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# Exploration on and of networks

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# CHAPTER 3

# Random walks on dynamic configuration models: a trichotomy

This chapter is based on a joint article with Luca Avena, Remco van der Hofstad and Frank den Hollander [13].

**Abstract**

We consider a dynamic random graph on $n$ vertices that is obtained by starting from a random graph generated according to the configuration model with a prescribed degree sequence and at each unit of time randomly rewiring a fraction $\alpha_n$ of the edges. We are interested in the mixing time of a random walk without backtracking on this dynamic random graph in the limit as $n \to \infty$, when $\alpha_n$ is chosen such that $\lim_{n\to\infty} \alpha_n(\log n)^2 = \beta \in [0, \infty]$. In [12] we found that, under mild regularity conditions on the degree sequence, the mixing time is of order $1/\sqrt{\alpha_n}$ when $\beta = \infty$. In the present paper we investigate what happens when $\beta \in [0, \infty)$. It turns out that the mixing time is of order $\log n$, with the scaled mixing time exhibiting a one-sided cutoff when $\beta \in (0, \infty)$ and a two-sided cutoff when $\beta = 0$. The occurrence of a one-sided cutoff is a rare phenomenon. In our setting it comes from a competition between the time scales of mixing on the static graph, as identified by Ben-Hamou and Salez [16], and the regeneration time of first stepping across a rewired edge.

# §3.1 Introduction

## §3.1.1 Background

The goal of the present paper is to study the mixing time of a random walk *without backtracking* on a dynamic version of the configuration model. The *static* configuration model is a random graph with a prescribed degree sequence. For random walk on the static configuration model, with or without backtracking, the asymptotics of the associated mixing time, and related properties such as the presence of the so-called cutoff phenomenon, were derived recently by Berestycki, Lubetzky, Peres and Sly [21], and by Ben-Hamou and Salez [16]. In particular, under mild assumptions on the degree sequence, guaranteeing that the graph is an *expander* with high probability, the mixing time was shown to be of order $\log n$, with $n$ the number of vertices.

In an earlier paper [12], we consider a *discrete-time dynamic* version of the configuration model, where at each unit of time a fraction $\alpha_n$ of the edges is sampled and rewired uniformly at random. Our dynamics *preserves the degrees* of the vertices. Consequently, when considering a random walk on this dynamic configuration model, its *stationary distribution remains constant over time* and the analysis of its mixing time is a well-posed question. It is natural to expect that, due to the graph dynamics, the random walk *mixes faster* than the $\log n$ order known for the static model. Under very mild assumptions on the prescribed degree sequence (Condition 3.1.2 below), we have shown that this is indeed the case when $(\alpha_n)_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} \alpha_n(\log n)^2 = \infty$, which corresponds to a regime of 'fast enough' graph dynamics. In particular, we have shown that for every $\varepsilon \in (0,1)$ the $\varepsilon$-mixing time grows like $\sqrt{2\log(1/\varepsilon)/\alpha_n}$ as $n \to \infty$ (when also $\lim_{n\to\infty} \alpha_n = 0$), with high probability (in the sense of Definition 3.1.1 below).

In the present paper we look at a slower dynamics, namely, $(\alpha_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} \alpha_n(\log n)^2 = \beta \in [0,\infty)$. Our main result (Theorem 3.1.4 below) states that, under somewhat stronger assumptions on the prescribed degree sequence (Condition 3.1.3 below), the mixing time is of order $\log n$, as for the static model, but that there is an interesting difference between the cases $\beta \in (0,\infty)$ and $\beta = 0$. Our proof builds on the strategy developed in [12] for the regime of fast dynamics. However, the argument in [12] establishing the almost self-avoiding nature of the random walk cannot be immediately extended to the regime of slow dynamics. This difficulty is overcome by using a different proof, in combination with an annealing argument (see Section 3.3).

The rest of the paper is organised as follows. In Section 3.1.2 we define the model. This is a verbatim repetition of what was written in [12, Section 1.2], in which we introduce notation and set the stage. In Section 3.1.3 we state our main theorem, which is a *trichotomy* for the cases $\beta = \infty$, $\beta \in (0,\infty)$ and $\beta = 0$. In Section 3.1.4 we place this theorem in its proper context.

Throughout the sequel we use standard notations for the asymptotic comparison of functions $f, g\colon \mathbb{N} \to [0,\infty)$: $f(n) = O(g(n))$ or $g(n) = \Omega(f(n))$ when $\limsup_{n\to\infty} f(n)/g(n) < \infty$; $f(n) = o(g(n))$ or $g(n) = \omega(f(n))$ when $\lim_{n\to\infty} f(n)/g(n) = 0$; $f(n) = \Theta(g(n))$ when both $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

## §3.1.2 Model

Since this section was a verbatim repetition of Section 2.1.2, we remove it and refer to Section 2.1.2 whenever necessary.

## §3.1.3 A trichotomy

We are interested in the behaviour of the total variation distance between the distribution of $X_t$ and the uniform distribution

$$\mathcal{D}_{\eta,x}(t) := \|\mathbb{P}_{\eta,x}(X_t \in \cdot\,) - U_H(\cdot)\|_{\mathrm{TV}}. \tag{3.1}$$

Note that $\mathcal{D}_{\eta,x}(t)$ depends on the initial configuration $\eta$ and half-edge $x$. We will prove statements that hold for *typical* choices of $(\eta, x)$ under the uniform distribution $\mu_n$ (recall that $H$ depends on the number of vertices $n$) given by

$$\mu_n := \mathrm{Conf}_H \times U_H \quad \text{ on } \mathit{Conf}_H \times H, \tag{3.2}$$

where *typical* is made precise through the following definition:

**Definition 3.1.1 (With high probability).** *A statement that depends on the initial configuration $\eta$ and initial half-edge $x$ is said to hold* with high probability (whp) *in $\eta$ and $x$ if the $\mu_n$-measure of the set of pairs $(\eta, x)$ for which the statement holds tends to 1 as $n \to \infty$.*

### Regularity conditions

In Theorem 3.1.4 below we use two sets of regularity conditions on the degree sequence:

**Condition 3.1.2.** ***(Regularity of degrees)***

(R1) *$\ell$ is even and $\ell = \Theta(n)$ as $n \to \infty$.*

(R2) *$\limsup_{n\to\infty} \nu_n < \infty$, where*

$$\nu_n := \frac{\sum_{z\in H} \deg(z)}{\ell} = \frac{\sum_{v\in V} d(v)[d(v)-1]}{\sum_{v\in V} d(v)} \tag{3.3}$$

*denotes the expected forward degree of a uniformly chosen half-edge.*

(R3) *$d(v) \geq 2$ for all $v \in V$.*

**Condition 3.1.3.** ***(Regularity of degrees (Cont.))***

(R1*) *$d_{\max} = \ell^{o(1)}$ as $n \to \infty$, where*

$$d_{\max} := \max_{v\in V} d(v). \tag{3.4}$$

(R2*) *As* $n \to \infty$,

$$\frac{\lambda_2}{\lambda_1^3} = \omega\left(\frac{(\log\log \ell)^2}{\log \ell}\right), \qquad \frac{\lambda_2^{3/2}}{\lambda_3\sqrt{\lambda_1}} = \omega\left(\frac{1}{\sqrt{\log \ell}}\right), \tag{3.5}$$

*where*

$$\lambda_1 := \frac{1}{\ell}\sum_{z\in H} \log(\deg(z)), \qquad \lambda_m := \frac{1}{\ell}\sum_{z\in H} |\log(\deg(z)) - \lambda_1|^m, \quad m = 2, 3. \tag{3.6}$$

(R3*) $d(v) \geq 3$ *for all* $v \in V$.

Condition 3.1.2 was used in [12] to deal with the regime of 'fast graph dynamics'. Conditions (R1) and (R2) are minimal requirements to guarantee that the graph is locally tree-like. Condition (R3) ensures that the random walk without backtracking is well-defined. Condition 3.1.3 was used in [16] to deal with the regime of no graph dynamics, i.e., the static graph. Condition (R1*) provides control on the large degrees. Condition (R2*) is technical and states that the degrees vary neither too little nor too much. Condition (R3*) ensures that the graph is connected with high probability and that there are no nodes where the random walk without backtracking moves deterministically.

Below, we will work under the Conditions (R1)–(R3) as well as (R1*)–(R3*). If $D_n = d(V_n)$ denotes the degree of a random vertex, then Condition (R2*) is implied by the often used condition that $D_n \to D$ in distribution (when $\mathbb{P}(D \geq 3) > 0$), together with $\mathbb{E}[D_n] \to \mathbb{E}[D]$ (see e.g. van der Hofstad [93, Chapter 7]). Thus, Condition (R2*) is rather mild. Condition (R1*) excludes vertices with a degree that is a positive power of $n$, which is claimed to be realistic for real-world networks (see e.g. [93, Chapter 1] for an extensive introduction). We have a truncation argument, along the lines of the one in Berestycki, van der Hofstad and Salez [22], showing that the degrees can be truncated and the random walk is unlikely to notice this truncation. However, the truncated graph may have vertices of degree 2, so that it is not clear how to apply the results in Ben-Hamou and Salez [16]. Furthermore, we believe that Condition (R3*) is unnecessary for our results. We state it here because we rely on the work of [16], which considers random walk without backtracking started from the worst-possible starting point. When there is a positive proportion of vertices of degree 2, the configuration model is bound to contain a long path of such vertices. On such a stretch, the walk moves deterministically, but it slows down the mixing because it takes time $\omega(\log n)$ to leave the stretch. Thus, mixing would occur at a time that is $\omega(\log n)$ larger than that when the walk starts from a uniform vertex, which makes worst-case and average-case mixing different. Still, since our walk starts from the uniform measure on half-edges, it is unlikely to encounter such a stretch. We refrain from investigating this issue further.

### Main theorem

Define the proportion of rewired edges per unit of time as

$$\alpha_n := k/m, \qquad n \in \mathbb{N}, \tag{3.7}$$

where $m = \ell/2$ is the total number of edges and $k$ is the number of edges that get rewired per unit of time. For the static model ($\alpha_n \equiv 0$), under Condition 3.1.3, the $\varepsilon$-mixing time $\inf\{t \in \mathbb{N}_0 \colon \mathcal{D}_{\eta,x}(t) \leq \varepsilon\}$ is known to scale like $[1 + o(1)]\, c_{n,\mathrm{stat}} \log n$ for all $\varepsilon \in (0,1)$, with $c_{n,\mathrm{stat}} = 1/\lambda_1 \in (0,\infty)$ (Ben-Hamou and Salez [16]). If Condition 3.1.2 holds too, then $n \mapsto c_{n,\mathrm{stat}}$ is bounded away from 0 and $\infty$. If also the degree distribution tends to a limit, then $\lim_{n\to\infty} c_{n,\mathrm{stat}} = c_{\mathrm{stat}} \in (0,\infty)$.

Our main theorem shows that the above behaviour turns into a *trichotomy* for the dynamic model:

**Theorem 3.1.4 (Scaled mixing profiles).** *Suppose that* $\lim_{n\to\infty} \alpha_n (\log n)^2 = \beta \in [0,\infty]$. *The following hold whp in* $\eta$ *and* $x$*:*

(1) *Subject to Condition* 3.1.2*, if* $\beta = \infty$*, then*

$$\mathcal{D}_{\eta,x}\big(c\alpha_n^{-1/2}\big) = \mathrm{e}^{-c^2/2} + o(1), \quad c \in [0,\infty). \tag{3.8}$$

(2) *Subject to Condition* 3.1.2(R1) *and Condition* 3.1.3*, if* $\beta \in (0,\infty)$*, then*

$$\mathcal{D}_{\eta,x}\big(c \log n\big) = \begin{cases} \mathrm{e}^{-\beta c^2/2} + o(1), & c \in [0, c_{n,\mathrm{stat}}), \\ o(1), & c \in (c_{n,\mathrm{stat}}, \infty). \end{cases} \tag{3.9}$$

(3) *Subject to Condition* 3.1.2(R1) *and Condition* 3.1.3*, if* $\beta = 0$*, then*

$$\mathcal{D}_{\eta,x}\big(c \log n\big) = \begin{cases} 1 - o(1), & c \in [0, c_{n,\mathrm{stat}}), \\ o(1), & c \in (c_{n,\mathrm{stat}}, \infty). \end{cases} \tag{3.10}$$

The proof of Theorem 3.1.4 is organised as follows. Theorem 3.1.4(1) was already proved in [12]. In Section 3.2 we show that Theorems 3.1.4(2)–(3) follow from a key proposition (Proposition 3.2.1 below), which will be proved in Sections 3.3–3.4. In Section 3.3 we show that on scale $\log n$ with high probability the random walk is self-avoiding, i.e., does not visit the same vertex twice, and that the same holds for a version of the random walk with random resets. In Section 3.4 we compute probabilities of rewiring histories and of self-avoiding paths conditional on rewiring histories.

## §3.1.4 Discussion

**1**. Theorem 3.1.4 gives the sharp asymptotics of the mixing profiles in three regimes, which we refer to as *supercritical* ($\beta = \infty$), *critical* ($\beta \in (0,\infty)$) and *subcritical* ($\beta = 0$). The latter includes the case of the static configuration model. While in the supercritical regime the mixing time is of order $1/\sqrt{\alpha_n} = o(\log n)$, in the critical and the subcritical regime it is of order $\log n$ (see Fig. 3.1). Note that for $\beta = \infty$ the scaling does not depend on the degrees, while for $\beta \in [0,\infty)$ it does via the constant $c_{n,\mathrm{stat}}$.

**2**. For the static model, because the scaling of the $\varepsilon$-mixing time does not depend on $\varepsilon \in (0,1)$ (Ben-Hamou and Salez [16]) there is *two-sided cutoff*, i.e., the total variation

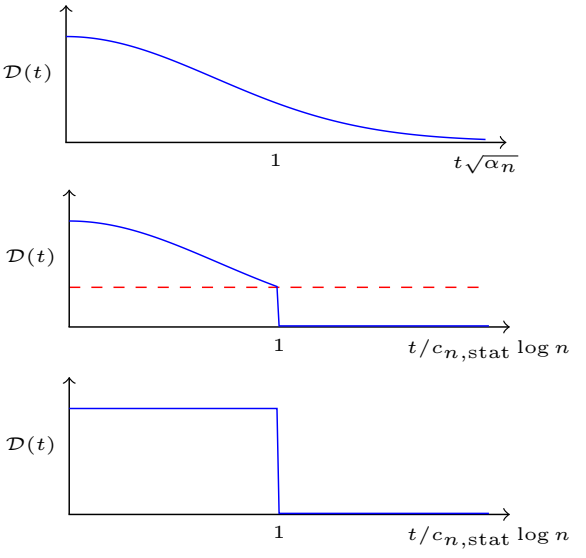


*Figure 3.1: Plot of $\mathcal{D}(t)$ on time scale $1/\sqrt{\alpha_n}$ for $\beta = \infty$, respectively, on time scale $c_{n,\mathrm{stat}}\log n$ for $\beta \in (0,\infty)$ and $\beta = 0$. Because the scaling holds whp in $\eta$ and $x$, we have suppressed these indices.*

distance drops from 1 to 0 in a time window of width $o(\log n)$. Theorem 3.1.4 shows that this behaviour persists throughout the subcritical regime, but that in the critical regime the drop is not from height 1 but from height $< 1$, i.e., there is *one-sided cut-off*. In contrast, in the supercritical regime there is *no cutoff*, i.e., the total variation distance drops from 1 to 0 gradually on scale $1/\sqrt{\alpha_n}$.

**3**. We emphasize that we look at the mixing times for 'typical' initial conditions and at the distribution of the random walk averaged over the trajectories of the graph process: the 'annealed' model. It would be interesting to investigate different setups, such as 'worst-case' mixing, in which the maximum of the mixing time over all initial conditions is considered, or the 'quenched' model, in which the entire trajectory of the graph process is fixed instead of just the initial configuration. In such setups the results can be drastically different. For example, we might consider an initial configuration in which every vertex has a maximal number of self-loops, which would give a maximal component size of 2, and the initial position is a half-edge of an isolated vertex with small degree. In such a situation, we have to wait at least until one of the half-edges of the isolated vertex is rewired, and this time can be of order of $1/\alpha_n$, which is much larger than $1/\sqrt{\alpha_n}$. Another interesting example is to consider a uniformly sampled initial configuration, with a worst-case starting location for the random walk. We may expect our results to carry over because the mixing-time estimates of Ben-Hamou and Salez [16] hold for worst-case initial positions. However, to show that this is true we would require more sophisticated techniques, since the underlying graph changes at each step of the dynamics.

**4**. It would be of interest to extend our results to random walk *with backtracking*. This is much harder. Indeed, because the configuration model is locally tree-like and random walk without backtracking on a tree is the same as self-avoiding walk, in our proof we can exploit the fact that typical walk trajectories are self-avoiding. In contrast, for the random walk with backtracking, after it jumps over a rewired edge, which in our model serves as a randomized stopping time, it may jump back over the same edge, in which case it has not mixed. This problem remains to be resolved.

## §3.2 Stopping time decomposition

As in [12], the proof is based on a randomized stopping time argument. Let

$$\tau \coloneqq \min\{t \in \mathbb{N}\colon\ X_{t-1} \in R_{\leq t}\}. \tag{3.11}$$

where $R_{\leq t} \coloneqq \cup_{s=1}^{t} R_s$ is the set of rewired edges up to time $t$. By the triangle inequality, we have

$$\begin{aligned}\mathcal{D}_{\eta,x}(t) \leq\ & \mathbb{P}_{\eta,x}(\tau > t)\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H\|_{\text{TV}} \\ & + \mathbb{P}_{\eta,x}(\tau \leq t)\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t) - U_H\|_{\text{TV}}\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\mathcal{D}_{\eta,x}(t) \geq\ & \mathbb{P}_{\eta,x}(\tau > t)\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H\|_{\text{TV}} \\ & - \mathbb{P}_{\eta,x}(\tau \leq t)\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t) - U_H\|_{\text{TV}}.\end{aligned} \tag{3.13}$$

**Proposition 3.2.1 (Closeness to stationarity and stopping time tails).** *Suppose that Condition* 3.1.2(R1) *and Condition* 3.1.3 *hold and that* $\beta \in [0,\infty)$. *If* $t = t(n) = [1 + o(1)]\, c \log n$ *for some* $c \in (0,\infty)$, *then whp in* $\eta$ *and* $x$,

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H(\cdot)\|_{\text{TV}} = \begin{cases} 1 - o(1), & c \in [0, c_{n,\text{stat}}), \\ o(1), & c \in (c_{n,\text{stat}}, \infty), \end{cases} \tag{3.14}$$

$$\mathbb{P}_{\eta,x}(\tau > t) = (1 - \alpha_n)^{t(t+1)/2} + o(1). \tag{3.15}$$

*If, in addition,* $k = k(n) = \omega((\log n)^2)$, *then*

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t) - U_H(\cdot)\|_{\text{TV}} = o(1). \tag{3.16}$$

We show how Theorems 3.1.4(2)–(3) follow from Proposition 3.2.1:

*Proof of Theorem* 3.1.4(2)–(3). First we prove (3.9). Under the condition $\lim_{n\to\infty} \alpha_n (\log n)^2 = \beta \in (0,\infty)$, since $m = \Theta(n)$ we have $k = \omega((\log n)^2)$, and so we can use all three items of Proposition 3.2.1. From (3.12), (3.13) and (3.16) it follows that, for any $t = [1 + o(1)]\, c \log n$,

$$\mathcal{D}_{\eta,x}(t) = \mathbb{P}_{\eta,x}(\tau > t)\|\, \mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H\|_{\text{TV}} + o(1). \tag{3.17}$$

Since $\lim_{n\to\infty} \alpha_n = 0$ and $t\alpha_n = o(1)$, by (3.15) also

$$\mathbb{P}_{\eta,x}(\tau > t) = (1-\alpha_n)^{t(t+1)/2} + o(1) = \exp(-\alpha_n t^2/2) + o(1). \tag{3.18}$$

Since $\alpha_n = [1+o(1)]\,\beta/(\log n)^2$, (3.18) together with (3.14) gives us

$$\mathcal{D}_{\eta,x}(t) = \begin{cases} \exp(-\beta c^2/2) + o(1), & c \in [0, c_{n,\mathrm{stat}}), \\ o(1), & c \in (c_{n,\mathrm{stat}}, \infty). \end{cases} \tag{3.19}$$

Next, we prove (3.10). If $\lim_{n\to\infty} \alpha_n (\log n)^2 = \beta = 0$, then by (3.15), for any $t = [1+o(1)]\,c\log n$,

$$\mathbb{P}_{\eta,x}(\tau > t) = \exp(-\alpha_n t^2/2) + o(1) = 1 - o(1), \qquad \mathbb{P}_{\eta,x}(\tau \le t) = o(1). \tag{3.20}$$

Inserting (3.20) into (3.12) and (3.13), we get

$$\mathcal{D}_{\eta,x}(t) = [1-o(1)]\,\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H\|_{\mathrm{TV}} + o(1). \tag{3.21}$$

Using (3.14), we obtain

$$\mathcal{D}_{\eta,x}(t) = \begin{cases} 1 - o(1), & c \in [0, c_{n,\mathrm{stat}}), \\ o(1), & c \in (c_{n,\mathrm{stat}}, \infty). \end{cases} \tag{3.22}$$

□

## §3.3 Self-avoiding trajectories

In this section, we show that the random walk trajectories are self-avoiding on the relevant time scales with high probability. We let $\mathsf{SA}_t$ denote the event $\{v(X_s) \neq v(X_{s'})$ for any $0 \le s < s' \le t\}$, i.e., no two half-edges are incident to the same vertex along the trajectory up to time $t$.

Along the way we need a random walk on the static model that is a slightly modified version of the random walk without backtracking. This version will be instrumental in the proof of our main theorem. For fixed $t \in \mathbb{N}$, let $[t] := \{1, \dots, t\}$ and define the $t$-step *modified random walk* starting from configuration $\eta$ and half-edge $x$ as follows:

(a) Let $\mathcal{T}$ be a random subset of $[t]$ drawn according to a probability mass function $(p_t(T))_{T\subset[t]}$ with $p_t(\varnothing) \in (0,1)$ for all $t$ (to be defined later on).

(b) At each time $s \in [t]$, if $s \notin \mathcal{T}$, then the random walk makes a non-backtracking move in configuration $\eta$, while if $s \in \mathcal{T}$, then it jumps to a uniformly chosen half-edge (possibly the half-edge it is on).

This is a random walk without backtracking that resets its position to a uniformly chosen half-edge at certain random times. We denote its law by $\mathbb{P}^{\mathrm{mod}}_{\eta,x}$, and put $\mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_0 = x) = 1$. Note that, although the distribution of this random walk depends on $t$ and on the distribution of $\mathcal{T}$, we suppress these from the notation.

If we condition on the event that $\mathcal{T} \neq \varnothing$, then the modified random walk makes a uniform jump at some time in $[t]$ after which it becomes stationary, and so

$$\mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t \in \cdot \mid \mathcal{T} \neq \varnothing) = U_H(\cdot). \tag{3.23}$$

On the other hand, if we condition on the event that $\mathcal{T} = \varnothing$, then the modified random walk is the same as the random walk without backtracking on the static graph given by configuration $\eta$ starting from $x$. Denoting the law of the latter by $\mathbb{P}^{\mathrm{stat}}_{\eta,x}$, we have

$$\mathbb{P}^{\mathrm{mod}}_{\eta,x}(\cdot \mid \mathcal{T} = \varnothing) = \mathbb{P}^{\mathrm{stat}}_{\eta,x}(\cdot). \tag{3.24}$$

The main result of this section is the following lemma:

**Lemma 3.3.1.** *Suppose that Condition* 3.1.2(R1) *and Condition* 3.1.3(R1*) *hold and that* $t = [1 + o(1)]\, c \log n$ *for some* $c \in (0, \infty)$. *Then whp in* $\eta$ *and* $x$,

$$\mathbb{P}_{\eta,x}(\mathsf{SA}_t) = 1 - o(1), \qquad \mathbb{P}^{\mathrm{mod}}_{\eta,x}(\mathsf{SA}_t) = 1 - o(1). \tag{3.25}$$

*Proof.* The proof uses two exploration processes on the graph with the help of the two random walks in the annealed setting. Recall that $\mu_n = \mathit{Conf}_H \times U_H$. The annealed measures for the two random walks are defined as

$$\mathbb{P}(\cdot) := \sum_{\eta,x} \mu_n(\eta, x)\, \mathbb{P}_{\eta,x}(\cdot), \qquad \mathbb{P}^{\mathrm{mod}}(\cdot) := \sum_{\eta,x} \mu_n(\eta, x)\, \mathbb{P}^{\mathrm{mod}}_{\eta,x}(\cdot). \tag{3.26}$$

First, we describe the exploration process for the random walk on the dynamic configuration model. To compute the probability of a self-avoiding path, we keep track of already explored half-edges. The exploration process proceeds as follows:

(a) At time $s = 0$, choose $x$ uniformly at random from $H$, set $X_0 = x$ and set $A_0$ to be the set containing $x$ and all its siblings (the set of 'active' half-edges at time 0).

(b) At each time $s \in [t]$, reveal the pair of $X_{s-1} = x_{s-1}$ in $C_s$, say $y_{s-1}$. Denote the edge $\{x_{s-1}, y_{s-1}\}$ by $e_s$. Add $y_{s-1}$ and all its siblings to $A_{s-1}$ to obtain $A_s$ (the set of 'active' half-edges at time $s$); some siblings may already have been added in a previous step.

(c) Choose one of the siblings of $y_{s-1}$ uniformly at random, say $x_s$, and set $X_s = x_s$.

This procedure builds up the trajectory of the random walk while ignoring what happens in the rest of the graph. Note that we only pair the half-edges along the trajectory, while the siblings of the half-edges along the trajectory are not paired until they are visited by the random walk.

Under this construction, the first time the random walk is not self-avoiding is the first time the revealed pair at step 2 is in the set of active half-edges. Hence we want to bound the probability

$$\mathbb{P}\big(C_s(x_{s-1}) \in A_{s-1} \mid e_i \in C_i, i \in [s-1]\big), \tag{3.27}$$

where $e_1, \ldots, e_{s-1}$ form a self-avoiding path. For any $y \in H \setminus \{x_{s-1}\}$, if $y$ is not paired up to time $s$, then it can be paired to $x_{s-1}$ through the initial pairing at

time 0 or through rewiring at later times. Since the initial pairing is uniform and this distribution is stationary under the graph dynamics, for all such $y$ the above conditional probability is the same, and so we have

$$\mathbb{P}\big(C_s(x_{s-1}) = y \mid e_i \in C_i, i \in [s-1]\big) \leq \frac{1}{\ell - 2s + 1}, \tag{3.28}$$

where we note that $2(s-1)$ half-edges are paired before time $s$. On the other hand, if $y \in H \setminus \{x_{s-1}\}$ is already paired before time $s$, then it can be paired to $x_{s-1}$ only through rewiring. Hence the same probability is less than it is for an unpaired half-edge, and so we have the same upper bound. Summing over $A_{s-1}$, we get

$$\mathbb{P}\big(C_s(x_{s-1}) \in A_{s-1} \mid e_i \in C_i, i \in [s-1]\big) \leq \frac{|A_{s-1}| - 1}{\ell - 2s + 1} \leq \frac{s d_{\max}}{\ell - 2s + 1}, \tag{3.29}$$

where we use that at each time we activate at most $d_{\max} = \max_{v \in V} d(v)$ half-edges. Finally, since $d_{\max} = n^{o(1)}$ by Condition 3.1.3(R1*), $t = [1+o(1)]\, c \log n$ and $\ell = \Theta(n)$ by Condition 3.1.2(R1), via a union bound and summing over $s \in [t]$, we get

$$\mathbb{P}(\mathsf{SA}_t^c) \leq \frac{d_{\max} t(t+1)/2}{\ell - 2t + 1} = o(1), \tag{3.30}$$

which establishes the left-hand side of (3.25). Indeed, by the Markov inequality, for any $(w_n)_{n \in \mathbb{N}}$ that tends to zero arbitrarily slowly, we have

$$\mu_n(\mathbb{P}_{\eta,x}(\mathsf{SA}_t^c) > w_n) \leq \frac{\mathbb{P}(\mathsf{SA}_t^c)}{w_n}, \tag{3.31}$$

which implies that, with $\mu_n$-probability at least $1 - o(1)$,

$$\mathbb{P}_{\eta,x}(\mathsf{SA}_t) = 1 - o(1). \tag{3.32}$$

Next, we describe the exploration process for the modified random walk. Again, we let $A_t$ denote the set of active half-edges. Now, instead of random rewirings, we have a static configuration chosen randomly according to the configuration model, and we have a set of random times $\mathcal{T} \subset [t]$ at which the random walk makes uniform jumps. The exploration process proceeds as follows:

(a) At time $s = 0$, choose $x$ uniformly at random from $H$, set $X_0 = x$ and let $A_0$ be the set containing $x$ and all its siblings. Choose also $\mathcal{T} \subset [t]$ randomly with probabilities $(p_t(T))_{T \subset [t]}$.

(b) At each time $s \in [t]$, we do the following:

  (a) If $s \in [t] \setminus \mathcal{T}$, then reveal the pair of $X_{s-1} = x_{s-1}$ in $\eta$, say $y_{s-1}$. Add $y_{s-1}$ and all its siblings to $A_{s-1}$ to obtain $A_s$. Choose one of the siblings of $y_{s-1}$ uniformly at random, say $x_t$, and set $X_s = x_s$.

  (b) If $s \in \mathcal{T}$, then choose $x_s$ uniformly at random from $H$, set $X_s = x_s$, add $x_s$ and all its siblings to $A_{s-1}$ to obtain $A_s$.

Under this construction, the first time the random walk is not self-avoiding is the first time we either have that the revealed pair at step 2(a) is in the set of active half-edges or that the random walk jumps to an active half-edge at step 2(b). We look at the probability

$$\mathbb{P}^{\mathrm{mod}}(X_s \in A_{s-1} \mid X_{[0,s-1]} = x_{[0,s-1]}), \tag{3.33}$$

where $x_{[0,s-1]}$ is a self-avoiding segmented path. We see that if $s \in \mathcal{T}$, then this probability is $|A_{s-1}|/\ell$. Otherwise it is at most $(|A_{s-1}| - 1)/(\ell - 2s + 1)$, and so we get

$$\mathbb{P}^{\mathrm{mod}}\big(X_s \in A_{s-1} \mid X_{[0,s-1]} = x_{[0,s-1]}\big) \leq \frac{|A_{s-1}|}{\ell - 2s + 1} \leq \frac{sd_{\max}}{\ell - 2s + 1}. \tag{3.34}$$

This bound agrees with (3.30), so we get the same conclusion for $\mathbb{P}^{\mathrm{mod}}$. Hence, with $\mu_n$-probability at least $1 - o(1)$,

$$\mathbb{P}^{\mathrm{mod}}_{\eta,x}(\mathsf{SA}_t) = 1 - o(1). \tag{3.35}$$

□

The proof for the modified random walk can easily be adapted to the random walk without backtracking on the static graph, simply by removing step 2(b) in the exploration process for the modified random walk. Hence we also have, whp in $\eta$ and $x$,

$$\mathbb{P}^{\mathrm{stat}}_{\eta,x}(\mathsf{SA}_t) = 1 - o(1). \tag{3.36}$$

## §3.4 Proof of the main proposition

In this section, we prove Proposition 3.2.1. We use the notation introduced in [12] and recall some of the definitions that are needed along the way.

For a fixed sequence of half-edges $x_{[0,t]}$ with $x_0 = x$ and a fixed set of times $T \subseteq [t]$, we use the short-hand notation

$$A(x_{[0,t]}; T) \coloneqq \big\{x_{i-1} \in R_{\leq i} \ \forall\, i \in T,\, x_{j-1} \notin R_{\leq j} \ \forall\, j \in [1,t] \setminus T\big\}, \tag{3.37}$$

where $R_{\leq i}$ denotes the set of half-edges that are rewired up to time $i$. This event gives us the rewiring history for the sequence of half-edges $x_{[0,t]}$. More precisely, it is the event that, for $i \in [t] \setminus T$, the half-edge $x_{i-1}$ in not rewired until time $i$, and, for $i \in T$, the half-edge $x_{i-1}$ is rewired at some time before or at time $i$.

We say that a sequence $x_{[0,t]}$ of half-edges of length $t$ is a *self-avoiding segmented path* in the configuration $\eta$ with respect to $T = \{t_1, \ldots, t_r\} \subset [t]$ if $x_{[0,t]}$ is self-avoiding, meaning that no two half-edges in $x_{[0,t]}$ are siblings, and each subsequence $x_{[t_{i-1},t_i-1]}$ induces a path in $\eta$ for $i \in [r+1]$ with $t_0 = 0$ and $t_{r+1} = t + 1$. We denote by $\mathsf{SP}^{\eta}_t(x, y; T)$ the set of all self-avoiding segmented paths in $\eta$ with respect to $T$ with $x_0 = x$ and $x_t = y$ (see Fig. 3.2) and by $\mathsf{SP}^{\eta}_t(x; T)$ the set of all self-avoiding segmented paths in $\eta$ with respect to $T$ with $x_0 = x$. Note that for $T = \varnothing$ these are simply the sets of self-avoiding paths.

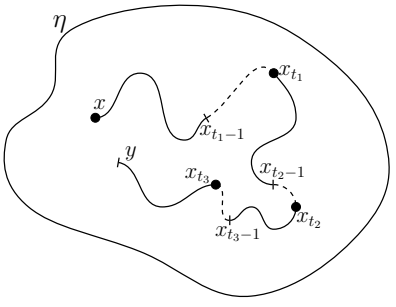


*Figure 3.2: An element of* $\mathsf{SP}_t^{\eta}(x, y; T)$ *with* $T = \{t_1, t_2, t_3\}$.

Lemmas 3.4.1 and 3.4.2 below are slight modifications of [12, Lemmas 3.1–3.2] and will be instrumental in the proof of Proposition 3.2.1. The first lemma is concerned with the probabilities of the rewiring histories of self-avoiding segmented paths:

**Lemma 3.4.1 (Rewiring histories of self-avoiding segmented paths).** *Fix* $t \in \mathbb{N}$, $T \subseteq [t]$ *and* $\eta, \zeta \in \mathrm{Conf}_H$. *Suppose that* $x_{[0,t]}$ *and* $y_{[0,t]}$ *are two self-avoiding segmented paths in* $\eta$ *and* $\zeta$, *respectively, of length* $t+1$. *Then*

$$\mathbb{P}_{\eta,x}\big(A(x_{[0,t]}; T)\big) = \mathbb{P}_{\zeta,y}\big(A(y_{[0,t]}; T)\big). \tag{3.38}$$

*Proof.* The proof follows the same line of argument as in the proof of [12, Lemma 3.1] and uses a coupling between two dynamic configuration models. Let $f$ be a one-to-one map from $H$ to itself with the property that it maps $x_i$ to $y_i$ for all $i \in [0, t]$, and preserves the edges between two configuration $\eta$ and $\zeta$, i.e., $f(\eta(x)) = \zeta(f(x))$ for all $x \in H$. The Markovian coupling $(C_t^x, C_t^y)_{t\in\mathbb{N}_0}$, where $C_0^x = \eta$ and $C_0^y = \zeta$, proceeds at every step $t \in \mathbb{N}$ as follows:

(a) Choose $k$ edges from $C_{t-1}^x$ uniformly at random without replacement, say $\{z_1, z_2\}$, $\ldots, \{z_{2k-1}, z_{2k}\}$. Choose the edges $\{f(z_1), f(z_2)\}, \ldots, \{f(z_{2k-1}), f(z_{2k})\}$ from $C_{t-1}^y$.

(b) Rewire the half-edges $z_1, \ldots, z_{2k}$ uniformly at random to obtain $C_t^x$ and set $C_t^y(f(z_i)) = f(C_t^x(z_i))$.

Since under the coupling the event $A(x_{[0,t]}; T)$ on $\eta$ is the same as the event $A(y_{[0,t]}; T)$ on $\zeta$, we get the desired result. □

From this lemma we see that the probability of a specific rewiring history for a self-avoiding segmented path does not depend on the path itself nor on the configuration: it only depends on $t$ and $T$. In what follows we set $p_t(T) = \mathbb{P}_{\eta,x}(A(x_{[0,t]}, T))$ for which it can be easily seen that

$$\mathbb{P}_{\eta,x}(A(x_{[0,t]}, T)) > 0 \text{ and } \sum_{T\subset[t]} \mathbb{P}_{\eta,x}(A(x_{[0,t]}, T)) = 1 \text{ for all } T \subset [t].$$

When we refer to the modified random walk we will use these probabilities as the distribution for the random times $\mathcal{T}$.

The second lemma is concerned with path probabilities for the random walk conditioned on the rewiring history:

**Lemma 3.4.2 (Paths estimate given rewiring history).** *Suppose that* $t = t(n) = [1 + o(1)]\, c \log n$ *for some* $c \in (0, \infty)$, $k = k(n) = \omega((\log n)^2)$ *and* $T = \{t_1, \dots, t_r\} \subseteq [t]$. *Let* $x_0 \cdots x_t \in \mathsf{SP}_t^\eta(x, y; T)$ *be a self-avoiding segmented path in* $\eta$ *that starts at* $x$ *and ends at* $y$. *Then*

$$\mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]} \mid A(x_{[0,t]}; T)\big) \geq \frac{1 - o(1)}{\ell^r} \prod_{i \in [1,t] \setminus T} \frac{1}{\deg(x_i)}. \tag{3.39}$$

*Proof.* The proof follows the same line of argument as the proof of [12, Lemma 3.2]. □

We continue with the proof of Proposition 3.2.1. We start by proving the result on the tail probabilities of $\tau$, since this is easier.

▷ *Proof of* (3.15). Using (3.25), we see that whp in $\eta$ and $x$

$$\mathbb{P}_{\eta,x}(\tau > t) - o(1) \leq \mathbb{P}_{\eta,x}(\mathsf{SA}_t, \tau > t) \leq \mathbb{P}_{\eta,x}(\tau > t). \tag{3.40}$$

On the other hand, by considering all possible self-avoiding paths,

$$\begin{aligned}
\mathbb{P}_{\eta,x}(\mathsf{SA}_t, \tau > t) &= \sum_{x_{[0,t]} \in \mathsf{SP}_t^\eta(x;\varnothing)} \mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]}, A(x_{[0,t]}; \varnothing)\big) \\
&= \sum_{x_{[0,t]} \in \mathsf{SP}_t^\eta(x;\varnothing)} \left( \prod_{i=1}^{t} \frac{1}{\deg(x_i)} \right) \mathbb{P}_{\eta,x}\big(A(x_{[0,t]}; \varnothing)\big) \\
&= p_t(\varnothing)\, \mathbb{P}_{\eta,x}^{\mathrm{stat}}(\mathsf{SA}_t),
\end{aligned} \tag{3.41}$$

where in the second line we use that

$$\mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]} \mid A(x_{[0,t]}; \varnothing)\big) = \prod_{i=1}^{t} \frac{1}{\deg(x_i)} \tag{3.42}$$

and in the third line that these are the path probabilities for the random walk without backtracking in the static model. By following the proof of [12, Eq. (2.6)], we also get

$$\mathbb{P}_{\eta,x}\big(A(x_{[0,t]}; \varnothing)\big) = (1 - \alpha_n)^{t(t+1)/2} + o(1). \tag{3.43}$$

Combining this with (3.36), we obtain

$$\mathbb{P}_{\eta,x}(\mathsf{SA}_t, \tau > t) = (1 - \alpha_n)^{t(t+1)/2} + o(1), \tag{3.44}$$

and the claim follows from (3.40). □

▷ *Proof of* (3.14). Fix $y \in H$. We have

$$\begin{aligned}\mathbb{P}_{\eta,x}(X_t = y, \mathsf{SA}_t, \tau > t) &= \sum_{x_{[0,t]} \in \mathsf{SP}_t^{\eta}(x,y;\varnothing)} \mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]}, \tau > t\big) \\ &= \sum_{x_{[0,t]} \in \mathsf{SP}_t^{\eta}(x,y;\varnothing)} \left(\prod_{i=1}^{t} \frac{1}{\deg(x_i)}\right) \mathbb{P}_{\eta,x}\big(A(x_{[0,t]};\varnothing)\big) \qquad (3.45) \\ &= p_t(\varnothing)\, \mathbb{P}^{\mathrm{stat}}_{\eta,x}(X_t = y, \mathsf{SA}_t).\end{aligned}$$

Using the third line of (3.41), we obtain

$$\mathbb{P}_{\eta,x}(X_t = y \mid \mathsf{SA}_t, \tau > t) = \mathbb{P}^{\mathrm{stat}}_{\eta,x}(X_t = y \mid \mathsf{SA}_t). \qquad (3.46)$$

On the other hand, by partitioning according to $\mathsf{SA}_t$ and $\mathsf{SA}_t^c$ and using that $\mathbb{P}_{\eta,x}(\mathsf{SA}_t) = 1 - o(1)$, we obtain

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - \mathbb{P}_{\eta,x}(X_t \in \cdot \mid \mathsf{SA}_t, \tau > t)\|_{\mathrm{TV}} = o(1), \qquad (3.47)$$

and

$$\|\mathbb{P}^{\mathrm{stat}}_{\eta,x}(X_t \in \cdot\,) - \mathbb{P}^{\mathrm{stat}}_{\eta,x}(X_t \in \cdot \mid \mathsf{SA}_t)\|_{\mathrm{TV}} = o(1). \qquad (3.48)$$

Combining these relations with (3.46), we obtain

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - \mathbb{P}^{\mathrm{stat}}_{\eta,x}(X_t \in \cdot)\|_{\mathrm{TV}} = o(1). \qquad (3.49)$$

Using the results of [16] for the random walk without backtracking in the static configuration model, we see that if $t = [1 + o(1)]c \log n$ with $c \in (0, c_{n,\mathrm{stat}})$, then

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H\|_{\mathrm{TV}} = 1 - o(1), \qquad (3.50)$$

while if $t = [1 + o(1)]\, c \log n$ with $c \in (c_{n,\mathrm{stat}}, \infty)$, then

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H\|_{\mathrm{TV}} = o(1). \qquad (3.51)$$

□

▷ *Proof of* (3.16). Fix $y \in H$ and suppose that $k = k(n) = \omega((\log n)^2)$. Using Lemmas 3.4.1 and 3.4.2,

$$\begin{aligned}&\mathbb{P}_{\eta,x}(X_t = y, \mathsf{SA}_t, \tau \le t) \\ &\quad = \sum_{r=1}^{t} \sum_{\substack{T \subseteq [1,t] \\ |T| = r}} \sum_{x_{[0,t]} \in \mathsf{SP}(x,y;T)} \mathbb{P}_{\eta,x}\big(X_{[0,t]} = x_{[0,t]} \mid A(x_{[0,t]};T)\big)\, \mathbb{P}_{\eta,x}\big(A(x_{[0,t]};T)\big) \\ &\quad \ge [1 - o(1)] \sum_{r=1}^{t} \sum_{\substack{T \subseteq [1,t] \\ |T| = r}} p_t(T) \sum_{x_{[0,t]} \in \mathsf{SP}(x,y;T)} \left(\prod_{i \in [t] \setminus T} \frac{1}{\deg(x_i)}\right) \frac{1}{\ell^r}. \qquad (3.52)\end{aligned}$$

We immediately note that

$$\mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t = y, \mathsf{SA}_t \mid \mathcal{T} = T) = \sum_{x_{[0,t]} \in \mathsf{SP}(x,y;T)} \left( \prod_{i \in [t] \setminus T} \frac{1}{\deg(x_i)} \right) \frac{1}{\ell^{|T|}}, \tag{3.53}$$

and so

$$\mathbb{P}_{\eta,x}(X_t = y, \mathsf{SA}_t, \tau \leq t) \geq [1 - o(1)]\, \mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t = y, \mathsf{SA}_t, \mathcal{T} \neq \varnothing). \tag{3.54}$$

Using (3.24), (3.25) and (3.41), whp in $\eta$ and $x$, we also have

$$\begin{aligned} \mathbb{P}_{\eta,x}(\mathsf{SA}_t, \tau \leq t) &= \mathbb{P}_{\eta,x}(\mathsf{SA}_t) - \mathbb{P}_{\eta,x}(\mathsf{SA}_t, \tau > t) \\ &\leq \mathbb{P}^{\mathrm{mod}}_{\eta,x}(\mathsf{SA}_t) + o(1) - p_t(\varnothing)\, \mathbb{P}^{\mathrm{stat}}_{\eta,x}(\mathsf{SA}_t) \\ &= \mathbb{P}^{\mathrm{mod}}_{\eta,x}(\mathsf{SA}_t) + o(1) - \mathbb{P}^{\mathrm{mod}}_{\eta,x}(\mathsf{SA}_t, \mathcal{T} = \varnothing) \\ &= \mathbb{P}^{\mathrm{mod}}_{\eta,x}(\mathsf{SA}_t, \mathcal{T} \neq \varnothing) + o(1). \end{aligned} \tag{3.55}$$

Combining this with (3.54) we get, for any $y \in H$,

$$\mathbb{P}_{\eta,x}(X_t = y \mid \mathsf{SA}_t, \tau \leq t) \geq [1 - o(1)]\, \mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t = y \mid \mathsf{SA}_t, \mathcal{T} \neq \varnothing). \tag{3.56}$$

which in turn gives

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \mathsf{SA}_t, \tau \leq t) - \mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t \in \cdot \mid \mathsf{SA}_t, \mathcal{T} \neq \varnothing)\|_{\mathrm{TV}} = o(1). \tag{3.57}$$

On the other hand, (3.25) gives

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \mathsf{SA}_t, \tau \leq t) - \mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t)\|_{\mathrm{TV}} = o(1), \tag{3.58}$$

and

$$\|\mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t \in \cdot \mid \mathsf{SA}_t, \mathcal{T} \neq \varnothing) - \mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t \in \cdot \mid \mathcal{T} \neq \varnothing)\|_{\mathrm{TV}} = o(1). \tag{3.59}$$

Finally, from the latter two relations in combination with (3.23) and (3.57), we get

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t) - U_H(\cdot)\|_{\mathrm{TV}} = o(1), \tag{3.60}$$

which is the desired result. □

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