Hydrodynamics and the quantum butterfly effect in black holes and large N quantum field theories

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1 Introduction

How does a closed quantum system thermalize? And why do black holes emit thermal radiation? Over the last decades, these two seemingly unrelated questions have attracted the attention of researchers in physics with backgrounds as different as condensed matter, quantum information, statistical mechanics and high-energy physicists. The reason behind this renewed interest is the idea that an underlying unknown mechanism might explain several puzzles that are still open problems in these fields, starting from the black hole information paradox, to the properties of cold atomic systems, till the very basic foundations of quantum statistical mechanics. It all relates to an essential feature of quantum mechanics: the role and the dynamics of quantum information. Since quantum mechanics is unitary, the information must be preserved. Nonetheless, experiments on closed quantum systems and theoretical predictions on black hole dynamics indicate that these systems have a thermal behaviour. Regardless of the unitary time evolution, quantum information seems to be degraded, dissipated. How to reconcile these contradictions is one of the most exciting challenges of today's physics. Beyond the purely theoretical appeal, the solution of these questions may heavily affect the near future technologies in light of the recent progresses towards building a quantum computer.

The dynamics of quantum information has recently been recast in terms of the information spreading in a quantum system. This information scrambling has some properties that reminds one of quantum chaos and is often referred to as the quantum butterfly effect. Whether these two concepts are indistinguishable or not is an open question we are not going to directly address in this thesis, but we believe it deserves caution.

In this thesis we try to understand the microscopic origin of scrambling in two opposite limits: weakly coupled field theories and strongly coupled field theories with holographic duals. By doing so, we push forward a very fascinating idea, namely that this microscopic quantum butterfly effect leaves imprints in the late-time physics, by affecting transport properties of the hydrodynamical excitations of the system. This could provide new techniques to study scrambling and shed some light on the above-mentioned problems.

As required by the object of investigation, this thesis contains topics from different fields within physics. For this reason, in this chapter we try to provide the basic elements to enable the reader to understand the motivations that led our research. In the following two sections, we review the concepts of classical and quantum thermalization. In section 1.3 we present a recently proposed observable to study scrambling. We conclude this chapter by summarizing the results of this thesis and by giving an overview to the following chapters.

1.1 Classical thermalization

When we focus on a closed classical system, for example a gas of particles, we know that after some time the velocities of the particles will be described by the Maxwell-Boltzmann distribution, even though they were randomized at the very beginning. This property, the fact that everywhere in the box the distribution function is the same, can be considered as a definition of classical thermalization. Classical systems reach this state in a dynamical way. Because of the nonlinear, and hence chaotic, equations of motion, each particle starts to explore the full phase space manifold allowed by energy conservation. This *ergodization* of the motion is such that, after some time, particles with different velocities will likely have a velocity very close to the center of the Boltzmann distribution.[1] Therefore it is clear that, in classical physics, one of the main drivers to ergodicity and so to thermalization is *chaos*[2]. A convenient definition of classical chaos is an extreme sensitivity to the initial conditions. Given two trajectories on the phase space which are very close to each other at a given time, they will quickly depart from each other with a rate set by the Lyapunov exponent

$$\lambda_{\mathrm{L}} = \lim_{t\to\infty,\delta\mathbf{X}_0\to\mathbf{0}} \frac{1}{t}\mathrm{Log}\left(\frac{|\delta\mathbf{X}(t)|}{|\delta\mathbf{X}(0)|}\right). \tag{1.1}$$

This property of dynamical systems has been brought to the general public under the suggestive name of butterfly effect, which is nowadays used also in the scientific community.

[1] During the time evolution, a finite size system eventually returns arbitrarily close to its initial state. This property, representing the statement of the Poincaré recurrence theorem [1], does not represent a problem for thermalization since the recurrence time is exponentially long in the system size. Moreover, statistical mechanics allows for atypical configuration as far as they have exponentially small probability.

[2] The connection between chaotic dynamics and ergodization is still an open problem. While chaotic systems are ergodic, the opposite does not necessarily have to be true.

1.2 Quantum thermalization

The dynamical thermalization, occurring in classical mechanics, cannot be extended in a simple way to quantum mechanics, because this means that a pure initial state would evolve into a mixed state (thermal), in contrast with the unitarity of the time evolution. Nevertheless, we expect the out-of-equilibrium dynamics of closed quantum systems to drive it into a state whose properties are very similar to what we would naively define as a thermal state: a stationary value of macroscopic quantities and stability over a wide range of initial conditions. This expectation is driven not only by physical intuition, but also by numerical results on isolated quantum systems [2].[3] One definition of thermalization involves the expectation value of the observable and can be stated as follows. Let's consider a closed quantum system driven out-of-equilibrium and let it evolve in time. If the system was initially prepared in a state with a well defined mean energy and, during the evolution, the expectation value of the observable can be well approximated by the microcanonical expression, we can consider the system as having thermalized [3] . Given an initial state $\phi = \sum_n c_n|n\rangle$ and the observable $\mathcal{O}$, the expectation value evolves with time as follows

$$\langle\phi(t)|\mathcal{O}|\phi(t)\rangle = \sum_n |c_n|^2\mathcal{O}_{nn} + \sum_{n,n\neq m} c_m^* c_n e^{i(E_m-E_n)t}\mathcal{O}_{mn}. \tag{1.2}$$

Requiring that this correlation function, after some thermalization time τ_t, matches a microcanonical result is a highly non trivial constraint. A first reason is that the time independent part of (1.2), $|c_n|^2\mathcal{O}_{nn}$, should match the microcanonical result. Furthermore, we should also impose the second term to vanish. At first sight, the latter requirement might seem very easy to satisfy, as in long time limit the second term averages to zero. But clearly it is very important to estimate the time scale at which it starts to hold. This is related to the spacing of the energy levels, a property which discerns whether a system possesses quasiparticle excitations or not. Indeed, the high energy spectrum of a generic many-body system is characterised by a level spacing exponentially suppressed in the system size[4]. The low energy spectrum, instead, changes. If a system has quasiparticle excitations, the

[3] A comprehensive list of the literature on the numerical studies can be found in [3].

[4] This can be easily seen for a quasiparticle system by considering a simple model of metal (with quasiparticles) with N sites. The energy of the system is described by $E = \sum_{\alpha=1}^{N} \epsilon_\alpha n_\alpha + ...$, where ϵ_α are the single particle energies and $n_\alpha = 0, 1$ the occupation numbers. As there are 2^N many-body levels, the spacing is proportional to 2^{-N}. This property holds also for systems without quasiparticle excitations [4].

single-particle level spacing behaves like $1/N$. But in a system without quasiparticles, the spacing is still exponentially suppressed in N.

This implies that the time at which the second term of (1.2) starts to vanish, proportional to the inverse level spacing $t^* \sim (E_n - E_m)^{-1}$, in a many body system can be exponentially long. Moreover the study of thermalization highly depends on the nature of excitations in the system, since in absence of quasiparticles even the analysis of the low energy sector of the spectrum can be non trivial.

A step forward in understanding the quantum thermalization was the work of Deutsch [5], who used Random Matrix Theory (RMT) to show that, for a random Hamiltonian, the first term of (1.2) indeed coincides with the microcanonical result. RMT, though, is a crude approximation since it washes away all the state dependence of the result, for example the energy of the state (which is crucial in the microcanonical description). These results are equivalent to the infinite temperature limit [3, 5].

The more refined explanation for the thermal expectation value of a local observable is provided by the Eigenstate Thermalization Hypothesis, conjectured by Srednicki in a series of seminal papers [6, 7]. For quantum systems that thermalize, the Hypothesis states that the spectrum of the Hamiltonian H is such that the expectation value of a local observable over the eigenstates n and m of H is of the following form[5]

$$\langle m|\mathcal{O}|n\rangle = \mathcal{O}(\bar{E})\delta_{mn} + e^{-S(\bar{E})/2} f_{\mathcal{O}}(\bar{E},\omega) R_{mn}. \tag{1.3}$$

In the above equation, $\bar{E}$ is the average energy of the states n and m, $\bar{E} = \frac{E_n+E_m}{2}$, and ω the difference, $\omega = E_m - E_n$. Moreover, $\mathcal{O}(\bar{E})$ and $f_{\mathcal{O}}(\bar{E},\omega)$ are smooth functions of the arguments and $\mathcal{O}(\bar{E})$ corresponds to the microcanonical expectation value at energy $\bar{E}$ of the operator $\mathcal{O}$. R_{mn} are random numbers with zero mean and unit variance and $S(\bar{E})$ is the microcanonical entropy.

Despite the successes of the ETH ansatz in describing the thermal behaviour of the correlation functions of local operators, there are still many open questions. In quantum mechanics a pivotal role is played by quantum information, which also in this case seems to be crucial [3]. Nevertheless, what ETH is not able to describe is the dynamics behind thermalization. Thanks to the collective effort of the last decade, we now understand that, under time evolution, the information spreads and

[5] The ETH can be formulated also in terms of operators which are not strictly local, but still subextensive with respect to the number of degrees of freedom, as in [8]. This allows to study ETH also in intrinsically non local systems, as for example SYK model.

delocalizes (it *scrambles*) over the system, becoming inaccessible to local experimental measurements which locally only probe an effectively thermal state.

This can be considered one of the qualitative explanations behind ETH and justifies the use of local operators. In more clear terms, it is important to understand the dynamical process that underlies the ETH ansatz. By drawing an analogy with the classical case, this might be connected to some mechanism that naively could be defined quantum chaos. Furthermore, as we see from equation (1.3), checking thermalization with the ETH ansatz requires the knowledge of the spectrum and the eigenstate of the system, which in a many-body systems is remarkably hard to compute.

The problems listed in the previous paragraphs raise the question whether it is possible to study the dynamics of quantum thermalization using some new observables or new techniques. In the coming section, we will review the out-of-time correlation function (OTOC), which has recently attracted lots of interested in the study of quantum chaos. Afterwards we will discuss a seemingly exotic idea, representing one of the main drives for this thesis, which is to understand quantum chaos by looking at properties of the late-time physics encoded in the hydrodynamical excitations.

1.3 Information scrambling and out-of-time ordered correlators (OTOC)

In the previous section we reviewed some essential features of the ETH and stressed how it seems to release the tension between thermalization and the unitary evolution of quantum mechanics. We can think of it as a precise understanding of thermalization in energy space, but it would be interesting to see what it means in the position space and in time. The way the community understands quantum thermalization nowadays is highly connected to the concept of local operators. When we perform an experimental measurement, in many cases we are probing the system locally, and we have no access to the degrees of freedom in regions far from the probe. This operation corresponds to tracing those degrees of freedom out, giving rise to the thermal spectrum. Clearly, if the information in a quantum system was not subjected to dynamics, *i.e.* it stayed localized, it would be easily detected in an experiment and we would not see thermalization. Our understanding of quantum thermalization strongly indicates that in quantum systems information has a dynamics and, consequently, it spreads over the degrees of freedom of the whole system. This process is called information scrambling, and has acquired an essential role in the studies of Black Hole information paradox, quantum thermalization and quantum information theory.

In order to probe the information scrambling, a so-called out-of-time correlations function (OTOC) has been put forward. This 4-point function was first introduced in the context of superconductivity by Larkin and Ovchinnikov [9] . There, this correlation function was not put in relation with quantum chaos but it was shown to measure the difference between the classical and the quantum results. Only subsequently the connection with quantum chaos arose [10–16]; since then, this correlation function has appeared in the context of black holes physics and the Sachdev-Ye-Kitaev (SYK) model [17] [10, 18] and there has been a big effort to create experimental protocols to measure it [19–29]. Moreover, several techniques were used to compute it either numerically or analytically [10, 11, 14, 30–49] and its connections with operator growth were studied in [31, 32, 38, 50–54]

The OTOC is defined as follows: given two operators V and W, opportunely normalized, it is

$$C(\mathbf{x}, t) = \langle [V(\mathbf{x}, t), W(0)]^{\dagger} [V(\mathbf{x}, t), W(0)] \rangle. \tag{1.4}$$

We can understand the information scrambling in terms of the time evolution of an operator, in this case V, initially located at the position $\mathbf{x}$. As a consequence of time evolution, this operator will start spreading over the system. The spreading of the operator can be easily visualized in a spin chain where V at time zero is a single site spin operator [55]. In the Heisenberg picture, the time evolution can be written in terms of nested commutators of the operator V with the full Hamiltonian H and, because of these commutators, the time evolution will contain spin operators of other sites.

We can probe the spreading by considering the commutator with an operator inserted in $\mathbf{0}$, for example with $[V(\mathbf{x}, t), W(0)]$. At time zero the commutator vanishes because of causality. However, with time evolution the operator $V(\mathbf{x}, t)$ will become more and more delocalized, and at some time t^* its front will hit the insertion $W(0)$, developing a non trivial value of the commutator. This simple picture would suggest the following

$$C'(\mathbf{x}, t) = \langle [V(\mathbf{x}, t), W(0)] \rangle \tag{1.5}$$

as a good observable for operator spreading. Such correlation function corresponds to the retarded (advanced) Green's function for t positive (negative). Unfortunately, the time ordered 2-point correlation functions,

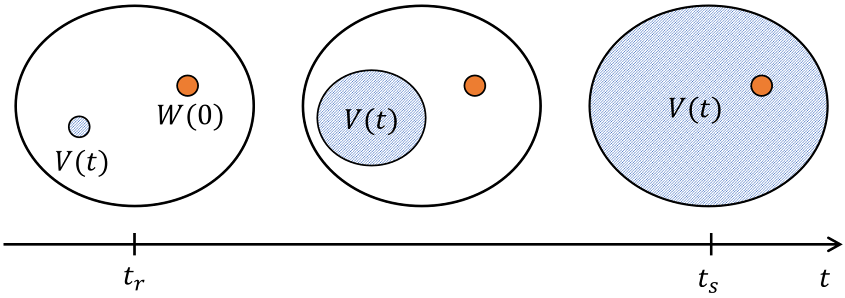

Figure 1.1: Representation of operator spreading. Figure taken from [56].

computed on a thermal state, decay very quickly and they are too constrained to carry information about scrambling. In order to extract such information, (1.4) turns up to be a fruitful choice.

If we focus on hermitian operators W and V, (1.4) can be rewritten as

$$C(\mathbf{x},t) = 2 - 2\,\mathrm{Re}F(\mathbf{x},t) \tag{1.6}$$

where

$$F(\mathbf{x},t) = \langle V(\mathbf{x},t)W(0)V(\mathbf{x},t)W(0)\rangle. \tag{1.7}$$

The latter expression represents the out-of-time order contribution to $C(\mathbf{x},t)$ and it contains all the physics about scrambling. The time dependence of this correlation function for chaotic system can be parametrized as

$$F(\mathbf{x},t) = 1 - \epsilon\, e^{\lambda_{\mathrm{L}}(t-g(\mathbf{x}))} \tag{1.8}$$

where λ_{L} is conjectured to be the highest Lyapunov exponent of the system, ϵ is a small term inversely proportional to the local number of degrees of freedom and $g(\mathbf{x})$ is a function that represents the spatial profile. For many large N field theories, $g(\mathbf{x})$ has a linear behaviour $g(\mathbf{x}) = \frac{|\mathbf{x}|}{v_{\mathrm{B}}}$, where v_{B} is the speed at which the front depicted in Fig. (1.1) moves and it is called the *butterfly velocity*.

On a thermal state, the OTOC is often defined as follows

$$C(\mathbf{x},t) = -\langle \rho[V(\mathbf{x},t),W(0)]^{\dagger}[V(\mathbf{x},t),W(0)]\rangle, \tag{1.9}$$

ρ being the thermal density matrix. A slightly different definition involves a symmetric insertion of the density matrix in the correlation function

$$C^{\mathrm{S}}(\mathbf{x},t) = -\langle \rho^{1/2}[V(\mathbf{x},t),W(0)]^{\dagger}\rho^{1/2}[V(\mathbf{x},t),W(0)]\rangle. \tag{1.10}$$

That (1.9) and (1.10) might have different properties is a topic that has not been considered in the last few years and will be discussed in chapter 5. With this configuration, the out-of-time ordered function acquires a new fundamental property. Under mild hypotheses, such as analyticity of correlation functions and unitarity of time evolution, Maldacena, Shenker and Stanford proved in [16] that, if the correlation function (1.10) presents an exponential growth regime, the Lyapunov exponent satisfies the following upper bound

$$\lambda_{\mathrm{L}} \leq \frac{2\pi \mathrm{k_B} T}{\hbar}. \tag{1.11}$$

The bound is saturated by systems which have a gravitational dual; this proves the *fast scrambling conjecture*, introduced in [11], which states that Black Holes are the fastest scramblers in nature. Nevertheless it is not known yet whether the saturation of the bound represents a sufficient condition for a theory to have a holographic dual. The right hand side of (1.11) is intimately connected to the nature of black holes and, in particular, to their event horizon. It can be shown [12, 14, 15] that, in a theory with a holographic dual, the OTOC corresponds to the effect on the geometry of few particles moving from the boundary towards the bulk. Once in the proximity of the event horizon, the energy of the particle, in the local frame, will be highly boosted, creating a shock wave along the horizon. The Lyapunov exponent is a measure of such a boost, which for any Black Hole results in the value $\frac{2\pi \mathrm{k_B} T}{\hbar}$.

The bound (1.11) appears like a fundamental property of quantum mechanics, and its possible interpretations and consequences are very fascinating. For the moment, we can rewrite the bound (1.11) in terms of the Lyapunov time $\tau_{\mathrm{L}} = 1/\lambda_{\mathrm{L}}$

$$\tau_{\mathrm{L}} \geq \frac{1}{2\pi} \frac{\hbar}{\mathrm{k_B} T}. \tag{1.12}$$

In this form it will soon be clear why the bound has been intensively studied in the last years, and it has to do with transport in strongly coupled systems without quasiparticles. This is one of the main open problems of today's physics, both from the theoretical and experimental point of view and in the next section we will try to highlight its connection with the above mentioned bound.

1.4 Quantum systems without quasiparticles

In this section we take an apparent departure from the topics of the previous pages, which hopefully will be soon clear to the reader. Our current understanding of transport in ordinary metals is provided by the Landau theory of Fermi liquids [57], which is built on the concept of quasiparticles. There, the relevant excitations are long lived and have a pivotal property: the energy of a state made of quasiparticles is simply the sum of their energies. These quasiparticles have the same quantum numbers as electrons, and interact with each other. When the system is perturbed, the interactions restore thermal equilibrium after a local equilibration time, which we indicate with τ_e. If the system is gapless, τ_e has a temperature dependence that, in the $T \to 0$ limit, goes as $\tau_e \sim 1/T^2$. In the presence of a gap Δ, instead, the equilibration time is even longer and scales as $\tau_e \sim e^{\Delta/T}$. Despite its success, there are still materials which Landau theory is not able to describe, such as cuprates, heavy fermions, ruthenates, pnictides, vanadium dioxide, fullerenes and organics. The electrons in these materials are strongly correlated and present a linear in T resistivity in a wide region of their phase space [58, 59]. The resistivity scaling suggests that the local equilibration time is exceptionally short and scales as $\tau_e \sim 1/T$. The surprising feature of this scaling is that, besides being very robust against disorder, it is present in systems with very different microscopic details. It seems thus that some universal mechanisms underlies the physics in this regime. This mechanism has been named Planckian dissipation [60–62] and states that strongly coupled systems without quasiparticles are the fastest in thermalizing. Furthermore their local equilibration time saturates the lower "dimensional analysis" bound

$$\tau_e \geq C\frac{\hbar}{k_B T}, \tag{1.13}$$

where C is some temperature independent constant of order one. More recently, measurements on thermal diffusivities [63] showed Planckian dissipation in YBCO samples.

The similarities between (1.12) and (1.13) might be interpreted as the existence of a connection among the Lyapunov time and the local equilibration time. Moreover, in this light, it was proposed by Blake [64, 65] that charge and thermal diffusivities of critical systems could be expressed as

$$D_{\mathrm{C/T}} \propto v_{\mathrm{B}}^2 \tau_L, \tag{1.14}$$

where the identity holds up to a numerical prefactor. This proposal was tested in several systems [66] but soon it became evident that the connection with charge diffusion was not as robust as thermal diffusion. The latter connectio is quite robust. Indeed in [67–69] it was shown that, in the infrared, the thermal diffusivity of a generic strongly coupled system with holographic dual satisfies relation (1.14) regardless of charge density, periodic potential strength, or magnetic field. Those results were obtained by using the holographic duality, which we are going to quickly review in the following section.

1.5 Holographic duality

Since its discovery in 1997 by Maldacena [70], AdS/CFT has profoundly changed the way we look at gravity and strongly coupled field theories. In this section we will try to give a very general overview on the topic, mainly focusing on the aspects that are necessary to understand some of the results of this thesis, and the reason why we have tried to address certain questions.[6]

The idea of holography can be traced back to two important results. In 1974, 't Hooft realized that $SU(N_c)$ gauge theories greatly simplify in the large N_c limit, keeping the 't Hooft coupling $\lambda = g_{YM}^2 N_c$ fixed [75]. In this limit, organized as a perturbative expansion in λ, the leading contribution is given by planar diagrams. In the same work, t'Hooft noticed that this expansion could be connected to a string theory path integral. On a completely different perspective, the work of Bekenstein on black holes thermodynamics [76] showed that the entropy of a Black Hole depends on its surface, and not on its volume. This indicates that in a gravitational theory gravitational degrees of freedom rearrange the information on a surface of codimension one, making possible (in principle) a description of $d+1$ dimensional gravity in terms of a field theory defined in d dimensions [77, 78].

AdS/CFT is an exact realization of this idea. In its most essential definition, the AdS/CFT correspondence is a duality between a conformal large N supersymmetric Yang-Mills theory (Large N SYM) defined in d dimensions and a classical gravitational theory defined on a $d+1$ spacetime with a negative cosmological constant, namely Anti-de-Sitter. Its origins lies in string theory, where large N SYM is equivalent to string theory in AdS. It is a strong-weak duality, *i.e.* it relates a strongly coupled theory to a weakly coupled one. This means that it allows to quantitatively

[6] For a review on the topic, see [71]. Applications to condensed matter physics are reviewed in the following books [72–74].

understand properties of strongly coupled field theories by studying classic gravitational dynamics.

Bekenstein's argument gives a hint about the holographic nature of gravity, but a natural question to ask is what does this extra dimension correspond to from the QFT point of view. Given a QFT defined on a d dimensional spacetime, we know since Wilson [79] that the theory changes according to the energy (or spacetime) scale μ we focus on. The way the theory changes by varying this scale is described by the Renormalization Group (RG) equation

$$\mu \frac{dg(\mu)}{d\mu} = \beta(g(\mu)), \tag{1.15}$$

where $g(\mu)$ schematically represent the coupling constants of the theory. A remarkable fact of equation (1.15) describing the renormalization group flow is that it is local in the energy scale. This might seem a minor aspect but has strong implications, specially in AdS/CFT. In holography, the renormalization scale not only becomes dynamical, but also geometrical. Its dynamics is governed by the Einstein action with a negative cosmological constant $-2\Lambda = \frac{d(d-1)}{L^2}$

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \left(R + \frac{d(d-1)}{L^2} \right), \tag{1.16}$$

with $\kappa = 8\pi G_N$ and G_N the Newton's constant and L is the AdS radius. The zero temperature solution in vacuum is described by the metric

$$ds^2 = \left(\frac{L}{z}\right)^2 (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2), \tag{1.17}$$

where $\mu, \nu = 0, ..., d-1$. The direction z can be considered as a renormalization group scale and the excitation of the theory are rearranged along z according to their wavelength, as represented in Fig. (1.2). We see that motion in $z \to \lambda z$ precisely can be accounted for by scaling $x \to \lambda x$.

On a more quantitative ground, the holographic correspondence can be stated in terms of an identity of two partition functions, called the GKPW rule [81, 82]

$$\langle e^{\int d^d x \sum_i J_i(x) \mathcal{O}_i(x)} \rangle_{\text{QFT}} = \int \mathcal{D}g \mathcal{D}\phi_i \, e^{i S_{bulk}[g, \phi_i(x,z)]|_{\phi(x,z=0)=J(x)}}. \tag{1.18}$$

The left hand side of the equality is the partition function of a QFT deformed by a source J. The right hand side corresponds to the gravitational partition function, dominated by the saddle point value, where

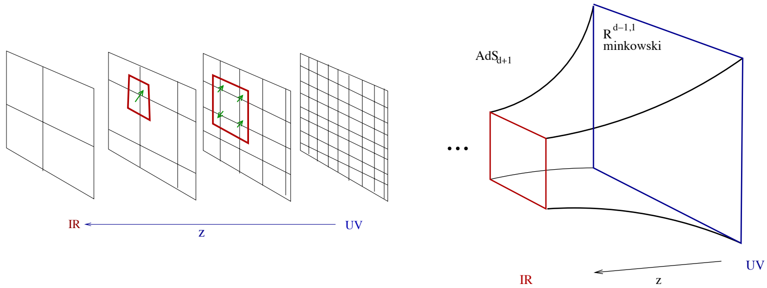

Figure 1.2: The emerging dimension in the AdS/CFT correspondence can be though of as a renormalization scale. The excitations of the dual field theory are rearranged over the boundary according to their wavelenghts: the long wavelenght, low energy dynamics of the dual theory is captured by events in the bulk. Short distance, high energy excitations are described by the gravitational degrees of freedom close to the boundary. The duality rearranges the excitation. Figure taken from [80].

the boundary value (at $z \to 0$) of the field ϕ is set by J, the value of the source of its dual operator $\mathcal{O}$. Including this fundamental identity, the duality can be summarized in terms of a dictionary, where quantities of the two theories are related. Some of the entries of the dictionary are presented in the tabular below. The order used to list them is not casual since the connection between the gravitational dynamics in the bulk and stress tensor properties in the boundary will be the essential motivation to pursue the results of chapter 4 that connect quantum chaos to the energy-energy correlation function.

Another feature of gravity that holography incorporates naturally that deserves attention for this thesis is the fluid/gravity correspondence [83]. This states that the spacetime dynamics, on a AdS background, can be mapped into the dynamics of a fluid living on the *boundary*. Since the long wavelength of the fluid is described by hydrodynamics and, in the AdS/CFT correspondence, the IR of the boundary theory is described by the bulk gravitational degrees of freedom, the fluid/gravity correspondence connects the hydrodynamic behaviour of the fluid living on the boundary with the gravitational perturbation of the event horizon. After a perturbation, Einstein's equations reproduce hydrodynamical equations and allow to compute the conductivities of the fluid [83, 84]. This does not imply that the event horizon is purely hydrodynamical, because very far from equilibrium the hydrodynamic approximation is not valid. Nevertheless, Black Holes have the surprising feature that they hydrodynamize very quickly.

The idea of a connection between the dynamics of black hole horizons and the Navier-Stokes equations of hydrodynamics dates back to the 80's when Thorne, Price and MacDonald formulated the so-called membrane paradigm [85]. They realized that, for an external observer, the behaviour of black holes would resemble the behaviour of a fluid membrane, *surrounding their event horizon*, with well-defined properties as viscosity and conductivity.

Boundary (Operators)	**Bulk (fields)**
stress-energy tensor T_{ab}	metric field $g_{\mu\nu}$
global current J^a	Maxwell field A_a
scalar operator $\mathcal{O}$	scalar field ϕ
conformal dimension of the operator	mass of the field
source of the operator	boundary value of the field
VEV of the operator	boundary value of radial momentum of the field
global symmetry	local symmetry
temperature of the field theory	Black Hole temperature
phase transition	Black Hole instability

1.6 A hydrodynamical refresh

When focusing on the long-wavelength and late-time description of a given system, often the result is a very old theory, whose origins go back to the early works of Euler and Bernoulli in the XVIII century: hydrodynamics. Even though it is a very old theory, there are still many properties that are not understood, like turbulence for example, and in the last years we have witnessed a revival of interest about its applications. In the present-day view, hydrodynamics is an effective field theory that is conventionally formulated in terms of equations of motion. The action formulation has recently been addressed in[7] [87–98] and, as often happens going from the equations of motion to the action, it has revealed new symmetries and constraints. The equations of motions of (relativistic) hydrodynamics are simply the conservation laws of the systems; for example, if there are no more conserved currents than the stress energy tensor, the EOMs are

$$\partial_\mu T^{\mu\nu} = 0. \tag{1.19}$$

[7] For review on the topic see [86] and references therein.

As stated, this does not represent a well-posed problem in $d \geq 2$ since there are $d(d+1)/2$ degrees of freedom and d equations of motion[8]. A crucial step in deriving the hydrodynamical equation is the hypothesis of *local thermal equilibrium*, which reduces sensibly the number of degrees of freedom. When the system is perturbed at long wavelength, it is possible to consider as dynamical variable the temperature $T(\mathbf{x})$ as a function of space and time, and the velocity four vector $u^\mu(\mathbf{x})$, subject to the normalization condition $u^\mu u_\mu = -1$. This choice makes the problem well posed, since there are d unknowns and d equations of motion. As typical for effective field theories, the stress-energy tensor can be written in terms of a gradient expansion in spatial derivatives and as a function of the dynamical variables. At zero-th order the expansion gives

$$T^{\mu\nu}(x) = \epsilon\, u^\mu u^\nu + p\, \mathrm{P}^{\mu\nu}, \tag{1.20}$$

where ϵ is the energy density, p the pressure density and with $\mathrm{P}^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$ we have indicated the projectors along the direction transverse to the four-velocity u^μ. In this order, the hydrodynamical equations do not present any dissipation, as can be easily shown [99]. Entropy production appears in first order, with the introduction of the dissipation tensor and transport coefficients

$$T^{\mu\nu}(x) = \epsilon\, u^\mu u^\nu + p\, \mathrm{P}^{\mu\nu} - \sigma^{\mu\nu}. \tag{1.21}$$

The dissipation component of the stress energy tensor can be generally parametrized as follows

$$\sigma^{\mu\nu} = \mathrm{P}^{\mu\alpha}\mathrm{P}^{\nu\beta}\left[\eta\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta}\partial_\rho u^\rho\right) + \zeta g_{\alpha\beta}\partial_\rho u^\rho\right], \tag{1.22}$$

the coefficients η and ζ being respectively the shear and the bulk viscosity. The form (1.22) of the dissipation tensor is not unique, but is chosen in such a way to affect only the spatial component of the stress-energy tensor T^{ij}.

The analytical structure of the hydrodynamical correlators, namely the location of the poles, can be extracted from the normal modes of the linearised hydrodynamical equations, *i.e.* by looking at solutions of the form $e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$. For example, let's restrict the analysis to the stress-energy tensor, and choose a frame where the spatial momentum is aligned with the z direction, $\mathbf{k} = (0, 0, k)$. It is possible to show that the stress energy

[8] In CFT, the case $d = 2$ is well defined since there is an extra constraint on the stress energy tensor which has to be traceless.

tensor has two physically different hydrodynamical modes, the shear and the sound modes [99]. The components T^{0i} and T^{zi}, with $i = (x, y)$, satisfy the following linearised hydrodynamic equation

$$\partial_t T^{0i} = -\frac{\eta}{\epsilon + p}\partial_z^2 T^{0i}, \tag{1.23}$$

which can be immediately recognized as a diffusion equation. Indeed by looking for a solution of the form $e^{-i\omega t + ikz}$ we get

$$\omega(k) = -i\frac{\eta}{\epsilon + p}k^2, \tag{1.24}$$

with diffusion constant $D_s = \frac{\eta}{\epsilon+p}$. Besides shear modes, there are also propagating sound modes, represented by simultaneous excitations of T^{00}, T^{0z} and T^{zz}. After diagonalizing a system of coupled hydrodynamic equations, the condition these modes have to satisfy is the following [100]

$$\omega(k) = v_s k - \frac{i}{2}\left(\frac{4}{3}\eta + \zeta\right)\frac{k^2}{\epsilon + p}, \tag{1.25}$$

where $v_s = \sqrt{\frac{\partial p}{\partial \epsilon}}$ is the speed of sound.

1.7 Hydrodynamic transport coefficients at weak coupling and in holography

The late-time and long-scale physics of a generic classical or a quantum many-body system is governed by the long-lived quantities, which are nothing but the conserved currents. Their conservation laws become the equation of motion and the theory is called hydrodynamics, which indeed is nothing but the dynamics of the conserved quantities. This in turn expresses the universality of its prediction, which heavily rely on symmetries arguments. Although it has a high degree of universality, the hydrodynamic prediction is intimately connected to the underlying microscopic physics via some of the coefficients of this expansion: the transport coefficients like viscosities and diffusivities. Given a Lagrangian the computation of these coefficients represents a challenge even at weak coupling, since they often manifest a nonanalytic dependence on the coupling parameters. Traditionally, for high-energy applications like QCD or relativistic QFTs in $3 + 1$ dimensions, the main tool has been to solve the Boltzmann equation [101–108]. Another approach, based on quantum field theory, can be found in [109, 110].

Another way to compute transport coefficients, more used in low-energy many-body physics condensed matter, makes use of the Kubo formulae for specific correlation functions, like the following ones for shear viscosity η or electrical conductivity σ [111]

$$\eta = \frac{\beta}{20} \lim_{p^0, \mathbf{p} \to 0} \int d^4x e^{ip\cdot x} \langle T_{ij}(x) T^{ij}(0) \rangle, \tag{1.26}$$

$$\sigma = \frac{\beta}{6} \lim_{p^0, \mathbf{p} \to 0} \int d^4x e^{ip\cdot x} \langle J_i(x) J^i(0) \rangle, \tag{1.27}$$

where T_{ij} is the traceless stress-energy tensor, J_i is the $U(1)$ current and the sum over repeated indices is intended. In the weakly coupled regime, the formulae (1.26) and (1.27) might suggest that, by simply performing loops expansion of the correlation function in the integral, we could get the proper perturbative results for the shear viscosity. Unfortunately, this expansion is not reliable due to the appearance of infrared singularities [112], and a full resummation of a series of diagrams is required [101]. This becomes an even more challenging task when relativistic gauge theories are considered. The pioneering computation of the shear viscosity for a scalar field with ϕ^4 interaction

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4, \tag{1.28}$$

was performed in [101, 112]. There, by using a finite temperature optical theorem, Jeon derived the cutting rules at finite temperature and then showed that an infinite set of ladder diagrams had to be resummed in order to obtain the correct leading order result. Since then, several works addressed this question for more complicated field theories and using different techniques, such as the real time close time-path contour (CTP) [113, 114] and the Imaginary Time Formalism (ITF) [115] of thermal field theory. Within the latter approach, a non perturbative resummation provided by the two-Particle-Irreducible (2PI) formalism was introduced in the computation of transport quantities in [116, 117] .

In this thesis we will extensively make use of those techniques, developed to compute the Kubo formulae, to evaluate the following correlation function

$$iG_{\mathrm{R}}(x, p|y, q) = \theta(x^0 - y^0) \langle [\rho(x, p), \rho(y, q)] \rangle \tag{1.29}$$

where we have defined the Wigner transport of the bilocal density operator

$$\begin{aligned}\rho(x,p) &= \int_y e^{-ipy}\,\mathrm{Tr}\left[\phi(x-y/2)\phi(x+y/2)\right] \\ &= \int_k e^{ikx}\,\mathrm{Tr}\left[\phi(p+k/2)\phi(p-k/2)\right],\end{aligned} \tag{1.30}$$

and the trace is taken over the internal indices of the field ϕ.

While in a weakly coupled QFT the computation of correlation functions which appear in the Kubo formulae is very involved, in strongly coupled field theories with holographic dual things simplify remarkably. By using the dictionary, it is possible to show that the stress-energy tensor two-point functions can be extracted by a perturbative gravity computation in the bulk. This allows an exact numerical solution and, in the hydrodynamic limit, an analytical solution [99, 100, 118–120]. We will use these techniques in chapter 4 to derive one of the most important results of this thesis.

1.8 Quantum chaos and hydrodynamics

In the previous sections we have briefly reviewed the conundrum represented by the thermalization of a closed quantum system. We have also given the qualitative understanding on how this thermal behaviour emerges, namely because of the scrambling of quantum information. Nevertheless, several questions have no satisfactory answer yet. We still lack a comprehensive dynamical understanding of how this information-scrambling happens. Moreover, we don't know whether scrambling and quantum chaos are the same phenomenon. We believe that these are essential questions to address, especially in light of the fast scrambling conjecture. The latter is likely to be a fundamental result, able to classify different states of matter. Unfortunately, it does not explain the mechanisms that determine whether a system can be close to saturation of the bound or not.

On general grounds, we know that, for relativistic theories, the mechanisms underlying the way a system (nearly) out-of-equilibrium reaches the thermal state leaves clear imprints in the analytic structure of the conserved currents-correlation functions [121, 122]. Since the dynamics of conserved currents is by definition hydrodynamics, it seems reasonable to try to understand many-body chaos by looking for signatures in the analytical structure of hydrodynamic excitations.

There is further evidence in support of this idea. As already mentioned, the scrambling properties of strongly coupled theories with holographic duals at finite temperature are set by the gravitational dynamics in the bulk. By using the membrane paradigm, we can imagine that this involves the correlation functions of the stress-energy tensor, which very quickly assume a form dictated by hydrodynamics.

Furthermore, some insights come from the study of the Sachdev-Ye-Kitaev (SYK) model. This is a $(0+1)$ theory of fermions with quartic all-to-all interaction, and Lagrangian (in the case of Majorana fermions)

$$H = -\sum_{ijkl} J_{ijkl}\, \chi^i \chi^j \chi^k \chi^l. \tag{1.31}$$

This model saturates the bound on chaos and is supposed to have a gravity dual. In the infrared limit, it exhibits a $SL(2,\mathbb{R})$ symmetry, which is reminiscent of the AdS_2 geometry. In [123], Jensen wrote a hydrodynamic effective field theory which is maximally chaotic and captures some features of the AdS_2 geometry. The fact that the IR physics of the SYK model can be captured by a hydrodynamical mode which is also maximally chaotic should be considered as further evidence that, at least for (nearly) maximally chaotic systems, hydrodynamics carries some information about scrambling.

1.9 Summary of results

In this thesis we address some of the questions raised so far. In the first part, we try to understand what is the dynamics of scrambling, or of quantum chaos. We tackle this problem for weakly coupled quantum field theories (or large N) and we draw an analogy with the quantum Boltzmann equation (QBE). Even though the QBE relies on the concept of quasiparticles, in 1997 Damle and Sachdev showed [124] that the QBE can be used also to study conductivities and transport properties above the two-dimensional superfluid-insulator quantum critical point. This means that, by carefully controlling the QBE, it can give predictions about excitations of systems in the absence of quasiparticles. The QBE can be generally written in the form

$$(\partial_t + \mathbf{v}\cdot\nabla) f(t,\mathbf{x},\mathbf{p}) = -\mathcal{L}[f] \tag{1.32}$$

where $\mathcal{L}$ is the linearized collision integral, which clearly encodes all the relevant information about the physics.

The starting point of our results is that, while the QBE is usually extracted from two-point functions [125–127], it is possible to obtain the linearized QBE in a clean way from the 4-point function (1.29). As we explain with (hopefully sufficient) detail in chapter 2, the conceptual advantage is the immediate matching of the QFT expressions with their kinetic theory counterparts. In this approach, the operator $(\partial_t + \mathbf{v}\cdot\nabla) + \mathcal{L}$, which defines the QBE, is fully encoded in the analytical structure of this 4-point Greens function[9]. Indeed such correlation function $f(\omega, p, q)$, in the spatially homogeneous case, satisfies the following Bethe-Salpeter equation (BSE)

$$-i\omega\, f(\omega, p) = \delta(p_0^2 - E_{\mathbf{p}}^2)\left(1 + \int_l \hat{R}^{\mathrm{transp}}(p,l) f(\omega, l)\right). \tag{1.33}$$

where $f(\omega, p) = \int_q f(\omega, p, q)$. Once on-shell, the kernel $\hat{R}^{\mathrm{transp}}$ reproduces exactly the collision operator $\hat{C}$, determining the matching of the QFT result with the Boltzmann equation. Moreover, all the information about the relaxation times, eventual branch cuts and hydrodynamic modes are intrinsically hidden in $\hat{R}$, as the correlation function is formally obtained by inverting (1.33)

$$f(\omega, p) = \frac{\mathcal{I}_p}{-i\omega - \int_l \hat{R}^{\mathrm{transp}}(p,l)}. \tag{1.34}$$

Focusing on the OTOC, $f'(t, \mathbf{x})$, at weak coupling the computation requires a resummation of *ladder* diagrams and it is also performed by a Bethe-Salpeter equation (BSE). This BSE can be recast into the following integro-differential equation

$$-i\omega\, f'(\omega, p) = \delta(p_0^2 - E_{\mathbf{p}}^2)\left(1 + \int_l \hat{R}^{\mathrm{OTOC}}(p,l) C(\omega, l)\right). \tag{1.35}$$

$f'(\omega, p)$ being the Fourier transform in p and Laplace transform in ω of the OTOC. Clearly, since in the late time regime ($\omega \to 0$ limit) the OTOC has the form $f'(t) \propto e^{\lambda_{\mathrm{L}} t}$, the spectrum of the integral operator $\hat{R}'(p,l)$ contains all the informations about ergodicity, *i.e.* the Lyapunov spectrum is given by its positive eigenvalues.

In order to understand the scrambling dynamics, we can dissect the OTOC-BSE. The way we have phrased it highlights it on purpose, but the surprising result is that the OTOC-BSE is also a kinetic equation of the form

$$(\partial_t + \mathbf{v}\cdot\nabla) f'(t, \mathbf{x}, \mathbf{p}) = -\mathcal{L}'[f'] \tag{1.36}$$

[9] This can be considered as the analogue of what happens in hydrodynamics. There, as reviewed in section 1.6, the universality of the equation of motions is translated in the universal form of the pole structure.

where the collision integral is nearly the same as (1.32). Instead of measuring the net number of collisions, as the standard QBE, it counts the gross number of collisions. This is to say that the terms appearing in $\mathcal{L}$ and $\mathcal{L}'$ are the same, with one of the signs flipped. We showed this results for ϕ^4 matrix model (chapter 2), (bosonic) $O(N)$ vector model in $(2+1)$ dimensions and the Gross-Neveu model in $(2+1)$ dimensions (chapter 3) . Moreover, we also show that the linearised kinetic operator, $\hat{R}^{\text{transp}}(p, l)$, is analytically related to the kernel of the BSE of the OTOC, $\hat{R}^{\text{OTOC}}(p, l)$, as follows

$$\hat{R}^{\text{OTOC}}(p, l) = \sinh(\beta p_0/2)^{-1}\, \hat{R}^{\text{transp}}(p, l) \sinh(\beta l_0/2). \tag{1.37}$$

It is tempting to read the previous expression as a new fluctuation dissipation relation:

$$C(\omega, p) = \sinh(\beta p_0/2) f(\omega, p), \tag{1.38}$$

which would suggest that the OTOC can be obtained as analytical continuation of the hydrodynamic correlation function governing transport. However, in this thesis we will not succumb to this temptation and we will postpone this question to future work. Another reason why we wrote (1.37) in this form is that it makes clear that the relation is nothing but a similarity transformation, which preserves the spectrum and other properties. Nevertheless we warn the incautious reader that, both in (1.35) and (1.37), there are delta functions which project out some eigenvalues. Indeed we know that unitarity forbids eigenvalues with positive imaginary part for the collision integral of the (standard) Boltzmann equation.

We now comment about the consequences of (1.37). Since, as we have argued before, the full physics of scrambling is encoded in $\hat{R}^{\text{OTOC}}(p, l)$ and most of the hydrodynamical transport physics is encoded in $\hat{R}^{\text{transp}}(p, l)$, this relation has a profound meaning and should be considered as a starting point for any further attempt to find imprints of ergodicity in the hydrodynamic spectrum. Moreover, we stress that (1.37) not only holds for the ϕ^4 model (chapter 2), but also for models which describe the physics above a QCP (chapter 3), where there are no quasiparticle excitations.[10] Therefore this seems to be a quite general result for weakly coupled systems, or large N QFTs. We have repeatedly hinted the suggestive similarities between transport computations and scrambling computations. At infinite coupling and for theories with gravitational dual, the idea that quantum chaos constrains transport is indeed true and it is realized in

[10] For fermions, we prove in chapter 3 that (1.37) is modified by replacing $\sinh \to \cosh$.

the phenomenon discovered in chapter 4 of this thesis (based on [128]), and later termed *pole-skipping* [56]. In section 1.3, we sketched the classification of the hydrodynamical modes and we showed that the energy-energy correlation function contains sound modes, whose perturbative dispersion relation is given by (1.25). Let us define the following point in the analytically continued (ω, k) plane

$$k_c = i\frac{\lambda_L}{v_B}, \qquad \omega_c = \omega(k_c) = i\lambda_L, \tag{1.39}$$

where $\lambda_L = 2\pi T$ and v_B is the butterfly velocity, defined in the previous pages. Pole-skipping is the statement that the fully resummed hydrodynamic series (1.25), after the analytic continuation $\omega \to i\omega$ and $k \to ik$, passes through the point (ω_c, k_c). Since the dispersion relation (1.25) parametrizes the pole structure of the energy-energy two-point function, this implies that this correlator has the form

$$G_R^{EE}(\omega, k) = \frac{b(\omega, k)}{a(\omega, k)} \tag{1.40}$$

and $a(\omega_c, k_c) = 0$. Moreover, (2) at the point (ω_c, k_c) also the residue vanishes,

$$b(\omega_c, k_c) = 0. \tag{1.41}$$

This implicates that the pole in the correlator $G_R^{EE}(\omega, k)$ disappears. This phenomenon has been later understood in terms of an effective field theory [56] as the fact that both ergodicity and hydrodynamics are governed by the same mode, which survives at late time. The robustness of such modes is due to the exsistence of a new symmetry (a shift symmetry) of the effective action. Whether this is a feature of systems saturating the bound on chaos is still an open problem. Furthermore, we now know that mathematically pole-skipping, as described in holographic systems, is a manifestation of a degeneracy of Einstein equation [129]. Recently, it was also proven that this phenomenon is present not only for energy density correlators [130] but persists also away from infinite coupling [131]. New constraints on thermal correlators (in holographic systems) coming from pole-skipping (although not immediately relatable to chaos) have been recently discovered in [132].

What happens going towards the weak-coupling limit, and so departing from the bound, it is still unknown.

1.10 This Thesis

1.10.1 Chapter 2

We first write a Boltzmann-like equation for the gross energy exchange in a bosonic system. We do this by modifying the collision integral in order to take into account the gross number of interactions between particles instead of the net number of interaction, which corresponds to the standard Boltzmann equation. After that, we show how this kinetic equation matches the BSE satisfied by the OTOC. We then review how to compute the out-of-time ordered correlation function in quantum field theory. This can be done by using the Schwinger-Keldysh (SK) formalism over the closed time path (CTP) contour . After analyzing the properties of this modified SK contour, we show that the computation of the OTOC is remarkably simplified. We explicitly use this framework to compute the OTOC in a $N \times N$ matrix ϕ^4 bosonic theory, without taking any large N limit.

1.10.2 Chapter 3

In this chapter, we focus our attention to the interplay between chaos and hydrodynamics in systems close to the quantum critical point (QCP). In the quantum critical regime, there is no quasiparticle interpretation, but it was shown that many transport properties can be inferred by analytically extending the result in the symmetric phase to the quantum critical phase. This chapter is mainly based on [133]. To understand these questions, we make an extensive use of some results regarding transport obtained in the imaginary time formalism (ITF) and we use the two-particle-irreducible effective action. We first analyze the bosonic $O(N)$ vector model, and we study the 4-point correlation function which is relevant for transport in the hydrodynamic limit. This computation generalizes at small external frequency some previous results by Aarts et Martinez [134, 135] which were obtained at strict zero frequency. This allows us to compare the equations governing transport with the BSE for the OTOC, which was obtained by Chowdhury and Swingle in [136]. Again, the mapping between the two results holds and confirms our conjectured connection between chaos and hydrodynamics. By going into the kinetic theory limit, we observe that indeed the OTOC is nothing but a gross energy exchange kinetic equation. In the second part of this chapter, we address a similar analysys to a system in a fermionic quantum critical regime, namely the $2+1$ the Gross-Neveu model. Here we first use the 2PI formalism to compute the 4-point function relevant to transport and then we compare it with the OTOC computation performed in [137]. This case, in a even more clear way, shows how transport and chaos are described by the same BSE equation but correspond to two different boundary conditions. Such difference, in turn, determines the sign flip of the kinetic equation for chaos with respect to the QBE. We explicitly show how the linearized Boltzmann equations emerge from a QFT computation of a time ordered 4-point function. By inspecting these results, it is straightforward to see that indeed the kinetic theory for gross particle exchange reproduces the BSE for the out-of-time correlation function.

1.10.3 Chapter 4

In this chapter we investigate the connection between quantum chaos and hydrodynamics in theories with a holographic dual. These theories are strongly coupled large N theories, and the out-of-time order correlation function can be computed by means of the AdS/CFT correspondence. This result was obtained some years ago by Shenker and Stanford in [12], who showed that the Lyapunov exponent can be extracted by studying a shockwave geometry in the bulk. We show that this shockwave computation can be understood in terms of a sound-wave excitation of the bulk geometry. Since these excitations have a physical interpretation as hydrodynamical energy-energy correlation functions of the dual boundary theory, we prove that for holographic theories the Lyapunov exponent and the butterfly velocity characterize the response to an highly out-of-equilibrium perturbation. We moreover identify the imprinting of quantum chaos in the analytical properties of the energy-energy correlation function which is now known as *pole-skipping*.

1.10.4 Chapter 5

In this last chapter we address an important question regarding the out-of-time correlation function. We try to understand whether this correlation function is independent of the way it gets regularised on the thermal circle. This should have been one of the first questions to ask because of the high interest of the community in this *observable*. Over the last years, several authors have computed this correlation function using a particular regularization, claiming a regularization independence of the result. Drawing a comparison, it would be like computing a correlation function in a gauge theory without verifying it is gauge independent. Based on the results obtained in [138], we show both in weak and strongly coupled theories that the OTOC strongly depend on the regularization. For the weak-coupling analysis we use the ϕ^4 bosonic theory, while for the strong coupling limit we focus on the SYK model. We explain why this correlation function depends on the contour and we explain why most of the investigations by other authors have overlooked this. This result gives rise to the question of which of the correlation functions is physically sensible. By using the kinetic theory results of the previous chapters, we show that the only meaningful OTOC is the symmetrical one (separated by half the thermal circle), which corresponds to a gross energy exchange. All the other OTOCs correspond to kinetic equations which give too much weight to either the *gain* or the *loss* terms in the collision integral. This results indeed shows that the only meaningful OTOC (in QFT) is the one for which a bound on the Lyapunov exponent has been proven.