# Maximum entropy models for financial systems

Almog, A.

**Author**: Almog, A.
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# Chapter 2

# Economic Networks

The International Trade Network (ITN) is a complex network formed by the bilateral trade relations between world countries. The network complex structure reflects important economic processes such as globalization, and the propagation of shocks and instabilities. Both from the perspective of network theory and macroeconomics, understanding the structure of the ITN is of paramount importance: in particular, understanding which economic quantities play a role in shaping the ITN structure is crucial. Traditional macroeconomics has mainly used the Gravity Model to characterize the magnitude of trade volumes, using macroeconomic properties such as GDP and geographic distance. On the other hand, recent maximum-entropy network models successfully reproduce the complex topology of the ITN, but provide no information about trade volumes. In this chapter, we first discuss the role played by the countries GDP in determining both the presence and the amount of trade exchanges between world countries. Next, we make an effort to integrate these two currently incompatible approaches via the introduction of two GDP driven models. We introduce a novel ingredient that we denote as 'topological invariance', i.e. the invariance of the expected topology under an arbitrary change of units of trade volumes. Via this unified and principled mechanism, which is transparent enough to be generalized to any economic network, the models provide a new econometric framework wherein trade probabilities and trade volumes can be separately controlled by macroeconomic variables. The models successfully reproduce both the topology and the weights of the ITN, finally reconciling the conflicting approaches.

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A. Almog, T. Squartini, D. Garlaschelli *New Journal of Physics*, **17**: 013009 (2015).
A. Almog, T. Squartini, D. Garlaschelli *Int. J. Computational Economics and Econometrics*, in press, arXiv:1512.02454 (2016).
A. Almog, R. Bird, D. Garlaschelli *arXiv:1506.00348* (2016).

## 2.1 Introduction

The bilateral trade relationships existing between world countries form a complex network known as the International Trade Network (ITN). The observed complex structure of the system is at the same time the outcome and the determinant of a variety of underlying economic processes, including economic growth, integration, and globalization. Moreover, recent events such as the financial crisis clearly pointed out that the interdependencies between financial markets can lead to cascading effects which, in turn, can severely affect the real economy. International trade plays a significant role among the possible channels of interaction among countries [1, 2, 3, 4], thereby possibly further propagating these cascading effects worldwide and adding one more layer of contagion. Characterizing the global networked economy is, therefore, an important open problem and modelling the ITN represents a crucial step of this challenge [5, 11, 13, 16, 17, 24, 25].

Historically, macroeconomic models have mainly focused on modelling the trade volumes between countries. The Gravity Model, which was introduced in the early 60's by Jan Tinbergen [30], serves as a robust empirical model that aims at predicting the bilateral trade flow between any two (trading) countries based on the knowledge of their Gross Domestic Product (GDP) and mutual geographic distance. Although the model has been upgraded, over the years, to include other possible factors of macroeconomic relevance, like common language and trade agreements, GDP and distance remain the two factors with the largest explanatory power. The gravity model can reproduce the observed trade volumes between countries satisfactorily. However, at least in its simplest and most widespread implementation, the model cannot account for zero volumes, therefore predicting a fully-connected trade network. This outcome is entirely inconsistent with the observed, heterogeneous, topology of the ITN, which represents the backbone on which trade is made. Subsequent refinements of the gravity model allowing for zero trade flows succeeded only in reproducing the total number of missing links, not their position in the trade network, thereby producing sparser but still non-realistic topologies [14, 15].

The sharp contrast between the observed topological complexity of the ITN and the homogeneity of the network structure generated by the GM (including its recent extensions) calls for significant improvements in the modelling approach. Important steps have been made using network models [7, 38, 39, 23, 26], among which maximum-entropy techniques [18, 19, 20] have been proven to be particularly advantageous. Maximum-entropy models aim at reproducing higher-order structural properties of real-world networks using lower-order information (more precisely, node-specific), which is constrained to be reproduced [27, 28, 29]. Important examples of local properties that can be chosen as constraints are the *degree*, i.e. the number of links of a node (in the ITN case, this is the number of trade partners of a country) and the *strength*, i.e. the total weight of the links of a node (in the ITN case, this is the total trade volume of a country). Examples of higher-order properties that the method aims at reproducing are the *clustering coefficient*, which refers to the fraction of realised triangles around nodes, and the *degree correlations*. The studies have focused on both binary and weighted representations of the ITN, i.e. the two representations defined by the *existence* and by the *magnitude* of trade exchanges among countries, respectively. In principle, depending on which local properties are chosen as constraints, maximum-entropy models can either fail or succeed in replicating the higher-order properties of the ITN.

Importantly, the use of maximum-entropy models has lead to a counter-intuitive result about the structure of the trade network: contrary to what naive economic reasoning would predict, controlling for *purely binary* local properties (node degrees) is more informative than controlling for the corresponding weighted properties (node strengths). In other words, while the binary network reconstructed only from the knowledge of the degrees of all countries is topologically very similar to the real ITN, the weighted network reconstructed only from the strengths of all countries is very different (typically much denser) from the real network [21, 22, 19]. This is somewhat surprising, given that economic theory assumes that weighted properties (the total trade volume of a country) are *per se* more informative than the corresponding binary ones (the number of trade partners of a country). This empirical puzzle calls for a theoretical explanation and has generated further interest in the challenge of finding a unique mechanism predicting link probabilities and link weights simultaneously. In this chapter, we will propose different models that successfully implement such mechanism. The models can reproduce the observed properties of the ITN and finally highlight a clear mathematical explanation for the observed binary/weighted asymmetry.

Our approach builds on previous theoretical results. Recently, an improved reconstruction approach [32], based on an analytical maximum-likelihood estimation method [18], has been proposed in order to define more sophisticated maximum-entropy ensembles of weighted networks. This approach exploits previous mathematical results [34] characterizing a network ensemble where both the degree and the strength sequences are constrained. The graph probability is the so-called generalized Bose-Fermi distribution [34], and the resulting network model goes under the name of Enhanced Configuration Model (ECM) [32]. When used to reconstruct the properties of several empirical networks, the ECM shows a significant improvement with respect to the case where either only the degree sequence (Binary Configuration Model, BCM for short) or only the strength sequence (Weighted Configuration Model, WCM for short) is constrained. One, therefore, expects that combining the knowledge of strengths and degrees is precisely the ingredient required to reproduce the ITN from purely local information successfully. Indeed, a more recent study has shown that, when applied to international trade data (both aggregated and commodity-specific), the method successfully reproduces the key properties of the ITN, across different years and for different levels of aggregation (i.e. for various commodity-specific layers) [33].

However, in itself, the ECM is a general network reconstruction method, rather than a true structural model. To transform it into a proper network model for the ITN structure, it would be necessary to find a macroeconomic interpretation for the underlying variables involved in the method. This operation would correspond to what has already been separately performed in different studies. For the binary level, a strong relationship between the GDP and the variable controlling the degree of a country in the BCM [7, 36] was identified. At the purely weighted level, a similar relationship was found between the GDP and the variable controlling the strength of a country in the WCM [23], in the same spirit of the gravity model. Here, we aim at generalizing the results, to one model that is able to generate both strengths and degrees.

We start with a summarized review of previous maximum-entropy approaches to the characterization of the ITN, thus explaining the mathematical building blocks on which we build our unifying approach. The rest of the chapter is organized as follows. In section 2.4 we discuss the macroeconomic approach to model the ITN, in particular, the Gravity Model of Trade. We also present various empirical relations existing between the GDP and a range of country-specific properties. These results suggest a justification for the use of GDP as an empirical fitness to be used in maximum-entropy models. In section 2.5 we introduce a novel, GDP-driven, two-step model that successfully reproduces the binary *and* the weighted higher-order properties of the ITN simultaneously. In section 2.6 we introduce what we call the Enhanced Gravity Model (EGM) of trade, which represents a new, generalized model combining maximum-entropy network models with economically established gravity-like models. The model overcomes the limitations of the existing approaches and retains the power of the popular GM in reproducing trade volumes via geographic distances and GDPs. While at the same time dramatically improving its network properties by reproducing both first-order properties, such as node degree and node strength, and higher-order properties, such as assortativity and clustering (in both binary and weighted representations of the network). Finally, in section 2.7 we summarize our results and provide some conclusions.

## 2.2 Data

We have used data from the Gleditsch database which spans over the years 1950-2000 [9], and from the United Nations Commodity Trade Database (UN COMTRADE) [10] from the year 1992 to 2004. We use yearly bilateral data on exports and imports Here we analyze the aggregated level, which results in yearly temporal snapshots of undirected total trade flows. The data sets are available in the form of weighted matrices of bilateral trade flows $w_{ij}$, the associated adjacency matrices $a_{ij}$ and vectors of GDPs. There are approximately 200 countries in the data set covering the considered 51 years; the GDP is measured in U.S. dollars.

This data set was the subject of many studies exploring both purely the binary representation, and its full weighted representation [18, 32, 33]. Another data set which is widely used to represent the ITN network is the trade data collected by Gleditsch [9]. The data contain the detailed list of bilateral import and export volumes, for each country in the period 1950-2000.

## 2.3 Maximum-entropy approaches to the international trade network

Since our results are a generalization of previous maximum-entropy approaches to the characterization of the ITN, in this section we first briefly review the main results of those approaches, while our new findings are presented in the next section. In doing so, we gradually introduce the mathematical building blocks of our analysis and illustrate our main motivations. Moreover, since previous studies have used different data sets, we also recalculate the quantities of interest on the same data set that we will use later for our own investigation. This allows us to align the results of previous approaches and properly compare them with our new findings.

### 2.3.1 Binary structure

If one focuses solely on the binary undirected projection of the ITN, then the Binary Configuration Model (BCM) represents a very successful maximum-entropy model. In the BCM, the local knowledge of the number of trade partners of each country, i.e. the degree sequence, is specified. It has been shown that higher-order properties of the ITN can be simply traced back to the knowledge of the degree sequence [21]. This result adds considerable information to the standard results of traditional macroeconomic analyses of international trade. In particular, it suggests that the degree sequence, which is a purely topological property, needs to be considered as an important target quantity that international trade models, in contrast with the mainstream approaches in economics, should aim at reproducing [19].

Let us first represent the observed structure of the ITN as a weighted undirected network specified by the square matrix $\mathbf{W}^*$, where the specific entry $w^*_{ij}$ represents the weight of the link between country $i$ and country $j$. Then, let us represent the binary projection of the network in terms of the binary adjacency matrix $\mathbf{A}^*$, with entries defined as $a^*_{ij} = \Theta[w^*_{ij}]$, $\forall\, i, j$, where $\Theta$ is a Heaviside step function. A maximum-entropy ensemble of networks is a collection of graphs where each graph is assigned a probability of occurrence determined by the choice of some constraints. The BCM is a maximum-entropy ensemble of binary graphs, each denoted by a generic matrix $\mathbf{A}$, where the chosen constraint is the degree sequence. In the canonical formalism [18], the latter can be constrained by writing the following Hamiltonian:

$$H(\mathbf{A}) = \sum_{i=1}^{N} \theta_i k_i(\mathbf{A}) \tag{2.1}$$

where the degree sequence is defined as $k_i(\mathbf{A}) = \sum_{j\neq i}^{N} a_{ij} = \sum_{j\neq i}^{N} \Theta[w_{ij}]$, $\forall\, i$, and $\theta_i$ are the free parameters (Lagrange multipliers) [18]. As a result of the constrained maximization of the entropy [18], the probability of a given configuration $\mathbf{A}$ can be written as

$$P(\mathbf{A}) = \frac{e^{-H(\mathbf{A})}}{\sum_{\mathbf{A}'} e^{-H(\mathbf{A}')}} = \prod_{i<j} \left[\frac{z_i z_j}{1+z_i z_j}\right]^{a_{ij}} \left[\frac{1}{1+z_i z_j}\right]^{1-a_{ij}} \tag{2.2}$$

where $z_i \equiv e^{-\theta_i}$ and $p_{ij} \equiv \frac{z_i z_j}{1+z_i z_j}$. The latter represents the probability of forming a link between nodes $i$ and $j$, which is also the expected value

$$\langle a_{ij} \rangle = \frac{z_i z_j}{1+z_i z_j} = p_{ij}. \tag{2.3}$$

According to the maximum-likelihood method proposed in [18], the vector of unknowns $\vec{z}$ can be numerically found by solving the system of $N$ coupled equations

$$\langle k_i \rangle = \sum_{j\neq i}^{N} p_{ij} = k_i(\mathbf{A}^*) \quad \forall i \tag{2.4}$$

where the expected value of each degree $k_i$ is matched to the observed value $k_i(\mathbf{A}^*)$ in the real network $\mathbf{A}^*$. The (unique) solution will be indicated as $\vec{z}^*$. When inserted back into eq.(2.3), this solution allows us to analytically describe the binary ensemble matching the observed constraints. Being the result of the maximization of the entropy, this ensemble represents the least biased estimate of the network structure, based only on the knowledge of the empirical degree sequence.

In fig. 2.3 we plot some higher-order topological properties of the ITN as a function of the degree of nodes, for the 2002 snapshot. These properties are the so-called average nearest neighbour degree and the clustering coefficient. For both quantities, we plot the observed values (red points) and the corresponding expected values predicted by the BCM (blue points). The exact expressions for both empirical and expected quantities are provided in the Appendix. We see that the expected values are in very close agreement with the observed properties. These results replicate recent findings [19] based on the same UN COMTRADE

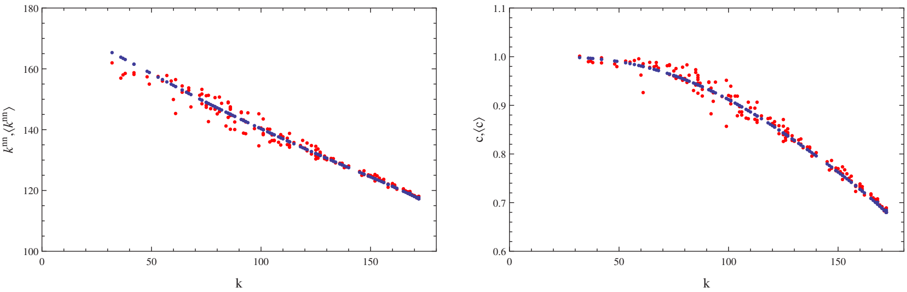


Figure 2.1: **Binary Configuration Model, reconstruction of higher-order properties.** between the observed undirected binary properties (red points) and the corresponding ensemble averages of the BCM (blue points) for the aggregated ITN in the 2002 snapshot. Left panel: Average Nearest Neighbor Degree $k^{nn}$ versus degree $k_i$. Right panel: Binary Clustering Coefficient $C_i$ versus degree $k_i$. The figure shows that the expected values are in very close agreement with the observed properties.

data. They show that at a binary level, the degree correlations (disassortativity) and clustering structure of the ITN are excellently reproduced by the BCM. As we also confirmed in the present analysis, these results were found to be very robust, as they hold true over time and for various resolutions (i.e., for different levels of aggregation of traded commodities) [19].

### Relation with the fitness Model

It should be noted that eq.(2.3) can be thought of as a particular case of the so-called *Fitness Model* [35], which is a popular model of binary networks where the connection probability $p_{ij}$ is assumed to be a function of the values of some 'fitness' characterizing each vertex. Indeed, the variables $\vec{z}^*$ can be treated as fitness parameters [7, 36] which control the probability of forming a link. A very interesting correlation between a fitness parameter of a country (assigned by the model) and the $GDP$ of the same country has been found [36]. This relation is replicated here in fig. 2.4, where the rescaled $GDP$ of each country ($g_i \equiv \frac{GDP_i}{\sum_i GDP_i}$) is compared to the value of the fitness parameter $z_i^*$ obtained by solving eq.(2.4). The red line is a linear fit of the type

$$z_i = \sqrt{a} \cdot g_i. \tag{2.5}$$

This leads to a more economic interpretation where the fitness parameters can be replaced (up to a proportionality constant) with the $GDP$ of countries, and used to reproduce the properties of the network. This procedure, first adopted in [7], can give predictions for the network based only on macroeconomic properties

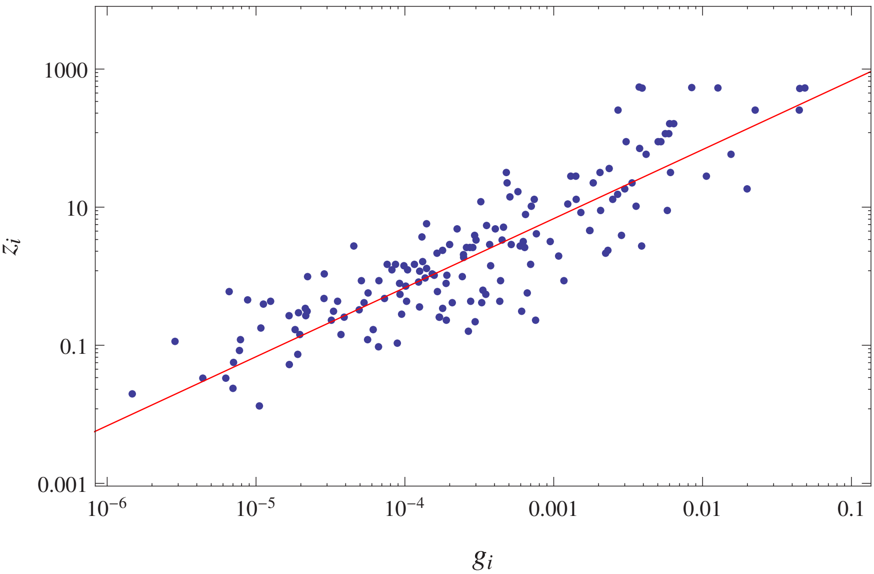


Figure 2.2: **Correlation between Lagrange multipliers and countries GDP.** The calculated $z_i$, compared with the $g_i$ (re-scaled GDP) for each country for the undirected binary aggregated ITN in the 2002 snapshot, with the linear fit $z_i = \sqrt{a} \cdot g_i$ (red line).

of countries, and reveals the importance of the *GDP* to the binary structure of the ITN. Importantly, this observation was the first empirical evidence in favour of the fitness model as a powerful network model [7]. Likewise, other studies have shown that the observed topological properties turn out to be important in explaining macroeconomics dynamics [1, 2].

### 2.3.2 Weighted structure

Despite the importance of the topology, the latter is only the backbone over which goods are traded, and the knowledge of the volume of such trade is critical. To be able to give predictions about the weight of connections, one needs to switch from an ensemble of binary graphs to one of weighted graphs.

The simplest weighted counterpart of the BCM is the WCM, which is a maximum-entropy ensemble of weighted networks where the constraint is the strength sequence, i.e. the total trade of each country in the case of the ITN. In the canonical formalism [18], the latter can be constrained by writing the following Hamiltonian:

$$H(\mathbf{W}) = \sum_{i=1}^{N} \theta_i s_i(\mathbf{W}) \tag{2.6}$$

where the strength sequence is defined as $s_i(\mathbf{W}) = \sum_{j \neq i}^{N} w_{ij}$, $\forall\, i$, and $\theta_i$ are the free parameters (Lagrange multipliers) [18]. As a result of the constrained maximization of the entropy [18], the probability of a given configuration $\mathbf{W}$ can be written as

$$P(\mathbf{W}) = \frac{e^{-H(\mathbf{W})}}{\sum_{\mathbf{W}'} e^{-H(\mathbf{W}')}} = \prod_{i<j} (y_i y_j)^{w_{ij}} (1 - y_i y_j) \tag{2.7}$$

where $y_i \equiv e^{-\theta_i}$.

Recent studies have shown that the higher-order binary quantities predicted by the WCM, as well as the corresponding weighted quantities, are very different from the observed counterparts [18, 19]. More specifically, the main limitation of the model is that of predicting a mostly homogeneous and very dense (sometimes fully connected) topology. Roughly speaking, the model excessively 'dilutes' the total trade of each country by distributing it to almost all other countries. This failure in correctly replicating the purely topological projection of the real network is the root of the bad agreement between expected and observed higher-order properties.

**Relation with the Gravity Model**

Just like the BCM has been related to the Fitness Model [7], a variant of the WCM has been related to the Gravity Model [23]. The variant is actually a continuous version of the WCM, where the strength sequence is constrained, and the weights are real numbers instead of integers. When applied to the ITN, the model gives the following expectation for the weight of the links:

$$\langle w_{ij} \rangle = T \cdot g_i g_j \qquad \forall i, j \tag{2.8}$$

where $T$ is the total strength in the network, and $g_i$ is the re-scaled $GDP$ as before [23]. In essence, the above expression identifies again a relationship between the $GDP$ and the hidden variable (analogous to the fitness in the binary case) specifying the strength of a node.

Equation (2.8) coincides with eq.(2.9) where $\beta = 1$ and $\gamma = 0$. The model, therefore, corresponds to a particularly simple version of the Gravity Model. Indeed, the model reproduces reasonably well the observed non-zero weights of the ITN [23]. However, just like the Gravity Model, the model predicts a complete graph where $a_{ij} = 1 \;\; \forall i, j$, and dramatically fails in reproducing the binary architecture of the network. This effect can be easily shown by realizing that the continuous nature of edge weights, which can take non-negative real values in the model, implies that there is a zero probability of generating zero weights (i.e. missing links). We will show the prediction of this model in comparison with our results later on in the chapter (when compared to the two-step model).

## 2.4 Macroeconomic approaches to the international trade network

In this section, after covering the maximum-entropy approaches, we briefly review the macroeconomic approaches to the characterization of the ITN, mainly focusing on the popular Gravity model of trade. Next, we discuss the role played by the countries GDP in determining both the presence and the amount of trade exchanges between world countries. Identifying the specific relations between the GDP and network properties, will enable us later to introduce models, which converge the two approaches.

### 2.4.1 The gravity model of trade

Traditionally, macroeconomic models have mainly focused on the weighted representation, because economic theory perceives the latter as being *a priori* more informative than the purely binary representation. The focus is on the expected volume of trade between two countries, given certain dyadic and country-specific macroeconomic properties. Jan Tinbergen, the physics-educated[1] Dutch economist who was awarded the first Nobel memorial prize in economics introduced the so-called Gravity Model (GM) of trade [41]. The GM aims at inferring the volume of trade between any two (trading) countries from the knowledge of their Gross Domestic Product, geographic distance, and other possible dyadic quantities of macroeconomic relevance (such as common currency, trade agreements, bordering conditions, common language, etc.) [58]. In one of its simplest forms, the gravity model predicts that the expected volume of trade between countries $i$ and $j$ is

$$\langle w_{ij} \rangle = \alpha \; GDP_i^{\beta} \; GDP_j^{\beta} \; R_{ij}^{\gamma}, \tag{2.9}$$

where $GDP_k$ is the Gross Domestic Product of country $k$, $R_{ij}$ is the geographic distance between countries $i$ and $j$, and $\alpha, \beta, \gamma$ are free parameters to be estimated by fitting the model to the data [30, 15, 42, 43].[2] More complicated variants of

[1] Jan Tinbergen studied physics in Leiden, where he carried out a Ph.D. under the supervision of the famous theoretical physicist Paul Ehrenfest. Tinbergen defended his thesis in 1929, and then started a career as an economist. He was awarded the first Nobel memorial prize in economics in 1969.

[2] Note that eq.(2.9), by assuming for simplicity the same exponent $\beta$ for the GDPs of both $i$ and $j$, predicts $\langle w_{ij} \rangle = \langle w_{ji} \rangle$ and should therefore be interpreted as a model for the undirected version of the network. In this representation, the trade from country $i$ to country $j$ and the trade from country $j$ to country $i$ are combined into a single value of bilateral trade. Given the highly symmetric structure of the ITN at the aggregate level (i.e. when all traded products are combined), this simplification

eq.(2.9) use additional explanatory factors (with associated free parameters) either favoring or resisting trade. These additional factors can be country-specific like the GDP (e.g. population) or dyadic like the geographic distances (e.g. common currency, trade agreements, etc.). In general, if we collectively denote with $\mathbf{n}_i$ the set of *node-specific* factors and with $\mathbf{D}_{ij}$ the set of *dyad-specific* factors used, eq.(2.9) can be generalized to

$$\langle w_{ij} \rangle = F(\mathbf{n}_i, \mathbf{n}_j, \mathbf{D}_{ij}), \tag{2.10}$$

where, in general, the functional form of $F(\mathbf{n}_i, \mathbf{n}_j, \mathbf{D}_{ij})$ need not be of the same type as in eq.(2.9), and each component of $\mathbf{n}_i$ and $\mathbf{D}_{ij}$ may have a corresponding free parameter to be fitted. In fact, despite the fact we are focusing on the gravity model applied to the international trade network as our main application, our discussion applies to many other models of (socio-economic) networks as well. For instance, the recent Radiation Model (RM) [45], which improves the predictions of the GM when applied to mobility (rather than trade) networks, is also described by eq.(2.32), where $\mathbf{n}_i$ and $\mathbf{D}_{ij}$ are certain geographical and demographical variables. Our following discussion applies to both the GM and the RM, as well as any more general model described by eq.(2.10).

Equation (2.10) refers to the expected value of $w_{ij}$. The full probability distribution from which this expected value is calculated depends on the particular implementation of the model. In the GM case, this distribution can be Gaussian (implying that the expected weights can be fitted to the observed ones via a simple linear regression [46, 47]), log-normal (requiring a linear regression of log-transformed weights [49]), Poisson [49], or more sophisticated [48] (see [42] for a review). The non-uniqueness of the weight distribution already highlights a fundamental arbitrariness in the model. This is only one of the limitations of the GM and similar models.

The GM can successfully reproduce the observed trade volume between trading countries. However, at least in its simplest and most widespread implementation, the model cannot generate zero volumes and therefore predicts a fully connected network. From a model fitting perspective, this means that the GM can be fitted only to the *non-zero* weights, i.e. the strictly positive volumes existing between pairs of *connected* countries. Therefore the model effectively disregards the empirical topology of the network, thus making predictions on the basis of incomplete data. Operatively, the GM can predict a realistic trade volume only *after* the presence of the trade relation itself has been established independently [42]. This problem is particularly critical, since, depending on the datasets, up to approximately half of the possible links in the real ITN are *not* realized [21, 22, 7, 19]. If the total trade is disaggregated into commodity-specific trade flows, the resulting commodity-specific networks are even sparser [50, 51]. Clearly, the same problem

retains all the basic network properties of the system [38, 21, 22, 44, 59, 60]. Throughout the chapter, we will stick to this undirected (bilateral trade) representation, but the extension to the directed case is straightforward.

holds in general for the RM and other models of the form (2.10).

While there are variants and extensions of the GM that do generate zero weights (e.g. the so-called Poisson pseudo-maximum likelihood models [49] and 'zero-inflated' gravity models [48]), these variants can mainly reproduce the empirical link density (the realized fraction of connections), but not the observed topology [42, 43]. Indeed, even in these generalized forms, the GM predicts a largely homogeneous topology, while the empirical topological backbone of the ITN is much more heterogeneous and complex [15, 42]. For instance, the distribution of node degree (number of connections of a country) is very broad, as for the distribution of node strength (total trade volume of a country) [5, 38, 11, 39, 13, 14, 15]. A small number of rich countries dominate the trade patterns and account for most of the trade. Clustering and mixing patterns exhibit the rich-club phenomenon [55, 56], where well-connected nodes also connected to each other. The higher-order correlations are disassortative in the binary representation (nodes of low degree are more likely to be connected to nodes of a higher degree than expected by chance [21]), but assortative in the weighted representation (nodes are more likely to be connected to nodes of similar strength than expected by chance[22]). These structural properties are remarkably stable over time: despite the fourfold increase in trade volume over the last 65 years, the overall topology of international trade has remained largely constant [57].

In the next section, we will move forward, by trying to detect similarities between the two approaches. We explore the various empirical relations between the GDP of a country and specific country (network) properties like degree and strength. This simple empirical analysis reveals the GDP as a "macroeconomic fitness", i.e. a powerful predictor of the number and strength of country's trade relations.

### 2.4.2 The GDP as macroeconomic fitness

Let us start with an empirical analysis of the GDP. We first define new rescaled quantities of the GDP: $g_i$ and $\tilde{g}_i$

$$g_i \equiv \frac{\mathrm{GDP}_i}{\sum_j \mathrm{GDP}_j},\ \forall\, i \qquad \tilde{g}_i \equiv \frac{\mathrm{GDP}_i}{\mathrm{GDP}_{mean}},\ \forall\, i, \tag{2.11}$$

where $\mathrm{GDP}_{mean} \equiv \frac{\sum_i^N \mathrm{GDP}_i}{N}$ is the average GDP for an observed year. The two quantities adjust the values of the countries GDPs for both the size of the network and the growth, and are a connected by a simple relation $\tilde{g}_i = N \cdot g_i$. We use the two quantities of the rescaled GDP throughout our analysis, mainly using $g_i$ for the reason that the quantity is bounded $0 \leq g_i \leq 1$ which coincides with our model.

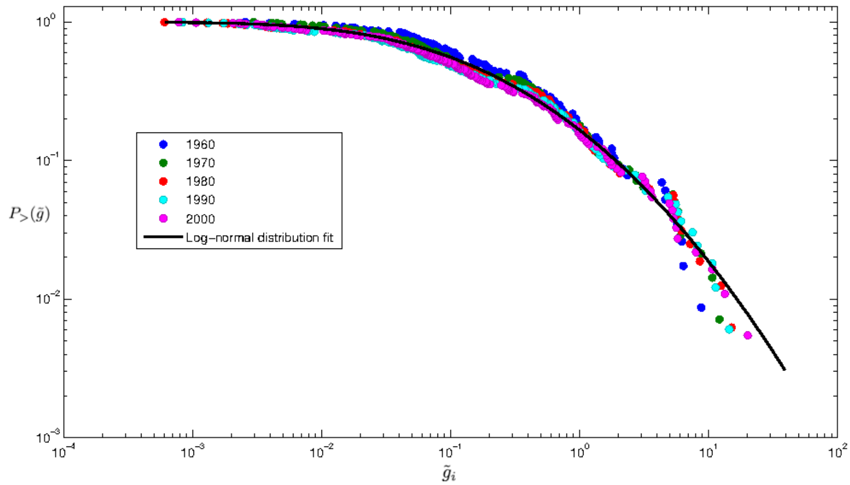


Figure 2.3: **Cumulative Distribution of countries GDP for different decades.** Empirical cumulative distributions $P_>(\tilde{g})$ of the GDP rescaled to the mean, for different years. The curve is a log-normal distribution fitted to the data.

In fig. 2.3 we plot the cumulative distribution of the rescaled GDP $\tilde{g}_i$ with $i$ indexing the countries for the different decades collected into our data set. What emerges is that the distributions of the rescaled GDPs can be described by log-normal distribution characterized by similar values of the parameters. The log-normal curve is fitted to all the values (from the different decades). This suggests that the rescaled GDPs are quantities which do not vary much with the evolution of the system, thus potentially representing the (constant) hidden macroeconomic fitness ruling the entire evolution of the system itself. This, in turn, implies understanding the functional dependence of the key topological quantities on the countries rescaled GDP.

As already pointed out by a number of results [19], the topological quantities which play a major role in determining the ITN structure are the countries degrees (i.e. the number of their trading partners) and the countries strengths (i.e. the total volume of their trading activity). Thus, the first step to understanding the role of the rescaled GDP in shaping the ITN structure is quantifying the dependence of degrees and strengths on it. Since we want to analyse each snapshot at a time (correction for size is not needed), here we will use the bounded rescaled GDP $g_i$. Moreover, this form of the rescaled GDP coincides with a bounded macroeconomic fitness value, which is consistent with the models presented in the next sections.

To this aim, let us explicitly plot $k_i$ versus $g_i$ and $s_i$ versus $g_i$ for a particular decade, as shown in fig. 2.4. The red points represent the relations between the two pairs of observed quantities for the 2000 snapshot. Interestingly, the rescaled GDP is directly proportional to the strength (on a log-log scale), thus indicating that *the wealth of countries is strongly correlated to the total volume of trade they participate in*. Such evidence provides the empirical basis for the definition of the gravity model, stating that *the trade between any two countries is directly proportional to the (product of the) countries GDP*.

On the other hand, the functional dependence of the degrees on the $g_i$ values is less simple to decipher. Generally speaking, the relation is monotonically increasing, and this means that countries with high GDP also have a high degree, i.e. are strongly connected with the others; coherently, countries characterized by a low value of the GDP have also a low degree, i.e. are less connected to the rest of the world. Moreover, while for low values of the GDP there seems to exist a linear relation (on a log-log scale) between $k_i$ and $g_i$, as the latter rises a saturation effect is observed (in correspondence of the value $k_{max} = N - 1$), due to the finite size of the network under analysis. Roughly speaking, richest countries lie on the vertical trait of the plot, while poorest countries lie on the linear trait of the same plot: in other words, *the degree of countries represents a purely topological indicator of the countries wealth*.

To sum up, fig. 2.4 shows that countries GDP plays a double role in shaping the ITN structure: first, it controls for the number of trading channels each country establishes; second, it controls for the volume of trade each country participates in, via the established connections. The blue points in fig. 2.4, instead, represent the relation between $\langle k_i \rangle$ versus $g_i$ and $\langle s_i \rangle$ versus $g_i$, where the quantities in brackets are the predicted values for degrees and strengths generated by our model, which we will discuss later.

The obvious question that arises from these findings is can we extend the result shown in section 2.3 to create a GDP-driven model for both the binary and the weighted representation of the ITN. In the next section, we tackle exactly this problem using a recent maximum-entropy model [33] which is able to reproduce both representations.

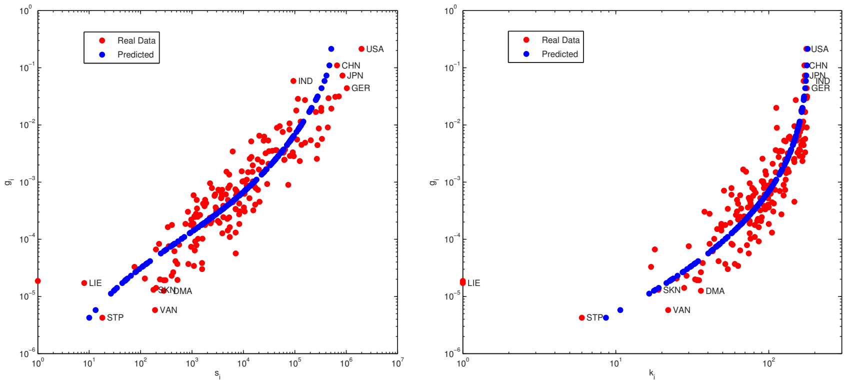


Figure 2.4: **The relation of the GDP with local network properties.** Comparison between observed (red points) and expected (blue points) degrees and strengths for the aggregated ITN in the 2000 snapshot. Right panel: degree $k_i$ versus normalized GDP $g_i$ and expected degree $\langle k_i \rangle$ versus normalized GDP $g_i$. Left panel: strength $s_i$ versus normalized GDP $g_i$ and expected strength $\langle s_i \rangle$ versus normalized GDP $g_i$.

## 2.5 A GDP-driven model of the ITN

Motivated by the challenge to satisfactorily model both the topology and the weights of the ITN, the ECM has been recently proposed as an improved model of this network [33]. The ECM focuses on weighted networks, and can enforce the degree and strength sequence simultaneously [32]. It builds on the so-called generalized Bose-Fermi distribution that was first introduced as a null model of networks with coupled binary and weighted constraints [34].

In the ECM, the degree and strength sequence can be constrained by writing the following Hamiltonian:

$$H(\mathbf{W}) = \sum_{i=1}^{N} [\alpha_i k_i(\mathbf{W}) + \beta_i s_i(\mathbf{W})] \tag{2.12}$$

where the strength sequence is defined as $s_i(\mathbf{W}) = \sum_{j \neq i}^{N} w_{ij}$, $\forall\, i$ and the degree sequence as $k_i(\mathbf{W}) = \sum_{j \neq i}^{N} a_{ij} = \sum_{j \neq i}^{N} \Theta[w_{ij}]$, $\forall\, i$. As a result, the probability of a given configuration $\mathbf{W}$ can be written as

$$P(\mathbf{W}) = \frac{e^{-H(\mathbf{W})}}{\sum_{\mathbf{W}'} e^{-H(\mathbf{W}')}} = \prod_{i<j} \frac{(x_i x_j)^{a_{ij}} (y_i y_j)^{w_{ij}} (1 - y_i y_j)}{1 - y_i y_j + x_i x_j y_i y_j} \tag{2.13}$$

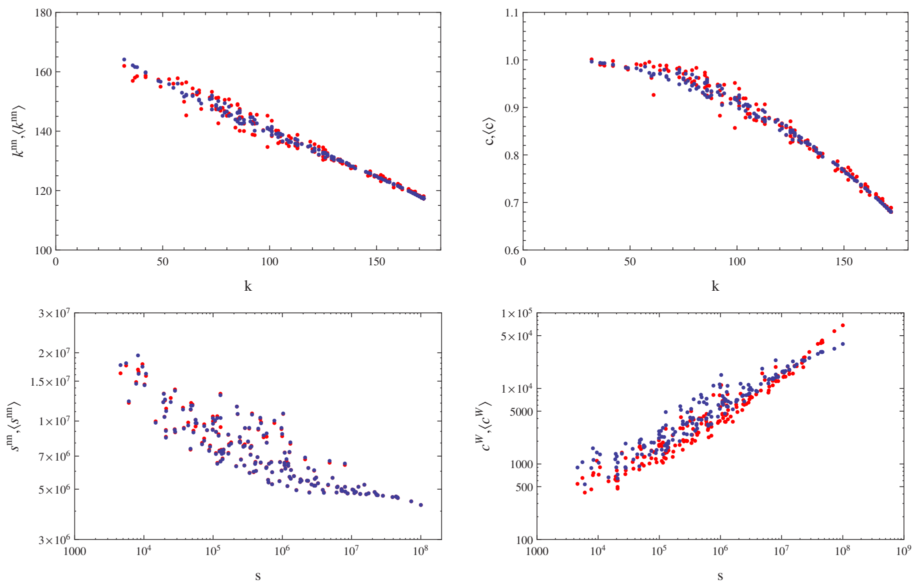


Figure 2.5: **Enhanced Configuration Model, reconstruction of higher-order properties.** Comparison between the observed undirected binary and weighted properties (red points) and the corresponding ensemble averages of the the ECM (blue points) for the aggregated ITN in the 2002 snapshot. Top left panel: Average Nearest Neighbour Degree $k^{nn}$ versus degree $k_i$; Top right panel: Binary Clustering Coefficient $C_i$ versus degree $k_i$ ; Bottom left panel: Average Nearest Neighbour Strength $s^{nn}$ versus strength $s_i$ ; Bottom right panel: Weighted Clustering Coefficient $C^W$ versus strength $s_i$ .

with $x_i \equiv e^{-\alpha_i}$ and $y_i \equiv e^{-\beta_i}$. The ECM gives the following predictions about the probability of a link ($\langle a_{ij} \rangle$) and the expected weight of the link ($\langle w_{ij} \rangle$):

$$\langle a_{ij} \rangle = \frac{x_i x_j y_i y_j}{1 - y_i y_j + x_i x_j y_i y_j} = p_{ij} \tag{2.14}$$

$$\langle w_{ij} \rangle = \frac{x_i x_j y_i y_j}{(1 - y_i y_j + x_i x_j y_i y_j)(1 - y_i y_j)} = \frac{p_{ij}}{1 - y_i y_j}. \tag{2.15}$$

According to the maximum-likelihood method proposed in [32], the vectors of unknowns $\vec{x}$ and $\vec{y}$ can be numerically found by solving the system of $2N$ coupled equations

$$\langle k_i(\mathbf{W})\rangle = \sum_{j\neq i}^{N} p_{ij} = k_i(\mathbf{W}^*) \qquad \forall\, i \tag{2.16}$$

$$\langle s_i(\mathbf{W})\rangle = \sum_{j\neq i}^{N} \langle w_{ij}\rangle = s_i(\mathbf{W}^*) \qquad \forall\, i \tag{2.17}$$

and will be indicated as $\vec{x}^*$ and $\vec{y}^*$. These unknown parameters can be treated as fitness parameters which control the probability of forming a link and the expected weight of that link simultaneously.

The application of the ECM to various real-world networks shows that the model can accurately reproduce the higher-order empirical properties of these networks [32]. When applied to the ITN in particular, the ECM replicates both binary and weighted empirical properties, for different levels of disaggregation, and for several years (temporal snapshots) [33]. Indeed, in fig. 2.5 we show the higher-order binary quantities (average nearest neighbour degree and clustering coefficient) as well as their weighted ones (average nearest neighbour strength and weighted clustering coefficient) for the 2002 snapshot of the ITN. We compare the observed values (red points) and the corresponding quantities predicted by the ECM (blue points). The mathematical expressions for all these quantities are provided in the Appendix. We find a very good agreement between data and model, confirming the recent results in [33] for the data set we are using here. We also confirmed that these results are robust for several temporal snapshots [33].

### 2.5.1 From Lagrange multipliers to macroeconomic properties

Considering the promising results of the ECM and the results from section 2.4.2, we now make a step forward and check whether the hidden variables $x_i$ and $y_i$, which effectively reproduce the observed ITN, can be thought of as 'fitness' parameters having a clear economic interpretation. This amounts to checking whether the relation shown previously in fig. 2.4 for the purely binary case can be generalized in order to find a macroeconomic interpretation to the abstract fitness parameters in the general weighted case as well.

In fig. 2.6 we show the relationship between the two parameters $x_i$ and $y_i$ and the rescaled $GDP$ ($g_i$) for each country of the ITN in the 2002 snapshot. We find strong correlations between these quantities. The fitness parameter $x_i$ turns out to be in a roughly linear relation with the rescaled GDP $g_i$, fitted by the curve

$$x_i = \sqrt{a} \cdot g_i \tag{2.18}$$

where $\sqrt{a}$ is the fitted constant, and $g_i = \frac{GDP_i}{\sum_i GDP_i}$ (all the $GDP$s are relative to that specific year). It should be noted that this relation is similar to that

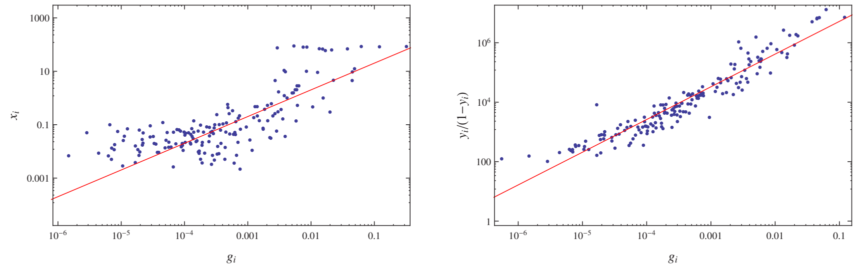


Figure 2.6: **Relation between Lagrange multipliers of the ECM and countries GDP.** calculated $x_i$ (left panel) and $\frac{y_i}{1-y_i}$ (right panel) compared with the $g_i$ (rescaled GDP) for each country for the undirected binary aggregated ITN in the 2002 snapshot, with the linear fits (in log-log scale) $x_i = \sqrt{a} \cdot g_i$, and $\frac{y_i}{1-y_i} = b \cdot g_i^c$ (red lines), where a, b, and c are the fitted constant parameters per year.

found between $z_i$ and $g_i$ in the BCM and shown previously in fig.2.4, but less accurate. This observation will be useful later. By contrast, since the $GDP$ is an unbounded quantity, while the fitness parameter $y_i$ is bounded between 0 and 1 (this is a mathematical property of the model [34, 32]), the relation between $y_i$ and $g_i$ is necessarily highly nonlinear. A simple functional form for such a relationship is given by

$$y_i = \frac{b \cdot g_i^c}{1 + b \cdot g_i^c}. \tag{2.19}$$

Indeed, fig. 2.6 confirms that the above expression provides a very good fit to the data.

We checked that the above results hold systematically over time, for each snapshot of the ITN in our data set. This implies that, in a given year, we can insert eqs.(2.18) and (2.19) into eqs.(2.38) and (2.37) to obtain a GDP-driven model of the ITN structure for that year. Such a model highlights that the $GDP$ has a crucial role in shaping both the binary and the weighted properties of the ITN. While this was already expected on the basis of the aforementioned results obtained using the BCM and the WCM (or the corresponding simplified gravity model) separately, finding the appropriate way to explicitly combine these results into a unified description of the ITN has remained impossible so far. Rather than exploring in more detail the predictions of the GDP-driven model in the form described above, we first make some considerations leading to a simplification of the model itself.

### 2.5.2 Reduced two-step model

At this point, it should be noted that we arrived at two seemingly conflicting results. We showed that both the BCM and ECM give a very good prediction for the binary topology of the network. However, eqs.(2.3) and (2.38), which specify the connection probability $p_{ij}$ in the two models, are significantly different. The comparable performance of the BCM and the ECM at a binary level (see figs.2.3 and 2.5) makes us expect that, when the specific values $\vec{z}^*$ and $\vec{x}^*$ are inserted into eqs.(2.3) and (2.38) respectively, the values of the connection probability become comparable in the two models, despite the different mathematical expressions.

In fig. 2.7 we compare the the two probabilities for the ITN in the 2002 snapshot. Note that each point refers to the probability of creating a link between a pair of countries, which results in $\frac{N(N-1)}{2}$ points. Indeed, we can see that the values are scattered along the identity line, confirming the expectation that the connection probability has similar value in the two models.

The above result allows us to make a remarkable simplification. In eqs.(2.38) and (2.37), we can replace the expression for $p_{ij}$ provided by the ECM with that provided by the BCM in eq.(2.3). To avoid confusion, we denote the new probability with $p_{ij}^{ts}$, where $ts$ stands for 'two-step', for a reason that will be clear immediately. This results in the following equations for the expected network properties:

$$\langle a_{ij}\rangle^{ts} = \frac{z_i z_j}{1+z_i z_j} \equiv p_{ij}^{ts}, \tag{2.20}$$

$$\langle w_{ij}\rangle^{ts} = \frac{p_{ij}^{ts}}{1-y_i y_j}. \tag{2.21}$$

where the $z_i$'s, and therefore the $p_{ij}^{ts}$'s, depend only on the degrees through eq.(2.4), while the $y_i$'s and the $\langle w_{ij}\rangle^{ts}$'s depend on both strengths and degrees through eqs.(2.16) and (2.17).

In this simplified model the connection probability, which fully specifies the topology of the ensemble of networks, no longer depends on the strengths as in the ECM, while the weights still do. This implies that we can specify the model via a two-step procedure where we first solve the $N$ equations determining $p_{ij}^{ts}$ via the degrees, and then find the remaining variables determining $\langle w_{ij}\rangle^{ts}$ through the ECM. For this reason, we denote the model as the Two-Step (TS) model.

The probability of a configuration $\mathbf{W}$ reads

$$P^{ts}(\mathbf{W}) = \prod_{i<j} q_{ij}^{ts}(w_{ij}) \tag{2.22}$$

where

$$q_{ij}^{ts}(w_{ij}) = \frac{(z_i z_j)^{a_{ij}}(y_i y_j)^{w_{ij}-a_{ij}}(1-y_i y_j)^{a_{ij}}}{1+z_i z_j} \tag{2.23}$$

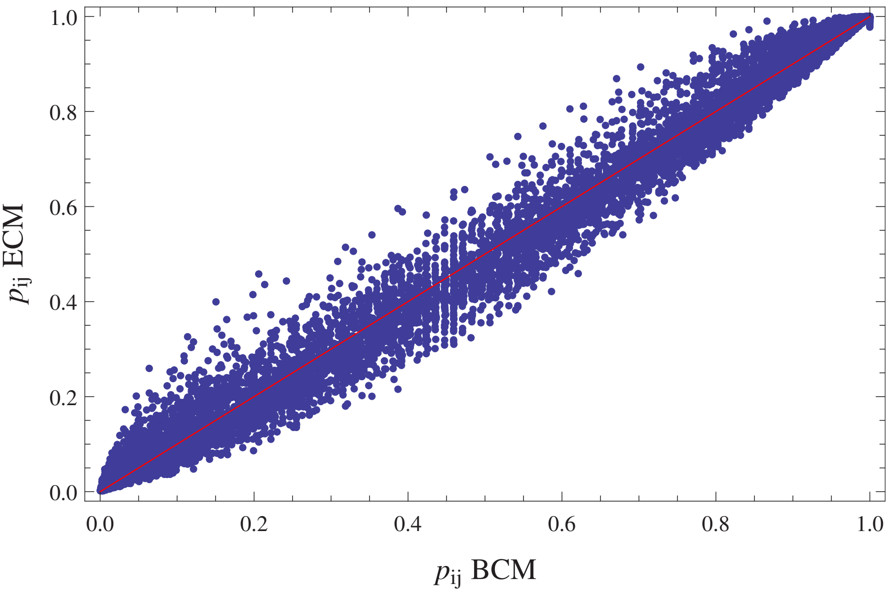


Figure 2.7: **Link formation probability for the different models.** The probability of forming a link in the ECM $p_{ij}$ $ECM$ compared to the probability in the BCM $p_{ij}$ $BCM$ for the undirected binary aggregated ITN in the 2002 snapshot. The red line describes the identity line.

is the probability that a link of weight $w_{ij}$ connects the nodes (countries) $i$ and $j$. The above probability has the same general expression as in the original ECM [32], but here $z_i$ comes from the estimation of the simpler BCM. It is instructive to rewrite (2.23) as

$$q_{ij}^{ts}(0) = \frac{1}{1+z_i z_j} = (1-p_{ij}^{ts}); \tag{2.24}$$

$$q_{ij}^{ts}(w) = p_{ij}^{ts}(y_i y_j)^{w-1}(1-y_i y_j),\ \forall\, w>0 \tag{2.25}$$

to highlight the random processes creating each link. As a first step, one determines whether a link is created or not with a probability $p_{ij}^{ts}$. If a link (of unit weight) is indeed established, a second attempt determines whether the weight of the same link is increased by another unit (with probability $y_i y_j$) or whether the process stops (with probability $1-y_i y_j$). Iterating this procedure, the probability that an edge with weight $w$ is established between nodes $i$ and $j$ is given precisely by $q_{ij}^{ts}(w)$ in eq.(2.25). The expected weight $\langle w_{ij}\rangle^{ts}$ is then correctly retrieved as $\sum_{w=0}^{+\infty} w \cdot q_{ij}^{ts}(w)$.

Using the relations found in eqs.(2.5) and (2.19), we can input the $g_i$ as the fitness parameters into eqs.(2.20) and (2.21) to get the following expressions that mathematically characterize our GDP-driven specification of the TS model:

$$\langle a_{ij}\rangle^{ts} = \frac{ag_i g_j}{1+ag_i g_j} \equiv p_{ij}^{ts} \tag{2.26}$$

$$\langle w_{ij}\rangle^{ts} = p_{ij}^{ts}\frac{(1+bg_i^c)(1+bg_j^c)}{(1+bg_i^c+bg_j^c)}. \tag{2.27}$$

The above equations can be used to reverse the approach used so far: rather than using the $2N$ free parameters of the ECM ($\vec{x}$ and $\vec{y}$) or of the TS model ($\vec{z}$ and $\vec{y}$) to fit the models on the observed values of the degrees and strengths, we can now use the knowledge of the GDP of all countries to obtain a model that only depends on the three parameters $a$, $b$, $c$. Assigning values to these parameters can be done using two techniques: maximization of the likelihood function and non-linear curve fitting. Since the model is a two-step one, we can first assign a value to the parameter $a$, and only in the second step (once $a$ is set) we fit the parameters $b$ and $c$.

We chose to fix $a$ by maximizing the likelihood function [36], which results in constraining the expected number of links to the observed number ($\langle L\rangle = L$), as in [7]. Fixing the values of $b$ and $c$ is slightly more complicated. Since the model uses the approximated expressions of the TS model, rather than those of the original ECM model, maximizing the likelihood function in the second step no longer yields the desired condition $\langle T\rangle = T$, where $T$ is the total strength in the network. Similarly, extracting the parameters from the fit as shown in fig. 2.6 does not maintain the total strength in the network. In absence of any a-priori preference, we chose the latter procedure, due to its relative numerical simplicity with respect to the former one.

In fig. 2.8 we show a comparison between the higher-order observed properties of the ITN in 2002 and their expected counterparts predicted by the GDP-driven TS model. Again, the mathematical expressions of these properties are provided in the Appendix. As a baseline comparison, we also show the predictions of the *GDP*-driven WCM model with continuous weights described by eq.(2.8) [23], which coincides with a simplified version of the gravity model as we mentioned.

We see that the GDP-driven TS model reproduces the observed trends very well. Of course, as expected, the predictions in fig.2.8 (which use only three free parameters) are noisier than those in fig.2.5 (which use $2N$ free parameters). This is due to the fact that eqs.(2.5) and (2.19) describe fitting curves rather than exact relationships. Importantly, our model performs significantly better than the WCM/gravity model in replicating both binary and weighted properties. Again, the drawback of these models lies in the fact that they predict a fully connected topology and a relatively homogeneous network.

It should also be noted that the plot of average nearest neighbour strength ($s^{nn}$) predicted by our model is slightly shifted with respect to the observed points. This effect is due to the fact that, as we mentioned, the total strength $T$ (hence the average trend of the $s^{nn}$) is only approximately reproduced by our model, as a result of the simplification from the ECM to the TS model.

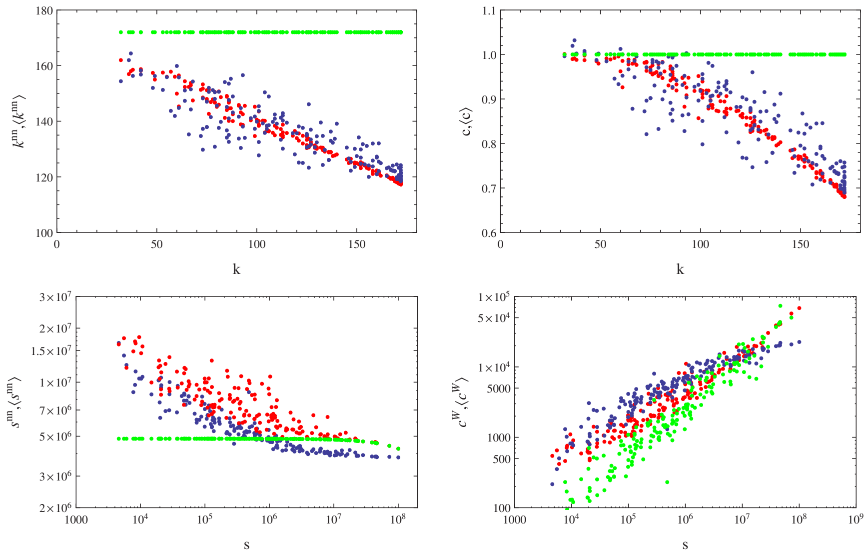

Figure 2.8: **Two-Step Model, reconstruction of higher-order properties.** Comparison between the observed properties (red points), the corresponding ensemble averages of the *GDP*-driven two-steps model (blue points) and the *GDP*-driven WCM model (green points), of the aggregated ITN in 2002. Top left: Average Nearest Neighbour Degree $k_i^{nn}$ versus degree $k_i$. Top right: Binary Clustering Coefficient $c_i$ versus degree $k_i$. Bottom left: Average Nearest Neighbour Strength $s_i^{nn}$ versus strength $s_i$. Bottom right: Weighted Clustering Coefficient $c_i^w$ versus strength $s_i$. The GDP-driven TS model reproduces the empirical trends very well with respect to the WCM.

As for all the other results in this chapter, we checked that our findings are robust over the entire time span of our data set. We, therefore, conclude that the ECM model, as well as its simplified TS variant, can be successfully turned into a fully GDP-driven model that simultaneously reproduces both the topology and the weights of the ITN.

The success of the TS model has a meaningful interpretation. Looking back at eqs.(2.20) and (2.26), we recall that the effect of the TS approximation is the fact that the connection probability $p_{ij}^{ts}$ can be estimated separately from the weights $\langle w_{ij} \rangle^{ts}$, using only the knowledge of the degree sequence if eq.(2.20) is used, or the GDP and total number of links if eq.(2.26) is used, while discarding that of the strengths. By contrast, the estimation of the expected weights $\langle w_{ij} \rangle^{ts}$ cannot be carried out separately, as it requires that the connection probability $p_{ij}^{ts}$ appearing in eqs.(2.21) and (2.27) is estimated first. This asymmetry of the model means that the topology of the ITN can be successfully inferred without any information about the weighted properties, while the weighted structure cannot be inferred without topological information. The expressions defining the TS model provide a mathematical explanation for this otherwise puzzling effect that has already been documented in previous analyses of the ITN [19, 33].

## 2.6 The enhanced gravity model

In the previous section, we have shown that the two-step model can reproduce the higher-order properties of both representations of the ITN. However, despite the vast improvements the model represent, it still has some definite limitations. Firstly, the model does not allow to introduce additional macroeconomic parameters like geographic distance or other pair-countries information like trade relations or common borders. This type of information is frequently being used in macroeconomic models. Furthermore, the model cannot reproduce the specific weights $w_{ij}$ of the network. In this section, using a different methodology than before, we start with the gravity model and reformulate it according to maximum-entropy principles. This maximum-entropy generalization is aimed at creating a model that combines the advantages of the gravity model (accurate link expectation) with the benefits of the maximum-entropy models (topology and higher-order reconstruction).

As we discussed before, the gravity model in eq.(2.10) (which includes eq.(2.9) as a particular case) is successful in reproducing link weights only *after* the existence of the links themselves has been preliminarily established. This means that eq.(2.10) in actually incorrect and should by rather reformulated as a model for the *conditional expectation* of the weight $w_{ij}$, given that $w_{ij} > 0$.

To do so, we need to introduce $q_{ij}(w)$ as the probability that the volume of trade between countries $i$ and $j$ takes a value $w$, with $w$ being, without loss of generality, a non-negative integer number (the event $w = 0$ indicates the absence of a trade link). The probability $q_{ij}(w)$ is the fundamental quantity that fully specifies the model. In particular, it controls both the topology and the link weights of the network. Our aim is to find the form of $q_{ij}(w)$ that produces the desired gravity-like conditional expectation for link weights, as well as a realistically expected topology. The search for the form of $q_{ij}(w)$ will be guided by the important requirement that *the expected topology should not depend on the (arbitrary) units of measure chosen to measure the link weights.* The latter requirement will be referred to as *topological invariance.*

Our first requirement is that $q_{ij}(w)$ produces eq.(2.10), once the latter has been rewritten as an expression for the conditional weights. We perform this rewriting first. Note that the probability $p_{ij}$ that countries $i$ and $j$ are connected (irrespective of the volume of trade) is given by

$$p_{ij} = 1 - q_{ij}(0) = \langle a_{ij} \rangle, \tag{2.28}$$

where $a_{ij} = \Theta(w_{ij})$ is the entry of the binary adjacency matrix of the network (equal to 1 if a link between countries $i$ and $j$ exists and 0 otherwise). Note that $p_{ij}$ does not depend on $q_{ij}(w)$ for $w > 0$. By contrast, the expected trade volume (irrespective of whether a connection is established) is given by

$$\langle w_{ij} \rangle \equiv \sum_{w>0} w \; q_{ij}(w), \tag{2.29}$$

which does not depend on $q_{ij}(0)$. Apparently, the fact that $\langle a_{ij} \rangle$ and $\langle w_{ij} \rangle$ depend on different quantities implies that they can be defined separately, thus allowing one to reproduce the topology and the weights simultaneously. However, we will show that this is not the case: to enforce topological invariance, an explicit dependence between $\langle a_{ij} \rangle$ and $\langle w_{ij} \rangle$ should be introduced.

To rewrite eq.(2.10) as a conditional expectation, we define the *conditional expected weight* of the link between nodes $i$ and $j$ as

$$\langle w_{ij} | a_{ij} = 1 \rangle \equiv \sum_{w>0} w \; q_{ij}(w | a_{ij} = 1) = \frac{\langle w_{ij} \rangle}{p_{ij}} \tag{2.30}$$

where

$$q_{ij}(w | a_{ij} = 1) = \frac{q_{ij}(w)}{\sum_{u>0} q_{ij}(u)} = \frac{q_{ij}(w)}{p_{ij}} \tag{2.31}$$

is the (correctly renormalized) conditional probability that $w_{ij}$ equals $w$ given that $a_{ij} = 1$ (i.e., given that the link is realized). We can now replace Eq. (2.10) with the intended expression

$$\langle w_{ij} | a_{ij} = 1 \rangle = F(\mathbf{n}_i, \mathbf{n}_j, \mathbf{D}_{ij}). \tag{2.32}$$

Our next requirement is that $q_{ij}(w)$ enforces topological invariance. To ensure that this is done without making *ad hoc* assumptions and using only empirical information, we are going to formulate the problem within a maximum-entropy framework. In doing so, we will generalize previous maximum-entropy formulations of the GM by making them manifestly topologically invariant. For clarity, in the next section we first briefly review these previous formulations, before providing our extension.

### 2.6.1 Maximum-entropy reformulation of the gravity model

*Per se*, the standard GM is not a micro-founded model. However, various micro-founded models admit a gravity-like relation as their equilibrium outcome [64, 65, 66, 67]. Notably, the GM can be obtained from the maximum-entropy principle [68], and this result has been reformulated recently in the context of maximum-entropy ensembles of networks [23]. The maximum-entropy framework is in some sense the most general (i.e. requiring the minimal set of assumptions) context

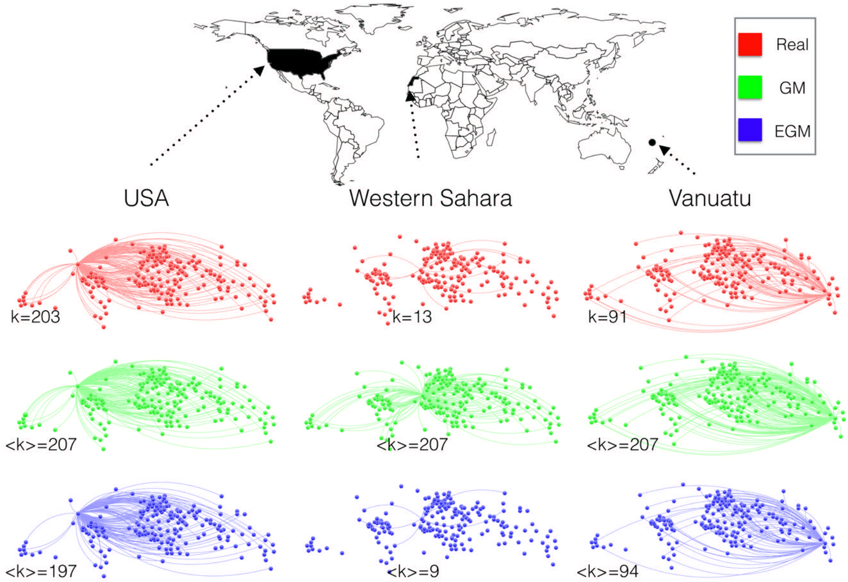


Figure 2.9: **The EGM enhanced reconstruction of the network topology.** The heterogeneous binary topology of the observed ITN (red points) and the expected values from the EGM (blue points). Left: the ANND as a function of the observed degree $k$. Right: the clustering coefficient $C$ as a function of the observed degree $k$.

wherein the GM emerges naturally as an outcome. It also leads to the least biased predictions, as it makes no other hypothesis than consistence with a certain aggregation of the data. We first shortly review this approach, then slightly modify it in a form that assumes discrete rather than continuous trade volumes, and finally, generalize it to a novel model that fixes the main issue of the GM.

The maximum-entropy approach starts by considering the space of all networks with $N$ nodes, where in our case $N$ is the number of countries.

Generalizing the results in [18, 22], we preliminarily require that all the empirical edge weights $\{w_{ij}\}$ are reproduced on average by a maximum-entropy model. This leads to the graph probability $P(\mathbf{W})$ that maximizes Shannon's entropy $S \equiv -\sum_{\mathbf{W}} P(\mathbf{W}) \ln P(\mathbf{W})$ (where the sum runs over all possible weighted networks with the same number of nodes as the real network).

If the weight of a link between two nodes is denoted (in some units) as the entry $w_{ij}$ of a non-negative integer matrix $\mathbf{W}$, then the corresponding entry in the (purely binary) adjacency matrix $\mathbf{A}$ of the graph is

$$a_{ij} \equiv \mathbf{\Theta}(w_{ij}), \tag{2.33}$$

where $\mathbf{\Theta}$ is a Heaviside step function.

The result is $P(\mathbf{W}) = \prod_{i<j} y_{ij}^{w_{ij}}/Z$, where $y_{ij}$ is now a 'dyadic' (as opposed to node-specific) hidden variable and where $Z$ is the normalizing constant, or partition function [18]. This model leads to expected weights of the form $\langle w_{ij}\rangle = \frac{y_{ij}}{1-y_{ij}}$ and $\langle w_{ij}|a_{ij} = 1\rangle = \frac{1}{1-y_{ij}}$. Compared with previous maximum-entropy approaches to the ITN [18, 22, 19, 33], this model has the big advantage that $y_{ij}$ can be chosen such that $\langle w_{ij}|a_{ij} = 1\rangle = F(\mathbf{n}_i, \mathbf{n}_j, \mathbf{D}_{ij})$ where $F$ is any function of dyadic ($\mathbf{D}_{ij}$) and node-specific ($\mathbf{n}_i$) variables, like in eq.(2.32). Note that $F(\mathbf{n}_i, \mathbf{n}_j, \mathbf{D}_{ij})$ has the same units of measure as $w_{ij}$. At the same time, the model fixes the entire probability $q_{ij}(w)$ that nodes $i$ and $j$ are connected by a link of weight $w$, which here has the geometric [18, 22] form $q_{ij}(w) = y_{ij}^{w}(1 - y_{ij})$, thus removing the aforementioned undesired arbitrariness of the weight distribution in the GM. Therefore this model can be regarded as the 'canonical' specification of the GM within a maximum-entropy framework. Note in particular that fixing $\langle w_{ij}|a_{ij} = 1\rangle = F(\mathbf{n}_i, \mathbf{n}_j, \mathbf{D}_{ij})$ automatically fixes all the other moments of the geometric weight distribution, eliminating the extra parameters introduced by the addition of additive or multiplicative noise to eq. (2.9).

Enforcing topological invariance means that, if we express the trade volumes $\{w_{ij}\}$ in terms of millions of dollars rather than dollars, we want to obtain the same expectations for the topology $\{a_{ij}\}$, even if the expected volumes $\{\langle w_{ij}\rangle\}$ should instead scale accordingly. Unfortunately, in the above model the connection probability $p_{ij} \equiv \langle a_{ij}\rangle = y_{ij} = \langle w_{ij}\rangle/(1 + \langle w_{ij}\rangle)$ is not invariant under a change of units of measure for $w_{ij}$. While $\langle w_{ij}\rangle$ scales with $w_{ij}$ as desired, $\langle a_{ij}\rangle$ should not scale with $\langle w_{ij}\rangle$, because the observed $a_{ij}$ is independent of the scale of $w_{ij}$: we do not want the expected number of 'zeroes' to be affected by the arbitrary units of measure of the non-zero weights. Incidentally, we note that choosing a realistically small unit (e.g. one dollar like in many trade databases) implies that $w_{ij}$ (and therefore $\langle w_{ij}\rangle$) is a large number, which implies $p_{ij} \to 1$: as the unit becomes smaller, this model predicts a denser network, asymptotically complete like in the traditional GM. The dependence of the link density on arbitrary units of weight, as well as its asymptotic saturation to a unit value, is a general explanation for the failure of an entire class of weighted network models (like the GM itself) that do not target purely topological properties in their construction. Indeed, we argue that a simple and general theoretical reason for the empirical failure of previous models is their lack of topological invariance.

### 2.6.2 The complete model

Since the maximum-entropy framework requires minimal assumptions (as it does not postulate mechanistic or behavioral rules), it represents the most general and transparent setting to reformulate the GM in such a way that topological invariance is enforced as an additional ingredient, while keeping the other more

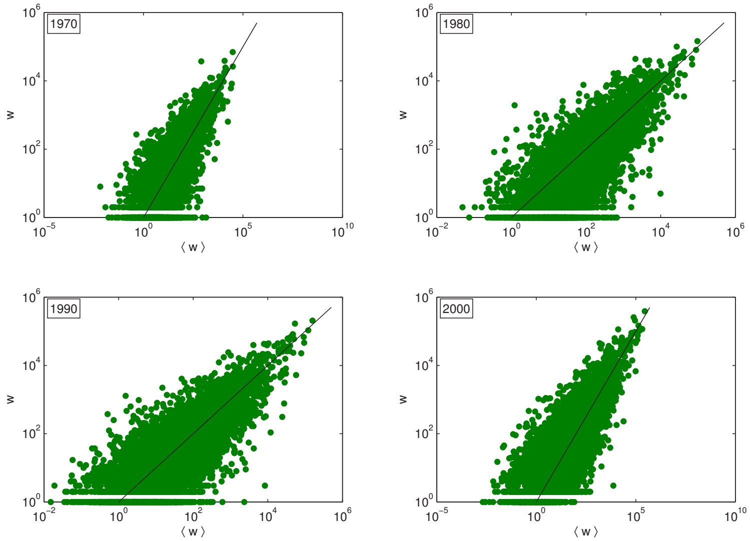


Figure 2.10: **The Gravity model, trade flow expectation.** Comparison between the observed weights of the ITN on the y-axis and the GM expected weights (green points) on the x-axis. The black line is the identity line.

traditional specifications unchanged.

We generalize a recent model [32], based on an analytical maximum-likelihood estimation method [18], that has been recently proposed in order to construct advanced maximum-entropy ensembles of weighted networks that significantly improve the fit to real data. The approach is particularly successful when applied to the ITN [33]. In its standard formulation, the model enforces the degree and strength sequence simultaneously [32]. It builds on the so-called generalized Bose-Fermi distribution that was first introduced as a null model of networks with coupled binary and weighted constraints [34].

Here, we reformulate the model more generally as a model that can flexibly reproduce the edge weights of a network using both dyadic and node-specific factors, while at the same time enforcing *topological invariance*, i.e. the desired invariance of the (expected) binary structure under a change in the (arbitrary) units of weight.

We therefore introduce a model that, besides the empirical weights $\{w_{ij}\}$, enforces the empirical topology $\{a_{ij}\}$, thus manifesting topological invariance from the very beginning. As shown in sec. Materials and Methods, this model yields a

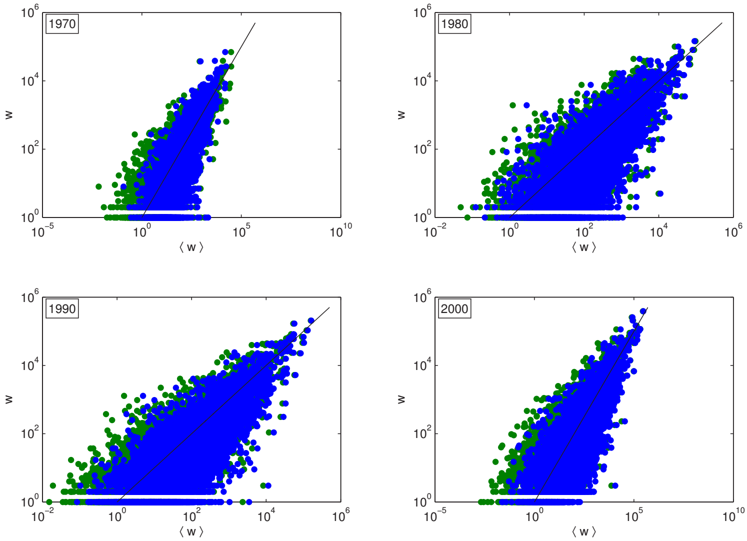


Figure 2.11: **The Enhanced Gravity model, trade flow expectation.** Comparison between the observed weights of the ITN on the y-axis and the EGM expected weights (blue points) on the x-axis, in green are the expectations of the original GM. The black line is the identity line.

weight probability

$$q_{ij}(w) \equiv \frac{(x_{ij})^{a_{ij}} (y_{ij})^{w_{ij}} (1 - y_{ij})}{1 - y_{ij} + x_{ij} y_{ij}}, \tag{2.34}$$

where $\{x_{ij}\}$ and $\{y_{ij}\}$ are two arrays of dyadic hidden variables. The conditional expected weight is

$$\langle w_{ij} | a_{ij} = 1 \rangle = \frac{1}{1 - y_{ij}} \tag{2.35}$$

and the connection probability is

$$p_{ij} = 1 - q_{ij}(0) = \frac{x_{ij} y_{ij}}{1 - y_{ij} + x_{ij} y_{ij}}. \tag{2.36}$$

As we illustrate below, now the presence of the extra variable $x_{ij}$ allows to keep $p_{ij}$ fixed (thus enforcing topological invariance) while varying $y_{ij}$ as a result of a possible change of scale in the original weights $\{w_{ij}\}$.

The dyadic nature of the model allows us to combine the successful ingredients of the traditional GM, which satisfactorily reproduces the conditional weights of the ITN, with those of more recent network models, which accurately reproduce the topology. One one hand, we require that (2.35) has the generic structure of the GM as in eq.(2.32):

$$\langle w_{ij}|a_{ij}=1\rangle = \frac{1}{1-y_{ij}} \equiv F(\mathbf{n}_i,\mathbf{n}_j,\mathbf{D}_{ij}). \tag{2.37}$$

On the other hand, we require that eq.(2.36) has the structure of a binary maximum-entropy model reproducing the topology of the ITN [7, 36, 21, 31, 20]:

$$p_{ij} = \frac{x_{ij}y_{ij}}{1-y_{ij}+x_{ij}y_{ij}} \equiv \frac{z_{ij}}{1+z_{ij}}. \tag{2.38}$$

The above two expressions define our topology-enhanced model in general form. Importantly, the conditional weights are independent of the topology, while the opposite is not true. Also note that $F(\mathbf{n}_i,\mathbf{n}_j,\mathbf{D}_{ij})$ depends on the chosen currency or money unit, while we require $z_{ij}$ to be invariant (and dependent uniquely on the binary structure of the network). This implies that the general solution of the model is

$$y_{ij} = 1-\frac{1}{F(\mathbf{n}_i,\mathbf{n}_j,\mathbf{D}_{ij})} \tag{2.39}$$

$$x_{ij} = z_{ij}\frac{1-y_{ij}}{y_{ij}} = \frac{z_{ij}}{F(\mathbf{n}_i,\mathbf{n}_j,\mathbf{D}_{ij})-1} \tag{2.40}$$

In general, both $F(\mathbf{n}_i,\mathbf{n}_j,\mathbf{D}_{ij})$ and $z_{ij}$ allow for any combination of dyadic and country-specific properties. In what follows, we make (the simplest) particular choices for these quantities. When fitting the standard GM to empirical data, the typical result is that the main factor determining trade volume is GDP. Adding geographical distances improves the fit significantly, while adding other dyadic properties is generally a small refinement. For these reason, we choose the node specific variable $\mathbf{n}_i$ to be the GDP of country, rescaled to the total world GDP $\mathbf{n}_i = g_i \equiv \frac{GDP_i}{\sum_j GDP_j}$ (as shown in Figure 2.4). Next, we choose the dyadic variable $\mathbf{D}_{ij}$ to be solely the distance between the two countries $\mathbf{D}_{ij}\equiv R_{ij}$. Thus,

$$F(\mathbf{n}_i,\mathbf{n}_j,\mathbf{D}_{ij}) \equiv \alpha\ (g_i\ g_j)^{\beta}\ R_{ij}^{\gamma}. \tag{2.41}$$

In this formation, $g_i$ is adimensional and does not depend on the chosen currency or unit of money. On the other hand, it has been shown [7, 31] that the binary structure of the ITN can be excellently reproduced by setting

$$z_{ij} \equiv \delta g_i g_j. \tag{2.42}$$

Although in principle one can add the geographic distances into $z_{ij}$ as well, empirical evidence shows that, contrary to the weighted case, the effect of distances on the binary structure of the ITN is not very strong [53]. This result motivates our choice above. In any case, adding extra factors is straightforward.

Putting the pieces together, our model is fully specified by the weight probability (2.34) with parameters given by eqs.(2.39) and (2.40), which in turn depend only on GDP and distance through eqs.(2.41) and (2.42). As a result we get the following expected values:

$$p_{ij} = \langle a_{ij} \rangle = \frac{\delta g_i g_j}{1 + \delta g_i g_j}, \tag{2.43}$$

$$\langle w_{ij} | a_{ij} = 1 \rangle = \alpha \ (g_i \ g_j)^{\beta} \ R_{ij}^{\gamma}, \tag{2.44}$$

$$\langle w_{ij} \rangle = p_{ij} \langle w_{ij} | a_{ij} = 1 \rangle = \frac{\alpha \delta (g_i g_j)^{\beta+1}}{1 + \delta g_i g_j} \ R_{ij}^{\gamma}. \tag{2.45}$$

Note that the presence of $p_{ij}$ in the expression for $\langle w_{ij} \rangle$ implies that the latter takes the standard gravity form (2.9) only for those pairs of countries with large GDP, which are surely connected in the model. For lower values of the GDP, the expression is instead different, and for pair of countries with very low GDP one gets $\langle w_{ij} \rangle \approx \alpha \delta (g_i g_j)^{\beta+1} \ R_{ij}^{\gamma}$, i.e. the exponent of $g_i g_j$ is increased by one.

The model can also be interpreted as constructed from two random processes. As a first step, one determines whether a link is present or not with a probability $p_{ij}$. If a link (of unit weight) is indeed established, a second attempt determines whether the weight of the same link is increased by another unit (with probability $y_{ij}$) or whether the process stops (with probability $1 - y_{ij}$). Iterating this procedure, the probability that an edge with weight $w$ is established between nodes $i$ and $j$ is given precisely by $q_{ij}(w)$ in eq.(2.34). The presence of these two degrees of freedom allows us to make a parallel, like in [33], with the economic literature about the so-called extensive and intensive margins of trade [61, 62, 63], defined as the preference for the network to evolve by either establishing new connections or strengthening the intensity of existing ones respectively.

### 2.6.3 Results

In this section we will compare the performance of the two models, in reproducing the observed properties (low-order and high-order) of the real complex trade network. We start with the trading volumes between countries, i.e. the weights of the existing links in the network. This property has been the main focus of the empirical economic literature on international trade, and it is the only property the classical gravity model is designed to reproduce.

In Fig 2.10 we can see a typical log-log plot of the expected values versus the real observed weights of the GM (as defined in eq. (2.9)). The three parameters of the model, $\alpha, \beta$, and $\gamma$, are first fitted to the data, then plotted with the identity line (black line). The picture shows the typical good agreement between the

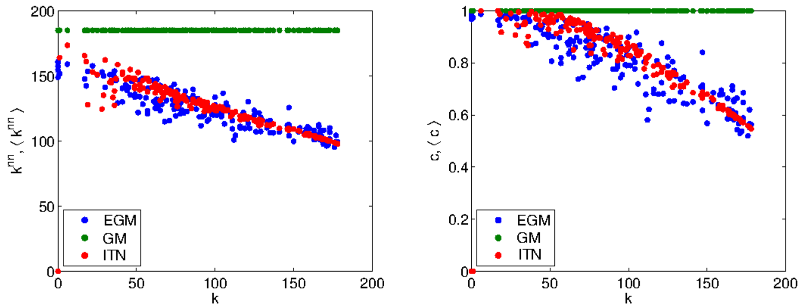


Figure 2.12: **The Enhanced Gravity Model, reconstruction of higher-order binary properties.** The heterogeneous binary topology of the observed ITN (red points) and the expected topology from the EGM (blue points) and the GM (green points). Left: the average nearest neighbour degree $k^{nn}$ as a function of the observed degree $k$. Right: the clustering coefficient $C$ as a function of the observed degree $k$.

predictions of the GM and the empirical non-zero trade volumes [15, 42, 43].

In Fig. 2.11 we compare the observed weights to both the gravity model (GM) and enhanced gravity model (EGM) conditional weights. Overall there is a very good agreement for both models; note that the EGM expectations are shifted to the right compared to the gravity model values, compensating for the higher expected number of zeroes. Despite the fact that in this study we use the classical gravity model in its simplest form, the EGM model can support more sophisticated models which incorporate additional dyadic information. Thus, the model has maximum flexibility when picking an expression for the conditional weights, and can be improved by using more refined models.

### Binary Structure

In the binary representation, the main first-order property is the number of trade partners (connections) of each country, i.e. the degree sequence of the network. This simple yet important representation provides an added layer of considerable information to the standard results of traditional macroeconomic analyses of international trade. Recent studies have shown that higher-order binary properties, like the degree correlations (disassortativity) and clustering structure, of the ITN can be traced back to the knowledge of the degree sequence [21, 19]. This result indicates that the degree sequence, which is a purely topological property, needs to be considered as an important target quantity that international trade models, in contrast with the mainstream approaches in economics, should aim at reproducing.

In Fig. 2.12 we plot some higher-order topological properties of the ITN as a function of the degree of nodes, for the 2000 snapshot. These properties are the so-called average nearest neighbour degree and the clustering coefficient. For both quantities, we plot the observed values (red points) and the corresponding expected values predicted by the EGM (blue points) and the GM (green points). The exact expressions for both empirical and expected quantities are provided in sec. Materials and Methods. We see that the expected values of the EGM are in very close agreement with the observed properties, as opposed to the classical GM which consistently predicts a complete network. They show that at a binary level, the degree correlations (disassortativity) and clustering structure of the ITN are excellently reproduced by the EGM.

**Weighted Structure**

Despite the importance of the topology, the latter is only the backbone over which goods are traded, and the knowledge of the volume of such trade is imperative. The simplest weighted counterpart of the degree sequence is the strength sequence, i.e. the total trade of each country in the case of the ITN. Recent studies have shown that the higher-order binary quantities inferred from the strength sequence, as well as the corresponding weighted quantities, are very different from the observed counterparts [21, 22]. More specifically, the main limitation of models targeting only weighted properties, just like the gravity model, is that of predicting a mostly homogeneous and very dense (sometimes fully connected) topology. Roughly speaking, the models excessively 'dilute' the total trade of each country by distributing it to almost all other countries. This failure in correctly replicating the purely topological projection of the real network is the root of the bad agreement between expected and observed higher-order properties.

In fig. 2.13 we plot some higher-order weighted properties of the ITN as a function of the strength of nodes, for the 2000 snapshot. These properties are the so-called average nearest neighbour strength and the weighted clustering coefficient. For both quantities, we plot the observed values (red points) and the corresponding expected values predicted by the EGM (blue points) and the GM (green points). The exact expressions for both empirical and expected quantities are provided in sec. Materials and Methods. Again, we see that the expected values of the EGM are in very close agreement with the observed properties, as opposed to the classical GM. The bad performance of the traditional GM results directly from the unrealistically expected topology (complete network). This result highlights the importance of added structural information (degree sequence) to the models.

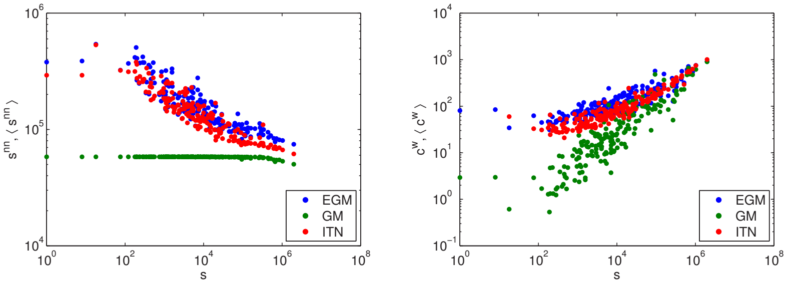


Figure 2.13: **The Enhanced Gravity Model, reconstruction of higher-order weighted properties.** Comparison between the observed weighted properties of the observed ITN (red points), the corresponding expected values of the EGM (blue points) and the gravity model (green points). Left: the average nearest neighbour strength $s^{nn}$ as a function of the observed strength $s$. Right: the weighted clustering coefficient $C^w$ as a function of the strength $s$.

## 2.7 Conclusions

In this chapter, we introduced a novel $GDP$-driven model which successfully reproduces both the binary and weighted properties of the ITN. The model uses the $GDP$ of countries as a sort of macroeconomic fitness, and reveals the existence of strong relations between the $GDP$ and the model parameters controlling the formation and the volume of trade relations. In the light of the limitations of the existing models (most notably the binary-only nature of the fitness model and the weighted-only nature of the gravity model), these results represent a promising step forward in the development of a unified model of the ITN structure. Later, we have introduced the EGM, which further improves these results and expand them by introducing additional macroeconomic parameters like geographic distance, aiming at bridging the gap between network-based and gravity-based approaches to the structure of international trade.

Theoretically, the EGM model originates within a maximum-entropy framework from a simple requirement of topological invariance under a change of money units. The maximum-entropy nature fixes the form of the weight distribution, thus removing an arbitrary ingredient of the original GM. Phenomenologically, the EGM allows us to reconcile two very different approaches that have remained incompatible so far: on one hand, the established GM which successfully reproduces non-zero trade volumes in terms of GDP and distance, while failing in predicting the correct topology [42]; on the other hand, network models which have been successful in reproducing the topology [7] but are more limited in their weighted structure [31]. Empirically, the EGM is the first model that can successfully reproduce the binary and the weighted empirical properties of the ITN simultaneously.

The EGM can be thought of as endowing the standard GM with a novel topologically invariant structure calibrated to replicate the binary properties of the ITN. Just like the standard GM, the EGM can accommodate additional economic factors in terms of extra-dyadic and country-specific properties.

Our results have strong implications for the theoretical foundations of trade models and the resulting policy implications. It is known that the traditional GM is consistent with a number of (possibly conflicting) micro-founded model specifications [64, 65, 66, 67]. For instance, a gravity-like relation can emerge as the equilibrium outcome of models of trade specialization and monopolistic competition with intra-industry trade [69, 70]. The empirical failure of the standard GM, which we ultimately traced back to its lack of topological invariance, implies a previously unrecognized limitation of these micro-founded models (and their policy implications) as well. At the same time, our results suggest a natural way to overcome this limitation via a topologically invariant reformulation of micro-founded models of trade, in such a way that a change in the units of trade volume has no impact on the resulting probability of trade among countries. How the policy implications of a model change as the mere result of this reformulation is an important point in the future research agenda. In general, we envisage the need for a new generation of micro-founded models that are consistent with the EGM. We, therefore, believe that the EGM can represent a novel benchmark supporting improved theories of trade and refined policy scenarios.

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