Hydrodynamics and the quantum butterfly effect in black holes and large N quantum field theories

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3 Towards the Quantum Critical Point

In weakly coupled or large N QFTs, we show the existence of an analytical relation between the *off-shell* out-of-time ordered correlation function (OTOC) and the correlation function which determines hydrodynamic transport. We explicitly exhibit such relation for a ϕ^4 matrix model and for systems close to a quantum critical point (QCP), respectively the bosonic $O(N)$ vector model and the Gross-Neveu model in $(2+1)$ dimensions. This result opens a new and precise direction to understand how information is scrambled in QFTs and which imprints this leaves on the physics of the long-lived excitations governed by hydrodynamics. A Boltzmann-like interpretation of many-body quantum chaos readily follows from this result, showing that also in the quantum critical regime many-body chaos can be understood as the counting of a gross (energy) exchange.

3.1 Introduction

Traditionally the QBE is obtained from the statistical two-point function [125–127],. The starting point of our results is that it is possible to obtain the linearized QBE in a very clean way from a 4-point function.

This 4-point function must also be resummed; this resummation is expressed in terms of a BSE which, in the spatial homogeneous case and long time limit ($\omega \to 0$), reads

$$-i\omega\, f(\omega, p) = \delta(p_0^2 - E_{\mathbf{p}}^2)\left(1 + \int_l \hat{R}^{\text{transp}}(p, l) f(\omega, l)\right). \tag{3.1}$$

where $f(\omega, p) = \int_q f(\omega, p, q)$. Once on-shell, the kernel $\hat{R}^{\text{transp}}$ reproduces exactly the collision operator $\hat{C}$. All the information about the relaxation times, eventual branch cuts and hydrodynamic and non hydrodynamical modes are intrinsically hidden in $\hat{R}^{\text{transp}}$.

The similarity of this equation with the BSE for chaos,

$$-i\omega\, C(\omega,p) = \delta(p_0^2 - E_{\mathbf{p}}^2)\left(1 + \int_l \hat{R}^{\mathrm{OTOC}}(p,l)C(\omega,l)\right), \tag{3.2}$$

is *not* coincidental. We can now summarize the most important results of this chapter. We show that, for a broad class of theories, ϕ^4 matrix model, (bosonic) vector model in $(2+1)$ dimension and the Gross-Neveu model in $(2+1)$ dimensions (for fermions), the linearised kinetic operator, $\hat{R}^{\mathrm{transp}}(p,l)$, is analytically related to the kernel of the BSE of the OTOC (3.31), $\hat{R}^{\mathrm{OTOC}}(p,l)$, as follows

$$\hat{R}^{\mathrm{OTOC}}(p,l) = \sinh(\beta p_0/2)^{-1}\, \hat{R}^{\mathrm{transp}}(p,l)\sinh(\beta l_0/2) \qquad \text{bosons,} \tag{3.3}$$

$$\hat{R}^{\mathrm{OTOC}}(p,l) = \cosh(\beta p_0/2)^{-1}\, \hat{R}^{\mathrm{transp}}(p,l)\cosh(\beta l_0/2) \qquad \text{fermions.} \tag{3.4}$$

This form in which we wrote (3.3) moreover makes clear that the relation is nothing but a similarity transformation which naively preserves the spectrum and other properties, even though the OTOC should have exponentially growing modes while the QBE only relaxing ones. The correct eigenvalues for either chaos or Boltzmann transport are only obtained after a *projection* out of some eigenvectors.

For the QBE, these precisely project out the growing modes whereas the OTOC contains the complementary spectrum. As our models are sufficiently generic, we are confident to put forward that this holds for all perturbative QFTs.

This is the significant finding we wish to present. Naively, the full physics of scrambling is encoded in $\hat{R}^{\mathrm{OTOC}}(p,l)$ and most of the hydrodynamical transport physics is encoded in $\hat{R}^{\mathrm{transp}}(p,l)$, but in truth they are literally the same. This relation has a profound meaning and should be considered as a starting point for any further attempt to find imprints of ergodicity in the hydrodynamic spectrum. Moreover, as we show in the present chapter, (3.3) not only holds for the ϕ^4 model, but also for models which describe the physics above a QCP, where there are no quasiparticle excitations. Therefore this is a quite general result for weakly coupled systems or systems studied in the large N limit.

Our presentation of the results (3.3) starts in section 3.2, where we show some formal similarities between the out-of-time ordered correlation function and the correlation function which defines transport. Those similarities can be summarized in two fundamental properties:

1 using the real time formalism, with a doubling of fields, by performing

the Keldysh rotation of the fields, only one of the 2^4 correlation functions contributes to the late time limit;

2 the BSE a priori couples all different 2^4 Keldysh components of the correlation function; in the late time limit a single 4-point function decouples and the BSE can be written in closed form.

For the case of transport, those results were shown in a series of papers first by Jeon [101], then in real time formalism by Wang and Heinz [113, 114] and later in the imaginary time formalism by Valle Basagoiti [115]. In section 3.2.2 we prove (1) for the OTOC, and in section 3.2.3 (2). In each case we only do so for $N \times N$ matrix scalar ϕ^4 field theory, but the results clearly extend to the other models.

In section 3.3 we review the derivation of the quantum Boltzmann equation. We summarize what the complementary solution is and argue that this other kinetic equation is a kinetic equation for chaos. In the following sections, we will give the full background for the connection between scrambling and transport. We show how this works for a bosonic $O(N)$ vector model in $2+1$ dimensions and for the Gross-Neveu model in $(2+1)$ dimension, respectively in section 3.4 and 3.5. Moreover we show that the relations (3.3) hold even in the proximity of the QCP. We do this by computing in both systems the BSE for transport and comparing the results with the studies of the OTOC performed in [136, 137].

At the end of sections 3.4 and 3.5 we derive from the BSE the kinetic theory equations both from chaos and transport. We prove that, in both cases, they agree with the kinetic equations for quantum chaos agree stated in section 3.3. Moreover, we show that the OTOC can be obtain from the transport BSE simply by a different choice of boundary conditions (and vice versa).

3.2 Hydrodynamic transport at weak coupling and scrambling: formal similarities

In this section we show some formal similarities ((1) and (2) in the above discussion) between the out-of-time-order correlation function and the late time limit of the density-density correlation functions, which are used to describe transport.

3.2.1 Relevant correlation function for transport in the hydrodynamic regime

The Wigner transform of the scalar density operator

$$\rho(x,p) = \int d^4y e^{-ipy} \phi\left(x + \frac{y}{2}\right) \phi\left(x - \frac{y}{2}\right), \tag{3.5}$$

corresponds to the quantum field theory analogue of the single particle distribution function which appears in the Boltzmann equation. Obviously $\int dp\, \rho(x,p) = \phi^2(x)$. If we write the Fourier transform of (3.5) with respect to the coordinate x, we see that

$$\rho(k,p) = \phi(p + k/2)\phi(-p + k/2). \tag{3.6}$$

Generally, all currents can be constructed out of this bilocal density operator. Consider for instance the contribution $\partial_i\phi\partial_j\phi$ that appears in the spatial components of the stress-energy tensor operator

$$\mathcal{F}[\partial_i\phi\partial_j\phi](k) = \int d^4p\, (p+k)_i\, \phi(p+k)\, p_j \phi(-p); \tag{3.7}$$

this can be written in terms of the Wigner transform defined above,

$$\begin{aligned} \mathcal{F}[\partial_i\phi\partial_j\phi](k) &= \int d^4p\, (p+k)_i\, p_j\, \rho(k, p+k/2) \\ &= \int d^4p\, (p+k/2)_i\, (p-k/2)_j\, \rho(k,p). \end{aligned} \tag{3.8}$$

This shows that, for $i \neq j$ we can express the stress energy tensor as

$$T^{ij}(k) = \int d^4p\, (p+k/2)^i\, (p-k/2)^j\, \rho(k,p). \tag{3.9}$$

In QFT, correlation functions can be obtained as variations of the path integral with respect to external sources. Such variation provides a well defined time ordering. When studying out-of-equilibrium physics, a convenient technique is given by the Schwinger-Keldysh path integral [173–176], which doubles the time branch and involves both time-ordered and anti time-ordered contributions (see Fig. 3.1 for the finite temperature case).

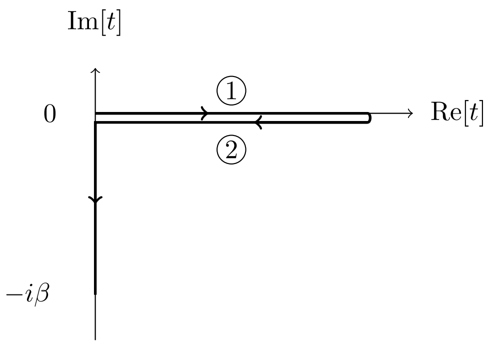

Figure 3.1: The Schwinger-Keldysh time contour includes two real time branches that are respectively labelled as 1 and 2. Correlation functions on this contour are always contour-ordered as shown by the arrows. Operators inserted in the branch 1 are time-ordered, while operators inserted in branch 2 are anti time-ordered.

By construction, the retarded Green's function of the stress energy tensor $G_R^{ij,lm}(x,y)$ is

$$G_R^{ij,lm}(x,y) = G_{11}^{ij,lm}(x,y) - G_{12}^{ij,lm}(x,y), \tag{3.10}$$

where the subscripts $1,2$ label the time branch where the stress-energy tensor is inserted. In Fourier transform, (3.10) is

$$G_R^{ij,lm}(k,-k) = G_{11}^{ij,lm}(k,-k) - G_{12}^{ij,lm}(k,-k). \tag{3.11}$$

We can now easily show that the retarded Green's function of the non-diagonal component of the stress-energy tensor is related to the retarded Green's function of the operator (3.5). By using (3.9), we have

$$\begin{aligned} -iG_R^{ij,lm}(k,-k) &= \langle T_1^{ij}(k)T_1^{lm}(-k)\rangle - \langle T_2^{lm}(-k)T_1^{ij}(k)\rangle \qquad (3.12)\\ &= \int d^4p d^4q\, (p+k/2)_i\, (p-k/2)_j\, (q-k/2)_l\, (q+k/2)_m \\ &\quad \times (\langle \rho_1(k,p)\rho_1(-k,q)\rangle - \langle \rho_2(-k,q)\rho_1(k,p)\rangle), \end{aligned}$$

with $i \neq j$ and $m \neq n$. We remind the reader that the correlation functions are always contour-ordered along the Keldysh contour, this explains the ordering $T_2^{lm}(-k)T_1^{ij}(k)$ in the expectation value. We can recast (3.12) in the more readable form

$$G_R^{T^{ij},T^{lm}}(k,-k) = \int d^4p d^4q\, (p+k/2)_i\, (p-k/2)_j\, (q-k/2)_l\, (q+k/2)_m \times G_R^{\rho\rho}(k|p,q).$$

The Kubo formula for shear viscosity involves the correlation functions of the shear channel components of the stress-energy tensor. For this reason, we choose the external momentum in the z direction, $k = (k_0, 0, 0, k_z)$, and $i = l = x$ and $j = m = y$

$$G_R^{ij,lm}(k, -k) = \int d^4p d^4q \, p_x p_y q_x q_y \, G_R^{\rho\rho}(k|p, q). \tag{3.13}$$

By definition (indicating with $\mathcal{T}_{SK}$ the contour ordering)

$$\begin{aligned} iG_R^{\rho\rho}(k|p, q) = &\langle \mathcal{T}_{SK} \left[\phi_1(p + k)\phi_1(-p)\phi_1(-q - k)\phi_1(q)\right]\rangle \\ &- \langle \mathcal{T}_{SK} \left[\phi_1(p + k)\phi_1(-p)\phi_2(-q - k)\phi_2(q)\right]\rangle \\ = &-i(G_{1111}(p + k, -p, -q - k, q) - G_{1122}(p + k, -p, -q - k, q)). \end{aligned} \tag{3.14}$$

The previous expression states that the $G_R^{\rho\rho}(k|p, q)$ corresponds to the difference of two 4-point functions. In order to simplify (3.14), we can try to perform a Keldysh rotation,

$$\phi_r = \frac{\phi_1 + \phi_2}{2}, \quad \phi_a = \phi_1 - \phi_2, \tag{3.15}$$

to basis where $G_\mathrm{R} = G_{ra}$ and $G_\mathrm{A} = G_{ar}$. After such rotation, the right-hand-side of (3.14) contains the linear combination of $2^4 = 16$ correlation functions.

However, in the limit of vanishing $\omega = k_0$ and $\mathbf{k} = (0, 0, k_z)$, Wang and Heinz showed in [113] that, for any bosonic field theory, the following holds

$$\begin{aligned} &G_{1111}(p + k, -p, -q - k, q) - G_{1122}(p + k, -p, -q - k, q) \\ &\overset{\omega\to 0}{\approx} \frac{1}{4}(N_{\mathbf{q}+\mathbf{k}} - N_{\mathbf{q}})G_{rraa}(p, q|k) = \frac{1}{4}(N_{\mathbf{p}} - N_{\mathbf{p}+\mathbf{k}})G^*_{aarr}(p, q|k). \end{aligned} \tag{3.16}$$

In the long time limit in which we are interested, the retarded 2-point function of the bilocal density operator, written in terms of fundamental fields in the Keldysh basis, $G_R^{\rho\rho}(k|p, q)$, thus assumes the simple form:

$$\begin{aligned} G_R^{\rho\rho}(k|p, q) &= \frac{1}{4}(N_{\mathbf{q}} - N_{\mathbf{q}+\mathbf{k}})G_{rraa}(p, q|k) \\ &= \frac{1}{4}(N_{\mathbf{p}+\mathbf{k}} - N_{\mathbf{p}})G^*_{aarr}(p, q|k). \end{aligned} \tag{3.17}$$

3.2.2 Decoupling of the OTOC in the extended Schwinger-Keldysh formalism

In this section we show how a similar simplification applies to the OTOC by using an extended version of the real time formalism of QFT. Considering an hermitian operator O, we will focus on the following out-of time correlation function

$$C(t,\mathbf{x}) = \langle \rho^{1/2}[O(t,\mathbf{x}),O(0,\mathbf{0}))]\rho^{1/2}[O(t,\mathbf{x}),O(0,\mathbf{0}))]\rangle. \tag{3.18}$$

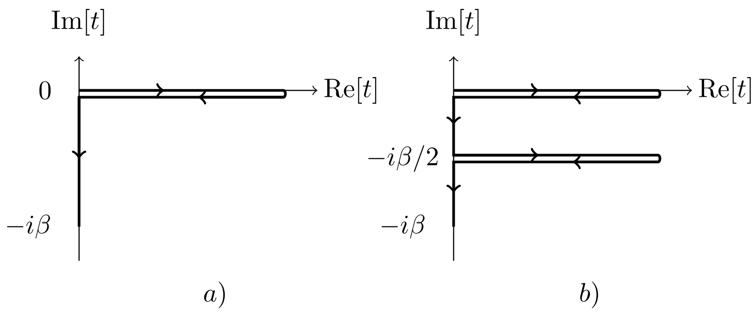

Figure 3.2: The different time contours. *a*) is the standard time contour in Schwinger Keldysh; *b*) is the extended contour necessary to compute the OTOC.

By expanding the commutators in (3.18), a new type of contribution appears besides the two-time ordered terms. The two terms are

$$-\langle \rho^{1/2}O(t,\mathbf{x})O(0,\mathbf{0}))\rho^{1/2}O(t,\mathbf{x})O(0,\mathbf{0}))\rangle - [(t,\mathbf{x}) \leftrightarrow (0,\mathbf{0})]. \tag{3.19}$$

In order to include these new terms, the contour of the path integral depicted in Fig. 3.1 has to be modified as shown in Fig. 3.2 *b*), by adding another time fold.

To preserve the ordering in the out-of-time correlation function,

$$\begin{aligned} C(x,y,w,z) =& \langle [O(x),O(y)][O(w),O(z)]\rangle = \langle O(x)O(y)O(w)O(z)\rangle \\ &+ \langle O(y)O(x)O(z)O(w)\rangle - \langle O(y)O(x)O(w)O(z)\rangle \\ &- \langle O(x)O(y)O(z)O(w)\rangle, \end{aligned} \tag{3.20}$$

we need to insert the operators in the correct branch. Labelling the 4 branches of the modified contour as in Fig. 3.3,

then (3.20) can be rewritten as

$$\begin{aligned} C(x,y,w,z) =& C^{4321}(x,y,w,z) + C^{3412}(x,y,w,z) - C^{3421}(x,y,w,z) \\ &- C^{4312}(x,y,w,z). \end{aligned} \tag{3.21}$$

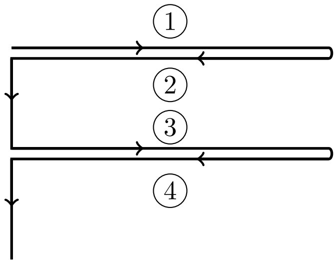

Figure 3.3: The insertions on the extended Keldysh contour can be labelled with an index $i = 1, \ldots, 4$.

In the last line, we have used the fact that, in the Schwinger-Keldysh formalism, correlation functions are contour-ordered. This means that the operator inserted in branch 4 will always appear on the most left side of the correlator. For example, as shown in fig. 3.4,

$$\begin{aligned} C^{3412}(x,y,w,z) &\equiv \langle \mathcal{T}_{SK}[O_3(x)O_4(y)O_1(w)O_2(z)]\rangle \\ &= \langle O(y)O(x)O(z)O(w)\rangle. \end{aligned} \tag{3.22}$$

Now we can perform the standard Keldysh rotation pairwise in the space of the operators, namely independently rotating the fields in the first and second time-fold,

$$O_r = \frac{O_1 + O_2}{2}, \;\; O_a = O_1 - O_2; \tag{3.23}$$

$$O_R = \frac{O_3 + O_4}{2}, \;\; O_A = O_3 - O_4. \tag{3.24}$$

Subscripts (a, r) are the Keldysh indices of the first time fold and (A, R) of the second time-fold. The basis change is implemented by the following block-diagonal matrix

$$\widetilde{Q} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}, \; Q^{i\alpha} = \begin{pmatrix} 1 & -1/2 \\ 1 & -1/2 \end{pmatrix}. \tag{3.25}$$

Then, the commutator-squared (3.21) can be expressed as follows

$$\begin{aligned} C(x,y,w,z) = (Q^{4\alpha}Q^{3\beta} - Q^{3\alpha}Q^{4\beta})(Q^{2\gamma}Q^{1\delta} - Q^{1\gamma}Q^{2\delta}) \\ \times C^{\alpha\beta\gamma\delta}(x,y,w,z). \end{aligned} \tag{3.26}$$

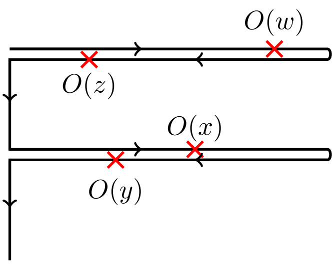

Figure 3.4: The evaluation of correlation functions is such that the insertions on the extended Keldysh contour are always contour ordered in the SK path integral.

Clearly, because of the block-diagonal structure of the $\widetilde{Q}$ matrix, the only non vanishing contribution is for $(\alpha, \beta) = \{(A, R), (R, A)\}$ and $(\gamma, \delta) = \{(a, r), (r, a)\}$. So

$$\begin{aligned} 4C(x, y, w, z) =& C^{RAra}(x, y, w, z) + C^{ARar}(x, y, w, z) - C^{RAar}(x, y, w, z) \\ &- C^{ARra}(x, y, w, z) \\ =& C^{RrAa}(x, w, y, z) + C^{AaRr}(x, w, y, z) - C^{RaAr}(x, w, y, z) \\ &- C^{ArRa}(x, w, y, z). \end{aligned} \tag{3.27}$$

So far we studied the correlator square with arbitrary insertions (x, y, w, z). The commutator-squared is defined by the choice $w = x$ and $y = z = 0$. For this choice, rotating back to the old basis, it is possible to show that, for any t, it holds [1]

$$C^{RaAr}(x, x, 0, 0) = C^{ArRa}(x, x, 0, 0) = 0. \tag{3.28}$$

The previous results remarkably simplifies the form of the commutator-squared

$$4C(x, 0, x, 0) = C^{RrAa}(x, x, 0, 0) + C^{AaRr}(x, x, 0, 0). \tag{3.29}$$

Equation (3.29) makes clear that the OTOC can in general be split into two channels, according to the sign of the time argument. Indeed it is possible to show that

$$\begin{aligned} C^{RrAa}(x, x, 0, 0) &= \theta(x^0) C^{RrAa}(x, x, 0, 0), \\ C^{AaRr}(x, x, 0, 0) &= \theta(-x^0) C^{AaRr}(x, x, 0, 0). \end{aligned}$$

[1] This result is reminiscent of the fact that the product of the advanced and the retarded Greens functions with the same time argument is zero

If we are interested in the late time regime of the OTOC, with $t > 0$, we can simply focus on the $C^{RrAa}(x, x, 0, 0)$. Consequently we can restrict our analysis to $f(x) \equiv C^{RrAa}(x, x, 0, 0)$ and

$$\begin{aligned} f(k) &= \int_x e^{ikx} f(x) \equiv \int_x e^{ikx} C^{RrAa}(x, x, 0, 0) \\ &= \int_{x,p_1,p_2,p_3,p_4} e^{i(k-p_1-p_2)x} C^{RrAa}(p_1, p_2, p_3, p_4)(2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \\ &= \int_{pq} C^{RrAa}(p + k, -p, -q - k, q). \end{aligned} \tag{3.30}$$

Thus the computation of OTOC reduces to the study of the 4-point function on a modified SK contour

$$f(k) = \int_{pq} C^{RrAa}(p, q|k), \tag{3.31}$$

where, for brevity, we indicated $(p, q|k) = (p+k, -p, -q-k, q)$. Comparing (3.31) with (3.17), we observe that the commutator square resembles an analytical continuation of the $G_R^{\rho\rho}(k|p, q)$ in the long time limit. Let us also observe that, by consistently reshuffling the momenta and the SK indices in the integral, we obtain

$$f(k) = \int_{pq} C^{RrAa}(p, q|k) = \int_{pq} C^{AaRr}(p, q| - k). \tag{3.32}$$

We will use (3.32) as a consistency check for our results.

In order to compute (3.31), we shall need some further knowledge of the structure of the Green's functions in this extended SK path integral. We first define the correlation functions in this extended SK contour as follows

$$G_{a_1 \dots a_n}(x_1, .., x_n) = (-i)^{n-1} \langle \mathcal{T}_{SK}[O(x_1)_{a_1} ... O(x_n)_{a_n}] \rangle \tag{3.33}$$

where the index a_i runs over the time brannches, $a_1 = 1, ..., 4$. After performing the rotation to the Keldysh basis, the correlation functions read

$$G_{\alpha_1 \dots \alpha_n}(x_1, .., x_n) = (-i)^{n-1} 2^{n_r - 1} 2^{n_R - 1} \langle \mathcal{T}_{SK}[O(x_1)_{\alpha_1} ... O(x_n)_{\alpha_n}] \rangle,$$

and n_r and n_R count respectively the the r and R indices among $\{\alpha_i\}$.

We now specialize to the bosonic case. It can be shown that

$$
\begin{aligned}
G_{ra}(k) &= G_{RA}(k), \\
G_{Ra}(k) &= G_{Ra}(k) = 0, \\
G_{ar}(k) &= G_{AR}(k), \\
G_{rR}(k) &= \frac{1}{2} e^{\beta k^0/2} (N(k^0) - 1) \left[G_{ra}(k) - G_{ar}(k) \right], \\
G_{Rr}(k) &= \frac{1}{2} e^{-\beta k^0/2} (N(k^0) + 1) \left[G_{ra}(k) - G_{ar}(k) \right],
\end{aligned}
$$

with $N(k^0) = 1 + 2n_B(k^0)$, $n_B(k^0)$ being the Bose-Einstein distribution function $n_B(k^0) = \frac{1}{e^{\beta k^0} - 1}$. Moreover, since $e^{\beta k^0/2}(N(k^0) - 1) = e^{-\beta k^0/2}(N(k^0) + 1)$, the (rR) and (Rr) components are all same

$$
\begin{aligned}
G_{rR}(k) &= G_{Rr}(k), \\
G_{rR}(-k) &= G_{Rr}(k) = G_{rR}(k).
\end{aligned}
$$

In this extended Keldysh basis, the set of Green's functions can be summarized as follows

$$
G = \left(\begin{array}{c|c} G_{\alpha\beta} & G_1 \\ \hline G_2 & G_{\alpha\beta} \end{array} \right), \tag{3.34}
$$

with

$$
G_{\alpha\beta} = \begin{pmatrix} G_{rr} & G_{ra} \\ G_{ar} & G_{aa} \end{pmatrix}, \quad G_1 = \begin{pmatrix} G_{rR} & 0 \\ 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} G_{Rr} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.35}
$$

Furthermore, many properties in this contour are remnant of the canonical SK path integral. By using that any n-point function with only a indices vanishes in standard SK, it is easy to see that any n-point functions with at least an index A (a), but without a R (r), vanishes. An example of this statement is the following

$$
\begin{aligned}
& G^{AA\alpha_3\alpha_4}(x, y, w, z) = G^{A\alpha_3 A\alpha_4}(x, y, w, z) = G^{A\alpha_3\alpha_4 A}(x, y, w, z) \\
&= G^{\alpha_3 AA\alpha_4}(x, y, w, z) = G^{\alpha_3 A\alpha_4 A}(x, y, w, z) = G^{\alpha_3\alpha_4 AA}(x, y, w, z) = 0,
\end{aligned} \tag{3.36}
$$

if $(\alpha_3, \alpha_4) \in \{a, r\}$.

3.2.3 Decoupling of the OTOC BSE: ϕ^4 matrix model example

The framework presented in the previous section is valid for any bosonic theory and can be easily generalized to the fermionic case. We shall stay with the bosonic theory, however. We will now show that the BSE that determines the exact expression for $G^{RrAa}(p,q|k)$ (3.31) remains a closed equation in the late time limit, decoupling from all the other Green's functions. We will specialize to the case of $N \times N$ Hermitian massive scalars Φ_{ab}, with a Φ^4 interaction in $(3+1)$ dimensions. The Lagrangian we are considering is

$$\mathcal{L} = \mathrm{tr}\left(\frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}(\nabla\Phi)^2 - \frac{m^2}{2}\Phi^2 - \frac{g^2}{4!}\Phi^4\right). \tag{3.37}$$

We are interested in the the following class of 4-point functions

$$G^{\alpha_1\alpha_2\alpha_3\alpha_4}(p,q|k) = i2^{n_r-1}\langle \mathcal{T}_{SK}[\phi^{ab}_{\alpha_1}(p+k)\phi^{ba}_{\alpha_2}(-p)\phi^{a'b'}_{\alpha_3}(-q-k)\phi^{b'a'}_{\alpha_4}(q)]\rangle. \tag{3.38}$$

This correlation function satisfies the following Bethe-Salpeter equation

$$\begin{aligned} G^{\alpha_1\alpha_2\alpha_3\alpha_4}(p,q|k) &= iG^{\alpha_1\alpha_3}(p+k)G^{\alpha_2\alpha_4}(-p)(2\pi)^4\delta^4(p-q) \\ &\quad -\frac{1}{2}G^{\alpha_1\beta_1}(p+k)G^{\alpha_2\gamma_1}(-p)\int_l K_{\beta_1\gamma_1\beta_4\gamma_4}(p,l|k)G^{\beta_4\gamma_4\alpha_3\alpha_4}(l,q|k) \end{aligned} \tag{3.39}$$

where the indices run on $\alpha = \{a, r\}$. (3.39) represents a nested set of equations which couples all the 2^4 correlation function. In order to compute hydrodynamical transport coefficients, as shear viscosity η, only G^{rraa} and its complex conjugate are needed. In the hydrodynamical limit, $\mathbf{k} \to 0$ and $k^0 = \omega \to 0$, this coupled system of BSE also considerably simplifies and the relevant components, G_{rraa} and G^*_{rraa}, decouple [114, 168]. In this limit, a crucial role is played by the *pinching-poles* approximation, which we discuss in Appendix 3.D. Building on the results for a purely scalar field [113, 169], for *any* values of N the BSE reads [44]

$$\begin{aligned} G^{aarr}(p,q|k) =& G^{ar}(p+k)G^{ar}(-p) \\ &\times\left[i(2\pi)^4\delta^4(p-q) - \int_l R^{\text{transp}}(p,l)\, G^{aarr}(l,q|k)\right] \end{aligned} \tag{3.40}$$

where the kernel is

$$R^{\text{transp}}(p,l) = -\frac{g^4}{2}\frac{N^2+5}{6}\frac{1+n(l_0)}{1+n(p_0)}\int_l n(s_0)(1+n(s_0-l_0+p_0))\rho(s)\rho(s-l+p). \tag{3.41}$$

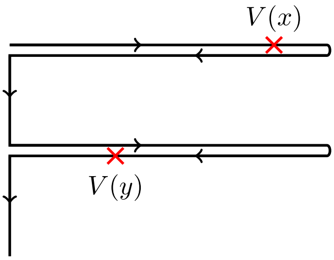

Figure 3.5: In the computation of the OTOC, each vertex insertion (V) can be inserted in one of the 4 time branches. The insertions in the same fold is already taken into account by using the dressed Green's functions on the rails. The new contribution comes from the insertions in the different folds.

The factor in front of the kernel is the two-to-two particle scattering amplitude $\frac{1}{2}|\mathcal{T}_{2\to 2}|^2 = \frac{g^4}{2}\frac{N^2+5}{6}$, which indeed reduce to the standard $|\mathcal{T}_{2\to 2}|^2 = g^4$ for the single scalar case ($N = 1$). If we are interested in computing the OTOC, the BSE (3.39) does not change form. We simply take the indices in 2-fold contour $\alpha = \{a, r, A, R\}$ and we need to evaluate the expression for the kernel $K_{\beta_1\gamma_1\beta_4\gamma_4}$.

$$K_{\alpha_1\beta_1\alpha_4\beta_4}(p, l|k) = \frac{N^2+5}{6}\lambda_{\alpha_1\alpha_2\alpha_3\alpha_4}\lambda_{\beta_1\beta_2\beta_3\beta_4}\int_s G_{\beta_2\alpha_2}(s)G_{\beta_3\alpha_3}(s-l+p), \tag{3.42}$$

with

$$\lambda_{\alpha_1\alpha_2\alpha_3\alpha_4} = g^2\frac{1-(-1)^{n_a}}{4}. \tag{3.43}$$

Each vertex can be inserted in only 2 of the 4 branches (either $1, 2$ or $3, 4$). This means that the indices $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ need to be either $\{A, R\}$ or $\{a, r\}$, as for example shown in fig. 3.5.

Now let's specialise the BSE to the commutator squared (3.31). The BSE becomes

$$\begin{aligned} G^{RrAa}(p, q|k) &= iG^{RA}(p+k)G^{ra}(-p)(2\pi)^4\delta^4(p-q)+ \\ &- \frac{1}{2}G^{R\alpha_1}(p+k)G^{r\beta_1}(-p)\int_l K_{\alpha_1\beta_1\alpha_4\beta_4}(p, l|k)C^{\alpha_4\beta_4 Aa}(l, q|k), \end{aligned} \tag{3.44}$$

which simplifies with the use of the relations (3.36),

$$G^{RrAa}(p,q|k) = iG^{RA}(p{+}k)G^{ra}(-p)(2\pi)^4\delta^4(p{-}q) - \frac{1}{2}G^{R\alpha_1}(p{+}k)G^{r\beta_1}(-p) \times \int_l \left(K_{\alpha_1\beta_1 Rr}(p,l|k)C^{RrAa}(l,q|k) + K_{\alpha_1\beta_1 rR}(p,l|k)C^{rRAa}(l,q|k)\right). \tag{3.45}$$

Focusing on first term in (3.45), the possible choices for the indices are $\alpha_1 \in \{A, R, r\}$ and $\beta_1 \in \{a, r, R\}$ since, expanding the product $G^{R\alpha_1}G^{r\beta_1}$, few terms vanish due to the identities $G^{Ra} = G^{rA} = 0$. Moreover, from the definition of the kernel (3.42), $K_{\alpha_1\beta_1\alpha_4\beta_4} = 0$ if α_1 and α_4 or β_1 and β_4 belong to different time folds. This reduces the combinations to $\alpha_1 \in \{A, R\}$ and $\beta_1 \in \{a, r\}$. By using $K_{RrRr} = K_{RaRr} = K_{ArRr} = 0$, we can furthermore simplify the first term in (3.45)

$$G^{R\alpha_1}(p+k)G^{r\beta_1}(-p)\int_l K_{\alpha_1\beta_1 Rr}(p,l|k)C^{RrAa}(l,q|k) = G^{RA}(p+k)G^{ra}(-p)\int_l K_{AaRr}(p,l|k)C^{RrAa}(l,q|k). \tag{3.46}$$

In a similar manner, it is possible to show that the second term in (3.45) vanishes

$$G^{R\alpha_1}(p+k)G^{r\beta_1}(-p)\int_l K_{\alpha_1\beta_1 rR}(p,l|k)C^{rRAa}(l,q|k) = G^{Rr}(p+k)G^{rR}(-p)\int_l K_{rRrR}(p,l|k)C^{RrAa}(l,q|k) = 0. \tag{3.47}$$

This means that the BSE for the OTOC is the following

$$G^{RrAa}(p,q|k) = G^{RA}(p+k)G^{ra}(-p)\Bigg(i(2\pi)^4\delta^4(p-q) - \frac{1}{2}\int_l K_{AaRr}(p,l|k)C^{RrAa}(l,q|k)\Bigg). \tag{3.48}$$

Thus, the kernel of the BSE is simply given by the product of Wightman functions connecting the two time folds

$$K_{AaRr}(p,l|k) = \frac{1}{4}g^4\frac{N^2+5}{6}\int_s G_{Rr}(s)G_{Rr}(s-l+p)$$
$$= g^4\frac{N^2+5}{6}e^{\beta(l^0-p^0)/2}\int_s n(s_0)(1+n(s_0-l_0+p_0))\rho(s)\rho(s-l+p). \tag{3.49}$$

The BSE of the commutator squared eventually reads

$$G^{RrAa}(p,q|k) = G^{RA}(p+k)G^{ra}(-p)\Bigg(i(2\pi)^4\delta^4(p-q) - \int_l R^{\mathrm{OTOC}}(p,l)C^{RrAa}(l,q|k)\Bigg) \tag{3.50}$$

and

$$R^{\mathrm{OTOC}}(p,l) = g^4\frac{N^2+5}{12}e^{\beta(l^0-p^0)/2} \times\int_s n(s_0)(1+n(s_0-l_0+p_0))\rho(s)\rho(s-l+p). \tag{3.51}$$

Our derivation of the G^{RrAa}, which is summarized by equations (3.50) and (3.51), is valid for any N, even $N=1$. Similarly to the case of transport, the BSE for the commutator squared decoupled, even though a priori the RHS of the BSE couples all the 4^4 4-point functions. By comparing (3.40) and (3.41) with (3.50) and (3.51), we can easily see that

$$R^{\mathrm{OTOC}}(p,l) = \frac{\sinh(\beta l_0/2)}{\sinh(\beta p_0/2)}R^{\mathrm{transp}}(p,l). \tag{3.52}$$

We observe that the term $e^{\beta(l^0-p^0)/2}$ is remnant of the regularization, as it comes from $e^{\sigma(l^0-p^0)}$, σ being the time width of the extended SK path integral. For an analysis of the consequences of the regularization dependence of the OTOC, we refer to chapter 5.

3.3 Kinetic theory of many-body chaos

In the previous section we have shown that, although a priori very different, once the late time limit is taken the commutator squared and the retarded

Green's function of the bilocal density field have many properties in common. In this section we make this connection more precise, within the quantum Boltzmann equation framework.

The BSE for the correlation function of the bilocal density operator is the QFT analogue of the Boltzmann equation. It was already noted in the literature that the collision integral entered the form of the BSE [101, 110, 126] and that it can be used to compute transport coefficients through Kubo relations. We reviewed this above. We now show that the BSE not only contains the collisional integral, but by appropriately retaining the first order in the external frequency, the BSE is nothing but the Fourier transform of the Boltzmann equation. This result, which is by itself of interest, acquires more appeal in light of the findings of the previous sections, namely the connection between the BSE that defines the OTOC and the BSE of the bilocal density operator. This thus allows us to derive the kinetic equation for many-body chaos which reproduces exactly the computation of the OTOC, and thus the Lyapunov spectrum. This kinetic equation shows that the OTOC, so the scrambling of information in a system, computes some gross (energy) exchange in contrast to net number exchange for transport.

In the subsequent sections we shall show that, since the relation holds also in the quantum critical limit of both bosonic and fermionic states, our kinetic equation unequivocally implies that, even in this critical regime, energy dynamics plays a crucial role in the information scrambling.

3.3.1 Quick review of the Boltzmann equation

The Boltzmann equation governs the time evolution of the single-particle distribution function $f(\mathbf{p}, \mathbf{r}, t)$. In terms of the change of particle number density per unit of phase space: $\delta n(t, \mathbf{p}) = n(t, \mathbf{p}) - n(E_{\mathbf{p}})$, the distribution function can be expressed as

$$f(t, \mathbf{p}) = \frac{\delta n(t, \mathbf{p})}{(1 + n(\mathbf{p}))n(\mathbf{p})} \tag{3.53}$$

where $n(\mathbf{p})$ is the equilibrium Bose-Einstein distribution $n(\mathbf{p}) = 1/(e^{\beta E(\mathbf{p})} - 1)$ which depends on the energy of the *on-shell* particle $E(\mathbf{p})$. Here we restrict the analysis to the spatially homogeneous case and consider all quantities as space-averaged (e.g. $n(t, \mathbf{p}) = \int d\mathbf{x}\, n(t, \mathbf{x}, \mathbf{p})$). Moreover we focus only on the contribution given by the two-to-two scattering to the dynamics of the phase-space. Higher order contributions require to go beyond the uncrossed ladder approximation, and we will not consider this case. The linearized Boltzmann equation is a homogeneous evolution equation for $f(t, \mathbf{p})$ (see e.g. [162–164]):[2]

$$\partial_t f(t, \mathbf{p}) = -\int_{\mathbf{l}} \mathcal{L}(\mathbf{p}, \mathbf{l}) f(t, \mathbf{l}), \tag{3.54}$$

where the kernel of the collision integral

$$\mathcal{L}(\mathbf{p}, \mathbf{l}) \equiv -\left[R^{\wedge}(\mathbf{p}, \mathbf{l}) - R^{\vee}(\mathbf{p}, \mathbf{l})\right] \tag{3.55}$$

measures the difference between the rates of scattering into the phase-space cell and scattering out the phase space cell. The term $R^{\wedge}$ takes into account the increases of the local density by scattering with a thermal excitation and it reads

$$R^{\wedge}(\mathbf{p}, \mathbf{l}) = \frac{1}{n(\mathbf{p})(1+n(\mathbf{p}))} \int\limits_{\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4} d\Sigma(\mathbf{p}, \mathbf{p}_2 | \mathbf{p}_3, \mathbf{p}_4) \left(\delta(\mathbf{p}_3 - \mathbf{l}) + \delta(\mathbf{p}_4 - \mathbf{l})\right). \tag{3.56}$$

The factor $R^{\vee}(\mathbf{p}, \mathbf{l})$, instead, involves the loss of the density in the phase cell occurred by the annihilation into the thermal bath or scattering from the bath into the same cell

$$R^{\vee}(\mathbf{p}, \mathbf{l}) = \frac{1}{n(\mathbf{p})(1+n(\mathbf{p}))} \int\limits_{\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4} d\Sigma(\mathbf{p}, \mathbf{p}_2 | \mathbf{p}_3, \mathbf{p}_4) \left(\delta(\mathbf{p} - \mathbf{l}) + \delta(\mathbf{p}_2 - \mathbf{l})\right). \tag{3.57}$$

Here, the infinitesimal cross section is weighted with the phase space contribution given by the equilibrium distribution function ((n_B) for initial states and $(1 + n_B)$ for final states)

$$d\Sigma(\mathbf{p}, \mathbf{p}_2 | \mathbf{p}_3, \mathbf{p}_4) = \frac{1}{2} |\mathcal{T}_{pp_2 \to p_3 p_4}|^2\, n(\mathbf{p})\, n(\mathbf{p}_2)\, (1 + n(\mathbf{p}_3))(1 + n(\mathbf{p}_4)) \times (2\pi)^4 \delta^4(p + p_2 - p_3 - p_4) \tag{3.58}$$

and $|\mathcal{T}_{pp_2 \to p_3 p_4}|^2$ the two-to-two transition amplitude squared. Some of the spectral properties of the Boltzmann equation can be studied by introducing the following inner product

$$\langle \psi' | \psi \rangle = \int_{\mathbf{p}} n(\mathbf{p})(1 + n(\mathbf{p})) \psi'^*(\mathbf{p}) \psi(\mathbf{p}). \tag{3.59}$$

[2] To make the formula easier to read, we indicate with $\int_p \equiv \int \frac{d^4p}{(2\pi)^4}$ and $\int_{\mathbf{p}} \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E(\mathbf{p})}$ for relativistic theories. For a non-relativistic system, $\int_{\mathbf{p}} \equiv \int \frac{d^3p}{(2\pi)^3}$. Similarly, $\int_x \equiv \int d^4x$ and $\int_{\mathbf{x}} \equiv \int d^3x$.

By using the symmetries of the cross-section

$$
\begin{aligned}
d\Sigma(\mathbf{p}_1,\mathbf{p}_2|\mathbf{p}_3,\mathbf{p}_4) &= d\Sigma(\mathbf{p}_2,\mathbf{p}_1|\mathbf{p}_3,\mathbf{p}_4) = d\Sigma(\mathbf{p}_3,\mathbf{p}_4|\mathbf{p}_1,\mathbf{p}_2)\\
&= d\Sigma(\mathbf{p}_1,\mathbf{p}_2|\mathbf{p}_4,\mathbf{p}_3)
\end{aligned}
\tag{3.60}
$$

it is possible to show that the operator $\mathcal{L}(\mathbf{p},\mathbf{l})$ is not only Hermitian on this inner product, but also positive semidefinite. This in turn means that all its eigenvalues are real and $\xi_n \geq 0$ and that the solutions to the Boltzmann equation are purely relaxational:

$$
f(\mathbf{p},t) = \sum_n A_n e^{-\xi_n t}\phi_n(\mathbf{p}). \tag{3.61}
$$

In the previous equation we have formally indicated with $\sum_n$ either a sum over discrete values or an integral over a continuum (see e.g. [162–165]). Another remarkable property of the spectrum is that every $\xi = 0$ eigenvalue corresponds to a symmetry and hence an associated conserved quantity (a collisional invariant). For the bosonic/fermionic field theory, the kinetic equation thus takes the form

$$
\begin{aligned}
\partial_t f(\mathbf{p},t) = &+\frac{1}{1\pm n_{B/F}(\mathbf{p})}\int_{\mathbf{l},\mathbf{p}_2,\mathbf{p}_4}\frac{(2\pi)^4\delta^4(\mathbf{p}^{\mathrm{os}}+\mathbf{p_2}^{\mathrm{os}}-\mathbf{l}^{\mathrm{os}}-\mathbf{p_4}^{\mathrm{os}})|\mathcal{T}|^2}{2E_{\mathbf{p}}}\\
&\qquad\times n_{B/F}(E_{\mathbf{p_2}})(1\pm n_{B/F}(E_{\mathbf{l}}))(1\pm n_{B/F}(E_{\mathbf{4}}))f(\mathbf{l},t)\\
&-\frac{1/2}{1\pm n_{B/F}(\mathbf{p})}\int_{\mathbf{l},\mathbf{p}_2,\mathbf{p}_4}\frac{(2\pi)^4\delta^4(\mathbf{p}^{\mathrm{os}}+\mathbf{l}^{\mathrm{os}}-\mathbf{p_2}^{\mathrm{os}}-\mathbf{p_4}^{\mathrm{os}})|\mathcal{T}|^2}{2E_{\mathbf{p}}}\\
&\qquad\times n_{B/F}(E_{\mathbf{l}})(1\pm n_{B/F}(E_{\mathbf{p_2}}))(1\pm n_{B/F}(E_{\mathbf{4}}))f(\mathbf{l},t)\\
&-\frac{1/2}{1\pm n_{B/F}(\mathbf{p})}\int_{\mathbf{l},\mathbf{p}_2,\mathbf{p}_4}\frac{(2\pi)^4\delta^4(\mathbf{p}^{\mathrm{os}}+\mathbf{l}^{\mathrm{os}}-\mathbf{p_2}^{\mathrm{os}}-\mathbf{p_4}^{\mathrm{os}})|\mathcal{T}|^2}{2E_{\mathbf{p}}}\\
&\qquad\times n_{B/F}(E_{\mathbf{l}})(1\pm n_{B/F}(E_{\mathbf{p_2}}))(1\pm n_{B/F}(E_{\mathbf{4}}))f(\mathbf{p},t).
\end{aligned}
\tag{3.62}
$$

where we denoted with $\mathbf{p}^{\mathrm{os}} = (E_{\mathbf{p}},\mathbf{p})$ the *on-shell* momenta and

$$
\begin{aligned}
\delta^4(\mathbf{p}^{\mathrm{os}}+\mathbf{p_2}^{\mathrm{os}}-\mathbf{l}^{\mathrm{os}}-\mathbf{p_4}^{\mathrm{os}}) = &\,\delta^3(\mathbf{p}^{\mathrm{os}}+\mathbf{p_2}^{\mathrm{os}}-\mathbf{l}^{\mathrm{os}}-\mathbf{p_4}^{\mathrm{os}})\\
&\times\delta(E_{\mathbf{p}}+E_{\mathbf{p_2}}-E_{\mathbf{l}}-E_{\mathbf{p_4}}).
\end{aligned}
$$

The first line corresponds to the *gain* term $R^{\wedge}(\mathbf{p},\mathbf{l})$,

$$
\begin{aligned}
R^{\wedge}(\mathbf{p},\mathbf{l}) = &+\frac{1}{1\pm n_{B/F}(\mathbf{p})}\int_{\mathbf{p}_2,\mathbf{p}_4}\frac{(2\pi)^4\delta^4(\mathbf{p}^{\mathrm{os}}+\mathbf{p_2}^{\mathrm{os}}-\mathbf{l}^{\mathrm{os}}-\mathbf{p_4}^{\mathrm{os}})|\mathcal{T}|^2}{4E_{\mathbf{p}}E_{\mathbf{l}}}\\
&\qquad\times n_{B/F}(E_{\mathbf{p_2}})(1\pm n_{B/F}(E_{\mathbf{l}}))(1\pm n_{B/F}(E_{\mathbf{4}})),
\end{aligned}
\tag{3.63}
$$

while the second and third line correspond to the *loss* term $R^{\vee}(\mathbf{p},\mathbf{l})$

$$R^{\vee}(\mathbf{p},\mathbf{l}) = \frac{1/2}{1 \pm n_{B/F}(\mathbf{p})} \int_{\mathbf{p_2},\mathbf{p_4}} \frac{(2\pi)^4 \delta^4(\mathbf{p}^{\text{os}} + \mathbf{l}^{\text{os}} - \mathbf{p_2}^{\text{os}} - \mathbf{p_4}^{\text{os}}) |\mathcal{T}|^2}{4E_{\mathbf{p}}E_{\mathbf{l}}} \\ \times n_{B/F}(E_{\mathbf{l}})(1 \pm n_{B/F}(E_{\mathbf{p_2}}))(1 \pm n_{B/F}(E_{\mathbf{4}})) \\ + \frac{1/2}{1 \pm n_{B/F}(\mathbf{p})} \int_{\mathbf{p_2},\mathbf{p_4}} \frac{(2\pi)^4 \delta^4(\mathbf{p}^{\text{os}} + \mathbf{l}^{\text{os}} - \mathbf{p_2}^{\text{os}} - \mathbf{p_4}^{\text{os}}) |\mathcal{T}|^2}{4E_{\mathbf{p}}E_{\mathbf{l}}} \\ \times n_{B/F}(E_{\mathbf{l}})(1 \pm n_{B/F}(E_{\mathbf{p_2}}))(1 \pm n_{B/F}(E_{\mathbf{4}})). \quad (3.64)$$

Moreover, we will see that the second term in $R^{\vee}$ is proportional the imaginary part of the self energy $2\,\Gamma_{\mathbf{p}}$.

3.3.2 From the BSE to the quantum Boltzmann equation

We now show how the quantum Boltzmann equation (3.62) can be derived from the BSE. For the sake of clarity, we first focus on the theory of $N \times N$ Hermitian matrix scalars. We then prove this results for the case of the bosonic $O(N)$ vector model (section 3.4) and the Gross-Neveu model (section 3.5). We start with the BSE for the 4-point Green's function $G^*_{aarr}(p+k,p)$ which, up to some thermal factor, coincides with $G^{\rho\rho}_{\text{R}}(p+k,p)$ as stated in (3.17). In the long wavelength and late time limit, $k = (\omega, \mathbf{0})$ and $\omega \to 0$, this correlation function satisfies the following BSE [44, 113]

$$G^*_{aarr}(p+k,p) = G_{\text{R}}(p+k)G_{\text{A}}(p)\left[iN^2 - \int_l R^{\text{transp}}(p,l)G^*_{aarr}(l+k,l)\right], \quad (3.65)$$

where $G_{\text{R/A}}$ are respectively the retarded/advanced Green's function and R^{transp} is the kernel of the BSE. Because of the long time limit, the product $G_{\text{R}}(p+k)G_{\text{A}}(p)$ suffers of the *pinching-pole* singularity, and can be approximated as follows [3]

$$G_{\text{R}}(p+k)G_{\text{A}}(p) = \frac{\pi}{E_{\mathbf{p}}} \frac{\delta(p_0^2 - E_{\mathbf{p}}^2)}{-i\omega + 2\Gamma_{\mathbf{p}}}. \quad (3.66)$$

The BSE has thus the form

$$G^*_{aarr}(p+k,p) = \frac{\pi}{E_{\mathbf{p}}} \frac{\delta(p_0^2 - E_{\mathbf{p}}^2)}{-i\omega + 2\Gamma_{\mathbf{p}}}\left[iN^2 - \int_l R^{\text{transp}}(p,l)G^*_{aarr}(l+k,l)\right]. \quad (3.67)$$

[3] For a discussion about the *pinching-pole* singularity, see Appendix 3.D.

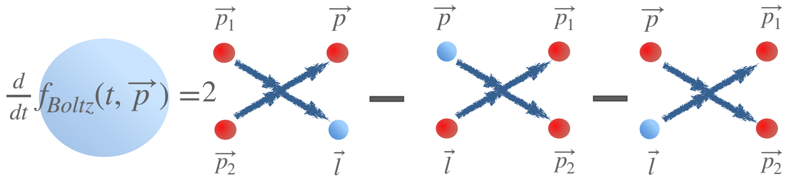

Figure 3.6: Pictorial representation of the linearised Boltzmann equation. The blue halo indicates the out of equilibrium distribution function, while the red the equilibrium distribution function.

In order to find the solution, we can pose an ansatz. We realize that the choice can be narrowed down to two different classes. They are

$$G^*_{aarr}(p+k,p) = G_1(\omega,\mathbf{p})\delta(p_0 - E_\mathbf{p}) \pm G_2(\omega,\mathbf{p})\delta(p_0 + E_\mathbf{p}). \quad (3.68)$$

Let's focus on the ansatz with plus,

$$G^*_{aarr}(p+k,p) = G_1(\omega,\mathbf{p})\delta(p_0 - E_\mathbf{p}) + G_2(\omega,\mathbf{p})\delta(p_0 + E_\mathbf{p}). \quad (3.69)$$

We now show that this choice of ansatz for G^*_{aarr} is correct, since it projects the solution into the physical subspace of relaxing modes described by the quantum Boltzmann equation. Substituting (3.69) into the BSE, we arrive to the following system of equations

$$(-i\omega + 2\Gamma_\mathbf{p})G_1(\mathbf{p},\omega) = \frac{\pi}{2E_\mathbf{p}^2} iN^2 - \int \frac{d\mathbf{l}}{(2\pi)^3}\frac{1}{4E_\mathbf{p}E_\mathbf{l}} \quad (3.70)$$
$$(R^{\text{transp}}(\mathbf{p},E_\mathbf{p}|\mathbf{l},E_\mathbf{l})G_1(\mathbf{l},\omega) + R^{\text{transp}}(\mathbf{p},E_\mathbf{p}|\mathbf{l},-E_\mathbf{l})G_2(\mathbf{l},\omega)),$$
$$(-i\omega + 2\Gamma_\mathbf{p})G_2(\mathbf{p},\omega) = \frac{\pi}{2E_\mathbf{p}^2} iN^2 - \int \frac{d\mathbf{l}}{(2\pi)^3}\frac{1}{4E_\mathbf{p}E_\mathbf{l}} \quad (3.71)$$
$$(R^{\text{transp}}(\mathbf{p},-E_\mathbf{p}|\mathbf{l},E_\mathbf{l})G_1(\mathbf{l},\omega) + R^{\text{transp}}(\mathbf{p},-E_\mathbf{p}|\mathbf{l},-E_\mathbf{l})G_2(\mathbf{l},\omega)).$$

By using the symmetries of the kernel,

$$R^{\text{transp}}(\mathbf{p},-E_\mathbf{p}|\mathbf{l},E_\mathbf{l}) = R^{\text{transp}}(\mathbf{p},E_\mathbf{p}|\mathbf{l},-E_\mathbf{l}), \quad (3.72)$$

we can define the variables $G^\pm(\mathbf{p},\omega) = G_1(\mathbf{p},\omega) \pm G_2(\mathbf{p},\omega)$ which satisfy the following equations

$$(-i\omega + 2\Gamma_\mathbf{p})G^\pm(\mathbf{p},\omega) + \int \frac{d\mathbf{l}}{(2\pi)^3}\frac{1}{4E_\mathbf{p}E_\mathbf{l}} \quad (3.73)$$
$$(R^{\text{transp}}(\mathbf{p},E_\mathbf{p}|\mathbf{l},E_\mathbf{l}) + R^{\text{transp}}(\mathbf{p},E_\mathbf{p}|\mathbf{l},-E_\mathbf{l}))G^\pm(\mathbf{l},\omega) = \frac{\pi}{2E_\mathbf{p}^2} iN^2(1 \pm 1).$$

The physical correlation function G^*_{aarr} corresponds to the solution G^+. Indeed, as we show in Appendix 3.G for the case of $N \times N$ hermitian matrix field (and in the following sections for the $O(N)$ vector model and the GN model), the terms appearing in the *on-shell* BSE kernel of G^+ can be identified with the *gain* and *loss* terms of the collision integral of the Boltzmann equation

$$R^{\wedge}(\mathbf{p},\mathbf{l}) = \frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}} R^{\text{transp}}(\mathbf{p}, E_{\mathbf{p}}|\mathbf{l}, E_{\mathbf{l}}), \tag{3.74}$$

$$R^{\vee}(\mathbf{p},\mathbf{l}) = \frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}} R^{\text{transp}}(\mathbf{p}, E_{\mathbf{p}}|\mathbf{l}, -E_{\mathbf{l}}) + 2(2\pi)^3\delta^3(\mathbf{p}-\mathbf{l})\Gamma_{\mathbf{p}}. \tag{3.75}$$

Thus, by comparing (3.74) and (3.75) to (3.73), G^+ can be formally solved as

$$G^+(\mathbf{p},\omega) = \left[\frac{1}{-i\omega + \mathcal{L}(\mathbf{p},\mathbf{l})}\right] \frac{\pi}{E_{\mathbf{p}}^2} iN^2, \tag{3.76}$$

the operator $\mathcal{L}(\mathbf{p},\mathbf{l})$ being the collision integral of the Boltzmann equation (3.55)

$$\mathcal{L}(\mathbf{p},\mathbf{l}) = -(R^{\wedge}(\mathbf{p},\mathbf{l}) - R^{\vee}(\mathbf{p},\mathbf{l})), \tag{3.77}$$

as depicted in fig. 3.6. This equation, in the strict $\omega = 0$ limit, equals the equation used in [101] to find the shear viscosity. Its spectrum is negative definite and gives the relaxation times of the theory. This proves our statement that the quantum Boltzmann equation can be derived from the BSE of the retarded Green's function of the bilocal density operator $G_{\mathrm{R}}^{\rho\rho}$. Moreover, from (3.76), we can deduce that the poles/branch cuts of $G_{\mathrm{R}}^{\rho\rho}$ corresponds to the relaxation times/branch cuts of the theory we can be easily addressed in this framework.

3.3.3 The Quantum Boltzmann equation for many-body chaos

In the previous section we have shown that, starting from the BSE for $G_{\mathrm{R}}^{\rho\rho}$, the solution is found by an appropriate choice of ansatz (plus in (3.68)) and it reproduces the quantum Boltzmann equation. We now want to study the physics of the other solution of the *same* BSE, corresponding to the minus in the ansatz (3.68). We show that the latter exactly reproduces the commutator squared correlation function.

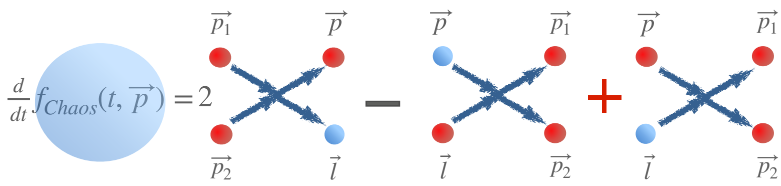

Figure 3.7: Pictorial representation of kinetic equation for the OTOC. The blue halo indicates the out of equilibrium distribution function, while the red the equilibrium distribution function.

By choosing a minus in (3.68), we obtain the following system of equation

$$(-i\omega + 2\Gamma_{\mathbf{p}})G_1(\mathbf{p},\omega) = \frac{\pi}{2E_{\mathbf{p}}^2} iN^2 - \int \frac{d\mathbf{l}}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}} \left(R^{\text{transp}}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},E_{\mathbf{l}})G_1(\mathbf{l},\omega) - R^{\text{transp}}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},-E_{\mathbf{l}})G_2(\mathbf{l},\omega)\right), \tag{3.78}$$

$$(-i\omega + 2\Gamma_{\mathbf{p}})G_2(\mathbf{p},\omega) = -\frac{\pi}{2E_{\mathbf{p}}^2} iN^2 - \int \frac{d\mathbf{l}}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}} \left(-R^{\text{transp}}(\mathbf{p},-E_{\mathbf{p}}|\mathbf{l},E_{\mathbf{l}})G_1(\mathbf{l},\omega) + R^{\text{transp}}(\mathbf{p},-E_{\mathbf{p}}|\mathbf{l},-E_{\mathbf{l}})G_2(\mathbf{l},\omega)\right). \tag{3.79}$$

As before, we can define $G_{\pm}(\mathbf{p},\omega) = G_1(\mathbf{p},\omega) \pm G_2(\mathbf{p},\omega)$ which satisfy

$$(-i\omega + 2\Gamma_{\mathbf{p}})G^{\pm}(\mathbf{p},\omega) + \int \frac{d\mathbf{l}}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}} \times \left(R^{\text{transp}}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},E_{\mathbf{l}}) - R^{\text{transp}}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},-E_{\mathbf{l}})\right)G^{\pm}(\mathbf{l},\omega) = \frac{\pi}{2E_{\mathbf{p}}^2} iN^2(1 \mp 1). \tag{3.80}$$

Since the commutator squared has an inhomogeneous term, the physical out-of-time correlation function $f(t,\mathbf{p})$ corresponds to the solution G^-. It can be formally written as

$$G^-(\mathbf{p},\omega) = \left[\frac{1}{-i\omega + \mathcal{L}'(\mathbf{p},\mathbf{l})}\right] \frac{\pi}{E_{\mathbf{p}}^2} iN^2. \tag{3.81}$$

In order to find the Lyapunov exponent(s), one usually study the poles of G^-. The operator $\mathcal{L}'(\mathbf{p},\mathbf{l})$ is

$$\mathcal{L}'(\mathbf{p},\mathbf{l}) = \int \frac{d\mathbf{l}}{(2\pi)^3} \frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}} \left(R^{\text{transp}}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},E_{\mathbf{l}}) - R^{\text{transp}}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},-E_{\mathbf{l}})\right)f(\mathbf{l}) - 2\Gamma_{\mathbf{p}}\, f(\mathbf{p}). \tag{3.82}$$

By using the same results of the previous section, (3.74) and (3.75), we can now give a clear interpretation to the physics captured by the Lyapunov exponent, which are the eigenvalue of the linearized collision integral $\mathcal{L}'$,

$$\mathcal{L}'(\mathbf{p}, \mathbf{l}) = -(R^{\wedge}(\mathbf{p}, \mathbf{l}) + R^{\vee}(\mathbf{p}, \mathbf{l}) - 4(2\pi)^3\delta^3(\mathbf{p} - \mathbf{l})\Gamma_{\mathbf{p}}), \tag{3.83}$$

depicted in fig. 3.7. Compared to the previous result for transport (3.77), here the kernel contains a sign difference which encodes the microscopic dynamics of scrambling, *i.e.* the OTOC counts the gross number of collision compared to the the net energy in the collision tracked by the standard quantum Boltzmann equation.

In order to understand the generality of our results, in the next sections we will focus on systems close to a quantum critical point, where the effects of entanglement and long range interaction becomes more important.

3.4 Bosonic $O(N)$ vector model at the quantum critical regime

In this section we provide further detailed evidences in support of our findings. We will show that our results even extend into the quantum critical regime. We focus on vector models with N components real fields ϕ_a in $(2+1)$ dimensions. Provided with a $O(N)$ symmetry, these theories have a quantum phase transition (QPT) at zero temperature [61], between the disordered phase with vanishing vacuum expectation value , $\langle\phi_a\rangle = 0$, and the order symmetry breaking phase $\langle\phi_a\rangle \neq 0$. They capture the relevant long wavelength degrees of freedom of many physical systems, such as the superfluid to bosonic Mott insulator transition [177] (realised for $N = 2$) and the paramagnet to Heisenberg antiferromagnet ($N = 3$) [17, 178, 179]. Although on both sides of the QCP the system is described by quasiparticle excitations, the finite temperature regime directly above the QPT diagram, often referred to as quantum critical regime, does not have such excitations. This observation might lead the reader to think that the kinetic theory developed in the previous section does not apply, since the pillar of kinetic theory is indeed the existence of quasiparticles.

This objection is too quick. Firstly, we stress that our results concerns hydrodynamical correlation functions. The kinetic theory limit can be considered as providing a microscopic picture of the physics underlying the curious relation between hydrodynamical transport correlation functions and out-of-time correlation functions, but this connection needs not to be limited to this case. Secondly and more concretely, in their seminal paper [124], Damle and Sachdev showed that one can use kinetic theory approaches in the quantum critical regime, despite the fact that there are no quasiparticle excitations. If we are interested in the dc conductivities, $\omega = 0$, at the quantum critical point, where by definition $T = 0$, there are two opposite limit that could be considered: the $T = 0$, $\omega \to 0$ *coherent* and *collisionless* regime, and the $\omega = 0$, $T \to 0$ *incoherent*, *collision-dominated* and *hydrodynamic* regime. They showed that the coherent regime does not yield the correct response, since the process considered are not relevant for the physical properties of the QCP. Instead, it is the hydrodynamic collision-dominated regime that provides the correct description of the dc conductivities and of transport. Moreover, such properties can be obtained by means of the Boltzmann equation, in a regime with quasiparticles and by continuity must hold in the non quasiparticle regime as well. As it is a collision-dominated regime, the collision integral plays the major role.

Now we can turn to our findings. In a recent paper [136], Chowdhury and Swingle studied scrambling in these theories, focusing particularly on the region of the phase diagram above the QCP. We will now review some of their findings and explain how they fit in the framework we have introduced in this chapter and [44]. The Lagrangian of the theory is

$$\mathcal{L} = \frac{1}{2}(\partial\phi_a)^2 - \frac{v}{2N}\left(\phi_a^2 - \frac{N}{g}\right)^2 , \tag{3.84}$$

where $v > 0$ is the self interaction coupling strength and and $g > 0$. In the strong coupling limit, corresponding to $v \to \infty$, the 2+1 dimensional theory is a conformal QFT. By introducing an Hubbard-Stratonovich field, $\lambda(t, \mathbf{x})$, to decouple the quartic term , the action becomes

$$\mathcal{L} = \frac{1}{2}(\partial\phi_a)^2 + \frac{\lambda(t,\mathbf{x})}{2\sqrt{N}}\left(\phi_a^2 - \frac{N}{g}\right) + \frac{\lambda(t,\mathbf{x})^2}{8v}. \tag{3.85}$$

In order to probe the onset of chaos, the authors studied the squared commutator

$$C(t,\mathbf{x}) = -\frac{1}{N^2}\sum_{a,b}\mathrm{Tr}\left[\rho^{1/2}[\phi_a(t,\mathbf{x}),\phi_b]\rho^{1/2}[\phi_a(t,\mathbf{x}),\phi_b]\right]. \tag{3.86}$$

The retarded and Wightman correlation functions involving the ϕ fields are

$$\begin{aligned} G_{\mathrm{R}}(t,\mathbf{x})\delta_{ab} &= -i\theta(t)\langle[\phi_a(t,\mathbf{x}),\phi_b]\rangle = -i\theta(t)\mathrm{Tr}\left(\rho[\phi_a(t,\mathbf{x}),\phi_b]\right), \\ G_W^{\beta/2}(t,\mathbf{x})\delta_{ab} &= \mathrm{Tr}\left(\rho^{1/2}\phi_a(t,\mathbf{x})\rho^{1/2}\phi_b(0)\right), \\ G_W(t,\mathbf{x})\delta_{ab} &= \mathrm{Tr}\left(\rho\phi_a(t,\mathbf{x})\phi_b(0)\right). \end{aligned}$$

The spectral function of the ϕ field, as usual related to the imaginary part of the retarded correlator, in the large N limit is

$$\rho(\omega,\mathbf{k}) = -2\mathrm{Im}[G_{\mathrm{R}}(\omega,\mathbf{k})] = \frac{\pi}{E_{\mathbf{k}}}(\delta(\omega - E_{\mathbf{k}}) - \delta(\omega + E_{\mathbf{k}})), \tag{3.87}$$

with $E_{\mathbf{k}})^2 = \mathbf{k}^2 + \mu^2$, μ being the thermal mass. The Wightman function is also intimately related to the spectral function, as follows

$$G_W(\omega,\mathbf{k}) = \frac{\rho(\omega,\mathbf{k})}{2\sinh(\beta\omega/2)}. \tag{3.88}$$

The Hubbard-Stratonovich field Green's functions in Euclidean time, which we will refer to with a λ subscript, in momentum space reads

$$G_\lambda(i\omega_n,\mathbf{k}) = \frac{1}{-1/4v - \Pi(i\omega_n,\mathbf{k})}. \tag{3.89}$$

and $\Pi(i\omega_n,\mathbf{k})$ is the one loop ϕ_i bubble in Euclidean time. The retarded Green's function can be obtained by analytic continuation of the Euclidean one, $G_{R,\lambda}(\omega,\mathbf{k}) = -G_{E,\lambda}(i\omega_n \to \omega + i\epsilon,\mathbf{k})$. $\Pi_{\mathrm{R}}(\omega,\mathbf{k})$ is obtained by the standard analytic continuation

$$\Pi_{\mathrm{R}}(\omega,\mathbf{k}) = \Pi(i\omega_n \to \omega + i0^+,\mathbf{k}), \tag{3.90}$$

with

$$\Pi(i\omega_n,\mathbf{k}) = \frac{1}{2}\sum\!\!\!\!\!\!\int_{\nu_m,\mathbf{k}} G(i\omega_n + i\nu_m)G(i\omega_n). \tag{3.91}$$

The Wightman function is

$$G^{\beta/2}_{W,\lambda}(\omega,\mathbf{k}) = \frac{\rho_\lambda(\omega,\mathbf{k})}{2\sinh(\beta\omega/2)} = e^{-\frac{\beta\omega}{2}} G^{21}_{W,\lambda}(\omega,\mathbf{k}), \tag{3.92}$$

where

$$G^{21}_{W,\lambda}(\omega,\mathbf{k}) = (1 + n(\omega))\rho_\lambda(\omega,\mathbf{k}). \tag{3.93}$$

To study the quantum critical regime, we will have to take the strong coupling limit, $v \to \infty$, in which the expression for the retarded Green's function of the Hubbard-Stratonovich field simplifies

$$G_{\mathrm{R},\lambda}(\omega,\mathbf{k}) = \frac{1}{\Pi_{\mathrm{R}}(\omega,\mathbf{k})}, \tag{3.94}$$

and the spectral density of the Hubbard-Stratonovich field can be approximated as

$$\rho_\lambda(\omega,\mathbf{k}) = -2\,\mathrm{Im}\left[\frac{1}{\Pi_{\mathrm{R}}(\omega,\mathbf{k})}\right] = \frac{2\,\mathrm{Im}\,[\Pi_{\mathrm{R}}]}{\mathrm{Im}\,[\Pi_{\mathrm{R}}]^2 + \mathrm{Re}\,[\Pi_{\mathrm{R}}]^2}. \tag{3.95}$$

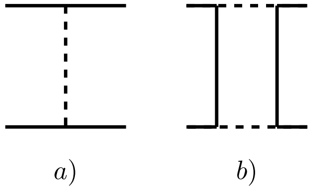

Figure 3.8: The contributions to the kernel of the OTOC. $a)$ corresponds to $G_{W,\lambda}$ while $b)$ corresponds to G_{eff} in (3.96).

The derivation of the Bethe-Salpeter equation for C (3.86) is given in [136] and the result is

$$C(\omega,p) = G_{\mathrm{R}}(k-p)G_{\mathrm{R}}(p)\left[1+\int_l R^{\mathrm{OTOC}}(p,l)C(\omega,l)\right], \tag{3.96}$$

whose kernel is given by the Wightman propagator of the auxiliary field and by

$$R^{\mathrm{OTOC}}(p,l) = \frac{1}{N}(G^{\beta/2}_{W,\lambda}(l-p)+G_{eff}(l,p)), \tag{3.97}$$

$$G_{eff}(l,p) = \int_{l'} G^{\beta/2}_{W}(l'-p)G^{\beta/2}_{W}(l-l')G^{\lambda}_{A}(l')G^{\lambda}_{R}(l'). \tag{3.98}$$

Similarly to the matrix model (3.66), in the low frequency limit $k=(\omega,\mathbf{0})$ and $\omega\to 0$, the product $G_{\mathrm{R}}(k-p)G_{\mathrm{R}}(p)$ can be approximated with [136]

$$G_{\mathrm{R}}(k-p)G_{\mathrm{R}}(p) \approx_{\omega\to 0} \frac{\pi}{E_{\mathbf{p}}}\frac{\delta(p_0^2-E_{\mathbf{p}}^2)}{(-i\omega+2\Gamma_{\mathbf{p}})}, \tag{3.99}$$

so the BSE (3.96) reads

$$(-i\omega+2\Gamma_{\mathbf{p}})C(p|\omega) = \frac{\pi}{E_{\mathbf{p}}}\delta(p_0^2-E_{\mathbf{p}}^2)\int\frac{d^3l}{(2\pi)^3}R^{\mathrm{OTOC}}(p,l)C(l|\omega). \tag{3.100}$$

By evaluating the delta function, (3.100) becomes

$$(-i\omega+2\Gamma_{\mathbf{p}})C(\omega,\mathbf{p}) - \int_{\mathbf{l}}\frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}}(R^{\mathrm{OTOC}}(E_{\mathbf{p}},\mathbf{p}|E_{\mathbf{l}},\mathbf{l})+R^{\mathrm{OTOC}}(E_{\mathbf{p}},\mathbf{p}|-E_{\mathbf{l}},\mathbf{l}))C(\omega,\mathbf{l}) = \frac{\pi}{2E_{\mathbf{p}}^2} \tag{3.101}$$

3.4.1 Transport in the O(N) vector model with the 2PI formalism

By computing the transport in this model we will show that , also in this case, there exists a mapping between the kernels of the OTOC and the kernels of transport,

$$R^{OTOC}(p,l) = \frac{\sinh(\beta l^0/2)}{\sinh(\beta p^0/2)} R^{\text{transp}}(p,l), \tag{3.102}$$

and that the interpretation of scrambling in terms of the kinetic theory equation depicted in fig. 3.7 holds also in the region above the quantum critical point.

Besides the methods we have already mentioned, *i.e.* the use of the finite temperature optical theorem by Jeon, and the more compact use of Schwinger-Keldysh formalism by Heinz and Wang, there is another way to approach the problem and it is by means of the two-particle irreducible $2PI$ effective action. The advantage of using this effective action is that, at the first non trivial truncation in the large N, or in weak coupling, it automatically provides the proper resummation of the relevant diagrams [117, 134, 180]. Moreover it can be proved that, in presence of gauge fields, the result obtained does not depend on the gauge fixing term and respects the Ward identities [116, 181]. In the present chapter we will focus only on a self-interacting spin-0 and spin-1/2 fields, but this formalism allows quite easily a generalization to gauge theories. We will use this method here.

In this section we will closely follow [134], though with the Lagrangian (3.84) in $(2+1)$ dimensions. The effective action in the bosonic case can be parametrized as follows [182]

$$\Gamma[G] = \frac{i}{2}\mathrm{Tr}\ln G^{-1} + \frac{i}{2}\mathrm{Tr}\, G_0^{-1}(G - G_0) + \Gamma_2[G], \tag{3.103}$$

where the 2PI part $\Gamma_2[G]$ can be expanded in $1/N$. For the model considered, (3.84), the expansion is [182]

$$\Gamma_2[G] = \Gamma_2[G]^{LO} + \Gamma_2[G]^{NLO} + ... \tag{3.104}$$

where the leading and the next to the leading order terms are depicted in

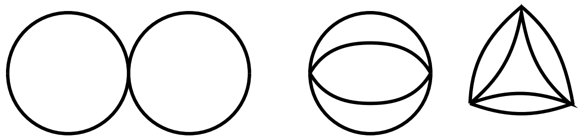

Figure 3.9: The contributions to the 2PI effective action in the $1/N$ expansion. $a)$ is the leading order contribution in $1/N$ and $b)$ the next-to-the leading order.

fig. 3.9 and corresponds to the terms

$$\Gamma_2[G]^{LO} = -\frac{v}{2N}\int_x G_{mm}(x,x)G_{nn}(x,x) \tag{3.105}$$

$$\Gamma_2[G]^{NLO} = \frac{i}{2}\mathrm{Tr}\left[-\frac{4iv}{N}\Pi - \left(\frac{4iv}{N}\Pi\right)^2 - \left(\frac{4iv}{N}\Pi\right)^3 - ...\right] = \frac{i}{2}\mathrm{Tr}\ln\mathbf{B}. \tag{3.106}$$

In (3.106) the bubble diagram $\Pi(x,y)$, depicted in 3.10, is

$$\Pi(x,y) = \frac{1}{2}G_{ab}(x,y)G_{ab}(x,y), \tag{3.107}$$

and we defined the auxiliary bilocal field $\mathbf{B}(x,y)$ as follows

$$\mathbf{B}(x,y) = \delta_{\mathcal{C}}(x-y) + \frac{4iv}{N}\,\Pi(x,y). \tag{3.108}$$

The inverse of $\mathbf{B}(x,y)$ has a very intuitive physical meaning: by defining

$$D(x,y) = -\frac{2iv}{N}\,\mathbf{B}(x,y)^{-1} \tag{3.109}$$

and using the identity $\int_y \mathbf{B}(x,y)\mathbf{B}(y,z)^{-1} = \delta_{\mathcal{C}}(x-z)$, we readily obtain that the correlator D,which we will now refer to as auxiliary field, satisfies the following equation, depicted in fig. 3.11,

$$D(x,y) = \frac{4iv}{N}\left(\delta(x-y) - \int_z \Pi(x,z)D(z,y)\right). \tag{3.110}$$

From the effective action (3.104) and the corresponding large N expansion (3.105), we can obtain the integral equation for the truncated 4-point

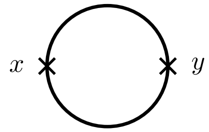

Figure 3.10: The bubble diagram $\Pi(x, y)$ contributions to the 2PI effective action in the $1/N$ expansion.

function

$$\Gamma^{(4)}_{ab;cd}(x, y; x', y') = \Lambda_{ab;cd}(x, y; x', y') + \frac{1}{2}\int_{ww'zz'} \Lambda_{ab;ef}(x, y; w, z)G_{ee'}(w, w')G_{ff'}(z, z')\Gamma^{(4)}_{e'f';cd}(w', z'; x', y') \tag{3.111}$$

The previous expression is quite general and can be used both in Imaginary Time Formalism (ITF) and in the Closed Time Path (CTP) formalism, without any restriction on the number of time folds. So, by properly adding the indices that parametrize the SK contour, the previous equation gives the BSE for any time-ordered or out-of-time ordered correlation function in the large N expansion for a bosonic theory. By imposing the extremization of the effective action (3.173), we obtain the Schwinger-Dyson equation

$$G^{-1}_{ab}(x, y) = G^{-1}_{0,ab}(x, y) - \Sigma_{ab}(x, y), \tag{3.112}$$

with the free propagator being

$$G^{-1}_{0,ab}(x, y) = -\delta_{ab}(\Box_x + 4v^2/g^2)\delta^4(x - y). \tag{3.113}$$

The self-energies are defined as functional derivatives of the 2PI effective action (3.104) with respect to the bilocal field G_{ab}

$$\Sigma_{ab}(x, y) = 2i\frac{\delta\Gamma_2[G]}{\delta G_{ab}(x, y)}, \tag{3.114}$$

and the kernels of the (3.111) can be obtained by a further functional derivative with respect to G

$$\Lambda_{ab;cd}(x, y; x', y') = \frac{\delta\Sigma_{ab}(x, y)}{\delta G_{cd}(x', y')}. \tag{3.115}$$

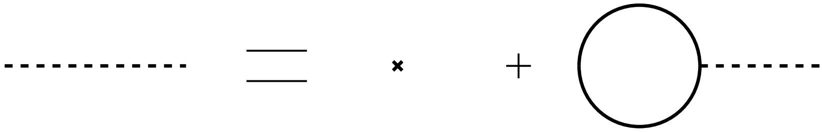

Figure 3.11: The diagrammatic recursive expression of the propagator $D(x, y)$ (3.110).

Up to this point, the expressions are still in real time and the BSE for the amputated 4-point function (3.111) is very lengthy and contains several terms. Nevertheless, since we are interested in the late time physics, most of these terms are negligible. This statement is the equivalent to the pinching pole approximation. In order to use it, it is convenient to move into momentum space and use the Matsubara formalisms, for which an analogue of the pinching pole approximation was derived in [115]. In momentum space, the correlation function is

$$G(i\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + 4\frac{v^2}{g^2} + \Sigma(i\omega_n, \mathbf{p})}. \tag{3.116}$$

The self energy contributions are computed using (3.114) and read

$$\Sigma^{LO}(i\omega_n, \mathbf{p}) = 2v \sum\!\!\!\!\!\!\!\int_{\nu_m, \mathbf{k}} G(i\nu_m, \mathbf{k}) \tag{3.117}$$

$$\Sigma^{NLO}(i\omega_n, \mathbf{p}) = - \sum\!\!\!\!\!\!\!\int_{\nu_m, \mathbf{k}} G(i\omega_n + i\nu_m, \mathbf{p} + \mathbf{k}) D(i\nu_n, \mathbf{k}). \tag{3.118}$$

The correlation function D is obtained by inverting the (3.110), choosing the Matsubara contour and going into momentum space

$$D(i\omega_n, \mathbf{p}) = \frac{1}{-\frac{N}{4v} - \Pi(i\omega_n, \mathbf{p})}. \tag{3.119}$$

Finally, the bubble diagram is

$$\Pi(i\omega_n, \mathbf{p}) = \frac{N}{2} \sum\!\!\!\!\!\!\!\int_{\nu_m, \mathbf{k}} G(i\omega_n + i\nu_m, \mathbf{p} + \mathbf{k}) G(i\nu_m, \mathbf{k}). \tag{3.120}$$

From now on we will use the following convention: with capital case momenta we indicate momenta in imaginary time formalism, $P = (i\omega_n, \mathbf{p})$, where $\omega_n = 2\pi nT$ is the Matsubara frequency. With lower case, instead, we indicate momenta after the analytic continuation to Lorentzian signature.

The amputated 4-point function is related to 3-point vertex function by the following identity

$$\Gamma_{ab}(P+Q,P) = 2\delta_{ab} + \sum_{n'} \int_{\mathbf{l}} G(L+Q)G(L)\Gamma^{(4)}_{cc,ab}(L,P;Q) \qquad (3.121)$$

By inserting the BSE for the $\Gamma^{(4)}_{cc,ab}$ into the previous expression, we obtain a BSE for the 3-point vertex

$$\Gamma_{ab}(P+Q,P) = 2\delta_{ab} + \frac{1}{2}\sum_{n'} \int_{\mathbf{l}} G(L+Q)\Gamma_{cd}(L+Q,L)G(L)\Lambda_{cd;ab}. \qquad (3.122)$$

Parametrizing the vertex as

$$\Gamma_{ab}(P+Q,P) = 2\delta_{ab}\Gamma(P+Q,P) \qquad (3.123)$$

we can thus derive the BSE for the diagonal part $\Gamma(P+Q,P)$ by simply substituting the previous expression in (3.122) and contracting with δ_{ab}

$$\Gamma(P+Q,P) = 1 + \frac{1}{2N}\sum_{n'} \int_{\mathbf{l}} G(L+Q)\Gamma(L+Q,L)G(L)\Lambda_{cc;aa}(L,P;Q). \qquad (3.124)$$

By means of (3.115), the leading and next-to-leading order contribution to the kernel of the integral equation are

$$\begin{aligned}\Lambda_{cd;ab}(L,P;Q) = &-\frac{4v}{N}\delta_{ab}\delta_{cd} + (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})D(L-P) \\ &+ 2\delta_{ab}\delta_{cd}D(R)D(R+Q)G(L-R)G(R-P),\end{aligned}$$

whose diagonal parts, depicted in fig. 3.12, are

$$\begin{aligned}&\Lambda^{LO}_{cc;aa}(L,P;Q) = -4vN, \\ &\Lambda^{NLO}_{cc;aa}(L,P;Q) = 2ND(L-P) + 2N^2 D(R)D(R+Q)G(L-R)G(R-P).\end{aligned}$$

First, in order to compare with the results of [136], we observe that it is convenient to take the N dependence out of D in (3.119) and of the bubble loop in (3.120), $D \to D/N$ and $\Pi \to N\Pi$. The correlation functions of the auxiliary field D are identical to the correlation functions of the Hubbard-Stratonovich field λ of the Lagrangian (3.85), as can be see by comparing (3.89) and (3.91) with (3.119) and (3.120)

$$D(i\omega_n, \mathbf{p}) = G_\lambda(i\omega_n, \mathbf{p}). \qquad (3.125)$$

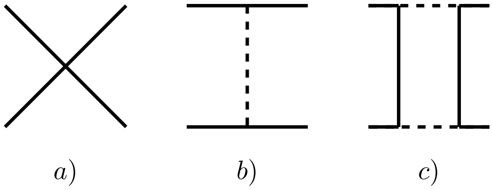

Figure 3.12: The contributions to the kernel of the 4-point function in the 2PI formalism. $a)$ corresponds to $\Lambda^{LO}_{cc;aa}(L,P;Q)$, while $b)$ and $c)$ to the two terms in $\Lambda^{NLO}_{cc;aa}(L,P;Q)$. As we will show, $a)$ will be subleading with respect to $b)$ and $c)$ after the analytic continuation to real frequencies.

Moreover, we include the factor of 1/2 in front of the kernels (3.124) and we have

$$\begin{aligned}\Lambda^{LO}(L,P;Q) \equiv \frac{1}{2}\Lambda^{LO}_{cc;aa}(L,P;Q) =& -2vN,\\ \Lambda^{NLO}(L,P;Q) \equiv \frac{1}{2}\Lambda^{NLO}_{cc;aa}(L,P;Q) =& G_\lambda(L-P)\\ &+ G_\lambda(R)G_\lambda(R+Q)G(L-R)G(R-P).\end{aligned}$$

The BSE for the 3-point vertex has thus the form

$$\Gamma(P+Q,P) = 1+\frac{1}{N}\sum_{n'}\int_{\mathbf{l}} G(L+Q)\Gamma(L+Q,L)G(L)\left[\Lambda^{LO}+\Lambda^{NLO}(L,P;Q)\right]. \tag{3.126}$$

Now, we multiply the bare vertex Γ with the propagators in the loop, as depicted in fig 3.13, which in Euclidean is

$$\begin{aligned}\int_p \tilde{G}(p+q,p) =& \sum\!\!\!\!\!\!\int_{p_n,\mathbf{p}} \tilde{G}(ip_n+i\nu_{n'},ip_n)|_{i\nu_{n'}\to q_0+i0^+}\\ =& \sum\!\!\!\!\!\!\int_{p_n,\mathbf{p}} G(ip_n+i\nu_{n'})\Gamma(ip_n+i\nu_{n'},ip_n)G(ip_n)|_{i\nu_{n'}\to q_0+i0^+}.\end{aligned} \tag{3.127}$$

This function satisfies the following BSE, which can be obtained from (3.126),

$$\begin{aligned}\tilde{G}(P+Q,P) =& G(P+Q)G(P)\\ &\left[1+\frac{1}{N}\sum_{n'}\int_{\mathbf{l}}\tilde{G}(L+Q,L)\left[\Lambda^{LO}+\Lambda^{NLO}(L,P;Q)\right]\right].\end{aligned} \tag{3.128}$$

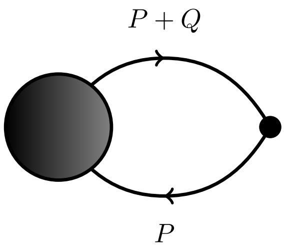

Figure 3.13: Diagrammatic representation of the full correlator $\widetilde{G}$ as a function of the vertex 3-point function $\Gamma(P+Q,P)$.

This integral equation is still expressed in imaginary time and it has to be analytically continued after performing the Matsubara sum. Following the techniques developped in [115, 183], and described in Appendix 3.E, we obtain the following result

$$\tilde{G}(p+q,p) = G_R(p+q)G_A(p) \left[1+\frac{1}{N}\int_l (n(l_0-p_0)-n(l_0))\Lambda^{NLO}(l,p)\tilde{G}(l+q,l)\right], \tag{3.129}$$

where

$$\Lambda^{NLO}(l,p) = \rho_\lambda(l-p) + \int_s (n(p_0-s_0)-n(l_0-s_0))\rho(p-s)\rho(l-s) \times G_{R,\lambda}(s)G_{A,\lambda}(s). \tag{3.130}$$

A closer look to the (3.129) shows that the leading order rung, $\Lambda^{LO}_{cc;aa}$, does not contribute to the BSE in real time. This is because the $\Lambda^{LO}_{cc;aa}$ does not contain any *pinching-pole* singularity and it is subleading with respect to $\Lambda^{NLO}_{cc;aa}$. As shown in appendix 3.B, massaging the product $(n(l_0-p_0)-n(l_0))\Lambda^{NLO}(l,p)$ gives a kernel

$$(n(l_0-p_0)-n(l_0))\Lambda^{NLO}(l,p) = \frac{n_B(l_0)}{n_B(p_0)} \times \left[G^{21}_\lambda(l-p) + \int_s G_{12}(p-s)G_{21}(l-s)G_{R,\lambda}(s)G_{A,\lambda}(s)\right]. \tag{3.131}$$

Thus the BSE for transport is

$$\tilde{G}(p+q,p) = G_R(p+q)G_A(p)\left[1+\int_l R^{\text{transp}}(l,p)\tilde{G}(l+q,l)\right], \tag{3.132}$$

with

$$R^{\text{transp}}(p,l) = \frac{1}{N}\frac{n_B(l_0)}{n_B(p_0)} \times \left[G_\lambda^{21}(l-p) + \int_s G_{12}(p-s)G_{21}(l-s)G_{R,\lambda}(s)G_{A,\lambda}(s)\right]. \tag{3.133}$$

Now we want to compare it with the kernel of the OTOC (3.97). By using (3.92) and

$$G^{\beta/2}(l-p) = e^{-\beta(l_0-p_0)/2}G_W^{21}(l-p) = e^{\beta(l_0-p_0)/2}G_W^{12}(l-p), \tag{3.134}$$

we obtain that the kernel R^{transp} and the kernel $R^{\text{OTOC}}(l,p)$ are related by the simple relation

$$R^{\text{OTOC}}(l,p) = \frac{\sinh(\beta l_0/2)}{\sinh(\beta p_0/2)}R^{\text{transp}}(l,p), \tag{3.135}$$

which proves our claim. Substituting the previous relation into the on-shell BSE for chaos, we obtain the following equation

$$(-i\omega + 2\Gamma_{\mathbf{p}})C(\omega,\mathbf{p}) - \int_{\mathbf{l}} \frac{\sinh(\beta E_{\mathbf{l}}/2)}{\sinh(\beta E_{\mathbf{l}}/2)}\frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}} \times (R^{\text{transp}}(E_{\mathbf{p}},\mathbf{p}|E_{\mathbf{l}},\mathbf{l}) - R^{\text{transp}}(E_{\mathbf{p}},\mathbf{p}|-E_{\mathbf{l}},\mathbf{l})))C(\omega,\mathbf{l}) = \frac{\pi}{2E_{\mathbf{p}}^2} \tag{3.136}$$

Similarly to section 3.3.2, we can find the following solution for the BSE for $\tilde{G}(p+q,p)$ (3.132)

$$(-i\omega + 2\Gamma_{\mathbf{p}})\tilde{G}(\omega,\mathbf{p}) - \int_{\mathbf{l}} \frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}} \times (R^{\text{transp}}(E_{\mathbf{p}},\mathbf{p}|E_{\mathbf{l}},\mathbf{l}) + R^{\text{transp}}(E_{\mathbf{p}},\mathbf{p}|-E_{\mathbf{l}},\mathbf{l})))\tilde{G}(\omega,\mathbf{l}) = \frac{\pi}{2E_{\mathbf{p}}^2}. \tag{3.137}$$

In the next section we explore the kinetic theory limit of both BSEs.

3.4.2 Kinetic theory analysis

In this section we show how the different the sign in (3.136) and (3.137) corresponds to a different counting in the collision integral of the kinetic equation, as depicted respectively in fig. 3.7 and fig. 3.6. We write the kernel (3.130) as follows

$$\Lambda^{NLO}(r,p) = \rho_\lambda(r-p) + \int_l (n(l^0) - n(r^0-p^0+l^0))\rho(l)\rho(r-p+l)|G_{\lambda,R}(p-l)|^2. \tag{3.138}$$

The imaginary part of the bosonic self-energy is given by (3.120) after performing the Matsubara sum and taking the analytical continuation to obtain the retarded bubble

$$\mathrm{Im}\Pi_R(r-p) = -\frac{1}{4}\int_L (n(l^0) - n(r^0 - p^0 + l^0))\rho(l)\rho(r-p+l). \quad (3.139)$$

Since

$$\rho_\lambda(p) = -2\,\mathrm{Im}\Pi_{R,\lambda}(p)|G_{\lambda,R}(p)|^2, \quad (3.140)$$

by inserting $\int_{L'} \delta^3(r+l-l'-p)$, (3.138) becomes

$$\Lambda^{NLO}(r,p) = \frac{1}{2}\int_{l,l'} (n(l^0) - n(l'^0))\rho(l)\rho(l')[|G_{\lambda,R}(r-p)|^2 + |G_{\lambda,R}(r-l')|^2 + |G_{\lambda,R}(r+l)|^2]. \quad (3.141)$$

We recognize the term inside parenthesis as the (*off-shell*) scattering amplitude for the process $(r,l) \to (p,l')$, as depicted in fig. 3.14

$$|\mathcal{T}_{(r,l)\to(p,l')}|^2 = \frac{1}{N}(|G_{\lambda,R}(r-p)|^2 + |G_{\lambda,R}(r-l')|^2 + |G_{\lambda,R}(r+l)|^2), \quad (3.142)$$

and we can rewrite the kernel as

$$\Lambda^{NLO}(r,p) = \frac{N}{2}\int_{l,l'} (n(l^0) - n(l'^0))\rho(l)\rho(l')|\mathcal{T}_{(r,l)\to(p,l')}|^2. \quad (3.143)$$

Now we express the kernel of the *on-shell* BSE for chaos (3.136) and transport (3.137) gain and loss processes. Thus we focus on

$$\frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}}(R^{\mathrm{transp}}(E_{\mathbf{p}},\mathbf{p}|E_{\mathbf{l}},\mathbf{l}) \pm R^{\mathrm{transp}}(E_{\mathbf{p}},\mathbf{p}|-E_{\mathbf{l}},\mathbf{l}))). \quad (3.144)$$

Using the definition of R^{transp} (3.133) we obtain

$$\begin{aligned}
&\int_{\mathbf{r},\mathbf{l},\mathbf{l}'} \frac{1}{2E_{\mathbf{p}}}(n(E_{\mathbf{r}} - E_{\mathbf{p}}) - n(E_{\mathbf{r}}))(n(E_{\mathbf{l}}) - n(E_{\mathbf{l}'}))\delta(E_{\mathbf{r}} + E_{\mathbf{l}} - E_{\mathbf{p}} - E_{\mathbf{l}'}) \\
&\qquad \times \frac{1}{N}(|G_{\lambda,R}(E_{\mathbf{r}} - E_{\mathbf{p}})|^2 + |G_{\lambda,R}(E_{\mathbf{r}} - E_{\mathbf{l}'})|^2 + |G_{\lambda,R}(E_{\mathbf{r}} + E_{\mathbf{l}})|^2) \\
&\mp \frac{1}{2}\int_{\mathbf{r},\mathbf{l},\mathbf{l}'} \frac{1}{2E_{\mathbf{p}}}(n(E_{\mathbf{r}} + E_{\mathbf{p}}) - n(E_{\mathbf{r}}))(n(E_{\mathbf{l}}) - n(-E_{\mathbf{l}'}))\delta(E_{\mathbf{l}} + E_{\mathbf{l}'} - E_{\mathbf{p}} - E_{\mathbf{r}}) \\
&\qquad \times \frac{1}{N}(|G_{\lambda,R}(E_{\mathbf{r}} + E_{\mathbf{p}})|^2 + |G_{\lambda,R}(E_{\mathbf{r}} - E_{\mathbf{l}'})|^2 + |G_{\lambda,R}(E_{\mathbf{r}} - E_{\mathbf{l}})|^2).
\end{aligned} \quad (3.145)$$

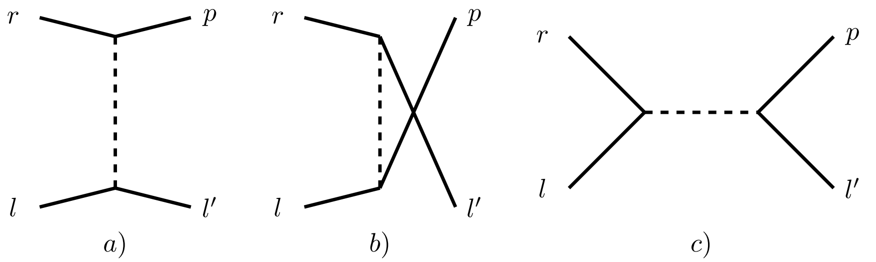

Figure 3.14: The contributions to the kernel of the kinetic equation of the creation of a particle with momentum p, first line in eq. (3.147).

By using the identities satisfied by the Bose-Einstein distribution function and listed in Appendix 3.B, we can rewrite the previous expression as follows

$$\begin{aligned}&\frac{1}{1+n(E_{\mathbf{p}})}\int_{\mathbf{r},\mathbf{l},\mathbf{l}'}\frac{1}{2E_{\mathbf{p}}}n(E_{\mathbf{l}'})(1+n(E_{\mathbf{r}}))(1+n(E_{\mathbf{l}}))\delta(E_{\mathbf{r}}+E_{\mathbf{l}}-E_{\mathbf{p}}-E_{\mathbf{l}'})\\&\qquad\times\frac{1}{N}(|G_{\lambda,R}(E_{\mathbf{r}}-E_{\mathbf{p}})|^2+|G_{\lambda,R}(E_{\mathbf{r}}-E_{\mathbf{l}'})|^2+|G_{\lambda,R}(E_{\mathbf{r}}+E_{\mathbf{l}})|^2)+\\&\mp\frac{1/2}{1+n(E_{\mathbf{p}})}\int_{\mathbf{r},\mathbf{l},\mathbf{l}'}\frac{1}{2E_{\mathbf{p}}}n(E_{\mathbf{r}})(1+n(E_{\mathbf{l}}))(1+n(E_{\mathbf{l}'}))\delta(E_{\mathbf{l}}+E_{\mathbf{l}'}-E_{\mathbf{p}}-E_{\mathbf{r}})\\&\qquad\times\frac{1}{N}(|G_{\lambda,R}(E_{\mathbf{r}}-E_{\mathbf{l}})|^2+|G_{\lambda,R}(E_{\mathbf{r}}-E_{\mathbf{l}'})|^2+|G_{\lambda,R}(E_{\mathbf{r}}+E_{\mathbf{p}})|^2).\end{aligned}\tag{3.146}$$

By comparison with (3.62), it is clear that the first two lines of (3.146) correspond to R^{gain}, while the third and fourth lines of (3.146) are identical to the second line of (3.62).

$$\begin{aligned}&\frac{1}{1+n(E_{\mathbf{p}})}\int_{\mathbf{r},\mathbf{l},\mathbf{l}'}\frac{1}{2E_{\mathbf{p}}}n(E_{\mathbf{l}'})(1+n(E_{\mathbf{r}}))(1+n(E_{\mathbf{l}}))\delta(E_{\mathbf{r}}+E_{\mathbf{l}}-E_{\mathbf{p}}-E_{\mathbf{l}'})\\&\qquad\qquad|\mathcal{T}_{(r,l)\to(p,l')}|^2\\&\pm\frac{1/2}{1+n(E_{\mathbf{p}})}\int_{\mathbf{r},\mathbf{l},\mathbf{l}'}\frac{1}{2E_{\mathbf{p}}}n(E_{\mathbf{r}})(1+n(E_{\mathbf{l}}))(1+n(E_{\mathbf{l}'}))\delta(E_{\mathbf{l}}+E_{\mathbf{l}'}-E_{\mathbf{p}}-E_{\mathbf{r}})\\&\qquad\qquad|\mathcal{T}_{(r,p)\to(l,l')}|^2.\end{aligned}\tag{3.147}$$

In order to complete the analysis of the BSE, we need to understand the $2\Gamma_{\mathbf{p}}$ contribution in (3.136) and (3.137). To do so, since $\Gamma_{\mathbf{p}} =$

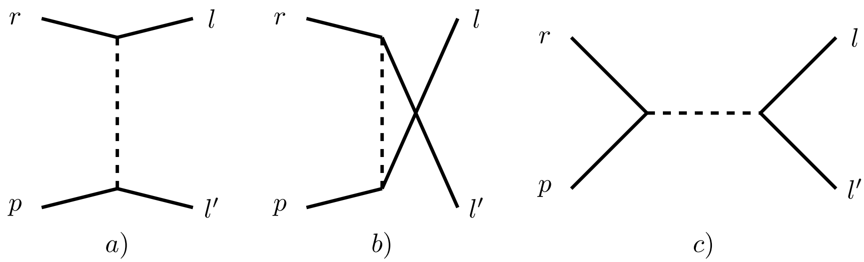

Figure 3.15: The contributions to the kernel of the kinetic equation of the annihilation of a particle with momentum p, last line in eq. (3.147).

$-\mathrm{Im}\Sigma^{NLO}(E_{\mathbf{p}},\mathbf{p})/E_{\mathbf{p}}$, we start by inspecting $\mathrm{Im}\Sigma^{NLO}(E_{\mathbf{p}},\mathbf{p})$

$$\mathrm{Im}\,\Sigma^{NLO}(P) = \frac{1}{2}\int_R \rho(R)\rho_\lambda(R-P)(n(r^0)-n(r^0-p^0)). \tag{3.148}$$

By means of (3.140), expanding $\mathrm{Im}\,\Pi_R$ with (3.139) and retaining only the kinematically allowed terms, we arrive to

$$\begin{aligned}\mathrm{Im}\Sigma^{NLO}(E_{\mathbf{p}},\mathbf{p}) = \frac{1}{4}\int_{\mathbf{l},\mathbf{l}',\mathbf{r}} & \\ \Big[(n(E_{\mathbf{l}})-n(E_{\mathbf{l}'}))(n(E_{\mathbf{r}})-n(E_{\mathbf{r}}-E_{\mathbf{p}}))|D_R(E_{\mathbf{r}}-E_{\mathbf{p}},\mathbf{r}-\mathbf{p})|^2 & \\ \times\,\delta(E_{\mathbf{r}}+E_{\mathbf{l}}-E_{\mathbf{p}}-E_{\mathbf{l}'}) & \\ +\,(n(E_{\mathbf{l}})-n(-E_{\mathbf{l}'}))(n(-E_{\mathbf{r}})-n(-E_{\mathbf{r}}-E_{\mathbf{p}}))|D_R(E_{\mathbf{r}}+E_{\mathbf{p}},\mathbf{r}+\mathbf{p})|^2 & \\ \times\,\delta(E_{\mathbf{r}}+E_{\mathbf{p}}-E_{\mathbf{l}}-E_{\mathbf{l}'}) & \\ +\,(n(-E_{\mathbf{l}})-n(-E_{\mathbf{l}'}))(n(E_{\mathbf{r}})-n(E_{\mathbf{r}}-E_{\mathbf{p}}))|D_R(E_{\mathbf{r}}-E_{\mathbf{p}},\mathbf{r}-\mathbf{p})|^2 & \\ \times\,\delta(E_{\mathbf{r}}+E_{\mathbf{l}'}-E_{\mathbf{p}}-E_{\mathbf{l}})\Big]. &\end{aligned} \tag{3.149}$$

We can now use the properties of the Bose-Einstein distribution function listed in Appendix 3.B. Thus,

$$\mathrm{Im}\Sigma^{NLO}(E_{\mathbf{p}},\mathbf{p}) = \frac{1}{4}\int_{\mathbf{l},\mathbf{l}',\mathbf{r}} \tag{3.150}$$
$$\Bigg[-\frac{1}{1+n(E_{\mathbf{p}})}n(E_{\mathbf{l}'})(1+n(E_{\mathbf{r}}))(1+n(E_{\mathbf{l}}))|D_R(E_{\mathbf{r}}-E_{\mathbf{p}},\mathbf{r}-\mathbf{p})|^2$$
$$\times\,\delta(E_{\mathbf{r}}+E_{\mathbf{l}}-E_{\mathbf{p}}-E_{\mathbf{l}'})$$
$$-\frac{1}{1+n(E_{\mathbf{p}})}n(E_{\mathbf{r}})(1+n(E_{\mathbf{l}}))(1+n(E_{\mathbf{l}'}))|D_R(E_{\mathbf{r}}+E_{\mathbf{p}},\mathbf{r}+\mathbf{p})|^2$$
$$\times\,\delta(E_{\mathbf{r}}+E_{\mathbf{p}}-E_{\mathbf{l}}-E_{\mathbf{l}'})$$
$$-\frac{1}{1+n(E_{\mathbf{p}})}n(E_{\mathbf{l}})(1+n(E_{\mathbf{r}}))(1+n(E_{\mathbf{l}'}))|D_R(E_{\mathbf{r}}-E_{\mathbf{p}},\mathbf{r}-\mathbf{p})|^2$$
$$\times\,\delta(E_{\mathbf{r}}+E_{\mathbf{l}'}-E_{\mathbf{p}}-E_{\mathbf{l}})\Bigg].$$

Finally, by properly relabelling the integration variables, we recognize the expression

$$\mathrm{Im}\Sigma^{NLO}(E_{\mathbf{p}},\mathbf{p}) = \tag{3.151}$$
$$-\frac{1/4}{1+n(E_{\mathbf{p}})}\int_{\mathbf{r},\mathbf{l},\mathbf{l}'} n(E_{\mathbf{r}})(1+n(E_{\mathbf{l}}))(1+n(E_{\mathbf{l}'}))\delta(E_{\mathbf{l}}+E_{\mathbf{l}'}-E_{\mathbf{p}}-E_{\mathbf{r}})$$
$$\times\,(|D_{\mathrm{R}}(E_{\mathbf{r}}-E_{\mathbf{l}})|^2+|D_{\mathrm{R}}(E_{\mathbf{r}}-E_{\mathbf{l}'})|^2+|D_{\mathrm{R}}(E_{\mathbf{r}}+E_{\mathbf{p}})|^2).$$

The thermal width is

$$2\Gamma_{\mathbf{p}} = \frac{1/2}{1+n(E_{\mathbf{p}})}\int_{\mathbf{r},\mathbf{l},\mathbf{l}'}\frac{1}{2E_{\mathbf{p}}}n(E_{\mathbf{r}})(1+n(E_{\mathbf{l}}))(1+n(E_{\mathbf{l}'})) \tag{3.152}$$
$$\delta(E_{\mathbf{l}}+E_{\mathbf{l}'}-E_{\mathbf{p}}-E_{\mathbf{r}})\times(|D_{\mathrm{R}}(E_{\mathbf{r}}-E_{\mathbf{l}})|^2+|D_{\mathrm{R}}(E_{\mathbf{r}}-E_{\mathbf{l}'})|^2+$$
$$|D_{\mathrm{R}}(E_{\mathbf{r}}+E_{\mathbf{p}})|^2).$$

Thus, also for the bosonic $O(N)$ vector model we have identified the gain and loss contribution in the kernel of the BSE

$$R^{\wedge}(\mathbf{p},\mathbf{l}) = \frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}}R^{\mathrm{transp}}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},E_{\mathbf{l}}), \tag{3.153}$$
$$R^{\vee}(\mathbf{p},\mathbf{l}) = -\frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}}R^{\mathrm{transp}}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},-E_{\mathbf{l}})+2(2\pi)^2\delta^2(\mathbf{p}-\mathbf{l})\Gamma_{\mathbf{p}} \tag{3.154}$$

We can now rewrite the BSE for transport (3.137) and chaos (3.136) as kinetic equation with the following collision integrals

$$-i\omega\,\tilde{G}(\omega,\mathbf{p}) = \int_{\mathbf{l}} (R^{\wedge}(\mathbf{p},\mathbf{l}) - R^{\vee}(\mathbf{p},\mathbf{l}))\,\tilde{G}(\omega,\mathbf{l}), \tag{3.155}$$

$$-i\omega\,C(\omega,\mathbf{p}) = \int_{\mathbf{l}} (R^{\wedge}(\mathbf{p},\mathbf{l}) + R^{\vee}(\mathbf{p},\mathbf{l}) - 4(2\pi)^2\delta^2(\mathbf{p}-\mathbf{l})\Gamma_{\mathbf{p}})C(\omega,\mathbf{l}) \tag{3.156}$$

3.4.3 Towards the bosonic Quantum Critical Point

In the previous sections we showed that also in the $O(N)$ model scrambling and transport are related at the level of Green's function. From this, it is possible to derive the kinetic theory interpretation of scrambling as in equation (3.156). The derivation of these identities merely rely on two hypothesis: the large N limit and the hydrodynamic limit. Consequently, our results hold as far as these hypothesis are satisfied. As mentioned in the introduction, close to the QCP transport can be studied by analytically continuing the hydrodynamic computation, performed outside the quantum critical regime, into the quantum critical regime. In the $O(N)$ model, this regime is obtained in the strong coupling limit ($v \to \infty$) of the Lagrangian (3.84),

$$\mathcal{L} = \frac{1}{2}(\partial\phi_a)^2 - \frac{v}{2N}\left(\phi_a^2 - \frac{N}{g}\right)^2, \tag{3.157}$$

at the value of the critical coupling g_c. This value can be obtained by imposing that the thermal mass vanishes, and equals to [136]

$$\frac{1}{g_c} = \frac{\Lambda}{4\pi}, \tag{3.158}$$

Λ being the physical cutoff. In the quantum critical region, away from zero temperature, the thermal mass becomes $\mu^2 \approx 0.962T$ [136]. The only changes in the BSE, both for transport and chaos, are thus the value of the thermal mass in the *on-shell* condition, $E_{\mathbf{p}} = \mathbf{p}^2 + \mu^2$ and the expression of the Hubbard-Stratonovich propagator, which enters the transition amplitude squared:

$$D(i\omega_n,\mathbf{k}) = \frac{1}{-1/4v - \Pi(i\omega_n,\mathbf{k})} \overset{v\to\infty}{=} -\frac{1}{\Pi(i\omega_n,\mathbf{k})}. \tag{3.159}$$

This completes our proof that, also in the proximity of a bosonic quantum critical point, scrambling can be microscopically understood in terms of counting the gross (energy) exchange.

3.5 Gross-Neveu model at the quantum critical point

Having discussed the case of bosonic quantum critical point in the previous section, we now turn our attention to fermionic quantum critical points. Although the great successes of the bosonic $O(N)$ vector model to capture critical phenomena within the Landau-Ginsburg-Wilson paradigm, whose critical regime can be described entirely in terms of fluctuations of a bosonic order parameter, it is nowadays clear that this does not cover all possible scenarios. Over the last years, it became evident that a plethora of interesting phenomena involve massless fermionic excitations at low energies coupled to vectorial [184–186], real [187, 188] or complex [189–195] order parameters. Those systems are described by fermionic quantum critical points which the bosonic $O(N)$ vector model fails to capture. In this context, a main role is played by the Gross-Neveu model (GN) [196] and the Gross-Neveu-Yukawa model (GNY) [197]. Specifically, here we study the Gross-Neveu model in $(2+1)$ dimension with N flavours of Dirac fermions. The Lagrangian is, in Euclidean time,

$$\mathcal{L}_{GN} = \psi^{\dagger}_{i,\alpha}(\partial_\tau - i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})_{\alpha\beta}\psi_{i,\beta} - \frac{g}{4N}(\psi^{\dagger}_{i,\alpha}\sigma^z_{\alpha\beta}\psi_{i,\beta})^2 \tag{3.160}$$

where we indicated with Latin letters the flavours indices, with Greek letters the spin indices and ψ_i is a two-component Dirac spinor. We also assume the summation over repeated indices. Moreover the Pauli matrices are defined as usual $\boldsymbol{\sigma} = (\sigma^x, \sigma^y)$. This action is symmetric under $x \to -x$ and $\psi_i \to i\sigma^x\psi_i$. This model has a quantum phase transition separating the Dirac semimetal phase and the gapped insulator with broken Z_2 symmetry. By introducing a Hubbard-Stratonovich field ϕ to decouple the quartic interaction,

$$\mathcal{L}_{GN} = \psi^{\dagger\,\alpha}_{i}(\partial_\tau - i\sigma\cdot\nabla)_{\alpha}^{\ \beta}\psi_{i,\beta} + \frac{1}{g}\phi^2 + \frac{1}{\sqrt{N}}\phi\,(\psi^{\dagger}\sigma^z\psi), \tag{3.161}$$

the action stays symmetric under $x \to -x$, $\psi_i \to i\sigma^x\psi_i$ and $\phi \to -\phi$. The expectation value of the field ϕ is thus the order parameter of the spontaneous Z_2 symmetry breaking. As the Z_2 symmetry can be related to the inversion symmetry of a honeycomb lattice, it captures the physics of graphene, graphene-like materials [198, 199] and cold atoms in optical lattice [200–202]. A related theory is the Gross-Neveu-Yukawa (GNY) model, whose matter content is represented by massless Dirac fermions and a massive Φ^4 boson, minimally coupled with Lagrangian

$$\mathcal{L}_{GNY} = \psi^\dagger_{i,\alpha}(\partial_\tau - i\sigma\cdot\nabla)_{\alpha\beta}\psi_{i,\beta} + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\nabla\Phi)^2 + \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!N}\Phi^4 + \frac{u}{\sqrt{N}}\Phi\,\psi^\dagger_{i,\alpha}\sigma^z_{\alpha\beta}\psi_{i,\beta}. \tag{3.162}$$

The GNY model describes also other symmetry classes, as chiral Heisenberg and chiral XY universality classes [203, 204]. Assuming small λ and as far as we focus on the long wavelength and low energy (compared to the mass m) degrees of freedom, we can integrate out the boson in (3.162). The result is (3.161) with the identification $g = \sqrt{2}u/m$. The low-energy, long wavelength limit of the GNY model thus coincide with the GN model [203] and the results of this section apply to all classes of systems previously mentioned.

3.5.1 Brief review of many-body chaos in GN

By means of the introduction of a scalar field to decouple the interaction in (3.161), the Lagrangian can be expressed as (3.161). The properties of many-body chaos for the GN model in $2+1$ dimensions, in the Lagrangian form of (3.161), were investigated in [137]. There, the authors computed the OTOC in the large N limit,

$$f^\beta_\alpha(t) = \frac{1}{N^2}\sum_{ij,\gamma}\int d^2x \times \mathrm{Tr}\left[\rho^{1/2}\{\psi_{i\alpha}(t,\mathbf{x}),\psi^{\dagger\,\gamma}_j(0,\mathbf{0})\}\rho^{1/2}\{\psi_{j\gamma}(0,\mathbf{0}),\psi^{\dagger\,\beta}_i(t,\mathbf{x}),\}\right], \tag{3.163}$$

by deriving an integro-differential equation (BSE),

$$f_\alpha^{\ \beta}(\nu;\omega,\mathbf{p}) = \frac{1}{N}S_R(\omega+\nu,\mathbf{p})_\alpha^{\ \gamma}S_A(\omega,\mathbf{p})_\delta^{\ \beta} \times\left[\delta^\delta_\gamma + \int_{\omega',\mathbf{r}}\Lambda^{OTOC}(\nu;\omega,\mathbf{p},\omega',\mathbf{r})^{\delta\gamma'}_{\gamma\delta'}f_{\gamma'}^{\ \delta'}(\nu;\omega',\mathbf{r})\right]. \tag{3.164}$$

The kernel of the BSE is

$$\Lambda^{OTOC}(r,p)^{\delta\gamma'}_{\gamma\delta'} = (\sigma^z)^{\gamma'}_\gamma(\sigma^z)^\delta_{\delta'}D^{\beta/2}_W(p-r) + \int_l(\sigma^z\,S^{\beta/2}_W(p-l)\sigma^z)^\delta_\gamma\,(\sigma^z\,S^{\beta/2}_W(r-l)\sigma^z)^{\gamma'}_{\delta'}\,D^R(l+q)D^A(l), \tag{3.165}$$

where the retarded Greens function of the fermion and the boson are respectively[4]

$$S_R(t,\mathbf{x})^{\beta}_{\alpha}\delta_{ij} = -i\theta(t)\langle\rho\{\psi_{i\alpha}(t,\mathbf{x}),\psi^{\dagger\,\beta}_{j}\}\rangle, \tag{3.166}$$

$$D_R(t,\mathbf{x}) = -i\theta(t)\langle\rho[\phi(t,\mathbf{x}),\phi]\rangle, \tag{3.167}$$

and the Wightman functions are computed with insertion separated by half of the thermal circle

$$S_W(t,\mathbf{x})^{\beta}_{\alpha}\delta_{ij} = -i\langle\sqrt{\rho}\,\psi_{i\alpha}(t,\mathbf{x})\,\sqrt{\rho}\psi^{\dagger\,\beta}_{j}\rangle, \tag{3.168}$$

$$D_W(t,\mathbf{x}) = -i\langle\sqrt{\rho}\,\phi(t,\mathbf{x})\sqrt{\rho}\,\phi\rangle. \tag{3.169}$$

In momentum space, the Wightman functions can be expressed in terms of the spectral function

$$S_W(\omega,\mathbf{p})^{\beta}_{\alpha}\delta_{ij} = \frac{\rho^{\beta}_{\alpha}(\omega,\mathbf{p})}{2\cosh(\beta\omega/2)}, \tag{3.170}$$

$$D_W(\omega,\mathbf{p}) = \frac{\rho_D(\omega,\mathbf{p})}{2\sinh(\beta\omega/2)}, \tag{3.171}$$

which satisfy $\rho(\omega,\mathbf{p}) = -2\mathrm{Im}G_R(\omega,\mathbf{p})$ and $\rho_D(\omega,\mathbf{p}) = -2\mathrm{Im}D_R(\omega,\mathbf{p})$.

3.5.2 Hydrodynamic transport in GN model

In this section we use the 2PI formalism to derive the transport equation for the GN model in $2+1$ dimensions to compare with the OTOC. The effective action for the original GN Lagrangian

$$\mathcal{L}_{GN} = \psi^{\dagger}_{i,\alpha}(\partial_\tau - i\sigma\cdot\nabla)_{\alpha\beta}\psi_{i,\beta} - \frac{g}{4N}(\psi^{\dagger}_{i,\alpha}\sigma^{z}_{\alpha\beta}\psi_{i,\beta})^2 \tag{3.172}$$

can be parametrized as follows [115, 182]

$$\Gamma[S] = -i\mathrm{Tr}\ln S^{-1} - i\mathrm{Tr}\ln S_0^{-1}(S-S_0) + \Gamma_2[S], \tag{3.173}$$

S_0 being the free propagator in the Euclidean time and S the full dressed 2-point function, satisfying

$$S^{-1} = S_0^{-1} - \Sigma. \tag{3.174}$$

In (3.173), $\Gamma_2[S]$ includes the contribution of all the amputated 2-particle irreducible diagrams (2PI) with exact propagators on the internal lines.

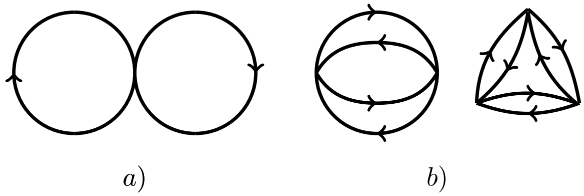

Figure 3.16: The contributions to the 2PI effective action in the $1/N$ expansion. a) is the leading order contribution Γ^{LO} and b) the first terms in the series of next-to-the leading order Γ^{NLO}.

In the 2PI formalism, the self energies can be derived as functional derivative of the 2PI effective action

$$\Sigma_{ij|\alpha\beta}(x,y) \equiv -i\frac{\delta\Gamma_2[S]}{\delta S_{ji|\beta\alpha}(y,x)}. \tag{3.175}$$

In the following, unless differently specified, we will use a condensed notation, using Latin letters both flavour, spin and space-time indices. This simplifies the previous expression as

$$\Sigma_{ij} = -i\frac{\delta\Gamma_2[S]}{\delta S_{ji}}. \tag{3.176}$$

The 4-point vertex function is defined as the amputated connected 4-point function and satisfies the following functional equation

$$\Gamma^{(4)}_{ij,kl} = \Lambda_{ij,kl} - \Lambda_{ij,ef}S^{ff'}S^{e'e}\Gamma^{(4)}_{f'e',kl}, \tag{3.177}$$

in which the kernel is by definition the functional derivative of the self energy with respect to the bilocal field S

$$\Lambda_{ij,kl} \equiv i\frac{\delta^2\Gamma_2[S]}{\delta S_{ji}\delta S_{lk}} = -\frac{\delta\Sigma_{kl}}{\delta S_{ji}}. \tag{3.178}$$

Now, following [182], we perform the large N expansion of the effective action considering the leading and the next-to-leading order, which diagrammatically are expressed in fig. 3.16

[4]For a review on the convention for fermions at finite temperature, see the Appendix 3.A.

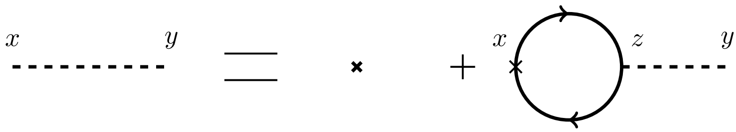

Figure 3.17: The diagrammatic recursive expression of the propagator $D(x, y)$.

$$\Gamma_2^{LO}[S] = \frac{g}{4N} \int_x \mathrm{Tr}[\sigma^z S(x,x) \sigma^z S(x,x)], \tag{3.179}$$

$$\Gamma_2^{NLO}[S] = \frac{i}{2} \mathrm{Tr} \ln \mathcal{B}, \tag{3.180}$$

and

$$\mathcal{B}(x,y) = \delta_{\mathcal{C}}(x-y) - \frac{ig}{2N} \Pi(x,y), \tag{3.181}$$

$$\Pi(x,y) = -\mathrm{Tr}[S(x,y)\sigma^z S(y,x)\sigma^z] = -S(x,y)_{ab|\alpha\beta}\, \sigma^z_{\beta\gamma}\, S(y,x)_{ba|\gamma\delta}\, \sigma^z_{\delta\alpha}. \tag{3.182}$$

In the last line we have explicited the indices to stress the structure of the Hilbert space we are considering and we note that the functional derivative of the bubble diagram satifies

$$\frac{\delta \Pi(x',y')}{\delta S_{lk|\beta\alpha}(y,x)} = -2\delta(x'-y)\delta(y'-x)(\sigma^z_{\alpha\gamma}\, S(x,y)_{kl|\gamma\delta}\, \sigma^z_{\delta\beta}). \tag{3.183}$$

Let's now compute the leading and next-to-the leading order contribution to the self-energies (for the sake of clarity we will write the indices in the intermediate steps)

$$\Sigma^{LO}_{kl|\gamma\delta}(x',y') = -i \frac{\delta \Gamma_2^{LO}[S]}{\delta S_{lk|\delta\gamma}(y',x')} = -i\frac{g}{2N} \delta(x'-y')(\sigma^z\, S(x',y')_{kl}\, \sigma^z)_{\gamma\delta}, \tag{3.184}$$

$$\begin{aligned} \Sigma^{NLO}_{kl|\gamma\delta}(x',y') &= -i \frac{\delta \Gamma_2^{NLO}[S]}{\delta S_{lk|\delta\gamma}(y',x')} = \frac{-ig}{4N} \int_{w,z} \mathcal{B}^{-1}(z,w) \frac{\delta \Pi(w,z)}{\delta S_{lk|\delta\gamma}(y',x')} \\ &= \frac{ig}{2N} \mathcal{B}^{-1}(x',y')(\sigma^z\, S(x',y')_{kl}\, \sigma^z)_{\gamma\delta} \\ &= D(x',y')(\sigma^z\, S(x',y')_{kl}\, \sigma^z)_{\gamma\delta}, \end{aligned} \tag{3.185}$$

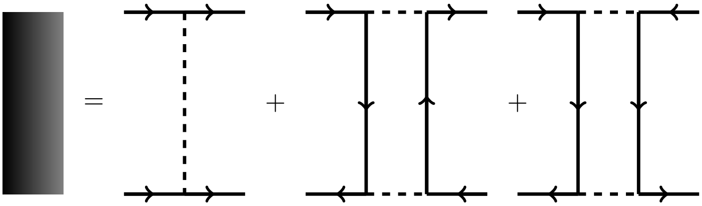

Figure 3.18: Diagrammatic representation of the kernel Λ^{NLO} computed in (3.188).

where we have defined $D(x', y') = \frac{ig}{2N}\mathcal{B}^{-1}(x', y')$. Inserting (3.181) into the identity $\mathcal{B}^{-1}\mathcal{B} = 1$, we obtain an integral equation for $D(x, y)$, depicted in fig. 3.17

$$D(x, y) = \frac{ig}{2N}\left[\delta_{\mathcal{C}}(x - y) + \int_z \Pi(x, z)D(z, y)\right]. \tag{3.186}$$

This two-point function is an effective description of the polarization bubble and, as we will see, corresponds to the propagator of an Hubbard-Stratonovich field introduced to linearise the quadratic interaction in the action. Now we compute the kernel using (3.178). The leading order contribution is simply

$$\Lambda^{LO}_{ij;kl|\alpha\beta;\gamma\delta}(x, y; x', y') = i\frac{\delta\Sigma^{LO}_{kl|\gamma\delta}(x', y')}{\delta S_{ji|\beta\alpha}(y, x)} \\ = \frac{g}{2N}\delta_{kj}\delta_{li}\delta(x' - y')\delta(x' - y)\delta(y' - x)(\sigma^z)_{\gamma\beta}(\sigma^z)_{\alpha\delta}. \tag{3.187}$$

Similarly to the bosonic $O(N)$ case, this will not affect the final BSE after the Matsubara sum. Going to the NLO term,

$$\Lambda^{NLO}_{ij;kl|\alpha\beta;\gamma\delta}(x, y; x', y') = i\frac{\delta\Sigma^{NLO}_{kl|\gamma\delta}(x', y')}{\delta S_{ji|\beta\alpha}(y, x)} \\ = \delta_{kj}\delta_{li}\delta(x' - y)\delta(y' - x)(\sigma^z)_{\gamma\beta}(\sigma^z)_{\alpha\delta}D(x', y') - \int_{w'z'} \delta(w' - y)\delta(z' - x) \\ (D(x', w')D(z', y') + D(y', w')D(z', x'))(\sigma^z\, S(x, y)_{ij}\, \sigma^z)_{\alpha\beta}(\sigma^z\, S(x', y')_{kl}\, \sigma^z)_{\gamma\delta} \\ = \delta_{kj}\delta_{li}\delta(x' - y)\delta(y' - x)(\sigma^z)_{\gamma\beta}(\sigma^z)_{\alpha\delta}D(x', y') \\ - 2(\sigma^z\, S(x, y)_{ij}\, \sigma^z)_{\alpha\beta}(\sigma^z\, S(x', y')_{kl}\, \sigma^z)_{\gamma\delta}\, D(x', y)D(x, y'). \tag{3.188}$$

We now go to momentum space and we observe that the the 4-point function $\Gamma^{(4)}_{kl,ij}(R, P; Q)$ is related to the 3-point vertex $\Gamma_{ij}(P + Q, P)$, as shown in fig. 3.19.

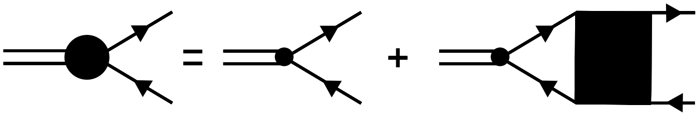

Figure 3.19: Representation of the relation among the 3-vertex and the amputated 4-point connected Green's function.

Since more convenient, we will focus on the latter

$$\Gamma_{ij}(P+Q,P) = \Gamma^0_{ij}(\mathbf{p}) - \sum\!\!\!\!\!\!\int_R S(R+Q)\Gamma^0(\mathbf{r})_{kl}S(R)\Gamma^{(4)}_{kl,ij}(R,P;Q) \quad (3.189)$$

where $\Gamma^0_{ij}(\mathbf{p})$ is the coupling between the fermionic fields and the external operator. For instance, in the shear viscosity the operator is the stress energy and we have $\Gamma^0_{ij}(\mathbf{p}) = \frac{1}{2}(\sigma^i p^j + \sigma^j p^i - \delta^{ij}\sigma\cdot\mathbf{p})$. If we want to focus on the density operator, defined as (3.6)

$$\rho(x,p) = \int_y e^{-ipy}\mathrm{Tr}(\psi(x-y/2)\bar{\psi}(x+y/2)) = \int_k e^{ikx}\mathrm{Tr}(\psi(p+k/2)\bar{\psi}(p-k/2)), \quad (3.190)$$

such insertion is just δ_{ij}. From now on we will focus on the latter. As shown in fig. 3.20, this vertex satisfies the following integral equation

$$\Gamma_{ij}(P+Q,P) = \Gamma^0_{ij}(\mathbf{p}) - \sum\!\!\!\!\!\!\int_R S(R+Q)\Gamma(R+Q,R)S(R)\Lambda_{kl,ij}(R,P;Q) \quad (3.191)$$

The previous equation is written in imaginary time. To obtain the real time value, a sum over Matsubara frequencies is required, which corresponds to the proper choice of the analytical continuation. It can be shown by induction that, because of the form of the BSE, $\Gamma_{ij}(P+Q,P)$ has branch cuts both in $\mathrm{Im}(p_0) = 0$ and $\mathrm{Im}(p_0+q^0) = 0$. An analysis similar to the previous section on the $O(N)$ model can be carried out. Also in this case, in the long time limit corresponding to $q_0 \to 0$, the pinching pole approximation selects only one analytic continuation of the vertex in the imaginary time formalism, which corresponds to $i(p_0+q_0) \to p_0+q_0+i0$ and $ip_0 \to p_0 - i0$. Consequently, the shear viscosity which is obtained by the resummed skeleton diagram we have focused on but with a different insertion $\Gamma^{(0)ij}$

$$G_{\pi\pi}(Q) = -\sum\!\!\!\!\!\!\int_P \mathrm{Tr}(S(P+Q)\Gamma_{ij}(P+Q,P)S(P)\Gamma^{(0)ij}(\mathbf{p})). \quad (3.192)$$

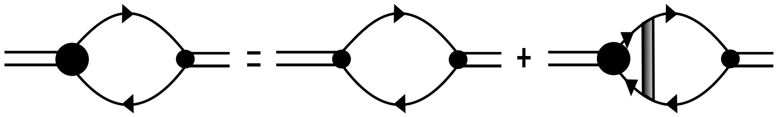

Figure 3.20: BSE for the 3-point vertex function.

In the limit of vanishing external momentum, it is given by

$$\lim_{q_0,\mathbf{q}\to 0} G_{\pi\pi}(q_0,\mathbf{q}) = -\lim_{q_0,\mathbf{q}\to 0}\int_p \mathrm{Tr}(S_R(p+q)\Gamma_{ij}(p_0+q_0+i0,p_0-i0)S_A(p)\Gamma^{(0)ij}(\mathbf{p})), \tag{3.193}$$

and the shear viscosity is determined by

$$\eta = \frac{1}{20}\lim_{q_0,\mathbf{q}\to 0}\frac{\partial}{\partial q^0}\mathrm{Re}G_{\pi\pi}(q_0,\mathbf{0}) \tag{3.194}$$

Now, let's go back to the evaluation of the density-density correlation function and define the vertex as

$$\tilde{\Gamma}^{\beta}_{\alpha}(p+q,p) = S_R(p+q)^{\gamma}_{\alpha}\Gamma^{\delta}_{\gamma}(p_0+q_0+i0,p_0-i0)S_A(p)^{\beta}_{\delta}. \tag{3.195}$$

The BSE for the (3.195), after the analytical continuation we discussed above is

$$\tilde{\Gamma}^{\beta}_{\alpha}(p+q,p) = S_R(p+q)^{\gamma}_{\alpha}S_A(p)^{\beta}_{\delta}\left[\delta^{\delta}_{\gamma} + \int_r \tilde{\Lambda}(r,p)^{\delta\gamma'}_{\gamma\delta'}\tilde{\Gamma}(r+q,r)^{\delta'}_{\gamma'}\right] \tag{3.196}$$

where the analytically continued kernel reads

$$\begin{aligned}\tilde{\Lambda}(r,p)^{\delta\gamma'}_{\gamma\delta'} = (n_B(r^0-p^0)+n_F(r^0))\Big[&(\sigma^z)^{\gamma'}_{\gamma}(\sigma^z)^{\delta}_{\delta'}\rho_B(r-p) \\ &+2\int_l (n_F(p^0-l^0)-n_F(r^0-l^0))(\sigma^z\rho_F(p-l)\sigma^z)^{\delta}_{\gamma} \\ &\times(\sigma^z\rho_F(r-l)\sigma^z)^{\gamma'}_{\delta'}D^R(l+q)D^A(l)\Big].\end{aligned} \tag{3.197}$$

This result is similar to the kernel of the BSE for the shear viscosity obtained in [181] for $q=0$ for the large N_f QCD. The main difference, besides the dimensionality, can be reduced to the presence of vector boson in QCD, while here the boson is simply a scalar. This suggests that the analysis performed in this section can be extended to the case of QCD in the large N_f limit. After some algebra, and using the identities written in Appendix 3.B, we arrive to the final expression

$$\tilde{\Gamma}^{\beta}_{\alpha}(p+q,p) = S_R(p+q)^{\gamma}_{\alpha} S_A(p)^{\beta}_{\delta} \left[\delta^{\delta}_{\gamma} + \int_r \tilde{\Lambda}(r,p)^{\delta\gamma'}_{\gamma\delta'} \tilde{\Gamma}(r+q,r)^{\delta'}_{\gamma'}\right], \quad (3.198)$$

where the kernel is

$$\begin{aligned}\tilde{\Lambda}(r,p)^{\delta\gamma'}_{\gamma\delta'} &= (\sigma^z)^{\gamma'}_{\gamma}(\sigma^z)^{\delta}_{\delta'} D_W(p-r) \\ &+ \int_l (\sigma^z\, S_W(p-l)\sigma^z)^{\delta}_{\gamma}\, (\sigma^z\, S_W(r-l)\sigma^z)^{\gamma'}_{\delta'}\, D^R(l+q) D^A(l).\end{aligned}$$

We now proceed by showing the relation between the BSE for transport and the BSE for scrambling. If we express the Wightman functions D_W and S_W in terms of the symmetrized ones, after some simplification we find

$$\tilde{\Gamma}^{\beta}_{\alpha}(p+q,p) = S_R(p+q)^{\gamma}_{\alpha} S_A(p)^{\beta}_{\delta} \quad (3.199)$$

$$\left[\delta^{\delta}_{\gamma} + \int_r \frac{\cosh(\beta \mathrm{p}^0/2)}{\cosh(\beta \mathrm{r}^0/2)} \Lambda^{OTOC}(r,p)^{\delta\gamma'}_{\gamma\delta'} \tilde{\Gamma}(r+q,r)^{\delta'}_{\gamma'}\right],$$

$$\Gamma^{\beta}_{OTOC,\alpha}(p+q,p) = S_R(p+q)^{\gamma}_{\alpha} S_A(p)^{\beta}_{\delta} \quad (3.200)$$

$$\left[\delta^{\delta}_{\gamma} + \int_r \Lambda^{OTOC}(r,p)^{\delta\gamma'}_{\gamma\delta'}\, \Gamma^{\delta'}_{OTOC,\gamma'}(r+q,r)\right]$$

where $\Lambda^{OTOC}(r,p)$ is the kernel of the BSE for OTOC in the Gross-Neveu model (3.165) derived in [137]

$$\begin{aligned}\Lambda^{OTOC}(r,p)^{\delta\gamma'}_{\gamma\delta'} &= (\sigma^z)^{\gamma'}_{\gamma}(\sigma^z)^{\delta}_{\delta'} D^{\beta/2}_W(p-r) \\ &+ \int_l (\sigma^z\, S^{\beta/2}_W(p-l)\sigma^z)^{\delta}_{\gamma}\, (\sigma^z\, S^{\beta/2}_W(r-l)\sigma^z)^{\gamma'}_{\delta'}\, D^R(l+q) D^A(l).\end{aligned}$$

This proves our claim that relates the kernel of the BSE of the OTOC to the kernel of the BSE of the bilocal density operator.

3.5.3 The kernel in the helicity basis

Now, lets' try to understand the physics behind the factor $\frac{\cosh(\beta \mathrm{p}^0/2)}{\cosh(\beta \mathrm{r}^0/2)}$ for the fermionic case. In the bosonic case, the factor $\frac{\sinh(\beta \mathrm{p}^0/2)}{\sinh(\beta \mathrm{r}^0/2)}$ automatically provided the natural weighting for the ansatz (and the consequent sign flip in the kernel). The fermion case it is slightly more elaborate, since this factor is even with respect to the momenta p_0 and l_0.

In this section we show which are the proper ansätze that give the correct solutions respectively for the case of transport and chaos. For this we need to rewrite the kernel (3.197) in a way that allows us to interpret the physics

$$\tilde{\Lambda}(r,p)^{\delta\gamma'}_{\gamma\delta'} = (n_B(r^0-p^0)+n_F(r^0))\Bigg[(\sigma^z)^{\gamma'}_{\gamma}(\sigma^z)^{\delta}_{\delta'}\rho_B(r-p) \\ + 2\int_{L,L'}(n_F(l^0)-n_F(l'^0))(\sigma^z\,\rho_F(l)\sigma^z)^{\delta}_{\gamma}(\sigma^z\,\rho_F(l')\sigma^z)^{\gamma'}_{\delta'}|D^R(p-l)|^2\Bigg]. \tag{3.201}$$

The fermionic spectral density, which is

$$\rho_F(p^0,\mathbf{p}) = \frac{\not{p}\pi}{|\mathbf{p}|}(\delta(p^0-|\mathbf{p}|)-\delta(p^0+|\mathbf{p}|)) = \not{p}\rho(p^0,\mathbf{p}), \tag{3.202}$$

can be written in the helicity basis as follows

$$\rho_F(l^0,\mathbf{l}) = 2\pi\sum_a \mathcal{P}_a(\mathbf{l})\delta(l^0-a|\mathbf{l}|), \tag{3.203}$$

where we introduced the projector into the helicity basis

$$\mathcal{P}_a(\mathbf{k}) = \frac{1+a\sigma\cdot\hat{\mathbf{k}}}{2}. \tag{3.204}$$

Substituting this in the second line of the expression of (3.201), with the additional substitution

$$(\sigma^z\,\rho_F(l)\sigma^z)^{\delta}_{\gamma} = 2\pi\sum_a(\sigma^z\mathcal{P}_a(\mathbf{l})\sigma^z)^{\delta}_{\gamma}\delta(l^0-a|\mathbf{l}|), \tag{3.205}$$

allows us to rewrite the kernel as

$$\tilde{\Lambda}(r,p)^{\delta\gamma'}_{\gamma\delta'} = (n_B(r^0-p^0)+n_F(r^0))\Bigg[(\sigma^z)^{\gamma'}_{\gamma}(\sigma^z)^{\delta}_{\delta'}\rho_B(r-p) \tag{3.206}$$

$$+ 2\sum_{a,b}\int_{l,l'}(n_F(l^0)-n_F(l'^0))\,(\sigma^z\mathcal{P}_b(\mathbf{l})\sigma^z)^{\delta}_{\gamma}(\sigma^z\mathcal{P}_a(\mathbf{l}')\sigma^z)^{\gamma'}_{\delta'}$$

$$(2\pi)^2\delta(l'^0-a|\mathbf{l}'|)\delta(l^0-b|\mathbf{l}|)|D^R(p-l)|^2\Bigg]. \tag{3.207}$$

Similarly as before, the spectral function of the auxiliary field, ρ_B, encodes the fermionic bubble diagram and can be rewritten as

$$\rho_B(r-p) = -2\,|D_R(r-p)|^2\,\mathrm{Im}\Pi_R(r-p). \tag{3.208}$$

In order to find the expression for the imaginary part of the bosonic self energy $\mathrm{Im}\Pi_R(p)$, we can use formula (3.182) in Fourier space and perform the Matsubara sum, which is done in Appendix 3.C. After analytically continuing to obtain the retarded contribution, we can take the imaginary part and the expression reads

$$\begin{aligned}\mathrm{Im}\Pi_R(p) = -\frac{1}{2}\sum_{a,b}\int_{\mathbf{l}} K_{ab}(\mathbf{l},\mathbf{l}+\mathbf{p})\,(n_F(a|\mathbf{l}|) - n_F(b|\mathbf{l}+\mathbf{p}|))& \\ (2\pi)\delta(p^0 + a|\mathbf{l}| - b|\mathbf{l}+\mathbf{p}|),&\end{aligned} \tag{3.209}$$

where we have defined the following quantity

$$K_{ab}(\mathbf{p},\mathbf{l}) = \mathrm{Tr}[\sigma^z P_a(\mathbf{p})\sigma^z P_b(\mathbf{l})] = \frac{1 - ab\,\hat{p}\cdot\hat{l}}{2} \tag{3.210}$$

and use it to simplify the expression of the $\mathrm{Im}\Pi_R(r-p)$ by introducing the momentum $\mathbf{l}'$ and imposing the energy conservation:

$$\begin{aligned}\mathrm{Im}\Pi_R(r-p) =& -\frac{1}{2}\sum_{a,b}\int_{\mathbf{l}} K_{ab}(\mathbf{l},\mathbf{l}+\mathbf{r}-\mathbf{p})\,(n_F(a|\mathbf{L}|) - n_F(b|\mathbf{l}+\mathbf{r}-\mathbf{p}|)) \\ &\times (2\pi)\delta(r^0 - p^0 + a|\mathbf{l}| - b|\mathbf{l}+\mathbf{r}-\mathbf{p}|) \\ =& -\frac{1}{2}\sum_{a,b}\int_{\mathbf{l},\mathbf{l}'} K_{ab}(\mathbf{l},\mathbf{l}')\,(n_F(a|\mathbf{l}|) - n_F(b|\mathbf{l}'|))(2\pi)^2 \\ &\delta^2(\mathbf{l}'+\mathbf{p}-\mathbf{r}-\mathbf{l}) \times (2\pi)\delta(r^0 - p^0 + a|\mathbf{l}| - b|\mathbf{l}'|).\end{aligned} \tag{3.211}$$

We can now write the kernel (3.207) as

$$\begin{aligned}\tilde{\Lambda}(r,p)^{\delta\gamma'}_{\gamma\delta'} =& (n_B(r^0-p^0) + n_F(r^0))\sum_{a,b}\int_{l,l'} (n_F(l^0) - n_F(l'^0)) \\ &\times \delta(l^0 - a|\mathbf{l}|)\delta(l'^0 - b|\mathbf{l}'|)(2\pi)^3\delta^3(r+l-p-l')\Bigg((\sigma^z)^{\gamma'}_{\gamma}(\sigma^z)^{\delta}_{\delta'}K_{ab}(\mathbf{l},\mathbf{l}') \\ &\times |D^R(r-p)|^2 + 2(\sigma^z\mathcal{P}_b(\mathbf{l})\sigma^z)^{\delta}_{\gamma}(\sigma^z\mathcal{P}_a(\mathbf{l}')\sigma^z)^{\gamma'}_{\delta'}|D^R(p-l)|^2\Bigg),\end{aligned}$$

which, by relabelling the momenta l and l', can be rewritten in terms of the s, t and u-channels

$$\tilde{\Lambda}(r,p)^{\delta\gamma'}_{\gamma\delta'} = (n_B(r^0-p^0)+n_F(r^0))\sum_{c,d}\int_{l,l'}(n_F(l^0)-n_F(l'^0)) \tag{3.212}$$

$$\times\,\delta(l^0-c|\mathbf{l}|)\delta(l'^0-d|\mathbf{l}'|)(2\pi)^3\delta^3(r+l-p-l')$$
$$\times\Bigg((\sigma^z)^{\gamma'}_{\gamma}(\sigma^z)^{\delta}_{\delta'}K_{cd}(\mathbf{l},\mathbf{l}')|D^R(r-p)|^2+(\sigma^z\mathcal{P}_c(\mathbf{l})\sigma^z)^{\delta}_{\gamma}(\sigma^z\mathcal{P}_d(\mathbf{l}')\sigma^z)^{\gamma'}_{\delta'}$$
$$\times\,|D^R(r-l')|^2+(\sigma^z\mathcal{P}_d(\mathbf{l}')\sigma^z)^{\delta}_{\gamma}(\sigma^z\mathcal{P}_c(\mathbf{l})\sigma^z)^{\gamma'}_{\delta'}|D^R(r+l)|^2\Bigg). \tag{3.213}$$

We have massaged the kernel in a way that it is easy to project into helical basis. This is a convenient way to analyze both the *on-shell* BSEs, which we do in the next section.

3.5.4 The physics behind the analytic continuation

To recapitulate, in the previous section we saw that the off-shell BSE for transport are, up to a similarity transformation, the same

$$\Gamma^{\beta}_{OTOC,\alpha}(p+q,p) = S_R(p+q)^{\gamma}_{\alpha}S_A(p)^{\beta}_{\delta}\Bigg[\delta^{\delta}_{\gamma}+\int_r\frac{\cosh(\beta r_0/2)}{\cosh(\beta p_0/2)} \tag{3.214}$$
$$\times\,\Lambda(r,p)^{\delta\gamma'}_{\gamma\delta'}\,\Gamma^{\delta'}_{OTOC,\gamma'}(r+q,r)\Bigg],$$
$$\Gamma^{\beta}_{\alpha}(p+q,p) = S_R(p+q)^{\gamma}_{\alpha}S_A(p)^{\beta}_{\delta}\Bigg[\delta^{\delta}_{\gamma}+\int_r\Lambda(r,p)^{\delta\gamma'}_{\gamma\delta'}\,\Gamma^{\delta'}_{\gamma'}(r+q,r)\Bigg],$$

with a kernel that can be expressed as (3.212). Now we want to explicitly take the late time limit, $q_0\to 0$, and project both the BSEs (3.214) into the helical basis. To do so, we observe that the product of retarded and advanced Green's function in the pinching pole approximation can be written as [137]

$$S_R(p+q)^{\gamma}_{\alpha}S_A(p)^{\beta}_{\delta}\approx 2\pi\sum_a\mathcal{P}_a(\mathbf{p})^{\gamma}_{\alpha}\mathcal{P}_a(\mathbf{p})^{\beta}_{\delta}\frac{\delta(p^0-a|\mathbf{p}|)}{-iq^0+2\Gamma_{\mathbf{p},a}}. \tag{3.215}$$

Since the homogeneous equation is defined with momentum p on-shell, an obvious ansatz is,

$$\Gamma_\alpha^\beta(p+q,p)=\sum_a f_a(q^0,p_0,\mathbf{p})\mathcal{P}_a(\mathbf{p})_\alpha^\beta\delta(p^0-a\mathbf{p}). \tag{3.216}$$

This will turn out to be the solution that describes chaos. Substituting into Equation (3.214), one has

$$\begin{aligned}\sum_a f_a(q^0,p_0,\mathbf{p})\mathcal{P}_a(\mathbf{p})_\alpha^\beta\delta(p^0-a\mathbf{p}) &= 2\pi\sum_a \mathcal{P}_a(\mathbf{p})_\alpha^\gamma\mathcal{P}_a(\mathbf{p})_\delta^\beta\frac{\delta(p^0-a|\mathbf{p}|)}{-iq^0+2\Gamma_{\mathbf{p},a}} \\ &\times\left[\delta_\gamma^\delta+\int_r\frac{\cosh(\beta r_0/2)}{\cosh(\beta r_0/2)}\Lambda(r,p)_{\gamma\delta'}^{\delta\gamma'}\right. \\ &\left.\times\sum_b f_b(q^0,r_0,\mathbf{r})\mathcal{P}_b(\mathbf{r})_{\gamma'}^{\delta'}\delta(r^0-b\mathbf{r})\right].\end{aligned} \tag{3.217}$$

We now study the a component of the above BSE and trace over the spin indices α,β

$$\begin{aligned}f_a(q^0,p_0,\mathbf{p})\delta(p^0-a\mathbf{p}) &= 2\pi\frac{\delta(p^0-a|\mathbf{p}|)}{-iq^0+2\Gamma_{\mathbf{p},a}}+\frac{\delta(p^0-a|\mathbf{p}|)}{-iq^0+2\Gamma_{\mathbf{p},a}} \\ &\times\int_r\frac{\cosh(\beta\mathbf{r}/2)}{\cosh(\beta\mathbf{p}/2)}\mathcal{P}_a(\mathbf{p})_\alpha^\gamma\mathcal{P}_a(\mathbf{p})_\delta^\beta\Lambda(r,p)_{\gamma\delta'}^{\delta\gamma'}\mathcal{P}_b(\mathbf{r})_{\gamma'}^{\delta'} \\ &\sum_b f_b(q^0,r_0,\mathbf{r})\delta(r^0-b\mathbf{r}).\end{aligned} \tag{3.218}$$

We can define the effective kernel as $\Lambda_{ab}(r,p)=\mathcal{P}_a(\mathbf{p})_\alpha^\gamma\mathcal{P}_a(\mathbf{p})_\delta^\alpha\Lambda(r,p)_{\gamma\delta'}^{\delta\gamma'}\mathcal{P}_b(\mathbf{r})_{\gamma'}^{\delta'}$,

$$\begin{aligned}\Lambda_{ab}(r,p) &= (n_B(r^0-p^0)+n_F(r^0))\sum_{c,d}\int_{l,l'}(n_F(l^0)-n_F(l'^0))\delta(l^0-c|\mathbf{l}|) \\ &\times\delta(l'^0-d|\mathbf{l}'|)(2\pi)^3\delta^3(r+l-p-l') \\ &\times\Big(K_{ab}(\mathbf{p},\mathbf{r})K_{cd}(\mathbf{l},\mathbf{l}')|D^R(r-p)|^2+K_{ac}(\mathbf{p},\mathbf{l})K_{bd}(\mathbf{r},\mathbf{l}')|D^R(r-l')|^2 \\ &+K_{bc}(\mathbf{r},\mathbf{l})K_{ad}(\mathbf{p},\mathbf{l})|D^R(r+l)|^2\Big).\end{aligned} \tag{3.219}$$

In (3.219), momenta p and r are not on shell yet. We can use (3.219) to rewrite (3.218)

$$-iq^0 f_a(q^0, a\mathbf{p}, \mathbf{p}) = 2\pi + \sum_b \int_{\mathbf{r}} \frac{\cosh(\beta \mathbf{r}/2)}{\cosh(\beta \mathbf{p}/2)} \times \left(\Lambda_{ab}(\mathbf{r}, \mathbf{p}) - 2\Gamma_{\mathbf{r},a}(2\pi)^2 \delta^2(\mathbf{p} - \mathbf{r})\delta_{ab} \right) f_b(q^0, b\mathbf{r}, \mathbf{r}), \tag{3.220}$$

having defined $\Lambda_{ab}(\mathbf{r}, \mathbf{p}) = \Lambda_{ab}(b|\mathbf{r}|, \mathbf{r}, a|\mathbf{p}|, \mathbf{p})$. By means of (3.219), it is easy to check the following properties of the kernel

$$\begin{aligned} \Lambda_{++}(\mathbf{r}, \mathbf{p}) &= \Lambda_{--}(\mathbf{r}, \mathbf{p}), \\ \Lambda_{-+}(\mathbf{r}, \mathbf{p}) &= \Lambda_{+-}(\mathbf{r}, \mathbf{p}), \end{aligned} \tag{3.221}$$

which are again consequences of the particle-hole symmetry. We observe that, at this stage, the ratios $\frac{\cosh(\beta \mathbf{r}/2)}{\cosh(\beta \mathbf{p}/2)}$ represents a similarity transformation, so it can be neglected since it does not affect the spectrum. Because of this, from now on we will drop this factor.

Before we show that (3.220) is the chaos BSE, we now consider another possible ansatz, which we will see is correct one for transport, that corresponds to (3.216) with the choice

$$f_a(q^0, p_0, \mathbf{p}) = a\tilde{f}_a(q^0, p_0, \mathbf{p}), \tag{3.222}$$

where the a labels the helicity. This leads to the following BSE (after multiplying for a and using $a^2 = 1$)

$$-iq^0 \tilde{f}_a(q^0, a\mathbf{p}, \mathbf{p}) = 2\pi a \sum_b \int_{\mathbf{r}} \left(ab\Lambda_{ab}(b\mathbf{r}, \mathbf{r}, a\mathbf{p}, \mathbf{p}) - 2\Gamma_{\mathbf{r},a}(2\pi)^2 \delta(\mathbf{p} - \mathbf{r})\delta_{ab} \right) \tilde{f}_b(q^0, b\mathbf{r}, \mathbf{r}). \tag{3.223}$$

In the previous equation, the factor ab plays the analogous role of the factor $\frac{\sinh(\beta r_0/2)}{\sinh(\beta p_0/2)}$ for the bosonic case. We now prove the statement that (3.223) leads to the Boltzmann equation, while the other reproduces the OTOC. As before, the difference in the kernel is simply given by a different counting of the contribution of scattering processes, as shown in the cartoon in fig.

3.6 and 3.7. We start by expanding and using the identities

$$
\begin{aligned}
-iq^0 f_+(q^0,\mathbf{p}) =& 2\pi \\
+& \int_{\mathbf{r}} \Lambda_{++} f_+(q^0,\mathbf{r}) + \Lambda_{+-} f_-(q^0,\mathbf{r}) - 2\Gamma_{\mathbf{r}}(2\pi)^2\delta^2(\mathbf{p}-\mathbf{r}) f_+(q^0,\mathbf{r}), \\
-iq^0 f_-(q^0,\mathbf{p}) =& 2\pi \\
+& \int_{\mathbf{r}} \Lambda_{+-} f_+(q^0,\mathbf{r}) + \Lambda_{++} f_-(q^0,\mathbf{r}) - 2\Gamma_{\mathbf{r}}(2\pi)^2\delta^2(\mathbf{p}-\mathbf{r}) f_-(q^0,\mathbf{r})
\end{aligned}
$$

The sum $f(q^0,\mathbf{p}) = f_+(q^0,\mathbf{p}) + f_-(q^0,\mathbf{p})$ satisfies

$$
-iq^0 f(q^0,\mathbf{p}) - \int_{\mathbf{r}} \Big(\Lambda_{++} + \Lambda_{+-} - 2\Gamma_{\mathbf{r}}(2\pi)^2\delta^2(\mathbf{p}-\mathbf{r}) \Big) f(q^0,\mathbf{r}) = 4\pi. \tag{3.224}
$$

The eigenvalue of this integral equation correspond to the Lyapunov exponent of the theory. To derive the transport relaxation time, instead, we have

$$
\begin{aligned}
-iq^0 \tilde{f}_+(q^0,\mathbf{p}) &= \int_{\mathbf{r}} \Lambda_{++} \tilde{f}_+(q^0,\mathbf{r}) - \Lambda_{+-} \tilde{f}_-(q^0,\mathbf{r}) - 2\Gamma_{\mathbf{r}}(2\pi)^2\delta^2(\mathbf{p}-\mathbf{r}) \tilde{f}_+(q^0,\mathbf{r}), \\
-iq^0 \tilde{f}_-(q^0,\mathbf{p}) &= \int_{\mathbf{r}} -\Lambda_{+-} \tilde{f}_+(q^0,\mathbf{r}) + \Lambda_{++} \tilde{f}_-(q^0,\mathbf{r}) - 2\Gamma_{\mathbf{r}}(2\pi)^2\delta^2(\mathbf{p}-\mathbf{r}) \tilde{f}_-(q^0,\mathbf{r}).
\end{aligned}
$$

The sum $\tilde{f}(q^0,\mathbf{p}) = \tilde{f}_+(q^0,\mathbf{p}) + \tilde{f}_-(q^0,\mathbf{p})$ satisfies

$$
-iq^0 \tilde{f}(q^0,\mathbf{p}) = \int_{\mathbf{r}} \Big(\Lambda_{++} - \Lambda_{+-} - 2\Gamma_{\mathbf{r}}(2\pi)^2\delta^2(\mathbf{p}-\mathbf{r}) \Big) \tilde{f}(q^0,\mathbf{r}). \tag{3.225}
$$

The eigenvalues of this integral equation corresponds to the relaxation times of the theory, *i.e.* the integral operator is nothing but the collision integral of the Boltzmann equation. We now can continue with the interpretation of the ansatz (3.216) and (3.222) in terms of kinetic equations.

3.5.5 Kinetic theory analysis

We want to show that, using the notation of the previous sections,

$$
\begin{aligned}
R^{\wedge}(\mathbf{p},\mathbf{r}) &= \Lambda_{++}(\mathbf{p},\mathbf{r}), \\
R^{\vee}(\mathbf{p},\mathbf{r}) &= \Lambda_{+-}(\mathbf{p},\mathbf{r}) + 2\Gamma_{\mathbf{r}}(2\pi)^2\delta^2(\mathbf{p}-\mathbf{r}).
\end{aligned} \tag{3.226}
$$

The first line is straightforward to verify. Indeed, using the identity (valid for $a|\mathbf{p}| + d|\mathbf{l}'| = c|\mathbf{l}| + b|\mathbf{r}|$)

$$(n_B(b|\mathbf{r}| - a|\mathbf{p}|) + n_F(b|\mathbf{r}|))(n_F(c|\mathbf{l}|) - n_F(d|\mathbf{l}'|)) = \frac{1 - n_F(b|\mathbf{r}|)}{1 - n_F(a|\mathbf{p}|)} \times n_F(d|\mathbf{l}'|)(1 - n_F(c|\mathbf{l}|)).$$

For $a = b = +$,

$$\Lambda_{++} = + \frac{1}{1 - n_F(\mathbf{p})} \sum_{cd} \int_{\mathbf{r},\mathbf{l},\mathbf{p}_4} (2\pi)^3 \delta(\mathbf{p} + \mathbf{l}' - \mathbf{r} - \mathbf{l}) \delta(E_\mathbf{p} + cE_\mathbf{l} - E_\mathbf{r} - dE_{\mathbf{l}'}) n_F(dE_{\mathbf{l}'})(1 - n_F(E_\mathbf{r}))(1 - n_F(cE_\mathbf{l})) |\mathcal{T}_{bcad}^{\mathbf{r},\mathbf{l}\to\mathbf{p},\mathbf{l}'}|^2. \tag{3.227}$$

In the previous expressions we defined the scattering amplitude, depicted in Fig. 3.21

$$\left|\mathcal{T}_{bcad}^{(\mathbf{r},\mathbf{l})\to(\mathbf{p},\mathbf{l}')}\right|^2 = K_{ab}(\mathbf{p},\mathbf{r}) K_{cd}(\mathbf{l},\mathbf{l}') |D_R(b|\mathbf{r}| - a|\mathbf{p}|, \mathbf{r} - \mathbf{p})|^2 + K_{ac}(\mathbf{p},\mathbf{l}) K_{bd}(\mathbf{r},\mathbf{l}') \times |D_R(b|\mathbf{r}| - d|\mathbf{l}'|, \mathbf{r} - \mathbf{l}')|^2 + K_{ad}(\mathbf{p},\mathbf{l}') K_{bc}(\mathbf{r},\mathbf{l}) \times |D_R(b|\mathbf{r}| + c|\mathbf{l}|, \mathbf{r} + \mathbf{l})|^2 \tag{3.228}$$

For $a = +$, $b = -$, the kernel provides the contribution to a different scattering process, depicted in fig. 3.22, when the particle with momentum p from the thermal bath is annihilated,

$$\Lambda_{+-} = + \frac{1/2}{1 - n_F(\mathbf{p})} \sum_{cd} \int_{\mathbf{r},\mathbf{p}_2,\mathbf{p}_4} (2\pi)^3 \delta(\mathbf{p} + \mathbf{r} - \mathbf{l} - \mathbf{l}') \delta(E_\mathbf{p} + E_\mathbf{r} - cE_\mathbf{l} - dE_{\mathbf{l}'}) n_F(E_\mathbf{r})(1 - n_F(cE_\mathbf{l}))(1 - n_F(dE_{\mathbf{l}'})) |\mathcal{T}_{-+cd}^{\mathbf{r},\mathbf{p}\to\mathbf{l},\mathbf{l}'}|^2. \tag{3.229}$$

To conclude, we need to rewrite the expression of $2\Gamma_\mathbf{p}$,

$$\Gamma_{\mathbf{p},a} = \frac{1}{2N} \sum_b \int_\mathbf{r} [n(b|\mathbf{r}| - a|\mathbf{p}|) + n_F(b|\mathbf{r}|)] K_{ab}(\mathbf{p},\mathbf{r}) \rho_D(b|\mathbf{r}| - a|\mathbf{p}|, \mathbf{r} - \mathbf{p}) \tag{3.230}$$

which satisfies $\Gamma_{\mathbf{p},+} = \Gamma_{\mathbf{p},-}$, as a consequence of the particle-hole symmetry.

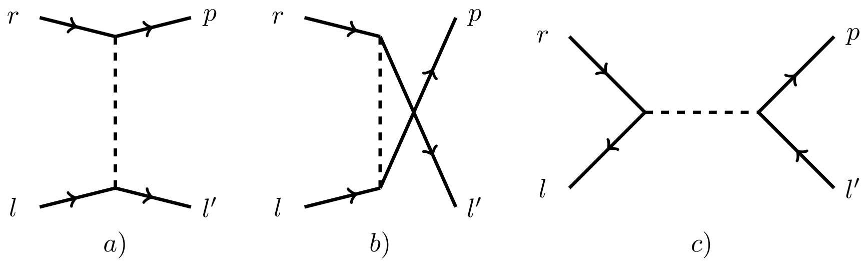

Figure 3.21: The contributions to the kernel of the kinetic equation of Λ_{++} in (3.227). The helicity indices are suppressed and are (r,d), (l,c), (l',d) and (p,a).

By using $\rho_D(b|\mathbf{r}|-a|\mathbf{p}|,\mathbf{r}-\mathbf{p}) = -2\,\mathrm{Im}\Pi_R(b|\mathbf{r}|-a|\mathbf{p}|,\mathbf{r}-\mathbf{p})\,|D_R(b|\mathbf{r}|-a|\mathbf{p}|,\mathbf{r}-\mathbf{p})|^2$ and (3.211),

$$
\begin{aligned}
\mathrm{Im}\Pi_R(b|\mathbf{r}|-a|\mathbf{p}|,\mathbf{r}-\mathbf{p}) = -\frac{1}{2}\sum_{c,d}\int_{\mathbf{L},\mathbf{L}'} K_{cd}(\mathbf{L},\mathbf{L}')\,(n_F(c|\mathbf{L}|)-n_F(d|\mathbf{L}'|))\\
(2\pi)\delta(b|\mathbf{r}|-a|\mathbf{p}|+c|\mathbf{L}|-d|\mathbf{L}'|)(2\pi)^2\delta^2(\mathbf{L}'+\mathbf{p}-\mathbf{r}-\mathbf{L}),
\end{aligned}
$$

we get

$$
\begin{aligned}
&\Gamma_{\mathbf{p},a} = \\
&= -\frac{1}{N}\sum_b\int_{\mathbf{r}}[n(b|\mathbf{r}|-a|\mathbf{p}|)+n_F(b|\mathbf{r}|)]K_{ab}(\mathbf{p},\mathbf{r})\mathrm{Im}\Pi_R(b|\mathbf{r}|-a|\mathbf{p}|,\mathbf{r}-\mathbf{p})\\
&\qquad\times|D_R(b|\mathbf{r}|-a|\mathbf{p}|,\mathbf{r}-\mathbf{p})|^2 =\\
&= \frac{1}{2N}\sum_{bcd}\int_{\mathbf{rll}'}[n_B(b|\mathbf{r}|-a|\mathbf{p}|)+n_F(b|\mathbf{r}|)](n_F(c|\mathbf{L}|)-n_F(d|\mathbf{L}'|))\\
&\qquad\times K_{ab}(\mathbf{p},\mathbf{r})K_{cd}(\mathbf{L},\mathbf{L}')\\
&\quad(2\pi)\delta(b|\mathbf{r}|-a|\mathbf{p}|+c|\mathbf{L}|-d|\mathbf{L}'|)(2\pi)^2\delta^2(\mathbf{L}'+\mathbf{p}-\mathbf{r}-\mathbf{L})\\
&\qquad\times|D_R(b|\mathbf{r}|-a|\mathbf{p}|,\mathbf{r}-\mathbf{p})|^2.
\end{aligned}
$$

By considering the only allowed kinematic contribution, the previous expression can be written as follows

$$2\Gamma_{\mathbf{p},a} = \frac{1/2}{1-n_F(aE_{\mathbf{p}})} \sum_{bcd} \int_{\mathbf{r},\mathbf{l},\mathbf{l}'} (2\pi)^3 \delta(cE_{\mathbf{l}} + dE_{\mathbf{l}'} - E_{\mathbf{p}} - bE_{\mathbf{r}}) \delta(\mathbf{l}+\mathbf{l}'-\mathbf{p}-\mathbf{r})$$
$$\times n_F(bE_{\mathbf{r}})(1-n_F(cE_{\mathbf{l}}))(1-n_F(dE_{\mathbf{l}'})) \left| \mathcal{T}_{abcd}^{(P,R)\to(L,L')} \right|^2 . \tag{3.231}$$

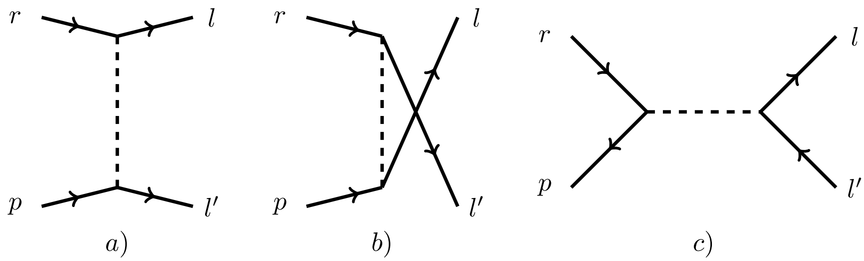

Figure 3.22: The contributions to the kernel of the kinetic equation of Λ_{+-} in (3.229). The helicity indices are suppressed and are (r,d), (l,b), (l',c) and (p,a).

This complete the proof of (3.226). Thus,

$$-iq^0 \tilde{f}(q^0,\mathbf{p}) = \int_{\mathbf{r}} (R^{\wedge}(\mathbf{p},\mathbf{l}) - R^{\vee}(\mathbf{p},\mathbf{l}))\tilde{f}(q^0,\mathbf{r}),$$
$$-iq^0 f(q^0,\mathbf{p}) = \int_{\mathbf{r}} (R^{\wedge}(\mathbf{p},\mathbf{l}) + R^{\vee}(\mathbf{p},\mathbf{l})) - 4(2\pi)^2\delta^2(\mathbf{p}-\mathbf{l})\Gamma_{\mathbf{l}})f(q^0,\mathbf{r}), \tag{3.232}$$

also for critical fermions. This is a highly non trivial result, that firmly establish the validity of our microscopic interpretation to scrambling in terms of a kinetic equation.

3.5.6 Towards the fermionic Quantum Critical Point

As for the bosonic $O(N)$ model, also for the GN model our results rely on the large N and hydrodynamic limit. The Gross-Neveu model,

$$\mathcal{L}_{GN} = \psi_i^{\dagger\,\alpha}(\partial_\tau - i\sigma\cdot\nabla)_{\alpha}^{\ \beta}\psi_{i,\beta} + \frac{1}{g}\phi^2 + \frac{1}{\sqrt{N}}\phi\,(\psi^\dagger\sigma^z\psi), \tag{3.233}$$

has a phase transition at

$$\frac{1}{g_c} = \frac{\Lambda}{4\pi}, \tag{3.234}$$

where Λ is the momentum cutoff. The analytical continuation of our results to the QCP is simply performed by substituting the value of g_c into the Hubbard-Stratonovich Green's function (3.186),

$$D(i\omega_n, \mathbf{k}) = \frac{1}{-2/g_c - \Pi(i\omega_n, \mathbf{k})}, \tag{3.235}$$

This completes our proof that the relation between chaos and scrambling. Such relation holds also for fermionic systems and it extends naturally to the (fermionic) quantum critical point. Furthermore, this strengthen the microscopic interpretation of chaos as a gross (energy) exchange.

3.6 Conclusion

We have shown the existence of an analytical relation between the *off-shell* BSE that defines the out-of-time order correlation function and the *off-shell* BSE that is used to described hydrodynamic transport In weakly coupled or large N QFTs. The remarkable askect of this relation is that it can be extended also beyond the regime of the quasiparticle framework and close to the quantum critical points, as we have proved for the bosonic $O(N)$ vector model and the Gross-Neveu model in $2+1$ dimensions. A straightforward consequence of this result is the microscopic understanding of scrambling, which is described by a Boltzmann-like equation that takes into account the gross energy exchange (compared to the standard BE that consider the net energy exchange). Moreover, since scrambling is related to the information spreading, we expect that an interpretation of our results in terms of more standard information theory quantities, as thermodynamic entropy of Kolmogorov-Sinai entropy, is viable. We refer these questions to future work.

3.A Notation for fermions

Given a fermion ψ_α with α labelling both spin and vector indices, the finite temperature correlation function in the Close Time Path (CTP) formalism are

$$S_{12|\alpha\beta}(x,y) = -\langle\bar{\psi}_\beta(y)\psi_\alpha(x)\rangle \tag{3.236}$$

$$S_{21|\alpha\beta}(x,y) = \langle\psi_\alpha(x)\bar{\psi}_\beta(y)\rangle \tag{3.237}$$

By writing the correlation function in terms of their Fourier transform in momentum space, and allowing a separation in the thermal circle of σ between the two operators, we get

$$S^{\sigma}_{12}(x) = S_{12}(\mathbf{x}, t + i\sigma) = \int d^4p\, e^{-ipx} e^{\sigma p^0} S^{\sigma=0}_{12}(p) \tag{3.238}$$

$$S^{\sigma}_{21}(x) = S_{21}(\mathbf{x}, t - i\sigma) = \int d^4p\, e^{-ipx} e^{-\sigma p^0} S^{\sigma=0}_{21}(p). \tag{3.239}$$

where the expression of the Green's function in momentum space are

$$S_{21}(p) = (1 - n_F(p^0))\rho_F(p) \tag{3.240}$$

$$S_{12}(p) = -n_F(p^0)\rho_F(p). \tag{3.241}$$

where $\rho_F(p)$ is the fermionic spectral function and it is related to the Green's function

$$\rho^F_{\alpha\beta}(p) = S_{21|\alpha\beta}(p) - S_{12|\alpha\beta}(p). \tag{3.242}$$

The system of equations (3.238) can be recast in momentum space as

$$S^{\sigma}_{12}(p) = e^{\sigma p^0} S^{\sigma=0}_{12}(p) \tag{3.243}$$

$$S^{\sigma}_{21}(p) = e^{-\sigma p^0} S^{\sigma=0}_{21}(p). \tag{3.244}$$

It is important to stress that the relation (3.242) holds only with the $\sigma = 0$ Wightman functions.

3.B Some identities

In this section we state some useful identities that we need to prove the relation among the kernel of the BSE for the OTOC and of the retarded Green's function of the bilocal density operator.

Bosonic O(N) vector model

We first focus on the kernel of the bosonic $O(N)$ vector model, which is

$$\Lambda^{NLO}(l,p) = \rho_D(l-p) + N \int_s (n(p_0 - s_0) - n(l_0 - s_0))\rho(p-s)\rho(l-s)D_R(s)D_A(s), \tag{3.245}$$

multiplied by the factor $(n(l_0 - p_0) - n(l_0))$. It is easy to see that the following identities hold:

$$n_B(l_0 - p_0) - n_B(l_0) = \frac{n_B(l_0)}{n_B(p_0)}(1 + n_B(l_0 - p_0)) \tag{3.246}$$

$$n_B(p^0 - s^0) - n_B(l^0 - s^0) = \frac{n_B(p^0 - s^0)}{1 + n_B(l_0 - p_0)}(1 + n_B(l^0 - s^0)). \tag{3.247}$$

Consequently the rung is equal to

$$(n(l_0 - p_0) - n(l_0))\Lambda^{NLO}(l,p) = \tag{3.248}$$
$$\frac{n_B(l_0)}{n_B(p_0)}\left[D_{21}^{\sigma=0}(l-p) + N\int_s G_{12}^{\sigma=0}(p-s)G_{21}^{\sigma=0}(l-s)D_R(s)D_A(s)\right].$$

rewriting in the Wightman functions in terms of the symmetric ones we get

$$(n(l_0 - p_0) - n(l_0))\Lambda^{NLO}(l,p) = \frac{\sinh(\beta \mathrm{p}_0/2)}{\sinh(\beta \mathrm{l}_0/2)}$$
$$\times\left[D_{21}^{\sigma=\beta/2}(l-p) + N\int_s G_{12}^{\sigma=\beta/2}(p-s)G_{21}^{\sigma=\beta/2}(l-s)D_R(s)D_A(s)\right].$$

Gross-Neveu model

Given the bosonic/fermionic equilibrium distribution function $n_{B/F}(p_0) = (e^{\beta p_0} \mp 1)^{-1}$, the following relations hold

$$n_B(r^0 - p^0) + n_F(r^0) = \frac{n_F(r^0)}{n_F(p^0)}(1 + n_B(r^0 - p^0)), \tag{3.249}$$

$$n_F(p^0 - l^0) - n_F(r^0 - l^0) = \frac{n_F(p^0 - l^0)}{1 + n_B(r^0 - p^0)}(1 - n_F(r^0 - l^0)). \tag{3.250}$$

The kernel of the BSE for the Gross-Neveu model is (3.201)

$$\tilde{\Lambda}(r,p)_{\gamma\delta'}^{\delta\gamma'} = (\sigma^z)_\gamma^{\gamma'}(\sigma^z)_{\delta'}^{\delta}\rho_D(r-p)$$
$$+\int_l (n_F(p^0 - l^0) - n_F(r^0 - l^0))(\sigma^z\,\rho_F(p-l)\sigma^z)_\gamma^\delta\,(\sigma^z\,\rho_F(r-l)\sigma^z)_{\delta'}^{\gamma'}$$

times $(n_B(r^0 - p^0) + n_F(r^0))$. By means of the (3.249), the first term is

$$(n_B(r^0 - p^0) + n_F(r^0))\rho_D(r^0 - p^0) = \frac{n_F(r^0)}{n_F(p^0)}e^{\beta(r^0 - p^0)/2}D_W^{\beta/2}(r-p) \tag{3.251}$$

where with the label $\beta/2$ we have indicated the correlation function with operator inserted at a distance $\beta/2$ over the thermal circle. They are defined as

$$G_W^{\beta/2}(r-p) = e^{-\beta(r^0-p^0)}(1+n_B(r^0-p^0))\rho_B(r-p), \tag{3.252}$$

$$S_W^{\beta/2}(r-p) = e^{-\beta(r^0-p^0)}(1-n_F(r^0-p^0))\rho_F(r-p), \tag{3.253}$$

and they can be rewritten in terms of the Wightman function with operator insertion almost coincident in the thermal circle

$$G_W^{\beta/2}(r-p) = e^{-\beta(r^0-p^0)}G_W(r-p), \tag{3.254}$$

$$S_W^{\beta/2}(r-p) = e^{-\beta(r^0-p^0)}S_W(r-p). \tag{3.255}$$

The second contribution to the kernel, corresponding to the second and third diagram in fig. 3.18, is

$$\begin{aligned}&(n_B(r^0-p^0)+n_F(r^0))(n_F(p^0-l^0)-n_F(r^0-l^0))\rho_F(r-l)\rho_F(p-l)\\ &=\frac{n_F(r^0)}{n_F(p^0)}S_{21}(r-l)S_{12}(l-p)=\frac{n_F(r^0)}{n_F(p^0)}e^{\beta(r^0-p^0)/2}S_W^{\beta/2}(r-l)S_W^{\beta/2}(l-p).\end{aligned} \tag{3.256}$$

Since the products of thermal factor simplifies as

$$\frac{n_F(r^0)}{n_F(p^0)}e^{\beta(r^0-p^0)/2} = \frac{\cosh(\beta p^0/2)}{\cosh(\beta r^0/2)}, \tag{3.257}$$

we can rewrite the kernel of the BSE of the bilocal density operator as

$$\begin{aligned}(n_B(r^0-p^0)+n_F(r^0))\tilde{\Lambda}(r,p)^{\delta\gamma'}_{\gamma\delta'} &= \frac{\cosh(\beta p^0/2)}{\cosh(\beta r^0/2)}\Bigg[(\sigma^z)^{\gamma'}_{\gamma}(\sigma^z)^{\delta}_{\delta'}G_W^{\beta/2}(r-p)\\ &+\int_l(\sigma^z\, S_W^{\beta/2}(l-p)\sigma^z)^{\delta}_{\gamma}\,(\sigma^z\, S_W^{\beta/2}(r-l)\sigma^z)^{\gamma'}_{\delta'}D^R(l+q)D^A(l)\Bigg].\end{aligned}$$

3.C Imaginary part of the self energy in the GN model

We start with the expression in imaginary time

$$\Pi(i\nu_n,\mathbf{p}) = -\sum_{\omega_m}\int_{\mathbf{k}}\mathrm{Tr}[G(i\omega_m-i\nu_n,\mathbf{k}-\mathbf{p})\sigma^z G(i\omega_m,\mathbf{k})\sigma^z] \tag{3.258}$$

now we insert the expression of the propagator in the helicity basis

$$\Pi(i\nu_n, \mathbf{p}) = -\sum_{\omega_m} \int_{\mathbf{k}} K_{ab}(\mathbf{k}-\mathbf{p}, \mathbf{k}) \frac{1}{i\omega_m - i\nu_n - a|\mathbf{k}-\mathbf{p}|} \frac{1}{i\omega_m - b|\mathbf{k}|} \tag{3.259}$$

and perform the Matsubara sum

$$\Pi(i\nu_n, \mathbf{p}) = -\int_{\mathbf{k}} K_{ab}(\mathbf{k}-\mathbf{p}, \mathbf{k}) \frac{n_F(a|\mathbf{k}-\mathbf{p}|) - n_F(b|\mathbf{k}|)}{i\nu_n + a|\mathbf{k}-\mathbf{p}| - b|\mathbf{k}|}. \tag{3.260}$$

To obtain the retarded, we perform the analytic continuation $i\nu_n \to \nu + i\epsilon$ and we can extract the imaginary part by simply use $\mathrm{Im}\,[1/(x \pm i\epsilon)] = \mp i\pi\delta(x)$ and shifting the integration variable $\mathbf{k}-\mathbf{p} \to \mathbf{k}$

$$\mathrm{Im}\,[\Pi_R(\nu, \mathbf{p})] = -\frac{1}{2} \int_{\mathbf{k}} K_{ab}(\mathbf{k}, \mathbf{k}+\mathbf{p})(n_F(a|\mathbf{k}|) - n_F(b|\mathbf{k}+\mathbf{p}|)) \times (2\pi)\delta(\nu + a|\mathbf{k}| - b|\mathbf{k}+\mathbf{p}|)$$

3.D Pinching-poles approximation

Quantum field theories at finite temperature possess on-shell thermal excitations. The lifetime of such excitations is inversely proportional to the coupling constant and indeed, in a non-interacting theory, these excitations are stable and can live indefinitely long. This is the reason behind the appearance of the delta function in the spectral density. Besides this well-known effect, there is another consequence of the existence of those excitations and it is divergence of the product of two spectral functions with opposite-sign momentum, *i.e.* $\rho(k+p)\rho(-p)$, once the zero momentum ($\mathbf{k} = 0$) and vanishing frequency limit is taken ($k_0 \to 0$). The poles of the two spectral functions pinch the real axis in the complex energy plane both from below and above and cause a divergence. This is called pinching-pole divergence. Turning a coupling on, the lifetime becomes finite and regulates such divergence (which is commonly referred to as nearly-pinching pole divergence). Nevertheless it still provides the leading contributions in the weak coupling computations and allows to organize the diagrammatic expansion. In the latter case, the retarded and advanced Green's function take the form

$$G_{\mathrm{R}}(p) = \frac{1}{(p_0 + i\Gamma_{\mathbf{p}})^2 - E_{\mathbf{p}}^2}, \quad G_{\mathrm{A}}(p) = \frac{1}{(p_0 - i\Gamma_{\mathbf{p}})^2 - E_{\mathbf{p}}^2} \tag{3.261}$$

To understand the analytical structure of the terms $G_R(p+\omega)G_A(p)$, $G_R(p+\omega)G_R(p)$ and $G_A(p+\omega)G_A(p)^2$, we study the poles of the retarded

$(P_{1/2})$ and the advanced $(P_{3/4})$ Green's function. They are respectively located at

$$(P_1) : p_0 = -E_p - i\Gamma_p; \qquad (P_3) : p_0 = E_p + i\Gamma_p;$$
$$(P_2) : p_0 = E_p - i\Gamma_p; \qquad (P_4) : p_0 = -E_p + i\Gamma_p.$$

In the previous expressions, a non vanishing ω simply shifts the real part of $-\omega$. Since we will take the zero external momentum limit, we want to find the most divergent piece in ω.

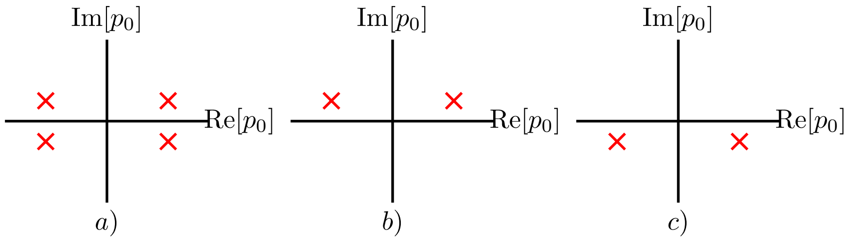

Figure 3.23: The pole structure of the product $G_R G_A$ (b), G_A^2 (c) and G_R^2 (a).

If we close the contour from below we get the following residues (times a $-2\pi i$ factor)

$$(P_1, P_3) : \frac{2\pi i}{4E_p^2}\frac{1}{\omega + 2E_p}; \;\; p_0 = -\omega - E_p;$$
$$(P_1, P_4) : \frac{2\pi i}{4E_p^2}\frac{1}{\omega + 2i\Gamma_p}; \;\; p_0 = -\omega - E_p;$$
$$(P_2, P_3) : \frac{2\pi i}{4E_p^2}\frac{1}{\omega + 2i\Gamma_p}; \;\; p_0 = -\omega + E_p;$$
$$(P_2, P_4) : \frac{2\pi i}{4E_p^2}\frac{1}{2E_p - \omega}; \;\; p_0 = -\omega + E_p;$$

If we close the contour above:

$$(P_1, P_3) : \frac{2\pi i}{4E_p^2}\frac{-1}{\omega + 2E_p}; \;\; p_0 = E_p; \qquad (P_1, P_4) : \frac{2\pi i}{4E_p^2}\frac{1}{\omega + 2i\Gamma_p}; \;\; p_0 = -E_p;$$
$$(P_2, P_3) : \frac{2\pi i}{4E_p^2}\frac{1}{\omega + 2i\Gamma_p}; \;\; p_0 = E_p; \qquad (P_2, P_4) : \frac{2\pi i}{4E_p^2}\frac{1}{2E_p - \omega}; \;\; p_0 = -E_p;$$

In the limit of vanishing ω, the most singular terms are (P_1, P_4) and (P_2, P_3), so that we can approximate

$$G_{ra}(p+k)G_{ra}(-p) \sim \frac{2\pi i}{4E_p^2}\frac{\delta(p_0-E_p)+\delta(p_0+E_p)}{\omega+2i\Gamma_p} = \frac{\pi}{E_p}\frac{\delta(p_0^2-E_p^2)}{-i\omega+2\Gamma_p} \tag{3.262}$$

Now let's study

$$G_{ra}(p+\omega)G_{ra}(p) = \frac{1}{4E_p^2}\left(\frac{1}{p^0+\omega+E_p+i\Gamma_p} - \frac{1}{p^0+\omega-E_p+i\Gamma_p}\right) \times \left(\frac{1}{p^0+E_p+i\Gamma_p} - \frac{1}{p^0-E_p+i\Gamma_p}\right)$$

The previous expression has 4 poles, respectively at

$$(P_1): p_0 = -\omega - E_p - i\Gamma_p; \qquad (P_3): p_0 = -E_p - i\Gamma_p;$$
$$(P_2): p_0 = -\omega + E_p - i\Gamma_p; \qquad (P_4): p_0 = E_p - i\Gamma_p;$$

If we close the contour above, the expression vanishes since there is no pole in the upper-half plane (this is equivalent to the statement that $G_R(t)^2 = 0$ for negative t). If we close the contour from below we get the following residues (times a $-2\pi i$ factor)

$$(P_1, P_3): 0; \;\; p_0 = -E_p;$$
$$(P_1, P_4): \frac{2\pi i}{4E_p^2}\frac{1}{-\omega+2i\Gamma_p}; \;\; p_0 = -E_p;$$
$$(P_2, P_3): \frac{2\pi i}{4E_p^2}\frac{1}{-\omega+2i\Gamma_p}; \;\; p_0 = E_p;$$
$$(P_2, P_4): \frac{2\pi i}{4E_p^2}\frac{1}{\omega-2E_p}; \;\; p_0 = +E_p;$$

By taking the complex conjugate of $G_{ra}(p+\omega)G_{ra}(p)$, we see that also the term $G_A(p+\omega)G_A(p)$ is subleading with respect to $G_R(p+\omega)G_A(p)$ in the limit of vanishing ω.

3.E Analytic continuation

Here we present the important steps to derive the analytical continuations to real time of the BSE. To do this, we closely follow the technique used in [115, 183]. There, the authors make extensive use of the formula, valid for a generic function $\tilde{G}(i\omega_n + i\nu_m, i\omega_n)$,

$$T\sum_n \tilde{G}(i\omega_n + i\nu_m, i\omega_n) = \sum_{\text{cuts}} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi i} n(\xi) \text{Disc}\, \tilde{G}(\xi + i\nu_m, \xi) - \sum_{\text{poles}} n(\xi_i) \text{Res}[\tilde{G}(\xi + i\nu_m, \xi), \xi_i]. \quad (3.263)$$

Because of the recursive BSE (3.128), the analytical structures of $\tilde{G}(i\omega_n + i\nu_m, i\omega_n)$ is represented by two branch cuts at $\text{Im}[\xi + i\nu_n] = 0$ and $\text{Im}[\xi] = 0$. The BSE resums the full series of correlation function with n rungs, relating the $n+1$ to the n, and taking the $n \to \infty$ limit. The $n = 0$ correlation function is simply $G(P+Q)G(P)$, so it has two branch cuts at $\text{Im}[\xi + i\nu_n] = 0$ and $\text{Im}[\xi] = 0$. By induction, it is easy to see that any n has the same analytical structure, and so for a general BSE of the form (3.128), the singularities correspond to the singularities of the product $G(P+Q)G(P)$ [115]. This means that the expression for the Matsubara sum is

$$T\sum_m \tilde{G}(i\omega_m + i\nu_n, i\omega_n) = \sum_{\text{cuts}} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi i} n(\xi)\, \text{Disc}\, \tilde{G}(\xi + i\nu_m, \xi) \quad (3.264)$$

$$= -\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i} n(\xi)\, [\tilde{G}(\xi + i\nu_n, \omega + i0^+) - \tilde{G}(\xi + i\nu_n, \xi + i0^-)] - \int_{-\infty}^{\infty} \frac{d\xi}{2\pi i} n(\xi)\, [\tilde{G}(\xi + i0^+, \xi - i\nu_n) - \tilde{G}(\xi + i0^-, \xi - i\nu_n)]$$

Plugging (3.127) in the previous equation and using $G(\omega + i0^+) = G_R(\omega)$ and $G(\omega + i0^-) = G_A(\omega)$, we get

$$T\sum_m \tilde{G}(i\omega_m + i\nu_n, i\omega_m) =$$
$$-\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i} n(\xi)[G(\xi + i\nu_m)\Gamma(\xi + i\nu_m, \xi + i0^+) + G(\xi - i\nu_n)\Gamma(\xi + i0^+, \xi - i\nu_n)]G_R(\xi)$$
$$+\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i} n(\xi)[G(\xi + i\nu_n)\Gamma(\xi + i\nu_n, \xi + i0^-) + G(\xi - i\nu_n)\Gamma(\xi + i0^-, \xi - i\nu_n)]G_A(\xi)$$

We now use the fact that we are interested in the following analytical continuation in the external frequency: $i\nu_n \to \nu + i0^+$. Thus

$$
\begin{aligned}
T\sum_m \tilde{G}(i\omega_m + \nu + i0^+, i\omega_m) =& \\
-\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i}\, n(\xi)\, &\big[G(\xi+\nu+i0^+)\Gamma(\xi+\nu+i0^+,\xi+i0^+) \\
&+G(\xi-\nu+i0^-)\Gamma(\xi+i0^+,\xi-\nu+i0^-)\big]\, G_R(\xi) \\
+\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i} n(\xi)\, &\big[G(\xi+\nu+i0^+)\Gamma(\xi+\nu+i0^+,\xi+i0^-) \\
&+G(\xi-\nu+i0^-)\Gamma(\xi+i0^-,\xi-\nu+i0^-)\big]\, G_A(\xi) = \\
-\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i}\, n(\xi)\, &\big[G_R(\xi+\nu)\Gamma(\xi+\nu+i0^+,\xi+i0^+) \\
&+G_A(\xi-\nu)\Gamma(\xi+i0^+,\xi-\nu+i0^-)\big]\, G_R(\xi) \\
+\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i} n(\xi)\, &\big[G_R(\xi+\nu)\Gamma(\xi+\nu+i0^+,\xi+i0^-) \\
&+G_A(\xi-\nu)\Gamma(\xi+i0^-,\xi-\nu+i0^-)\big]\, G_A(\xi).
\end{aligned}
$$

In the previous expression, we can shift the integration variable in the second and last term and we get

$$
\begin{aligned}
T\sum_m \tilde{G}(i\omega_m + \nu + i0^+, i\omega_m) =& \\
-\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i}\, &[n(\xi)G_R(\xi+\nu)\Gamma(\xi+\nu+i0^+,\xi+i0^+)G_R(\xi) \\
&- n(\xi+\nu)G_A(\xi)\Gamma(\xi+\nu+i0^-,\xi+i0^-)G_A(\xi+\nu)] \\
-\int_{-\infty}^{\infty} \frac{d\xi}{2\pi i}\, &(n(\xi+\nu)-n(\xi))[G_A(\xi)G_R(\xi+\nu)\Gamma(\xi+\nu+i0^+,\xi+i0^-)].
\end{aligned}
$$

As we are interested in the $\nu \to 0$ limit, we know that the product $G_A(\xi)G_R(\xi+\nu)$ dominates the sum. This allows us to write

$$
\begin{aligned}
T\sum_m \tilde{G}(i\omega_m + \nu + i0^+, i\omega_m) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi i}\, &(n(\xi+\nu)-n(\xi)) \\
&\times G_A(\xi)G_R(\xi+\nu)\Gamma(\xi+\nu+i0^+,\xi+i0^-).
\end{aligned}
$$

The previous analysis has to be carried out with all the Matsubara frequency terms in the RHS of the BSE. The full correlation function will involve the Matsubara sum over the frequencies p_n, which satisfies the BSE

$$\sum_{p_n} \int_{\mathbf{p}} \tilde{G}(ip_n + i\nu_{n'}, ip_n) = \sum_{p_n} \int_{\mathbf{p}} G(ip_n + i\nu_{n'}) G(ip_n) \times \left[1 + \frac{1}{N} \sum_m \int_{\mathbf{l}} \tilde{G}(il_m + i\nu_{n'}, il_m) \left[\Lambda^{LO} + \Lambda^{NLO}(il_m, ip_n; i\nu_{n'})\right]\right]. \tag{3.265}$$

A similar treatment was done in [135], where the authors considered a strict zero external frequency limit. We performed a similar analysis by retaining the small but still non zero external frequency. In both cases, the crucial consequences of the Matsubara sum are the following: since the leading order kernel Λ^{LO} does not present any singularity, it vanishes in the Matsubara sum. So we are left with only the Λ^{NLO} kernel in the BSE. Moreover, together with the pinching-pole approximation, the matsubara sum pick only a particular analytic continuation which can be shown reproduces the result obtained in real time formalism [115].

3.F Consistency of the result for the $N \times N$ matrix model

As we anticipated earlier, we can verify our result by means of the identity (3.32). We study the BSE for $G^{AaRr}(p, q|k)$ and then send $k \to -k$:

$$G^{AaRr}(p, q|k) = iG^{AR}(p + k)G^{ar}(-p)(2\pi)^4\delta^4(p - q) + \\ -G^{A\alpha_1}(p+k)G^{a\beta_1}(-p) \int_l K_{\alpha_1\beta_1\alpha_4\beta_4}(p+k, -p, -l-k, l)G^{\alpha_4\beta_4Rr}(p, q|k). \tag{3.266}$$

By expanding the sum over the extended SK indices, we observe that since $G^{AA} = G^{aa} = 0$, only one combination has a non vanishing contribution

$$\begin{aligned} G^{AaRr}(p, q|k) =& iG^{AR}(p + k)G^{ar}(-p)(2\pi)^4\delta^4(p - q) \\ &- G^{AR}(p + k)G^{ar}(-p) \int_l K_{Rr\alpha_4\beta_4}(p, l|k)G^{\alpha_4\beta_4Rr}(l, q|k) \\ =& iG^{AR}(p + k)G^{ar}(-p)(2\pi)^4\delta^4(p - q) \\ &- G^{AR}(p + k)G^{ar}(-p) \int_l K_{RrAa}(p, l|k)G^{AaRr}(l, q|k). \end{aligned}$$

By writing the kernel K_{RrAa},

$$K_{RrAa}(p,l|k) = \frac{1}{4}\frac{N^2+5}{6}\int_s G_{Rr}(s)G_{Rr}(s-l+p) = K_{AaRr}(p,l|k), \tag{3.267}$$

we arrive to the BSE for the $G^{AaRr}(p,q|k)$ Green's function

$$\begin{aligned} G^{AaRr}(p,q|k) =& G^{AR}(p+k)G^{ar}(-p)\Bigg(i(2\pi)^4\delta^4(p-q) \\ & - \frac{1}{2}\int_l K_{AaRr}(p,l|k)G^{AaRr}(l,q|k)\Bigg). \end{aligned} \tag{3.268}$$

By sending $k \to -k$ and using the pinching-pole approximation, we obtain the identity (3.32).

3.G From the BSE to the kinetic equation in the ϕ^4 matrix model

In this appendix we explicitly show how the kernel of the BSE for transport (3.41), once on-shell, reproduces the kinetic equation for transport (3.62). In the case of the bosonic matrix model, the scattering amplitude is independent of the momenta and equals, for any N,

$$|\mathcal{T}|^2 = g^4\frac{N^2+5}{6}. \tag{3.269}$$

In the following we will need some identity for the Bose-Einstein distribution function

$$\begin{aligned} & n(p_1)n(p_2)(1+n(p_3))(1+n(p_4))\delta(p_1^0+p_2^0-p_3^0-p_4^0) = \\ & \qquad (1+n(p_1))(1+n(p_2))n(p_3)n(p_4)\delta(p_1^0+p_2^0-p_3^0-p_4^0), \end{aligned} \tag{3.270}$$

together with the symmetry property of the Bose-Einstein distribution function and the spectral density with respect to the $s \to -s$ transformation

$$(1+n(-s)) = -n(s) \quad \Leftrightarrow \quad n(-s) = -(1+n(s)), \tag{3.271}$$

$$\rho^{free}(s) = \frac{2\pi}{2E_{\mathbf{s}}}\big(\delta(s^0-E_{\mathbf{s}}) - \delta(s^0+E_{\mathbf{s}})\big) = -\rho^{free}(-s). \tag{3.272}$$

Now, the kernel of the BSE equals (3.41)

$$R^{\text{transp}}(p,l) = -\frac{|\mathcal{T}|^2}{2}\frac{1+n(l_0)}{1+n(p_0)}\int\limits_{s_1,s_2}\rho(s_1)\rho(s_2)(1+n(s_1))n(s_2) \times (2\pi)^4\delta^4(s_1-s_2+l-p). \quad (3.273)$$

In the BSE, the structure of the expansion is such that the propagator occurring in the rungs are taken as free propagators. So we can insert the free spectral density in the previous equation and we get

$$R^{\text{transp}}(p,l) = -\frac{1}{2}|\mathcal{T}|^2(2\pi)^2\frac{1+n(l_0)}{1+n(p_0)}\int\limits_{s_1,s_2}\frac{(1+n(s_1))n(s_2)}{4E_{\mathbf{s_1}}E_{\mathbf{s_2}}}(2\pi)^4\delta^4(s_1-s_2+l-p) \times (\delta(s_1^0-E_{\mathbf{s_1}})-\delta(s_1^0+E_{\mathbf{s_1}}))(\delta(s_2^0-E_{\mathbf{s_2}})-\delta(s_2^0+E_{\mathbf{s_2}})).$$

We now have to evaluate the delta function, which constraints the dynamics

$$\int\limits_{s_1,s_2}\rho^{free}(s_1)\rho^{free}(s_2)(1+n(s_1))n(s_2)(2\pi)^4\delta^4(s_1-s_2+l-p) =$$

$$= \int\limits_{\mathbf{s_1},\mathbf{s_2}}(2\pi)^4\delta^3(s_1-s_2+l-p)\Big[(1+n(E_{\mathbf{s_1}}))n(E_{\mathbf{s_2}})\delta(E_{\mathbf{s_1}}-E_{\mathbf{s_2}}+l-p)+ (1+n(E_{\mathbf{s_2}}))n(E_{\mathbf{s_1}})\delta(-E_{\mathbf{s_1}}+E_{\mathbf{s_2}}+l-p)+n(E_{\mathbf{s_1}})n(E_{\mathbf{s_2}})\delta(-E_{\mathbf{s_1}}-E_{\mathbf{s_2}}+l-p) + (1+n(E_{\mathbf{s_1}}))(1+n(E_{\mathbf{s_2}}))\delta(E_{\mathbf{s_1}}+E_{\mathbf{s_2}}+l-p)\Big]$$

$$= \int\limits_{\mathbf{s_1},\mathbf{s_2}}(2\pi)^4\delta^3(s_1-s_2+l-p)\Big[2(1+n(E_{\mathbf{s_1}}))n(E_{\mathbf{s_2}})\delta(E_{\mathbf{s_1}}-E_{\mathbf{s_2}}+l-p)+ +n(E_{\mathbf{s_1}})n(E_{\mathbf{s_2}})\delta(-E_{\mathbf{s_1}}-E_{\mathbf{s_2}}+l-p)+(1+n(E_{\mathbf{s_1}}))(1+n(E_{\mathbf{s_2}}))\delta(E_{\mathbf{s_1}}+E_{\mathbf{s_2}}+l-p)\Big].$$

Thus

$$R^{\text{transp}}(p,l) = -\frac{1}{2}|\mathcal{T}|^2\frac{2\pi}{1+n(p_0)}\int\limits_{\substack{\mathbf{s_1},\mathbf{s_2}\\ \mathbf{l}=\mathbf{p}+\mathbf{s_2}-\mathbf{s_1}}} \quad (3.274)$$

$$[\delta(E_{\mathbf{s_1}}-E_{\mathbf{s_2}}+l-p)2(1+n(l_0))(1+n(E_{\mathbf{s_1}}))n(E_{\mathbf{s_2}}) + \delta(-E_{\mathbf{s_1}}-E_{\mathbf{s_2}}+l-p)(1+n(l_0))n(E_{\mathbf{s_1}})n(E_{\mathbf{s_2}}) +\delta(E_{\mathbf{s_1}}+E_{\mathbf{s_2}}+l-p)(1+n(l_0))(1+n(E_{\mathbf{s_1}}))(1+n(E_{\mathbf{s_2}}))]\,.$$

In the pinching pole approximation, the product of retarded and advanced Green's function gives a delta function which puts the momentum p *on-shell*. This in turn places *on-shell* also the momentum l, since the only non trivial solution is supported on physical states. Consequently

$$R^{\mathrm{transp}}(\mathbf{p}, E_{\mathbf{p}}; \mathbf{l}, E_{\mathbf{l}}) = -\frac{1}{2}|\mathcal{T}|^2 \frac{2\pi}{1+n(p_0)} \int_{\substack{\mathbf{s_1},\mathbf{s_2} \\ \mathbf{l}=\mathbf{p}+\mathbf{s_2}-\mathbf{s_1}}} [\delta(E_{\mathbf{s_1}} - E_{\mathbf{s_2}} + E_{\mathbf{l}} - p) 2(1+n(E_{\mathbf{l}}))(1+n(E_{\mathbf{s_1}}))n(E_{\mathbf{s_2}}) + \delta(-E_{\mathbf{s_1}} - E_{\mathbf{s_2}} + E_{\mathbf{l}} - p)(1+n(E_{\mathbf{l}}))n(E_{\mathbf{s_1}})n(E_{\mathbf{s_2}}) + \delta(E_{\mathbf{s_1}} + E_{\mathbf{s_2}} + E_{\mathbf{l}} - p)(1+n(E_{\mathbf{l}}))(1+n(E_{\mathbf{s_1}}))(1+n(E_{\mathbf{s_2}}))] \,.$$

We recognize the processes $ps_2 \to ls_1$ (with a factor 2 for $s_2 \leftrightarrow s_1$; $ps_2s_1 \to 1$ and $p \to s_1 s_2 l$). These are all "loss-"rates. The last two terms cancel, however, as it is kinematically forbidden for a *on-shell* particle of mass m to decay to three *on-shell* particles of the same mass m and vice versa. The previous term notably simplifies into

$$R^{\mathrm{transp}}(\mathbf{p}, E_{\mathbf{p}}; \mathbf{l}, E_{\mathbf{l}}) = -\frac{1}{2}|\mathcal{T}|^2 \frac{2\pi}{1+n(E_{\mathbf{p}})} \int_{\substack{\mathbf{s_1},\mathbf{s_2} \\ \mathbf{l}=\mathbf{p}+\mathbf{s_2}-\mathbf{s_1}}} [\delta(E_{\mathbf{s_1}} - E_{\mathbf{s_2}} + E_{\mathbf{l}} - E_{\mathbf{p}}) 2(1+n(E_{\mathbf{l}}))(1+n(E_{\mathbf{s_1}}))n(E_{\mathbf{s_2}})] \,. \tag{3.275}$$

The other "negative"-energy kernel gives – using $(1+n(-s)) = -n(s)$

$$R^{\mathrm{transp}}(\mathbf{p}, E_{\mathbf{p}} | \mathbf{l}, -E_{\mathbf{l}}) = \frac{1}{2}|\mathcal{T}|^2 \frac{2\pi}{1+n(E_{\mathbf{p}})} \int_{\substack{\mathbf{s_1},\mathbf{s_2} \\ \mathbf{l}=\mathbf{p}+\mathbf{s_2}-\mathbf{s_1}}} \tag{3.276}$$

$$[\delta(E_{\mathbf{s_1}} - E_{\mathbf{s_2}} - E_{\mathbf{l}} - E_{\mathbf{p}}) 2n(E_{\mathbf{l}})(1+n(E_{\mathbf{s_1}}))n(E_{\mathbf{s_2}}) + \delta(-E_{\mathbf{s_1}} - E_{\mathbf{s_2}} - E_{\mathbf{l}} - E_{\mathbf{p}}) n(E_{\mathbf{l}})n(E_{\mathbf{s_1}})n(E_{\mathbf{s_2}}) + \delta(E_{\mathbf{s_1}} + E_{\mathbf{s_2}} - E_{\mathbf{l}} - E_{\mathbf{p}}) n(E_{\mathbf{l}})(1+n(E_{\mathbf{s_1}}))(1+n(E_{\mathbf{s_2}}))] \,. \tag{3.277}$$

We recognize the processes $pls_2 \to s_1$, $pls_1s_2 \to 0$, $pl \to s_1s_2$. Using total energy conservation, these should be interpreted as "gains" $pls_2 \leftarrow s_1$, $pls_1s_2 \leftarrow 0$, $pl \leftarrow s_1s_2$. Again, the first two processes are kinematically not allowed. Thus

$$R^{\mathrm{transp}}(\mathbf{p}, E_{\mathbf{p}} | \mathbf{l}, -E_{\mathbf{l}}) = \frac{1}{2}|\mathcal{T}|^2 \frac{2\pi}{1+n(E_{\mathbf{p}})} \int_{\substack{\mathbf{s_1},\mathbf{s_2} \\ \mathbf{l}=\mathbf{p}+\mathbf{s_2}-\mathbf{s_1}}} [\delta(E_{\mathbf{s_1}} + E_{\mathbf{s_2}} - E_{\mathbf{l}} - E_{\mathbf{p}}) n(E_{\mathbf{l}})(1+n(E_{\mathbf{s_1}}))(1+n(E_{\mathbf{s_2}}))] \,. \tag{3.278}$$

The object to prove is that the collision kernel equals

$$\hat{C}(\mathbf{p},\mathbf{l}) = -\left[\int_{\mathbf{l}} 2\Gamma_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{l}) + \frac{1}{2E_{\mathbf{p}}}\left(R^{\text{transp}}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},E_{\mathbf{l}}) + R^{\text{transp}}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},-E_{\mathbf{l}})\right)\right]. \tag{3.279}$$

Comparing (3.275) and (3.278) to the Boltzmann equation (3.62) which, in a more compact form, reads

$$\begin{aligned}\partial_t f(\mathbf{p},t) =& \frac{1}{(1+n(\mathbf{p}))}\int_{\substack{\mathbf{l},\mathbf{s_1},\mathbf{s_2}\\ \mathbf{l}=\mathbf{p}+\mathbf{s_2}-\mathbf{s_1}}} \frac{|\mathcal{T}|^2/2}{2E_{\mathbf{p}}} \\ &\times [(2\pi)\delta(E_{\mathbf{s_1}}-E_{\mathbf{s_2}}+E_{\mathbf{l}}-p)2(1+n(E_{\mathbf{l}}))(1+n(E_{\mathbf{s_1}}))n(E_{\mathbf{s_2}}) \\ &-(2\pi)\delta(E_{\mathbf{s_1}}+E_{\mathbf{s_2}}-E_{\mathbf{l}}-p)n(E_{\mathbf{l}})(1+n(E_{\mathbf{s_1}}))(1+n(E_{\mathbf{s_2}})) \\ &\left.-\int_{\mathbf{l}'}(2\pi)\delta(E_{\mathbf{s_1}}+E_{\mathbf{s_2}}-E_{\mathbf{l}}-p)n(E_{\mathbf{l}})(1+n(E_{\mathbf{s_1}}))(1+n(E_{\mathbf{s_2}}))\delta(\mathbf{p}-\mathbf{l}')f(\mathbf{l}',t)\right],\end{aligned} \tag{3.280}$$

we realize that we are missing the last term. As we are going to show, this is represented by imaginary part of the self-energy $2\Gamma_{\mathbf{p}}$. We thus wish to check that

$$-2\Gamma_{\mathbf{p}} = -\frac{2}{1+n(\mathbf{p})}f(\mathbf{p},t)\int_{\mathbf{l},\mathbf{p}_2,\mathbf{p}_4}\frac{(2\pi)^4\delta^4(\mathbf{p}^{\text{os}}+\mathbf{l}^{\text{os}}-\mathbf{p_2}^{\text{os}}-\mathbf{p_4}^{\text{os}})|\mathcal{T}|^2}{2E_{\mathbf{p}}} n(E_{\mathbf{l}})(1+n(E_{\mathbf{p_2}}))(1+n(E_{\mathbf{4}})). \tag{3.281}$$

The imaginary part of the self-energy can be easily expressed in terms of the kernel as

$$\Gamma_{\mathbf{p}} = -\frac{1}{6}\int\frac{d^3l}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}}(R^{\text{transp}}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},E_{\mathbf{l}}) - R^{\text{transp}}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},-E_{\mathbf{l}})). \tag{3.282}$$

Now expanding the kernel, we obtain

$$
\begin{aligned}
-2\Gamma_{\mathbf{p}} = & \frac{1}{3}\int \frac{d^3 l}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}E_{\mathbf{l}}}(\mathcal{K}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},E_{\mathbf{l}}) - \mathcal{K}(\mathbf{p},E_{\mathbf{p}}|\mathbf{l},-E_{\mathbf{l}})) = \\
= & -\frac{1}{3}\frac{|\mathcal{T}|^2}{2}\frac{1}{1+n(E_{\mathbf{p}})}\int_{\mathbf{l},\mathbf{s_1},\mathbf{s_2}}\frac{(2\pi)^4\delta^4({\mathbf{s_1}}^{\mathrm{os}} - {\mathbf{s_2}}^{\mathrm{os}} + \mathbf{l}^{\mathrm{os}} - \mathbf{p}^{\mathrm{os}})}{2E_{\mathbf{p}}} \\
& 2(1+n(E_{\mathbf{l}}))(1+n(E_{\mathbf{s_1}}))n(E_{\mathbf{s_2}}) + \\
& -\frac{1}{3}\frac{|\mathcal{T}|^2}{2}\frac{1}{1+n(E_{\mathbf{p}})}\int_{\mathbf{l},\mathbf{s_1},\mathbf{s_2}}\frac{(2\pi)^4\delta^4({\mathbf{s_1}}^{\mathrm{os}} - {\mathbf{s_2}}^{\mathrm{os}} + \mathbf{l}^{\mathrm{os}} - \mathbf{p}^{\mathrm{os}})}{2E_{\mathbf{p}}} \\
& n(E_{\mathbf{l}})(1+n(E_{\mathbf{s_1}}))(1+n(E_{\mathbf{s_2}})). \qquad (3.283)
\end{aligned}
$$

By relabeling $s_2 \leftrightarrow \ell$ in the first line (and switching $\mathbf{p} \to -\mathbf{p}$ and $\mathbf{s_1} \to -\mathbf{s_1}$), we note that the two lines add into

$$
\begin{aligned}
-2\Gamma_{\mathbf{p}} = & -\frac{|\mathcal{T}|^2}{2}\frac{1}{1+n(E_{\mathbf{p}})}\int_{\mathbf{l},\mathbf{s_1},\mathbf{s_2}}\frac{(2\pi)^4\delta^4({\mathbf{s_1}}^{\mathrm{os}} - {\mathbf{s_2}}^{\mathrm{os}} + \mathbf{l}^{\mathrm{os}} - \mathbf{p}^{\mathrm{os}})}{2E_{\mathbf{p}}} \\
& n(E_{\mathbf{l}})(1+n(E_{\mathbf{s_1}}))(1+n(E_{\mathbf{s_2}})) = \\
= & -\frac{1/2}{1+n(\mathbf{p})}\int_{\mathbf{l},\mathbf{s_1},\mathbf{s_2}}\frac{(2\pi)^4\delta^4(\mathbf{p}^{\mathrm{os}} + \mathbf{l}^{\mathrm{os}} - {\mathbf{s_1}}^{\mathrm{os}} - {\mathbf{s_2}}^{\mathrm{os}})|\mathcal{T}|^2}{2E_{\mathbf{p}}} \\
& n(E_{\mathbf{l}})(1+n(E_{\mathbf{s_1}}))(1+n(E_{\mathbf{s_2}})). \qquad (3.284)
\end{aligned}
$$

The last equation completes the proof that in the late time limit, the BSE for the retarded Green's function of the bilocal density operator precisely reproduces the Boltzmann equation. Thus we conclude that, in terms of our earlier notation, (3.74) and (3.75) are satisfied.