Exploration on and of networks

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CHAPTER 4

Mixing times of random walks with random rewirings

Abstract

We consider a random walk without backtracking on a general class of dynamic random graphs with n vertices, where the vertices and their degrees are fixed but the edges are rewired according to a prescribed rule. In previous works [12, 13], we considered the special case in which, at each unit of time, a certain fraction of the edges, chosen uniformly and independently of the random walk, are rewired uniformly. We showed that there are three different regimes, depending on how the fraction of edges to be rewired decays as a function of the number of vertices. In this paper, we show, for a general class of rewiring rules, how the mixing time of the random walk on the dynamically rewired random graph is linked to the mixing time of the random walk on static random graphs, drawn according to the configuration model. Furthermore, we give an explicit example, called *local rewiring*, in which the edges are rewired only along the random walk path, and using the above link, we show that, for this model, we have the same trichotomy as in [13] but on a different time scale. In our proof, we use a coupling argument where the random walk on the dynamically rewired random graph is coupled to a modified version of the random walk on the static random graph.

§4.1 Introduction

We consider a random walk on a dynamic random graph in which the vertices are fixed but the edges are randomly rewired at each unit of time according to a prescribed rule. By *rewiring* we mean an operation on the graph that changes the edges while keeping the degrees of the vertices fixed. This type of graph dynamics was considered in the context of uniform sampling of simple graphs with given degree sequences [34, 46, 54, 53, 69]. The main purpose of these works is to construct Markov chains on the set of simple graphs with a given degree sequence whose stationary distribution is uniform on this set. If the convergence to the stationary distribution of the Markov chain is sufficiently fast, i.e. *the mixing time* is sufficiently small, then it is possible to obtain approximately uniform samples in an efficient way, simply by simulating the Markov chain.

In [34, 54, 53], the authors consider a so-called *switch* chain in which, at each time unit, two edges (i, j) and (k, l), are chosen uniformly at random and their end-points are switched to obtain the edges (i, k) and (j, l), provided that the resulting graph is simple. In [34], the authors consider the switch chain in the context of simple regular graphs and show that the mixing time is polynomial in the size of the graph. Their results were later extended to the case of simple graphs with irregular degree sequences [54] and to directed graphs [53]. In [46, 69], the authors consider a so-called *flip* chain, which is a modified version of the switch chain in which a switch is performed if the two randomly chosen edges have a common neighbor, i.e., if (i, l) is an edge. In [46], the authors consider the flip chain in the context of simple regular graphs and they show that the mixing time is polynomial in the size of the graph by comparing the flip chain to a switch chain and using the results of [34].

In the present paper, we are interested in the behaviour of a random walk on a dynamically rewired random graph, rather than in the behaviour of the random graph dynamics itself. Namely, we study the mixing times of random walks on dynamically rewired random graphs, where the initial graph is drawn according to the configuration model. Our results are in the same spirit as those in [12, 13], in which random walks on a dynamic version of the configuration model with a specific rewiring mechanism were considered. In fact, we extend the results of [12, 13] to a more general class of dynamically rewired versions of the configuration model, which includes the dynamic configuration model of [12] as a special case.

The mixing times of random walks on static random graphs has been studied in the last few decades for a wide range of random graph models. For an overview, we refer to [12, 13] and references therein. In contrast, there are relatively few studies on the mixing times of random walks on dynamic random graphs. This line of research was started recently in [83], which considers random walks on dynamical percolation on $\mathbb{Z}^d$ in the subcritical regime. In [82], the results in [83] were extended to the supercritical regime. In [89], the authors consider random walks on a dynamic version of Erdős-Rényi random graph model and show that the joint chain of the random walk and the dynamic random graph exhibits cut-off phenomenon. Since there are two layers of randomness, the random walk and the graph dynamics, in all these works, several distinct notions of mixing times are considered, such as the annealed case vs. the quenched case and the mixing time of the random walk vs. the mixing time of the joint chain. In our work, we consider the mixing time of the random walk component annealed over the graph dynamics, which will be made clear in the sequel.

The remainder of this paper is organised as follows. In Section 4.1.1, we introduce the model and set the notation. In Section 4.1.2, we state our main theorem (Theorem 4.1.5). Section 4.2 is devoted to the introduction of some core ingredients. In Section 4.3, we give the proof of the main theorem. In Section 4.4, we introduce a specific model within the framework of random walks on dynamically rewired random graph models and show that it exhibits the same trichotomy found in [13] but on a different time scale. In Section 4.5, we put our work in the proper context by discussing several issues in more detail and suggesting possible extensions.

Throughout the sequel we use standard notations for the asymptotic comparison of functions $f, g\colon \mathbb{N} \to [0,\infty)$: $f(n) = O(g(n))$ or $g(n) = \Omega(f(n))$ when $\limsup_{n\to\infty} f(n)/g(n) < \infty$; $f(n) = o(g(n))$ or $g(n) = \omega(f(n))$ when $\lim_{n\to\infty} f(n)/g(n) = 0$; $f(n) = \Theta(g(n))$ when both $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

§4.1.1 Model

We denote by V the set of vertices of the graph and by $\deg(v)$ the degree of vertex $v \in V$. To each vertex $v \in V$ we associate $\deg(v)$ half-edges and by H we denote the set of all half-edges, i.e.,

$$H = \{(v,i) : v \in V \text{ and } 1 \le i \le \deg(v)\}.$$

If a half-edge $x \in H$ is associated to a vertex $v \in V$, then we say that x *is incident to* v. We denote by $v(x) \in V$ the vertex to which $x \in H$ is incident and by $H(v) \coloneqq \{x \in H\colon v(x) = v\} \subset H$ the set of half-edges incident to $v \in V$. If $x, y \in H(v)$ with $x \neq y$, then we write $x \sim y$ and say that x and y are siblings of each other. The degree of a half-edge $x \in H$ is defined as

$$\deg(x) \coloneqq \deg(v(x)) - 1. \tag{4.1}$$

We consider graphs on n vertices, so that $|V| = n$, with m edges, so that $|H| = \sum_{v\in V} \deg(v) = 2m =: \ell$.

We view the set of edges as a pairing of half-edges. A pairing of half-edges ξ, called a *configuration*, is a bijection of H to itself without fixed points and with the property that $\xi(\xi(x)) = x$ for all $x \in H$. With a slight abuse of notation, we will use the same symbol ξ to denote the set of pairs of half-edges in ξ, so $\{x, y\} \in \xi$ means that $\xi(x) = y$ and $\xi(y) = x$. Each pair of half-edges in ξ will also be called an edge. The set of all configurations on H will be denoted by Conf_H, and the uniform distribution on Conf_H will be denoted by Conf_H.

We note that each configuration gives rise to a (multi-)graph that may contain self-loops (edges having the same vertex on both ends) or multiple edges (between the same pair of vertices). The distribution of the random graph corresponding to a uniformly distributed configuration is called the configuration model (see [93, Chapter 7]). On the other hand, a graph can be obtained via several distinct configurations.

We will consider asymptotic statements in the sense of $|V| = n \to \infty$. Quantities like $V, H, \deg, m$ and ℓ all depend on n. In order to lighten the notation, we often suppress n from the notation.

The central object of this study is a Markov chain $(X, C) = (X_t, C_t)_{t\in\mathbb{N}_0}$, where $X_t \in H$ and $C_t \in \mathit{Conf}_H$ for all $t \in \mathbb{N}_0$. Here, X denotes the random walk component and C denotes the random configuration component. The configuration component gives rise to a graph sequence in which each graph has the same degree sequence. At each time $t \in \mathbb{N}$, we first update the configuration and then let the walk move.

Random walk. We consider a random walk on a dynamic random graph in which some half-edges are rewired at each step. The random walk is not allowed to back-track, in the sense that it cannot traverse the same edge twice in a row. Since in our model the underlying graph is dynamic and the edges change over time, it is more conveniently defined as a random walk on the set of half-edges H. Suppose that at time $t \in \mathbb{N}$ we updated the configuration to $C_t = \xi$. Then the random walk moves, according to the transition probabilities

$$P_\xi(x, y) := \begin{cases} \frac{1}{\deg(y)} & \text{if } \xi(x) \sim y \text{ and } \xi(x) \neq y, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

In words, when the random walk is at half-edge x in configuration ξ, it jumps to one of the siblings of the half-edge it is paired to uniformly at random (see Fig. 4.1). The transition probabilities are symmetric with respect to the pairing given by ξ, i.e., $P_\xi(x, y) = P_\xi(\xi(y), \xi(x))$, in particular, the matrix of transition probabilities is doubly stochastic, and so the uniform distribution on H, denoted by U_H, is stationary for P_ξ for any $\xi \in \mathit{Conf}_H$. In the sequel, when we use the term *random walk* we always refer to this model.

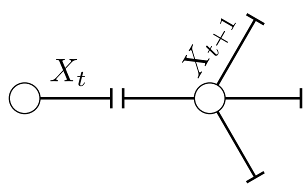

Figure 4.1: The random walk moves from half-edge X_t to half-edge X_{t+1}, one of the siblings of the half-edge that X_t is paired to.

Graph dynamics. We consider a general class of graph dynamics in which some edges are randomly rewired at each unit of time according to a presribed rule. A subset of edges to be rewired is chosen randomly, these edges are broken into half-edges and the resulting half-edges are paired randomly according to a prescribed distribution. The set of half-edges involved in the rewiring at time $t \in \mathbb{N}$ is denoted by R_t.

Suppose that $X_{t-1} = x$ and $C_{t-1} = \xi$. Then, at time t, the above dynamics gives rise to a distribution $Q_x(\xi, \cdot)$ on Conf_H. In [12, 13], a specific choice of dynamics was considered, in which $Q_x(\xi, \cdot)$ did not actually depend on x. In such a situation, the configuration component forms a Markov chain itself.

Joint chain. The law of the joint chain $(X, C) = (X_t, C_t)_{t\in\mathbb{N}_0}$, starting from initial half-edge x and initial configuration ξ, is given by the conditional probabilities

$$\begin{aligned}&\mathbb{P}_{x,\xi}(X_t = z, C_t = \zeta \mid X_{t-1} = y, C_{t-1} = \eta)\\&= Q_y(\eta, \zeta)P_\zeta(y, z), \quad t \in \mathbb{N}\end{aligned} \tag{4.3}$$

with

$$\mathbb{P}_{x,\xi}(X_0 = x, C_0 = \xi) = 1. \tag{4.4}$$

While the joint chain is Markov, the marginal chains $X = (X_t)_{t\in\mathbb{N}}$ and $C = (C_t)_{t\in\mathbb{N}}$ are not necessarily Markov.

We note that when the graph dynamics does not depend on the random walk, i.e., $Q_x(\cdot,\cdot) = Q_y(\cdot,\cdot)$ for all $x, y \in H$, the uniform distribution U_H is a stationary distribution for the random walk, i.e., for all $\xi \in Conf_H$ and $t \in \mathbb{N}$,

$$\sum_{x\in H} \frac{1}{\ell}\mathbb{P}_{x,\xi}(X_t \in \cdot) = U_H(\cdot).$$

This can be easily seen by noting that the random walk conditioned on a realization of the graph dynamics is a time-inhomogeneous Markov chain for which U_H is a stationary distribution.

§4.1.2 Main theorem

We are interested in the behaviour of the total variation distance between the distribution of the random walk component and the uniform distribution on the set of half-edges U_H, i.e.,

$$\mathcal{D}_{x,\xi}(t) := \|\mathbb{P}_{x,\xi}(X_t \in \cdot) - U_H(\cdot)\|_{\text{TV}}. \tag{4.5}$$

The total variation distance between two probability measures μ and ν on the same finite state space S is defined by

$$\|\mu - \nu\|_{\text{TV}} := \sum_{x\in S} |\mu(x) - \nu(x)| = \sum_{x\in S}[\mu(x) - \nu(x)]_+ = \sup_{A\subseteq S}[\mu(A) - \nu(A)]. \tag{4.6}$$

We emphasize that the marginal chain $X = (X_t)_{t\in\mathbb{N}}$ is not Markov and the total variation distance $\|\mathbb{P}_{x,\xi}(X_t \in \cdot) - U_H(\cdot)\|_{\text{TV}}$ is not guaranteed to be decreasing in t, even when it converges to 0.

Theorem 4.1.5 below concerns the behaviour of $\mathcal{D}_{x,\xi}(t)$ for *"typical"* choices of x and ξ. We formalize the notion of typicality now:

Definition 4.1.1 (With high probability). *Let $\mu = \mu_n := U_H \times \mathrm{Conf}_H$. A statement that depends on the initial half-edge x and configuration ξ is said to hold* with high probability (whp) *in x and ξ if the μ-measure of the set of pairs (x, ξ) for which the statement holds tends to 1 as $n \to \infty$.*

One of the key objects of our study will be a randomized stopping time, namely, the first time the random walk steps along a previously rewired edge. Let $R_{\le t} := \cup_{s=1}^t R_s$, and let I_t denote the indicator of the event that the random walk steps along a previously rewired edge at time t, i.e., if $X_{t-1} \in R_{\leq t}$, then $I_t = 1$, and otherwise $I_t = 0$. We define the randomized stopping time τ as follows:

$$\tau := \min\{t \in \mathbb{N} : I_t = 1\}. \tag{4.7}$$

Theorem 4.1.5 below will be stated in terms of the tail probabilities of τ, written $\mathbb{P}_{x,\xi}(\tau > t)$, and only holds under certain conditions. First, we give the conditions that concerns the degree sequences of the random graphs that we deal with:

Condition 4.1.2. *(Regularity of degrees)*

(R1) *ℓ is even and $\ell = \Theta(n)$ as $n \to \infty$.*

(R2) $\max_{v \in V} \deg(v) =: d_{\max} = o(n/(\log n)^2)$ *as $n \to \infty$.*

(R3) $\deg(v) \geq 2$ *for all $v \in V$.*

Condition 4.1.2(R1) ensures that the underlying graph is sparse, and together with Condition 4.1.2(R2) ensures that random walk paths are with high probability self-avoiding, as we will see in the proof of Lemma 4.2.2. Condition 4.1.2(R3) ensures that random walk is well-defined. These are the minimal conditions under which Theorem 4.1.5 is true. Next, we give additional conditions which allow us to use results of Ben-Hamou and Salez [16] on the mixing times of random walks on static configuration models:

Condition 4.1.3. *(Additional regularity of degrees)*

(R1*) $\max_{v \in V} \deg(v) =: d_{\max} = n^{o(1)}$ *as $n \to \infty$.*

(R2*)

$$\frac{\lambda_2}{\lambda_1^3} = \omega\left(\frac{(\log\log \ell)^2}{\log \ell}\right), \qquad \frac{\lambda_2^{3/2}}{\lambda_3\sqrt{\lambda_1}} = \omega\left(\frac{1}{\sqrt{\log \ell}}\right), \quad n \to \infty,$$

where

$$\lambda_1 := \frac{1}{\ell}\sum_{z \in H} \log(\deg(z)), \qquad \lambda_m := \frac{1}{\ell}\sum_{z \in H} |\log(\deg(z)) - \lambda_1|^m, \quad m = 2, 3.$$

(R3*) $\deg(v) \geq 3$ *for all $v \in V$.*

Conditions 4.1.3(R1*) and (R2*) are technical and it might be possible to relax them via a truncation argument [22]. Condition 4.1.3(R3*) ensures that the random walk does not behave deterministically, and under this condition the configuration model is connected with high probability. Condition 4.1.3 will not be used in Theorem 4.1.5 below, but will be needed to use results of Ben-Hamou and Salez [16] to refine Theorem 4.1.5 in Corollary 4.1.6 below.

Next, we give the conditions that concern the graph dynamics. To do so we need more notation. We define the annealed distribution by

$$\mathbb{P} := \sum_{\substack{x \in H, \\ \xi \in Conf}} \mu(x, \xi)\mathbb{P}_{x,\xi}, \tag{4.8}$$

which is the distribution of the random walk on the dynamically rewired graph annealed over the initial half-edge and the initial configuration. We will look at the annealed distribution of the graph dynamics at time t conditional on the walk before time t and on some partial information about the rewiring history before time t.

For $t \in \mathbb{N}$, let $[t] := \{1, \dots, t\}$, and for $s \in \mathbb{N}$ with $s < t$, and let $[s, t] := \{s, \dots, t\}$. Fix $t \in \mathbb{N}$, let $T = \{t_1, \dots, t_r\}$ be a subset of $[t-1]$. Consider four sequences of half-edges, $x_{[0,t-1]} = x_0 x_1 \dots x_{t-1}$, $\bar{x}_{[0,t-1]} = \bar{x}_0 \bar{x}_1 \dots \bar{x}_{t-1}$, $\hat{x}_{[r]} = \hat{x}_1 \hat{x}_2 \dots \hat{x}_r$ and $\tilde{x}_{[r]} = \tilde{x}_1 \tilde{x}_2 \dots \tilde{x}_r$, such that

- $\bar{x}_{s-1} \sim x_s$ for $s \in [t-1] \setminus T$,
- $\hat{x}_i \sim x_{t_i}$ for $i = 1, \dots, r$,
- the vertices $v(x_0), v(x_1), \dots, v(x_{t-1})$, $v(\bar{x}_{t_1-1}), \dots, v(\bar{x}_{t_r-1}), v(\bar{x}_{t-1})$, $v(\tilde{x}_1), \dots,$ $v(\tilde{x}_r)$ are all distinct.

We call such sequences *dynamically self-avoiding with respect to T*. We will look at:

- the set T: the times up to time $t-1$ at which the random walk steps along a previously rewired edge,
- the sequence $x_0 \dots x_{t-1}$: the path of the random walk up to time $t-1$,
- the sequence $\bar{x}_0 \dots \bar{x}_{t-1}$: the pairs of the latter in the initial configuration,
- the sequence $\hat{x}_1 \dots \hat{x}_r$: the pairs of $x_{t_1-1} \dots x_{t_r-1}$ at the times $t_1, \dots, t_r$ respectively,
- the sequence $\tilde{x}_1, \dots \tilde{x}_r$: the pairs of $\hat{x}_1 \dots \hat{x}_r$ in the initial configuration.

For fixed $t \in \mathbb{N}$, $T = \{t_1, \dots, t_r\} \subset [t-1]$, and fixed sequences of half-edges $x_{[0,t-1]}$, $\bar{x}_{[0,t-1]}$, $\hat{x}_{[r]}$ and $\tilde{x}_{[r]}$, let $H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]})$ be the event that

- $I_s = 1$ for $s \in T$ and $I_s = 0$ for $s \in [t-1] \setminus T$,
- $C_0(x_s) = \bar{x}_s$ for $s = 0, \dots, t-1$,
- $C_{t_i}(x_{t_i-1}) = \hat{x}_i$ for $i = 1, \dots, r$,
- $C_0(\hat{x}_i) = \tilde{x}_i$ for $i = 1, \dots, r$,
- $X_s = x_s$ for $s = 0, \dots, t-1$.

When this event occurs we say that the *the history of the walk on the dynamically rewired graph up to time t is dynamically self-avoiding.*

With these definitions in hand, we can state the conditions on the graph dynamics:

Condition 4.1.4. *(Regularity of graph dynamics) For all $t = t(n) = O(\log n)$ and all $T = \{t_1, \dots, t_r\} \subset [t-1]$,*

(D1) $\mathbb{P}(I_t = 1 \mid H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]}))$ *is the same for all choices of $x_{[0,t-1]}$, $\bar{x}_{[0,t-1]}$, $\hat{x}_{[r]}$, $\tilde{x}_{[r]}$ that are dynamically self-avoiding with respect to T.*

(D2) $\|\mathbb{P}(C_t(x_{t-1}) \in \cdot \mid H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]}) \cap \{I_t = 1\}) - U_H(\cdot)\|_{\text{TV}} = o(1/\log n)$ *for all choices of* $x_{[0,t-1]}$, $\bar{x}_{[0,t-1]}$, $\hat{x}_{[r]}$, $\tilde{x}_{[r]}$ *that are dynamically self-avoiding with respect to* T.

For $x \in H$ and $\xi \in Conf_H$, we denote by $\mathbb{P}^{\text{stat}}_{x,\xi}(X_t \in \cdot)$ the law of the random walk on the static graph given by the configuration ξ, and by $\mathcal{D}^{\text{stat}}_{x,\xi}(t)$ its total variation distance to the uniform distibution U_H at time t. Our main result reads as follows:

Theorem 4.1.5. *Suppose that* $t = t(n) = O(\log n)$. *Subject to Conditions 4.1.2 and 4.1.4, the following holds for the random walk on the dynamically rewired graph whp in* x *and* ξ*:*

$$\mathcal{D}_{x,\xi}(t) = \mathbb{P}_{x,\xi}(\tau > t)\mathcal{D}^{\text{stat}}_{x,\xi}(t) + o(1). \tag{4.9}$$

For the static model, under Condition 4.1.3, the ε-mixing time $\inf\{t \in \mathbb{N}_0 \colon \mathcal{D}^{\text{stat}}_{x,\xi}(t) \leq \varepsilon\}$ is known to scale like $t^{\text{stat}}_{\text{mix}} = t^{\text{stat}}_{\text{mix}}(n) := [1 + o(1)]\, c_{n,\text{stat}} \log n$ for all $\varepsilon \in (0, 1)$, with $c_{n,\text{stat}} = 1/\lambda_1 \in (0, \infty)$, where λ_1 is as defined in Condition 4.1.3(R2*)(Ben-Hamou and Salez [16]). This holds whp in ξ and uniformly in the starting position x. Using this relation we can refine Theorem 4.1.5:

Corollary 4.1.6. *Suppose* $t = t(n) = O(\log n)$. *Subject to Conditions 4.1.2(R1), 4.1.3 and 4.1.4, the following hold for the random walk on dynamically rewired graphs whp in* x *and* ξ*:*

$$\mathcal{D}_{x,\xi}(t) = \begin{cases} \mathbb{P}_{x,\xi}(\tau > t) + o(1) & \textit{if } \limsup_{n\to\infty} t/t^{\text{stat}}_{\text{mix}} < 1, \\ o(1) & \textit{if } \liminf_{n\to\infty} t/t^{\text{stat}}_{\text{mix}} > 1. \end{cases} \tag{4.10}$$

Proof. By the results in [16], whp in ξ we have

$$\mathcal{D}^{\text{stat}}_{x,\xi}(t) = \begin{cases} 1 - o(1) & \text{if } \limsup_{n\to\infty} t/t^{\text{stat}}_{\text{mix}} < 1, \\ o(1) & \text{if } \liminf_{n\to\infty} t/t^{\text{stat}}_{\text{mix}} > 1. \end{cases}$$

Combining these with Theorem 4.1.5 we get the desired result. □

The proof of Theorem 4.1.5 will be given in Section 4.3. In the next section (Section 4.2), we introduce the key ingredients of the proof. After proving Theorem 4.1.5, we introduce a specific example of a random walk on dynamically rewired random graph, which we call *'random walk with local rewiring'* and prove a mixing time result for this model in Section 4.4, by using Corollary 4.1.6.

§4.2 Coupling to the modified random walk

We define *the modified random walk*, denoted by $(Y_t)_{t\in\mathbb{N}_0}$, as the random walk on the static graph that at certain random times makes uniform jumps. The distribution of the jump times does not depend on the random walk path. More formally, we have a sequence $(J_t)_{t\in\mathbb{N}}$ of random variables adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$, taking values in $\{0, 1\}$ according to a given distribution on $\{0, 1\}^{\mathbb{N}}$. For fixed $t \in \mathbb{N}$, J_t is seen as the indicator of the event that the modified random walk makes a uniform jump at time t. The law of the modified random walk $(Y_t)_{t\in\mathbb{N}_0}$ on ξ that starts from the initial half-edge x, which is also adapted to $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$, is given by the conditional probabilities

$$\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t = z \mid Y_{t-1} = y, J_1 = j_1, \ldots, J_t = j_t) \tag{4.11}$$

$$= \mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t = z \mid Y_{t-1} = y, J_t = j_t) = \begin{cases} P_\xi(y,z) & \text{if } j_t = 0, \\ \frac{1}{\ell} & \text{if } j_t = 1, \end{cases} \quad t \in \mathbb{N}, \tag{4.12}$$

with

$$\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_0 = x) = 1. \tag{4.13}$$

We note that, according to the definition, neither $(J_t)_{t\in\mathbb{N}}$ nor the pair $(Y_t, J_t)_{t\in\mathbb{N}}$ needs to be Markov but $(Y_t)_{t\in\mathbb{N}_0}$ is Markov conditional on a realisation of $(J_t)_{t\in\mathbb{N}}$.

Uniform jumps of the modified random walk can be rephrased in the following form. Let Y'_t be a uniformly chosen half-edge, independent of the random walk path and the jump times. If $J_t = 1$, then we choose a uniform sibling of Y'_t, say y, and set $Y_t = y$. Since Y'_t is uniform and one of its siblings is chosen uniformly at random, the resulting half-edge is distributed uniformly on H. In the following we use this formulation, since it makes the exposition more clear.

As an analogue of τ, we define σ to be the first time that the modified random walk makes a uniform jump, i.e.,

$$\sigma \coloneqq \inf\{t \in \mathbb{N} : J_t = 1\} \tag{4.14}$$

Coupling of two random walks. We couple the law $\mathbb{P}_{x,\xi}(X_t \in \cdot)$ of the random walk on the dynamic random graph, with initial half-edge x and initial configuration ξ, to the law $\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot)$ of the modified random walk. We want the coupled random walks to stick together as much as possible. When the two random walks make different steps, we say that the coupling of the two random walks has *failed*, and we denote the first time that this happens by F. Until the coupling fails, the times at which the random walk on the dynamically rewired graph makes a step over a previously rewired edge correspond to the times at which the modified random walk makes a uniform jump.

We define an auxiliary random set A_t, called the set of *active* half-edges, which is constructed by adding half-edges at each unit of time. This set will keep track of the half-edges traversed by the two random walks, the half-edges that are rewired at the position of the random walk, and their pairs in the initial configuration. Note that A_0 consists of x and its siblings, i.e., $A_0 = H(v(x))$. The coupling is as follows:

(a) At time $t \in \mathbb{N}$, if the coupling has not failed yet and neither $\xi(X_{t-1})$ nor any of its siblings belongs to A_{t-1}, then maximally couple the distribution of I_t, conditional on the history of the random walk and the rewirings seen by the random walker, to the distribution of J_t, conditional on the values of indicators $J_1, \ldots, J_{t-1}$:

  (a) If the coupling of the conditional distributions of I_t and J_t is successful and $I_t = J_t = 0$, then add $\xi(X_{t-1})$ and all of its siblings to A_{t-1} to obtain A_t, let X make a random walk move, and set $Y_t = X_t$.

(b) If the coupling of the conditional distributions of I_t and J_t is successful and $I_t = J_t = 1$, then maximally couple the distribution of the pair of X_{t-1} in C_t, $C_t(X_{t-1})$, conditional on the history of the random walk and the event that $I_t = 1$, to the distribution of Y_t':

  (a) If the coupling of $C_t(X_{t-1})$ and Y_t' is succesful, and neither $C_t(X_{t-1})$ nor any of its siblings is in A_{t-1}, then add $\xi(X_{t-1})$ and all of its siblings, $C_t(X_{t-1})$ and all of its siblings to A_{t-1} to obtain A_t, let X make a random walk move, and set $Y_t = X_t$.

  (b) Otherwise, declare the coupling of the two random walks as failed.

(c) If the coupling of the conditional distributions of I_t and J_t is not succesful, i.e., $I_t \neq J_t$, then declare the coupling of the two random walks as failed.

(b) At time $t \in \mathbb{N}$, if the coupling has failed before, then let X and Y evolve independently. If the coupling has not failed yet but either $\xi(X_{t-1})$ or some of its siblings belong to A_{t-1}, then declare the coupling of the two random walks as failed, and let X and Y evolve independently.

Remark 4.2.1. *At each time $t \in \mathbb{N}$, the random walks try to avoid stepping on the active half-edges A_{t-1}. The coupling of the two random walks fails in three cases:*

(a) *if the coupling of $C_t(X_{t-1})$ and Y_t' fails, or the two random walks step over a half-edge in A_{t-1} in step* (b)*,*

(b) *if the coupling of I_t and J_t fails in step* (c)*,*

(c) *if the pair of X_{t-1} in the starting configuration is already in A_{t-1} as in step* (b)*.*

The second case in item 1, as well as item 3, correspond to the situation in which the random walks are not dynamically self-avoiding. We want to avoid this situation, since it might lead to a previously rewired half-edge that was stepped over previously. This implies that the random walks are dynamically self-avoiding before the coupling of the two random walks fail. The first case in item 1 corresponds to the situation in which the conditional distribution of $C_t(X_{t-1})$ is far from the uniform distribution in total variation distance. Item 3 corresponds to the situation in which the conditional distribution of the times at which the random walk on the dynamically rewired graph and the conditional distribution of the times at which the modified random walk makes uniform jumps are far from each other in total variation distance.

The next lemma states that these events are unlikely up to logarithmic times when Conditions 4.1.2 and 4.1.4 hold for the random walk on the dynamically rewired graph:

Lemma 4.2.2. *Suppose that $t = t(n) = O(\log n)$, and that Conditions 4.1.2 and 4.1.4 hold for the random walk on the dynamically rewired graph. For all $s \leq t$ and all $T = \{s_1, \ldots, s_r\} \subset [s-1]$, fix a group of sequences $x^{s,T}_{[0,s-1]}, \bar{x}^{s,T}_{[0,t-1]}, \hat{x}^{s,T}_{[r]}, \tilde{x}^{s,T}_{[r]}$ that is dynamically self-avoiding with respect to T, and consider the modified random walk fpr which the jump distribution has conditional distribution*

$$
\begin{aligned}
&\mathbb{P}^{\mathrm{mod}}_{x,\xi}(J_s = 1 \mid J_{s'} = 0 \text{ for } s' \in [s-1] \setminus T, J_{s''} = 1 \text{ for } s'' \in T) \\
&= \mathbb{P}(I_s = 1 \mid H(T, x^{s,T}_{[0,s-1]}, \bar{x}^{s,T}_{[0,s-1]}, \hat{x}^{s,T}_{[r]}, \tilde{x}^{s,T}_{[r]})). \qquad (4.15)
\end{aligned}
$$

Then, whp in x and ξ,

$$\|\mathbb{P}_{x,\xi}(X_t \in \cdot) - \mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot)\|_{\text{TV}} = o(1) \tag{4.16}$$

and

$$\mathbb{P}_{x,\xi}(\tau > t) = \mathbb{P}^{\text{mod}}_{x,\xi}(\sigma > t) + o(1). \tag{4.17}$$

Proof. Let $\mathbb{P}^{\text{couple}}_{x,\xi}$ denote the law of the coupling of the two random walks with $X_0 = x$ and $C_0 = \xi$. Since the two random walks agree up to the time the coupling fails, we have

$$\|\mathbb{P}_{x,\xi}(X_t \in \cdot) - \mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot)\|_{\text{TV}} \leq \mathbb{P}^{\text{couple}}_{x,\xi}(F \leq t). \tag{4.18}$$

So, in order to prove our claim, it suffices to show that, whp in x and ξ,

$$\mathbb{P}^{\text{couple}}_{x,\xi}(F \leq t) = o(1). \tag{4.19}$$

To achieve this, we will use an annealing argument on the initial graph and the initial location. Recall that $\mu = U_H \times Conf_H$, and let

$$\mathbb{P}^{\text{couple}} = \sum_{x,\xi} \mu(x,\xi)\mathbb{P}^{\text{couple}}_{x,\xi}. \tag{4.20}$$

We will show that

$$\mathbb{P}^{\text{couple}}(F \leq t) = o(1) \tag{4.21}$$

by exploring the initial configuration through the coupled random walk paths until time F, the time at which the coupling fails. The exploration proceeds as follows:

(a) At time $s = 0$, choose a half-edge uniformly at random from H, say x, set $X_0 = Y_0 = x$ and $A_0 = H(v(x))$, the subset of H consisting of x and its siblings.

(b) At time $s \in \mathbb{N}$, first explore the pair of $X_{s-1} = Y_{s-1}$ in the initial configuration ξ, then make the coupled random walks move until the coupling fails, and update A_s accordingly.

According to this description, the exploration process explores the part of the graph seen by the random walks, as well as the parts changed by the rewiring at the positions of the random walks, and it stops as soon as the coupling of the two random walks fails. Suppose that the coupling of the two random walks has not failed before time s. Then it can fail at time s in the following three cases:

(a) if coupling of I_s and J_s fails in step (c),

(b) if coupling of $C_s(X_{s-1})$ and Y'_s fails in step (b),

(c) if the random walks step over a half-edge that is in A_{s-1} in step (b) or step (b).

By (4.15), I_s and J_s can be coupled perfectly, so the probability of the event in case 1 is 0.

For case 2 we note that, by Remark 4.2.1, before the coupling of the two random walks fails, the history of the random walk is dynamically self-avoiding. By Condition 4.1.4(D2), the total variation distance between the conditional distribution of

$C_s(X_{s-1})$ and the uniform distribution U_H is $o(1/\log n)$. Since Y'_s is also distributed uniformly on H, the probability of the event in case 2 is $o(1/\log n)$.

For case 3, we first need an upper bound on the size of A_{s-1}. Each time we explore the initial configuration, we add at most $d_{\max}$ half-edges to the set of active half-edges. If, in addition, a rewiring occurs, then we add at most $2d_{\max}$ half-edges to the set of active half-edges. This gives us

$$|A_{s-1}| \leq 3sd_{\max}. \tag{4.22}$$

For a fail event in step (b), we see that the probability that $C_s(X_{s-1}) \in A_{s-1}$ is smaller than

$$\frac{|A_{s-1}|}{\ell} + o(1/\log n) \leq \frac{3sd_{\max}}{\ell} + o(1/\log n), \tag{4.23}$$

since the random walk is dynamically self-avoiding before the coupling of the two random walks fails (see Remark 4.2.1), so the total variation distance between the conditional distribution of $C_s(X_{s-1})$ and the uniform distribution U_H is $o(1/\log n)$, by Condition 4.1.4(D2). For a fail event in step (b), we see that the probability that $C_0(X_{s-1}) \in A_{s-1}$ is smaller than

$$\frac{|A_{s-1}|}{\ell - 4s + 4} \leq \frac{3sd_{\max}}{\ell - 4s + 4}, \tag{4.24}$$

since up to time s we form at most $2s - 2$ pairs in C_0, $s - 1$ of them on the random walk path and an additional $s - 1$ if rewiring occurs at each step up to time s.

The above estimates give us

$$\mathbb{P}^{\text{couple}}(F = s \mid F > s - 1) \leq \frac{6sd_{\max}}{\ell - 4s + 4} + o(1/\log n). \tag{4.25}$$

Taking a union bound up to time t, and using that $t = O(\log n)$, $d_{\max} = o(n/(\log n)^2)$ and $\ell = \Theta(n)$, we get

$$\mathbb{P}^{\text{couple}}(F \leq t) \leq \frac{3t(t+1)d_{\max}}{\ell - 4t} + o(1) = o(1), \tag{4.26}$$

which in turn implies that, with μ-probability $1 - o(1)$,

$$\mathbb{P}^{\text{couple}}_{x,\xi}(F \leq t) = o(1). \tag{4.27}$$

Indeed, letting $\mathbb{P}^{\text{couple}}(F \leq t) = p_n$ and $B = \{(x, \xi) \in H \times \mathit{Conf}_H : \mathbb{P}^{\text{couple}}_{x,\xi}(F \leq t) > p_n^{1/2}\}$, we see that

$$\mathbb{P}^{\text{couple}}(F \leq t) = p_n > \mu(B)p_n^{1/2}, \tag{4.28}$$

and hence $\mu(B) < p_n^{1/2}$. So, with μ-probability at least $1 - p_n^{1/2}$, we have $\mathbb{P}^{\text{couple}}_{x,\xi}(F \leq t) \leq p_n^{1/2} = o(1)$.

Since the I_s's and J_s's are perfectly coupled until the coupling of the two random walks fails, we also have, whp in x and ξ,

$$|\mathbb{P}_{x,\xi}(\tau > t) - \mathbb{P}^{\text{mod}}_{x,\xi}(\sigma > t)| \leq \mathbb{P}^{\text{couple}}_{x,\xi}(F \leq t) = o(1). \tag{4.29}$$

□

§4.3 Link between the dynamic and the static models

In this section, we prove Theorem 4.1.5. Consider the modified random walk given in the statement of Lemma 4.2.2 and sample uniform jump times up to time t. For any fixed $T = \{t_1, \dots, t_r\} \subset [t]$, we see that the modified random walk conditional on the event $J(T) \coloneqq \{J_s = 0 \text{ for } s \in [t] \setminus T, J_s = 1 \text{ for } s \in T\}$ is a time-inhomogeneous Markov chain that makes random-walk moves at times $s \in [t] \setminus T$ and uniform jumps at times $s \in T$. Since this Markov chain becomes stationary when it makes a uniform jump, for any $\emptyset \neq T \subset [t]$, $x \in H$ and $\xi \in Conf_H$,

$$\mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot \mid J(T)) = U_H(\cdot), \tag{4.30}$$

which gives us

$$\mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot \mid \sigma \leq t) = \frac{\sum_{T\subset[t],T\neq\emptyset} \mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot \mid J(T))}{\sum_{T\subset[t],T\neq\emptyset} \mathbb{P}^{\text{mod}}_{x,\xi}(J(T))} = U_H(\cdot). \tag{4.31}$$

On the other hand, since the modified random walk up to time t conditional on the event $\{\sigma > t\}$ is the same as the random walk on the static graph, for any $x \in H$ and $\xi \in Conf_H$ we have

$$\|\mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot \mid \sigma > t) - U_H(\cdot)\|_{\text{TV}} = \mathcal{D}^{\text{stat}}_{x,\xi}(t). \tag{4.32}$$

Using the triangle inequality twice, we obtain

$$\begin{aligned}\|\mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot) - U_H(\cdot)\|_{\text{TV}} \leq &\mathbb{P}^{\text{mod}}_{x,\xi}(\sigma > t)\|\mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot \mid \sigma > t) - U_H(\cdot)\|_{\text{TV}} \\ &+ \mathbb{P}^{\text{mod}}_{x,\xi}(\sigma \leq t)\|\mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot \mid \sigma \leq t) - U_H(\cdot)\|_{\text{TV}}\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}\|\mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot) - U_H(\cdot)\|_{\text{TV}} \geq &\mathbb{P}^{\text{mod}}_{x,\xi}(\sigma > t)\|\mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot \mid \sigma > t) - U_H(\cdot)\|_{\text{TV}} \\ &- \mathbb{P}^{\text{mod}}_{x,\xi}(\sigma \leq t)\|\mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot \mid \sigma \leq t) - U_H(\cdot)\|_{\text{TV}}.\end{aligned} \tag{4.34}$$

Inserting (4.31) and (4.32), we obtain

$$\|\mathbb{P}^{\text{mod}}_{x,\xi}(Y_t \in \cdot) - U_H(\cdot)\|_{\text{TV}} = \mathbb{P}^{\text{mod}}_{x,\xi}(\sigma > t)\mathcal{D}^{\text{stat}}_{x,\xi}(t). \tag{4.35}$$

Now using Lemma 4.2.2, we see that, whp in x and ξ,

$$\mathcal{D}_{x,\xi}(t) = \mathbb{P}_{x,\xi}(\tau > t)\mathcal{D}^{\text{stat}}_{x,\xi}(t) + o(1). \tag{4.36}$$

§4.4 Random walk with local rewiring

In this section, we consider a specific example of a random walk on a dynamically rewired graph in which the graph dynamics depends on the position of the random walk. We call this model the *random walk with local rewiring*. The rewiring mechanism works as follows:

(a) At each time $t \in \mathbb{N}$, we draw a Bernoulli random variable Z_t with parameter α, independent of each other and independent of the random walk and the configuration,

(b) If $Z_t = 0$, then the configuration does not change, $C_t = C_{t-1}$, and X_t makes a random-walk move,

(c) If $Z_t = 1$, then we draw a half-edge uniformly at random from $H \setminus \{X_{t-1}\}$, say y, we pair X_{t-1} to y and $C_{t-1}(X_{t-1})$ to $C_{t-1}(y)$ to obtain the new configuration C_t, and X_t makes a random walk move on C_t.

More fomally, let

$$Q_x^R(\xi,\eta) = Q_x^R(\eta,\xi) := \begin{cases} \frac{1}{\ell-1} & \text{if } \xi(\eta(x)) = \eta(\xi(x)) \text{ and } |\xi \setminus \eta| \leq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.37}$$

Within the framework of Section 4.1.1, the above mechanism corresponds to the model in which

$$Q_x(\xi,\eta) = (1-\alpha)I(\xi,\eta) + \alpha Q_x^R(\xi,\eta), \tag{4.38}$$

where $I(\xi,\eta) = 1$ if $\eta = \xi$, and $I(\xi,\eta) = 0$ otherwise, i.e., I is the identity matrix. Since Q_x^R is symmetric for all $x \in H$, we see that the distribution Conf_H is a stationary distribution for Q_x^R for all $x \in H$. This implies that Conf_H is a stationary distribution for Q_x for all $x \in H$.

A direct calculation shows that $U_H \times \mathrm{Conf}_H$ is a stationary distribution of this dynamics:

Proposition 4.4.1. *$U_H \times Conf_H$ is a stationary distribution for the random walk with local rewiring with parameter α, for any $\alpha \in [0,1]$.*

Proof. Since U_H is stationary for P_η for any $\eta \in Conf_H$, and Conf_H is stationary for Q_x for any $x \in H$, for any $y \in H$ and $\eta \in Conf_H$,

$$\begin{aligned}
&\sum_{x\in H}\sum_{\xi\in Conf_H} U_H(x)\mathrm{Conf}_H(\xi)\mathbb{P}_{x,\xi}(X_1 = y, C_1 = \eta) \\
&\quad= \sum_{x\in H}\sum_{\xi\in Conf_H} U_H(x)\mathrm{Conf}_H(\xi)Q_x(\xi,\eta)P_\eta(x,y) \\
&\quad= \sum_{x\in H} U_H(x)P_\eta(x,y) \sum_{\xi\in Conf_H} \mathrm{Conf}_H(\xi)Q_x(\xi,\eta) \\
&\quad= \mathrm{Conf}_H(\eta)\sum_{x\in H} U_H(x)P_\eta(x,y) = \mathrm{Conf}_H(\eta)U_H(y),
\end{aligned}$$

which shows that $U_H \times \mathrm{Conf}_H$ is a stationary distribution for the random walk with local rewiring model. □

It is not easily seen that the Markov chain is irreducible and aperiodic. In Section 4.4.1 we show that this is indeed the case when $\alpha \in (0,1)$, and so the distribution of the joint chain converges to $U_H \times \mathrm{Conf}_H$ as $t \to \infty$. An important implication is that the distribution of the random walk alone converges to U_H as $t \to \infty$. Indeed, for any $x \in H$, $\xi \in \mathit{Conf}_H$ and $t \in \mathbb{N}$ we have

$$\mathcal{D}_{x,\xi}(t) \le \|\mathbb{P}_{x,\xi}((X_t, C_t) \in \cdot) - U_H \times \mathrm{Conf}_H(\cdot)\|_{\mathrm{TV}},$$

and since the right-hand side tends to 0 as $t \to \infty$, $\mathcal{D}_{x,\xi}(t)$ also tends to 0 as $t \to \infty$. On the other hand, this argument does not automatically imply that $\mathcal{D}_{x,\xi}(t)$ is non-increasing in t.

§4.4.1 Irreducibility and aperiodicity

In this section we show that the random walk with local rewiring model is irreducible and aperiodic, which ensures that the total variation distance $\mathcal{D}_{x,\xi}(t)$ converges to 0 as $t \to \infty$ for fixed $x \in H$, $\xi \in \mathit{Conf}_H$ and $\alpha \in (0, 1)$. Our proof builds on the proof of irreducibility of the switch chain on multigraphs given by Eggleton and Holton [40].

Proposition 4.4.2. *The rewiring random walk $(X_t, C_t)_{t\in\mathbb{N}_0}$ is irreducible and aperiodic for any initial state $(x, \xi) \in H \times \mathrm{Conf}_H$ and any choice of $\alpha \in (0, 1)$.*

Proof. Let $V = \{v_1, \dots, v_n\}$ and assume that $\deg(v_1) \le \deg(v_2) \le \cdots \le \deg(v_n)$. Identify the set of half-edges H with $[\ell] = \{1, \dots, \ell\}$ such that the half-edges $1, \dots,$ $\deg(v_1)$ are associated to v_1, the half-edges $\deg(v_1) + 1, \dots, \deg(v_1) + \deg(v_2)$ to v_2, and so on. Let $v'_1, \dots, v'_{2k} \in V$ be the odd-degree vertices. We fix a configuration $\xi_0 \in \mathit{Conf}_H$ such that each vertex has the maximum number of self-loops, i.e., each vertex $v \in V$ with even degree has $\deg(v)/2$ self-loops, each vertex $v \in V$ with odd degree has $(\deg(v) - 1)/2$ self-loops, and there is exactly one edge between every pair of odd-degree vertices v'_{2i-1}, v'_{2i} for $i = 1, \dots, k$ (see Figure 4.2). We will show that the pair $(1, \xi_0) \in H \times \mathit{Conf}_H$ is accessible from any pair $(x, \xi) \in H \times \mathit{Conf}_H$ by allowed moves in the random walk with local rewiring model.

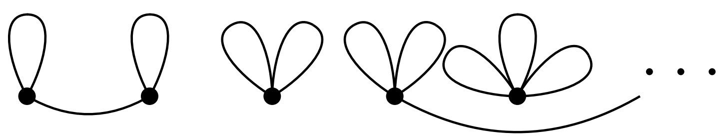

Figure 4.2: The configuration ξ_0.

First we show that, for any $x \in H$, $(1, \xi_0)$ is accessible from (x, ξ_0), by considering two different scenarios:

(a) Suppose that x is on a self-loop and $\xi_0(x) = x'$. We first move to $(1, \xi_1)$ from $(1, \xi_0)$ by rewiring the half-edges $x, x', 1$ and 2 where ξ_0 and ξ_1 agree on all the edges except that $\xi_1(1) = x'$ and $\xi_1(2) = x$. After that we again move to $(1, \xi_0)$ from $(1, \xi_1)$ by rewiring $1, 2, x$ and x' (see Figure 4.3).

(b) Suppose that x is not on a self-loop, i.e., it is on an edge between two odd-degree vertices. We first move to (x', ξ_0) without rewiring, where $x' \in H$ is on a self-loop. After that we apply the procedure in the item 1 to (x', ξ_0).

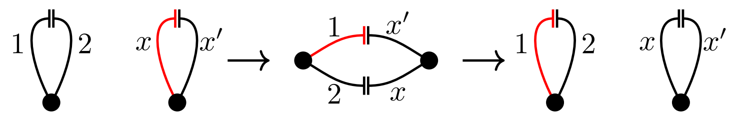

Figure 4.3: Move from half-edge x on a self-loop to half-edge 1 in ξ_0. The red color indicates the position of the walk.

Next, we show that for any $(x, \xi) \in H \times \mathit{Conf}_H$ with $\xi \neq \xi_0$ we have access from (x, ξ) to (y, ξ_0), for some $y \in H$. To do this, we show that we can move from (x, ξ) to some $(y, \eta) \in H \times \mathit{Conf}_H$ such that the configuration η has more edges in common with ξ_0 than ξ has, i.e., $|\xi \cap \xi_0| < |\eta \cap \xi_0|$, by considering the two scenarios:

(a) Suppose that x is on an edge that is not in ξ_0, i.e., $\xi(x) \neq \xi_0(x)$. Then we move to (y, η) by rewiring the half-edges $x, \xi(x), \xi_0(x)$ and $\xi(\xi_0(x))$, where ξ and η agree on all the edges except that $\eta(x) = \xi_0(x)$ and $\eta(\xi(x)) = \xi(\xi_0(x))$ and $y \sim \xi_0(x)$. Since $\eta(x) = \xi_0(x)$, we have that $|\xi \cap \xi_0| \leq |\eta \cap \xi_0| - 1$.

(b) Suppose that x is on an edge that is in ξ_0, i.e., $\xi(x) = \xi_0(x)$. Let $y \in H$ be a half-edge such that $\xi(y) \neq \xi_0(y)$, $\xi(x) = x'$ and $\xi(y) = y'$. Since $\deg(v) \geq 2$ for all $v \in V$, in the graph given by ξ there is a cycle of edges $\{y, y'\}, \{y_1, y_1'\}, \ldots, \{y_K, y_K'\}$ with $v(y') = v(y_1)$, $v(y_K') = v(y)$ and $v(y_i') = v(y_{i+1})$ for $i = 1, \ldots, K-1$. Let $\eta \in \mathit{Conf}_H$ be the configuration that agrees with ξ on all the edges except that $\eta(x) = y'$ and $\eta(y) = x'$, so that the edges $\{y_1, y_1'\}, \ldots, \{y_K, y_K'\}$ are present in η as well as in ξ. First we move from (x, ξ) to (y_1, η) by rewiring x, x', y and y'. Then we make K moves, from (y_i, η) to (y_{i+1}, η) for $i = 1, \ldots, K$, where $y_{K+1} = y$ without rewiring. After that we move from (y, η) to (y_1, ξ) by rewiring x, x', y and y', and finally we traverse the cycle again without rewiring to reach (y, ξ) from (y_1, ξ) (see Figure 4.4). Now y is on an edge that is not in ξ_0, so by applying the procedure in item 1 we can increase the number of edges we have in common with ξ_0.

By applying these procedures, we can reduce the number of edges that are not in ξ_0, so we can go from any $(x, \xi) \in H \times \mathit{Conf}_H$ to (y, ξ_0) for some $y \in H$, and then apply the above procedure to reach $(1, \xi_0)$.

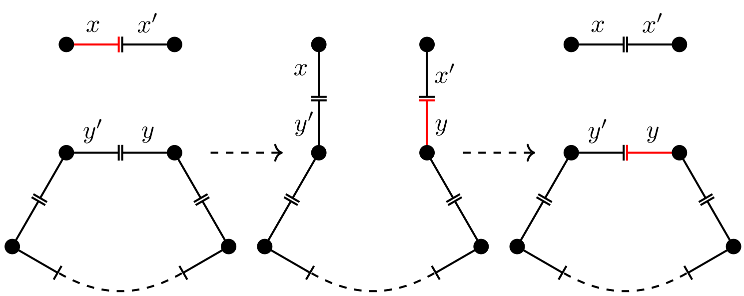

Figure 4.4: Moving from (x, ξ) to (y, η) by using a cycle. The red color indicates the position of the walk.

To show that we can access an arbitrary state (x, ξ) from $(1, \xi_0)$, we first note that we can access (y, ξ_0), for any y, from $(1, \xi_0)$ by relabelling the half-edges and using the first argument above. Then we see that we can access (x, ξ) from (y, ξ_0) for any y using the above strategy of reducing the edges and using the cycles to move around. Hence, the Markov chain is irreducible. Since, by traversing the self-loop without rewiring, we can reach $(1, \xi_0)$ from itself in one step, we see that the Markov chain is also aperiodic. □

§4.4.2 The mixing time of the random walk with local rewiring

In this section, we study the quantity $\mathcal{D}_{x,\xi}(t)$ for the random walk with local rewiring and show that we have the same trichotomy as for the random walk on the dynamic configuration model [13]:

Theorem 4.4.3 (**Scaled mixing profiles**). *Suppose that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \alpha_n \log n = \beta \in [0, \infty]$*, and consider the rewiring random walk with parameter* α_n*. Subject to Condition* 4.1.2(R1) *and Condition* 4.1.3*, the following hold whp in* x *and* ξ*:*

(1) *If* $\beta = \infty$*, then*

$$\mathcal{D}_{x,\xi}\big(\lfloor c\alpha_n^{-1} \rfloor\big) = \mathrm{e}^{-c} + o(1), \quad c \in [0, \infty). \tag{4.39}$$

(2) *If* $\beta \in (0, \infty)$*, then*

$$\mathcal{D}_{x,\xi}\big(\lfloor c \log n \rfloor\big) = \begin{cases} \mathrm{e}^{-\beta c} + o(1), & c \in [0, c_{n,\mathrm{stat}}), \\ o(1), & c \in (c_{n,\mathrm{stat}}, \infty). \end{cases} \tag{4.40}$$

(3) *If* $\beta = 0$*, then*

$$\mathcal{D}_{x,\xi}\big(\lfloor c \log n \rfloor\big) = \begin{cases} 1 - o(1), & c \in [0, c_{n,\mathrm{stat}}), \\ o(1), & c \in (c_{n,\mathrm{stat}}, \infty). \end{cases} \tag{4.41}$$

Proof. We show that Condition 4.1.4 holds and then use Corollary 4.1.6 to prove the claim. For fixed $t = O(\log n)$, fix some $T = \{t_1, \dots, t_r\} \subset [t-1]$ and some $x_{[0,t-1]}$, $\bar{x}_{[0,t-1]}$, $\hat{x}_{[r]}$ and $\tilde{x}_{[r]}$ that are dynamically self-avoiding with respect to T. Conditioned on the event $H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]})$, x_{t-1} cannot be rewired before time t. Indeed, by construction the half-edges that are rewired before time t are $x_{t_1-1}, \dots, x_{t_r-1}$, $\bar{x}_{t_1-1}, \dots, \bar{x}_{t_r-1}$, $\hat{x}_1, \dots, \hat{x}_r$ and $\tilde{x}_1, \dots, \tilde{x}_r$, and x_{t-1} is not equal to any of these. So we have

$$\begin{aligned} &\mathbb{P}(I_t = 1 \mid H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]})) \\ &= \mathbb{P}(Z_t = 1 \mid H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]})) = \alpha_n, \end{aligned} \tag{4.42}$$

and $\mathbb{P}(C_t(x_{t-1}) \in \cdot \mid H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]}) \cap \{I_t = 1\})$ is the uniform distribution on $H \setminus \{x_{t-1}\}$, which gives

$$\|\mathbb{P}(C_t(x_{t-1}) \in \cdot \mid H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]}) \cap \{I_t = 1\}) - U_H(\cdot)\|_{\mathrm{TV}} = \frac{1}{\ell}. \tag{4.43}$$

Since this holds for any choice of $x_{[0,t-1]}$, $\bar{x}_{[0,t-1]}$, $\hat{x}_{[r]}$ and $\tilde{x}_{[r]}$, Condition 4.1.4 holds.

On the other hand, the event $\{\tau = t\}$ is the same as the event $\{\min\{s \in \mathbb{N} : R_s = 1\} = t\}$, since when a rewiring occurs the random walk steps over a rewired edge with probability 1. This implies that for any x and ξ, and since $\lim_{n\to\infty} \alpha_n = 0$,

$$\mathbb{P}_{x,\xi}(\tau > t) = (1 - \alpha_n)^t = \exp(-\alpha_n t) + o(1). \tag{4.44}$$

So we have

$$\mathbb{P}_{x,\xi}(\tau > t) = \exp(-c) + o(1) \text{ when } \lim_{n\to\infty} \alpha_n \log n = \infty \text{ and } t = \lfloor c\alpha_n^{-1} \rfloor, \tag{4.45}$$

$$\mathbb{P}_{x,\xi}(\tau > t) = \exp(-\beta c) + o(1) \text{ when } \lim_{n\to\infty} \alpha_n \log n = \beta \text{ and } t = \lfloor c \log n \rfloor, \tag{4.46}$$

$$\mathbb{P}_{x,\xi}(\tau > t) = 1 - o(1) \text{ when } \lim_{n\to\infty} \alpha_n \log n = 0 \text{ and } t = \lfloor c \log n \rfloor. \tag{4.47}$$

Combining these with Corollary 4.1.6, we obtain the desired result. □

§4.5 Discussion

1. Coupling between the two random walks: The core ingredient of the proof of the main result, which is the coupling between the random walk on the dynamically rewired graph and the modified random walk, is best visualised as follows: imagine we are looking at the random walk on the dynamically rewired graph from the point of view of the initial configuration. Then it looks as if the random walk performs an ordinary random walk on the static initial graph (when it walks on the parts that are not changed by the dynamics), with the exception that at some random times it makes uniform jumps (when it encounters a previously rewired edge). This suggests that the random walk on the dynamically rewired graph can be coupled to a random walk that exactly does this.

The framework of the coupling to a modified random walk introduced in this paper is based on the ideas developed in [13]. In fact, the coupling of the random walk on the dynamically rewired random graph and the modified random walk is implicit in the proof of the main theorem of [13]. There the main idea was that the path probabilities under the two random walk models coincide for self-avoiding paths, and it was shown that the random walk paths are with high probability self-avoiding.

The crucial observation is that the random walk paths on a typical configuration are self-avoiding with high probability under the law of the configuration model. The particular form of Condition 4.1.4 is motivated by this observation. This also suggests that the same results should hold when the distribution of the initial graph is replaced by some other distribution on graphs on which random walk paths are *'typically'* self-avoiding.

2. One-sided cut-off: It is easy to construct examples of one-sided cut-off in the more general framework of Markov chains. Suppose that P is the matrix of transition probabilities of an ergodic Markov chain on a state space $\mathcal{X}$ with a stationary distribution π, and let Π be the matrix whose rows are all equal to π. Fix $\alpha \in (0, 1]$ and consider the Markov chain where at each step transitions are made according to matrix P with probability $1 - \alpha$ and according to matrix Π with probability α, and these choices are made independently at each step. This corresponds to the Markov chain with transition probabilities given by $(1 - \alpha)P + \alpha\Pi$. Note that, as soon as Π is used for transition, the Markov chain becomes stationary. If we let σ be the first time Π is used for a transition then σ is a strong stationary time for the Markov chain, and hence the total variation distance can be bounded by tail probabilities of σ. In fact, for any $x \in \mathcal{X}$ and $t \in \mathbb{N}$ we have

$$\|Q^t(x, \cdot) - \pi\|_{\mathrm{TV}} = (1 - \alpha)^t \|P^t(x, \cdot) - \pi\|_{\mathrm{TV}}, \tag{4.48}$$

since the probability of the event $\{\sigma > t\}$ is $(1-\alpha)^t$ and the Markov chain is stationary at time t conditioned on the event $\{\sigma \leq t\}$.

Now, suppose $(P_n)_{n\in\mathbb{N}}$ is a sequence of ergodic Markov chains indexed by the size n of the state space, π_n is the stationary distribution and T_n is the mixing time of P_n with $T_n \to \infty$ as $n \to \infty$. Let Π_n be the matrix of transition probabilities whose rows are all equal to π_n, and consider the Markov chain whose transition probabilities are given by the matrix $Q_n = (1 - \alpha_n)P_n + \alpha_n\Pi_n$. If $(P_n)_{n\in\mathbb{N}}$ exhibits cut-off, then we have the same trichotomy as in Theorem 4.4.3:

- $\lim_{n\to\infty} \alpha_n T_n = \infty$: the mixing time is of order α_n^{-1} without cut-off,
- $\lim_{n\to\infty} \alpha_n T_n = \beta \in (0, \infty)$: the mixing time is of order T_n with one-sided cut-off,
- $\lim_{n\to\infty} \alpha_n T_n = 0$: the mixing time is of order T_n with two-sided cut-off (the same as for P_n).

3. Regularity of the graph dynamics: Simple modifications to the random walk with local rewiring model can lead to violations of Condition 4.1.4. Let us consider a modification in which the rewiring mechanism is slightly changed: When $Z_t = 1$ we choose an edge, say $\{y, z\}$, uniformly at random from the set of all edges of C_{t-1} except the edge $\{X_{t-1}, C_{t-1}(X_{t-1})\}$, and we pair the half-edges $X_{t-1}, C_{t-1}(X_{t-1}), y, z$ uniformly at random to obtain the new configuration C_t. In this case, the probability that X_{t-1} is paired to its previous pair $C_{t-1}(X_{t-1})$ is $1/3$, and hence Condition 4.1.4(D2) is not satisfied. Another possibility is to let α_n depend on X_{t-1}. Suppose that we are given a sequence $(\alpha_{n,x})_{x\in H}$, and $Z_t = 1$ with probability $\alpha_{n,x}$ conditioned on $X_{t-1} = x$. In this case Condition 4.1.4(D1) is violated.

4. Local vs. global rewiring mechanisms: The rewiring mechanism of the random walk with local rewiring model can be seen as a *'local-to-global'* rewiring mechanism: one end of the rewired edge is selected *'locally'* at the position of the random walk, while the other end is selected *'globally'* from the set of all possible half-edges. On the other hand, the rewiring mechanism of the random walk on the dynamic configuration model introduced in [12], can be seen as a *'global-to-global'* rewiring mechanism, in the same sense. The effects of local versus global choices are best seen in the tail probabilities of the randomised stopping time τ. In the random walk on the dynamic configuration model, we had $\mathbb{P}_{x,\xi}(\tau > t) = (1-\alpha)^{t(t+1)/2} + o(1)$ whp in x and ξ, where the $t(t+1)/2$ term comes from the cumulative effect of doing a global rewiring at each step.

It would be interesting to study rewiring mechanisms that interpolate between these two examples. One possibility is to consider a model in which some of the half-edges in a neighborhood of the random walk are paired to randomly chosen half-edges. Formally, let $B^r_\xi(x)$ be the set of half-edges that can be reached from x by a random walk of at most r steps on the configuration ξ. Suppose that, at each time t, every half-edge in $B^r_{C_{t-1}}(X_{t-1})$ is rewired independently with probability α. The case $r = 0$ would correspond to the random walk with local rewiring model, while the case $r = \infty$ would correspond to a global-to-global rewiring mechanism similar to the rewiring mechanism of the dynamic configuration model. In between these two extremes, we expect to see that tail probabilities of τ interpolating between that of the random walk with local rewiring model and the random walk on the dynamic configuration model.

5. Comparison with the switch chain: The rewiring mechanism of the random walk with local rewiring model can be seen as a variation of the switch chain of [34]. There are two main differences:

- in the switch Markov chain, the switching edges are chosen uniformly at random from all possible pairs, while in the random walk with local rewiring model one of the switching edges is chosen according to the random walk,
- in the switch Markov chain, the underlying graph is forced to be simple, while in the random walk with local rewiring model, multiple edges and self-loops are allowed.

It would be interesting to study a variation of the random walk with local rewiring model in which the simplicity of the graph is preserved. The main challenge would be to deal with the combinatorial contraints that are imposed by the preservation of the simplicity.