 = h_z h_z^* = h_z h_{z^*}$ and similarly for $|h_{zz}|^2$. Note also that the third-order cumulants, β, γ and δ are close to zero when h is close to being Gaussian, which we assume. On the other hand, σ and α are nonzero in general.

Note that it is not generally true that correlations and cumulants are identical. It is true though in this special case of correlations / cumulants up to third order between derivatives of a homogeneous field, as shall be demonstrated with γ as an example. Expressing the cumulant in correlations gives

$$\gamma = C_3(h_{zz^*}, h_{zz^*}, h_{zz^*}) = \langle h_{zz^*}^3\rangle - 3\langle h_{zz^*}\rangle\langle h_{zz^*}^2\rangle + 2\langle h_{zz^*}\rangle^3. \tag{4.7}$$

Note that, with the exception of the first, all terms carry a factor $\langle h_{zz^*}\rangle$, which can be expressed as

$$\langle h_{zz^*}\rangle = \partial_{z_1}\partial_{z_1^*}\langle h(\boldsymbol{r})\rangle. \tag{4.8}$$

Since we consider h to be homogeneous, $\langle h(\boldsymbol{r})\rangle$ is constant and thus we have $\langle h_{zz^*}\rangle = 0$. This trick applies not only to γ, but to all five correlations in eq. (4.6).

We can now construct the logarithm of the generating function as prescribed by eq. (3.3),

$$\begin{aligned}\log\chi = &-\sigma|\lambda_z|^2 - \alpha|\lambda_{zz}|^2 - \tfrac{1}{2}\alpha\lambda_{zz^*}^2 \\ &- i\beta|\lambda_z|^2\lambda_{zz^*} + i\beta(\lambda_z^2\lambda_{z^*z^*} + \lambda_{z^*}^2\lambda_{zz}) \\ &- \tfrac{i}{6}\gamma\lambda_{zz^*}^3 - i\delta|\lambda_{zz}|^2\lambda_{zz^*}. \end{aligned} \tag{4.9}$$

Recall that the number of permutations of the λ's needs to be accounted for, as per eq. (3.3); this explains why for instance the term $\lambda_{zz^*}^3$ has a prefactor $i/6$ whereas the prefactor of $|\lambda_{zz}|^2\lambda_{zz^*} = \lambda_{zz}\lambda_{z^*z^*}\lambda_{zz^*}$ is i (due to the 6 distinct permutations of the λ's).

We see that χ features an exponential of a third-degree polynomial, making the inverse Fourier transform (to be performed in order to get the probability distribution) nontrivial. Remember however that the cubic terms are small owing to the near-Gaussianity of h, allowing us to make the expansion

$$\begin{aligned}\chi = &\Big(1 - i\beta|\lambda_z|^2\lambda_{zz^*} + i\beta(\lambda_z^2\lambda_{z^*z^*} + \lambda_{z^*}^2\lambda_{zz}) \\ &- \tfrac{i}{6}\gamma\lambda_{zz^*}^3 - i\delta|\lambda_{zz}|^2\lambda_{zz^*}\Big) \\ &\times \exp\Big(-\sigma|\lambda_z|^2 - \alpha|\lambda_{zz}|^2 - \tfrac{1}{2}\alpha\lambda_{zz^*}^2\Big). \end{aligned} \tag{4.10}$$

The inverse Fourier transform of this gives[2]

$$\begin{aligned}p(h_z, h_{zz}, h_{zz^*}) = &\Big(1 + \frac{\beta}{\alpha\sigma^2}h_{zz^*}(|h_z|^2 - \sigma) - \frac{\beta}{\alpha\sigma^2}(h_z^2 h_{z^*z^*} + h_{z^*}^2 h_{zz}) \\ &+ \frac{\gamma}{6\alpha^3}(h_{zz^*}^3 - 3\alpha h_{zz^*}) + \frac{\delta}{\alpha^3}h_{zz^*}(|h_{zz}|^2 - \alpha)\Big) \\ &\times \frac{1}{\pi^2\sqrt{2\pi}\sigma\alpha^{3/2}}e^{-|h_z|^2/\sigma - |h_{zz}|^2/\alpha - h_{zz^*}^2/2\alpha}. \end{aligned} \tag{4.11}$$

Now that the joint probability distribution of the relevant derivatives is obtained, we can set $h_z = h_{z^*} = 0$; this condition defines a critical point. As we have seen before in both chapter 2 and 3, we need to multiply p by the Jacobian

$$J = \left|\frac{\partial(h_z, h_{z^*})}{\partial(z, z^*)}\right| = \Big||h_{zz}|^2 - h_{zz^*}^2\Big|, \tag{4.12}$$

in order to get the appropriate probability distribution with respect to z and z^* (representing x and y).

2 A factor of π^2 rather than $(2\pi)^2$ is associated with the complex variables h_z and h_{zz} in the Fourier transform due to our normalization; see also eq. (3.34).

Now we are ready to set $h_z = h_{z^*} = 0$ and integrate pJ over h_{zz} and h_{zz^*}. The range is determined by the type of critical point of interest; focus on the minima first. For these we must have $|h_{zz}| < |h_{zz^*}|$ and $h_{zz^*} > 0$. The integration over h_{zz} is done by integrating over its real and imaginary part. Since the integrand depends only on the modulus of h_{zz}, we move to polar coordinates. Let us define $r = |h_{zz}|$ and $s = h_{zz^*}$. The integration range is then $0 < r < s$, and with eq. (4.11) we get

$$\begin{aligned} n_{min} = {} & \frac{1}{\pi^2\sqrt{2\pi}\sigma\alpha^{3/2}} \\ & \times \int_0^\infty \mathrm{d}s \int_0^s 2\pi r \,\mathrm{d}r \,(s^2 - r^2) e^{-r^2/\alpha - s^2/2\alpha} \\ & \times \left(1 - \frac{\beta}{\alpha\sigma}s + \frac{\gamma}{6\alpha^3}(s^3 - 3\alpha s) + \frac{\delta}{\alpha^3}s(r^2 - \alpha)\right). \end{aligned} \tag{4.13}$$

This integration is pretty straightforward: Although the range of r is finite, the integrand is a Gaussian multiplied by a polynomial that has only odd degrees of r, hence it does not give rise to error functions. The resulting integral over s is also standard. The final result reads

$$n_{min} = \frac{\alpha}{2\sqrt{3}\pi\sigma} - \frac{1}{\pi\sigma}\sqrt{\frac{\alpha}{2\pi}}\left(\frac{4}{3}\frac{\beta}{\sigma} + \frac{4}{9}\frac{\delta}{\alpha} - \frac{10}{27}\frac{\gamma}{\alpha}\right). \tag{4.14}$$

For a Gaussian field, we would have $\beta = \gamma = \delta = 0$, $\sigma = \frac{1}{4}K_2$ and $\alpha = \frac{1}{16}K_4$. This would give us $n_{min} = K_4/(8\sqrt{3}\pi K_2)$, exactly as noted before (eqs. (1.40) and (2.34)).

To get the density of maxima, the same integrand as in eq. (4.13) needs to be integrated over the range $s < 0$ and $0 < r < -s$. However, note that if we make the transformation $s \to -s$, the range of integration is the same as in eq. (4.13). Furthermore, note that the transformation $s \to -s$ in the integrand is equivalent to $\beta \to -\beta$, $\gamma \to -\gamma$ and $\delta \to -\delta$. With this insight, we easily find that the expression for n_{max} is the same as the above, except with a plus in place of the first minus.

With this result, the imbalance between maxima and minima is found to be

$$\Delta n \equiv \frac{n_{max} - n_{min}}{n_{max} + n_{min}} = \sqrt{\frac{6}{\pi\alpha}}\left(\frac{4}{3}\frac{\beta}{\sigma} + \frac{4}{9}\frac{\delta}{\alpha} - \frac{10}{27}\frac{\gamma}{\alpha}\right). \tag{4.15}$$

This is the main result of this chapter. As an illustration, we shall now use this result to understand the evolution of maxima and minima in the context of a differential equation describing surface growth.

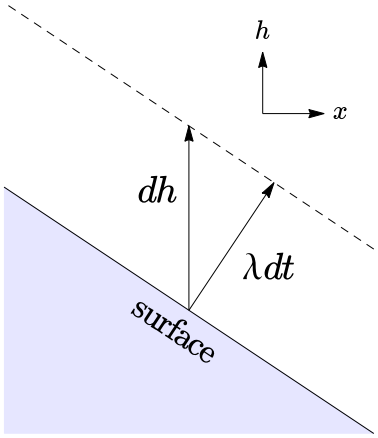


Figure 4.1: A two-dimensional (excluding the y direction) geometrical interpretation of the second term of the KPZ equation (eq. (4.16)) applied to a growing surface. The surface is assumed to grow perpendicularly at a constant rate λ. Measured vertically, the growth rate is $\mathrm{d}h/\mathrm{d}t = \lambda\sqrt{1+(\mathrm{d}h/\mathrm{d}x)^2}$. In three dimensions, the derivative is replaced by a gradient (see eq. (4.17)).

4.2 KPZ EQUATION

The deterministic Kardar-Parisi-Zhang (KPZ) equation [32] is given by

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2. \tag{4.16}$$

This equation is often used to describe the height profile of a growing surface: The first term on the right-hand side describes the diffusion of particles along the surface, while the second term accounts for the assumption that the growth is perpendicular to the slope of the surface, while h describes the height along the universal up direction [1]. This leads to (see fig. 4.1)

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \lambda\sqrt{1+(\nabla h)^2} = \lambda + \frac{\lambda}{2}(\nabla h)^2 + \dots \tag{4.17}$$

The leading term λ is ignored since it is just a constant that does not affect the profile of the surface.

Another interpretation of eq. (4.16) is obtained by taking the gradient on both sides, which yields

$$\frac{\partial \boldsymbol{v}}{\partial t} = \nu\nabla^2 \boldsymbol{v} + \lambda \boldsymbol{v}\boldsymbol{\nabla}\boldsymbol{v}, \tag{4.18}$$

where $\boldsymbol{v} = \boldsymbol{\nabla} h$ is a velocity field. This is a vector Burger's equation which arises in fluid mechanics. The maxima and minima of h correspond to sources and sinks of v.

We will take $h(\boldsymbol{r}, t)$ to be a Gaussian field at $t = 0$, and use our result eq. (4.15) to determine how the non-Gaussianities, which arise and evolve due to the KPZ equation, influence the densities of maxima and minima.

First note that if we would set $\lambda = 0$ in eq. (4.16), we retrieve the heat equation, which preserves the Gaussianity of a field: If we enter $h(\boldsymbol{r}, t = 0) = H(\boldsymbol{r})$, where $H(\boldsymbol{r})$ is a Gaussian field as given by eq. (1.9), we find that the solution is

$$h(\boldsymbol{r}, t) = \sum_{\boldsymbol{k}} A(k) e^{-k^2 \nu t} \cos(\boldsymbol{k} \cdot \boldsymbol{r} + \phi_k). \tag{4.19}$$

We find that the amplitudes pick up a factor $\exp(-k^2 \nu t)$, but the phases remain independent. Therefore, even though its amplitude spectrum changes, $h(\boldsymbol{r}, t)$ remains Gaussian at any time t and the density of maxima and minima remains the same, since this is a general property of Gaussian fields.

If we have $\lambda \neq 0$, $h(\boldsymbol{r}, t)$ no longer remains Gaussian. In fact, as we will see, the density of maxima and minima is no longer the same. We shall assume λ to be small in comparison to ν, and find out how these densities differ as a function of time, using eq. (4.15). For this, we need to determine the two- and three-point correlations σ, α, β, γ and δ.

First, we substitute $u = \exp((\lambda/2\nu)h)$. Note that, since this is a monotonically increasing function of h, the maxima and minima of u are exactly the same points as those of h. In terms of u, the KPZ equation becomes:

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u, \tag{4.20}$$

which is simply the heat equation. However, $u(\boldsymbol{r}, t = 0) = \exp((\lambda/2\nu)h_0)$ is now not a Gaussian field. If we assume that $\lambda \ll \nu$, we have:

$$u_0 = 1 + \frac{\lambda}{2\nu} h_0 + \frac{\lambda^2}{8\nu^2} h_0^2 + O((\lambda/\nu)^3). \tag{4.21}$$

Since the leading term, equal to one, has no influence on either the maxima and minima or the heat equation, we can ignore it. The same applies to the prefactor $\frac{\lambda}{2\nu}$ of the second term. Hence we make a final transformation

$$v \equiv \frac{2\nu}{\lambda}(u - 1), \tag{4.22}$$

$$v_0 = h_0 + \frac{\lambda}{4\nu} h_0^2 + O((\lambda/\nu)^2). \tag{4.23}$$

Note that v still obeys the heat equation and also shares the same maxima and minima with h and u. Moreover, we now have $v(\boldsymbol{r}, t = 0)$ in the desired form of a Gaussian h_0 plus a perturbation. Since v obeys the heat equation, we can use the corresponding Green's function to write down the general solution

$$\begin{aligned} v(r,t) &= \int \mathrm{d}^2\tilde{\boldsymbol{r}}\, G(\boldsymbol{r}, \tilde{\boldsymbol{r}}, t) v_0(\tilde{\boldsymbol{r}}) \\ &= \int \mathrm{d}^2\tilde{\boldsymbol{r}}\, \frac{1}{4\pi\nu t} e^{-\frac{(\boldsymbol{r}-\tilde{\boldsymbol{r}})^2}{4\nu t}} \Big(h_0(\tilde{\boldsymbol{r}}) + \frac{\lambda}{4\nu} h_0(\tilde{\boldsymbol{r}})^2 \Big), \end{aligned} \tag{4.24}$$

where $v_0(\tilde{\boldsymbol{r}}) = v(\boldsymbol{r}, t = 0)$.

We can now calculate the five correlations needed to determine Δn. We will demonstrate the procedure using $\sigma = \langle v_z(\boldsymbol{r}, t) v_{z^*}(\boldsymbol{r}, t) \rangle$ as an example.

$$\begin{aligned} \sigma &= \langle v_z(r,t) v_{z^*}(r,t) \rangle \\ &= \iint \mathrm{d}^2\tilde{\boldsymbol{r}}_{\mathbf{1}} \mathrm{d}^2\tilde{\boldsymbol{r}}_{\mathbf{2}}\, \partial_{z_1} G(\boldsymbol{r}_{\mathbf{1}}, \tilde{\boldsymbol{r}}_{\mathbf{1}}, t) \partial_{z_2^*} G(\boldsymbol{r}_{\mathbf{2}}, \tilde{\boldsymbol{r}}_{\mathbf{2}}, t) \\ &\qquad \times \langle v_0(\tilde{\boldsymbol{r}}_{\mathbf{1}}) v_0(\tilde{\boldsymbol{r}}_{\mathbf{2}}) \rangle \Big|_{\boldsymbol{r}_{\mathbf{1}} = \boldsymbol{r}_{\mathbf{2}} = \boldsymbol{r}}. \end{aligned} \tag{4.25}$$

The brackets represent averaging over all $\phi_{\boldsymbol{k}}$ that define v_0, while the spatial derivatives act only on the respective Green's function. The latter gives

$$\begin{aligned} \partial_{z_1} G(\boldsymbol{r}_{\mathbf{1}}, \tilde{\boldsymbol{r}}_{\mathbf{1}}, t) &= \partial_{z_1} \left(\frac{1}{4\pi\nu t} e^{-\frac{(\boldsymbol{r}_{\mathbf{1}} - \tilde{\boldsymbol{r}}_{\mathbf{1}})^2}{4\nu t}} \right) \\ &= \frac{1}{\pi (4\nu t)^2} ((x_1 - \tilde{x}_1) - i(y - \tilde{y}_1)) e^{-\frac{(\boldsymbol{r}_{\mathbf{1}} - \tilde{\boldsymbol{r}}_{\mathbf{1}})^2}{4\nu t}}. \end{aligned} \tag{4.26}$$

The moment present in eq. (4.25) is

$$\begin{aligned} \langle v_0(\tilde{\boldsymbol{r}}_{\mathbf{1}}) v_0(\tilde{\boldsymbol{r}}_{\mathbf{2}}) \rangle &= \Big\langle \Big(h_0(\tilde{\boldsymbol{r}}_{\mathbf{1}}) + \frac{\lambda}{4\nu} h_0(\tilde{\boldsymbol{r}}_{\mathbf{1}})^2 \Big) \Big(h_0(\tilde{\boldsymbol{r}}_{\mathbf{2}}) + \frac{\lambda}{4\nu} h_0(\tilde{\boldsymbol{r}}_{\mathbf{2}})^2 \Big) \Big\rangle \\ &= \langle h_0(\tilde{\boldsymbol{r}}_{\mathbf{1}}) h_0(\tilde{\boldsymbol{r}}_{\mathbf{2}}) \rangle \\ &\quad + \frac{\lambda}{4\nu} \Big(\langle h_0(\tilde{\boldsymbol{r}}_{\mathbf{1}}) h_0(\tilde{\boldsymbol{r}}_{\mathbf{2}})^2 \rangle + \langle h_0(\tilde{\boldsymbol{r}}_{\mathbf{1}})^2 h_0(\tilde{\boldsymbol{r}}_{\mathbf{2}}) \rangle \Big) \\ &\quad + \left(\frac{\lambda}{4\nu} \right)^2 \langle h_0(\tilde{\boldsymbol{r}}_{\mathbf{1}})^2 h_0(\tilde{\boldsymbol{r}}_{\mathbf{2}})^2 \rangle. \end{aligned} \tag{4.27}$$

Note that the second term (the one linear in $\lambda/4\nu$) is a three-point correlation, and therefore zero due to the symmetry of the Gaussian field h_0.

We will ignore the last term since our analysis is restricted to first order in $\lambda/4\nu$. All that remains is the two-point correlation, which with the help of eq. (1.9) is seen to be

$$\begin{aligned}\langle v_0(\tilde{\boldsymbol{r}}_{\mathbf{1}})v_0(\tilde{\boldsymbol{r}}_{\mathbf{2}})\rangle &= \langle h_0(\tilde{\boldsymbol{r}}_{\mathbf{1}})h_0(\tilde{\boldsymbol{r}}_{\mathbf{2}})\rangle \\ &= \sum_{\boldsymbol{k}} \tfrac{1}{2}A(k)^2\cos(\boldsymbol{k}\cdot(\tilde{\boldsymbol{r}}_{\mathbf{1}}-\tilde{\boldsymbol{r}}_{\mathbf{2}})).\end{aligned} \tag{4.28}$$

We will now plug our intermediate results, eqs. (4.26) and (4.28), back into eq. (4.25). For convenience, we will set $\boldsymbol{r}=\mathbf{0}$, which we are allowed to do thanks to the homogeneity of v. We find

$$\begin{aligned}\sigma = \sum_{\boldsymbol{k}} \tfrac{1}{2}A(k)^2 \iint \mathrm{d}^2\tilde{\boldsymbol{r}}_{\mathbf{1}}\mathrm{d}^2\tilde{\boldsymbol{r}}_{\mathbf{2}}\,\pi^{-2}(4\nu t)^{-4}(\tilde{\boldsymbol{r}}_{\mathbf{1}}\cdot\tilde{\boldsymbol{r}}_{\mathbf{2}}) \\ \times e^{-(|\tilde{\boldsymbol{r}}_{\mathbf{1}}|^2+|\tilde{\boldsymbol{r}}_{\mathbf{2}}|^2)/(4\nu t)}\cos(\boldsymbol{k}\cdot(\tilde{\boldsymbol{r}}_{\mathbf{1}}-\tilde{\boldsymbol{r}}_{\mathbf{2}})).\end{aligned} \tag{4.29}$$

Note that based on eq. (4.26) we should have put $(\tilde{x}_1-i\tilde{y}_1)(\tilde{x}_2+i\tilde{y}_2)$ instead of $(\tilde{\boldsymbol{r}}_{\mathbf{1}}\cdot\tilde{\boldsymbol{r}}_{\mathbf{2}})$; the latter is merely the real part of the former. However, since we already know that the final answer is real (since $\sigma=\langle|v_z|^2\rangle$), we can conclude that the imaginary part would not give a contribution.

After performing the integrals in eq. (4.29) we get the result given below. The three-point correlations β, γ and δ give rise to six-dimensional integrals involving four-point correlations (which are first order in $\lambda/4\nu$). These correlations can be factorized into two two-point correlations by Wick's theorem, resulting in a sum over two wave vectors $\boldsymbol{k}_{\mathbf{1}}$ and $\boldsymbol{k}_{\mathbf{2}}$, as opposed to the one we had in the case of σ.

All the relevant correlations are

$$\sigma = \sum_{\boldsymbol{k}} \frac{1}{2} A(k)^2 \frac{1}{4} k^2 e^{-2k^2\nu t}, \tag{4.30a}$$

$$\alpha = \sum_{\boldsymbol{k}} \frac{1}{2} A(k)^2 \frac{1}{16} k^4 e^{-2k^2\nu t}, \tag{4.30b}$$

$$\beta = -\frac{\lambda}{4\nu} \sum_{\boldsymbol{k_1}} \sum_{\boldsymbol{k_2}} \frac{1}{4} A(k_1)^2 A(k_2)^2 \frac{1}{4} \Big(k_1^2 k_2^2 - (\boldsymbol{k_1} \cdot \boldsymbol{k_2})^2 \Big) \times e^{-2(k_1^2 + k_2^2 + \boldsymbol{k_1} \cdot \boldsymbol{k_2})\nu t}, \tag{4.30c}$$

$$\gamma = -\frac{\lambda}{4\nu} \sum_{\boldsymbol{k_1}} \sum_{\boldsymbol{k_2}} \frac{1}{4} A(k_1)^2 A(k_2)^2 \frac{3}{32} k_1^2 k_2^2 (\boldsymbol{k_1} + \boldsymbol{k_2})^2 \times e^{-2(k_1^2 + k_2^2 + \boldsymbol{k_1} \cdot \boldsymbol{k_2})\nu t}, \tag{4.30d}$$

$$\delta = -\frac{\lambda}{4\nu} \sum_{\boldsymbol{k_1}} \sum_{\boldsymbol{k_2}} \frac{1}{4} A(k_1)^2 A(k_2)^2 \frac{1}{32} \Big(-k_1^2 k_2^2 (k_1^2 + k_2^2) + ((\boldsymbol{k_1} + \boldsymbol{k_2})^4 - k_1^4 - k_2^4)(\boldsymbol{k_1} \cdot \boldsymbol{k_2}) \Big) \times e^{-2(k_1^2 + k_2^2 + \boldsymbol{k_1} \cdot \boldsymbol{k_2})\nu t}. \tag{4.30e}$$

For a continuous spectrum, the sums can be replaced by integrals.

We see that the parameters depend on the spectrum of h_0 in a nontrivial way. Especially the presence of $\boldsymbol{k_1} \cdot \boldsymbol{k_2}$ (which is also present in terms like $(\boldsymbol{k_1} + \boldsymbol{k_2})^2$) in the relations for β, γ and δ complicates matters, as it introduces a dependence on the angle between $\boldsymbol{k_1}$ and $\boldsymbol{k_2}$. An exact analytical evaluation is therefore only realizable for a few spectra of a convenient form. Even for the so-called ring spectrum, with $A(k) \propto \delta(k - k_0)$, arguably the simplest spectrum one can have, the angular dependence introduces nontrivial functions. In this case, eq. (4.15) reads

$$\Delta n = \frac{\lambda}{4\nu} \frac{8}{9} \sqrt{\frac{6}{\pi}} \frac{e^{-\tau}}{\tau} \Big(-(2 + \tau) I_0(2\tau) - 5\tau I_1(2\tau) + (2 + \tau + 6\tau^2)\, {}_0F_1(2; \tau^2) \Big), \tag{4.31}$$

where $\tau \equiv k_0^2 \nu t$; I_0 and I_1 are modified Bessel functions of the first kind and ${}_0F_1$ is the confluent hypergeometric function. Recall that we set $K_0 = \langle h_0^2 \rangle = 1$ for convenience; for the general case, a factor of $\sqrt{K_0}$ needs to be added.

Another, more elegant case in which an exact evaluation of eq. (4.15) is possible is the Gaussian spectrum $A(k) \propto \exp(-k^2/(4k_0^2))$, for which

$$\Delta n = \frac{\lambda}{4\nu} \frac{64\tau^3(1+4\tau)^{7/2}}{\sqrt{3\pi}(1+2\tau)^3(1+6\tau)^4}, \tag{4.32}$$

where again $\tau \equiv k_0^2 \nu t$ and a factor of $\sqrt{K_0}$ needs to be added for our result to apply in general.

Going back to the general case of an unspecified power spectrum, it is convenient to expand Δn in t. The result is

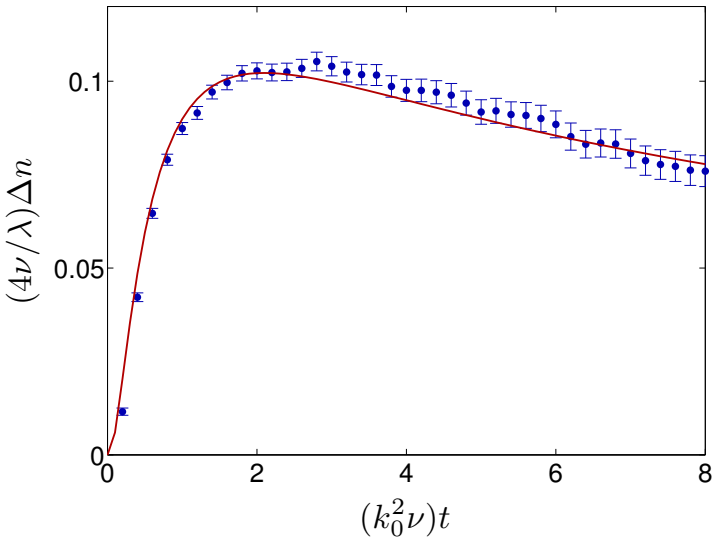
$$\Delta n = \frac{\lambda}{4\nu} \frac{4}{9} \sqrt{\frac{6}{\pi}} \frac{2K_2K_6 - 3K_4^2}{K_2\sqrt{K_4}} (\nu t)^2 + O(t^3), \tag{4.33}$$

for all K_0. One may note that for a Gaussian spectrum, there is no quadratic order in eq. (4.32), which is confirmed by the above formula, since $2K_2K_6 - 3K_4^2 = 0$ in this case.

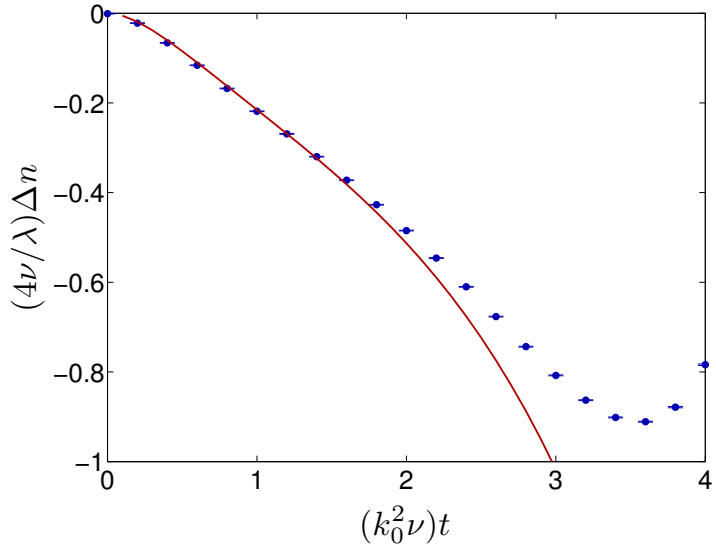
The analytical results for Δn above are compared to results from numerical simulations (with $K_0 = 1$ and $\lambda/4\nu = 0.1$) in fig. 4.2. The general method is the same as seen in chapter 2 and appendix A. We start with a Gaussian field h_0 defined on a finite square grid with periodic boundary conditions. We then transform to v_0 and use the alternating direction implicit (ADI) method to simulate the heat equation, collecting statistics on the maxima and minima at every time step. The results are averaged over for tens of thousands of h_0's, each with the same spectrum but random phases.

In general, if a field evolves under a nonlinear equation for a long time, the non-Gaussianity can become large, even when the perturbation is small, because it will add up over time. Thus we may expect a breakdown of our predictions after some time, as in fig. 4.2b. However, the KPZ equation has a special mapping to a diffusion equation (eq. (4.20)), and this implies that the non-Gaussian perturbations never build up. Eq. (4.24) shows that the nonlinear correction diffuses outward but does not grow over time. Therefore, for the KPZ equation, our approximations should remain accurate for arbitrarily long times. This is indeed what we see in fig. 4.2a, where $h(t = 0)$ is Gaussian field with a Gaussian spectral function.

In fig. 4.2b however there is a breakdown for the ring spectrum. This spectrum is special because it has zero weight at $k = 0$. This implies that the leading Gaussian term in eq. (4.24) is suppressed exponentially, decaying as $\exp(-k_0^2\nu t)$ (see eq. (4.19)). Thus after a long time, the second term



(a)



(b)

Figure 4.2: The imbalance between maxima and minima Δn of $h(\boldsymbol{r}, t)$, where h obeys the KPZ equation (with $\lambda/4\nu = 0.1$), as a function of time. At $t = 0$, $h(\boldsymbol{r})$ was taken to be a Gaussian field with (a) a Gaussian spectrum $A(k) \propto \exp(-k^2/(4k_0^2))$; (b) a ring spectrum $A(k) \propto \delta(k - k_0)$. Shown are our theoretical perturbative result (eq. (4.15)) and data from simulations.

dominates, and our approximation that v is close to a Gaussian no longer holds. Whenever the spectral function has a weight at $k = 0$ (as in fig. 4.2a), the approximation works for a longer time.

4.3 CONCLUSIONS

We have found a general perturbative formula, eq. (4.15), for determining the imbalance between maxima and minima of an isotropic random field that is almost Gaussian. It allows one to attack the reverse problem, namely, to determine the size of the phenomenon that causes the non-Gaussianity, by measuring the relative densities of maxima and minima. In the case of the deterministic KPZ equation for instance, the imbalance can reveal the size of the nonlinear parameter λ relative to the diffusion coefficient ν.

In chapter 2, we investigated the imbalance between maxima and minima as a result of non-Gaussianity. Although we arrived at an exact result, it applied only to the special case of a local perturbation, i.e. for a field given by $h(\boldsymbol{r}) = F_{NL}(H(\boldsymbol{r}))$ where H is a Gaussian field and F_{NL} any (nonlinear) function. The result in the present study, although perturbative, also accommodates *nonlocal* perturbations, provided that the resulting field is still homogeneous and isotropic.

For local perturbations, we found that the size of the imbalance is exponentially small in the size of the perturbation. Nonlocal perturbations however allow for a power-law relation. This is apparent in eq. (4.33), which shows that the KPZ equation can cause an imbalance that grows quadratically with time. As a result, the densities of maxima and minima can prove to be a sensitive test to not only detect non-Gaussianity, but also to distinguish local from nonlocal perturbations that induce non-Gaussian statistics.