Exploration on and of networks

Güldas, H.

Author: Güldas, H.
Title: Exploration on and of networks
Issue Date: 2019-07-03

CHAPTER 5

Union complexity of random disk regions

This chapter is based on joint work with Mark de Berg.

Abstract

We study the union complexity of a set of n disks when disk centers are sampled uniformly and independently at random in a convex compact region S. We consider the case where all the disks have a common radius $R = \mathsf{diam}(S)$ and prove that if S is a square or a disk, then the expected union complexity is $\Theta(n^{1/3})$. Our proofs are based on the arguments used by Har-Peled [55] for the expected complexity of convex hulls of random points. We also show a connection between the union complexity of disk regions and the complexity of convex hull of a set of points.

§5.1 Introduction and main results

The introduction to this chapter was given in Section 1.2. Nevertheless, we repeat the setting and the definitions for ease of reading. Let S be a fixed convex compact region in $\mathbb{R}^2$, and $X = \{X_1, \ldots, X_n\}$ be a set of n points sampled independently and uniformly at random from S. Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a collection of n disks, where D_i is the closed disk centered at X_i with a fixed radius R such that $\mathsf{diam}(S) \leq R < \infty$, where $\mathsf{diam}(S)$ is the diameter of S, for $i = 1, \ldots, n$. By choosing the radius large enough such that any disk covers S completely, we make sure that the boundary of the disks always lie outside of S and this makes the analysis easier. The set of boundary disks of $\mathcal{D}$, denoted by $\mathsf{BD}(\mathcal{D})$, is the set of disks in $\mathcal{D}$ whose boundaries are not completely covered by other disks, i.e.,

$$\mathsf{BD}(\mathcal{D}) = \{D \in \mathcal{D} : \partial D \setminus \cup_{D' \in \mathcal{D} \setminus \{D\}} D' \neq \emptyset\},$$

where ∂D denotes the boundary of D. We are interested in union complexity of $\mathcal{D}$ which is the number of boundary arcs of $\mathcal{D}$. This number is linear in the number of disks in $\mathsf{BD}(\mathcal{D})$. Let B_n denote the number of boundary disks of $\mathcal{D}$ when $\mathcal{D}$ contains n disks. B_n is a random variable, since disk centers are random, and we are interested in the expected value of B_n as a function of n. We consider two cases: the case where S is a unit square and all the disks have radius $R = \sqrt{2}$, and the case where S is a unit disk and all the disks have radius $R = 2$.

In what follows, we use the notation for asymptotic comparison of functions $f, g : \mathbb{N} \to [0, \infty)$: $f(n) = O(g(n))$ or $g(n) = \Omega(f(n))$ when $\limsup_{n\to\infty} f(n)/g(n) < \infty$; $f(n) = o(g(n))$ or $g(n) = \omega(f(n))$ when $\lim_{n\to\infty} f(n)/g(n) = 0$; $f(n) = \Theta(g(n))$ when both $f(n) = O(g(n))$ and $g(n) = O(f(n))$. We denote by $d(x, y)$ the Euclidean distance between $x, y \in \mathbb{R}^2$, and with a slight abuse of notation we write $d(x, A) = \inf\{d(x, y) : y \in A\}$ for $x \in \mathbb{R}^2$ and $A \subset \mathbb{R}^2$. Our main result is given in the following theorem.

Theorem 5.1.1. *Suppose that*

(a) *either S is the unit square $[0, 1] \times [0, 1] \in \mathbb{R}^2$ and each disk has radius $R = \sqrt{2}$;*

(b) *or S is the unit disk $\{x \in \mathbb{R}^2 : d(x, o) \leq 1\}$, where o is the origin, and each disk has radius $R = 2$.*

Then

$$\mathbb{E}(B_n) = \Theta(n^{1/3}).$$

For the case of the unit square, the problem appears in the context of conflict-free colouring as discussed in Section 1.2.1. We present the unit-disk case as a generalisation. The union-complexity problem is related to the problem of the complexity of the convex hull, as we pointed out in Section 1.2.2. In fact, our proof follows some ideas developed for tackling convex-hull problems [39, 55]. In Section 5.2 we give the proof of Theorem 5.1.1 for the case of a unit square and in Section 5.3 for the case of a unit disk. In Section 5.4, we discuss several extensions of the boundary complexity problem.

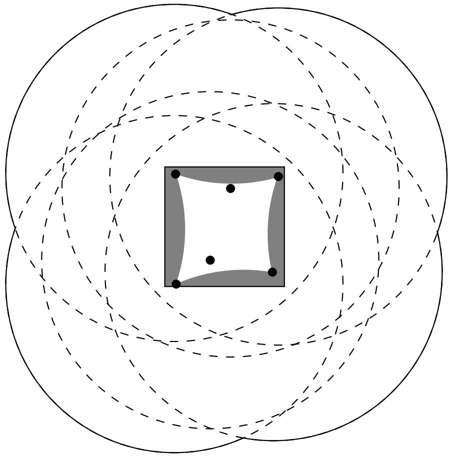


Figure 5.1: The halo for a set of 6 disks with centers inside the unit square is shown as the shaded region in the square.

Before proceeding with the proof of the Theorem 5.1.1, we introduce some further notation and state a lemma that will be crucial. Consider the general setting: S is a convex compact region and the disks have radius $\mathsf{diam}(S) \leq R < \infty$. Let $\mathsf{Cov}(\mathcal{D}) := \cup_{D \in \mathcal{D}} D$ denote the coverage area of $\mathcal{D}$, i.e., the subset of $\mathbb{R}^2$ covered by the disks in $\mathcal{D}$. Let $\mathsf{Halo}(\mathcal{D}) = \{x \in S : d(x, \partial(\mathsf{Cov}(\mathcal{D})) \leq R\}$ be the set of points in S whose distance to the boundary of the coverage area is less than R (see Figure 5.1), and let $A_n = \mathbb{E}(\mathsf{Area}(\mathsf{Halo}(\mathcal{D})))$ be the expected area of the halo. To compute the expected number of boundary disks, we use the area of the halo. The two are related through the following lemma, which is analogous to Efron's Theorem for the convex hull [39].

Lemma 5.1.2. *Suppose that S is a convex compact region in $\mathbb{R}^2$ with unit area, and let $\mathcal{D}$ be a collection of n disks with a fixed radius R such that any single disk covers S completely and such that the centers are sampled uniformly and independently from S. Then $\mathbb{E}(B_n) = nA_{n-1}$, where A_n is the expected area of the halo of a set of n points sampled uniformly and independently at random from S.*

Proof. First we note that, for any $i = 1, \dots, n$, the disk D_i is a boundary disk if and only if its center falls inside the halo of $\overline{\mathcal{D}}_i$, where $\overline{\mathcal{D}}_i := \mathcal{D} \setminus \{D_i\}$. This gives

$$B_n = \sum_{i=1}^{n} \mathbb{1}_{\{D_i \in \mathsf{BD}(\mathcal{D})\}} = \sum_{i=1}^{n} \mathbb{1}_{\{X_i \in \mathsf{Halo}(\overline{\mathcal{D}}_i)\}},$$

so

$$\begin{aligned}\mathbb{E}(B_n) &= \sum_{i=1}^{n} \mathbb{P}(X_i \in \mathsf{Halo}(\overline{\mathcal{D}}_i)) \\ &= \sum_{i=1}^{n} \int_{S^{n-1}} \mathbb{P}(X_i \in \mathsf{Halo}(\overline{\mathcal{D}}_i) \mid X_j = x_j, j \in [n] \setminus \{i\}) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n.\end{aligned}$$

Note that the conditional probability $\mathbb{P}(X_i \in \mathsf{Halo}(\overline{\mathcal{D}}_i) \mid X_j = x_j, j \in [n] \setminus \{i\})$ is equal to $\mathsf{Area}(\mathsf{Halo}(\overline{\mathcal{D}}_i))$, and so by symmetry we have

$$\begin{aligned}\mathbb{E}(B_n) &= \sum_{i=1}^{n} \int_{S^{n-1}} \mathsf{Area}(\mathsf{Halo}(\overline{\mathcal{D}}_i)) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n \\ &= \sum_{i=1}^{n} \mathbb{E}(\mathsf{Area}(\mathsf{Halo}(\overline{\mathcal{D}}_i))) = nA_{n-1}.\end{aligned}$$

□

§5.2 Case of the unit square

In this section we give the proof of Theorem 5.1.1 for the case of the unit square. Thanks to Lemma 5.1.2, in order to compute the expected boundary complexity we only need to compute the expected area of the halo A_n. The next proposition gives an upper bound. The proof follows the arguments in [55] for the convex hull of a point set sampled in the unit square.

Proposition 5.2.1. *Suppose that S is the unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$. Then $A_n = O(n^{-2/3})$.*

Proof. We divide the unit square into n rows and n columns each of width $1/n$, which gives n^2 small squares of size $1/n \times 1/n$. We derive an upper bound for the number of squares that intersect the halo and multiply this by n^{-2} to get an upper bound for the area.

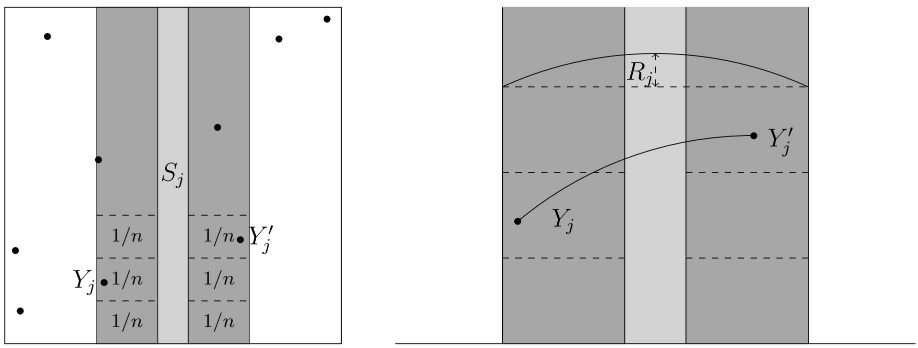
Let $S_{i,j} = [(i-1)/n, i/n] \times [(j-1)/n, j/n]$ be the jth square of the ith column, $C_i = \cup_{j=1}^{n} S_{i,j}$ be the ith column, and $C(k,l) = \cup_{i=k}^{l} C_i$. Let $X = \{X_1, \ldots, X_n\}$ be the random set of disk centres. Let $m = \lfloor n^{2/3} \rfloor$ and for $j = m+1, \ldots, n-m$ define $Y_j \coloneqq \min\{k \in [n] : X \cap (\cup_{i=j-m}^{j-1} S_{i,k}) \neq \emptyset\}$, i.e., Y_j is the index of the lowest row that contains a point from X in $C(j-m, j-1)$. Define Y'_j analogously for $C(j+1, j+m)$ (see Figure 5.2a).

The squares at the bottom of the jth column that intersect the halo stay below or intersect the circle arc with radius $\sqrt{2}$ that passes through the lowest disk centers in $C(j-m, j-1)$ and $C(j+1, j+m)$. Furthermore, this arc stays below the arc that passes through the upper-left corner of the square $S_{j-m,\max\{Y_j, Y'_j\}}$ and the upper-right corner of the square $S_{j+m,\max\{Y_j,Y'_j\}}$. The latter arc has cord length $(2n^{2/3}+1)/n = 2n^{-1/3} + n^{-1}$, so the distance between the highest point of the arc and the cord is $\sqrt{2} - \sqrt{2 - \left(2n^{-1/3} + n^{-1}\right)^2} = O(n^{-2/3})$ as n tends to ∞. Let R_j denote the number of squares that stay between the chord and the highest point of the arc. Then R_j is of order $O(n^{1/3})$ (see Figure 5.2b).

Clearly, the number of small squares in C_j that intersects the halo is less than $\max\{Y_j, Y'_j\} + R_j < Y_j + Y'_j + R_j$. Next we compute the $\mathbb{E}(Y_j)$ and $\mathbb{E}(Y'_j)$. For Y_j, we divide the area $C(j-m, j-1)$ into rectangles of area $1/n$, so that each rectangle is m squares wide and n/m squares high. Let Z_j be the index of the lowest rectangle that contains a point from X. Then $Y_j \leq n^{1/3} Z_j$. Now, observe that $\mathbb{P}(Z_j \geq k) \leq (1-(k-1)/n)^n \leq e^{-(k-1)}$. Hence

$$\mathbb{E}(Z_j) \leq \sum_{k=1}^{n^{2/3}} k\mathbb{P}(Z_j = k) \leq \sum_{k=1}^{\infty} \mathbb{P}(Z_j \geq k) \leq \sum_{k=1}^{\infty} ke^{-(k-1)} = O(1). \tag{5.1}$$

From this we get $\mathbb{E}(Y_j) = O(n^{1/3})$ and similarly $\mathbb{E}(Y_j') = O(n^{1/3})$. Summing over $j = m, \ldots, n-m$, we see that the expected number of small squares that fall into the halo at the bottom of the square between the columns $m+1$ and $n-m$ is $O(n^{4/3})$. Doing the same for the upper, left and right sides, we get a total number of $O(n^{4/3})$ small squares contributing to the halo. We have not accounted for the four squares with side length $m = n^{2/3}$ at the corners, but these contain a total number of $O(n^{2/3})$ small squares. So in total the halo has $O(n^{4/3})$ small squares. Since each small square has area n^{-2}, we get $A_n = O(n^{-2/3})$. □



(a) (b)

Figure 5.2: Illustration of proof of Proposition 5.2.1

Using similar arguments, we next prove that $n^{2/3}$ is the correct order for the expected area of the halo.

Proposition 5.2.2. *Suppose that S is the unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$. Then $A_n = \Omega(n^{-2/3})$.*

Proof. As in the proof of Proposition 5.2.1, consider the window of width $2m + 1 = 2\lfloor n^{2/3} \rfloor + 1$ around the jth column. Consider the arc whose endpoints are $((j-m-1)/n, 0)$ and $((j+m)/n, 0)$ and whose center lies below the unit square. The cord length of this arc is $(2m+1)/n$, so y-coordinate of the highest point of this arc is $\sqrt{2} - \sqrt{2 - ((2m+1)/n)^2} = \Omega(n^{-2/3})$ and hence the latter point lies in a row with index $\Omega(n^{1/3})$. The expected number of small squares on jth column that stay in the halo is bounded from below by the minimum of the row index of the highest point of the latter arc and $\mathbb{E}(\min\{Y_j, Y_j'\}) - 1$. Dividing $C(j-m, j-1)$ and $C(j+1, j+m)$ into rectangles of area $1/n$, and defining Z_j and Z_j' as in the proof of Proposition 5.2.1, we see that $Y_j - 1 \geq n^{1/3}(Z_j - 1)$ and $Y_j' - 1 \geq n^{1/3}(Z_j' - 1)$,

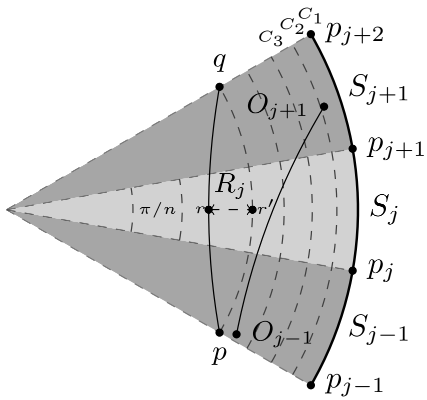


Figure 5.3: Illustration of proof of Proposition 5.3.1.

so $\mathbb{E}(\min\{Y_j, Y_j'\}) - 1 \geq n^{1/3}\mathbb{E}(\min\{Z_j, Z_j'\} - 1)$. Note that $\mathbb{E}(\min\{Z_j, Z_j'\} - 1) \geq \mathbb{P}(Z_j > 1, Z_j' > 1) = (1 - 2/n)^n \geq e^{-3}$ for large enough n. From this we conclude that $\mathbb{E}(\min\{Y_j, Y_j'\}) - 1 = \Omega(n^{1/3})$, so the expected number of squares on jth column that stay in the halo is $\Omega(n^{1/3})$, which gives us the desired result. □

Combining the last two propositions with Lemma 5.1.2, we obtain the result of Theorem 5.1.1 for the unit square.

§5.3 Case of the unit disk

In this section we give the proof of Theorem 5.1.1 for the case of the unit disk. As in the case of the unit square, we obtain upper and lower bounds for the expected area of the halo A_n, then we combine these bounds with Lemma 5.1.2 to obtain the result of Theorem 5.1.1. The next proposition gives an upper bound. Again, the proof follows the arguments in [55] for the convex hull.

Proposition 5.3.1. *Suppose that S is the unit disk $\{x \in \mathbb{R}^2 : d(x, o) \leq 1\} \subset \mathbb{R}^2$. Then $A_n = O(n^{-2/3})$.*

Proof. Assuming without loss of generality that $n = m^3$ for some $m \in \mathbb{N}$, we divide the unit disk S into n tiles of equal area as follows: divide S into m slices, $S_1, \dots, S_m$, by drawing m lines from the center to m equally spaced points $p_1, \dots, p_m$ on ∂S. Then divide each slice into m^2 tiles of equal area as follows: consider m^2 concentric rings given by m^2 concentric circles $C_1 = \partial S, C_2, \dots, C_{m^2}$, with radii $r_1 = 1, r_2, \dots, r_{m^2}$ respectively, such that the intersection of each slice and ring gives a tile of area π/n (see Figure 5.3). Let $S_{i,j}$ be the ith outermost tile in S_j for $i = 1, \dots, m^2$ and $j = 1, \dots, m$, i.e., $S_{i,j}$ is the intersection of S_j and the ring between the circles C_i and C_{i+1}. We compute the expected number of tiles that intersects the halo and multiply the result by $1/n$ to get an upper bound for the expected area of the halo. We do this by computing the expected number of tiles that intersect the halo for each slice.

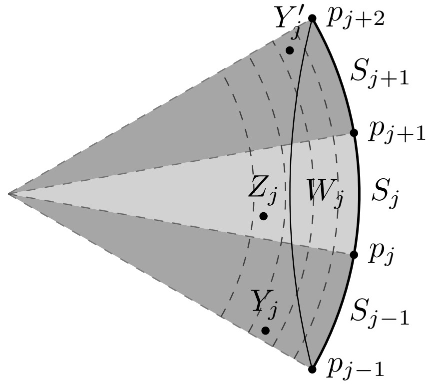


Figure 5.4: Illustration of proof of Proposition 5.3.2.

Let Y_j be the index of the outermost tile of the slice S_{j-1} that contains a disk center, i.e., $Y_j = \min\{i : S_{i,j-1} \cap X \neq \emptyset\}$, and analogously define Y'_j for the slice S_{j+1}. Let O_{j-1} and O_{j+1} be the outermost disk centers, that are the disk centers furthest away from the origin, in S_{j-1} and S_{j+1} respectively. Consider the arc with radius 2 that passes through O_{j-1} and O_{j+1} whose center lies away from the origin relative to the line passing through O_{j-1} and O_{j+1}. The tiles of S_j that intersect the halo stay outside this arc. Furthermore, the latter arc stays outside the arc a with radius 2 that passes through points p and q and whose center lies away from the origin relative to the line passing through p and q, where p and q are the extreme points of the arc $a' = C_{Z_j+1} \cap (S_{j-1} \cup S_j \cup S_{j+1})$ and $Z_j = \max\{Y_j, Y'_j\}$. Let r and r' be the midpoints of the arcs a and a', respectively, and let R_j be the number of tiles between r and r' (see Figure 5.3). Then the number of tiles in S_j that intersect the halo is bounded from above by $R_j + Z_j \leq R_j + Y_j + Y'_j$.

The length of the line segment connecting r and r' is

$$d(o,p) - d(o,p)\sin\left(\frac{3\pi}{m}\right) + 2\left(1 - \cos\left(\arcsin\left(\frac{|op|}{2}\sin\left(\frac{3\pi}{m}\right)\right)\right)\right) = O(m^{-2}),$$

On the other hand, the radial length of every tile is greater than or equal to $r_1 - r_2 \geq 1/(2m^2)$, so we have $R_j = O(1)$. To compute $\mathbb{E}(Y_j)$, we note that $\mathbb{P}(Y_j \geq k) = (1-(k-1)/n)^n \leq \exp(-(k-1))$. This gives

$$\mathbb{E}(Y_j) = \sum_{k=1}^{\infty} \mathbb{P}(Y_j \geq k) \leq \sum_{k=1}^{\infty} e^{-(k-1)} = O(1). \tag{5.2}$$

Thus, the expected number of tiles in S_j that intersect the halo is $O(1)$ and the expected total number of tiles that intersect the halo is $O(m)$, which gives $A_n = O(m^{-2}) = O(n^{-2/3})$. □

The next proposition gives a lower bound for the expected area of the halo A_n:

Proposition 5.3.2. *Suppose S is the unit disk $\{x \in \mathbb{R}^2 : d(x,o) \leq 1\} \subset \mathbb{R}^2$. Then $A_n = \Omega(n^{-2/3})$.*

Proof. As in the proof of Proposition 5.3.1, we divide the disk into tiles and obtain a lower bound for the number of tiles of S_j that stay in the halo for $j = 1, \dots, m$. Let Y_j and Y_j' be as defined in the proof of Proposition 5.3.1, Z_j be the index of the outermost tile of the slice S_j, and W_j be the index of the innermost tile of S_j that does not intersect the arc with radius 2 that passes through the points p_{j-1} and p_{j+2} (see Figure 5.4). Then a lower bound for the number of tiles of S_j that stay in the halo is $\min\{Y_j - 1, Y_j' - 1, Z_j - 1, W_j\}$. We note that W_j is not random, and a calculation similar to that of R_j in the proof of Proposition 5.3.1 gives that $W_j > 0$. We also note that $\mathbb{E}(\min\{Y_j, Y_j', Z_j\} - 1) \geq \mathbb{P}(Y_j > 1, Y_j' > 1, Z_j > 1) = (1 - 3/n)^n \geq e^{-4} = \Omega(1)$ for large enough n. So the expected number of tiles of S_j that stay in the halo is $\Omega(1)$. Taking the sum over $j = 1, \dots, m = n^{1/3}$, we see that the expected number of tiles that intersect the halo is $\Omega(n^{1/3})$ and multiplying by the area of each tile, which is $1/n$, we obtain the desired result.

□

§5.4 Discussion

In this section, we briefly discuss several extensions of the boundary complexity problem. One possible extension is where the radius of the random disks depends on the number of disks n. For instance the random disks have radius r_n with $\lim_{n\to\infty} r_n = 0$. In this case, the expected number of boundary disks is a function of n and r_n, and its behaviour depends on how fast r_n tends to 0. For example, when $r_n = O(n^{-2})$, the number of isolated disks, i.e., the disks that have no intersection with any other disk, is of order n, which tells us that the number of boundary disks is of order n as well. When $r_n = \Omega(n^{-2})$, however, the problem becomes more complicated and we will address it in future work. Another interesting regime is the case where $\lim_{n\to\infty} r_n = \infty$. For example, in this case Proposition 1.2.3 suggests that if r_n tends to infinity fast enough, then the boundary complexity is the same as the complexity of the convex hull. Also this will be the subject of future work.

Another possible extension is to replace S by an arbitrary convex polygon or convex compact region with smooth boundary. By following the proofs in [55] for the convex hull, the proofs for the unit square and the unit disk can be adapted to arbitrary polygons and regions with smooth boundary, to show that the order of the number of boundary disks is again $n^{1/3}$. The leading order coefficient can be different for different shapes and its computation requires a more detailed analysis.