

# The Cuntz-Pimsner extension and mapping cone exact sequences

Francesca Arici†, Adam Rennie‡

†Institute for Mathematics, Astrophysics and Particle Physics, FNWI,  
Radboud University Nijmegen, Heyendaalseweg 135, 6525AJ Nijmegen, The Netherlands

‡School of Mathematics and Applied Statistics, University of Wollongong,  
Northfields Ave 2522, Australia

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## Abstract

For Cuntz-Pimsner algebras of bi-Hilbertian bimodules of finite Jones-Watatani index satisfying some side conditions, we give an explicit isomorphism between the  $K$ -theory exact sequences of the mapping cone of the inclusion of the coefficient algebra into a Cuntz-Pimsner algebra, and the Cuntz-Pimsner exact sequence. In the process we extend some results by the second author and collaborators from finite projective bimodules to finite index bimodules, and also clarify some aspects of Pimsner's ‘extension of scalars’ construction.

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## 1 Introduction

Mapping cones play an important role in studying the properties of  $KK$ -theory, [9, 17], and have likewise been used to further the study of non-commutative topology and dynamics [8, 19]. The aim of this note is to make explicit, in a specific case, the abstract relationship between extensions of  $C^*$ -algebras and mapping cone extensions.

Motivated by direct calculations with mapping cone and Cuntz-Pimsner exact sequences, as in [2, 3], we investigate the relationship between the defining extension of the Cuntz-Pimsner algebra  $O_E$

$$0 \longrightarrow \text{End}_A^0(F_E) \longrightarrow \mathcal{T}_E \xrightarrow{\pi} O_E \longrightarrow 0, \quad (3.1)$$

(here  $F_E$  is the Fock module,  $\mathcal{T}_E$  the Toeplitz-Pimsner algebra and  $\text{End}_A^0$  denotes the algebra of compact endomorphisms) and the exact sequence of the mapping cone  $M(A, O_E)$  of the inclusion of the coefficient algebra  $A$  into  $O_E$

$$0 \longrightarrow SO_E \longrightarrow M(A, O_E) \longrightarrow A \longrightarrow 0,$$

where  $SO_E$  is the suspension. We show that we can construct an explicit isomorphism of the associated  $K$ -theory sequences at the level of unbounded  $KK$ -cycles.

Abstractly, the existence of such an isomorphism follows from the fact that the  $KK$ -category is a triangulated category whose exact triangles are mapping cone triangles, with isomorphisms given by  $KK$ -equivalence (cf. [17]). Indeed, for every semi-split extension with quotient map  $\pi$ , by

[9], one has an isomorphism of triangles making the extension triangle equivalent to the mapping cone triangle of  $\pi$ , i.e. one has a commutative diagram of triangles where all “vertical” arrows are  $KK$ -equivalences.

More specifically, in the case of Cuntz-Pimsner algebras, when the coefficient algebra  $A$  is nuclear, the defining extension is semi-split, and hence one obtains an isomorphism of the extension triangle with the mapping cone triangle for  $\pi$

$$SO_E \longrightarrow M(\mathcal{T}_E, O_E) \longrightarrow \mathcal{T}_E \xrightarrow{\pi} O_E. \quad (1.1)$$

Using the  $KK$ -equivalence between  $A$  and  $\mathcal{T}_E$  and the natural Morita equivalence between  $A$  and  $\text{End}_A^0(\mathcal{T}_E)$ , one can show that the mapping cone triangle

$$SO_E \longrightarrow M(A, O_E) \longrightarrow A \longrightarrow O_E$$

for the inclusion of the coefficient algebra  $A$  into  $O_E$  is in turn isomorphic to (1.1). This follows from the axioms of a triangulated category which imply that the mapping cone of  $A \rightarrow O_E$  is unique up to a (non-canonical) isomorphism in  $KK$ . Combining the two isomorphisms of triangles, one obtains the isomorphism of exact triangles

$$\begin{array}{ccccccc} SO_E & \longrightarrow & M(A, O_E) & \longrightarrow & A & \longrightarrow & O_E \\ \downarrow = & & \downarrow & & \downarrow \alpha & & \downarrow = \\ SO_E & \longrightarrow & \text{End}_A^0(\mathcal{T}_E) & \longrightarrow & \mathcal{T}_E & \longrightarrow & O_E \end{array}$$

which induces an isomorphism of the corresponding  $KK$ -exact sequences.

In this paper we provide the isomorphism between the associated six-term exact sequences explicitly at the level of unbounded  $KK$ -cycles. This allows one to exploit these mapping cones in concrete computations. We indicate how this works in the case of  $C^*$ -algebras of non-singular graphs.

Many of the constructions we rely on from [11, 21] were proved for finitely generated bimodules over unital algebras. In order to deal with suspensions we extend these results to handle the more general case of bimodules with finite right Watatani index. Our main result is as follows.

**Theorem 6.2** *Let  $E$  be a bi-Hilbertian  $A$ -bimodule of finite right Watatani index, full as a right module with injective left action, and satisfying Assumptions 1 and 2 on pages 8 and 9 respectively. Let  $(O_E, \Xi_A, \mathcal{D})$  be the unbounded representative of the defining extension of  $O_E$ , and  $(M(A, O_E), \hat{\Xi}_A, \hat{\mathcal{D}})$  the lift to the mapping cone. Then*

$$\cdot \otimes_{M(A, O_E)} [(M(A, O_E), \hat{\Xi}_A, \hat{\mathcal{D}})] : K_*(M(A, O_E)) \rightarrow K_*(A)$$

*is an isomorphism that makes diagrams in  $K$ -theory commute. If furthermore the algebra  $A$  belongs to the bootstrap class, the Kasparov product with the class  $[(M(A, O_E), \hat{\Xi}_A, \hat{\mathcal{D}})] \in KK(M(A, O_E), A)$  is a  $KK$ -equivalence. Together with the identity map,  $\cdot \otimes_{M(A, O_E)} [(M(A, O_E), \hat{\Xi}_A, \hat{\mathcal{D}})]$  induces an isomorphism of  $KK$ -theory exact sequences.*

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## 2 Finite index bi-Hilbertian bimodules for non-unital algebras

We start by recalling the basic setup of [21] and [11], and show how it extends to non-unital algebras using more refined constructions from [12]. In [21] and [11], the basic data was a unital separable nuclear  $C^*$ -algebra  $A$ , and a bi-Hilbertian bimodule  $E$  over  $A$  in the sense of [12, Definition 2.3], which is finitely generated and projective for both the right and left module structures.

In this paper we will dispense with the unitality of the algebra  $A$ , and consequently also the finitely generated and projective hypotheses on the module  $E$ . So we will assume throughout the paper that  $E$  is a countably generated bimodule over  $A$ , which carries both left and right  $A$ -valued inner products  ${}_A(\cdot|\cdot)$ ,  $(\cdot|\cdot)_A$  for which the respective actions are injective and adjointable, and  $E$  is complete. The two inner products automatically yield equivalent norms (see, for instance [21, Lemma 2.2]). We write  ${}_AE$  for  $E$  when we wish to emphasise its left module structure and  $E_A$  for  $E$  when emphasising the right module structure.

Thus a bi-Hilbertian bimodule is a special case of a  $C^*$ -correspondence  $(E, \phi)$  over  $A$ , which is a right Hilbert  $A$ -module  $E$  endowed with a  $*$ -homomorphism  $\phi : A \rightarrow \text{End}_A^*(E)$ , where  $\text{End}_A^*(E)$  is the algebra of adjointable operators on  $E$ . For  $x$  and  $y$  in  $E_A$ , we denote the associated rank-one operator by  $\Theta_{x,y} := x(y|\cdot)_A$ . The algebra of compact operators  $\text{End}_A^0(E)$  is the closed linear span of the rank-one operators  $\Theta_{x,y}$ . The algebra  $\text{End}_A^*(E)$  is the multiplier algebra  $\text{Mult}(\text{End}_A^0(E))$  of the compact endomorphisms  $\text{End}_A^0(E)$ .

Since  $E$  is countably generated (as a right module) there are vectors  $\{e_j\}_{j \geq 1} \subset E$  such that

$$\sum_{j \geq 1} \Theta_{e_j, e_j} = \text{Id}_E,$$

where the convergence is in the strict topology of  $\text{End}_A^*(E)$ . Such a collection of vectors is called a frame, and [12, Theorem 2.22] proves that

$$e^\beta := \sum_{j \geq 1} {}_A(e_j|e_j) \tag{2.1}$$

is a well-defined (central positive) element of the multiplier algebra of  $A$  if and only if the left action of  $A$  on  $E$  is by compact endomorphisms. The injectivity of the left action which we assume ensures that  $e^\beta$  is invertible (justifying the notation). Equation (2.1) expresses the finiteness of the right Watatani index of  $E$ , which is then independent of the choice of frame. This finiteness condition seems to be the correct replacement for the finitely generated hypothesis in the unital case, since a module over a unital algebra with finite right Watatani index is finitely generated (and so projective). As further evidence for this, and for later use, we record the following result.

**Proposition 2.1.** *Let  $E$  be a bi-Hilbertian  $A$ -bimodule with finite right Watatani index  $e^\beta$ . Define the suspended bi-Hilbertian  $SA$ -bimodule  $SE$  over the suspension  $SA := C_0(\mathbb{R}) \otimes A$  as follows. Define  $SE := C_0(\mathbb{R}) \otimes E$ , with the operations  $(f_j, g_j \in C_0(\mathbb{R}), a_j \in A, e_j \in E)$*

$$\begin{aligned} (g_1 \otimes a_1) \cdot (f \otimes e) \cdot (g_2 \otimes a_2) &= g_1 f g_2 \otimes a_1 e a_2 \\ (f_1 \otimes e_1 | f_2 \otimes e_2)_{SA} &:= f_1^* f_2 \otimes (e_1 | e_2)_A \\ {}_{SA}(f_1 \otimes e_1 | f_2 \otimes e_2) &= f_1 f_2^* \otimes {}_A(e_1 | e_2). \end{aligned}$$

*Then  $SE$  has finite right Watatani index given by  $1 \otimes e^\beta$  where  $1 \in C_b(\mathbb{R})$  is the constant function with value 1 and  $e^\beta \in \text{Mult}(A)$  is the right Watatani index of  $E$ . If  $E_A$  is full so too is  $SE_{SA}$  and if the left action of  $A$  on  $E$  is injective, so too is the left action of  $SA$  on  $SE$ .*

*Proof.* The proof that  $\mathcal{S}E$  is bi-Hilbertian is a routine check of the conditions, and so too the statements about fullness and injectivity. The right Watatani index must be finite by [12, Theorem 2.22], since  $\mathcal{S}A$  acts by compacts on  $\mathcal{S}E$ , and so it only remains to determine the value of the index.

We let  $\{e_j\}_{j \geq 1}$  be a (countable) frame for  $E$  and pick a partition of unity  $(\phi_k)_{k \in \mathbb{Z}}$  subordinate to the intervals  $(k - \epsilon, 1 + k + \epsilon)$  for some fixed  $0 < \epsilon < 1$ . Then by a direct computation we find that  $(\sqrt{\phi_k} \otimes e_j)_{j,k}$  is a frame for  $\mathcal{S}E_{\mathcal{S}A}$  and similarly that

$$\sum_{j,k} {}_{\mathcal{S}A}(\sqrt{\phi_k} \otimes e_j | \sqrt{\phi_k} \otimes e_j) = 1 \otimes e^\beta \in \text{Mult}(\mathcal{S}A) \simeq C_b(\mathbb{R}) \otimes \text{Mult}(A). \quad \square$$

An important class of examples are the self-Morita equivalence bimodules (SMEBs) over  $A$ . A self-Morita equivalence bimodule is a bi-Hilbertian  $A$ -bimodule for which

$${}_A(e|f)g = e(f|g)_A.$$

We do not require this compatibility condition in the definition of bi-Hilbertian bimodule.<sup>1</sup> We will see in Proposition 3.2 that, upon changing the algebra of scalars, we can always construct a self-Morita equivalence bimodule out of a bi-Hilbertian bimodule. This implies in particular that the Cuntz-Pimsner algebra of a bi-Hilbertian bimodule can always be interpreted as a generalised crossed product in the sense of [1] for a self-Morita equivalence bimodule over a different algebra.

### 3 Cuntz-Pimsner algebras

We start from a bi-Hilbertian  $A$ -bimodule  $E$  with finite right Watatani index. We assume that the left action of  $A$  (which is necessarily by compacts) is also injective, and that the right module  $E_A$  is full. Regarding  $E$  as a right module with a left  $A$ -action by adjointable operators (a correspondence) we can construct the Cuntz-Pimsner algebra  $O_E$ . This we do concretely in the Fock representation. The algebraic Fock module is the algebraic direct sum

$$F_E^{\text{alg}} = \bigoplus_{k \geq 0}^{\text{alg}} E^{\otimes_A k} = \bigoplus_{k=0}^{\text{alg}} E^{\otimes k} = A \oplus E \oplus E^{\otimes 2} \oplus \dots$$

where the copy of  $A$  is the trivial  $A$ -correspondence. The Fock module  $F_E$  is the Hilbert  $C^*$ -module completion of  $F_E^{\text{alg}}$ . For  $\nu \in F_E^{\text{alg}}$ , we define the creation operator  $T_\nu$  by the formula

$$T_\nu(e_1 \otimes \dots \otimes e_k) = \nu \otimes e_1 \otimes \dots \otimes e_k, \quad e_j \in E.$$

The expression  $T_\nu$  extends to an adjointable operator on  $F_E$ , whose adjoint  $T_\nu^*$  acts (when  $\nu$  is homogenous with  $\nu \in E^{\otimes |\nu|}$ ) by

$$T_\nu^*(e_1 \otimes \dots \otimes e_k) = \begin{cases} (\nu | e_1 \otimes \dots \otimes e_{|\nu|})_A \cdot e_{|\nu|+1} \otimes \dots \otimes e_k & k \geq |\nu| \\ 0 & \text{otherwise} \end{cases},$$

and so is called an annihilation operator. The  $C^*$ -algebra generated by the set of creation operators  $\{T_e : e \in E\}$  is the Toeplitz-Pimsner algebra  $\mathcal{T}_E$ . It is straightforward to show that  $\mathcal{T}_E$  contains

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<sup>1</sup>Our notion of Hilbert bimodule is different from the one of [6, Definition 1.8], which was used in [1] in the construction of generalised crossed products.

the algebra  $\text{End}_A^0(F_E)$  of compact endomorphisms on the Fock module as an ideal. The defining extension for the Cuntz-Pimsner algebra  $O_E$  is the short exact sequence

$$0 \longrightarrow \text{End}_A^0(F_E) \longrightarrow \mathcal{T}_E \xrightarrow{\pi} O_E \longrightarrow 0. \quad (3.1)$$

It should be noted that Pimsner [18] in his general construction uses an ideal that in general is smaller than  $\text{End}_A^0(F_E)$ . In our case,  $A$  acts from the left on  $E_A$  by compact endomorphisms, ensuring that Pimsner's ideal coincides with  $\text{End}_A^0(F_E)$ . For  $\nu \in F_E^{\text{alg}}$ , we let  $S_\nu$  denote the class of  $T_\nu$  in  $O_E$ . If  $\nu \in E^{\otimes k}$  we write  $|\nu| := k$ .

Since we assume  $A$  to be separable and nuclear, by [16, Theorem 2.7] (see also [15, Theorem 7.3]) the algebra  $O_E$  is separable and nuclear. By [5, Corollary IV.3.2.5]  $C^*$ -algebra extensions with separable and nuclear quotients are semi-split, hence the defining extension (3.1) is semi-split, i.e. it admits a completely positive cross section  $s : O_E \rightarrow \mathcal{T}_E$ . As a consequence, the above extension will induce six terms exact sequences in  $KK$ -theory.

Using the natural Morita equivalence between  $\text{End}_A^0(F_E)$  and  $A$ , the  $KK$ -equivalence between  $A$  and  $\mathcal{T}_E$  proved in [18, Theorem 4.4] and [18, Lemma 4.7], the six term exact sequences can be simplified to a great extent. Specialising to the case of  $K$ -theory we obtain

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-[E]} & K_0(A) & \xrightarrow{\iota_*} & K_0(O_E) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(O_E) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{1-[E]} & K_1(A) \end{array}$$

where  $\iota_* := \iota_{A, O_E*}$  is the map in  $K$ -theory induced by the inclusion  $\iota_{A, O_E} : A \hookrightarrow O_E$  of the coefficient algebra into the Pimsner algebra and  $1 - [E]$  denotes the Kasparov product  $\cdot \otimes_A ([\text{Id}_{KK(A, A)}] - [E])$ .

Similarly, the corresponding six term exact sequence for  $K$ -homology reads

$$\begin{array}{ccccc} K^0(A) & \xleftarrow{1-[E]} & K^0(A) & \xleftarrow{\iota_*} & K^0(O_E) \\ \partial \downarrow & & & & \uparrow \partial \\ K^1(O_E) & \xrightarrow{\iota_*} & K^1(A) & \xrightarrow{1-[E]} & K^1(A) \end{array}$$

### 3.1 Pimsner's extension of scalars

Before tackling the extension, its  $KK$ -class and the relation to mapping cones, we examine the relationship of the Cuntz-Pimsner construction to the generalised crossed product set up of [1]. Pimsner [18] showed that by changing the scalars the completely positive cross section mentioned above can be obtained explicitly, though this is at the expense of changing the exact sequence (3.1) and the coefficient algebra.

We will recall these constructions, and a little background, with a view to proving that Pimsner's extension of scalars realises  $O_E$  as the Cuntz-Pimsner algebra of a SMEB. While at least some of the content of this statement is folklore, we could find nothing more explicit than Pimsner's original construction in the literature. We provide both a precise statement and proof below.

The formula

$$z \cdot S_\nu := z^{|\nu|} S_\nu, \quad \forall \nu \in E^{\otimes k},$$

is easily seen to extend to a  $U(1)$ -action on  $O_E$ . We denote the fixed point algebra for this action by  $O_E^\gamma$ . Averaging over the circle action defines a conditional expectation

$$\rho : O_E \rightarrow O_E^\gamma, \quad \rho(x) := \int_{U(1)} z \cdot x \, dz,$$

where  $dz$  denotes the normalized Haar measure on  $U(1)$ . The infinitesimal generator of the circle action defines a closed operator  $N$  on the completion  $X_{O_E^\gamma}$  of  $O_E$  as a  $O_E^\gamma$ -Hilbert module in the inner product defined from  $\rho$ . Under the spectral subspace assumption (see [7, Definition 2.2]),  $N$  is a self-adjoint, regular operator with locally compact resolvent whose commutators with  $\{S_\nu : \nu \in F_E^{\text{alg}}\}$  are bounded. In particular,

$$(\mathcal{O}_E, X_{O_E^\gamma}, N) \tag{3.2}$$

defines an unbounded  $(O_E, O_E^\gamma)$ -Kasparov module, where  $\mathcal{O}_E$  is the polynomial algebra in the creation and annihilation operators  $S_e, S_e^*, e \in E_A$ .

With these reminders in place, we turn to the extension of scalars. First, the SMEB case is precisely when we do not need to extend the scalars, for those  $C^*$ -correspondences  $(E, \phi)$  over  $A$  for which  $O_E^\gamma = A$  can be characterised as follows.

**Proposition 3.1** (cf. [15, Proposition 5.18]). *Let  $(E, \phi)$  be a  $C^*$ -correspondence over  $A$  with left action given by compact operators, and let  $O_E$  be the corresponding Pimsner algebra. Then  $E$  is a self-Morita equivalence bimodule if and only if the fixed point algebra  $O_E^\gamma$  coincides with the coefficient algebra  $A$ .*

In general,  $O_E^\gamma$  is substantially larger than  $A$  and the generator of the circle action is insufficient for constructing an unbounded  $(\mathcal{O}_E, A)$ -Kasparov module representing our original extension (3.1).

The unbounded Kasparov module in (3.2) gives a class in  $KK^1(O_E, O_E^\gamma)$ , and when  $E$  is a self-Morita equivalence bimodule, this class represents the extension (3.1), [21]. In the more general case when  $O_E^\gamma \neq A$ , Pimsner considered the right  $O_E^\gamma$ -module  $E' := E \otimes_A O_E^\gamma$ , [18, pp 195-196]. Under some additional assumptions this enlargement of the scalars puts us back into the self-Morita equivalence bimodule case, where Cuntz-Pimsner algebras are known to correspond to the generalised crossed-products of [1] by [14, Theorem 3.7].

**Proposition 3.2.** *Given a correspondence  $(E, \phi)$ , suppose that the module  $E_A$  is full and the left action  $\phi$  is essential<sup>2</sup>. Then the module  $E' := E \otimes_A O_E^\gamma$  is a bi-Hilbertian bimodule over  $O_E^\gamma$  which is left and right full and which satisfies the compatibility condition*

$$_A(\xi|\eta)\zeta = \xi(\eta|\zeta)_A,$$

*hence is a self-Morita equivalence bimodule over  $O_E^\gamma$ . The Cuntz-Pimsner algebra  $O_E \cong O_{E'}$  agrees with the generalised crossed product  $O_E^\gamma \rtimes_{E'} \mathbb{Z}$ .*

We again thank Jens Kaad and Bram Mesland for fruitful discussions that lead to the formulation and proof of this result.

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<sup>2</sup>Recall that the action of  $A$  on  $E$  is said to be *essential* if the linear span of  $\phi(A)E$  is dense in  $E$ .

*Proof.* By its very definition,  $E'$  is a right Hilbert  $O_E^\gamma$  module, with right action and inner product given by the interior tensor product construction. In particular, the right inner  $O_E^\gamma$ -valued product is given by

$$(e_1 \otimes f_1 | e_2 \otimes f_2)_{O_E^\gamma} := f_1^*(e_1 | e_2)_A f_2 = (f_1 | (e_1 | e_2)_A f_2)_{O_E^\gamma}, \quad e_1, e_2 \in E, \quad f_1, f_2 \in O_E^\gamma.$$

If the left action of  $A$  on  $E$  is essential and the right inner product is full, then  $E'$  is a right-full Hilbert  $O_E^\gamma$ -module by the following argument. Using the right fullness of  $E_A$ , [20, Lemma 5.53] shows that there exists a sequence  $y_j \in E$  such that for all  $b \in A$

$$\lim_{k \rightarrow \infty} \sum_{j=0}^k (y_j | y_j)_A b = b,$$

and thus because the left  $A$  action is essential  $\lim_{k \rightarrow \infty} \sum_{j=0}^k (y_j | y_j)_A y = y$  for all  $y \in E$ . Now let  $S_{\mu_1 \dots \mu_n} S_{\nu_1 \dots \nu_n}^* \in O_E^\gamma$ . We want to show that this element of the fixed point algebra  $O_E^\gamma$  can be approximated by inner products. By rewriting the inner product

$$\begin{aligned} (S_{y_j} S_{\nu_1 \dots \nu_n} S_{\mu_1 \dots \mu_n}^* | S_{y_j})_{O_E^\gamma} &= S_{\mu_1 \dots \mu_n} S_{y_j \nu_1 \dots \nu_n}^* S_{y_j} = S_{\mu_1 \dots \mu_n} (S_{y_j}^* S_{y_j} S_{\nu_1 \dots \nu_n})^* \\ &= S_{\mu_1 \dots \mu_n} ((y_j | y_j)_A S_{\nu_1 \dots \nu_n})^* = S_{\mu_1 \dots \mu_n} (S_{(y_j | y_j)_A \nu_1 \dots \nu_n})^*, \end{aligned}$$

we see that

$$\lim_{k \rightarrow \infty} \sum_{j=0}^k (S_{y_j} S_{\nu_1 \dots \nu_n} S_{\mu_1 \dots \mu_n}^* | S_{y_j})_{O_E^\gamma} = \lim_{k \rightarrow \infty} \sum_{j=0}^k S_{\mu_1 \dots \mu_n} (S_{(y_j | y_j)_A \nu_1 \dots \nu_n})^* = S_{\mu_1 \dots \mu_n} S_{\nu_1 \dots \nu_n}^*,$$

and so  $E \otimes_A O_E^\gamma$  is right full.

The non-trivial part is the left module structure. We define a left action  $\tilde{\phi} : O_E^\gamma \rightarrow \text{End}_{O_E^\gamma}^*(E')$  by using the natural inclusion  $E' \hookrightarrow O_E$  given on simple tensors by  $e \otimes f \mapsto S_e \cdot f$ ,  $e \in E$  and  $f \in O_E^\gamma$ . The core  $O_E^\gamma$  is generated by elements of the form  $S_\mu S_\nu^*$ , with  $|\mu| = |\nu| = n$ . Such elements act on simple tensors by

$$\tilde{\phi}(S_\mu S_\nu^*)(e \otimes f) = \mu_1 \otimes (S_{\mu_2 \dots \mu_n} S_{\nu_2 \dots \nu_n}^* (\nu_1 | e)_A f), \quad e \in E, \quad f \in O_E^\gamma,$$

since  $S_{\mu_2 \dots \mu_n} S_{\nu_2 \dots \nu_n}^*$  is again an element of the fixed point algebra  $O_E^\gamma$ .

In order to define a left inner product, we again use the above identification and define

$$_{O_E^\gamma} (e_1 \otimes f_1 | e_2 \otimes f_2) := S_{e_1} f_1 f_2^* S_{e_2}^*.$$

We now show this inner product is left-full. This can be done by choosing a frame  $(x_i)_{i=1}^N$  for  $E_A$  ( $N$  can be infinity). Then

$$\sum_{i=1}^N S_{x_i} S_{x_i}^* S_{\mu_1 \dots \mu_n} S_{\nu_1 \dots \nu_n}^* = S_{\mu_1 \dots \mu_n} S_{\nu_1 \dots \nu_n}^*,$$

and at the same time writing  $\nu = \nu_1 \bar{\nu}$  and  $\mu = \mu_1 \bar{\mu}$  we have

$$S_{x_i} S_{x_i}^* S_{\mu_1 \dots \mu_n} S_{\nu_1 \dots \nu_n}^* = S_{x_i O_E^\gamma} ((x_i | \mu_1)_A | S_{\bar{\nu}} S_{\bar{\mu}}^*) S_{\nu_1}^* = O_E^\gamma (x_i \otimes (x_i | \mu_1)_A | \nu_1 \otimes S_{\bar{\nu}} S_{\bar{\mu}}^*),$$

which shows that the left inner product is full. We conclude by checking the compatibility condition by computing that

$$\begin{aligned} \tilde{\phi} \left( _{O_E^\gamma} (e_1 \otimes f_1 | e_2 \otimes f_2) \right) e_3 \otimes f_3 &= \tilde{\phi}(S_{e_1} f_1 f_2^* S_{e_2}^*) e_3 \otimes f_3 = e_1 \otimes (f_1 f_2^* S_{e_2}^* S_{e_3} f_3) \\ &= e_1 \otimes f_1 (f_2^* S_{e_2}^* S_{e_3} f_3) = e_1 \otimes f_1 (e_2 \otimes f_2 | e_3 \otimes f_3)_{O_E^\gamma}. \quad \square \end{aligned}$$

### 3.2 The extension class

A Kasparov module representing the class of the extension (3.1) was constructed in [21], under the assumption that  $A$  is unital and  $E$  finitely generated, and a further assumption discussed below. Here we recall the salient points, and extend the discussion to handle the non-unital situation.

So we suppose that  $E$  is a bi-Hilbertian  $A$ -bimodule with finite right Watatani index, full as a right module and with injective left action of  $A$ . We choose a frame  $(e_i)_{i \geq 1}$  for  $E_A$ . The frame  $(e_i)_{i \geq 1}$  induces a frame for  $E^{\otimes A^k}$ , namely  $(e_\rho)_{|\rho|=k}$  where  $\rho$  is a multi-index and  $e_\rho = e_{\rho_1} \otimes \cdots \otimes e_{\rho_k}$ . We define

$$\Phi_k : \text{End}_A^{00}(E^{\otimes A^k}) \rightarrow A, \quad \Phi_k(T) = \sum_{|\rho|=k} A(Te_\rho | e_\rho).$$

Here  $\text{End}_A^{00}(E^{\otimes A^k})$  denotes the finite rank operators on  $E^{\otimes A^k}$ . It follows from [12, Lemma 2.16] that  $\Phi_k$  does not depend on the choice of frame and extends to a norm continuous map on  $\text{End}_A^0(E^{\otimes k})$ , [12, Corollary 2.24]. By [12, Proposition 2.27], the functionals  $\Phi_k$  extend to strictly continuous maps  $\Phi_k : \text{End}_A^*(E^{\otimes k}) \rightarrow \text{Mult}(A)$ .

In particular, we denote by  $e^{\beta_k}$  the element  $\Phi_k(\text{Id}_{E^{\otimes A^k}}) = \sum_{|\rho|=k} A(e_\rho | e_\rho) \in \text{Mult}(A)$ . Since  $\Phi_k$  is independent of the choice of frame, so is  $e^{\beta_k}$ . Note that  $e^{\beta_k}$  is a positive, central, invertible element of  $\text{Mult}(A)$ , [12, Corollaries 2.24, 2.28]. Therefore  $\beta_k$  is a well-defined self-adjoint central element in  $\text{Mult}(A)$ .

We further extend the functional  $\Phi_k$  to  $\Phi_k : \text{End}_A^*(F_E) \rightarrow \text{Mult}(A)$  by defining  $\Phi_k(T) := \Phi_k(P_k T P_k)$  for  $T \in \text{End}_A^*(F_E)$ , where  $P_k : F_E \rightarrow E^{\otimes k}$  is the projection. Naively, we would like to define

$$\Phi_\infty(T) \text{ “} := \text{” } \text{res}_{s=1} \sum_{k=0}^{\infty} \Phi_k(T) e^{-\beta_k} (1 + k^2)^{-s/2}, \quad \text{for suitable } T \in \text{End}_A^*(F_E). \quad (3.3)$$

Indeed,  $\Phi_k(T) e^{-\beta_k}$  is easily shown to be bounded, and so it is tempting to try to define  $\Phi_\infty$  using some ‘generalised residue’ in the sense of generalised limits and Dixmier traces. In general, problems arise since  $\Phi_\infty$  (if well-defined) is not a numerical functional, but  $A$ -valued. Worse still, in the non-unital setting we only have the strict continuity of the  $\Phi_k$  in general. The lack of norm continuity is handled as follows.

**Lemma 3.3.** *Suppose that  $T \in \mathcal{T}_E \subset \text{End}_A^*(F_E)$ . Then for each  $k = 0, 1, 2, \dots$ , the compression  $P_k T P_k$  is a compact endomorphism on  $E^{\otimes k}$ , and hence  $\Phi_k : \mathcal{T}_E \rightarrow A$  is norm continuous.*

*Proof.* We approximate  $T \in \mathcal{T}_E$  in norm by a finite sum of generators  $T_\xi T_\eta^*$  for  $\xi, \eta \in F_E$  homogeneous. If  $|\xi| \neq |\eta|$  then  $P_k T_\xi T_\eta^* P_k = 0$ , and so we suppose that  $|\xi| = |\eta|$ .

In that case, for  $k < |\xi|$  we again have  $P_k T_\xi T_\eta^* P_k = 0$ , while for  $k \geq |\xi|$  the endomorphism  $P_k T_\xi T_\eta^* P_k$  coincides with a compact endomorphism of  $E^{\otimes k}$  by [18, Corollary 3.7] and the injectivity of the left action of  $A$ . Since  $P_k T P_k$  is approximated in norm by finite sums of endomorphisms  $P_k T_\xi T_\eta^* P_k$ ,  $P_k T P_k$  is a compact endomorphism of  $E^{\otimes k}$ .  $\square$

Thus for  $\text{Re}(s) > 1$ , since  $\|\Phi_k(T) e^{-\beta_k}\| \leq \|T\|$ , the map

$$\mathcal{T}_E \ni T \mapsto \sum_{k=0}^{\infty} \Phi_k(T) e^{-\beta_k} (1 + k^2)^{-s/2}$$

is norm continuous. The only remaining problem with the tentative definition in Equation (3.3) is the existence of the residue. Following [21], we work under the following assumption guaranteeing that the residue exists for  $T \in \mathcal{T}_E$ .

**Assumption 1.** *We assume that for every  $k \in \mathbb{N}$ , there is a  $\delta > 0$  such that whenever  $\nu \in E^{\otimes k}$  there exists a  $\tilde{\nu} \in E^{\otimes k}$  satisfying*

$$\|e^{-\beta n} \nu e^{\beta n - k} - \tilde{\nu}\| = O(n^{-\delta}), \quad \text{as } n \rightarrow \infty.$$

When Assumption 1 holds, Equation (3.3) defines an  $A$ -bilinear functional  $\Phi_\infty : \mathcal{T}_E \rightarrow A$ , which is a continuous  $A$ -bilinear positive expectation, which in addition vanishes on  $\text{End}_A^0(E) \subset \mathcal{T}_E$ . Hence  $\Phi_\infty$  descends to a positive  $A$ -bilinear expectation  $\Phi_\infty : O_E \rightarrow A$ . The details of this construction can be found in [21, Section 3.2], and the only change in the non-unital case is the norm continuity, which follows from Lemma 3.3. This functional furnishes us with an  $A$ -valued inner product  $(S_1|S_2)_A := \Phi_\infty(S_1^* S_2)$  on  $O_E$ , and the completed module is denoted  $\Xi_A$ .

**We assume that Assumption 1 holds for the remainder of the paper.**

**Theorem 3.4** (Theorem 3.14 of [21]). *If the bi-Hilbertian bimodule  $E$  satisfies Assumption 1, then the tuple  $(O_E, \Xi_A, 2Q - 1)$  is an odd Kasparov module representing the class of the extension (3.1). The projection  $Q$  has range isometrically isomorphic to the Fock module  $F_E$ .*

**Example 3.5.** *When  $E$  is a self-Morita equivalence bimodule,  $\Phi_\infty : O_E \rightarrow A$  coincides with the expectation  $\rho : O_E \rightarrow O_E^\gamma$  discussed prior to Equation (3.2). Therefore*

$$\Xi_A = \bigoplus_{n \in \mathbb{Z}} E^{\otimes n}$$

*with the convention that  $E^{\otimes(-|n|)} = \overline{E}^{\otimes|n|}$ , where  $\overline{E}$  is the conjugate module, which agrees with the  $C^*$ -algebraic dual of  $E$ . In this case we can define the **number operator**  $N$  on the module  $\Xi_A$  by  $N\rho = n\rho$  for  $\rho \in E^{\otimes n}$ . Then  $(O_E, \Xi_A, N)$  is an unbounded Kasparov module representing the class of the extension (3.1) in  $KK^1(O_E, A)$ , by [21, Theorem 3.1].*

In [11], under an additional assumption, Theorem 3.4 was extended, presenting an unbounded representative of the class defined by  $(O_E, \Xi_A, 2Q - 1)$ . In order to construct the unbounded representative  $\mathcal{D}$  we need an additional assumption on the bimodule. Under Assumption 1, we can define the operator  $\mathfrak{q}_k : E^{\otimes k} \rightarrow E^{\otimes k}$  by

$$\mathfrak{q}_k \nu := \tilde{\nu} = \lim_{n \rightarrow \infty} e^{-\beta n} \nu e^{\beta n - k}.$$

By [11, Lemma 2.2], each  $\mathfrak{q}_k$  is adjointable for both module structures, a bimodule map and positive. Then in order to construct  $\mathcal{D}$  we need to assume

**Assumption 2.** *For any  $k$ , we can write  $\mathfrak{q}_k = c_k R_k = R_k c_k$  where  $R_k \in \text{End}_A^*(E^{\otimes k})$  is a projection and  $c_k$  is given by left-multiplication by an element in  $\text{Mult}(A)$ .*

Both Assumptions 1 and 2 hold for a wide variety of examples, as shown in [21] and [11].

When  $A$  is unital and Assumption 2 holds, [11, Theorem 2.10] proves that the module  $\Xi_A$  decomposes as a direct sum of bi-Hilbertian  $A$ -bimodules  $\Xi_{n,r}$  of finite right Watatani index.

To check this in the non-unital case means computing the index directly in terms of the frame for  $\Xi_A$  presented in [11, Lemmas 2.8, 2.9]. The construction of the frame begins with a frame  $\{e_j\}_{j \geq 1}$  for  $E_A$  and a frame  $\{f_k\}_{k \geq 1}$  for  ${}_A E$ , and produces a frame  $\{W_{e_\rho, c_{|\sigma|}^{-1/2} f_\sigma}\}_{\rho, \sigma} \subset \Xi_A$  for multi-indices  $\rho, \sigma$ , and where  $c_{|\sigma|}$  is as in Assumption 2. For fixed values of  $|\rho|, |\sigma|$  we have

$$\begin{aligned}
\sum_{|\rho|=r, |\sigma|=s} A(W_{e_\rho, c_s^{-1/2} f_\sigma} | W_{e_\rho, c_s^{-1/2} f_\sigma}) &= \sum_{|\rho|=r, |\sigma|=s} \Phi_\infty(S_{e_\rho} S_{c_s^{-1/2} f_\sigma}^* S_{c_s^{-1/2} f_\sigma} S_{e_\rho}^*) \\
&= \sum_{|\rho|=r, |\sigma|=s} \Phi_\infty(S_{e_\rho} (c_s^{-1/2} f_\sigma | c_s^{-1/2} f_\sigma)_A S_{e_\rho}^*) \\
&\leq \|c_s^{-1}\| \sum_{|\rho|=r, |\sigma|=s} \Phi_\infty(S_{e_\rho} (f_\sigma | f_\sigma)_A S_{e_\rho}^*) \\
&\leq \|c_s^{-1}\| \sum_{|\rho|=r} \Phi_\infty(S_{e_\rho} \ell_s S_{e_\rho}^*) \leq \|c_s\|^{-1} \ell_s \sum_{|\rho|=r} \Phi_\infty(S_{e_\rho} S_{e_\rho}^*) \\
&= \|c_s\|^{-1} \ell_s e^{\beta_r}
\end{aligned}$$

where  $\ell_s$  is the *left numerical* Watatani index of  $E^{\otimes s}$ , which is finite by [12, Theorem 4.8]. This computation shows (in particular) that the summands  $\Xi_{n,r}$  in the decomposition

$$\Xi_A = \bigoplus_{n \in \mathbb{Z}, r \geq \max\{0, n\}} \Xi_{n,r} \quad (3.4)$$

are bi-Hilbertian  $A$ -bimodules of finite right Watatani index, and we denote the projections onto these sub-modules by  $P_{n,r}$ .

Then one defines  $\mathcal{D} = \sum_{n,r} \psi(n,r) P_{n,r}$  where  $\psi$  is a suitable function, [11, Definition 2.12]. By [11, Lemma 2.14] the projection  $Q$  appearing in Theorem 3.4 has the form

$$Q = \sum_{n=0}^{\infty} P_{n,n},$$

with respect to the above decomposition (3.4).

*Remark 3.6.* Note that in the self-Morita equivalence bimodule case, a suitable choice of the operator  $\mathcal{D}$  along with the decomposition of the module  $\Xi_A$  coincides with the number operator and the decomposition described in Example 3.5.

**We assume that Assumption 2 holds for the remainder of the paper.**

**Theorem 3.7** (Theorem 2.16 of [11]). *If the bi-Hilbertian  $A$ -bimodule  $E$  satisfies Assumptions 1 and 2, then the tuple  $(O_E, \Xi_A, \mathcal{D})$  is an odd unbounded Kasparov module representing the class of the extension (3.1). The spectrum of  $\mathcal{D}$  can be chosen to consist of integers with bi-Hilbertian  $A$ -bimodule eigenspaces of finite right Watatani index, and non-negative spectral projection  $Q$ .*

The only difference arising in the non-unital case is that the resolvent of  $\mathcal{D}$  is not compact, but only locally compact. This follows since, just as in Lemma 3.3, the compression  $P_{m,s} S P_{n,r}$  of  $S \in O_E$  is a compact endomorphism. Since the eigenvalues of  $\mathcal{D}$  are chosen to have  $\pm$ infinity as their only limit points, we find that  $S(1 + \mathcal{D}^2)^{-1/2}$  is a norm convergent sum of compacts.

Our last task before turning to the mapping cone exact sequence is to show that the class of modules we consider is stable under suspension. Proposition 2.1 gives us most of what we want, and we just need to check that if  $E$  satisfies Assumptions 1 and 2 then so too does  $SE$ .

**Proposition 3.8.** *If  $E$  is a bi-Hilbertian  $A$ -bimodule of finite right Watatani index which is full as a right module and with injective left action satisfying Assumptions 1 and 2, then the suspended module  $SE$  is a bi-Hilbertian  $SA$ -bimodule of finite right Watatani index which is full as a right module and with injective left action satisfying Assumptions 1 and 2.*

*Proof.* This follows from Proposition 2.1 and the fact that the right Watatani index of  $(SE)^{\otimes k}$  is  $1 \otimes e^{\beta_k}$  where  $e^{\beta_k}$  is the right Watatani index of  $E^{\otimes k}$ .  $\square$

## 4 Comparing the mapping cone and Cuntz-Pimsner exact sequences

In addition to the defining exact sequence for  $O_E$ , we can look at the mapping cone extension for the inclusion  $\iota_{A,O_E} : A \hookrightarrow O_E$  of the scalars into the Cuntz-Pimsner algebra. Recall that the mapping cone  $M(A, O_E)$  of the inclusion  $\iota_{A,O_E}$  is the  $C^*$ -algebra

$$M(A, O_E) := \{f \in C([0, \infty), O_E) : f(0) \in A, f(\infty) = 0, f \text{ continuous.}\}$$

We will frequently abbreviate  $M(A, O_E)$  to  $M$ . The algebra  $M$  fits into a short exact sequence involving the suspension  $SO_E \simeq C_0((0, \infty), A)$ . Due to the use of the mapping cone, we will often write suspensions as  $C_0((0, \infty)) \otimes \cdot$  instead of  $C_0(\mathbb{R}) \otimes \cdot$ . The sequence is

$$0 \longrightarrow SO_E \xrightarrow{j_*} M(A, O_E) \xrightarrow{\text{ev}} A \longrightarrow 0,$$

where  $\text{ev}(f) = f(0)$  and  $j(g \otimes a)(t) = g(t)a$ . The mapping cone extension is semi-split and induces six term exact sequences in  $KK$ -theory.

Specialising to  $K$ -theory yields the exact sequence

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\partial'} & K_0(SO_E) & \xrightarrow{j_*} & K_1(M) \\ \text{ev}_* \uparrow & & & & \downarrow \text{ev}_* \\ K_0(M) & \xleftarrow{j_*} & K_1(SO_E) & \xleftarrow{\partial'} & K_1(A) \end{array}$$

By [8, Lemma 3.1] the boundary map  $\partial' : K_j(A) \rightarrow K_{j+1}(O_E)$  is given, up to the Bott map  $\text{Bott} : K_j(O_E) \rightarrow K_{j+1}(SO_E)$ , by minus the inclusion of  $A$  in  $O_E$ , i.e.  $\partial' = -\text{Bott} \circ \iota_{A,O_E*}$ . Similar considerations hold for the dual  $K$ -homology exact sequence.

We now compare the defining short exact sequence for  $O_E$  and the mapping cone sequence for the inclusion  $\iota_{A,O_E} : A \hookrightarrow O_E$ . To do so, we use the identification  $\text{Bott} : K_j(O_E) \rightarrow K_{j+1}(SO_E)$  to define a map  $j_*^B : K_i(O_E) \rightarrow K_{i+1}(M)$  given by  $j_* \circ \text{Bott}$ . Then we have the partial comparison with two out of three maps given by the identity:

$$\begin{array}{ccccccccccccccc} \cdots & \xrightarrow{\iota_*} & K_0(O_E) & \xrightarrow{j_*^B} & K_1(M) & \xrightarrow{\text{ev}_*} & K_1(A) & \xrightarrow{\iota_*} & K_1(O_E) & \xrightarrow{j_*^B} & K_0(M) & \xrightarrow{\text{ev}_*} & K_0(A) & \xrightarrow{\iota_*} & \cdots \\ & & \downarrow = & & \downarrow ? & & \downarrow = & & \downarrow = & & \downarrow ? & & \downarrow = & & \\ \cdots & \xrightarrow{\iota_*} & K_0(O_E) & \xrightarrow{\partial} & K_1(A) & \xrightarrow{1-[E]} & K_1(A) & \xrightarrow{\iota_*} & K_1(O_E) & \xrightarrow{\partial} & K_0(A) & \xrightarrow{1-[E]} & K_0(A) & \xrightarrow{\iota_*} & \cdots \end{array}$$

Thus the question we seek to address is whether there is a map that can be put in place of  $?$  which makes the diagram commute (and so provide an isomorphism of six-term sequences).

*Remark 4.1.* As pointed out in the introduction, the existence of an isomorphism between the two exact sequences follows from the fact that the  $KK$ -category is triangulated, with exact triangles the mapping cone triangles. The missing map can be easily constructed as a Kasparov product with the class

$$[\tilde{\alpha}] \otimes_{M_\pi} [u] \otimes_{\text{End}_A^0(\mathcal{F}_E)} [\mathcal{F}_E] \in KK(M, A), \quad (4.1)$$

where  $M_\pi$  denotes the mapping cone  $M(\mathcal{T}_E, O_E)$ ,  $\tilde{\alpha} : M(A, O_E) \rightarrow M(\mathcal{T}_E, O_E)$  is the inclusion of mapping cones induced by the natural inclusion  $\alpha : A \rightarrow \mathcal{T}_E$ ,  $[\mathcal{F}_E] \in KK(\text{End}_A^0(\mathcal{F}_E), A)$  is the class of the Morita equivalence, and  $[u] \in KK(M(\mathcal{T}_E, O_E), \text{End}_A^0(\mathcal{F}_E))$  is the  $KK$ -equivalence given by [9, Corollary 2.4].

In the following we will provide an unbounded representative for a class that makes diagrams in  $K$ -theory commute, by lifting the unbounded representative of the extension class to the mapping cone, as we describe below. The axioms of triangulated categories do not guarantee the uniqueness of such a class, hence we leave it as an open problem to verify that our unbounded Kasparov module is a representative for the class in (4.1)

The map  $\partial$  is implemented by the Kasparov product with the class of the defining extension. Now we are working under Assumptions 1 and 2, and so we have an explicit unbounded representative  $(\mathcal{O}_E, \Xi_A, \mathcal{D})$  for the defining extension. As noted earlier,  $\mathcal{D}$  has discrete spectrum and commutes with the left action of  $A$ , hence we have  $\iota_{A, O_E}^*[(\mathcal{O}_E, \Xi_A, \mathcal{D})] = 0$ . In particular there is a class  $[\hat{\mathcal{D}}] \in KK(M(A, O_E), A)$  such that  $j^{B*}[\hat{\mathcal{D}}] = [(\mathcal{O}_E, \Xi_A, \mathcal{D})]$ . As the notation suggests, there is an explicit unbounded representative for the class  $[\hat{\mathcal{D}}]$ , provided by the main result of [8]. Subject to some further hypotheses, the class  $[\hat{\mathcal{D}}]$  can be used to help compute index pairings, [8, Theorem 5.1], because of the explicit unbounded representative. The even unbounded Kasparov module representing the class  $[\hat{\mathcal{D}}]$  is denoted

$$(M(A, O_E), \hat{\Xi}_A = X \oplus X^\sim, \hat{\mathcal{D}}). \quad (4.2)$$

The module  $X$  is a completion of  $L^2([0, \infty)) \otimes \Xi_A$  while  $X^\sim$  also contains functions with a limit at infinity. The operator is given by

$$\hat{\mathcal{D}} = \begin{pmatrix} 0 & -\partial_t + \mathcal{D} \\ \partial_t + \mathcal{D} & 0 \end{pmatrix},$$

together with suitable APS-type boundary conditions, [8, Section 4.1]. The details will not influence the following discussion, but we stress that the operator is concrete, and so index pairings are explicitly computable.

Trying  $\cdot \hat{\otimes}_M \hat{\mathcal{D}}$  in place of  $?$  we find that the squares to the left of each instance of  $\hat{\mathcal{D}}$  in the diagram

$$\begin{array}{ccccccccccccccc} \cdots & \xrightarrow{\iota_*} & K_0(O_E) & \xrightarrow{j_*^B} & K_1(M) & \xrightarrow{\text{ev}_*} & K_1(A) & \xrightarrow{\iota_*} & K_1(O_E) & \xrightarrow{j_*^B} & K_0(M) & \xrightarrow{\text{ev}_*} & K_0(A) & \xrightarrow{\iota_*} & \cdots \\ & & \downarrow = & & \downarrow \cdot \hat{\otimes} [\hat{\mathcal{D}}] & & \downarrow = & & \downarrow = & & \downarrow \cdot \hat{\otimes} [\hat{\mathcal{D}}] & & \downarrow = & & \\ \cdots & \xrightarrow{\iota_*} & K_0(O_E) & \xrightarrow{\partial} & K_1(A) & \xrightarrow{1-[E]} & K_1(A) & \xrightarrow{\iota_*} & K_1(O_E) & \xrightarrow{\partial} & K_0(A) & \xrightarrow{1-[E]} & K_0(A) & \xrightarrow{\iota_*} & \cdots \end{array} \quad (4.3)$$

commute. Now what about the squares to the right?

## 5 The $K$ -theory of the mapping cone of a Cuntz-Pimsner algebra

We use the characterisation of the  $K$ -theory group  $K_0(M)$  due to [19]. Classes in  $K_0(M)$  can be realised as (stable homotopy classes of) partial isometries  $v \in M_k(O_E)$  with range and source projections  $vv^*, v^*v \in M_k(A)$ . In the usual projection picture, the class of the partial isometry  $v$  corresponds to the class [8, Section 5]

$$[e_v] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right], \quad e_v(t) = \begin{pmatrix} 1 - \frac{1}{1+t^2}vv^* & \frac{-it}{1+t^2}v \\ \frac{it}{1+t^2}v^* & \frac{1}{1+t^2}v^*v \end{pmatrix}.$$

It is important to note for later use that this characterisation of  $K_0(M)$  does not depend on having unital algebras: for  $A \subset B$  not necessarily unital, we need only consider partial isometries  $v \in M_k(\tilde{B})$  over the unitisation  $\tilde{B}$  with  $vv^*, v^*v \in M_k(\tilde{A})$  satisfying  $vv^* - v^*v \in M_k(A)$  or even more generally,  $[vv^*] - [v^*v] \in K_0(A)$ . In the following discussion one can just replace  $v$  over  $O_E$  with  $v$  over  $\widehat{O_E}$  satisfying  $[v^*v] - [vv^*] \in K_0(A)$ , and even take  $v \in \widetilde{\mathcal{K} \otimes O_E}$ .

Returning to the exact sequence, we again let  $v$  be a partial isometry over  $O_E$ , say  $v \in M_k(O_E)$ , with  $v^*v$  and  $vv^*$  projections over  $\iota_{A,O_E}(A)$ . Then we have  $\text{ev}_*([v]) = [v^*v] - [vv^*]$ . In the other direction, we need to evaluate the product  $[v] \otimes_{O_E} [\hat{\mathcal{D}}] \otimes_A ([\text{Id}_{KK(A,A)}] - [E])$ .

Our strategy is to use [8, Theorem 5.1], to find that the latter product is given by

$$- \text{Index}(Q_k v Q_k : v^* v F_E^k \rightarrow vv^* F_E^k) \otimes_A ([\text{Id}_{KK(A,A)}] - [E]), \quad (5.1)$$

where  $Q_k = Q \otimes 1_k$ , and  $Q \Xi_A = F_E$ , the Fock module. Here  $[E]$  is short hand for the class in  $KK(A, A)$  of  $(A, E_A, 0)$ , and similarly  $[\text{Id}_{KK(A,A)}]$  can be represented by  $(A, A_A, 0)$ .

In order to be able to use this formula, we need to check the hypotheses of [8, Theorem 5.1], and then actually compute the product in Equation (5.1). The precise statement of [8, Theorem 5.1] in our case is

**Theorem 5.1.** *Let  $(\mathcal{O}_E, \Xi_A, \mathcal{D})$  be the unbounded Kasparov module for the (pre-)  $C^*$ -algebras  $\mathcal{O}_E \subset O_E$  and  $A$  representing the extension class. Let  $(M, \hat{\Xi}_A, \hat{\mathcal{D}})$  be the unbounded Kasparov  $M(A, O_E)$ - $A$  module of Equation (4.2). Then for any unitary  $u \in M_k(A)$  such that  $Q_k$  and the projection  $(\ker \mathcal{D}) \otimes \text{Id}_k$  both commute with  $u(\mathcal{D} \otimes \text{Id}_k)u^* =: u\mathcal{D}_k u^*$  and  $u^*\mathcal{D}_k u$  we have the following equality of index pairings with values in  $K_0(A)$ :*

$$\begin{aligned} \langle [u], [(\mathcal{O}_E, \Xi_A, \mathcal{D})] \rangle &:= \text{Index}(Q_k u^* Q_k) = \text{Index}(e_u(\hat{\mathcal{D}}_k \otimes 1_2)e_u) - \text{Index}(\hat{\mathcal{D}}_k) \\ &=: \left\langle [e_u] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right], [(M, \hat{\Xi}_A, \hat{\mathcal{D}})] \right\rangle \in K_0(A). \end{aligned}$$

Moreover, if  $v$  is a partial isometry,  $v \in M_k(\mathcal{O}_E)$ , with  $vv^*, v^*v \in M_k(A)$  and such that  $Q_k$  and  $(\ker \mathcal{D})_k$  both commute with  $v\mathcal{D}_k v^*$  and  $v^*\mathcal{D}_k v$  we have

$$\left\langle [e_v] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right], [(M, \hat{\Xi}_A, \hat{\mathcal{D}})] \right\rangle = -\text{Index}(Q_k v Q_k : v^* v F_E^k \rightarrow vv^* F_E^k) \in K_0(A).$$

It is important to observe that when we consider  $v\mathcal{D}_k v^*$ , we are suppressing the representation of  $v$  on  $\Xi_A^k$ , but it makes a difference in what follows. For this reason we temporarily introduce the notation  $\varphi : O_E \rightarrow \text{End}_A^*(\Xi_A)$  for the representation. This representation naturally extends to a

representation  $\varphi_k : M_k(O_E) \rightarrow \text{End}_A^*(\oplus_{i=1}^k \Xi_A)$ , and to  $\mathcal{K} \otimes O_E \rightarrow \text{End}_A^*(\mathcal{H} \otimes \Xi_A)$  in the countably generated case.

The next result, that we state here in the particular case of Cuntz-Pimsner algebras of bimodules satisfying Assumptions 1 and 2, holds in general for any representation of an algebra on a bimodule, for which there exists a decomposition of the type in Equation (3.4).

**Lemma 5.2.** *Given  $v \in M_k(O_E)$  or  $\mathcal{K} \otimes O_E$ , define*

$$v_{m,s} := \sum_{n \in \mathbb{Z}, r \geq \max\{0, n\}} P_{n+m, r+s} \varphi_k(v) P_{n,r}.$$

*Then  $\varphi_k(v) = \sum_{m,s} v_{m,s}$  where the sum converges strictly. If  $v$  is a partial isometry with range and source projections in  $A$ , the  $v_{n,r}$  are partial isometries with  $v_{n,r}^* v_{m,s} = \delta_{n,m} \delta_{r,s} v_{n,r}^* v_{n,r}$  and  $v_{n,r} v_{m,s}^* = \delta_{n,m} \delta_{r,s} v_{n,r} v_{n,r}^*$ . Hence the projections  $v_{n,r}^* v_{n,r}$  are pairwise orthogonal, and likewise the projections  $v_{n,r} v_{n,r}^*$  are pairwise orthogonal.*

*Proof.* The first statement follows from the definition of  $v_{m,s}$ , the orthogonal decomposition  $\Xi_A = \oplus \Xi_{n,r}$  together with the fact that  $\sum_{n,r} P_{n,r}$  converges to  $\text{Id}_{\Xi_A}$  strictly.

Now suppose that we have  $v \in O_E$  a partial isometry with range and source projections in  $A$  (the following argument adapts to partial isometries in matrix algebras). Then we have

$$\varphi(v) = \sum v_{m,s}.$$

Since  $vv^*$  and  $v^*v$  are matrices over  $A$ , we see, in particular, that they commute with  $\mathcal{D}$ . Hence

$$vv^* = \sum v_{m,s} v_{n,r}^* \in M_N(A) \Rightarrow v_{m,s} v_{n,r}^* = \delta_{m,n} \delta_{s,r} v_{m,s} v_{m,s}^*.$$

Similarly  $v_{m,s}^* v_{n,r} = \delta_{m,n} \delta_{s,r} v_{m,s}^* v_{m,s}$ . Now we recall that  $vv^*$  is a projection and consider

$$vv^* = \sum v_{m,s} v_{m,s}^* = (vv^*)^2 = \left( \sum v_{m,s} v_{m,s}^* \right)^2 = \sum v_{m,s} v_{m,s}^* v_{n,r} v_{n,r}^* = \sum (v_{m,s} v_{m,s}^*)^2$$

the last equality following from  $v_{m,s}^* v_{n,r} = \delta_{m,n} \delta_{s,r} v_{m,s}^* v_{m,s}$ . Since  $v_{m,s} v_{m,s}^* v_{n,r} v_{n,r}^* = 0$  for  $(m,s) \neq (n,r)$ , we see that each  $v_{m,s} v_{m,s}^*$  is a projection over  $A$ , and the various  $v_{m,s} v_{m,s}^*$  are mutually orthogonal. Similarly, the  $v_{m,s}^* v_{m,s}$  form a set of mutually orthogonal projections.  $\square$

We deduce the commutation relation  $v_{m,s} P_{l,t} = P_{l+m, t+s} v_{m,s}$  for all  $l, m \in \mathbb{Z}$ ,  $t \geq \max\{0, l\}$ ,  $s \geq \max\{0, m\}$ . This seems surprising given the more complicated commutation relation of [11, Lemma 2.15], but they are reconciled by the following observation (proved in the Lemma below). If  $\mu \in F_E$  is homogenous of degree  $|\mu|$  then for  $n$  sufficiently large and positive

$$P_{n+|\mu|, n+|\mu|} S_\mu P_{n,n} \neq 0.$$

Hence  $S_\mu = \sum_{j=0}^{|\mu|} (S_\mu)_{|\mu|, j}$  has  $S_{\mu, |\mu|} \neq 0$ , and we see that the decomposition in the Lemma uses much more information than just the degree given by the gauge action.

**Lemma 5.3.** *Suppose that  $S \in O_E$  satisfies  $S_{n,r} \neq 0$  for some  $n \in \mathbb{Z}$  and  $r \geq \max\{0, n\}$ . Then  $S_{n,n} \neq 0$ .*

*Proof.* We approximate  $S$  by a finite sum of monomials  $S_\alpha S_\beta^*$ . Then  $S_{n,r}$  is approximated by monomials  $S_\alpha S_\beta^*$  with  $|\alpha| = r$  and  $|\alpha| - |\beta| = n$ .

For such monomials, and  $m > |\beta|$ , we have  $S_\alpha S_\beta^* P_{m,m} \Xi_A \subset P_{m+|\alpha|-|\beta|, m+|\alpha|-|\beta|} S_\alpha S_\beta^* \Xi_A$ , and by considering  $[S_\beta S_\gamma] \in \Xi_A$  we see that  $S_\alpha S_\beta^* P_{m,m} \neq 0$ . Hence  $P_{m+|\alpha|-|\beta|, m+|\alpha|-|\beta|} S_\alpha S_\beta^* P_{m,m} \neq 0$  for  $m > |\beta|$ . Hence  $(S_\alpha S_\beta^*)_{|\alpha|-|\beta|, |\alpha|-|\beta|} = (S_\alpha S_\beta^*)_{n,n} \neq 0$  and so also  $S_{n,n} \neq 0$ .  $\square$

**Lemma 5.4.** *Let  $v \in O_E$  (or  $\widetilde{O}_E$ ) be a partial isometry with range and source projections in  $\widetilde{A}$ . Then  $\varphi(v)$  is a finite sum of ‘homogenous’ components  $v_{m,s}$ .*

*Remark 5.5.* We are effectively repeating the argument of [7, Lemmas 4.4 and 4.5] for modular unitaries and partial isometries. We know that  $e^{it\mathcal{D}} \varphi(S) e^{-it\mathcal{D}} \in \varphi(O_E)$  for partial isometries  $S$  with range and source in  $A$ , and that is all we will need.

*Proof.* We first suppose that in fact  $v$  is unitary, and define  $w_t = \varphi(v^*) e^{it\mathcal{D}} \varphi(v) e^{-it\mathcal{D}}$ . It follows from Lemma 5.2 that  $w_t$  commutes (strongly) with  $\mathcal{D}$  for all  $t$ . Then

$$\begin{aligned} w_{t+s} &= \varphi(v^*) e^{i(t+s)\mathcal{D}} \varphi(v) e^{-i(t+s)\mathcal{D}} \\ &= \varphi(v^*) e^{it\mathcal{D}} \varphi(v) e^{-it\mathcal{D}} e^{is\mathcal{D}} \varphi(v^*) e^{-is\mathcal{D}} e^{i(t+s)\mathcal{D}} \varphi(v) e^{-i(t+s)\mathcal{D}} \\ &= w_t e^{it\mathcal{D}} (\varphi(v^*) e^{is\mathcal{D}} \varphi(v) e^{-is\mathcal{D}}) e^{-it\mathcal{D}} \\ &= w_t e^{it\mathcal{D}} w_s e^{-it\mathcal{D}} \\ &= w_t w_s. \end{aligned}$$

Hence  $w_t$  is a norm continuous path of unitaries in  $A$ , whence  $w_t = e^{ita}$  for some  $a = a^* \in A$ . Thus  $e^{it\mathcal{D}} \varphi(v) e^{-it\mathcal{D}} = \varphi(v) e^{ita}$ . Recall now that we can choose  $\mathcal{D}$  to have only integral eigenvalues, and so  $\varphi(v) = \varphi(v) e^{i2\pi a}$ . Hence  $a$  has spectrum a finite subset of  $\mathbb{Z}$ , and we then easily see that only finitely many components  $v_{m,s}$  can be non-zero. In the general case we replace  $v$  by the unitary

$$\begin{pmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{pmatrix}$$

and argue as above.  $\square$

**Lemma 5.6.** *For any partial isometry  $v$  over  $\widetilde{O}_E$  with range and source projections in  $\widetilde{A}$ , the operators  $\varphi(v)\mathcal{D}\varphi(v^*)$  and  $\varphi(v^*)\mathcal{D}\varphi(v)$  commute with both the kernel projection of  $\mathcal{D}$  and the non-negative spectral projection of  $\mathcal{D}$ , given by  $Q$ .*

*Proof.* We assume for simplicity that  $v \in \widetilde{O}_E$ . Using Lemmas 5.2 and 5.4, we see that the following computation is justified and yields the first claim:

$$\begin{aligned} v\mathcal{D}v^* &= \sum_{m,s} v P_{m,s} \psi(m,s) v^* = \sum_{m,s,n,r} v_{n,r} P_{m,s} \psi(m,s) v^* = \sum_{m,s,n,r} P_{m+n,s+r} \psi(m,s) v_{n,r} v^* \\ &= \sum_{m,s,n,r} P_{m+n,s+r} \psi(m,s) v_{n,r} v_{n,r}^*. \end{aligned}$$

The claims about  $v^*\mathcal{D}v$  follow in the same way.  $\square$

## 6 The isomorphism $K_*(M(A, O_E)) \rightarrow K_*(A)$ and the $KK$ -equivalence

To prove that  $\cdot \otimes_M \hat{D} : K_*(M(A, O_E)) \rightarrow K_*(A)$  gives an isomorphism we need only show that  $\cdot \otimes_M \hat{D}$  makes the diagram commute, since it then follows from the five lemma that taking the Kasparov product with  $\hat{D}$  is an isomorphism.

To prove that  $\cdot \otimes_M \hat{D} : K_*(M(A, O_E)) \rightarrow K_*(A)$  yields a  $KK$ -equivalence requires much more in general, but follows relatively easily in the bootstrap case. We go further, and provide an explicit inverse when it exists, and conjecture that in fact it is an inverse in all generality.

### 6.1 The isomorphism in $K$ -theory

We now know enough to prove the commutation of the diagram in Equation (4.3). We first consider  $\cdot \otimes_M [\hat{D}] : K_0(M(A, O_E)) \rightarrow K_0(A)$ , and in this situation begin by considering  $v = v_{m,s}$  ‘homogenous’.

By [11, Lemma 2.14], the range of  $Q$  is the range of  $\sum_{n \geq 0} P_{n,n}$ , hence  $QvQ = 0$  unless  $s = m$ . For  $s = m$  we have

$$\begin{aligned} & \text{Index}(QvQ : v^*vF_E \rightarrow vv^*F_E) \otimes_A (\text{Id}_{KK(A,A)} - [E]) \\ &= \begin{cases} \left( [\oplus_{j=0}^{-m-1} v^*vE^{\otimes j}] \right) \otimes_A ([\text{Id}_{KK(A,A)}] - [E]) & m < 0 \\ \left( -[\oplus_{j=0}^{m-1} vv^*E^{\otimes j}] \right) \otimes_A ([\text{Id}_{KK(A,A)}] - [E]) & m \geq 0 \end{cases} \\ &= \begin{cases} [v^*vA] - [v^*vE^{\otimes -m}] & m < 0 \\ -[vv^*A] + [vv^*E^{\otimes m}] & m \geq 0 \end{cases} \end{aligned}$$

where the last equality follows from a telescopic argument. So to prove that

$$\text{Index}(QvQ : v^*vF_E \rightarrow vv^*F_E) \otimes_A (\text{Id}_{KK(A,A)} - [E]) = \text{ev}_*([v]) = [v^*v] - [vv^*]$$

we are reduced to proving the isomorphisms of  $A$ -modules

$$vv^*E^{\otimes m} \simeq v^*vA \quad \text{for } m > 0 \quad \text{and} \quad v^*vE^{\otimes |m|} \simeq vv^*A \quad \text{for } m < 0.$$

This is straightforward though, by the following argument.

For  $m < 0$ , the map  $v : v^*vE^{\otimes |m|} \rightarrow vE^{\otimes |m|} \subset A$  is a one-to-one  $A$ -module map, which is onto its image, which is contained in  $vv^*A$ . Hence  $v^*vE^{\otimes |m|}$  and  $vE^{\otimes |m|}$  are isomorphic.

For  $m > 0$ , the map  $v^* : vv^*E^{\otimes |m|} \rightarrow v^*E^{\otimes |m|} \subset A$  is a one-to-one  $A$ -module map, which is onto its image, which is contained in  $v^*vA$ . Hence  $vv^*E^{\otimes |m|}$  and  $v^*E^{\otimes |m|}$  are isomorphic.

Thus the result is true for homogenous partial isometries, and likewise for direct sums of homogenous partial isometries, and by Lemma 5.4 this is enough. This gives commutativity of the diagram

$$\begin{array}{ccc} K_0(M) & \xrightarrow{\text{ev}_*} & K_0(A) \\ \downarrow \cdot \otimes [\hat{D}] & & \downarrow = \\ K_0(A) & \xrightarrow{\cdot \otimes (\text{Id}_A - [E])} & K_0(A) \end{array}$$

and hence an isomorphism  $\cdot \otimes_M [\hat{\mathcal{D}}] : K_0(M(A, O_E)) \rightarrow K_0(A)$ . To complete the argument, we need to consider suspensions.

If  $f \in \mathcal{SM}(A, O_E)$  we let  $f(t) = g_t$  with  $g_t \in M(A, O_E)$  for all  $t \in \mathbb{R}$ . Then define

$$\Psi : \mathcal{SM}(A, O_E) \rightarrow M(\mathcal{S}A, \mathcal{S}O_E), \quad (\Psi(f)(s))(t) = g_t(s), \quad s \in [0, \infty), \quad t \in \mathbb{R},$$

and check that  $\Psi$  is an isomorphism. Hence, in particular,  $K_1(M(A, O_E)) \cong K_0(M(\mathcal{S}A, \mathcal{S}O_E))$ .

Next we observe that  $O_{\mathcal{S}E} \cong \mathcal{S}O_E$ . The isomorphism is defined on generators by  $\varphi(S_{f \otimes e}) = f \otimes S_e$ , and using the gauge invariant uniqueness theorem, as in [15, Theorem 6.4], we see that the map is injective, and then since the range contains the generators of  $\mathcal{S}O_E$ , it is an isomorphism.

The unitary isomorphism of Kasparov modules  $(\mathcal{S}A, \mathcal{S}E_{\mathcal{S}A}, 0) = (C_0(\mathbb{R}), C_0(\mathbb{R})_{C_0(\mathbb{R})}, 0) \otimes_{\mathbb{C}} (A, E_A, 0)$  shows that the suspension of the map  $\cdot \otimes_A ((A, A_A, 0) - (A, E_A, 0))$  is the map  $\cdot \otimes_{\mathcal{S}A} ((\mathcal{S}A, \mathcal{S}A_{\mathcal{S}A}, 0) - (\mathcal{S}A, \mathcal{S}E_{\mathcal{S}A}, 0))$ . A similar but easier statement holds for the suspension of the evaluation map, and so combining these various facts we find that the diagram

$$\begin{array}{ccc} K_1(M(A, O_E)) & \xrightarrow{\text{sev}_*} & K_1(A) \\ \downarrow \cdot \otimes [\hat{\mathcal{D}}] & & \downarrow = \\ K_1(A) & \xrightarrow{\mathcal{S} \otimes (\text{Id}_A - [E])} & K_1(A) \end{array}$$

is given by

$$\begin{array}{ccc} K_0(M(\mathcal{S}A, \mathcal{S}O_E)) & \xrightarrow{\text{ev}_*} & K_0(\mathcal{S}A), \\ \downarrow \cdot \otimes [\hat{\mathcal{D}}] & & \downarrow = \\ K_0(\mathcal{S}A) & \xrightarrow{\otimes (\text{Id}_{\mathcal{S}A} - [\mathcal{S}E])} & K_0(\mathcal{S}A) \end{array}$$

where now  $\text{ev}_*$  is the evaluation map corresponding to the inclusion of  $\mathcal{S}A$  into  $\mathcal{S}O_E$ . Now by Propositions 2.1 and 3.8,  $\mathcal{S}E$  satisfies all the assumptions that  $E$  does. Thus our proof that the ‘even part’ of the diagram commutes now holds verbatim to show that the ‘odd part’ of the diagram commutes.

## 6.2 The $KK$ -equivalence and the main theorem

We conclude by showing that the class of  $[\hat{\mathcal{D}}]$  not only implements an isomorphism in  $K$ -theory, but an actual  $KK$ -equivalence when  $A$  is in the bootstrap class.

First observe that by [4, Proposition 23.10.1], if  $A, B$  are two  $C^*$ -algebras in the bootstrap class, then  $\alpha \in KK(A, B)$  is a  $KK$ -equivalence if and only if the induced map  $\cdot \otimes_A \alpha : K_*(A) \rightarrow K_*(B)$  is invertible. This follows from the Universal Coefficient Theorem of [22].

Next, whenever the coefficient algebra  $A$  of the correspondence  $E_A$  belongs to the bootstrap class, so does the algebra  $O_E$  (cf. [15, Proposition 8.8]), and we obtain a  $KK$ -equivalence in this case.

Hence, provided that the coefficient algebra is contained in the bootstrap class, the class  $[\hat{\mathcal{D}}] \in KK(O_E, A)$  is a  $KK$ -equivalence. The problem with this approach is two-fold. On the one hand we need to know or verify that the coefficient algebra is in the bootstrap class. While this is often possible, knowing that  $\hat{\mathcal{D}}$  is a  $KK$ -equivalence does not provide a representative of the other half of the equivalence.

We ameliorate both these problems by providing an explicit representative for the other half of the  $KK$ -equivalence, when it is one.

In our particular situation, we can choose a countable frame  $\{e_i\}_{i \geq 1}$  for the right module  $E_A$ , and define the (possibly infinite) matrix over  $\widetilde{O_E}$  by

$$w = \begin{pmatrix} S_{e_1}^* & 0 & \cdots & 0 \\ S_{e_2}^* & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$

Then  $w^*w = \text{Id}_{\widetilde{O_E}} \oplus 0_\infty = \iota_{A, O_E}(\text{Id}_{\widetilde{A}}) \oplus 0_\infty$  and

$$ww^* = \begin{pmatrix} (e_1|e_1)_A & (e_1|e_2)_A & \cdots \\ (e_2|e_1)_A & (e_2|e_2)_A & \cdots \\ \vdots & \ddots & \ddots \end{pmatrix} = (e_i|e_j)_{i,j \geq 1} =: p_E \in M_\infty(A).$$

When the left action of  $A$  on  $E$  is injective and  $E$  has finite right Watatani index, the projection  $p_E$  lies in  $\iota_{A, O_E}(\widetilde{A})$ , and so  $w$  defines a class in  $K_0(M)$ . We will prove this fact below.

We can explicitly realise  $[w]$  as a difference of classes of projections over  $\widetilde{M}(A, O_E)$ .<sup>3</sup> Making the identification  $[w] = [p_w] - [1_N]$ , we have

$$p_w(t) = \begin{pmatrix} 1_N - \frac{1}{1+t^2}p_E & \frac{-it}{1+t^2}w \\ \frac{it}{1+t^2}w^* & \frac{1}{1+t^2}\text{Id}_{O_E} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+t^2}(1_N - p_E) + \frac{t^2}{1+t^2}1_N & \frac{-it}{1+t^2}w \\ \frac{it}{1+t^2}w^* & \frac{1}{1+t^2}\text{Id}_{O_E} \end{pmatrix}.$$

The frame  $\{e_i\}_{i \geq 1}$  gives a stabilisation map  $\psi : E \rightarrow H_A = H \otimes A$  (for any separable Hilbert space  $H$ ) by defining  $\psi(e) = ((e_j|e)_A)_{j \geq 1}$ . Since  $E_A$  carries a left action of  $A$ , say  $\phi : A \rightarrow \text{End}_A(E)$ , so too does  $p_E H_A$  with  $\psi \circ \phi(a) \circ \psi^{-1} = ((e_i|\phi(a)e_j)_A)_{i,j}$ .

There are two important features of the representation of  $A$  on  $p_E H_A$ . The first is that  $w$  defines a class in  $K_0(M(A, O_E))$ . Let  $(u_n)_{n \geq 1}$  be an approximate identity for  $A$ , and recall that  $A$  acts injectively and by compacts to see that  $((e_i|\phi(u_n)e_j)_A)_{i,j}$  converges strictly to  $p_E$ . Extending the representation of  $A$  to  $\widetilde{A}$ , we see that  $p_E = \psi \circ \phi(a) \circ \psi^{-1}(\text{Id}_{\widetilde{A}})$ . Thus  $p_E$  is a class over  $\widetilde{A}$ .

The second important feature is the ability to inflate from  $K$ -theory classes to  $KK$ -classes. Since  $w(0_{N-1} \oplus \phi(a))w^* = \psi \circ \phi(a) \circ \psi^{-1}$  and  $w^*\psi \circ \phi(a) \circ \psi^{-1}w = 0_{N-1} \oplus \phi(a)$ , it is straightforward to check that for all  $t \in [0, \infty)$

$$p_w(t) \begin{pmatrix} ((e_i|\phi(a)e_j)_A)_{i,j} & 0 \\ 0 & \phi(a) \end{pmatrix} = \begin{pmatrix} ((e_i|\phi(a)e_j)_A)_{i,j} & 0 \\ 0 & \phi(a) \end{pmatrix} p_w(t)$$

as operators on  $O_E^{2N}$  (or  $A^{2N}$  for  $t = 0$ ). Hence we can enrich the  $K$ -theory class

$$[w] = \left[ \left( \mathbb{C}, \begin{pmatrix} p_w \widetilde{M}(A, O_E)^{2N} \\ \widetilde{M}(A, O_E)^N \end{pmatrix}, 0 \right) \right] \in KK(\mathbb{C}, M(A, O_E))$$

to a class

$$[W] = \left[ \left( A, \begin{pmatrix} p_w \widetilde{M}(A, O_E)^{2N} \\ \widetilde{M}(A, O_E)^N \end{pmatrix}, 0 \right) \right] \in KK(A, M(A, O_E)).$$

---

<sup>3</sup>Here  $\widetilde{M}(A, O_E)$  is the minimal unitisation of  $M(A, O_E)$ . As usual, the equality of the classes of  $p_w(\infty)$  and  $1_N$  gives us classes in the  $KK$  groups for  $M(A, O_E)$ . See [13, Corollary 1, Section 7]

These two classes are related, via the natural inclusion  $\iota_{\mathbb{C},A} : \mathbb{C} \rightarrow A$ , by  $[w] = \iota_{\mathbb{C},A}^*([W])$ .

**Lemma 6.1.** *Let  $[\mathcal{D}] \in KK^1(O_E, A) = KK(SO_E, A)$  be the class of the defining extension for  $O_E$ ,  $(M(A, O_E), \hat{\Xi}_A, \hat{\mathcal{D}})$  the lift to the mapping cone, and  $[W] \in KK(A, M(A, O_E))$  the class defined above. Then*

$$[W] \otimes_{M(A, O_E)} [\hat{\mathcal{D}}] = -\text{Id}_{KK(A, A)}.$$

*Proof.* Applying [8, Theorem 5.1] gives

$$[w] \otimes_{M(A, O_E)} [\hat{\mathcal{D}}] = -\text{Index}(P \otimes 1_N w P \otimes 1_N : w^* w(\Xi)^N \rightarrow w w^*(\Xi)^N)$$

where  $P$  is the non-negative spectral projection of  $\mathcal{D}$ . Since the non-negative spectral projection of  $\mathcal{D}$  is the projection onto a copy of the Fock space, we have

$$\ker(P \otimes 1_N w P \otimes 1_N) = A_A = E_A^{\otimes 0}, \quad \ker(P \otimes 1_N w^* P \otimes 1_N) = \{0\}.$$

We can interpret the index not just as a difference of right  $A$ -modules, but as a difference of  $A$ -bimodules. This works because the left action of  $A$  commutes with  $\mathcal{D}$  and so  $P$ . Hence

$$[W] \otimes_M [\hat{\mathcal{D}}] = -[(A, A_A, 0)] = -\text{Id}_{KK(A, A)}$$

as was to be shown.  $\square$

From Lemma 6.1, we know that  $-\hat{\mathcal{D}} \otimes_A [W] \in KK(M(A, O_E), M(A, O_E))$  is an idempotent element. In particular,  $[\hat{\mathcal{D}}] \otimes_A \cdot$  is always injective and  $[W] \otimes_M \cdot$  is always surjective and injective on the image of  $[\hat{\mathcal{D}}] \otimes_A \cdot$ . Thus as soon as  $[\hat{\mathcal{D}}] \otimes_A \cdot$  is surjective,  $[\hat{\mathcal{D}}]$  is a  $KK$ -equivalence. Similarly, the map  $\cdot \otimes_A [W]$  is always injective and surjective on the range of  $\cdot \otimes_M [\hat{\mathcal{D}}]$ .

One approach to showing that  $[W]$  is in fact an inverse for  $[\hat{\mathcal{D}}]$  would be to show that the diagram

$$\begin{array}{ccccccccccccccc} \cdots & \xrightarrow{\iota_*} & K_0(O_E) & \xrightarrow{j_*^B} & K_1(M) & \xrightarrow{\text{ev}_*} & K_1(A) & \xrightarrow{\iota_*} & K_1(O_E) & \xrightarrow{j_*^B} & K_0(M) & \xrightarrow{\text{ev}_*} & K_0(A) & \xrightarrow{\iota_*} & \cdots \\ & & \uparrow = & & \uparrow \cdot \otimes [W] & & \uparrow = & & \uparrow = & & \uparrow \cdot \otimes [W] & & \uparrow = & & \\ \cdots & \xrightarrow{\iota_*} & K_0(O_E) & \xrightarrow{\partial} & K_1(A) & \xrightarrow{1-[E]} & K_1(A) & \xrightarrow{\iota_*} & K_1(O_E) & \xrightarrow{\partial} & K_0(A) & \xrightarrow{1-[E]} & K_0(A) & \xrightarrow{\iota_*} & \cdots \end{array} \quad (6.1)$$

commutes. The composition  $[W] \otimes_{M(A, O_E)} [\text{ev}]$  is just the module

$$\left[ \left( A, \begin{pmatrix} (1_N - p_E) \tilde{A}^N \\ A \\ \tilde{A}^N \end{pmatrix}, 0 \right) \right]$$

with grading  $(1_N - p_E) \oplus 1 \oplus -1_N$ . So  $[W] \otimes_{M(A, O_E)} [\text{ev}] = [A] - [p_E A^N] = [A] - [E]$ . Thus  $-[W]$  makes one square commute, and we could try to show that the square to the left of  $W$  commutes (up to sign) as well.

This means showing that  $-\mathcal{D} \otimes_A [W] = [j] \in KK(SO_E, M(A, O_E))$ , which is implied by the stronger condition  $-\hat{\mathcal{D}} \otimes_A [W] = \text{Id}_{KK(M, M)} \in KK(M(A, O_E), M(A, O_E))$ . We have not been able to prove this equality in general, and leave it as an open problem.

If  $A$  is in the bootstrap class, then so too is  $O_E$  and so  $M(A, O_E)$ . In this case  $\cdot \otimes_M [\hat{\mathcal{D}}]$  is an isomorphism, hence the map  $\cdot \otimes_A [W]$ , which is always injective on the range of  $\cdot \otimes_M [\hat{\mathcal{D}}]$ , is an isomorphism as well: the inverse of  $\cdot \otimes_M [\hat{\mathcal{D}}]$ .

**Theorem 6.2.** *Let  $E$  be a bi-Hilbertian  $A$ -bimodule of finite right Watatani index, full as a right module with injective left action, and satisfying Assumptions 1 and 2 on pages 8 and 9 respectively. Let  $(O_E, \Xi_A, \mathcal{D})$  be the unbounded representative of the defining extension of  $O_E$ , and  $(M(A, O_E), \hat{\Xi}_A, \hat{\mathcal{D}})$  the lift to the mapping cone. Then*

$$\cdot \otimes_{M(A, O_E)} [(M(A, O_E), \hat{\Xi}_A, \hat{\mathcal{D}})] : K_*(M(A, O_E)) \rightarrow K_*(A)$$

*is an isomorphism that makes diagrams in  $K$ -theory commute. If furthermore the algebra  $A$  belongs to the bootstrap class, the Kasparov product with the class  $[(M(A, O_E), \hat{\Xi}_A, \hat{\mathcal{D}})] \in KK(M(A, O_E), A)$  is a  $KK$ -equivalence. Together with the identity map,  $\cdot \otimes_{M(A, O_E)} [(M(A, O_E), \hat{\Xi}_A, \hat{\mathcal{D}})]$  induces an isomorphism of  $KK$ -theory exact sequences.*

Applying the result to the graph  $C^*$ -algebra of a locally finite directed graph with no sources nor sinks yields a well-known exact sequence for computing the  $K$ -theory. With  $G = (G^0, G^1, r, s)$  the directed graph ( $G^0$ =vertices,  $G^1$ =edges),  $A = C_0(G^0)$  and  $E = C_0(G^1)$  we have  $O_E = C^*(G)$  and we have the exact sequence.

$$0 \longrightarrow K_1(C^*(G)) \longrightarrow K_0(M(A, C^*(G))) \xrightarrow{\text{ev}_*} K_0(A) \longrightarrow K_0(C^*(G)) \longrightarrow 0.$$

Using  $K_0(A) = \oplus^{|G^0|} \mathbb{Z}$ , together with the isomorphism  $K_0(M(A, C^*(G))) \cong K_0(A)$  given by  $\hat{\mathcal{D}}$ , gives

$$0 \longrightarrow K_1(C^*(G)) \longrightarrow \bigoplus^{|G^0|} \mathbb{Z} \xrightarrow{1-V^T} \bigoplus^{|G^0|} \mathbb{Z} \longrightarrow K_0(C^*(G)) \longrightarrow 0.$$

where  $V$  is the vertex matrix of the graph  $G$ , given by

$$V(i, j) := \{e \in G^1 : s(e) = v_i, r(e) = v_j\}.$$

Similarly, since  $A = C_0(G^0)$  is in the bootstrap class, in  $K$ -homology we find

$$0 \longrightarrow K^0(C^*(G)) \longrightarrow \prod^{|G^0|} \mathbb{Z} \xrightarrow{1-V} \prod^{|G^0|} \mathbb{Z} \longrightarrow K^1(C^*(G)) \longrightarrow 0,$$

and these results recapture the results of [10] for non-singular graphs.

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