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MEASURE-VALUED MASS EVOLUTION PROBLEMS WITH FLUX BOUNDARY CONDITIONS AND SOLUTION-DEPENDENT VELOCITIES*

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Abstract. In this paper we prove well-posedness for a measure-valued continuity equation with solution-dependent velocity and flux boundary conditions, posed on a bounded one-dimensional domain. We generalize the results of an earlier paper [*J. Differential Equations*, 259 (2015), pp. 1068–1097] to settings where the dynamics are driven by interactions. In a forward-Euler-like approach, we construct a time-discretized version of the original problem and employ those results as a building block within each subinterval. A limit solution is obtained as the mesh size of the time discretization goes to zero. Moreover, the limit is independent of the specific way of partitioning the time interval $[0, T]$. This paper is partially based on results presented in Chapter 5 of [*Evolution Equations for Systems Governed by Social Interactions*, Ph.D. thesis, Eindhoven University of Technology, 2015], while a number of issues that were still open there are now resolved.

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1. Introduction.

A considerable amount of recent mathematical literature has been devoted to evolution equations formulated in terms of measures. Such equations are used to describe systems that occur in, e.g., biology (animal aggregations [CFRT10, CCR11], crowds of pedestrians [CPT14], structured populations [DG05, GLMC10, CCGU12, AI05]) and material science (defects in metallic crystals [vMM14]). Many interesting and relevant scenarios take place in *bounded* domains. Apart from the examples mentioned above, these include intracellular transport processes (cf. [EHM15b, section 1]) and manufacturing chains [GHS$^+$14]. However, most works that deal with well-posedness of measure-valued equations and properties of their solutions treat these equations in the full space; see, for instance, [BGCG06, CDF$^+$11, TF11, CLM13, CCS15]. The present work explicitly focuses on bounded domains and the challenge of defining mathematically and physically "correct" boundary conditions.

In [EHM15a], we derived boundary conditions for a one-dimensional measure-valued transport equation on the unit interval $[0, 1]$ with prescribed velocity field v.

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A shorthand notation for this equation is

$$\frac{\partial}{\partial t}\mu_t + \frac{\partial}{\partial x}(v\,\mu_t) = f \cdot \mu_t. \tag{1.1}$$

We focused on the well-posedness of this equation, in the sense of *mild solutions*, and the convergence of solutions corresponding to a sequence $(f_n)_{n\in\mathbb{N}}$ in the right-hand side. Some specific choices for $(f_n)_{n\in\mathbb{N}}$ represent, for instance, effects in a boundary layer that approximate, as $n \to \infty$, sink or source effects localized on the boundary. The boundary layer corresponds to the regions in $[0,1]$ where the functions f_n are nonzero.

There are several reasons *why* we consider mild solutions rather than weak solutions. First of all, the mild formulation in terms of the variation of constants formula—see (2.18)—follows directly from a probabilistic interpretation, as was shown in [EHM15a, section 6]. Therefore the choice for mild solutions is justified by a modeling argument. Second, usually uniqueness of weak solutions cannot be expected to hold, while mild solutions are unique when the perturbation $(\mu \mapsto f \cdot \mu)$ is Lipschitz. In [EHM15a], where the perturbation even has discontinuities, we still obtain uniqueness of the mild solution. This is one of the main results of [EHM15a]. In the works [AI05, GLMC10, CCGU12, GJMC12] a specific weak solution is constructed that is precisely the mild solution that we obtain by different means. Finally, there is a technical advantage of using mild solutions. Most of our estimates are in terms of the dual bounded Lipschitz norm $\|\cdot\|^*_{\mathrm{BL}}$, which will be introduced in section 2.1. Because test functions do not appear explicitly, our calculations are often simpler than when weak solutions are considered. Moreover, our estimates are in fact uniform over test functions in a bounded set.

In the present work, we propose and investigate a procedure to generalize the former results to include velocity fields that depend on the solution itself. Such generalization makes it possible to model in a bounded domain the dynamics governed by interactions between the "particles"; in particular we will be concerned with interaction terms of convolution type that are given by a weighted average over the whole population.

The results in this paper hold for a source-sink right-hand side that is based on a function f that is an element of the space $\mathrm{BL}([0,1])$ of bounded Lipschitz functions on $[0,1]$. In [EHM15a], we worked with $f : [0,1] \to \mathbb{R}$ that is *piecewise* bounded Lipschitz, though. Hence, here we are able to describe absorption in a boundary layer, but not yet absorption on the boundary alone. In the discussion section of this paper (see section 5.1), we comment on the possibilities to extend our results to f that is piecewise bounded Lipschitz.

We consider (1.1) for velocity fields that are no longer *fixed* elements of $\mathrm{BL}([0,1])$. Instead of v, we write $v[\mu]$ for the velocity field that depends functionally on the measure μ. The transport equation on $[0,1]$ becomes

$$\frac{\partial}{\partial t}\mu_t + \frac{\partial}{\partial x}(v[\mu_t]\,\mu_t) = f \cdot \mu_t. \tag{1.2}$$

The aim of this paper is to ensure the well-posedness of (1.2), in a suitable sense. Because (1.2) is a nonlinear equation, establishing well-posedness is not straightforward. Here, we employ a *forward-Euler-like approach* that builds on the fundamentals constructed in [EHM15a]. We partition the time interval $[0,T]$ and fix the velocity on each subinterval. That is, restricted to a subinterval, the velocity depends only on the spatial variable and not on the solution measure. Within each subinterval the

measure-valued solution evolves according to the fixed velocity and the evolution fits in the framework set in [EHM15a]. A more detailed description of our approach is given in section 3. We decrease the mesh size in the partition of $[0, T]$ and estimate the difference between Euler approximations. The main result of this paper is the fact that this procedure converges.

A forward-Euler scheme similar to ours is used in [PR13] for measures absolutely continuous with respect to the Lebesgue measure. Their results are extended to general measures in [CPT14, Chapter 7]. The difference between their work and ours is twofold: they use the Wasserstein distance and they work in unbounded domains.

The references that directly inspired us are [CG09, Hoo13, GLMC10]. The approach presented in this paper deviates from [Hoo13], since we restrict ourselves to evolution on the interval $[0, 1]$, while [Hoo13] considers $[0, \infty)$. Furthermore, our regularity conditions on the velocity—given in Assumption 3.1—are weaker than in [Hoo13]; cf. Remark 3.3. Moreover, [Hoo13] restricts to velocity fields that point inward at 0. In this way, no mass is allowed to flow out of the domain $[0, \infty)$. In our approach, the fact that the flow is stopped at the boundary is encoded in the semigroup $(P_t)_{t\geqslant 0}$, irrespective of the sign of the velocity there; cf. section 2.2. We consider it too restrictive to have a condition on the sign of the velocity at 0 or 1; in practice it is very difficult to make sure that such condition is satisfied when the velocity $v[\mu]$ depends on the solution (like in, e.g., Example 3.2).

In this paper we limit our attention to a one-dimensional state space, $[0, 1]$, because in this case the (global) Lipschitz continuous dependence of the stopped flow on the time-invariant velocity field v is a rather straightforward property (see section 2.2, Lemma 2.2). In higher dimensional (bounded) state spaces this seems much more delicate to establish. We comment on this in more detail after the proof of Lemma 2.2. One should note, however, that the results on convergence of the forward-Euler-like approach that we present do not depend on the dimensionality other than through the mentioned Lipschitzian property as presented in Lemma 2.2.

This paper is organized as follows. Within each subinterval of the Euler approximation the dynamics are given by a fixed velocity, like in [EHM15a]. Therefore, we start in section 2 by collecting the results of [EHM15a] that we require here: a number of properties of the semigroup $(P_t)_{t\geqslant 0}$ and of the solution operator, called $(Q_t)_{t\geqslant 0}$. The forward-Euler-like approach to construct solutions is introduced in section 3, where we also state the main results of this paper: Theorems 3.10 and 3.12 and Corollary 3.11. In plain words and combined into one pseudo-theorem, these results read as follows.

THEOREM. *The proposed forward-Euler-like approach converges as the mesh size of the time discretization goes to zero. The limit is independent of the specific way in which the time domain is partitioned. This approximation procedure yields existence and uniqueness of mild solutions to the nonlinear problem, and solutions depend continuously on initial data.*

A more precise formulation follows later. We prove these results in section 4 using estimates between two Euler approximations of (1.1). In section 5 we reflect on the achievements of this paper, discuss open issues, and provide directions for further research.

2. Preliminaries. This section contains a summary of the results obtained in [EHM15a] on which we shall build. Moreover, we mention the technical preliminaries needed for the arguments in this paper.

2.1. Basics of measure theory. If S is a topological space, we denote by $\mathcal{M}(S)$ the space of finite Borel measures on S and by $\mathcal{M}^+(S)$ the convex cone of positive measures included in it. For $x \in S$, δ_x denotes the Dirac measure at x. Let

$$\langle \mu, \phi \rangle := \int_S \phi \, d\mu \tag{2.1}$$

denote the natural pairing between measures $\mu \in \mathcal{M}(S)$ and bounded measurable functions ϕ. The *push-forward* or *image measure* of μ under Borel measurable $\Phi : S \to S$ is the measure $\Phi\#\mu$ defined on Borel sets $E \subset S$ by

$$(\Phi\#\mu)(E) := \mu\big(\Phi^{-1}(E)\big). \tag{2.2}$$

One easily verifies that $\langle \Phi\#\mu, \phi \rangle = \langle \mu, \phi \circ \Phi \rangle$.

We denote by $C_b(S)$ the Banach space of real-valued bounded continuous functions on S equipped with the supremum norm $\|\cdot\|_\infty$. The *total variation norm* $\|\cdot\|_{\mathrm{TV}}$ on $\mathcal{M}(S)$ is defined by

$$\|\mu\|_{\mathrm{TV}} := \sup\left\{ \langle \mu, \phi \rangle \;\middle|\; \phi \in C_b(S),\ \|\phi\|_\infty \leqslant 1 \right\}.$$

It follows immediately that for $\Phi : S \to S$ continuous, $\|\Phi\#\mu\|_{\mathrm{TV}} \leqslant \|\mu\|_{\mathrm{TV}}$. In our setting, S is a Polish space (separable, completely metrizable topological space; cf. [Dud04, p. 344]). It is well-established (cf. [Dud66, Dud74]) that in this case the weak topology on $\mathcal{M}(S)$ induced by $C_b(S)$ *when restricted to the positive cone* $\mathcal{M}^+(S)$ is metrizable by a metric derived from a norm, e.g., the Fortet–Mourier norm or the Dudley norm. The latter is also called the *dual bounded Lipschitz norm*, which we shall introduce now. To that end, let d be a metric on S that metrizes the topology, such that (S, d) is separable and complete. Let $\mathrm{BL}(S, d) = \mathrm{BL}(S)$ be the vector space of real-valued bounded Lipschitz functions on (S, d). For $\phi \in \mathrm{BL}(S)$, let

$$|\phi|_{\mathrm{L}} := \sup\left\{ \frac{|\phi(x) - \phi(y)|}{d(x, y)} \;\middle|\; x, y \in S,\ x \neq y \right\}$$

be its Lipschitz constant. Now

$$\|\phi\|_{\mathrm{BL}} := \|\phi\|_\infty + |\phi|_{\mathrm{L}} \tag{2.3}$$

defines a norm on $\mathrm{BL}(S)$ for which this space is a Banach space [FM53, Dud66]. In fact, with this norm $\mathrm{BL}(S)$ is a Banach algebra for pointwise product of functions:

$$\|\phi \cdot \psi\|_{\mathrm{BL}} \leq \|\phi\|_{\mathrm{BL}} \, \|\psi\|_{\mathrm{BL}}. \tag{2.4}$$

Alternatively, one may define on $\mathrm{BL}(S)$ the equivalent norm

$$\|\phi\|_{\mathrm{FM}} := \max\big(\|\phi\|_\infty, |\phi|_{\mathrm{L}}\big),$$

where FM stands for Fortet–Mourier (see below). Let $\|\cdot\|^*_{\mathrm{BL}}$ be the dual norm of $\|\cdot\|_{\mathrm{BL}}$ on the dual space $\mathrm{BL}(S)^*$, i.e., for any $x^* \in \mathrm{BL}(S)^*$ its norm is given by

$$\|x^*\|^*_{\mathrm{BL}} := \sup\{ |\langle x^*, \phi \rangle| \mid \phi \in \mathrm{BL}(S),\ \|\phi\|_{\mathrm{BL}} \leqslant 1 \}.$$

The map $\mu \mapsto I_\mu$ with $I_\mu(\phi) := \langle \mu, \phi \rangle$ defines a linear *embedding* of $\mathcal{M}(S)$ into $\mathrm{BL}(S)^*$; see [Dud66, Lemma 6]. Thus $\|\cdot\|^*_{\mathrm{BL}}$ induces a norm on $\mathcal{M}(S)$, which is denoted by

the same symbols. It is called the dual bounded Lipschitz norm or Dudley norm. Generally, $\|\mu\|^*_{\mathrm{BL}} \leqslant \|\mu\|_{\mathrm{TV}}$ for all $\mu \in \mathcal{M}(S)$. For positive measures the two norms coincide:

$$\|\mu\|^*_{\mathrm{BL}} = \mu(S) = \|\mu\|_{\mathrm{TV}} \qquad \text{for all } \mu \in \mathcal{M}^+(S). \tag{2.5}$$

One may also consider the restriction to $\mathcal{M}(S)$ of the dual norm $\|\cdot\|^*_{\mathrm{FM}}$ of $\|\cdot\|_{\mathrm{FM}}$ on $\mathrm{BL}(S)^*$. This yields an equivalent norm on $\mathcal{M}(S)$ that is called the Fortet–Mourier norm (see, e.g., [LMS02, Zah00]):

$$\|\mu\|^*_{\mathrm{BL}} \leqslant \|\mu\|^*_{\mathrm{FM}} \leqslant 2\|\mu\|^*_{\mathrm{BL}}. \tag{2.6}$$

This norm also satisfies $\|\mu\|^*_{\mathrm{FM}} \leqslant \|\mu\|_{\mathrm{TV}}$, so (2.5) holds for $\|\cdot\|^*_{\mathrm{FM}}$ too. Moreover (cf. [HW09, Lemma 3.5]), for any $x, y \in S$,

$$\|\delta_x - \delta_y\|^*_{\mathrm{BL}} = \frac{2d(x,y)}{2 + d(x,y)} \leqslant \min(2, d(x,y)) = \|\delta_x - \delta_y\|^*_{\mathrm{FM}}. \tag{2.7}$$

In general, the space $\mathcal{M}(S)$ is not complete for $\|\cdot\|^*_{\mathrm{BL}}$. We denote by $\overline{\mathcal{M}}(S)_{\mathrm{BL}}$ its completion, viewed as closure of $\mathcal{M}(S)$ within $\mathrm{BL}(S)^*$. The space $\mathcal{M}^+(S)$ is complete for $\|\cdot\|^*_{\mathrm{BL}}$ and hence closed in $\mathcal{M}(S)$ and $\overline{\mathcal{M}}(S)_{\mathrm{BL}}$.

The $\|\cdot\|^*_{\mathrm{BL}}$-norm is convenient also for integration. In Appendix C of [EHM15a] some technical results about integration of measure-valued maps were collected. These will also be used in this paper. The continuity of the map $x \mapsto \delta_x : S \to \mathcal{M}^+(S)_{\mathrm{BL}}$ together with (C.2) in [EHM15a] yields the identity

$$\mu = \int_S \delta_x \, d\mu(x) \tag{2.8}$$

as a Bochner integral in $\overline{\mathcal{M}}(S)_{\mathrm{BL}}$; for basic results on Bochner integration, the reader is referred to, e.g., [DU77]. The observation (2.8) will essentially link continuum (μ) and particle description (δ_x) for our equation on $[0,1]$.

2.2. Properties of the stopped flow. Let $v \in \mathrm{BL}([0,1])$ be fixed. We assume that a single particle ("individual") is moving in the domain $[0,1]$ deterministically, described by the differential equation for its position $x(t)$ at time t:

$$\begin{cases} \dot{x}(t) = v(x(t)), \\ x(0) = x_0. \end{cases} \tag{2.9}$$

A solution to (2.9) is unique; it exists for time up to reaching the boundary 0 or 1 and depends continuously on initial conditions. Let $x(\,\cdot\,; x_0)$ be this solution and I_{x_0} be its maximal interval of existence. Define

$$\tau_\partial(x_0) := \sup I_{x_0} \in [0, \infty],$$

i.e., $\tau_\partial(x_0)$ is the time at which the solution starting at x_0 reaches the boundary (if it happens) when x_0 is an interior point. Note that $\tau_\partial(x_0) = 0$ when x_0 is a boundary point where v points outward, while $\tau_\partial(x_0) > 0$ when x_0 is a boundary point where v vanishes or points inward.

The *individualistic stopped flow* on $[0,1]$ associated to v is the family of maps $\Phi_t : [0,1] \to [0,1]$, $t \geqslant 0$, defined by

$$\Phi_t(x_0) := \begin{cases} x(t; x_0) & \text{if } t \in I_{x_0}, \\ x(\tau_\partial(x_0); x_0) & \text{otherwise.} \end{cases} \tag{2.10}$$

To lift the dynamics to the space of measures, we define $P_t : \mathcal{M}([0,1]) \to \mathcal{M}([0,1])$ by means of the push-forward under Φ_t: for all $\mu \in \mathcal{M}([0,1])$,

$$P_t\mu := \Phi_t \# \mu = \mu \circ \Phi_t^{-1}; \tag{2.11}$$

see (2.2). Clearly, P_t maps positive measures to positive measures and P_t is mass preserving on positive measures. Since the family of maps $(\Phi_t)_{t\geqslant 0}$ forms a semigroup, so do the maps P_t in the space $\mathcal{M}([0,1])$. That is, $(P_t)_{t\geqslant 0}$ is a *Markov semigroup* on $\mathcal{M}[0,1]$ (cf. [LMS02]). The basic estimate

$$\|P_t\mu\|_{\mathrm{TV}} \leqslant \|\mu\|_{\mathrm{TV}} \tag{2.12}$$

holds for $\mu \in \mathcal{M}([0,1])$.

In the rest of this section we summarize those properties of $(P_t)_{t\geqslant 0}$ that are needed in this paper. We first recall Lemma 2.2 from [EHM15a].

LEMMA 2.1 (see [EHM15a, Lemma 2.2]). *Let $\mu \in \mathcal{M}([0,1])$ and $t, s \in \mathbb{R}^+$. Then*
(i) $\|P_t\mu - P_s\mu\|^*_{\mathrm{BL}} \leqslant \|v\|_\infty \|\mu\|_{\mathrm{TV}} |t-s|$,
(ii) $\|P_t\mu\|^*_{\mathrm{BL}} \leqslant \max(1, |\Phi_t|_{\mathrm{L}}) \|\mu\|^*_{\mathrm{BL}} \leqslant e^{|v|_{\mathrm{L}} t} \|\mu\|^*_{\mathrm{BL}}$.

To distinguish between the semigroups on $\mathcal{M}([0,1])$ associated to $v, v' \in \mathrm{BL}([0,1])$, we write P^v and $P^{v'}$, respectively. Analogously, we distinguish between the semigroups $(\Phi_t^v)_{t\geqslant 0}$ and $(\Phi_t^{v'})_{t\geqslant 0}$ on $[0,1]$ and between the intervals of existence $I^v_{x_0}$ and $I^{v'}_{x_0}$ associated to (2.9).

LEMMA 2.2. *For all $\mu \in \mathcal{M}([0,1])$, $v, v' \in \mathrm{BL}([0,1])$, and $t \in \mathbb{R}^+_0$*

$$\|P^v_t\mu - P^{v'}_t\mu\|^*_{\mathrm{BL}} \leqslant \|v - v'\|_\infty \, t \, \|\mu\|_{\mathrm{TV}} \, e^{L\,t}, \tag{2.13}$$

where $L := \min(|v|_{\mathrm{L}}, |v'|_{\mathrm{L}})$.

Proof. For any $\phi \in \mathrm{BL}([0,1])$, we have

$$|\left\langle \phi, P^v_t\mu - P^{v'}_t\mu \right\rangle| = |\left\langle \phi \circ \Phi^v_t - \phi \circ \Phi^{v'}_t, \mu \right\rangle| \leqslant |\phi|_{\mathrm{L}} \|\Phi^v_t - \Phi^{v'}_t\|_\infty \|\mu\|_{\mathrm{TV}}, \tag{2.14}$$

hence

$$\|P^v_t\mu - P^{v'}_t\mu\|^*_{\mathrm{BL}} \leqslant \|\Phi^v_t - \Phi^{v'}_t\|_\infty \|\mu\|_{\mathrm{TV}}. \tag{2.15}$$

Let $x \in [0,1]$.

Case 1. $t \in I^v_x \cap I^{v'}_x$.

$$\begin{aligned} |\Phi^v_t(x) - \Phi^{v'}_t(x)| &= \left| \int_0^t v(\Phi^v_s(x)) - v'(\Phi^{v'}_s(x)) \, ds \right| \\ &\leqslant |v|_{\mathrm{L}} \int_0^t |\Phi^v_s(x) - \Phi^{v'}_s(x)| \, ds + \|v - v'\|_\infty \, t. \end{aligned}$$

Gronwall's lemma yields

$$|\Phi^v_t(x) - \Phi^{v'}_t(x)| \leqslant \|v - v'\|_\infty \, t \, e^{|v|_{\mathrm{L}}\, t} \tag{2.16}$$

for all $x \in [0,1]$. Due to the symmetry of (2.16) in v and v', the same estimate (2.16) can be obtained with $|v'|_{\mathrm{L}}$ instead of $|v|_{\mathrm{L}}$, and hence we can write $\min(|v|_{\mathrm{L}}, |v'|_{\mathrm{L}})$ in the exponent. This observation yields, together with (2.15), the statement of the lemma.

Case 2. $t \notin I_x^v$. We extend $v : [0,1] \to \mathbb{R}$ to $\bar{v} : \mathbb{R} \to \mathbb{R}$ by defining $\bar{v}(x) := v(0)$ if $x < 0$ and $\bar{v}(x) := v(1)$ if $x > 1$. Then $\bar{v}$ is a bounded Lipschitz extension of v such that $\|\bar{v}\|_\infty = \|v\|_\infty$ and $|\bar{v}|_{\mathrm{L}} = |v|_{\mathrm{L}}$. Let $\Phi_t^{\bar{v}} : \mathbb{R} \to \mathbb{R}$ be the solution semigroup associated to the unique (global) solution to (2.9) with v replaced by $\bar{v}$ and with initial condition to be taken from the whole of $\mathbb{R}$. We extend v' analogously to $\bar{v}'$.

Irrespective of whether $t \in I_x^{v'}$ or $t \notin I_x^{v'}$, and whether in the latter case $\Phi_t^v(x) = \Phi_t^{v'}(x)$ or $\Phi_t^v(x) \neq \Phi_t^{v'}(x)$, the following estimate holds:

$$|\Phi_t^v(x) - \Phi_t^{v'}(x)| \leqslant |\Phi_t^{\bar{v}}(x) - \Phi_t^{\bar{v}'}(x)| \tag{2.17}$$

for all $x \in [0,1]$. Estimate $|\Phi_t^{\bar{v}}(x) - \Phi_t^{\bar{v}'}(x)|$ using the same ideas as in (2.15) and (2.16) and obtain

$$\|P_t^{\bar{v}}\mu - P_t^{\bar{v}'}\mu\|_{\mathrm{BL}}^* \leqslant \|\bar{v} - \bar{v}'\|_\infty \, t \, \|\mu\|_{\mathrm{TV}} \, \exp(\min(|\bar{v}|_{\mathrm{L}}, |\bar{v}'|_{\mathrm{L}})\, t).$$

The statement of the lemma follows from the equalities $|\bar{v}|_{\mathrm{L}} = |v|_{\mathrm{L}}$, $|\bar{v}'|_{\mathrm{L}} = |v'|_{\mathrm{L}}$, $\|\bar{v} - \bar{v}'\|_\infty = \|v - v'\|_\infty$ and (2.17). The case $t \notin I_x^{v'}$ is analogous. ∎

Remark 2.3. The definition of stopped flow in state spaces of dimension two and higher and establishing elementary properties of its lift to measures is more delicate than the one-dimensional case presented above. Consider an open domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ (with sufficiently smooth boundary). Let $\overline{\Omega}$ be its closure and let $v \in \mathrm{BL}(\overline{\Omega}, \mathbb{R}^n)$ be a velocity field on $\overline{\Omega}$. Solutions to the initial value problem (2.9) with $x_0 \in \Omega$ still exist for some positive time, but in this higher dimensional setting it may happen that trajectories of the flow in $\overline{\Omega}$ defined by v are partially contained in the boundary $\partial\Omega$ or even only "touch" $\partial\Omega$. So reaching the boundary in finite time is not equivalent to "leaving the domain."

It is even possible to give a simpler counterexample, in which Ω is the unit disk in $\mathbb{R}^2$. For any $x = (x_1, x_2) \in \mathbb{R}^2$, the orthogonal vector is defined as $x^\perp := (x_2, -x_1)$. Let the velocities $v, v' \in \mathrm{BL}(\bar{\Omega})$ be given by $v(x) = x^\perp$ and $v'(x) = x^\perp + \varepsilon\, x$ for all $x \in \bar{\Omega}$, where $0 < \varepsilon \ll 1$ is arbitrary. Then $\|v - v'\|_\infty = \varepsilon$. Let $\mu = \delta_{x_0}$, with $x_0 := (0, 1 - \varepsilon^2)$, whence $\|\mu\|_{\mathrm{TV}} = 1$. Now $P_t^v\mu$ is a Dirac measure that performs (eternal) circular motion on the circle of radius $1 - \varepsilon^2$, at speed $1 - \varepsilon^2$. On the other hand, because the semigroup P encodes the *stopped flow*, $P_t^{v'}\mu$ is a Dirac that hits $\partial\Omega$ in $\mathcal{O}(\varepsilon)$ time and remains at that position afterward.

Since $P_t^v\mu$ is a rotation on a circle of $\mathcal{O}(1)$ radius, with $\mathcal{O}(1)$ speed, there must be a time t of order $\mathcal{O}(1)$ (in particular, after $P_t^{v'}\mu$ hits the boundary), at which the two Diracs $P_t^v\mu$ and $P_t^{v'}\mu$ are located an $\mathcal{O}(1)$ distance apart. Hence, $\|P_t^v\mu - P_t^{v'}\mu\|_{\mathrm{BL}}$ is of order $\mathcal{O}(1)$. The right-hand side of (2.13) is of order $\mathcal{O}(\varepsilon)$ at $\mathcal{O}(1)$ times, though, while ε is arbitrarily small. Thus, (2.13) cannot be satisfied, and Lemma 2.2 does not hold in $\mathbb{R}^2$.

A possible way to circumvent the observed problems is to redefine the semigroup Φ such that when particles hit the boundary, they do not stop but their velocity is projected to the tangential direction. In this case Lemma 2.2 might hold. However, by redefining Φ new difficulties may be introduced in the later proofs of this paper. Therefore, we restrict ourselves to the one-dimensional case here.

2.3. Properties of the solution for prescribed velocity. We consider mild solutions to (1.1) that are defined in the following sense.

DEFINITION 2.4 (see [EHM15a, Definition 2.4]). *A* measure-valued mild solution *to the Cauchy problem associated to* (1.1) *on* $[0,T]$ *with initial value* $\nu \in \mathcal{M}([0,1])$ *is*

a continuous map $\mu : [0,T] \to \mathcal{M}([0,1])_{\mathrm{BL}}$ *that is* $\|\cdot\|_{\mathrm{TV}}$*-bounded and that satisfies the variation of constants formula*

$$\mu_t = P_t\,\nu \;+\; \int_0^t P_{t-s} F_f(\mu_s)\,ds \qquad \text{for all } t \in [0,T]. \tag{2.18}$$

Here, the perturbation map $F_f : \mathcal{M}([0,1]) \to \mathcal{M}([0,1])$ *is given by* $F_f(\mu) := f \cdot \mu$.

We showed in [EHM15a] that mild solutions in the sense of Definition 2.4 exist, are unique, and depend continuously on the initial data. We repeat those results in the following theorem.

THEOREM 2.5. *Let* $f : [0,1] \to \mathbb{R}$ *be a piecewise bounded Lipschitz function such that* $v(x) \neq 0$ *at any point* x *of discontinuity of* f. *Then for each* $T \geqslant 0$ *and* $\mu_0 \in \mathcal{M}([0,1])$ *there exists a unique continuous and locally* $\|\cdot\|_{\mathrm{TV}}$*-bounded solution to* (2.18). *Moreover, there exists* $C_T > 0$ *such that for all initial values* $\mu_0, \mu_0' \in \mathcal{M}([0,1])$ *the corresponding mild solutions* μ *and* μ' *satisfy*

$$\|\mu_t - \mu_t'\|_{\mathrm{BL}}^* \leqslant C_T \|\mu_0 - \mu_0'\|_{\mathrm{BL}}^* \tag{2.19}$$

for all $t \in [0,T]$.

Proof. See [EHM15a, Propositions 3.1, 3.3, and 3.5] for details. ∎

In this paper, we restrict ourselves to those functions f that are bounded Lipschitz on $[0,1]$; see section 5.1 for further discussion on the need of this restriction. Let $v \in \mathrm{BL}([0,1])$ and $f \in \mathrm{BL}([0,1])$ be arbitrary. For all $t \geqslant 0$, we define $Q_t : \mathcal{M}([0,1]) \to \mathcal{M}([0,1])$ to be the operator that maps the initial condition to the solution in the sense of Definition 2.4. Theorem 2.5 guarantees that this operator is well-defined and continuous for $\|\cdot\|_{\mathrm{BL}}^*$. Moreover, Q preserves positivity, due to [EHM15a, Corollary 3.4].

In the rest of this section, we give an overview of the properties of the solution operator Q.

LEMMA 2.6 (semigroup property). *The set of operators* $(Q_t)_{t\geqslant 0}$ *satisfies the semigroup property. That is,*

$$Q_t\, Q_s\, \mu = Q_{t+s}\, \mu$$

for all $s, t \geqslant 0$ *and for all* $\mu \in \mathcal{M}([0,1])$.

Proof. The proof follows the lines of the argument of [Š94, p. 283]. We consider

$$\begin{aligned} Q_{t+s}\mu - Q_t\, Q_s\, \mu &= P_{t+s}\mu + \int_0^{t+s} P_{t+s-\sigma}\, F_f(Q_\sigma\, \mu)\, d\sigma \\ &\quad - P_t\, Q_s\, \mu - \int_0^t P_{t-\sigma}\, F_f(Q_\sigma\, Q_s\, \mu)\, d\sigma \end{aligned} \tag{2.20}$$

and observe that

$$\begin{aligned} P_t\, Q_s\, \mu &= P_t\, P_s\, \mu + P_t \int_0^s P_{s-\sigma}\, F_f(Q_\sigma\, \mu)\, d\sigma \\ &= P_{t+s}\, \mu + \int_0^s P_{t+s-\sigma}\, F_f(Q_\sigma\, \mu)\, d\sigma. \end{aligned} \tag{2.21}$$

Because $f \in \mathrm{BL}([0,1])$, the map $\sigma \mapsto P_{s-\sigma}\, F_f(Q_\sigma\, \mu)$ is continuous and hence it is measurable. Therefore, the second equality in (2.21) holds due to [EHM15a, (C.3)]. A combination of (2.20) and (2.21) yields that

$$\begin{aligned} Q_{t+s}\mu - Q_t\, Q_s\, \mu &= \int_s^{t+s} P_{t+s-\sigma}\, F_f(Q_\sigma\, \mu)\, d\sigma - \int_0^t P_{t-\sigma}\, F_f(Q_\sigma\, Q_s\, \mu)\, d\sigma \\ &= \int_0^t P_{t-\sigma}\, \left(F_f(Q_{\sigma+s}\, \mu) - F_f(Q_\sigma\, Q_s\, \mu)\right)\, d\sigma. \end{aligned} \tag{2.22}$$

To obtain the last step in (2.22), we use the coordinate transformation $\tau := \sigma - s$ in the first integral and subsequently rename the new variable τ as σ. We estimate the total variation norm of (2.22) in the following way:

$$\begin{aligned} \|Q_{t+s}\mu - Q_t\, Q_s\, \mu\|_{\mathrm{TV}} &\leqslant \int_0^t \|P_{t-\sigma}\, \left(F_f(Q_{\sigma+s}\, \mu) - F_f(Q_\sigma\, Q_s\, \mu)\right)\|_{\mathrm{TV}}\, d\sigma \\ &\leqslant \int_0^t \|F_f(Q_{\sigma+s}\, \mu) - F_f(Q_\sigma\, Q_s\, \mu)\|_{\mathrm{TV}}\, d\sigma \\ &\leqslant \|f\|_\infty \int_0^t \|Q_{\sigma+s}\, \mu - Q_\sigma\, Q_s\, \mu\|_{\mathrm{TV}}\, d\sigma. \end{aligned}$$

Here, we used [EHM15a, Proposition C.2(iii)] (noting that the integrands are continuous with respect to σ) in the first line, (2.12) in the second line, and the fact that $f \in \mathrm{BL}([0,1]) \subset C_b([0,1])$ in the last line. Gronwall's lemma now implies that $\|Q_{t+s}\mu - Q_t\, Q_s\, \mu\|_{\mathrm{TV}} = 0$ for all $s, t \geqslant 0$. ∎

LEMMA 2.7. *For all $\mu \in \mathcal{M}([0,1])$ and $s, t \geqslant 0$, we have that*

$$\|Q_t\mu - Q_s\mu\|_{\mathrm{BL}}^* \leqslant \|\mu\|_{\mathrm{TV}} \cdot \left(\|f\|_\infty + \|v\|_\infty\right) \cdot e^{\|f\|_\infty \max(t,s)} \cdot |t-s|.$$

Proof. The statement of this lemma is part of the result of [EHM15a, Proposition 3.3]. ∎

LEMMA 2.8. *For all $\mu \in \mathcal{M}([0,1])$ and $t \geqslant 0$, we have that*

(i) $\|Q_t\mu\|_{\mathrm{TV}} \leqslant \|\mu\|_{\mathrm{TV}}\, \exp(\|f\|_\infty\, t)$ *and*

(ii) $\|Q_t\mu\|_{\mathrm{BL}}^* \leqslant \|\mu\|_{\mathrm{BL}}^*\, \exp(|v|_{\mathrm{L}}\, t + \|f\|_{\mathrm{BL}}\, t\, e^{|v|_{\mathrm{L}}\, t})$.

Proof. (i) This estimate is given in [EHM15a, Proposition 3.3].

(ii) By applying [EHM15a, (C.1)] and Lemma 2.1(ii) we obtain from (2.18) the estimate

$$\|Q_t\, \mu\|_{\mathrm{BL}}^* \leqslant \exp(|v|_{\mathrm{L}}\, t)\, \|\mu\|_{\mathrm{BL}}^* + \int_0^t \exp(|v|_{\mathrm{L}}\, (t-s))\|f\|_{\mathrm{BL}}\, \|Q_s\mu\|_{\mathrm{BL}}^*\, ds.$$

Gronwall's lemma now yields the statement of part (ii) of the lemma. ∎

COROLLARY 2.9. *For all $\mu, \nu \in \mathcal{M}([0,1])$ and $t \geqslant 0$, we have that*

$$\|Q_t\, \mu - Q_t\, \nu\|_{\mathrm{BL}}^* \leqslant \|\mu - \nu\|_{\mathrm{BL}}^*\, \exp(|v|_{\mathrm{L}}\, t + \|f\|_{\mathrm{BL}}\, t\, e^{|v|_{\mathrm{L}}\, t}).$$

Proof. Apply part (ii) of Lemma 2.8 to the measure $\mu - \nu \in \mathcal{M}([0,1])$. ∎

We write Q^v and $Q^{v'}$ to distinguish between the semigroups Q on $\mathcal{M}([0,1])$ associated to $v \in \mathrm{BL}([0,1])$ and $v' \in \mathrm{BL}([0,1])$, respectively.

LEMMA 2.10. *For all $v, v' \in \mathrm{BL}([0,1])$, $\mu \in \mathcal{M}([0,1])$, and $t \geqslant 0$, the following estimate holds:*

$$\|Q_t^v \mu - Q_t^{v'} \mu\|_{\mathrm{BL}}^* \leqslant \|v - v'\|_\infty \, \|\mu\|_{\mathrm{TV}} \, \exp(L\, t + \|f\|_{\mathrm{BL}}\, t\, e^{L\, t}) \cdot [t + t^2 \|f\|_\infty \, e^{\|f\|_\infty \, t}],$$

where $L := \min(|v|_{\mathrm{L}}, |v'|_{\mathrm{L}})$.

Proof. We have

$$\|Q_t^v \mu - Q_t^{v'} \mu\|_{\mathrm{BL}}^* \leqslant \|P_t^v \mu - P_t^{v'} \mu\|_{\mathrm{BL}}^* + \int_0^t \|P_{t-s}^v F_f(Q_s^v \mu) - P_{t-s}^{v'} F_f(Q_s^{v'} \mu)\|_{\mathrm{BL}}^* \, ds. \tag{2.23}$$

Lemma 2.2 provides an appropriate estimate of the first term on the right-hand side. For the integrand in the second term, we have

$$\begin{aligned} \|P_{t-s}^v F_f(Q_s^v \mu) - P_{t-s}^{v'} F_f(Q_s^{v'} \mu)\|_{\mathrm{BL}}^* &\leqslant \|P_{t-s}^v F_f(Q_s^v \mu) - P_{t-s}^{v'} F_f(Q_s^v \mu)\|_{\mathrm{BL}}^* \\ &\quad + \|P_{t-s}^{v'} F_f(Q_s^v \mu) - P_{t-s}^{v'} F_f(Q_s^{v'} \mu)\|_{\mathrm{BL}}^* \\ &\leqslant \|v - v'\|_\infty \, (t-s) \, \|F_f(Q_s^v \mu)\|_{\mathrm{TV}} \, e^{L(t-s)} \\ &\quad + e^{|v'|_{\mathrm{L}} \, (t-s)} \, \|F_f(Q_s^v \mu) - F_f(Q_s^{v'} \mu)\|_{\mathrm{BL}}^*, \end{aligned} \tag{2.24}$$

due to Lemma 2.2 and Lemma 2.1(ii). We proceed by estimating the right-hand side of (2.24) and obtain

$$\begin{aligned} \|P_{t-s}^v F_f(Q_s^v \mu) - P_{t-s}^{v'} F_f(Q_s^{v'} \mu)\|_{\mathrm{BL}}^* &\leqslant \|v - v'\|_\infty \, (t-s) \, \|f\|_\infty \, \|\mu\|_{\mathrm{TV}} \, e^{\|f\|_\infty \, s} \, e^{L(t-s)} \\ &\quad + e^{|v'|_{\mathrm{L}} \, (t-s)} \, \|f\|_{\mathrm{BL}} \, \|Q_s^v \mu - Q_s^{v'} \mu\|_{\mathrm{BL}}^*, \end{aligned} \tag{2.25}$$

where we use part (i) of Lemma 2.8 in the first term on the right-hand side. Since the estimate in (2.25) is symmetric in v and v', we can replace $|v'|_{\mathrm{L}}$ by L.

Substitution of the result of Lemma 2.2 and (2.25) in (2.23) yields

$$\begin{aligned} \|Q_t^v \mu - Q_t^{v'} \mu\|_{\mathrm{BL}}^* &\leqslant \|v - v'\|_\infty \, t \, \|\mu\|_{\mathrm{TV}} \, e^{Lt} \, (1 + t \, \|f\|_\infty \, e^{\|f\|_\infty \, t}) \\ &\quad + e^{Lt} \, \|f\|_{\mathrm{BL}} \int_0^t \|Q_s^v \mu - Q_s^{v'} \mu\|_{\mathrm{BL}}^* \, ds. \end{aligned}$$

The statement of the lemma follows from Gronwall's lemma. ∎

3. Measure-dependent velocity fields: Main results. This section contains the main results of the present work. We generalize the assumptions on v from [EHM15a] in the following way to measure-dependent velocity fields.

Assumption 3.1 (assumptions on the measure-dependent velocity field). Assume that $v : \mathcal{M}([0,1]) \times [0,1] \to \mathbb{R}$ is a mapping such that

(i) $v[\mu] \in \mathrm{BL}([0,1])$ for each $\mu \in \mathcal{M}([0,1])$.

Furthermore, assume that for any $R > 0$ there are constants K_R, L_R, M_R such that for all $\mu, \nu \in \mathcal{M}([0,1])$ satisfying $\|\mu\|_{\mathrm{TV}} \leqslant R$ and $\|\nu\|_{\mathrm{TV}} \leqslant R$, the following estimates hold:

(ii) $\|v[\mu]\|_\infty \leqslant K_R$,
(iii) $|\, v[\mu]\,|_{\mathrm{L}} \leqslant L_R$, and
(iv) $\|v[\mu] - v[\nu]\|_\infty \leqslant M_R \, \|\mu - \nu\|_{\mathrm{BL}}^*$.

Example 3.2. An example of a function v satisfying Assumption 3.1 is

$$v[\mu](x) := \int_{[0,1]} \mathcal{K}(x - y) \, d\mu(y) = (\mathcal{K} * \mu)(x) \tag{3.1}$$

for each $\mu \in \mathcal{M}([0,1])$ and $x \in [0,1]$ with $\mathcal{K} \in \mathrm{BL}([-1,1])$. This is a relevant choice, because it models interactions among individuals.

Remark 3.3. Parts (ii) and (iii) of Assumption 3.1 are an improvement compared to [Hoo13]. There, the infinity norm and Lipschitz constant are assumed to hold uniformly for all $\mu \in \mathcal{M}([0,1])$; cf. Assumption (F1) in [Hoo13, p. 40]. We note that the convolution in Example 3.2 satisfies Assumption 3.1 but does not satisfy Assumption (F1) in [Hoo13]. They require a uniform Lipschitz constant because their Lemma 4.3 is an estimate in the $\|\cdot\|^*_{\mathrm{BL}}$-norm for which part (ii) of our Lemma 2.1 is used. Our counterpart of Lemma 4.3 in [Hoo13] is Lemma 3.4. We give an estimate in terms of the $\|\cdot\|_{\mathrm{TV}}$-norm using (2.12) which does not involve the Lipschitz constant.

Our aim is to prove well-posedness (in some sense yet to be defined) of (1.2). That is,

$$\frac{\partial}{\partial t}\mu_t + \frac{\partial}{\partial x}(v[\mu_t]\,\mu_t) = f \cdot \mu_t$$

on $[0,1]$. As said in section 2.3, we restrict ourselves to f that is bounded Lipschitz on $[0,1]$.

We now introduce the aforementioned forward-Euler-like approach to construct approximate solutions. Let $T > 0$ be given. Let $N \geqslant 1$ be fixed and define a set $\alpha \subset [0,T]$ as follows:

$$\alpha := \big\{ t_j \in [0,T] \ : \ 0 \leqslant j \leqslant N,\ t_0 = 0,\ t_N = T,\ t_j < t_{j+1} \big\}. \tag{3.2}$$

A set α of this form is called a *partition* of the interval $[0,T]$ and N denotes the number of *subintervals* in α.

Let $\mu_0 \in \mathcal{M}([0,1])$ be fixed. For a given partition $\alpha := \{t_0, \ldots, t_N\} \subset [0,T]$, define a measure-valued trajectory $\mu \in C([0,T]; \mathcal{M}([0,1]))$ by

$$\begin{cases} \mu_t := Q^{v_j}_{t-t_j}\,\mu_{t_j} & \text{if } t \in (t_j, t_{j+1}]; \\ v_j := v[\mu_{t_j}]; \\ \mu_{t=0} = \mu_0, \end{cases} \tag{3.3}$$

for all $j \in \{0, \ldots, N-1\}$. Here, $(Q^v_t)_{t \geqslant 0}$ denotes the semigroup introduced in section 2.3 associated to an arbitrary $v \in \mathrm{BL}([0,1])$. Note that by Assumption 3.1, part (i), $v_j = v[\mu_{t_j}] \in \mathrm{BL}([0,1])$ for each j.

We call this a forward-Euler-like approach, because it is the analogon of the forward Euler method for ODEs (cf., e.g., [But03, Chapter 2]). Consider the ODE $dx/dt = v(x)$ on $\mathbb{R}$ for some (Lipschitz continuous) $v : \mathbb{R} \to \mathbb{R}$. The forward Euler method approximates the solution on some interval $(t_j, t_{j+1}]$ by evolving the approximate solution at time t_j, named x_j, due to a *constant* velocity $v(x_j)$. That is, $x(t) \approx x_j + (t - t_j) \cdot v(x_j)$ for all $t \in (t_j, t_{j+1}]$.

In (3.3), we introduce the approximation μ_t, where μ_t results from μ_{t_j} by the evolution due to the constant velocity field $v[\mu_{t_j}]$. The word *constant* here does not refer to v being the same for all $x \in [0,1]$ but to the fact that v corresponding to the same μ_{t_j} is used throughout $(t_j, t_{j+1}]$.

The conditions in parts (ii)–(iv) of Assumption 3.1 are only required to hold for measures in a TV-norm bounded set, in view of the following lemma.

LEMMA 3.4. *Let $\mu_0 \in \mathcal{M}([0,1])$ be given and let $v : \mathcal{M}([0,1]) \times [0,1] \to \mathbb{R}$ satisfy Assumption* 3.1(i). *For a given partition $\alpha := \{t_0, \ldots, t_N\} \subset [0,T]$, let*

$\mu \in C([0,T];\mathcal{M}([0,1]))$ be defined by (3.3)*. Then the set of all timeslices of μ, that is,*

$$\mathcal{A} := \{\mu_t : t \in [0,T]\},$$

is bounded in both $\|\cdot\|_{\mathrm{TV}}$ and $\|\cdot\|^*_{\mathrm{BL}}$. The bounds are independent of the choice of α.

Proof. Fix $j \in \{0,\dots,N-1\}$ and let $t \in (t_j, t_{j+1}]$. By part (i) of Lemma 2.8, we have that

$$\begin{aligned}\|\mu_t\|_{\mathrm{TV}} = \|Q^{v_j}_{t-t_j}\,\mu_{t_j}\|_{\mathrm{TV}} &\leqslant \|\mu_{t_j}\|_{\mathrm{TV}}\,\exp(\|f\|_\infty\,(t-t_j))\\ &\leqslant \|\mu_{t_j}\|_{\mathrm{TV}}\,\exp(\|f\|_\infty\,(t_{j+1}-t_j))\end{aligned}$$

for all $t \in (t_j, t_{j+1}]$. Iteration of the right-hand side with respect to j yields

$$\|\mu_t\|_{\mathrm{TV}} \;\leqslant\; \|\mu_0\|_{\mathrm{TV}} \prod_{i=0}^{j} \exp(\|f\|_\infty\,(t_{i+1}-t_i)) \;=\; \|\mu_0\|_{\mathrm{TV}}\,\exp(\|f\|_\infty\,(t_{j+1}-t_0)).$$

Hence, for all $t \in [0,T]$

$$\|\mu_t\|_{\mathrm{TV}} \;\leqslant\; \|\mu_0\|_{\mathrm{TV}}\,\exp(\|f\|_\infty\,(t_N-t_0)) \;=\; \|\mu_0\|_{\mathrm{TV}}\,\exp(\|f\|_\infty\,T).$$

This bound is in particular independent of t, N and the distribution of points within α. The bound in $\|\cdot\|^*_{\mathrm{BL}}$ follows from the inequality $\|\nu\|^*_{\mathrm{BL}} \leqslant \|\nu\|_{\mathrm{TV}}$ that holds for all $\nu \in \mathcal{M}([0,1])$. ∎

In this paper we construct sequences of Euler approximations, each following from a sequence of partitions $(\alpha_k)_{k\in\mathbb{N}}$ that satisfies the following assumption.

Assumption 3.5 (assumptions on the sequence of partitions). Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence of partitions of $[0,T]$ and let $(N_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ be the corresponding sequence such that each α_k is of the form

$$\alpha_k := \{t^k_j \in [0,T] \;:\; 0 \leqslant j \leqslant N_k,\; t^k_0 = 0,\; t^k_{N_k} = T,\; t^k_j < t^k_{j+1}\}. \tag{3.4}$$

Define

$$M^{(k)} := \max_{j\in\{0,\dots,N_k-1\}} t^k_{j+1} - t^k_j \tag{3.5}$$

for all $k \in \mathbb{N}$. Assume that the sequence $(M^{(k)})_{k\in\mathbb{N}}$ is nonincreasing and $M^{(k)} \to 0$ as $k \to \infty$.

Example 3.6. The following sequences of partitions satisfy Assumption 3.5:

- For all $k \in \mathbb{N}$, take $N_k := 2^k$, and let $t^k_j := jT/2^k$ for all $j \in \{0,\dots,N_k\}$. This implies that $M^{(k)} = T/2^k$ for all $k \in \mathbb{N}$. This specific sequence of partitions was used in [Eve15, Chapter 5].
- Fix $q \in \mathbb{N}^+$. For all $k \in \mathbb{N}$, take $N_k := q^k$, and let $t^k_j := jT/q^k$ for all $j \in \{0,\dots,N_k\}$. This implies that $M^{(k)} = T/q^k$ for all $k \in \mathbb{N}$. In the discussion section of [Eve15, Chapter 5], the results of the current paper were conjectured to hold for this case.
- For all $k \in \mathbb{N}$, take $N_k := k+1$, and let $t^k_j := jT/(k+1)$ for all $j \in \{0,\dots,N_k\}$. This implies that $M^{(k)} = T/(k+1)$ for all $k \in \mathbb{N}$. This is an elementary time discretization (with uniform mesh size) used frequently when proving the convergence of numerical methods.

- Let α_0 be a possibly nonuniform partition of $[0,T]$. Construct the sequence $(\alpha_k)_{k\in\mathbb{N}}$ in such a way that any α_{k+1} is a refinement of α_k. That is, $\alpha_{k+1} \subset \alpha_k$ for all $k \in \mathbb{N}$. Elements may be added in a nonuniform fashion to obtain α_{k+1} from α_k, as long as $M^{(k)} \to 0$ as $k \to \infty$. In this case $(N_k)_{k\in\mathbb{N}}$ is automatically nondecreasing.

Also, some less straightforward sequences of nonuniform partitions are admissible, in which subsequent partitions are not refinements. See, for example, Figure 1, in which two subsequent elements from the sequence $(\alpha_k)_{k\in\mathbb{N}}$ are given. These elements could indeed occur, since $M^{(k+1)} < M^{(k)}$. This example is rather counterintuitive, as there is a local growth of the mesh size at the left-hand side of the interval $[0,T]$ when we go from α_k to α_{k+1}. Note that even $N^{(k+1)} < N^{(k)}$. However, admissibility of a sequence of partitions is only determined by the local ordering of the *maximum* mesh spacing (i.e., the condition $M^{(k+1)} \leqslant M^{(k)}$) and its long-time behavior: $M^{(k)} \to 0$ as $k \to \infty$.

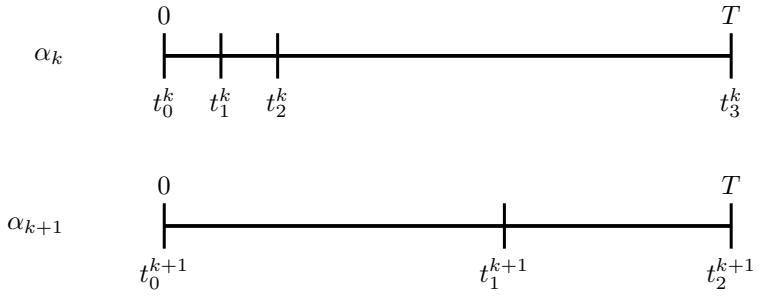

FIG. 1. *Two possible subsequent partitions in a sequence $(\alpha_k)_{k\in\mathbb{N}}$ satisfying Assumption* 3.5.

Remark 3.7. Assumption 3.5 implies that $N_k \to \infty$ as $k \to \infty$.

If $(M^{(k)})_{k\in\mathbb{N}}$ is *not* nonincreasing, but still $M^{(k)} \to 0$, then it is possible to extract a subsequence $(\alpha_{k_\ell})_{\ell\in\mathbb{N}}$ such that $(M^{(k_\ell)})_{\ell\in\mathbb{N}}$ *is* nonincreasing.

We define a mild solution in this context as follows.

DEFINITION 3.8 (mild solution of (1.2)). *Let the space of continuous maps from $[0,T]$ to $\mathcal{M}([0,1])_{\mathrm{BL}}$ be endowed with the metric defined for all $\mu,\nu \in C([0,T];\mathcal{M}([0,1]))$ by*

$$\sup_{t\in[0,T]} \|\mu_t - \nu_t\|^*_{\mathrm{BL}}. \tag{3.6}$$

Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence of partitions satisfying Assumption 3.5. *For each $k \in \mathbb{N}$, let $\mu^k \in C([0,T];\mathcal{M}([0,1]))$ be defined by* (3.3) *with partition α_k. Then, for any such sequence of partitions $(\alpha_k)_{k\in\mathbb{N}}$, any limit of a subsequence of $(\mu^k)_{k\in\mathbb{N}}$ is called a (measure-valued) mild solution of* (1.2).

The name *mild solutions* is appropriate, because they are constructed from piecewise mild solutions in the sense of Definition 2.4.

Remark 3.9. Consider the solution of (3.3) for any partition $\alpha \subset [0,T]$. Mass that has accumulated on the boundary can move back into the interior of the domain whenever the velocity changes direction from one time interval to the next. This is due to the definition of the maximal interval of existence I_{x_0} and the hitting time $\tau_\partial(x_0)$ in section 2.2.

In the rest of this paper we focus on *positive* measure-valued solutions, because these are the only physically relevant solutions in many applications. The main result of this paper is the following theorem.

THEOREM 3.10. *Let $\mu_0 \in \mathcal{M}^+([0,1])$ be given and let $v : \mathcal{M}([0,1]) \times [0,1] \to \mathbb{R}$ satisfy Assumption* 3.1. *Endow the space $C([0,T];\mathcal{M}([0,1]))$ with the metric defined by* (3.6). *Then, there is a unique element of $C([0,T];\mathcal{M}^+([0,1]))$ with initial condition μ_0, that is a mild solution in the sense of Definition* 3.8. *That is, for each sequence of partitions $(\alpha_k)_{k\in\mathbb{N}}$ satisfying Assumption* 3.5, *the corresponding sequence $(\mu^k)_{k\in\mathbb{N}}$ defined by* (3.3) *is a sequence in $C([0,T];\mathcal{M}^+([0,1]))$ and has a unique limit as $k \to \infty$.*

Moreover, this limit is independent of the choice of $(\alpha_k)_{k\in\mathbb{N}}$.

COROLLARY 3.11 (global existence and uniqueness). *For each $\mu_0 \in \mathcal{M}^+([0,1])$ and $v : \mathcal{M}([0,1]) \times [0,1] \to \mathbb{R}$ satisfying Assumption* 3.1, *a unique mild solution exists for all time $t \geqslant 0$.*

THEOREM 3.12 (continuous dependence on initial data). *For all $T > 0$ and $\tilde{R} > 0$ there is a constant $C_{\tilde{R},T}$ such that for all $\mu_0, \nu_0 \in \mathcal{M}^+([0,1])$ satisfying $\|\mu_0\|_{\mathrm{TV}} \leqslant \tilde{R}$ and $\|\nu_0\|_{\mathrm{TV}} \leqslant \tilde{R}$, the corresponding mild solutions $\mu, \nu \in C([0,T];\mathcal{M}^+([0,1]))$ satisfy*

$$\sup_{\tau\in[0,T]} \|\mu_\tau - \nu_\tau\|^*_{\mathrm{BL}} \leqslant C_{\tilde{R},T}\, \|\mu_0 - \nu_0\|^*_{\mathrm{BL}}.$$

The proofs of these theorems and this corollary are given in the next section, section 4. The key idea of the proof of Theorem 3.10 is to show that the sequence $(\mu^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in a complete metric space and hence converges. We use estimates between approximations μ^k and μ^m, $m \geqslant k$. Similar estimates are employed to obtain the result of Theorem 3.12. To prove Corollary 3.11, we show that a solution at time $t \geqslant 0$ is provided by Theorem 3.10 if $T > 0$ is chosen such that $t \in [0,T]$. Moreover, this solution at time t is independent of the exact choice of T.

4. Proofs of Theorems 3.10 and 3.12 and of Corollary 3.11. In this section we prove the main results of this paper: Theorem 3.10, Corollary 3.11, and Theorem 3.12. The essential part of the proof of Theorem 3.10 is provided by the following lemma.

LEMMA 4.1. *For fixed $\mu_0 \in \mathcal{M}^+([0,1])$ and $(\alpha_k)_{k\in\mathbb{N}}$ satisfying Assumption* 3.5, *the corresponding sequence $(\mu^k)_{k\in\mathbb{N}}$ defined by* (3.3) *is a Cauchy sequence in $C([0,T];\mathcal{M}^+([0,1]))$. In particular, there is a constant C such that*

$$\sup_{\tau\in[0,T]} \|\mu^k_\tau - \mu^m_\tau\|^*_{\mathrm{BL}} \leqslant C \max_{j\in\{0,\dots,N_k-1\}} \left(t^k_{j+1} - t^k_j\right)$$

for all $k, m \in \mathbb{N}$ satisfying $m \geqslant k$.

Proof. Fix $k, m \in \mathbb{N}$ with $m \geqslant k$, let $\tau \in [0,T]$ be arbitrary, and let $j \in \{0,\dots,N_k - 1\}$ be such that $\tau \in (t^k_j, t^k_{j+1}]$. Define, for appropriate $N^{(j)} \geqslant 1$, the ordered set

$$\{\tau_\ell \ : \ 0 \leqslant \ell \leqslant N^{(j)}\} := \{t^k_j\} \cup \left(\alpha_m \cap (t^k_j, t^k_{j+1}]\right) \cup \{t^k_{j+1}\}. \tag{4.1}$$

The set $\alpha_m \cap (t^k_j, t^k_{j+1}]$ contains all t^m_ℓ, $\ell \in \{1,\dots,N_m\}$ such that $t^k_j < t^m_\ell \leqslant t^k_{j+1}$. For the sake of being complete, we emphasize that any duplicate elements that might occur on the right-hand side of (4.1) are not "visible" in the set on the left-hand side.

Assume that $i \in \{0, \dots, N^{(j)} - 1\}$ is such that $\tau \in (\tau_i, \tau_{i+1}]$. To simplify notation, we write $v_\ell^\kappa := v[\mu_{\tau_\ell}^\kappa]$ for all $\kappa \in \mathbb{N}$ and $\ell \in \{0, \dots, N^{(j)}\}$. Define $i_0 \in \{0, \dots, N_m\}$ to be the smallest index such that $t_{i_0}^m \geqslant t_j^k$.

Case 1. $t_{i_0}^m = t_j^k$. In this case, there is a $q \in \{0, \dots, N_m - 1\}$ such that $\tau_i = t_q^m$. Hence,

$$\mu_\tau^k = Q_{\tau-\tau_i}^{v_0^k} \mu_{\tau_i}^k \qquad \text{and} \qquad \mu_\tau^m = Q_{\tau-\tau_i}^{v_i^m} \mu_{\tau_i}^m.$$

We estimate

$$\begin{aligned}
\|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* &\leqslant \|Q_{\tau-\tau_i}^{v_0^k}(\mu_{\tau_i}^k - \mu_{\tau_i}^m)\|_{\mathrm{BL}}^* + \|\big(Q_{\tau-\tau_i}^{v_0^k} - Q_{\tau-\tau_i}^{v_i^m}\big)\mu_{\tau_i}^m\|_{\mathrm{BL}}^* \\
&\leqslant \|\mu_{\tau_i}^k - \mu_{\tau_i}^m\|_{\mathrm{BL}}^* \exp\Big(|v_0^k|_{\mathrm{L}}\,(\tau - \tau_i) + \|f\|_{\mathrm{BL}}\,(\tau - \tau_i)\, e^{|v_0^k|_{\mathrm{L}}\,(\tau-\tau_i)}\Big) \\
&\quad + \|v_0^k - v_i^m\|_\infty\, \|\mu_{\tau_i}^m\|_{\mathrm{TV}} \exp\Big(L\,(\tau - \tau_i) + \|f\|_{\mathrm{BL}}\,(\tau - \tau_i)\, e^{L\,(\tau-\tau_i)}\Big) \\
&\quad \cdot \Big[(\tau - \tau_i) + (\tau - \tau_i)^2 \|f\|_\infty\, e^{\|f\|_\infty\,(\tau-\tau_i)}\Big] \qquad (4.2)
\end{aligned}$$

using Corollary 2.9 and Lemma 2.10. Here, L denotes $\min(|v_0^k|_{\mathrm{L}}\,, |v_i^m|_{\mathrm{L}})$. In view of Lemma 3.4, we define $R := \|\mu_0\|_{\mathrm{TV}} \cdot \exp(\|f\|_\infty T)$. From Lemma 2.7 (with $s = 0$), and parts (ii) and (iv) of Assumption 3.1 it follows that

$$\begin{aligned}
\|v_0^k - v_i^m\|_\infty &\leqslant M_R \Big(\|\mu_{\tau_0}^k - \mu_{\tau_0}^m\|_{\mathrm{BL}}^* + \sum_{\ell=1}^{i} \|\mu_{\tau_\ell}^m - \mu_{\tau_{\ell-1}}^m\|_{\mathrm{BL}}^*\Big) \\
&\leqslant M_R\, \|\mu_{\tau_0}^k - \mu_{\tau_0}^m\|_{\mathrm{BL}}^* + M_R \sum_{\ell=1}^{i} \|Q_{\tau_\ell-\tau_{\ell-1}}^{v_{\ell-1}^m}\, \mu_{\tau_{\ell-1}}^m - \mu_{\tau_{\ell-1}}^m\|_{\mathrm{BL}}^* \\
&\leqslant M_R\, \|\mu_{\tau_0}^k - \mu_{\tau_0}^m\|_{\mathrm{BL}}^* + M_R \sum_{\ell=1}^{i} R\,\big(\|f\|_\infty + K_R\big)\, e^{\|f\|_\infty\, T}\,(\tau_\ell - \tau_{\ell-1}) \\
&\leqslant M_R\, \|\mu_{\tau_0}^k - \mu_{\tau_0}^m\|_{\mathrm{BL}}^* + M_R\, R\,\big(\|f\|_\infty + K_R\big)\, e^{\|f\|_\infty\, T}\,(\tau_i - \tau_0). \qquad (4.3)
\end{aligned}$$

We combine (4.2) and (4.3) and use part (iii) of Assumption 3.1 and the basic estimates $\tau - \tau_i \leqslant \tau_{i+1} - \tau_i$ and $\tau_{i+1} - \tau_i \leqslant T$ (in suitable places) to obtain that

$$\begin{aligned}
\|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* &\leqslant \exp\big(A_1\,(\tau_{i+1} - \tau_i)\big)\, \|\mu_{\tau_i}^k - \mu_{\tau_i}^m\|_{\mathrm{BL}}^* \\
&\quad + A_2\,(\tau_{i+1} - \tau_i)\, \|\mu_{\tau_0}^k - \mu_{\tau_0}^m\|_{\mathrm{BL}}^* \\
&\quad + A_3\,(\tau_{i+1} - \tau_i)(\tau_i - \tau_0) \qquad (4.4)
\end{aligned}$$

for some positive constants A_1, A_2, and A_3 that depend on f, T, and R but not on i or j. This upper bound holds for all $\tau \in (\tau_i, \tau_{i+1}]$.

Case 2. $t_j^k < t_{i_0}^m$ *and* $i = 0$. Note that $j \neq 0$ and $i_0 \neq 0$ must hold. We recall the notation $v_\ell^\kappa := v[\mu_{\tau_\ell}^\kappa]$ for all $\kappa \in \mathbb{N}$ and $\ell \in \{0, \dots, N^{(j)}\}$. In this case,

$$\mu_\tau^k = Q_{\tau-\tau_0}^{v_0^k} \mu_{\tau_0}^k \qquad \text{and} \qquad \mu_\tau^m = Q_{\tau-\tau_0}^{\bar{v}} \mu_{\tau_0}^m,$$

where $\bar{v} := v[\mu^m_{t^m_{i_0-1}}]$. Similar to (4.2), we have

$$\begin{aligned}\|\mu^k_\tau - \mu^m_\tau\|^*_{\mathrm{BL}} &\leqslant \|Q^{v^k_0}_{\tau-\tau_0}(\mu^k_{\tau_0} - \mu^m_{\tau_0})\|^*_{\mathrm{BL}} + \|\big(Q^{v^k_0}_{\tau-\tau_0} - Q^{\bar{v}}_{\tau-\tau_0}\big)\mu^m_{\tau_0}\|^*_{\mathrm{BL}} \\ &\leqslant \|\mu^k_{\tau_0} - \mu^m_{\tau_0}\|^*_{\mathrm{BL}} \exp\left(|v^k_0|_{\mathrm{L}}\,(\tau-\tau_0) + \|f\|_{\mathrm{BL}}\,(\tau-\tau_0)\,e^{|v^k_0|_{\mathrm{L}}\,(\tau-\tau_0)}\right) \\ &\quad + \|v^k_0 - \bar{v}\|_\infty\,\|\mu^m_{\tau_0}\|_{\mathrm{TV}} \exp\left(L\,(\tau-\tau_0) + \|f\|_{\mathrm{BL}}\,(\tau-\tau_0)\,e^{L\,(\tau-\tau_0)}\right) \\ &\quad \cdot\Big[(\tau-\tau_0) + (\tau-\tau_0)^2\|f\|_\infty\,e^{\|f\|_\infty\,(\tau-\tau_0)}\Big], \end{aligned} \tag{4.5}$$

where $L = \min(|v^k_0|_{\mathrm{L}}\,, |\bar{v}|_{\mathrm{L}})$. We define $R := \|\mu_0\|_{\mathrm{TV}} \cdot \exp(\|f\|_\infty T)$; cf. Lemma 3.4. The analogon of (4.3) is

$$\begin{aligned}\|v^k_0 - \bar{v}\|_\infty &\leqslant M_R\left(\|\mu^k_{\tau_0} - \mu^m_{\tau_0}\|^*_{\mathrm{BL}} + \|\mu^m_{\tau_0} - \mu^m_{\bar{\tau}}\|^*_{\mathrm{BL}}\right) \\ &= M_R\,\|\mu^k_{\tau_0} - \mu^m_{\tau_0}\|^*_{\mathrm{BL}} + M_R\,\|Q^{\bar{v}}_{\tau_0-\bar{\tau}}\,\mu^m_{\bar{\tau}} - \mu^m_{\bar{\tau}}\|^*_{\mathrm{BL}} \\ &\leqslant M_R\,\|\mu^k_{\tau_0} - \mu^m_{\tau_0}\|^*_{\mathrm{BL}} + M_R\,R\,\big(\|f\|_\infty + K_R\big)\,e^{\|f\|_\infty\,T}\,(\tau_0 - \bar{\tau}) \end{aligned} \tag{4.6}$$

with $\bar{\tau} := t^m_{i_0-1}$. Together (4.5) and (4.6) yield

$$\begin{aligned}\|\mu^k_\tau - \mu^m_\tau\|^*_{\mathrm{BL}} \leqslant \big[\exp\big(A_1\,(\tau_1-\tau_0)\big) + A_2\,(\tau_1-\tau_0)\big]\,\|\mu^k_{\tau_0} - \mu^m_{\tau_0}\|^*_{\mathrm{BL}} \\ + A_3\,(\tau_1-\tau_0)(\tau_0-\bar{\tau}) \end{aligned} \tag{4.7}$$

for the same positive constants A_1, A_2, and A_3 as in (4.4). Here, we used part (iii) of Assumption 3.1 and the estimates $\tau - \tau_0 \leqslant \tau_1 - \tau_0$ and $\tau_1 - \tau_0 \leqslant T$. The upper bound (4.7) holds for all $\tau \in (\tau_0, \tau_1]$.

Case 3. $t^k_j < t^m_{i_0}$ *and* $i \geqslant 1$. In this case, $t^k_j < \tau_i < t^k_{j+1}$ and hence there is a $q \in \{1, \ldots, N_m - 1\}$ such that $\tau_i = t^m_q$. We have

$$\mu^k_\tau = Q^{v^k_0}_{\tau-\tau_i}\,\mu^k_{\tau_i} \qquad \text{and} \qquad \mu^m_\tau = Q^{v^m_i}_{\tau-\tau_i}\,\mu^m_{\tau_i}.$$

Estimate (4.2) also holds in this case. Because $t^m_{i_0} > t^k_j$ there is no $q \in \{0, \ldots, N_m - 1\}$ such that $\tau_0 = t^m_q$, and therefore $v[\,\cdot\,]$ is not to be evaluated at $\mu^m_{\tau_0}$. Consequently, we have instead of (4.3),

$$\begin{aligned}\|v^k_0 - v^m_i\|_\infty &\leqslant \|v^k_0 - \bar{v}\|_\infty + \|v^m_1 - \bar{v}\|_\infty + \sum_{\ell=2}^{i}\|v^m_\ell - v^m_{\ell-1}\|^*_{\mathrm{BL}} \\ &\leqslant \|v^k_0 - \bar{v}\|_\infty + M_R\,\|Q^{\bar{v}}_{\tau_1-\bar{\tau}}\,\mu^m_{\bar{\tau}} - \mu^m_{\bar{\tau}}\|^*_{\mathrm{BL}} \\ &\quad + M_R\sum_{\ell=2}^{i}\|Q^{v^m_{\ell-1}}_{\tau_\ell-\tau_{\ell-1}}\,\mu^m_{\tau_{\ell-1}} - \mu^m_{\tau_{\ell-1}}\|^*_{\mathrm{BL}} \end{aligned}$$

with $\bar{v} := v[\mu^m_{t^m_{i_0-1}}]$ and $\bar{\tau} := t^m_{i_0-1}$. Note that the sum on the right-hand side might be empty. Using the idea of (4.3) and the result of (4.6), we obtain

$$\begin{aligned}\|v^k_0 - v^m_i\|_\infty &\leqslant M_R\,\|\mu^k_{\tau_0} - \mu^m_{\tau_0}\|^*_{\mathrm{BL}} + M_R\,R\,\big(\|f\|_\infty + K_R\big)\,e^{\|f\|_\infty\,T}\,(\tau_0 - \bar{\tau}) \\ &\quad + M_R\,R\,\big(\|f\|_\infty + K_R\big)\,e^{\|f\|_\infty\,T}\,(\tau_i - \bar{\tau}) \\ &\leqslant M_R\,\|\mu^k_{\tau_0} - \mu^m_{\tau_0}\|^*_{\mathrm{BL}} + 2\,M_R\,R\,\big(\|f\|_\infty + K_R\big)\,e^{\|f\|_\infty\,T}\,(\tau_i - \bar{\tau}). \end{aligned} \tag{4.8}$$

Due to (4.2) and (4.8), we have

$$\begin{aligned}\|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \leqslant\ & \exp\big(A_1\,(\tau_{i+1}-\tau_i)\big)\,\|\mu_{\tau_i}^k - \mu_{\tau_i}^m\|_{\mathrm{BL}}^* \\ & +\ A_2\,(\tau_{i+1}-\tau_i)\,\|\mu_{\tau_0}^k - \mu_{\tau_0}^m\|_{\mathrm{BL}}^* \\ & +\ 2\,A_3\,(\tau_{i+1}-\tau_i)(\tau_i-\bar\tau)\end{aligned} \tag{4.9}$$

for all $\tau \in (\tau_i, \tau_{i+1}]$, where A_1, A_2, and A_3 are the same constants as in (4.4) and (4.7).

We now combine the estimates obtained in Cases 1, 2, and 3: it follows from (4.4), (4.7), and (4.9) that

$$\begin{aligned}\sup_{\tau\in(\tau_i,\tau_{i+1}]}\|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \leqslant\ & \exp\big(A_1\,(\tau_{i+1}-\tau_i)\big)\sup_{\tau\in(\tau_{i-1},\tau_i]}\|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \\ & +\ A_2\,(\tau_{i+1}-\tau_i)\,\|\mu_{\tau_0}^k - \mu_{\tau_0}^m\|_{\mathrm{BL}}^* \\ & +\ 4\,A_3\,M^{(k)}\,(\tau_{i+1}-\tau_i)\end{aligned}$$

for all $i \in \{1,\dots,N^{(j)}-1\}$, while for $i=0$

$$\begin{aligned}\sup_{\tau\in(\tau_0,\tau_1]}\|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \leqslant\ & \big[\exp\big(A_1\,(\tau_1-\tau_0)\big) + A_2\,(\tau_1-\tau_0)\big]\,\|\mu_{\tau_0}^k - \mu_{\tau_0}^m\|_{\mathrm{BL}}^* \\ & +\ A_3\,M^{(k)}\,(\tau_1-\tau_0).\end{aligned}$$

We have used that $\tau_i - \tau_0 \leqslant M^{(k)}$ in (4.4), $\tau_0 - \bar\tau \leqslant t_{i_0}^m - t_{i_0-1}^m \leqslant M^{(m)} \leqslant M^{(k)}$ in (4.7), and

$$\tau_i - \bar\tau \leqslant\ \tau_{N^{(j)}} - \tau_0 + \tau_0 - \bar\tau \leqslant\ M^{(k)} + t_{i_0}^m - t_{i_0-1}^m \leqslant\ M^{(k)} + M^{(m)} \leqslant\ 2\,M^{(k)}$$

in (4.9). This is the place where we use explicitly that partition α_m is "finer" (or "not coarser") than α_k in the sense that $M^{(m)} \leqslant M^{(k)}$; cf. Assumption 3.5. By an induction argument one can show that the upper bound

$$\begin{aligned}\sup_{\tau\in(\tau_i,\tau_{i+1}]}\|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \leqslant & \sum_{\ell=0}^{i}\left(\prod_{q=\ell+1}^{i}\exp\big(A_1\,(\tau_{q+1}-\tau_q)\big)\right) \\ & \cdot\Big[A_2\,(\tau_{\ell+1}-\tau_\ell)\,\|\mu_{\tau_0}^k - \mu_{\tau_0}^m\|_{\mathrm{BL}}^* + 4\,A_3\,M^{(k)}\,(\tau_{\ell+1}-\tau_\ell)\Big] \\ & +\ \left(\prod_{q=0}^{i}\exp\big(A_1\,(\tau_{q+1}-\tau_q)\big)\right)\,\|\mu_{\tau_0}^k - \mu_{\tau_0}^m\|_{\mathrm{BL}}^*\end{aligned} \tag{4.10}$$

holds for all $i \in \{0,\dots,N^{(j)}-1\}$. The products in brackets are equal to $\exp\big(A_1\,(\tau_{i+1}-\tau_{\ell+1})\big)$ and $\exp\big(A_1\,(\tau_{i+1}-\tau_0)\big)$, respectively. By using these explicit expressions and by taking the supremum over i on the left-hand and right-hand sides of (4.10), we obtain

$$\begin{aligned}\sup_{\tau\in(\tau_0,\tau_{N^{(j)}}]}\|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \leqslant & \sum_{\ell=0}^{N^{(j)}-1}\exp\big(A_1\,(\tau_{N^{(j)}}-\tau_{\ell+1})\big) \\ & \cdot\big[A_2\,(\tau_{\ell+1}-\tau_\ell)\,\|\mu_{\tau_0}^k - \mu_{\tau_0}^m\|_{\mathrm{BL}}^* + 4\,A_3\,M^{(k)}\,(\tau_{\ell+1}-\tau_\ell)\big] \\ & +\ \exp\big(A_1\,(\tau_{N^{(j)}}-\tau_0)\big)\,\|\mu_{\tau_0}^k - \mu_{\tau_0}^m\|_{\mathrm{BL}}^*.\end{aligned} \tag{4.11}$$

Since $\tau_{N^{(j)}} - \tau_{\ell+1} \leqslant \tau_{N^{(j)}} - \tau_0$ for all $\ell \in \{0, \dots, N^{(j)}-1\}$, and $\tau_0 = t_j^k$ and $\tau_{N^{(j)}} = t_{j+1}^k$, it follows from (4.11) that

$$\begin{aligned}
&\sup_{\tau\in(t_j^k,t_{j+1}^k]} \|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \\
&\quad\leqslant \left[1 + A_2 \sum_{\ell=0}^{N^{(j)}-1} (\tau_{\ell+1} - \tau_\ell)\right] \cdot \exp\left(A_1\,(t_{j+1}^k - t_j^k)\right) \cdot \|\mu_{t_j^k}^k - \mu_{t_j^k}^m\|_{\mathrm{BL}}^* \\
&\qquad + 4\,A_3\,M^{(k)}\,(t_{j+1}^k - t_j^k)\,\exp\left(A_1\,(t_{j+1}^k - t_j^k)\right) \sum_{\ell=0}^{N^{(j)}-1} (\tau_{\ell+1} - \tau_\ell) \\
&\quad= \left[1 + A_2\,(t_{j+1}^k - t_j^k)\right] \cdot \exp\left(A_1\,(t_{j+1}^k - t_j^k)\right) \cdot \|\mu_{t_j^k}^k - \mu_{t_j^k}^m\|_{\mathrm{BL}}^* \\
&\qquad + 4\,A_3\,M^{(k)}\,(t_{j+1}^k - t_j^k)\,\exp\left(A_1\,(t_{j+1}^k - t_j^k)\right). \qquad (4.12)
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
&\sup_{\tau\in(t_j^k,t_{j+1}^k]} \|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \\
&\quad\leqslant \left[1 + A_2\,(t_{j+1}^k - t_j^k)\right] \cdot \exp\left(A_1\,(t_{j+1}^k - t_j^k)\right) \cdot \sup_{\tau\in(t_{j-1}^k,t_j^k]} \|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \\
&\qquad + 4\,A_3\,M^{(k)}\,(t_{j+1}^k - t_j^k)\,\exp\left(A_1\,(t_{j+1}^k - t_j^k)\right) \qquad (4.13)
\end{aligned}$$

for all $j \in \{1, \dots, N_k - 1\}$, and for $j = 0$ we have

$$\sup_{\tau\in(t_0^k,t_1^k]} \|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \leqslant 4\,A_3\,M^{(k)}\,(t_1^k - t_0^k)\,\exp\left(A_1\,(t_1^k - t_0^k)\right), \qquad (4.14)$$

because $\mu_{t_0^k}^k = \mu_0 = \mu_{t_0^k}^m$. By an induction argument similar to the one leading to (4.10), we obtain that

$$\begin{aligned}
\sup_{\tau\in(t_j^k,t_{j+1}^k]} \|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \leqslant \sum_{\ell=0}^{j} &\left(\prod_{q=\ell+1}^{j} \left[1 + A_2\,(t_{q+1}^k - t_q^k)\right] \cdot \exp\left(A_1\,(t_{q+1}^k - t_q^k)\right)\right) \\
&\cdot 4\,A_3\,M^{(k)}\,(t_{\ell+1}^k - t_\ell^k)\,\exp\left(A_1\,(t_{\ell+1}^k - t_\ell^k)\right) \qquad (4.15)
\end{aligned}$$

for all $j \in \{0, \dots, N_k - 1\}$. Note that $\left[1 + A_2\,(t_{q+1}^k - t_q^k)\right] \leqslant \exp\left(A_2\,(t_{q+1}^k - t_q^k)\right)$ for all $q \in \{0, \dots, N_k - 1\}$. Define $A_4 := A_1 + A_2$. It follows from (4.15) that

$$\begin{aligned}
&\sup_{\tau\in(t_j^k,t_{j+1}^k]} \|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \\
&\quad\leqslant 4\,A_3\,M^{(k)}\,\exp(A_1\,T) \sum_{\ell=0}^{j} \left(\prod_{q=\ell+1}^{j} \exp\left(A_4\,(t_{q+1}^k - t_q^k)\right)\right) \cdot (t_{\ell+1}^k - t_\ell^k) \\
&\quad\leqslant 4\,A_3\,M^{(k)}\,\exp(A_1\,T) \sum_{\ell=0}^{j} \exp\left(A_4\,(t_{j+1}^k - t_{\ell+1}^k)\right) \cdot (t_{\ell+1}^k - t_\ell^k) \\
&\quad\leqslant 4\,A_3\,M^{(k)}\,\exp((A_1 + A_4)\,T)\,(t_{j+1}^k - t_0^k). \qquad (4.16)
\end{aligned}$$

We take the supremum over j on both sides of the inequality (4.16) and get

$$\sup_{\tau\in[0,T]} \|\mu_\tau^k - \mu_\tau^m\|_{\mathrm{BL}}^* \leqslant 4\,A_3\,M^{(k)}\,T\,\exp((A_1 + A_4)\,T). \qquad (4.17)$$

Note that we extended the supremum from $\tau \in (0,T]$ to $\tau \in [0,T]$, but this does not change the upper bound. Define $C := 4\,A_3\,T\,\exp((A_1+A_4)\,T)$ to get the result of the lemma. Because $M^{(k)} \to 0$ as $k \to \infty$, the estimate (4.17) implies that $(\mu^k)_{k\in\mathbb{N}}$ is a Cauchy sequence. ∎

Remark 4.2. It is crucial that we use $\|\cdot\|^*_{\mathrm{BL}}$ and not $\|\cdot\|_{\mathrm{TV}}$ in Lemma 4.1. The factor $\|v_0^k - v_i^m\|_\infty$ appears in (4.2) due to Lemma 2.10. Due to Assumption 3.1(iv) we subsequently obtain an estimate in which $\|\mu^k_{\tau_0} - \mu^m_{\tau_0}\|^*_{\mathrm{BL}}$ appears. Analogous estimates apply to $\|v_0^k - \bar{v}\|_\infty$ in (4.5). Note that Lemma 2.10 builds on Lemma 2.2. In (2.14) the Lipschitz property of the test functions is explicitly used and hence there is no direct way to formulate the result of Lemma 2.2 in terms of $\|\cdot\|_{\mathrm{TV}}$. Consequently, we do not have an estimate of $\|\mu^k_\tau - \mu^{k+1}_\tau\|_{\mathrm{TV}}$ against $\|v - v'\|_\infty$ comparable to (4.2).

We are now ready to prove Theorem 3.10.

Proof. By definition, $\overline{\mathcal{M}}([0,1])$ is complete in the metric induced by the norm $\|\cdot\|^*_{\mathrm{BL}}$. The space $\mathcal{M}^+([0,1])$ is a closed subspace of $\overline{\mathcal{M}}([0,1])$, so $\mathcal{M}^+([0,1])$ is complete. Hence, the space

$$\{\nu \in C([0,T];\mathcal{M}^+([0,1])) \,:\, \nu(0) = \mu_0\}$$

is complete for the metric defined for all $\mu, \nu \in C([0,T];\mathcal{M}^+([0,1]))$ by (3.6).

For each initial measure $\mu_0 \in \mathcal{M}^+([0,1])$ and for each $k \in \mathbb{N}$, consider the Euler approximation μ^k defined by (3.3) corresponding to partition α_k. This approximation μ^k is an element of $C([0,T];\mathcal{M}^+([0,1]))$, because the semigroup $(Q^v_t)_{t\geqslant 0}$ preserves positivity for all $v \in \mathrm{BL}([0,1])$; see [EHM15a, Corollary 3.4]. In Lemma 4.1, we showed that for given $(\alpha_k)_{k\in\mathbb{N}}$ the sequence $(\mu^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\{\nu \in C([0,T];\mathcal{M}^+([0,1])) \,:\, \nu(0) = \mu_0\}$, which is a complete space, as was argued above. Hence, the sequence $(\mu^k)_{k\in\mathbb{N}}$ converges in $\{\nu \in C([0,T];\mathcal{M}^+([0,1])) \,:\, \nu(0) = \mu_0\}$.

The limit is independent of the sequence of partitions chosen from the class characterized by Assumption 3.5. If $(\alpha_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ are two such sequences, then it is possible to construct a sequence $(\gamma_k)_{k\in\mathbb{N}}$ that has a subsequence that is also a subsequence of $(\alpha_k)_{k\in\mathbb{N}}$ and that has (another) subsequence that is a subsequence of $(\beta_k)_{k\in\mathbb{N}}$. Moreover, $(\gamma_k)_{k\in\mathbb{N}}$ can be constructed such that the corresponding sequence of maximal interval lengths is nondecreasing.

Let $(\mu^{\alpha_k})_{k\in\mathbb{N}}$, $(\mu^{\beta_k})_{k\in\mathbb{N}}$ and $(\mu^{\gamma_k})_{k\in\mathbb{N}}$ be the corresponding sequences of Euler approximations. The sequence $(\mu^{\gamma_k})_{k\in\mathbb{N}}$ can be shown to converge to the same limit as $(\mu^{\alpha_k})_{k\in\mathbb{N}}$ and to the same limit as $(\mu^{\beta_k})_{k\in\mathbb{N}}$. Hence, $(\mu^{\alpha_k})_{k\in\mathbb{N}}$ and $(\mu^{\beta_k})_{k\in\mathbb{N}}$ converge to the same limit. This finishes the proof. ∎

The proof of Corollary 3.11 builds on the result of Theorem 3.10.

Proof. Fix $t \geqslant 0$ and let $T > 0$ be such that $t \in [0,T]$. For given $\mu_0 \in \mathcal{M}^+([0,1])$ and $v : \mathcal{M}([0,1]) \times [0,1] \to \mathbb{R}$ satisfying Assumption 3.1, a unique mild solution $\mu \in C([0,T];\mathcal{M}^+([0,1]))$ exists, hence μ_t, the solution at time t, exists. We now show that this μ_t is independent of the choice of T.

Let $T_1, T_2 > 0$ and assume without loss of generality that $T_1 < T_2$. For the given $\mu_0 \in \mathcal{M}^+([0,1])$ and $v : \mathcal{M}([0,1]) \times [0,1] \to \mathbb{R}$, let μ denote the mild solution in $C([0,T_1];\mathcal{M}^+([0,1]))$ obtained by partitioning $[0,T_1]$. Take a sequence of partitions $(\alpha_k)_{k\in\mathbb{N}} \subset [0,T_1]$ satisfying Assumption 3.5 with corresponding Euler approximations $(\mu^k)_{k\in\mathbb{N}}$. Next, construct a sequence of partitions $(\beta_k)_{k\in\mathbb{N}} \subset [0,T_2]$ satisfying Assumption 3.5 such that $\alpha_k \subset \beta_k$ for each $k \in \mathbb{N}$. More specifically, restricted to $[0,T_1]$ each partition β_k coincides with α_k. Note that such $(\beta_k)_{k\in\mathbb{N}}$ exists. Let $(\nu^k)_{k\in\mathbb{N}}$ be the sequence of Euler approximations corresponding to $(\beta_k)_{k\in\mathbb{N}}$.

For each $k \in \mathbb{N}$, the restriction $\nu^k\big|_{[0,T_1]}$ is defined by (3.3) with respect to the partition $\beta_k \cap [0,T_1] = \alpha_k$. Hence $\nu^k\big|_{[0,T_1]}$ is defined in the same way as μ^k, and thus $\sup_{\tau\in[0,T_1]} \|\mu_\tau^k - \nu_\tau^k\|_{\mathrm{BL}}^* = 0$ or simply $\nu^k\big|_{[0,T_1]} = \mu^k$. Consequently, the same must hold in the limit as $k \to \infty$, because of the triangle inequality:

$$\begin{aligned}\sup_{\tau\in[0,T_1]} \|\mu_\tau - \nu_\tau\|_{\mathrm{BL}}^* &\leqslant \sup_{\tau\in[0,T_1]} \|\mu_\tau^k - \nu_\tau^k\|_{\mathrm{BL}}^* + \sup_{\tau\in[0,T_1]} \|\mu_\tau^k - \mu_\tau\|_{\mathrm{BL}}^* + \sup_{\tau\in[0,T_1]} \|\nu_\tau^k - \nu_\tau\|_{\mathrm{BL}}^* \\ &\leqslant \underbrace{\sup_{\tau\in[0,T_1]} \|\mu_\tau^k - \nu_\tau^k\|_{\mathrm{BL}}^*}_{=0} + \underbrace{\sup_{\tau\in[0,T_1]} \|\mu_\tau^k - \mu_\tau\|_{\mathrm{BL}}^*}_{\to 0} + \underbrace{\sup_{\tau\in[0,T_2]} \|\nu_\tau^k - \nu_\tau\|_{\mathrm{BL}}^*}_{\to 0}.\end{aligned}$$

So, $\sup_{\tau\in[0,T_1]} \|\mu_\tau - \nu_\tau\|_{\mathrm{BL}}^* = 0$. Hence, $\mu_\tau = \nu_\tau$ for all $\tau \in [0,T_1]$ and thus the solution at time τ is independent of the final time chosen. ∎

Finally, we prove Theorem 3.12.

Proof. Let the mild solutions μ and ν be given and let $(\alpha_k)_{k\in\mathbb{N}}$ be an arbitrary sequence of partitions of $[0,T]$ satisfying Assumption 3.5. Let $(\mu^k)_{k\in\mathbb{N}}$ and $(\nu^k)_{k\in\mathbb{N}}$ denote the sequences of Euler approximations defined by (3.3), both for the sequence of partitions $(\alpha_k)_{k\in\mathbb{N}}$ and with initial conditions μ_0 and ν_0, respectively.

Since μ and ν are mild solutions

$$\begin{aligned}\mu &= \lim_{k\to\infty} \mu^k \qquad \text{and} \\ \nu &= \lim_{k\to\infty} \nu^k\end{aligned}$$

hold, with convergence in the metric (3.6). It follows from Lemma 3.4 that all elements of

$$\{\mu_t^k : k \in \mathbb{N}, t \in [0,T]\} \cup \{\nu_t^k : k \in \mathbb{N}, t \in [0,T]\}$$

are bounded by $R := \tilde{R}\exp(\|f\|_\infty T)$ in both $\|\cdot\|_{\mathrm{TV}}$ and $\|\cdot\|_{\mathrm{BL}}^*$. Fix $k \in \mathbb{N}$, let $\alpha_k := \{t_0^k, \ldots, t_{N_k}^k\}$, and take j such that $\tau \in (t_j^k, t_{j+1}^k]$. Consider the difference $\|\mu_\tau^k - \nu_\tau^k\|_{\mathrm{BL}}^*$.

We use an estimate in the spirit of (4.2)–(4.3)–(4.4). Note that the proof of Lemma 4.1 also holds if $k = m$, which implies $N^{(j)} = 1$ and hence $i = 0$. It follows from (4.2)–(4.3), with $i = 0$ and with ν^k instead of μ^m, that

$$\|\mu_\tau^k - \nu_\tau^k\|_{\mathrm{BL}}^* \leqslant \left[1 + B_2\left(t_{j+1}^k - t_j^k\right)\right] \exp\left(B_1\left(t_{j+1}^k - t_j^k\right)\right) \|\mu_{t_j^k}^k - \nu_{t_j^k}^k\|_{\mathrm{BL}}^* \tag{4.18}$$

for some positive constants B_1 and B_2 that depend on f, T, and $\tilde{R}$ but not on j or k. This estimate holds for all $\tau \in (t_j^k, t_{j+1}^k]$ and resembles (4.4). We take the supremum over $\tau \in (t_j^k, t_{j+1}^k]$ on the left-hand side of (4.18), apply this relation recursively, and take the supremum over j to obtain that

$$\begin{aligned}\sup_{\tau\in[0,T]} \|\mu_\tau^k - \nu_\tau^k\|_{\mathrm{BL}}^* &\leqslant \left(\prod_{\ell=0}^{N_k-1} \left[1 + B_2\left(t_{\ell+1}^k - t_\ell^k\right)\right] \exp\left(B_1\left(t_{\ell+1}^k - t_\ell^k\right)\right)\right) \|\mu_0 - \nu_0\|_{\mathrm{BL}}^* \\ &\leqslant \left(\prod_{\ell=0}^{N_k-1} \exp\left(B_2\left(t_{\ell+1}^k - t_\ell^k\right)\right) \exp\left(B_1\left(t_{\ell+1}^k - t_\ell^k\right)\right)\right) \|\mu_0 - \nu_0\|_{\mathrm{BL}}^* \\ &\leqslant \exp\left(\left(B_1 + B_2\right)\left(t_{N_k}^k - t_0^k\right)\right) \|\mu_0 - \nu_0\|_{\mathrm{BL}}^* \\ &= \exp\left(\left(B_1 + B_2\right) T\right) \|\mu_0 - \nu_0\|_{\mathrm{BL}}^* \qquad (4.19)\end{aligned}$$

for all $k \in \mathbb{N}$. The triangle inequality yields

$$\sup_{\tau\in[0,T]} \|\mu_\tau - \nu_\tau\|^*_{\mathrm{BL}} \leqslant \sup_{\tau\in[0,T]} \|\mu^k_\tau - \nu^k_\tau\|^*_{\mathrm{BL}} + \underbrace{\sup_{\tau\in[0,T]} \|\mu^k_\tau - \mu_\tau\|^*_{\mathrm{BL}}}_{\to 0} + \underbrace{\sup_{\tau\in[0,T]} \|\nu^k_\tau - \nu_\tau\|^*_{\mathrm{BL}}}_{\to 0},$$

whence the same estimate as in (4.19) holds for $\sup_{\tau\in[0,T]} \|\mu_\tau - \nu_\tau\|^*_{\mathrm{BL}}$. ∎

Remark 4.3. We would have been inclined to use directly (2.19) on the interval $(t^k_j, t^k_{j+1}]$, instead of deriving (4.18). We need, however, the exact dependence on $(t^k_{j+1} - t^k_j)$ of the prefactor, to make sure that—after iteration over j—the prefactor in (4.19) is bounded. This dependence is not (directly) provided by (2.19), nor by the proof of [EHM15a, Proposition 3.5].

Remark 4.4. The result of Theorem 3.12 relies—via Corollary 2.9 and Lemma 2.10—on Gronwall's inequality. This is possible here because we restrict ourselves to Lipschitz perturbations. In our previous work [EHM15a] we considered the more general class of *piecewise* bounded Lipschitz perturbations. Hence, there we stated explicitly (see the paragraph before [EHM15a, Proposition 3.5]) that the standard approach did not work.

5. Discussion. In this paper we have generalized the results of [EHM15a] to measure-dependent velocity fields via a forward-Euler-like approach. Our motivation was to derive flux boundary conditions for situations in which the dynamics are driven by interactions. Such dynamics are in general more interesting than the dynamics that follow from prescribed velocity fields as in [EHM15a]. We managed to obtain a converging procedure, but only for bounded Lipschitz continuous right-hand sides. Hence, compared to [EHM15a], our results hold, e.g., for boundary layers in which mass decays, but not for the limit case of vanishing boundary layer. We start off this discussion section (see section 5.1) by commenting on the possibility to extend to piecewise bounded Lipschitz right-hand sides and to obtain the limit of vanishing boundary layer. Second, we point out (in section 5.2) how this paper generalizes the results of [Eve15, Chapter 5] and how a number of open problems mentioned in [Eve15, section 5.5] are now resolved. Ultimately, we suggest possible future research (section 5.3).

5.1. Piecewise bounded Lipschitz perturbations. To obtain the technical results in section 2.3, we explicitly used the assumption that the perturbation f is bounded Lipschitz on $[0, 1]$. Theorems 3.10 and 3.12 rely on the results in section 2.3. We would have liked to obtain these results for piecewise bounded Lipschitz f, in particular to model decay of mass at one of the boundaries only (cf. [EHM15a]). In [EHM15a] we circumvent the arising problems by providing the solution explicitly in [EHM15a, Proposition 3.3]. In the setting of the present paper, this explicit form would be given for each interval $(t^k_j, t^k_{j+1}]$, $k \in \mathbb{N}$, in (3.3) by

$$\mu^k_t := \int_{[0,1]} \exp\left(\int_0^{t-t^k_j} f(\Phi^{v^k_j}_s(x))\,ds\right) \cdot \delta_{\Phi^{v^k_j}_{t-t^k_j}(x)}\, d\mu^k_{t^k_j}(x), \tag{5.1}$$

where $v^k_j := v[\mu^k_{t^k_j}]$. In [EHM15a] we showed that it is possible to obtain the estimates needed to establish continuous dependence on initial data, because this explicit form has a regularizing effect on f and its discontinuities due to the integration in time. The key ingredient there, which is absent in the approach of the present work, is the

fact that the velocity field is the same for all time. If one wants to prove Theorem 3.12 using (5.1) instead of the properties of the semigroup Q, one encounters that at some point for any $\Delta t > 0$ fixed a Lipschitz estimate of the form

$$\left\| \int_0^{\Delta t} f(\Phi_s^v(\cdot))\, ds - \int_0^{\Delta t} f(\Phi_s^u(\cdot))\, ds \right\|_\infty \leqslant C \, \|u - v\|_\infty \tag{5.2}$$

is required for all u and v taken from a class of admissible velocity fields. One would then proceed to estimate $\|u - v\|_\infty$ against the bounded Lipschitz distance of the corresponding measures, using part (iv) of Assumption 3.1.

In view of [EHM15a], the restriction that the velocity should not be zero at discontinuities of f is reasonable, but even if we are willing to obey that condition, an estimate like (5.2) cannot be expected to hold. Let $f(x) = 0$ if $x \in [0, 1)$ and $f(1) = -1$. Take $\varepsilon > 0$ and take $v \equiv \varepsilon$, $u \equiv -\varepsilon$. Then (for $\varepsilon < 1/\Delta t$)

$$\begin{aligned} \left\| \int_0^{\Delta t} f(\Phi_s^v(\cdot))\, ds - \int_0^{\Delta t} f(\Phi_s^u(\cdot))\, ds \right\|_\infty &\geqslant \left| \int_0^{\Delta t} f(\Phi_s^v(1))\, ds - \int_0^{\Delta t} f(\Phi_s^u(1))\, ds \right| \\ &= \left| \int_0^{\Delta t} f(1)\, ds - \int_0^{\Delta t} f(1 - \varepsilon\, s)\, ds \right| = \Delta t. \end{aligned}$$

Since $\Delta t > 0$ is fixed and $\|u - v\|_\infty = 2\varepsilon$ can be made arbitrarily small, (5.2) cannot be satisfied.

An additional difficulty is that it remains to be seen how we can ensure that a condition like $v(1) \neq 0$ is satisfied by a velocity field that depends on the solution itself.

5.2. Uniqueness of mild solutions and generality of partitions. In [Eve15, section 5.5] we point out that there are two reasons why we obtained uniqueness of mild solutions there. On the one hand, this is because the constructed approximating sequence converges, thus inevitably each subsequence (cf. Definition 3.8) converges to the same limit. This statement still holds true for the present work. On the other hand, uniqueness holds in [Eve15, Chapter 5] because there we only constructed one approximating sequence, namely, by partitioning the interval $[0, T]$ into 2^k subintervals. In this respect, the present paper is a considerable improvement. The class of admissible partitions (see Assumption 3.5) includes partitions into q^k equal subintervals for arbitrary $q \in \mathbb{N}^+$; see Example 3.6. We conjectured in [Eve15, section 5.5] that the sequence of corresponding Euler approximations converges, and the results of this paper confirm that conjecture. The fact that, in this case, each interval $(t_j^k, t_{j+1}^k]$ is split into q^{m-k} subintervals $(t_\ell^m, t_{\ell+1}^m]$ is generically treated by introducing the number $N^{(j)}$ and using a recursion over index $i \in \{0, \dots, N^{(j)} - 1\}$ to obtain (4.11). In [Eve15, Chapter 5], however, we performed explicit calculations, using that each $(t_j^k, t_{j+1}^k]$ is split into two subintervals.

In [Eve15, section 5.5] anticipated that using a sequence of nonuniform partitions of $[0, T]$ would imply the need for a condition regularizing the variation in subinterval lengths to make sure that *all* subintervals become small sufficiently fast as $k \to \infty$. In the present work we show that it suffices to have for the maximum subinterval length $M^{(k)} \to 0$ as $k \to \infty$.

The iterative argument in [Eve15, Chapter 5] requires that the partition for index $k + 1$ is a refinement of the partition for index k (more particularly, a division of each subinterval into two). The complications expected to occur if subsequent partitions

are not refinements are resolved in the current work by introduction of the index i_0 in the proof of Lemma 4.1 and allowing for the case $t_j^k \neq t_{i_0}^m$.

The final contribution of the present work to be mentioned here is that in Theorem 3.10 we have positively answered the question posed in [Eve15, section 5.5] of whether the mild solutions obtained as limits of distinct sequences of partitions are actually identical.

5.3. Future directions. The extension of the results stated in section 3 to functions f with discontinuities would clear the way for an approximation procedure like the one treated in [EHM15a], that is, to have f nonzero *only* on the boundary of the domain and to approximate it with a sequence of bounded Lipschitz functions $(f_n)_{n\in\mathbb{N}} \subset \mathrm{BL}([0,1])$. In [EHM15a] we showed convergence of the corresponding solutions as $n \to \infty$ (for $v \in \mathrm{BL}([0,1])$ fixed). The challenge would be (i) to establish the well-posedness of the problem for discontinuous f and (ii) to show that the Euler approximation limit and the boundary layer limit commute.

Let us focus on the vanishing boundary layer like in [EHM15a]. Assume there are regions around 0 and 1 in which mass decays and that these regions shrink to zero width. That is, there is a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{BL}([0,1])$ and there is an f satisfying $f(x) = 0$ if $x \in (0,1)$ and, e.g., $f(0) = f(1) = -1$, such that $f_n \to f$ pointwise, and the Lebesgue measure of the set $\{x \in [0,1] : f_n(x) \neq f(x)\}$ tends to zero as $n \to \infty$. If we assume that we can extend the results of this paper to piecewise bounded Lipschitz f, then well-posedness for the limit case is guaranteed. It remains to be proven, however, that the solution for finite boundary layer actually converges to the solution of the limit problem.

This is the same question as asking whether the two limits that we take actually commute. The first limit is in the forward-Euler-like approach to obtain a mild solution. We assigned an index k to the elements in the approximating sequence and proved in Theorem 3.10 that the limit "$\lim_{k\to\infty}$" exists (for $f \in \mathrm{BL}([0,1])$). The second limit "$\lim_{n\to\infty}$" is the one involving the sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{BL}([0,1])$. Proving the well-posedness for f piecewise bounded Lipschitz is the same as proving that the limit "$\lim_{k\to\infty} \lim_{n\to\infty}$" exists. Proving that the sequence of solutions corresponding to each f_n actually converges to some limit in $C([0,T];\mathcal{M}^+([0,1]))$ is equivalent to proving that "$\lim_{n\to\infty} \lim_{k\to\infty}$" exists. To conclude that the two limits commute, an additional argument is needed. It requires a characterization of "$\lim_{n\to\infty} \lim_{k\to\infty}$" that can be compared to "$\lim_{k\to\infty} \lim_{n\to\infty}$." Both proving that "$\lim_{n\to\infty} \lim_{k\to\infty}$" exists and characterizing the limit can be a difficult task, however, since our current results do not provide an explicit expression for "$\lim_{k\to\infty}$." A possible way to characterize the limit "$\lim_{k\to\infty}$" could be to show that the mild solution obtained in this paper is actually a weak solution and to use the weak formulation of (1.2) as a characterization. If the solutions obtained in this paper are weak solutions, this is also a further justification of the terminology "mild solutions."

An additional result to be derived concerns the stability with respect to parameters, in particular with respect to f and the specific form of v. Stability statements are essential in view of parameter identification. It is important to know how measurement errors in the parameters affect the solution of our model. In fact, Lemma 2.10 already provides stability in v for the solution of [EHM15a], provided that $f \in \mathrm{BL}([0,1])$.

Moreover, we would like to study the long-term dynamics of the solutions $t \mapsto \mu_t$ for various initial conditions.

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