Exploration on and of networks

Güldas, H.

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Hakan Güldaş

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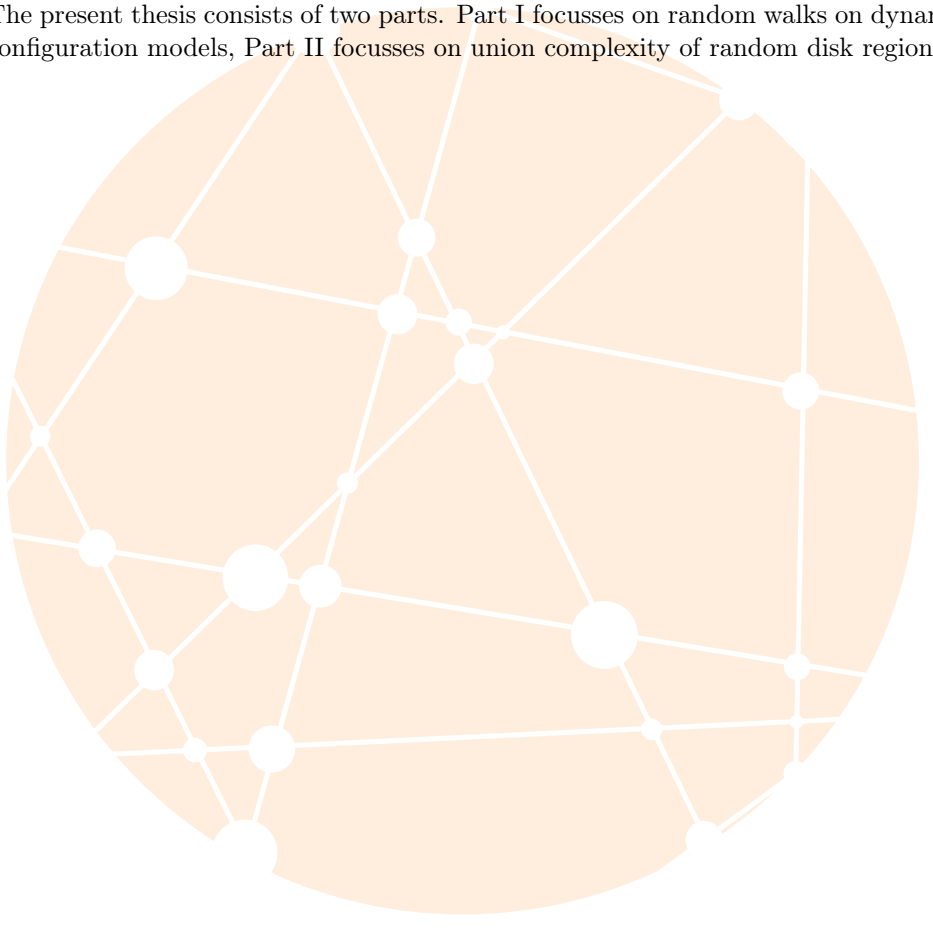
Contents

CHAPTER 1
Introduction

The present thesis consists of two parts. Part I focusses on random walks on dynamic configuration models, Part II focusses on union complexity of random disk regions.



§1.1 Introduction to Part I

The theory of random graphs provides abstract mathematical tools for studying real-world problems related to complex networks. Questions concerning the formation, structure and functionality of real-world networks have been studied via random graph models. Many questions involve also a random process on the network. Mathematical notions for modelling random processes on networks include random walks, percolation and interacting particle systems. These notions have been studied extensively in the context of *static* random graphs. However, many real-world networks are *dynamic* in nature: their structure changes over time. Thus, it is natural to extend static settings to dynamic settings. Part I of the thesis contains our work in this direction: it studies mixing properties of random walks on dynamic random graphs.

In Section 1.1.1, we give an overview of random graph theory in the context of complex networks. We start by discussing the historical development of the study of complex networks. After that, we review two basic random graph models, the Erdős-Rényi Random Graph model and the configuration model, which are among the most widely studied models. We close Section 1.1.1 by discussing, very briefly, other random graph models that were invented in the context of complex networks. In Section 1.1.2, we give a review of the theory of random walks on static random graphs, with a focus on mixing times. We introduce the notion of mixing times in the more general framework of Markov chains and discuss several aspects. We close Section 1.1.2 by reviewing the literature on mixing times of random walks on random graphs. In Section 1.1.3, we review the modest literature on random walks on dynamic random graphs that has accumulated so far and list our contributions.

§1.1.1 Complex networks and random graphs

Since 1999, there has been a surge of interest in the study of real-world networks, and network science has emerged as an independent academic discipline. A number of journals are devoted to the field, and many universities offer courses and special programs on network science and *complex networks*. The term *'complex networks'* was used by Steven Strogatz in his 2001 paper [90] to refer to networks with non-trivial topological features, unlike regular networks such as chains, grids, lattices and fully-connected graphs. Many real-world networks are complex in this sense.

The research on complex networks was pioneered in a series of papers by Barabási and co-authors [14, 15, 3], Watts and Strogatz [98], and Newman and Watts [76, 77]. Their aim was to explain the mechanisms behind the formation of networks that give rise to certain statistical features that are observed in real-world networks, but cannot be explained by randomness alone. Empirical studies of real-world networks have shown that many real-world networks have certain key features in common, such as:

- **Sparsity:** the average number of links per node does not grow too large with the network size.
- **Power-law degrees:** the empirical distribution of the number of edges per ver-

tex follows a *power-law* distribution. In a power-law distribution, the frequency of vertices with k edges is roughly proportional to $k^{-\gamma}$ for some $\gamma > 2$ and for large k.

- **Small-world phenomenon:** the distance between typical vertex pairs does not grow too fast with the size of the network.
- **High clustering:** if two vertices are connected to a third vertex, then it is likely that these two vertices are connected to each other as well.

The above properties are observed in networks in many different settings, such as the World Wide Web, social networks and biological networks. For a review of early studies on real-world networks and their statistical properties, see [2, 78].

Why is it that different networks have so many features in common and how do these features affect the functionality of the network, for example, the vulnerability of a computer network or the spread of a disease in a population? The mathematical tools that can tackle these problems are provided by the mathematical theory of random graphs. Before going into details, we review two random graph models that are widely studied in the mathematics community.

Erdős-Rényi random graph model

The theory of random graphs was initiated by Paul Erdős and Alfred Rényi in their seminal papers dating back to 1959–1961 [42, 43, 44]. Although Erdős and Rényi studied random graphs for their own sake, their random graph model was motivated by the use of the *probabilistic method* for solving problems in extremal graph theory; see for example [41]. The probabilistic method can be summarised as the use of probabilistic arguments to prove deterministic statements, often in the form of existence results of combinatorial structures with certain properties. The basic principle is as follows: if a random combinatorial structure has a certain property with a positive probability, then there exists at least one deterministic structure with that property. The canonical reference for the probabilistic method is Alon and Spencer's textbook on the subject [8].

The remarkable discovery of Erdős and Rényi was the *threshold phenomenon* (also called *phase transition*) exhibited by random graphs. Their goal was to determine the number of edges a random graph must have in order to acquire a certain property, such as containing a set of given subgraphs, containing a *'giant'* component or being connected. They showed that these properties emerge rather suddenly and formalised this emergence in mathematical terms. Let us denote by $\mathrm{ER}_n(m)$ the random graph distributed uniformly over the set of graphs on n vertices with m edges. In the formal setting, one considers a family of random graphs whose number of vertices n tends to infinity and whose number of edges $m = m(n)$ depends on the number of vertices. In [43], Erdős and Rényi showed that $\mathrm{ER}_n(m)$ acquires certain properties, with high probability, only when the number of edges m grows faster than some threshold function $A(n)$, and they call this *threshold phenomenon*.

The line of research initiated by Erdős and Rényi was later continued by others. The body of results with a similar flavor and of techniques used for proving these

results make up the core of the monograph entitled *Random Graphs* by Bollobás [25]. In fact, the majority of the works study a different but related model. For $p \in [0,1]$, let $\mathrm{ER}_n(p)$ be the random graph on n vertices where each of the $\binom{n}{2}$ edges is present with probability p, independently of each other. Although, in their original papers, Erdős and Rényi studied $\mathrm{ER}_n(m)$, the model with $\mathrm{ER}_n(p)$ is referred to as the *Erdős-Rényi Random Graph (ERRG)* model. The two models are practically interchangeable when m is close to $p\binom{n}{2}$ (see [25, Theorem 2.2]), and $\mathrm{ER}_n(p)$ also exhibits threshold phenomena for many properties. One particular feature of $\mathrm{ER}_n(p)$ in the sparse regime (where $p = \lambda/n$ for some $\lambda \in (0,\infty)$) is that the local neighborhoods of the vertices are tree-like in the following sense: the neighborhood of a uniformly chosen vertex in $\mathrm{ER}_n(p)$ can be coupled to the family tree of a branching process with a Poisson offspring distribution. This can be used to prove the phase transition for the emergence of a giant component in $\mathrm{ER}_n(p)$ (see [93, Chapter 4]).

While the ERRG model exhibits many mathematically interesting phenomena, it fails to capture many features exhibited by real-world networks. When $p = \lambda/n$ for some $\lambda \in (0,\infty)$, $\mathrm{ER}_n(p)$ is sparse, the average degree converges in probability to λ as n tends to ∞, and the distribution of the degree of a fixed vertex is close to a Poisson distribution with parameter λ for large n. Most real-world networks have power-law degree distributions, thus the ERRG model cannot capture the scale-free property of real-world networks. Another drawback of the model is that it is completely symmetric, in the sense that the vertices have the same degree distribution. To remedy these shortcomings, generalisations of ERRG model have been suggested, such as the generalised random graph model [29], the inhomogeneous random graph model [26], the Chung-Lu model [33] and the Norros-Reittu model [79]. For an extensive review, see [93, Chapter 6].

Configuration model

One of the problems with the ERRG model and its generalisations is that the resulting graph contains isolated vertices with positive probability in the sparse regime, which makes them impractical in many cases for the study of real-world networks. One possible solution is to fix the degrees of the graph beforehand and generate a random graph with the prescribed degree sequence. The *configuration model* is one such model. Let $\mathbf{d} = (d_1, \dots, d_n)$ be the given degree sequence on n vertices. The configuration model is constructed by attaching d_i half-edges to vertex v_i for each $i \in [n] \coloneqq \{1, \dots, n\}$ and then pairing these half-edges uniformly at random. A pairing of the half-edges is called a *configuration* and a uniformly distributed configuration is denoted by $\mathrm{Conf}_n(\mathbf{d})$. The resulting graph is denoted by $\mathrm{CM}_n(\mathbf{d})$, which need not be *simple*. A simple graph is a graph that does not contain any loop (an edge with the same vertex at both ends) or multiple edges between any pair of vertices. For graphs that are not necessarily simple, the term *multi*-graph is used. In the rest of this section, we use the term graph to refer to multi-graphs.

In the configuration model, each configuration is identified with a graph, and a graph can be obtained via several distinct configurations. Letting $\ell_n = \sum_{i\in[n]} d_i$, we see that there are $(\ell_n - 1)!! \coloneqq (\ell-1) \times (\ell-3) \times \cdots \times 3 \times 1$ distinct configurations in

total, so each configuration has probability $\frac{1}{(\ell_n-1)!!}$. We can identify a graph G with a matrix $(X_{ij})_{i,j\in[n]}$, where X_{ij} denotes the number of edges between vertices v_i and v_j for $i \neq j$ and X_{ii} the number of self-loops at v_i. This gives

$$d_i = X_{ii} + \sum_{j\in[n]} X_{ij}, \quad i \in [n].$$

The distribution of $\mathrm{CM}_n(\mathbf{d})$ is given by the following proposition:

Proposition ([93, Proposition 7.7]). *Let $G = (x_{ij})_{i,j\in[n]}$ be a graph on n vertices such that*

$$d_i = x_{ii} + \sum_{j\in[n]} x_{ij} \text{ for } i \in [n].$$

Then

$$\mathbb{P}(\mathrm{CM}_n(\mathbf{d}) = G) = \frac{1}{(\ell_n-1)!!} \frac{\prod_{i\in[n]} d_i!}{\prod_{i\in[n]} 2^{x_{ii}} \prod_{1\le i\le j\le n} x_{ij}!}.$$

From this, we see that $\mathrm{CM}_n(\mathbf{d})$ is not distributed uniformly over the set of graphs on n vertices with degree sequence $\mathbf{d}$. However, when G is a simple graph, i.e. when $x_{ii} = 0$ for all $i \in [n]$ and $x_{ij} \in \{0, 1\}$ for all $i, j \in [n]$, we have

$$\mathbb{P}(\mathrm{CM}_n(\mathbf{d}) = G) = \frac{\prod_{i\in[n]} d_i!}{(\ell_n-1)!!}. \tag{1.1}$$

This shows that the distribution of $\mathrm{CM}_n(\mathbf{d})$ conditionally on being simple is uniform on the set of simple graphs on n vertices with degree sequence $\mathbf{d}$.

The configuration model was introduced by Bollobás [24] to count labelled simple regular graphs. This work was inspired by ideas introduced by Bender and Canfield [17]. To count the labelled simple graphs, we can use the configuration model as follows. We know that, conditional on being simple, $\mathrm{CM}_n(\mathbf{d})$ is distributed uniformly on the set of simple graphs with degree sequence $\mathbf{d}$. Letting $N_n(\mathbf{d})$ denote the number of simple graphs with degree sequence $\mathbf{d}$, we see that for any simple graph G with degree sequence $\mathbf{d}$,

$$\mathbb{P}(\mathrm{CM}_n(\mathbf{d}) = G \mid \mathrm{CM}_n(\mathbf{d}) \text{ is simple}) = \frac{1}{N_n(\mathbf{d})}.$$

Combining this with (1.1), we see that

$$N_n(\mathbf{d}) = \frac{(\ell_n-1)!!}{\prod_{i\in[n]} d_i!} \mathbb{P}(\mathrm{CM}_n(\mathbf{d}) \text{ is simple}).$$

So, if we can estimate the probability that the resulting graph is simple, then we can estimate the number of simple graphs with the given degree sequence. This is exactly how Bollobás obtained his result on the number of simple regular graphs.

The configuration model and uniform simple graphs with given degree sequences were later studied by others. Łuczak showed that when the minimum degree is three, the resulting graph is connected with a probability tending to 1 as n tends to infinity

[68]. Molloy and Reed studied the emergence of a giant component and its size for sparse random graphs with nonregular degree sequences [72, 73]. They established the phase transition for the giant component, similar to that of the ERRG model. In fact, some results for $\mathrm{ER}_n(p)$ and $\mathrm{ER}_n(m)$ can be obtained through the results for uniform random graph with a given degree sequence, by noting that the latter has the same distribution as the former conditioned on the degree sequence. Fernholz and Ramachandran [48] established that the diameter of the uniform random graph with a given degree sequence on n vertices is $c \log n + o(\log n)$, for some $c > 0$ depending on the degree sequence, under certain sparsity conditions. Using this result and the above observation, they refined earlier results on the diameter of the $\mathrm{ER}_n(p)$ [32, 26]. The diameter and typical distances for the configuration model, especially for the scale-free degree sequences, was also studied by van der Hofstad and co-authors (see for example [31, 95, 96]).

When studying the configuration model, one typically considers fixed degree sequences indexed by the number of vertices n, i.e. $(\mathbf{d}_n = (d_i^n)_{i\in[n]})_{n\in\mathbb{N}}$ with certain statistical properties, and studies properties of $\mathrm{CM}_n(\mathbf{d}_n)$ as n tends to infinity. A typical condition on the degrees is the convergence of the empirical degree distribution to a deterministic limit distribution. Let D_n be the random variable whose distribution function is given by

$$F_n(x) = \sum_{i=1}^{n} \mathbf{1}_{\{d_i^n \le x\}},$$

i.e., D_n is the degree of a uniformly chosen vertex on the graph with n vertices whose degree sequence is given by $\mathbf{d}_n = (d_i^n)_{i\in[n]}$ and F_n is the empirical distibution of the degrees. The following regularity conditions on the degrees are common:

Condition 1.1.1 (Regularity of the degrees)**.** *There exists an integer-valued random variable D with $\mathbb{P}(D > 0) = 1$ such that*

(a) $D_n \xrightarrow{d} D$ *as* $n \to \infty$*, where* $\xrightarrow{d}$ *denotes convergence in distribution,*

(b) $\lim_{n\to\infty} \mathbb{E}[D_n] = \mathbb{E}[D] < \infty$*,*

(c) $\lim_{n\to\infty} \mathbb{E}[D_n^2] = \mathbb{E}[D^2] < \infty$*.*

In the context of complex networks, the limiting degree distribution D is usually assumed to follow a power-law distribution. Conditions 1.1.1(a)-(b) ensure that the degree sequences are sparse. Condition 1.1.1(c) controls the variance of the degrees and the maximum degree. Under Conditions 1.1.1(a)-(c), the asymptotics of the probability that $\mathrm{CM}_n(\mathbf{d_n})$ is a simple graph is given by the following theorem:

Theorem 1.1.2 ([93, Theorem 7.12])**.** *Assume that the sequence of degree sequences $(\mathbf{d}_n)_{n\in\mathbb{N}}$ satisfies Conditions 1.1.1(a)-(c). Then the probability that $\mathrm{CM}_n(\mathbf{d}_n)$ is simple converges to*

$$\mathrm{e}^{-\nu/2-\nu^2/4},$$

where $\nu = \mathbb{E}[D(D-1)]/\mathbb{E}[D]$.

Using this result and the above arguments about the number of simple graphs with a given degree sequence, we see that, under Conditions 1.1.1(a)-(c), $N_n(\mathbf{d_n})$ grows

asymptotically as

$$\frac{(\ell_n - 1)!!}{\prod_{i\in[n]} d_i^n!} \mathrm{e}^{-\nu/2-\nu^2/4}.$$

As in the ERRG model, one of the characteristics of the configuration model is its tree-like structure. The neighborhood of a uniformly chosen vertex can be coupled to the family tree of a branching process whose offspring distribution is related to the degree sequence. Again, this can be used to prove results about the phase transition for the giant component, typical distances and the diameter (see [94, Chapters 3 and 6]). The tree-like structure is best formalized in the framework of *local weak convergence*, which was developed in [20, 6, 5]. Under certain regularity conditions, local weak limits of configuration models and Erdős-Rényi Random Graph models are unimodular Galton-Watson trees [94]. We emphasize the tree-like structure, since it is also important for the study of random walks on these graphs. As we will see later, it plays a crucial role in the analysis of the mixing time of random walks.

Apart from the counting perspective, early studies on the configuration model also focused on the problem of uniformly sampling a graph with a given degree sequence. The simplest approach is to repeatedly sample from the configuration model until the resulting graph is simple. In [100], the author adopts this approach to generate a uniform simple regular graph. However, when the degrees are large this method is prohibitively inefficient, as is evident from Theorem 1.1.2, because ν is large. A sampling procedure is considered to be efficient when the expected running time of the underlying algorithm is polynomial in the size of the graph. Later, McKay and Wormald [71] relaxed the degree constraints by incorporating a *switching* mechanism into this procedure, and Gao and Wormald [52] relaxed the degree constraints further. If one is content with *approximately* uniform samples, then Markov Chain Monte Carlo methods are a fallback option. We will discuss this approach later in more detail, in the context of dynamic random graphs.

While the configuration model provides great flexibility in terms of degree sequences, particularly in the study of scale-free complex networks, it exhibits a low clustering compared to many real-world networks. This is mainly due to its tree-like structure. To overcome this shortcoming, extensions of the configuration model were proposed in which clustering information can be incorporated. Examples of these models include the configuration model with household structure [92], and the hierarchical configuration model [97]. For a more thorough discussion of these extensions, see [93, Chapter 7].

Complex networks and random graphs

A big leap in the study of random graphs came with the works on complex networks appearing in the physics literature in late 1990's. The small-world model of Watts and Strogatz [98] and the preferential attachment model of Barabási and Albert [14] led to an explosion of research in the field of random graphs. The study of complex networks within the physics community was largely based on the empirical study of real-world networks and on simulation of random graph models. These studues usually lack mathematical rigour. However, mathematicians were quick to fill in

the gaps and push the theory forward. For example, Bollobás, Riordan, Spencer and Tusnády [27] showed that the preferential attachment mechanism suggested by Barabási and Albert [14] indeed gives rise to a power-law degree sequence, as they predicted using simulations and a heuristic argument. There have been many works on this theme since 2000, on many different aspects of random graph models related to the phenomena occuring in real-world networks. For a detailed account of these, we refer to the book by Durrett [38] on random graph dynamics and the two books by van der Hofstad [93, 94] on random graphs and complex networks.

§1.1.2 Random walks on static random graphs

In this section, we give an overview of the study of random walks on static random graphs. The classical theory of random walks focuses on random walks on infinite graphs with a simple structure, such as $\mathbb{Z}^d$. The canonical example is the nearest-neighbour simple random walk on $\mathbb{Z}^d$, in which jumps to neighbouring vertices occur with equal probability at each time unit. Results on this type of random walks go back almost a hundred years. In a classical result for the nearest-neighbour simple random walk on $\mathbb{Z}^d$ [84], Pólya established that when $d \leq 2$ the random walk comes back to its initial position infinitely many times with probability one, i.e., the random walk is *recurrent*, while when $d \geq 3$ the random walk visits every vertex at most finitely many times with probability one, i.e., the random walk is *transient*. Also random walks in random environments have been considered. In this setting, the jump probabilities are themselves random. This line of research mostly focuses on qualitative behaviour of the random walk, such as recurrence versus transience, or laws of large numbers, central limit theorems and large deviation principles.

The study of random walks on more general but finite graphs has attracted much attention more recently. In contrast to the classical theory of random walks, this line of research focuses on the asymptotics of the finite-time behaviour of the random walk when the size of the graph tends to infinity. Within this framework, the main objects of study are hitting times, mixing times and cover times. The simplest example is again the simple random walk, in which jumps to neighbouring vertices occur with equal probabilities, as in the case of the simple random walk on $\mathbb{Z}^d$. However, since the degrees need not be the same for all vertices, the jump probabilities are inhomogeneous. These inhomogeneities make it harder to study random walks on general graphs in detail. In a more complicated setting, the underlying graph is also random. While random walks on random graphs can be viewed as an example of random walks in random environments, the research focuses on different settings and different questions, and the mathematical techniques used in the two areas differ greatly. One of the main reasons for this is that the natural questions in finite and infinite graph settings are vastly different.

In this thesis, we concentrate on the mixing time of random walks on random graphs. We first review the mixing time of general Markov chains.

Mixing time of general Markov chains

Let $(X_t)_{t\in\mathbb{N}_0}$ be a time-homogeneous, irreducible and aperiodic Markov chain on a finite state space $\mathcal{S}$ of size n, whose transition probabilities are given by a $n\times n$ matrix P whose rows and columns are indexed by the elements of $\mathcal{S}$, i.e.,

$$\mathbb{P}(X_t = y \mid X_{t-1} = x) = P(x,y) \text{ for all } t \in \mathbb{N} \text{ and } x, y \in \mathcal{S}.$$

Then there exists a unique probability measure π on $\mathcal{S}$, called the *stationary distribution*, which solves the equation

$$\pi P = \pi,$$

where π is viewed as a row vector of length n. Moreover, t-step transition probabilities converge to π as $t \to \infty$, in the following sense:

$$\lim_{t\to\infty} P^t(x,y) = \pi(y) \text{ for any } x, y \in \mathcal{S}.$$

In practice, one is interested in the speed of convergence towards π. The usual way of measuring how far the distribution of the Markov chain is from the stationary distribution is the *total variation* distance. The total variation distance between two probability measures μ and ν on a countable state space $\mathcal{S}$ is defined as

$$\|\mu - \nu\|_{\text{TV}} \coloneqq \frac{1}{2}\sum_{x\in\mathcal{S}} |\mu(x) - \nu(x)|.$$

In the literature, convergence to stationarity is usually studied in the *worst-case* setting as we explain next. Let

$$\mathcal{D}_x(t) \coloneqq \|P^t(x,\cdot) - \pi(\cdot)\|_{\text{TV}} \text{ and } \mathcal{D}(t) = \max_{x\in\mathcal{S}} \mathcal{D}_x(t).$$

It is easy to see that both $\mathcal{D}_x(t)$, for all $x \in \mathcal{S}$, and $\mathcal{D}(t)$, are non-decreasing in t. A classical result states that the convergence to the stationary distribution measured in total variation distance happens exponentially fast:

Theorem (Theorem 4.9 in [65]). *Suppose that P is the matrix of transition probabilities of an irreducible and aperiodic Markov chain on a finite state space $\mathcal{S}$ with stationary distribution π. Then there exist two constants $\alpha \in (0,1)$ and $C > 0$ such that*

$$\mathcal{D}(t) \le C\alpha^t.$$

Using this theorem, we see that the asymptotic rate of exponential convergence can be bounded from above

$$\limsup_{t\to\infty} \mathcal{D}(t)^{1/t} \le \alpha.$$

When the Markov chain is also time-reversible, i.e., if the detailed balance condition

$$\pi(x)P(x,y) = \pi(y)P(y,x), \quad x, y \in \mathcal{S},$$

holds, then the asymptotic rate of exponential convergence is given by

$$\lim_{t\to\infty} \mathcal{D}(t)^{1/t} = \lambda_*,$$

where

$$\lambda_* = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } P \text{ and } \lambda \neq 1\}$$

For a proof of this, see for example [65, Corollary 12.7].

While the above setting describes the asymptotic convergence properties of a fixed Markov chain P as $t \to \infty$, research in the past few decades has focused on the finite-time asymptotics of a family of Markov chains indexed by the size of the state space in the limit as the size tends to infinity. In this setting, a precision level ε is fixed and one usually looks at the first time when the distance to the stationary distribution falls below ε. The *ε-mixing time* is defined as

$$t_{\text{mix}}(\varepsilon) \coloneqq \min\{t \in \mathbb{N} : \mathcal{D}(t) \leq \varepsilon\}.$$

In fact, mixing times at different precision levels can be related to each other by using the sub-multiplicativity of $\mathcal{D}(t)$. Indeed, for all $s, t \in \mathbb{N}$, $\mathcal{D}(s+t) \leq \mathcal{D}(s)\mathcal{D}(t)$, and therefore it is possible to show that (see [65, Section 4.5])

$$t_{\text{mix}}(\varepsilon) \leq \lceil \log_2 \varepsilon^{-1} \rceil t_{\text{mix}}(1/4).$$

The techniques used to study mixing times vary greatly, and in specific examples one has to come up with ad hoc methods. However, there are a few techniques that can be used in more general settings. One such method is *coupling*. A coupling of two probability distributions μ and ν on state spaces $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, is a pair of random variables $(X, Y) \in \mathcal{S}_1 \times \mathcal{S}_2$ such that the marginal distribution of X is μ and the marginal distribution of Y is ν, i.e., $\mathbb{P}(X \in \cdot) = \mu(\cdot)$ and $\mathbb{P}(Y \in \cdot) = \nu(\cdot)$. The notion of coupling is related to total variation distance via the following relation (see [91, Chapter 1]):

$$\|\mu - \nu\|_{\text{TV}} = \inf\{\mathbb{P}(X \neq Y) : (X, Y) \text{ is a coupling of } \mu \text{ and } \nu\}. \tag{1.2}$$

To see how this can be used in the context of mixing times of Markov chains, we need to define another notion of distance to stationarity. Let $\overline{\mathcal{D}}(t) = \max_{x,y \in \mathcal{S}} \|P^t(x, \cdot) - P^t(y, \cdot)\|_{\text{TV}}$. Although the definition of this quantity does not involve the stationary distribution directly, it satisfies the inequalities (see [65, Lemma 4.10])

$$\mathcal{D}(t) \leq \overline{\mathcal{D}}(t) \leq 2\mathcal{D}(t). \tag{1.3}$$

Given a Markov chain on state space $\mathcal{S}$ with transition matrix P, a *Markovian coupling* of two copies of the same Markov chain is a Markov chain $(X_t, Y_t)_{t \in \mathbb{N}_0}$ on $\mathcal{S} \times \mathcal{S}$ such that, for all $x, y, x', y' \in \mathcal{S}$ and $t \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}(X_t = x' \mid X_{t-1} = x, Y_{t-1} = y) &= P(x, x'), \\ \mathbb{P}(Y_t = y' \mid X_{t-1} = x, Y_{t-1} = y) &= P(y, y'). \end{aligned}$$

Consider a Markovian coupling of two copies of the Markov chain given by P, with the additional property

$$\text{if } X_t = Y_t, \text{ then } X_s = Y_s \text{ for all } s \geq t,$$

so that whenever the two components meet they stick together. Let τ_{couple} be the first time the two components meet. Then $\{X_t \neq Y_t\} \subset \{\tau_{\text{couple}} > t\}$ and hence $\mathbb{P}(X_t \neq Y_t) \leq \mathbb{P}(\tau_{\text{couple}} > t)$. Using (1.2) and the first inequality of (1.3), we see that

$$\mathcal{D}(t) \leq \mathbb{P}(\tau_{\text{couple}} > t).$$

For examples of how to obtain upper bounds on mixing times via coupling, see [65, Chapter 5] and references therein.

Another useful tool to obtain upper bounds on mixing times is the notion of *strong stationary times*. A strong stationary time for a Markov chain on state space $\mathcal{S}$ starting from state $x \in \mathcal{S}$ with stationary distribution π is a randomised stopping time τ such that, for all $t \in \mathbb{N}_0$ and $y \in \mathcal{S}$,

$$\mathbb{P}_x(X_t = y, \tau = t) = \pi(y)\mathbb{P}_x(\tau = t),$$

or equivalently,

$$\mathbb{P}_x(X_t = y, \tau \leq t) = \pi(y)\mathbb{P}_x(\tau \leq t),$$

where $\mathbb{P}_x$ denotes the law of the Markov chain starting from $x \in \mathcal{S}$. In words, X_τ has distribution π and is independent of τ. The total variation distance to the stationary distribution can be bounded by the tail probability of the strong stationary time (see [65, Proposition 6.11]):

$$\mathcal{D}_x(t) \leq \mathbb{P}_x(\tau > t).$$

In fact, strong stationary times may be optimal in some sense. To explain this, we need another notion of distance to stationarity, called the *separation distance*, defined by

$$s_x(t) \coloneqq \max_{y \in \mathcal{S}} \left[1 - \frac{P^t(x,y)}{\pi(y)}\right].$$

Let $s(t) = \max_{x \in \mathcal{S}} s_x(t)$. The total variation distance to the stationary distribution can be bounded by the separation distance (see [65, Lemma 6.16])

$$\mathcal{D}_x(t) \leq s_x(t).$$

If the Markov chain is reversible, then (see [65, Lemma 6.17])

$$s(2t) \leq 1 - (1 - \bar{\mathcal{D}}(t))^2 \leq 2\bar{\mathcal{D}}(t) \leq 4\mathcal{D}(t).$$

There are strong stationary times that are optimal in the following sense:

Proposition ([65, Proposition 6.21])**.** *For every starting state $x \in \mathcal{S}$, there exists a strong stationary time τ such that, for all $t \in \mathbb{N}_0$*

$$s_x(t) = \mathbb{P}_x(\tau > t).$$

For examples of how to obtain upper bounds on mixing times via strong stationary times, see [65, Chapter 6] and references therein. In our proofs in Chapters 2,3 and 4, we use a randomised stopping time that is not a strong stationary time, but very close to one.

The usual setting in recent works on mixing times of Markov chains involves a family of Markov chains $P^{(n)}$, $n \in \mathbb{N}$, where $P^{(n)}$ is a Markov chain on a state space $\mathcal{S}^{(n)}$ and the size of the state space tends to infinity as n tends to infinity. Letting $t^{(n)}_{\text{mix}}(\varepsilon)$ denote the ε-mixing time of the Markov chain $P^{(n)}$, one is usually interested in finding an expression for $t^{(n)}_{\text{mix}}(\varepsilon)$ as a function of n. Within this setting, a remarkable phenomenon observed for many different families of Markov chains is the so-called *cutoff* phenomenon, in which $t^{(n)}_{\text{mix}}(\varepsilon)$ virtually does not depend on ε. More precisely, a family of Markov chains is said to exhibit a cutoff phenomenon if, for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{t^{(n)}_{\text{mix}}(\varepsilon)}{t^{(n)}_{\text{mix}}(1-\varepsilon)} = 1.$$

In this case, the distance to the stationary distribution stays near 1 for awhile and subsequently drops to near 0 in a relatively short time window. The notion of *cutoff window* is formalised in the following way. A family of Markov chains is said to have cutoff with a window of size $O(w_n)$ if $w_n = o(t^{(n)}_{\text{mix}}(\frac{1}{4}))$ and

$$\lim_{\alpha\to-\infty} \liminf_{n\to\infty} \mathcal{D}^{(n)}(t^{(n)}_{\text{mix}}(\tfrac{1}{4}) + \alpha w_n) = 1,$$
$$\lim_{\alpha\to\infty} \limsup_{n\to\infty} \mathcal{D}^{(n)}(t^{(n)}_{\text{mix}}(\tfrac{1}{4}) + \alpha w_n) = 0.$$

In early works, Diaconis and Shahshahani [36] established cutoff for uniform transposition random walks on permutation groups, and Aldous [7] for random walks on hypercubes. Later, cutoff was observed for many other Markov chains, including random walks on groups, random walks on random graphs, and interacting particle systems. For a more detailed discussion of the cutoff phenomenon, we refer to [65, Chapter 18].

Mixing times of random walks on static random graphs

The study of mixing times of random walks on random graphs has received considerable attention in last few decades. While these studies are interesting in their own right, they are motivated by problems in combinatorics and computer science [35]. A Markov chain on a state space of size n is said to be *rapidly mixing* if the mixing time is polynomial in $\log n$. Rapid mixing is especially useful in algorithmic applications, where one usually deals with very large state spaces, typically exponential in the system size. In such cases, rapid mixing is used to obtain polynomial time algorithms. Many examples of random walks on random graphs have been shown to be rapidly mixing.

In an early work on mixing times of random walks on random graphs [56], Hildebrand studied simple random walks on random regular graphs with degrees $d = (\log n)^a$ for some $a \geq 2$ and on $\text{ER}_n(p)$ with $p = d/n$, and showed that the mixing time is $\log n / \log d$ in both cases. Later, Benjamini and Mossel [19] studied simple random walk on the part of the infinite cluster of supercritical percolation on $\mathbb{Z}^d$ within the box $[-n, n]^d$. Using the average conductance method developed by Lovász and Kannan [66], they showed that the mixing time is of order n^2. In [51], Fountoulakis

and Reed improved the conductance method of Lovász and Kannan. Using improved estimates, they showed that the mixing time of simple random walk on the giant component of the supercritical Erdős-Rényi random graph is of order $\log^2 n$ [50].

Benjamini, Kozma and Wormald [18] obtained the same result on the mixing time of simple random walk on the giant component by using different techniques, namely, via expansion properties of the giant component. Nachmias and Peres [75] showed that the mixing time of simple random walk on the largest component of $\mathrm{ER}_n(p)$ is of order n within the critical window, i.e., when $p = (1 + \lambda n^{-1/3})/n$ with $\lambda \in \mathbb{R}$. Ding, Lubetzky and Peres [37] studied the near-critical case, where $p = (1+\varepsilon)/n$ with $n^{-1/3} \ll \varepsilon \ll 1$, and showed that the mixing time is of order $\varepsilon^{-3} \log^2(\varepsilon^3 n)$, which interpolates between the critical and the supercritical case.

In 2010, Lubetzky and Sly [67] established cutoff for simple random walk and nonbacktracking random walk on d-regular graphs with $d \geq 3$, and derived precise asymptotics of the total variation distance, via path counting arguments. Nonbacktracking random walk is the same as simple random walk, except that it cannot traverse the same edge twice in a row.

Theorem (Lubetzky and Sly [67])**.** *Let G be a uniformly random d-regular graph on n vertices with $d \geq 3$. Then, for all $\varepsilon \in (0,1)$, with high probability:*

(a) *the mixing time of nonbacktracking random walk on G satisfies*

$$t_{\mathrm{mix}}^{(n)}(\varepsilon) = \log_{d-1}(dn) + O_{\mathbb{P}}(1).$$

(b) *the mixing time of simple random walk on G satisfies*

$$t_{\mathrm{mix}}^{(n)}(\varepsilon) = \frac{d}{d-2} \log_{d-1} n + (\Lambda + o_{\mathbb{P}}(1))\Phi^{-1}(\varepsilon)\sqrt{\log_{d-1} n},$$

where $\Lambda = \frac{2\sqrt{d(d-1)}}{(d-2)^{3/2}}$ *and* $\Phi(x) = \frac{1}{2\pi}\int_x^\infty \exp(-u^2/2)du,\ x \in \mathbb{R}$.

Here, *with high probability* refers to the randomness of the underlying graph and means that the asymptotics holds with a probability that tends to 1 as n tends to infinity. A crucial ingredient in the proof in [67] is the locally tree-like structure of random regular graphs.

Extensions to the configuration model with non-regular degrees were later obtained by Berestycki, Lubetzky, Peres and Sly [21] (for simple and nonbacktracking random walks), and by Ben-Hamou and Salez [16] (for nonbacktracking random walks). In [21], the authors study the mixing time of random walks starting from a *typical* position, instead of a worst-case position. As in [67], they use the locally tree-like structure of the configuration model, but the presence of inhomogeneities in the degrees of vertices requires a more sophisticated approach. In [16], the authors obtain finer asymptotics for the mixing time of nonbacktracking random walk compared to [21], under less restrictive conditions, including the exact order of the cutoff window. The results of [16] are of special interest to us, since we use them in our works on random walks on dynamic random graphs that are presented in Chapters 3 and 4.

§1.1.3 Random walks on dynamic random graphs

The study of random walks on dynamic random graphs involves random walks on graphs that change over time in a random fashion. While this line of research can be considered to be part of the more general framework of random walks in dynamic random environments, the problems and techniques considered are very different. The typical setting in the study of random walks in dynamic random environments involves random walks on $\mathbb{Z}^d$ with the jump probabilities of the random walk changing over time in a random fashion. For a review, see [11, 23] and references therein. Another line of research focuses on random walks on general graphs in which edge or vertex weights change over time, rather than the graph structure itself, such as reinforced random walks (see [63, 81]). Our interest in this thesis lies mainly in models of random walks on graphs whose structure changes over time. For examples of such models see [9, 30, 49, 57, 86].

Mixing times of random walks on dynamic random graphs have been studied only very recently. In [83], Peres, Stauffer and Steif studied the mixing time of random walk on dynamical percolation on $\mathbb{Z}^d$ in the subcritical regime. In the dynamical percolation model, which was introduced by Häggström, Peres and Steif [80], each edge of $\mathbb{Z}^d$ gets refreshed independently at random times, given by exponential clocks, and when refreshed an edge becomes open with probability p and closed with probability $1-p$ independently of the state it had before. Peres, Stauffer and Steif considered the case $p < p_c(\mathbb{Z}^d)$, where $p_c(\mathbb{Z}^d)$ is the critical probability for bond percolation on $\mathbb{Z}^d$, and looked at the problem on the discrete torus with side length n and on the lattice $\mathbb{Z}^d$. By using a regeneration time argument, they estimated the mixing time for the joint Markov chain of the random walk and dynamical percolation.

More recently, Peres, Sousi and Steif [82] considered the same model in the supercritical regime $p > p_c(\mathbb{Z}^d)$. Using the evolving sets method developed by Morris and Peres [74], they obtain an upper bound for the mixing time of the random walk on a quenched realisation of dynamical percolation with the additional constraint that $\theta(p) > 1/2$, where $\theta(p)$ is the probability that the origin belongs to the infinite cluster in bond percolation on $\mathbb{Z}^d$ with probability p. In [89], Sousi and Thomas considered random walk on a dynamical Erdős-Rényi graph in the supercritical regime $p = \lambda/n$ with $\lambda > 1$. The dynamical Erdős-Rényi graph is the same as dynamical percolation, except that the underlying graph is the complete graph on n vertices K_n, instead of $\mathbb{Z}^d$. They showed that the joint Markov chain exhibits cutoff, by showing that the random walk component mixes much faster than the dynamic graph component. The proof was based on a regeneration time argument. More sophisticated variations of the dynamical Erdős-Rényi graph, without the random walk, was also studied very recently in different contexts. In [101], the authors study a variation of the dynamical Erdős-Rényi graph, in which the edge refresh rates also randomly change over time, in the context of complex networks. In [70], the authors study a similar but more complicated model. In particular, they consider the evolution of the number of edges, with explicit results for the corresponding moments, functional central limit theorems and large deviations asymptotics.

The main subject of this thesis is random walks on dynamic random graphs in

which the degrees do not change over time, but the edges are *rewired* randomly. Such dynamic random graph models, without the random walk, were studied earlier in the context of approximate uniform sampling of graphs with given degree sequences. As we have pointed out earlier, one method for sampling graphs is the Markov Chain Monte Carlo (MCMC) method. In MCMC, an ergodic Markov chain, whose stationary distribution is the target sampling distribution, is constructed and run for a sufficiently long time, i.e., longer than the mixing time, so that the resulting sample will be approximately distributed as the target distribution. In order to decide how long MCMC the must run, we need good estimates of mixing times.

The study of MCMC methods and their mixing times in the context of sampling graphs with given degree sequences goes back to the 1990's. In [60], Jerrum and Sinclair gave an algorithm for sampling regular graphs that is based on a Markov chain whose mixing time is polynomial in the number of vertices. Their algorithm works for a large class of degree sequences, in particular, for regular graphs. In [61], Kannan, Tetali and Vempala studied a Markov chain for sampling bipartite graphs and tournaments. Their Markov chain involved a *switch* mechanism, in which endpoints of two randomly chosen edges are switched, provided that the switch results in a simple graph. Using the canonical paths method of Jerrum and Sinclair [88], they showed that the mixing time is polynomial in the number of vertices.

In [34], Cooper, Dyer and Greenhill adapted the switch chain of Kannan et al. to the case of random regular graphs and gave an upper bound for the mixing time that is polynomial in the number of vertices. In [54], Greenhill studied the switch chain for non-regular degree sequences and, in [53], Greenhill and Sfragara studied the switch chain for non-regular simple graphs and directed graphs. They obtained upper bounds for the mixing time via the multicommodity flow argument of Sinclair [87]. In [69], Mahlmann and Schindelhauer introduced a variant of the switch chain, which they called *flip* chain, for sampling regular graphs. Their algorithm is more *local*, in the sense that the random choice of switching edges is made locally, namely, they have to be incident to the same edge. In [46], Feder, Guetz, Mihail and Saberi obtained an upper bound for the mixing time of the flip chain on regular graphs by using a Markov chain comparison argument.

Our contribution

The three chapters of Part I of this thesis are based on our three papers on the mixing time of random walks on dynamic random graphs. In Chapter 2, we present our work on the mixing time of random walks on dynamic configuration models. The dynamic configuration model is a dynamic version the configuration model, in which dynamic random graphs with a fixed degree sequences are generated. A fraction α_n of the edges is rewired at each unit of time, where n is the number of vertices. This model was introduced in [12], which forms the basis of Chapter 2. In Chapter 2, we study the mixing time of the random walk without backtracking on the dynamic configuration model in the *supercritical* regime. By supercritical we mean $\lim_{n\to\infty} \alpha_n(\log n)^2 = \infty$. We show that, under some regularity conditions on the degree sequence, the mixing time is of order $\alpha_n^{-1/2}$, which is of order $o(\log n)$ and hence is much smaller than

the mixing time of the random walk on the static configuration model. To prove our main result, we use a randomised stopping time argument. We consider the first time τ at which the random walk crosses an edge that was rewired before, and show that τ behaves like a strong stationary time. The key part of the proof involves an exploitation of the tree-like structure of the configuration model.

In Chapter 3 of the thesis, we present our work on the extension of the results of [12] to the *critical* and the *subcritical* regimes, which correspond to $\lim_{n\to\infty} \alpha_n(\log n)^2 \in (0,\infty)$ and $\lim_{n\to\infty} \alpha_n(\log n)^2 = 0$, respectively. The mixing time was analysed in [13], which forms the basis of Chapter 3. Together with the main result of [12], we see that there is *trichotomy* for the mixing time of random walks on dynamic configuration models:

- $\lim_{n\to\infty} \alpha_n(\log n)^2 = \infty$: the mixing time is of order $\alpha_n^{-1/2}$ and there is *no cutoff*,
- $\lim_{n\to\infty} \alpha_n(\log n)^2 \in (0,\infty)$: the mixing time is of order $\log n$ and there is *one-sided cutoff*,
- $\lim_{n\to\infty} \alpha_n(\log n)^2 = 0$: the mixing time is of order $\log n$ and there is *two-sided cutoff*.

The latter regime includes the random walk on the static configuration model.

We again use the randomised stopping time argument from [12]. However, because the distances over which the random walk can travel are now of order $\log n$, we cannot rely on the tree-like structure of the configuration model only. Instead, we show that the random walk path is with high probability self-avoiding, which in fact is a consequence of the locally tree-like structure of the configuration model, and this in turn ensures that the randomised stopping time τ behaves like a strong stationary time.

In Chapter 4 of the thesis, we present our work on the mixing time of random walks on more general dynamically rewired random graphs. We consider a more general setting where the vertices and vertex degrees are fixed as before, but the edges are rewired according to a more general prescribed rule. This setting includes the dynamic configuration model as a special case. We show that, under some conditions on the graph dynamics, the total variation distance for the random walk on the dynamically rewired random graph can be linked to the total variation distance for the random walk on the static configuration model. We also introduce a specific model that we call *the random walk with local rewirings*. In this model, the rewiring occurs only *along* the random walk path, with probability α_n for each time unit. Using our result for the general framework, we show that this random walk exhibits the same trichotomy as the random walk on the dynamic configuration model, but on a different time scale. More precisely, we identify three regimes:

- $\lim_{n\to\infty} \alpha_n \log n = \infty$: the mixing time is of order α_n^{-1} and there is *no cutoff*,
- $\lim_{n\to\infty} \alpha_n \log n \in (0,\infty)$: the mixing time is of order $\log n$ and there is *one-sided cutoff*,

- $\lim_{n\to\infty} \alpha_n \log n = 0$: the mixing time is of order $\log n$ and there is two-sided cutoff.

Again, the latter regime includes the random walk on the static configuration model.

To prove the above result, we show that the random walk on the dynamically rewired random graph can be coupled to a *modified* random walk on the static configuration model. While this coupling was implicit in the proof used in [13], we show that the same argument can be used in a more general setting. We show that the total variation distance for the modified random walk can be expressed in terms of the tail probability of the randomised stopping time τ (the first time the random walk crosses an edge that was rewired before) and the total variation distance for the random walk on the static configuration model.

§1.2 Introduction to Part II

In Part II of the thesis, we study the problem of *union complexity* of random disk regions. We have a collection of disks, whose centers are distributed randomly within a region of the plane, and we are interested in the expected number of boundary arcs. Formally, let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a set of n disks in $\mathbb{R}^2$. The set of boundary disks of $\mathcal{D}$, denoted by $\mathsf{BD}(\mathcal{D})$, is the set of disks in $\mathcal{D}$ whose boundaries are not completely covered by other disks, i.e.,

$$\mathsf{BD}(\mathcal{D}) = \{D \in \mathcal{D} : \partial D \setminus \cup_{D' \in \mathcal{D} \setminus \{D\}} D' \neq \emptyset\},$$

where ∂D denotes the boundary of D. A boundary arc of a boundary disk D is a connected component of the set $\partial D \setminus \cup_{D' \in \mathcal{D} \setminus \{D\}} D'$ and union complexity of $\mathcal{D}$ is the total number of boundary arcs of the boundary disks of $\mathcal{D}$. It was shown that, when $\mathcal{D}$ consists of disks, the total number of boundary arcs of $\mathcal{D}$ is bounded from above by $6|\mathsf{BD}(\mathcal{D})| - 12$ [62]. Hence, we study the union complexity by focusing on the number of boundary disks and we state our results in terms of the number of boundary disks.

The union complexity of geometric regions is important in several combinatorial and algorithmic problems in different fields ranging from linear programming and robotics to molecular modeling and geographic information systems [1]. A special case of the union-complexity problem is motivated by an algorithm to compute a *conflict-free colouring* for unit disks. Besides, the union-complexity problem is similar to the problem of the complexity of the convex hull, when the number of vertices of the convex hull of a random point set is studied. In Section 1.2.1, we motivate the union complexity problem in the context of conflict-free colouring of unit disk regions. In Section 1.2.2, we introduce the convex hull problem, review the relevant literature, and show its relation to the union complexity problem. In Section 1.2.3, we briefly report our contribution.

§1.2.1 Conflict-free colouring

The problem of conflict-free colouring was introduced by Even et al. [45] for simple geometric regions . In the general setting, we are given a set $\mathcal{D}$ of regions in the plane. The *coverage area* of $\mathcal{D}$ is defined as $\mathsf{Cov}(\mathcal{D}) := \cup_{D \in \mathcal{D}} D$. For a point $p \in \mathsf{Cov}(\mathcal{D})$, we let $\mathcal{D}(p) := \{D \in \mathcal{D} : p \in D\}$, i.e., the set of regions that contain p. A colouring of $\mathcal{D}$ is a function $\chi : \mathcal{D} \to \mathbb{N}$. A conflict-free (CF) colouring is defined as follows:

Definition 1.2.1. *A conflict-free colouring of $\mathcal{D}$ is a colouring χ of $\mathcal{D}$ such that for every $p \in \mathsf{Cov}(\mathcal{D})$, there exists a $D \in \mathcal{D}(p)$ with $\chi(D) \neq \chi(D')$ for any other $D' \in \mathcal{D}(p)$, i.e., for each point in the coverage area there is a region with a* unique *colour among the regions containing that point. A minimum CF colouring of $\mathcal{D}$ is a CF colouring of $\mathcal{D}$ that uses a minimum number of colours among all possible CF colourings of $\mathcal{D}$ (see Figure 1.1).*

Even et al. [45] show that when $\mathcal{D}$ consists of n unit disks (disks with unit radius), it is always possible to achieve a conflict-free colouring with $O(\log n)$ colours by giving

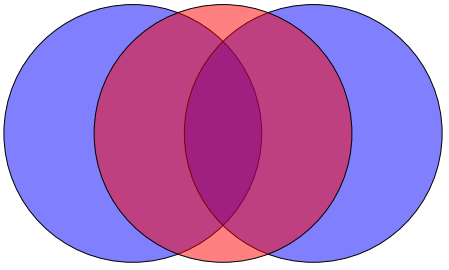


Figure 1.1: A minimum conflict-free colouring of three disks. Two outer disks are coloured blue and the middle disk is coloured red.

an algorithm that uses $O(\log n)$ colours for any set of unit disks in the plane. They also show that when the centres of the disks lie on a straight line and every pair of disks intersects, the minimum number of colours required is $\Omega(\log n)$. So, in the worst-case scenario, minimum CF colourings of unit disks in the plane use $\Theta(\log n)$ colours. In Chapter 5, we consider the average-case scenario for disks in the plane. We take the average over all possible configurations of disks whose centres are uniformly distributed in a convex compact region.

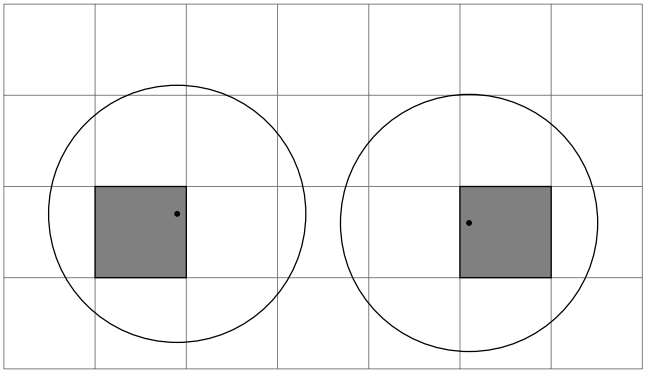


Figure 1.2: Disks whose centres lie at least 3 squares apart do not intersect.

Lev-Tov and Peleg [64] give a constant-factor approximation algorithm for the minimum CF colouring of a set of unit disks in the plane, i.e., an algorithm that achieves a solution that uses, for any given input, at most a constant times more colours than the minimum number of colours needed for that input. The algorithm proceeds by dividing the plane into squares and treating the disks whose centers lie in each square separately. The diagonals of the squares are of length 1, so if a square contains a disk center, then it is completely covered by that disk. Another useful property of the algorithm is that two disks whose centres lie in squares that are 3 squares apart do not intersect (see Figure 1.2). Hence the disks of every fourth square in a row and in a column can be coloured by using the same colour set.

The algorithm given by Lev-Tov and Peleg [64] relies on CF colourings of *chains* of regions. A chain is a set of regions that can be ordered in a natural way. The formal definition is as follows:

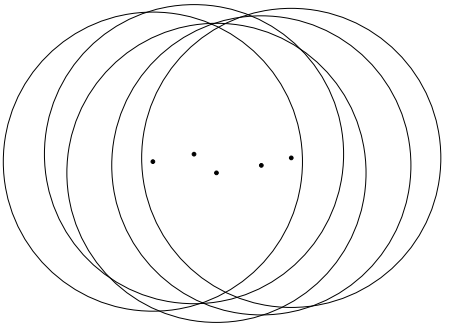


Figure 1.3: A chain of five disks whose centers are marked by crosses.

Definition 1.2.2. *A set $\mathcal{D}$ of n regions is called a* chain *if there exists an indexing of the regions $\mathcal{D} = \{D_1, \ldots, D_n\}$ such that, for every pair $i, j \in [n]$ with $i < j$, the set $\cap_{k\in[i,j]} D_k \setminus \cup_{k\notin[i,j]} D_k$ is non-empty and for every $p \in \mathsf{Cov}(\mathcal{D})$ there exist $i, j \in [n]$ with $i < j$ such that $p \in \cap_{k\in[i,j]} D_k \setminus \cup_{k\notin[i,j]} D_k$ (see Figure 1.3).*

A collection of disks whose centres lie on a line and for which every pair of disks intersect is an example of a chain. It is proven by Even et al. [45] that, for a chain of length n, minimum CF colourings always use $\Theta(\log n)$ colours.

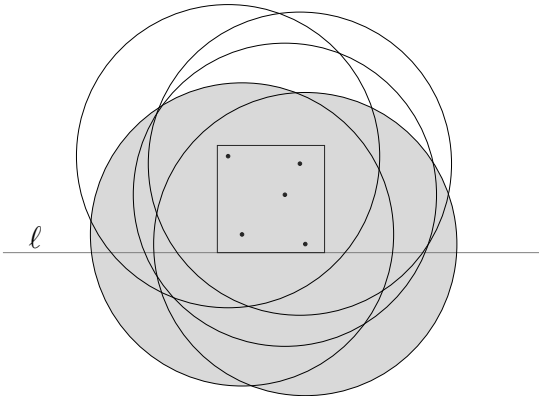


Figure 1.4: Lower boundary diks of five disks highlighted in grap.

In the algorithm given by Lev-Tov and Peleg [64], CF colourings of disks whose centers lie in a square involves CF colourings of chains. For a square Q, let $\mathcal{D}^Q$ denote the set of disks whose centres lie in Q. Let ℓ be the line passing through the lower side of Q. Let B be the intersection of the half plane below ℓ and $\mathsf{Cov}(\mathcal{D}^Q)$, and let $\mathcal{D}^Q_{\text{lower}}$ be the set of disk whose bounding arcs appear on the boundary of B. Disks in $\mathcal{D}^Q_{\text{lower}}$ are called *lower boundary disks* (see Figure 1.4). Upper, left and right boundary segments are defined similarly, and they are denoted by $\mathcal{D}^Q_{\text{upper}}$, $\mathcal{D}^Q_{\text{left}}$, $\mathcal{D}^Q_{\text{right}}$, respectively. A crucial observation is that each of these sets forms a chain (see [64, Lemma 2.5]) and their union covers $\mathsf{Cov}(\mathcal{D}^Q)$ entirely (since the diagonal of the square is 1). So the problem of minimum CF colourings of $\mathcal{D}^Q$ reduces to the

problem of minimum CF colourings of $\mathcal{D}^Q_{\text{bound}} = \mathcal{D}^Q_{\text{lower}} \cap \mathcal{D}^Q_{\text{upper}} \cap \mathcal{D}^Q_{\text{left}} \cap \mathcal{D}^Q_{\text{right}}$, the set of boundary disks. Thus, it is important to estimate the average number of boundary disks for the average-case analysis of the algorithm.

§1.2.2 Complexity of the convex hull of random point sets

Let $X = \{X_1, \ldots, X_n\}$ be a set of n independently sampled random points in the plane. The convex hull of X, denoted as $\mathsf{CH}(X)$, is the smallest convex set that contains X, which is also the set of all possible convex combinations of points of X, i.e.,

$$\mathsf{CH}(X) = \{x \in \mathbb{R}^2 : x = \sum_{i=1}^{n} \alpha_i X_i \text{ with } \sum_{i=1}^{n} \alpha_i = 1 \text{ and } \alpha_i \geq 0 \text{ for all } i = 1, \ldots, n\}.$$

Let $V(X)$ be the set of vertices of $\mathsf{CH}(X)$, i.e., the set of points of X that are at the boundary of its convex hull (note that, contrary to standard terminology, we also call a point from X a vertex when it lies in the interior of an edge of $\mathsf{CH}(X)$). The study of convex hulls of random point sets goes back to 1960's. Rényi and Sulanke [85] showed that, as n tends to infinity, the expectation of $|V(X)|$, where X is a set of n randomly sampled points, scales like $\sqrt{\log n}$ if the points are sampled according to the normal distribution, like $\log n$ if the points are sampled uniformly in a polygon, and like $n^{1/3}$ if the points are sampled uniformly in a convex compact region with a smooth boundary. Later studies focused on exact asymptotics for the expectation of $|V(X)|$ and its variance, on limit theorems and on generalization to higher dimensions. For a survey of results on convex hull problems, see [99].

The usual proof strategy involves computing the probability of two fixed points forming an edge of the convex hull and multiplying this probability by $\binom{n}{2}$. This gives the expected number of edges of the convex hull, which is the same as the number of vertices. Two points of X form an edge if all the other points fall on the same side of the line passing through these two points. Efron [39] devised a different approach that exploits the relation between the area of the convex hull and the number of vertices. Using this approach, Har-Peled [55] presented a simple proof for the asymptotic scaling of $|V(X)|$. His arguments can be viewed as a discretization of the classical arguments of Rényi and Sulanke in [85].

The convex-hull problem is directly related to the union-complexity problem. Let $\mathcal{D}(X, r) = \{D_1, \ldots, D_n\}$, where D_i is the disk with radius r centered at X_i. Let us use the shorthand notation $\mathsf{BD}(X, r) = \mathsf{BD}(\mathcal{D}(X, r))$ for the set of boundary disks of $\mathcal{D}(X, r)$. Although we do not need the following proposition in the proofs of Theorem 5.1.1 below, we state the connection between the convex hull and the boundary disks.

Proposition 1.2.3. *Suppose that $X = \{X_1, \ldots, X_n\}$ is a set of n points in $\mathbb{R}^2$. Then, for $i = 1, \ldots, n$, the point X_i is a vertex of the convex hull of X, i.e. $X_i \in V(X)$, if and only if for any $r > 0$ the disk with radius r centered at X_i is a boundary disk of $\mathcal{D}(X, r)$, i.e. $D_i \in \mathsf{BD}(X, r)$.*

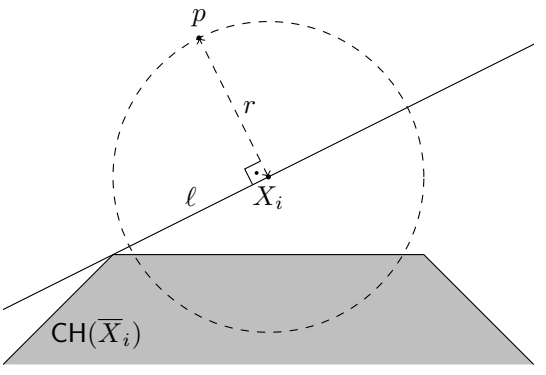


Figure 1.5: Illustration of the first part of the proof of Proposition 1.2.3.

Proof. Fix $i \in [n]$. First assume that $X_i \in V(X)$ and fix $r > 0$. Let ℓ be a line through X_i that is tangent to $\mathsf{CH}(\overline{X}_i)$ where $\overline{X}_i \coloneqq X \setminus \{X_i\}$. Let h^+ denote the closed half-plane bounded by ℓ such that $\mathsf{CH}(X) \subset h^+(\ell)$, and let h^- be the opposite half-plane. Finally, let $p \in h^-$ be the point at distance r from X_i such that the segment pX_i is perpendicular to ℓ (see Figure 1.5). Clearly, $p \in \partial D_i$ and all other points in X have distance greater than r to p. Hence, D_i is a boundary disk.

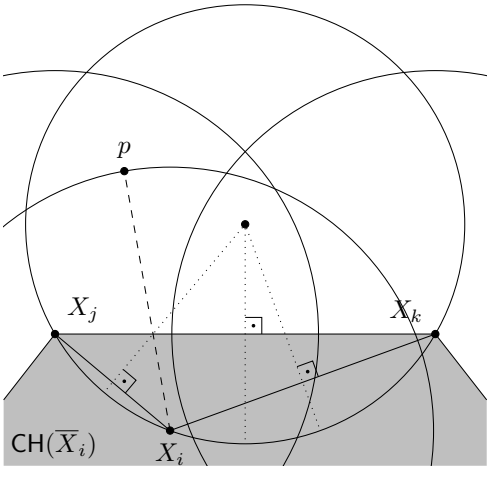


Figure 1.6: Illustration of the second part of the proof of Proposition 1.2.3. Dotted lines are perpendicular bisectors of the triangle $X_iX_jX_k$.

Next assume that $X_i \notin V(X)$. Then X_i is in the interior of $\mathsf{CH}(\overline{X}_i)$. For distinct $j, k \in [n]$ with $j, k \neq i$, let $r_{j,k}$ denote the radius of the circumscribed circle of the points X_i, X_j and X_k. Set $r = \max_{j,k\in[n]} r_{j,k} + \mathsf{diam}(X)$, where $\mathsf{diam}(X)$ is the diameter of X. Then the boundary of the disk D_i with radius r centered at X_i is not a boundary disk of $\mathcal{D}(X, r)$. Indeed, fix any point on the boundary of D_i, say p. If $p \in \mathsf{CH}(X)$, then p is covered by all the other disks, since $r > \mathsf{diam}(X)$. If $p \notin \mathsf{CH}(X)$, then consider the line segment between the points p and X_i. Suppose that this line

segment passes through the edge X_jX_k of the convex hull for some $j, k \in [n]$. Consider the perpendicular bisector ℓ_1 of the line segment X_iX_j and the perpendicular bisector ℓ_2 of the line segment X_iX_k. Assume without loss of generality that X_i lies to the right of ℓ_1 and to the left of ℓ_2. Since $r > r_{j,k}$, p lies to the left of ℓ_1 or to the right of ℓ_2. Then, $\min\{d(p, X_j), d(p, X_k)\} < d(p, X_i) = r$ and at least one of the disks D_j and D_k contains p (see Figure 1.6). So any point on the boundary of D_i is covered by some other disk in $\mathcal{D}(X, r)$, and hence D_i is not a boundary disk of $\mathcal{D}(X, r)$.

□

§1.2.3 Our contribution

In Chapter 5, we study the average-case union complexity for disks whose centers are sampled uniformly and independently at random in a region S in $\mathbb{R}^2$ and whose radii are greater than the diameter of S. We obtain union-complexity results by bounding the number of boundary disks. We consider two cases: the case where S is a square and the case where S is a disk. We show that, in both cases, the expected number of boundary disks scales like $n^{1/3}$ as n, the number of random disks, tends to infinity. Our proof is an adaptation of the proof of Har-Peled [55] for the convex hull problem.

Our result for the square case is important in the context of the average-case analysis of Lev-Tov and Peleg's algorithm [64] for the conflict-free colouring of disk regions [64], since their colouring scheme relies on the colouring of the boundary disks. Considering the connection between the union complexity and the convex-hull complexity, our results are surprising. For the convex-hull problem, the order of the expected number of vertices of the convex hull is different for the two cases: it is $\log n$ for the square case and $n^{1/3}$ for the disk case. However, for the union-complexity problem, the order of the number of boundary disks is $n^{1/3}$ in both cases.

PART I

RANDOM WALKS ON DYNAMIC RANDOM GRAPHS

CHAPTER 2

Mixing times of random walks on dynamic configuration models

This chapter is based on a joint article with Luca Avena, Remco van der Hofstad and Frank den Hollander [12].

Abstract

The mixing time of a random walk, with or without backtracking, on a random graph generated according to the configuration model on n vertices, is known to be of order $\log n$. In this paper we investigate what happens when the random graph becomes *dynamic*, namely, at each unit of time a fraction α_n of the edges is randomly rewired. Under mild conditions on the degree sequence, guaranteeing that the graph is locally tree-like, we show that for every $\varepsilon \in (0,1)$ the ε-mixing time of random walk without backtracking grows like $\sqrt{2\log(1/\varepsilon)/\log(1/(1-\alpha_n))}$ as $n \to \infty$, provided that $\lim_{n\to\infty} \alpha_n (\log n)^2 = \infty$. The latter condition corresponds to a regime of fast enough graph dynamics. Our proof is based on a randomised stopping time argument, in combination with coupling techniques and combinatorial estimates. The stopping time of interest is the first time that the walk moves along an edge that was rewired before, which turns out to be close to a strong stationary time.

§2.1 Introduction and main result

§2.1.1 Motivation and background

The *mixing time* of a Markov chain is the time it needs to approach its stationary distribution. For random walks on *finite graphs*, the characterisation of the mixing time has been the subject of intensive study. One of the main motivations is the fact that the mixing time gives information about the geometry of the graph (see the books by Aldous and Fill [4] and by Levin, Peres and Wilmer [65] for an overview and for applications). Typically, the random walk is assumed to be 'simple', meaning that steps are along edges and are drawn uniformly at random from a set of allowed edges, e.g. with or without backtracking.

In the last decade, much attention has been devoted to the analysis of mixing times for random walks on *finite random graphs*. Random graphs are used as models for real-world networks. Three main models have been in the focus of attention: (1) the Erdős-Rényi random graph (Benjamini, Kozma and Wormald [18], Ding, Lubetzky and Peres [37], Fountoulakis and Reed [50], Nachmias and Peres [75]); (2) the configuration model (Ben-Hamou and Salez [16], Berestycki, Lubetzky, Peres and Sly [21], Bordenave, Caputo and Salez [28], Lubetzky and Sly [67]); (3) percolation clusters (Benjamini and Mossel [19]).

Many real-world networks are dynamic in nature. It is therefore natural to study random walks on *dynamic finite random graphs*. This line of research was initiated recently by Peres, Stauffer and Steif [83] and by Peres, Sousi and Steif [82], who characterised the mixing time of a simple random walk on a dynamical percolation cluster on a d-dimensional discrete torus, in various regimes. The goal of the present paper is to study the mixing time of a random walk *without backtracking* on a dynamic version of the configuration model.

The *static* configuration model is a random graph with a prescribed degree sequence (possibly random). It is popular because of its mathematical tractability and its flexibility in modeling real-world networks (see van der Hofstad [93, Chapter 7] for an overview). For random walk on the static configuration model, with or without backtracking, the asymptotics of the associated mixing time, and related properties such as the presence of the so-called cutoff phenomenon, were derived recently by Berestycki, Lubetzky, Peres and Sly [21], and by Ben-Hamou and Salez [16]. In particular, under mild assumptions on the degree sequence, guaranteeing that the graph is an *expander* with high probability, the mixing time was shown to be of order $\log n$, with n the number of vertices.

In the present paper we consider a *discrete-time dynamic* version of the configuration model, where at each unit of time a fraction α_n of the edges is sampled and rewired uniformly at random. [A different dynamic version of the configuration model was considered in the context of *graph sampling*. See Greenhill [54] and references therein.] Our dynamics *preserves the degrees* of the vertices. Consequently, when considering a random walk on this dynamic configuration model, its *stationary distribution remains constant over time* and the analysis of its mixing time is a well-posed question. It is natural to expect that, due to the graph dynamics, the

random walk *mixes faster* than the $\log n$ order known for the static model. In our main theorem we will make this precise under mild assumptions on the prescribed degree sequence stated in Condition 2.1.2 and Remark 2.1.3 below. By requiring that $\lim_{n\to\infty} \alpha_n (\log n)^2 = \infty$, which corresponds to a regime of fast enough graph dynamics, we find in Theorem 2.1.7 below that for every $\varepsilon \in (0,1)$ the ε-mixing time for random walk *without backtracking* grows like $\sqrt{2\log(1/\varepsilon)/\log(1/(1-\alpha_n))}$ as $n \to \infty$, with high probability in the sense of Definition 2.1.5 below. Note that this mixing time is $o(\log n)$, so that the dynamics indeed speeds up the mixing.

§2.1.2 Model

We start by defining the model and setting up the notation. The set of vertices is denoted by V and the degree of a vertex $v \in V$ by $d(v)$. Each vertex $v \in V$ is thought of as being incident to $d(v)$ *half-edges* (see Fig. 2.1). We write H for the set of half-edges, and assume that each half-edge is associated to a vertex via incidence. We denote by $v(x) \in V$ the vertex to which $x \in H$ is incident and by $H(v) := \{x \in H\colon v(x) = v\} \subset H$ the set of half-edges incident to $v \in V$. If $x, y \in H(v)$ with $x \neq y$, then we write $x \sim y$ and say that x and y are siblings of each other. The degree of a half-edge $x \in H$ is defined as

$$\deg(x) := d(v(x)) - 1. \tag{2.1}$$

We consider graphs on n vertices, i.e., $|V| = n$, with m edges, so that $|H| = \sum_{v\in V} \deg(v) = 2m =: \ell$.

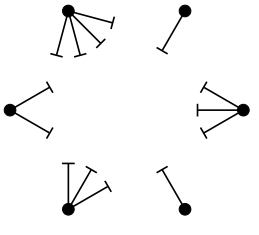


Figure 2.1: Vertices with half-edges.

The *edges* of the graph will be given by a *configuration* that is a *pairing of half-edges*. We denote by $\eta(x)$ the half-edge paired to $x \in H$ in the configuration η. A configuration η will be viewed as a bijection of H without fixed points and with the property that $\eta(\eta(x)) = x$ for all $x \in H$ (also called an involution). With a slight abuse of notation, we will use the same symbol η to denote the set of pairs of half-edges in η, so $\{x, y\} \in \eta$ means that $\eta(x) = y$ and $\eta(y) = x$. Each pair of half-edges in η will also be called an edge. The set of all configurations on H will be denoted by $Conf_H$.

We note that each configuration gives rise to a graph that may contain self-loops (edges having the same vertex on both ends) or multiple edges (between the same

pair of vertices). On the other hand, a graph can be obtained via several distinct configurations.

We will consider asymptotic statements in the sense of $|V| = n \to \infty$. Thus, quantities like $V, H, d, \deg$ and ℓ all depend on n. In order to lighten the notation, we often suppress n from the notation.

Configuration model

We recall the definition of the configuration model, phrased in our notation. Inspired by Bender and Canfield [17], the configuration model was introduced by Bollobás [24] to study the number of regular graphs of a given size (see also Bollobás [25]). Molloy and Reed [72], [73] introduced the configuration model with general prescribed degrees.

The configuration model on V with degree sequence $(d(v))_{v\in V}$ is the uniform distribution on $Conf_H$. We sometimes write $d_n = (d(v))_{v\in V}$ when we wish to stress the n-dependence of the degree sequence. Identify H with the set

$$[1, \ell] \coloneqq \{1, \dots, \ell\}.$$

A sample η from the configuration model can be generated by the following *sampling algorithm*:

1. Initialize $U = H, \eta = \varnothing$, where U denotes the set of unpaired half-edges.
2. Pick a half-edge, say x, uniformly at random from $U \setminus \{\min U\}$.
3. Update $\eta \to \eta \cup \{\{x, \min U\}\}$ and $U \to U \setminus \{x, \min U\}$.
4. If $U \neq \varnothing$, then continue from step 2. Else return η.

The resulting configuration η gives rise to a graph on V with degree sequence $(d(v))_{v\in V}$.

Remark 2.1.1. Note that in the above algorithm two half-edges that belong to the same vertex can be paired, which creates a self-loop, or two half-edges that belong to vertices that already have an edge between them can be paired, which creates multiple edges. However, if the degrees are not too large (as in Condition 2.1.2 below), then as $n \to \infty$ the number of self-loops and the number of multiple edges converge to two independent Poisson random variables (see Janson [58], [59], Angel, van der Hofstad and Holmgren [10]). Consequently, convergence in probability for the configuration model implies convergence in probability for the configuration model conditioned on being simple.

Let U_n be uniformly distributed on $[1, n]$. Then

$$D_n = d(U_n) \tag{2.2}$$

is the degree of a random vertex on the graph of size n. Write $\mathbb{P}_n$ to denote the law of D_n. Throughout the sequel, we impose the following mild regularity conditions on the degree sequence:

Condition 2.1.2. ***(Regularity of degrees)***

(R1) *Let $\ell = |H|$. Then ℓ is even and of order n, i.e., $\ell = \Theta(n)$ as $n \to \infty$.*

(R2) *Let*

$$\nu_n := \frac{\sum_{z\in H} \deg(z)}{\ell} = \frac{\sum_{v\in V} d(v)[d(v)-1]}{\sum_{v\in V} d(v)} = \frac{\mathbb{E}_n(D_n(D_n-1))}{\mathbb{E}_n(D_n)} \tag{2.3}$$

denote the expected degree of a uniformly chosen half-edge. Then $\limsup_{n\to\infty} \nu_n < \infty$.

(R3) $\mathbb{P}_n(D_n \geq 2) = 1$ *for all $n \in \mathbb{N}$.*

Remark 2.1.3. Conditions (R1) and (R2) are minimal requirements to guarantee that the graph is locally tree-like (in the sense of Lemma 2.4.2 below). They also ensure that the probability of the graph being simple has a strictly positive limit. Conditioned on being simple, the configuration model generates a random graph that is uniformly distributed among all the simple graphs with the given degree sequence (see van der Hofstad [93, Chapter 7], [94, Chapters 3 and 6]). Condition (R3) ensures that the random walk without backtracking is well-defined because it cannot get stuck on a dead-end.

Dynamic configuration model

We begin by describing the random graph process. It is convenient to take as the state space the set of configurations Conf_H. For a fixed initial configuration η and fixed $2 \leq k \leq m = \ell/2$, the graph evolves as follows (see Fig. 2.2):

(a) At each time $t \in \mathbb{N}$, pick k edges (pairs of half-edges) from C_{t-1} uniformly at random without replacement. Cut these edges to get $2k$ half-edges and denote this set of half-edges by R_t.

(b) Generate a uniform pairing of these half-edges to obtain k new edges. Replace the k edges chosen in step 1 by the k new edges to get the configuration C_t at time t.

This process rewires k edges at each step by applying the configuration model sampling algorithm in Section 2.1.2 restriced to k uniformly chosen edges. Since half-edges are not created or destroyed, the degree sequence of the graph given by C_t is the same for all $t \in \mathbb{N}_0$. This gives us a Markov chain on the set of configurations Conf_H. For $\eta, \zeta \in \mathit{Conf}_H$, the *transition probabilities* for this Markov chain are given by

$$Q(\eta,\zeta) = Q(\zeta,\eta) := \begin{cases} \frac{1}{(2k-1)!!} \frac{\binom{m-d_{\mathrm{Ham}}(\eta,\zeta)}{k-d_{\mathrm{Ham}}(\eta,\zeta)}}{\binom{m}{k}} & \text{if } d_{\mathrm{Ham}}(\eta,\zeta) \leq k, \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

where $d_{\mathrm{Ham}}(\eta,\zeta) := |\eta \setminus \zeta| = |\zeta \setminus \eta|$ is the Hamming distance between configurations η and ζ, which is the number of edges that appear in η but not in ζ. The factor $1/(2k-1)!!$ comes from the uniform pairing of the half-edges, while the factor

$\binom{m-d_{\mathrm{Ham}}(\eta,\zeta)}{k-d_{\mathrm{Ham}}(\eta,\zeta)}/\binom{m}{k}$ comes from choosing uniformly at random a set of k edges in η that contains the edges in $\eta \setminus \zeta$. It is easy to see that this Markov chain is irreducible and aperiodic, with stationary distribution the uniform distribution on $Conf_H$, denoted by Conf_H, which is the distribution of the configuration model.

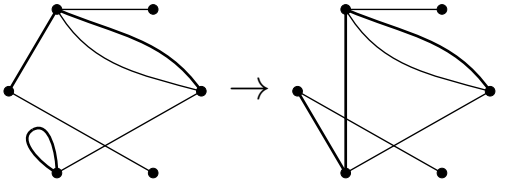


Figure 2.2: One move of the dynamic configuration model. Bold edges on the left are the ones chosen to be rewired. Bold edges on the right are the newly formed edges.

Random walk without backtracking

On top of the random graph process we define the random walk without backtracking, i.e., the walk cannot traverse the same edge twice in a row. As in Ben-Hamou and Salez [16], we define it as a random walk on the set of half-edges H, which is more convenient in the dynamic setting because the edges change over time while the half-edges do not. For a fixed configuration η and half-edges $x, y \in H$, the transition probabilities of the random walk are given by (recall (2.1))

$$P_\eta(x,y) \coloneqq \begin{cases} \frac{1}{\deg(y)} & \text{if } \eta(x) \sim y \text{ and } \eta(x) \neq y, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

When the random walk is at half-edge x in configuration η, it jumps to one of the siblings of the half-edge it is paired to uniformly at random (see Fig. 2.3). The transition probabilities are symmetric with respect to the pairing given by η, i.e., $P_\eta(x,y) = P_\eta(\eta(y),\eta(x))$, in particular, they are doubly stochastic, and so the uniform distribution on H, denoted by U_H, is stationary for P_η for any $\eta \in Conf_H$.

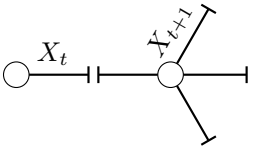


Figure 2.3: The random walk moves from half-edge X_t to half-edge X_{t+1}, one of the siblings of the half-edge that X_t is paired to.

Random walk on dynamic configuration model

The random walk without backtracking on the dynamic configuration model is the joint Markov chain $(M_t)_{t\in\mathbb{N}_0} = (C_t, X_t)_{t\in\mathbb{N}_0}$ in which $(C_t)_{t\in\mathbb{N}_0}$ is the Markov chain

on the set of configurations $Conf_H$ as described in (2.4), and $(X_t)_{t\in\mathbb{N}_0}$ is the random walk that at each time step t jumps according to the transition probabilities $P_{C_t}(\cdot,\cdot)$ as in (2.5).

Formally, for initial configuration η and half-edge x, the one-step evolution of the joint Markov chain is given by the conditional probabilities

$$\mathbb{P}_{\eta,x}\big(C_t=\zeta, X_t=z \mid C_{t-1}=\xi, X_{t-1}=y\big) = Q(\xi,\zeta)\,P_\zeta(y,z), \qquad t\in\mathbb{N}, \tag{2.6}$$

with

$$\mathbb{P}_{\eta,x}(C_0=\eta, X_0=x)=1. \tag{2.7}$$

It is easy to see that if $d(v)>1$ for all $v\in V$, then this Markov chain is irreducible and aperiodic, and has the unique stationary distribution $\mathrm{Conf}_H \times U_H$.

While the graph process $(C_t)_{t\in\mathbb{N}_0}$ and the joint process $(M_t)_{t\in\mathbb{N}_0}$ are Markovian, the random walk $(X_t)_{t\in\mathbb{N}_0}$ is not. However, U_H is still the stationary distribution of $(X_t)_{t\in\mathbb{N}_0}$. Indeed, for any $\eta\in Conf_H$ and $y\in H$ we have

$$\sum_{x\in H} U_H(x)\,\mathbb{P}_{\eta,x}(X_t=y) = \sum_{x\in H}\frac{1}{\ell}\,\mathbb{P}_{\eta,x}(X_t=y) = \frac{1}{\ell} = U_H(y). \tag{2.8}$$

The next to last equality uses that $\sum_{x\in H}\mathbb{P}_{\eta,x}(X_t=y)=1$ for every $y\in H$, which can be seen by conditioning on the graph process and using that the space-time inhomogeneous random walk has a doubly stochastic transition matrix (recall the remarks made below (2.5)).

§2.1.3 Main theorem

We are interested in the behaviour of the total variation distance between the distribution of X_t and the uniform distribution

$$\mathcal{D}_{\eta,x}(t) := \|\mathbb{P}_{\eta,x}(X_t\in\cdot\,) - U_H(\cdot)\|_{\mathrm{TV}}. \tag{2.9}$$

[We recall that the total variation distance of two probability measures μ_1,μ_2 on a finite state space S is given by the following equivalent expressions:

$$\|\mu_1-\mu_2\|_{\mathrm{TV}} := \sum_{x\in S}|\mu_1(x)-\mu_2(x)| = \sum_{x\in S}[\mu_1(x)-\mu_2(x)]_+ = \sup_{A\subseteq S}[\mu_1(A)-\mu_2(A)], \tag{2.10}$$

where $[a]_+ := \max\{a,0\}$ for $a\in\mathbb{R}$.] Since $(X_t)_{t\in\mathbb{N}_0}$ is not Markovian, it is not clear whether $t\mapsto\mathcal{D}_{\eta,x}(t)$ is decreasing or not. On the other hand,

$$\mathcal{D}_{\eta,x}(t) \le \|\mathbb{P}_{\eta,x}(M_t\in\cdot\,) - (U_H\times\mathrm{Conf}_H)(\cdot)\|_{\mathrm{TV}}, \tag{2.11}$$

and since the right-hand side converges to 0 as $t\to\infty$, so does $\mathcal{D}_{\eta,x}(t)$. Therefore the following definition is well-posed:

Definition 2.1.4 (Mixing time of the random walk). *For $\varepsilon\in(0,1)$, the* ε-mixing time of the random walk *is defined as*

$$t^n_{\mathrm{mix}}(\varepsilon;\eta,x) := \inf\big\{t\in\mathbb{N}_0\colon\ \mathcal{D}_{\eta,x}(t)\le\varepsilon\big\}. \tag{2.12}$$

Note that $t^n_{\text{mix}}(\varepsilon;\eta,x)$ depends on the initial configuration η and half-edge x. We will prove statements that hold for *typical* choices of (η,x) under the uniform distribution μ_n (recall that H depends on the number of vertices n) given by

$$\mu_n := \text{Conf}_H \times U_H \quad \text{ on } \mathit{Conf}_H \times H, \tag{2.13}$$

where *typical* is made precise through the following definition:

Definition 2.1.5 (**With high probability**). *A statement that depends on the initial configuration η and half-edge x is said to hold* with high probability (whp) *in η and x if the μ_n-measure of the set of pairs (η,x) for which the statement holds tends to* 1 *as $n\to\infty$.*

Below we sometimes write whp with respect to some probability measure other than μ_n, but it will always be clear from the context which probability measure we are referring to.

Throughout the paper we assume the following condition on

$$\alpha_n := k/m, \qquad n\in\mathbb{N}, \tag{2.14}$$

denoting *the proportion of edges involved in the rewiring* at each time step of the graph dynamics defined in Section 2.1.2:

Condition 2.1.6 (**Fast graph dynamics**). *The ratio α_n in* (2.14) *is subject to the constraint*

$$\lim_{n\to\infty} \alpha_n(\log n)^2 = \infty. \tag{2.15}$$

We can now state our main result.

Theorem 2.1.7 (**Sharp mixing time asymptotics**). *Suppose that Conditions* 2.1.2 *and* 2.1.6 *hold. Then, for every $\varepsilon>0$, whp in η and x,*

$$t^n_{\text{mix}}(\varepsilon;\eta,x) = [1+o(1)]\sqrt{\frac{2\log(1/\varepsilon)}{\log(1/(1-\alpha_n))}}. \tag{2.16}$$

Note that Condition 2.1.6 allows for $\lim_{n\to\infty}\alpha_n = 0$. In that case (2.16) simplifies to

$$t^n_{\text{mix}}(\varepsilon;\eta,x) = [1+o(1)]\sqrt{\frac{2\log(1/\varepsilon)}{\alpha_n}}. \tag{2.17}$$

§2.1.4 Discussion

1. Theorem 2.1.7 gives the sharp asymptotics of the mixing time in the regime where the dynamics is fast enough (as specified by Condition 2.1.6). Note that if $\lim_{n\to\infty}\alpha_n = \alpha\in(0,1]$, then $t^n_{\text{mix}}(\varepsilon;\eta,x)$ is of order one: at every step the random walk has a non-vanishing probability to traverse a rewired edge, and so it is qualitatively similar to a random walk on a complete graph. On the other hand, when $\lim_{n\to\infty}\alpha_n = 0$ the mixing time is of order $1/\sqrt{\alpha_n} = o(\log n)$, which shows that the

dynamics still *speeds up* the mixing. The regime $\alpha_n = \Theta(1/(\log n)^2)$, which is not captured by Theorem 2.1.7, corresponds to $1/\sqrt{\alpha_n} = \Theta(\log n)$, and we expect the mixing time to be *comparable* to that of the static configuration model. In the regime $\alpha_n = o(1/(\log n)^2)$ we expect the mixing time to be the *same* as that of the static configuration model. In a future paper we plan to provide a comparative analysis of the three regimes.

2. In the static model the ε-mixing time is known to scale like $[1 + o(1)]\, c \log n$ for some $c \in (0, \infty)$ that is independent of $\varepsilon \in (0, 1)$ (Ben-Hamou and Salez [16]). Consequently, there is *cutoff*, i.e., the total variation distance drops from 1 to 0 in a time window of width $o(\log n)$. In contrast, in the regime of fast graph dynamics there is *no cutoff*, i.e., the total variation distance drops from 1 to 0 gradually on scale $1/\sqrt{\alpha_n}$.

3. Our proof is robust and can be easily extended to variants of our model where, for example, $(k_n)_{n\in\mathbb{N}}$ is random with k_n having a first moment that tends to infinity as $n \to \infty$, or where time is continuous and pairs of edges are randomly rewired at rate α_n.

4. Theorem 2.1.7 can be compared to the analogous result for the static configuration model only when $\mathbb{P}_n(D_n \geq 3) = 1$ for all $n \in \mathbb{N}$. In fact, only under the latter condition does the probability of having a connected graph tend to one (see Luczak [68], Federico and van der Hofstad [47]). If (R3) holds, then on the dynamic graph the walk mixes on the whole of H, while on the static graph it mixes on the subset of H corresponding to the giant component.

5. We are not able to characterise the mixing time of the joint process of dynamic random graph and random walk. Clearly, the mixing time of the joint process is at least as large as the mixing time of each process separately. While the graph process helps the random walk to mix, the converse is not true because the graph process does not depend on the random walk. Observe that once the graph process has mixed it has an almost uniform configuration, and the random walk ought to have mixed already. This observation suggests that if the mixing times of the graph process and the random walk are not of the same order, then the mixing time of the joint process will have the same order as the mixing time of the graph process. Intuitively, we may expect that the mixing time of the graph corresponds to the time at which all edges are rewired at least once, which should be of order $(n/k)\log n = (1/\alpha_n)\log n$ by a coupon collector argument. In our setting the latter is much larger than $1/\sqrt{\alpha_n}$.

6. We emphasize that we look at the mixing times for 'typical' initial conditions and we look at the distribution of the random walk averaged over the trajectories of the graph process: the 'annealed' model. It would be interesting to look at different setups, such as 'worst-case' mixing, in which the maximum of the mixing times over all initial conditions is considered, or the 'quenched' model, in which the entire trajectory of the graph process is fixed instead of just the initial configuration. In such setups the results can be drastically different. For example, if we consider the quenched model for d-regular graphs, then we see that for any time t and any fixed realization of configurations up to time t, the walk without backtracking can reach at most $(d-1)^t$

half-edges. This gives us a lower bound of order $\log n$ for the mixing time in the quenched model, which contrasts with the $o(\log n)$ mixing time in our setup.

7. It would be of interest to extend our results to random walk with backtracking, which is harder. Indeed, because the configuration model is locally tree-like and random walk without backtracking on a tree is the same as self-avoiding walk, in our proof we can exploit the fact that typical walk trajectories are self-avoiding. In contrast, for the random walk with backtracking, after it jumps over a rewired edge, which in our model serves as a randomized stopping time, it may jump back over the same edge, in which case it has not mixed. This problem remains to be resolved.

§2.1.5 Outline

The remainder of this paper is organised as follows. Section 2.2 gives the main idea behind the proof, namely, we introduce a randomised stopping time $\tau = \tau_n$, the first time the walk moves along an edge that was rewired before, and we state a key proposition, Proposition 2.2.1 below, which says that this time is close to a strong stationary time and characterises its tail distribution. As shown at the end of Section 2.2, Theorem 2.1.7 follows from Proposition 2.2.1, whose proof consists of three main steps. The first step in Section 2.3 consists of a careful combinatorial analysis of the distribution of the walk given the history of the rewiring of the half-edges in the underlying evolving graph. The second step in Section 2.4 uses a classical exploration procedure of the static random graph from a uniform vertex to unveil the locally tree-like structure in large enough balls. The third step in Section 2.5 settles the closeness to stationarity and provides control on the tail of the randomized stopping time τ.

§2.2 Stopping time decomposition

We employ a *randomised stopping time* argument to get bounds on the total variation distance. We define the randomised stopping time $\tau = \tau_n$ to be the first time the walker makes a move through an edge that was rewired before. Recall from Section 2.1.2 that R_t is the set of half-edges involved in the rewiring at time step t. Letting $R_{\leq t} = \cup_{s=1}^{t} R_s$, we set

$$\tau := \min\{t \in \mathbb{N}\colon\ X_{t-1} \in R_{\leq t}\}. \tag{2.18}$$

As we will see later, τ behaves like a strong stationary time. We obtain our main result by deriving bounds on $\mathcal{D}_{\eta,x}(t)$ in terms of conditional distributions of the random walk involving τ and in terms of tail probabilities of τ. In particular, by the triangle inequality, for any $t \in \mathbb{N}_0$, $\eta \in \mathit{Conf}_H$ and $x \in H$,

$$\begin{aligned}\mathcal{D}_{\eta,x}(t) \leq\ & \mathbb{P}_{\eta,x}(\tau > t)\, \|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H(\cdot)\|_{\mathrm{TV}} \\ & + \mathbb{P}_{\eta,x}(\tau \leq t)\, \|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t) - U_H(\cdot)\|_{\mathrm{TV}}\end{aligned} \tag{2.19}$$

and

$$\mathcal{D}_{\eta,x}(t) \geq \mathbb{P}_{\eta,x}(\tau > t)\, \|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H(\cdot)\|_{\text{TV}} \\ - \mathbb{P}_{\eta,x}(\tau \leq t)\, \|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t) - U_H(\cdot)\|_{\text{TV}}. \tag{2.20}$$

With these in hand, we only need to find bounds for $\mathbb{P}_{\eta,x}(\tau > t)$, $\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H(\cdot)\|_{\text{TV}}$ and $\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t) - U_H(\cdot)\|_{\text{TV}}$.

The key result for the proof of our main theorem is the following proposition:

Proposition 2.2.1 (**Closeness to stationarity and tail behavior of stopping time**).
Suppose that Conditions 2.1.2 *and* 2.1.6 *hold. For* $t = t(n) = o(\log n)$*, whp in* x *and* η*,*

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t) - U_H(\cdot)\|_{\text{TV}} = o(1), \tag{2.21}$$

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H(\cdot)\|_{\text{TV}} = 1 - o(1), \tag{2.22}$$

$$\mathbb{P}_{\eta,x}(\tau > t) = (1-\alpha_n)^{t(t+1)/2} + o(1). \tag{2.23}$$

We close this section by showing how Theorem 2.1.7 follows from Proposition 2.2.1:

Proof. By Condition 2.1.6,

$$\sqrt{\frac{2\log(1/\varepsilon)}{\log(1/(1-\alpha_n))}} = O(\alpha_n^{-1/2}) = o(\log n). \tag{2.24}$$

Using the bounds in (2.19)–(2.20), together with (2.21)–(2.23) in Proposition 2.2.1, we see that for $t = o(\log n)$,

$$(1-\alpha_n)^{t(t+1)/2} + o(1) \leq \mathcal{D}_{\eta,x}(t) \leq (1-\alpha_n)^{t(t+1)/2} + o(1). \tag{2.25}$$

Choosing t as in (2.16) we obtain $\mathcal{D}_{\eta,x}(t) = \varepsilon + o(1)$, which is the desired result. □

The remainder of the paper is devoted to the proof of Proposition 2.2.1.

§2.3 Pathwise probabilities

In order to prove (2.21) of Proposition 2.2.1, we will show in (2.69) in Section 2.5 that the following crucial bound holds for *most* $y \in H$:

$$\mathbb{P}_{\eta,x}(X_t = y \mid \tau \leq t) \geq \frac{1-o(1)}{\ell}. \tag{2.26}$$

By *most* we mean that the number of y such that this inequality holds is $\ell - o(\ell)$ whp in η and x. To prove (2.26) we will look at $\mathbb{P}_{\eta,x}(X_t = y, \tau \leq t)$ by partitioning according to all possible paths taken by the walk and all possible rewiring patterns that occur on these paths. For a time interval $[s,t] \coloneqq \{s, s+1, \ldots, t\}$ with $s \leq t$, we define

$$x_{[s,t]} \coloneqq x_s \cdots x_t. \tag{2.27}$$

In particular, for any $y \in H$,

$$\begin{aligned}&\mathbb{P}_{\eta,x}(X_t = y, \tau \le t) \\ &= \sum_{T \subseteq [1,t]} \sum_{x_1,\dots,x_{t-1} \in H} \mathbb{P}_{\eta,x}\Big(X_{[1,t]} = x_{[1,t]},\ x_{i-1} \in R_{\le i}\ \forall\, i \in T,\\ &\qquad\qquad x_{j-1} \notin R_{\le j}\ \forall\, j \in [1,t] \setminus T\Big)\end{aligned} \tag{2.28}$$

with $x_0 = x$ and $x_t = y$. Here, r is the number of steps at which the walk moves along a previously rewired edge, and T is the set of times at which this occurs.

For a fixed sequence of half-edges $x_{[0,t]}$ with $x_0 = x$ and a fixed set of times $T \subseteq [1,t]$ with $|T| = r$, we will use the short-hand notation

$$A(x_{[0,t]}; T) \coloneqq \big\{x_{i-1} \in R_{\le i}\ \forall\, i \in T,\ x_{j-1} \notin R_{\le j}\ \forall\, j \in [1,t] \setminus T\big\}. \tag{2.29}$$

Writing $T = \{t_1, \dots, t_r\}$ with $1 \le t_1 < t_2 < \cdots < t_r \le t$, we note that the conditional probability $\mathbb{P}_{\eta,x}(X_{[1,t]} = x_{[1,t]} \mid A(x_{[0,t]}; T))$ can be non-zero only if each subsequence $x_{[t_{i-1},t_i-1]}$ induces a non-backtracking path in η for $i \in [2, r+1]$ with $t_0 = 0$ and $t_{r+1} = t+1$. The last sum in (2.28) is taken over such sequences in H, which we call *segmented paths* (see Fig. 2.4). For each $i \in [1, r+1]$ the subsequence $x_{[t_{i-1},t_i-1]}$ of length $t_i - t_{i-1}$ that forms a non-backtracking path in η is called a *segment*.

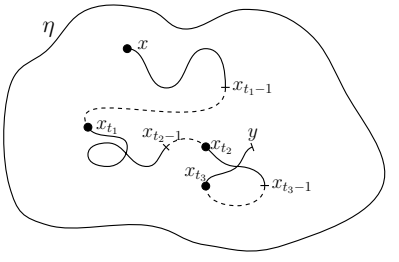


Figure 2.4: An example of a segmented path with 4 segments. Solid lines represent the segments, consisting of a path of half-edges in η, dashed lines indicate the succession of the segments. The latter do not necessarily correspond to a pair in η, and will later correspond to rewired edges in the graph dynamics.

We will restrict the last sum in (2.28) to the set of *self-avoiding segmented paths*. These are the paths where no two half-edges are siblings, which means that the vertices $v(x_i)$ visited by the half-edges x_i are distinct for all $i \in [0,t]$, so that if the random walk takes this path, then it does not see the same vertex twice. We will denote by $\mathsf{SP}_t^\eta(x, y; T)$ *the set of self-avoiding segmented paths* in η of length $t+1$ that start at x and end at y, where T gives the positions of the ends of the segments (see Fig. 2.5). Segmented paths $x_{[0,t]}$ have the nice property that the probability $\mathbb{P}_{\eta,x}(A(x_{[0,t]}; T))$ is the same for all $x_{[0,t]}$ that are isomorphic, as stated in the next lemma:

Lemma 2.3.1 (Isomorphic segmented path are equally likely). *Fix $t \in \mathbb{N}$, $T \subseteq [1,t]$ and $\eta \in \mathrm{Conf}_H$. Suppose that $x_{[0,t]}$ and $y_{[0,t]}$ are two segmented paths in η of length $t+1$ with $|x_{[s,s']}| = |y_{[s,s']}|$ for any $0 \le s < s' \le t$, where $|x_{[s,s']}|$ denotes the number of distinct half-edges in $x_{[s,s']}$. Then*

$$\mathbb{P}_{\eta,x}\big(A(x_{[0,t]}; T)\big) = \mathbb{P}_{\eta,x}\big(A(y_{[0,t]}; T)\big). \tag{2.30}$$

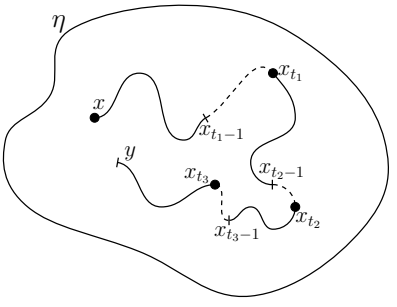


Figure 2.5: An element of $\mathsf{SP}_t^\eta(x, y; T)$ *with* $T = \{t_1, t_2, t_3\}$.

Proof. Fix $x, y \in H$. Consider the coupling $((C_t^x)_{t\in\mathbb{N}_0}, (C_t^y)_{t\in\mathbb{N}_0})$ of two dynamic configuration models with parameter k starting from η, defined as follows. Let $f\colon H \to H$ be such that

$$f(x) = \begin{cases} y_i & \text{if } x = x_i \text{ for some } i \in [0,t], \\ x_i & \text{if } x = y_i \text{ for some } i \in [0,t], \\ \eta(y_i) & \text{if } x = \eta(x_i) \text{ for some } i \in [0,t], \\ \eta(x_i) & \text{if } x = \eta(y_i) \text{ for some } i \in [0,t], \\ x & \text{otherwise.} \end{cases} \tag{2.31}$$

This is a one-to-one function because $|x_{[s,s']}| = |y_{[s,s']}|$ for any $0 \le s < s' \le t$. What f does is to map the half-edges of $x_{[0,t]}$ and their pairs in η to the half-edges of $y_{[0,t]}$ and their pairs in η, and vice versa, while preserving the order in the path. For the coupling, at each time $t \in \mathbb{N}$ we rewire the edges of C_{t-1}^x and C_{t-1}^y as follows:

(a) Choose k edges from C_{t-1}^x uniformly at random without replacement, say $\{z_1, z_2\}$, $\ldots$, $\{z_{2k-1}, z_{2k}\}$. Choose the edges $\{f(z_1), f(z_2)\}, \ldots, \{f(z_{2k-1}), f(z_{2k})\}$ from C_{t-1}^y.

(b) Rewire the half-edges $z_1, \ldots, z_{2k}$ uniformly at random to obtain C_t^x. Set $C_t^y(f(z_i)) = f(C_t^x(z_i))$.

Step 2 and the definition of f ensure that in Step 1 $\{f(z_1), f(z_2)\}, \ldots, \{f(z_{2k-1}), f(z_{2k})\}$ are in C_{t-1}^y. Since under the coupling the event $A(x_{[0,t]}; T)$ is the same as the event $A(y_{[0,t]}; T)$, we get the desired result. □

In order to prove the lower bound in (2.26), we will need two key facts. The first, stated in Lemma 2.3.2 below, gives a lower bound on the probability of a walk trajectory given the rewiring history. The second, stated in Lemma 2.4.3 below, is a lower bound on the number of relevant self-avoiding segmented paths, and exploits the locally tree-like structure of the configuration model.

Lemma 2.3.2 (Paths estimate given rewiring history). *Suppose that* $t = t(n) = o(\log n)$ *and* $T = \{t_1, \ldots, t_r\} \subseteq [1, t]$. *Let* $x_0 \cdots x_t \in \mathsf{SP}_t^\eta(x, y; T)$ *be a self-avoiding*

segmented path in η that starts at x and ends at y. Then

$$\mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]} \mid A(x_{[0,t]};T)\big) \geq \frac{1-o(1)}{\ell^r} \prod_{i\in[1,t]\setminus T} \frac{1}{\deg(x_i)}. \tag{2.32}$$

Proof. In order to deal with the dependencies introduced by conditioning on the event $A(x_{[0,t]};T)$), we will go through a series of conditionings. First we note that for the random walk to follow a specific path, the half-edges it traverses should be rewired *correctly* at the *right times*. Conditioning on $A(x_{[0,t]};T)$ accomplishes part of the job: since we have $x_{i-1} \notin R_{\leq i}$ for $i \in [1,t]\setminus T$ and $x_{[0,t]} \in \mathsf{SP}^{\eta}_t(x,y;T)$, we know that, at time i, x_{i-1} is paired to a sibling of x_i in C_i, and so the random walk can jump from x_{i-1} to x_i with probability $1/\deg(x_i)$ at time i for $i \in [1,t]\setminus T$.

Let us call the path $x_{[0,t]}$ *open* if $C_i(x_{i-1}) \sim x_i$ for $i \in [1,t]$, i.e., if x_{i-1} is paired to a sibling of x_i in C_i for $i \in [1,t]$. Then

$$\mathbb{P}_{\eta,x}(X_{[1,t]} = x_{[1,t]} \mid x_{[0,t]} \text{ is open}) = \prod_{i=1}^{t} \frac{1}{\deg(x_i)}, \tag{2.33}$$

and

$$\mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]} \mid x_{[0,t]} \text{ is not open}\big) = 0. \tag{2.34}$$

Using these observations, we can treat the random walk and the rewiring process separately, since the event $\{x_{[0,t]} \text{ is open}\}$ depends only on the rewirings. Our goal is to compute the probability

$$\mathbb{P}_{\eta,x}\big(x_{[0,t]} \text{ is open} \mid A(x_{[0,t]};T)\big). \tag{2.35}$$

Note that, by conditioning on $A(x_{[0,t]};T)$, the part of the path within segments is already open, so we only need to deal with the times the walk jumps from one segment to another. To have $x_{[0,t]}$ open, each x_{t_j-1} should be paired to one of the siblings of x_{t_j} for $j \in [1,r]$. Hence

$$\begin{aligned}&\mathbb{P}_{\eta,x}\big(x_{[0,t]} \text{ is open} \mid A(x_{[0,t]};T)\big)\\ &\quad= \sum_{\substack{z_1,\dots,z_r\in H\\ z_j\sim x_{t_j}\ \forall\, j\in[1,r]}} \mathbb{P}_{\eta,x}\big(C_{t_j}(x_{t_j-1}) = z_j\ \forall\, j\in[1,r] \mid A(x_{[0,t]};T)\big).\end{aligned} \tag{2.36}$$

Fix $z_1,\dots,z_r \in H$ with $z_j \sim x_{t_j}$, and let $y_j = x_{t_j-1}$ for $j \in [1,r]$. We will look at the probability

$$\mathbb{P}_{\eta,x}\big(C_{t_j}(y_j) = z_j\ \forall\, j\in[1,r] \mid A(x_{[0,t]};T)\big). \tag{2.37}$$

Conditioning on the event $A(x_{[0,t]};T)$ we impose that each y_j is rewired at some time before t_j, but do not specify at which time this happens. Let us refine our conditioning one step further by specifying these times. Fix $s_1,\dots,s_r \in [1,t]$ such that $s_j \leq t_j$ for each $j \in [1,r]$ (the s_j need not be distinct). Let $\widehat{A}$ be the event that $x_{i-1} \notin R_{\leq i}$ for $i \in [1,t]\setminus T$ and y_j is rewired at time s_j for the last time before time t_j for $j \in [1,r]$. Then $\widehat{A} \subseteq A(x_{[0,t]};T)$. Since s_j is the last time before t_j at which

y_j is rewired, the event $C_{t_j}(y_j) = z_j$ is the same as the event $C_{s_j}(y_j) = z_j$ when we condition on $\widehat{A}$. We look at the probability

$$\mathbb{P}_{\eta,x}\big(C_{s_j}(y_j) = z_j \ \forall\, j \in [1,r] \mid \widehat{A}\big). \tag{2.38}$$

Let $s'_1 < \cdots < s'_{r'} \in [1,t]$ be the distinct times such that $s'_i = s_j$ for some $j \in [1,r]$, and n_i^y the number of j's for which $s_j = s'_i$ for $i \in [1,r']$, so that by conditioning on $\widehat{A}$ we rewire n_i^y half-edges y_j at time s'_i. Letting also $D_i = \{C_{s'_i}(y_j) = z_j,\ \text{for } j \text{ such that } s_j = s'_i\}$, we can write the above conditional probability as

$$\prod_{i=1}^{r'} \mathbb{P}_{\eta,x}\big(D_i \mid \widehat{A},\, \cap_{j=1}^{i-1} D_j\big). \tag{2.39}$$

We next compute these conditional probabilities.

Fix $i \in [1,r']$ and $\eta' \in \mathit{Conf}_H$. We do one more conditioning and look at the probability

$$\mathbb{P}_{\eta,x}\big(D_i \mid \widehat{A},\, \cap_{j=1}^{i-1} D_j,\, C_{s'_i-1} = \eta'\big). \tag{2.40}$$

The rewiring process at time s'_i consists of two steps: (1) pick k edges uniformly at random; (2) do a uniform rewiring. Concerning (1), by conditioning on $\widehat{A}$, we see that the y_j's for which $s_j = s'_i$ are already chosen. In order to pair these to z_j's with $s_j = s'_i$, the z_j's should be chosen as well. If some of the z_j's are already paired to some y_j's already chosen, then they will be automatically included in the rewiring process. Let m'_i be m minus the number of half-edges in $\{x_0,\dots,x_t\} \cup \{z_1,\dots,z_r\}$, for which the conditioning on $\widehat{A}$ implies that they cannot be in $R_{s'_i}$. Then

$$\begin{aligned}
&\mathbb{P}_{\eta,x}\Big(z_j \in R_{s'_i} \text{ for } j \text{ such that } s_j = s'_i \;\Big|\; \widehat{A},\, \cap_{j=1}^{i-1} D_j,\, C_{s'_i-1} = \eta'\Big) \\
&\geq \frac{\binom{m'_i - 2n_i^y}{k-2n_i^y}}{\binom{m'_i - n_i^y}{k-n_i^y}} = \frac{\prod_{j=0}^{n_i^y-1}(k-n_i^y-j)}{\prod_{j=0}^{n_i^y-1}(m'_i-n_i^y-j)} \geq \frac{\prod_{j=0}^{n_i^y-1}(k-n_i^y-j)}{m^{n_i^y}}.
\end{aligned} \tag{2.41}$$

Concerning (2), conditioned on the relevant z_j's already chosen in (1), the probability that they will be paired to correct y_j's is

$$\frac{1}{\prod_{j=1}^{n_i^y}(2k-2j+1)}. \tag{2.42}$$

Since the last two statements hold for any η' with $\mathbb{P}_{\eta,x}(C_{s'_i-1} = \eta' \mid \widehat{A},\, \cap_{j=1}^{i-1} D_j) > 0$, combining these we get

$$\mathbb{P}_{\eta,x}\big(D_i \mid \widehat{A},\, \cap_{j=1}^{i-1} D_j\big) \geq \frac{\prod_{j=0}^{n_i^y-1}(k-n_i^y-j)}{m^{n_i^y}\prod_{j=1}^{n_i^y}(2k-2j+1)} = \left(\frac{1-O(n_i^y/k)}{2m}\right)^{n_i^y}. \tag{2.43}$$

Since $\sum_{i=1}^{r'} n_i^y = r$, substituting (2.43) into (2.39) and rolling back all the conditionings we did so far, we get

$$\mathbb{P}_{\eta,x}\big(C_{t_j}(x_{t_j-1}) = z_j \ \forall\, j \in [1,r] \mid A(x_{[0,t]};T)\big) \geq \frac{1-O(r^2/k)}{\ell^r} = \frac{1-o(1)}{\ell^r}, \tag{2.44}$$

where we use that $r^2/k \to 0$ since $r = o(\log n)$ and $k = \alpha_n n$ with $(\log n)^2 \alpha_n \to \infty$. Now sum over $z_1, \dots, z_r$ in (2.36), to obtain

$$\mathbb{P}_{\eta,x}\big(x_{[0,t]} \text{ is open} \mid A(x_{[0,t]}; T)\big) \geq \frac{(1-o(1)) \prod_{j=1}^{r} \deg(x_{t_j})}{\ell^r}, \tag{2.45}$$

and multiply with (2.33) to get the desired result. □

§2.4 Tree-like structure of the configuration model

In this section we look at the structure of the neighborhood of a half-edge chosen uniformly at random in the configuration model. Since we will work with different probability spaces, we will denote by $\mathbb{P}$ a generic probability measure whose meaning will be clear from the context.

For fixed $t \in \mathbb{N}$, $x \in H$ and $\eta \in \mathit{Conf}_H$, we denote by $B_t^\eta(x) := \{y \in H\colon \operatorname{dist}_\eta(x,y) \leq t\}$ the t-neighborhood of x in η, where $\operatorname{dist}_\eta(x,y)$ is *the length of the shortest non-backtracking path from x to y*. We start by estimating the mean of $|B_t^\eta(x)|$, the number of half-edges in $B_t^\eta(x)$.

Lemma 2.4.1 (Average size of balls of relevant radius). *Let ν_n be as in Condition 2.1.2 and suppose that $t = t(n) = o(\log n)$. Then, for any $\delta > 0$,*

$$\mathbb{E}(|B_t^\eta(x)|) = [1 + o(1)]\, \nu_n^{t+1} = o(n^\delta), \tag{2.46}$$

where the expectation is w.r.t. μ_n in (2.13).

Proof. We have

$$|B_t^\eta(x)| = \sum_{y \in H} \mathbb{1}_{\{\operatorname{dist}_\eta(x,y) \leq t\}}. \tag{2.47}$$

Putting this into the expectation, we get

$$\mathbb{E}(|B_t^\eta(x)|) = \frac{1}{\ell} \sum_{x,y \in H} \mathbb{P}(\operatorname{dist}_\eta(x,y) \leq t). \tag{2.48}$$

For fixed $x, y \in H$,

$$\begin{aligned}
\mathbb{P}(\operatorname{dist}_\eta(x,y) \leq t) &\leq \sum_{d=1}^{t} \sum_{x_1,\dots,x_{d-1} \in H} \mathbb{P}(x x_1 \cdots x_{d-1} y \text{ forms a self-avoiding path in } \eta) \\
&\leq \sum_{d=1}^{t} \sum_{x_1,\dots,x_{d-1} \in H} \left(\prod_{j=1}^{d-1} \frac{\deg(x_j)}{\ell - 2j + 1} \right) \frac{\deg(y)}{\ell - 2d + 1} \\
&= \frac{\deg(y)}{\ell} \sum_{d=1}^{t} \left(\prod_{i=1}^{d} \frac{\ell}{\ell - 2i + 1} \right) \sum_{x_1,\dots,x_{d-1} \in H} \left(\prod_{i=1}^{d-1} \frac{\deg(x_i)}{\ell} \right) \\
&= \frac{\deg(y)}{\ell} \sum_{d=1}^{t} \left(\prod_{i=1}^{d} \frac{\ell}{\ell - 2i + 1} \right) \left(\sum_{z \in H} \frac{\deg(z)}{\ell} \right)^{d-1}.
\end{aligned} \tag{2.49}$$

Since $t = o(\log n)$ and $\ell = \Theta(n)$, we have

$$\mathbb{P}(\mathrm{dist}_\eta(x, y) \le t) \le [1 + o(1)] \frac{\deg(y)}{\ell} (\nu_n)^t. \tag{2.50}$$

Substituting this into (2.48), we get

$$\mathbb{E}(|B_t^\eta(x)|) \le \frac{1 + o(1)}{\ell} \sum_{x,y \in H} \frac{\deg(y)}{\ell} (\nu_n)^t = [1 + o(1)] (\nu_n)^{t+1} = o(n^\delta), \tag{2.51}$$

where the last equality follows from (R2) in Condition 2.1.2 and the fact that $t = o(\log n)$. □

For the next result we will use an *exploration process* to build the neighborhood of a uniformly chosen half-edge. (Similar exploration processes have been used in [16],[21] and [67].) We explore the graph by starting from a uniformly chosen half-edge x and building up the graph by successive uniform pairings, as explained in the procedure below. Let $\mathsf{G}(s)$ denote the *thorny graph* obtained after s pairings as follows (in our context, a thorny graph is a graph in which half-edges are not necessarily paired to form edges, as shown in Fig. 2.6). We set $\mathsf{G}(0)$ to consist of x, its siblings, and the incident vertex $v(x)$. Along the way we keep track of *the set of unpaired half-edges at each time* s, denoted by $U(s) \subset H$, and the so-called *active* half-edges, $A(s) \subset U(s)$. We initialize $U(0) = H$ and $A(0) = \{x\}$. We build up the sequence of graphs $(\mathsf{G}(s))_{s \in \mathbb{N}_0}$ as follows:

(a) At each time $s \in \mathbb{N}$, take the *next* unpaired half-edge in $A(s-1)$, say y. Sample a half-edge uniformly at random from H, say z. If z is already paired or $z = y$, then reject and sample again. Pair y and z.

(b) Add the newly formed edge $\{y, z\}$, the incident vertex $v(z)$ of z, and its siblings to $\mathsf{G}(s-1)$, to obtain $\mathsf{G}(s)$.

(c) Set $U(s) = U(s-1) \setminus \{y, z\}$, i.e., remove y, z from the set of unpaired half-edges, and set $A(s) = A(s-1) \cup \{H(v(z))\} \setminus \{y, z\}$, i.e., add siblings of z to the set of active half-edges and remove the active half-edges just paired.

This procedure stops when $A(s)$ is empty. We think of $A(s)$ as a first-in first-out queue. So, when we say that we pick the *next* half-edge in Step 1, we refer to the half-edge on top of the queue, which ensures that we maintain the breadth-first order. The rejection sampling used in Step 1 ensures that the resulting graph is distributed according to the configuration model. This procedure eventually gives us the connected component of x in η, the part of the graph that can be reached from x by a non-backtracking walk, where η is distributed uniformly on $Conf_H$.

Lemma 2.4.2 (Tree-like neighborhoods). *Suppose that* $s = s(n) = o(n^{(1-2\delta)/2})$ *for some* $\delta \in (0, \frac{1}{2})$. *Then* $\mathsf{G}(s)$ *is a tree with probability* $1 - o(n^{-\delta})$.

Proof. Let F be the first time the uniform sampling of z in Step 1 fails at the first attempt, or z is a sibling of x, or z is in $A(s-1)$. Thus, at time F we either choose

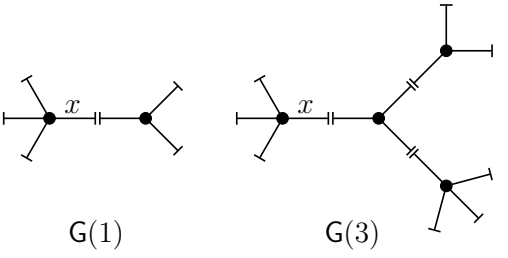


Figure 2.6: Example snapshots of $\mathsf{G}(s)$ *at times* $s = 1$ *and* $s = 3$.

an already paired half-edge or we form a cycle by pairing to some half-edge already present in the graph. We have

$$\mathbb{P}(\mathsf{G}(s) \text{ is not a tree}) \leq \mathbb{P}(F \leq s). \tag{2.52}$$

Let Y_i, $i \in \mathbb{N}$, be i.i.d. random variables whose distribution is the same as the distribution of the degree of a uniformly chosen half-edge. When we form an edge before time F, we use one of the unpaired half-edges of the graph, and add new unpaired half-edges whose number is distributed as Y_1. Hence the number of unpaired half-edges in $\mathsf{G}(u)$ is stochastically dominated by $\sum_{i=1}^{u+1} Y_i - u$, with one of the Y_i's coming from x and the other ones coming from the formation of each edge. Therefore the probability that one of the conditions of F will be met at step u is stochastically dominated by $(\sum_{i=1}^{u} Y_i + u - 2)/\ell$. We either choose an unpaired half-edge in $\mathsf{G}(u)$ or we choose a half-edge belonging to an edge in $\mathsf{G}(u)$, and by the union bound we have

$$\begin{aligned}&\mathbb{P}(\mathsf{G}(s) \text{ is not a tree} \mid (Y_i)_{i\in[1,s]}) \leq \mathbb{P}(F \leq s \mid (Y_i)_{i\in[1,s]})\\ &\leq \frac{\sum_{u=1}^{s}\sum_{i=1}^{u}(Y_i + u - 2)}{\ell} = \frac{\sum_{i=1}^{s}(s-i+1)Y_i + s(s-1)/2}{\ell}.\end{aligned} \tag{2.53}$$

Since $\mathbb{E}(Y_1) = \nu_n = O(1)$ and $s = o(n^{(1-2\delta)/2})$, via the Markov inequality we get that, with probability at least $1 - o(n^{-\delta})$,

$$s\sum_{i=1}^{s} Y_i < n^{1-\delta}. \tag{2.54}$$

Combining this with the bound given above and the fact that $\ell = \Theta(n)$, we arrive at

$$\mathbb{P}(\mathsf{G}(s) \text{ is not a tree}) = o(n^{-\delta}). \tag{2.55}$$

□

To further prepare for the proof of the lower bound in (2.26) and Proposition 2.2.1 in Section 2.5, we introduce one last ingredient. For $x \in H$ and $\eta \in \mathit{Conf}_H$, we denote by $\bar{B}_t^{\eta}(x)$ the set of half-edges from which there is a non-backtracking path to x of length at most t. For fixed $t \in \mathbb{N}$, $T = \{t_1, \ldots, t_r\} \subseteq [1, t]$ and $\eta \in \mathit{Conf}_H$, we say that an $(r+1)$-tuple $(x_0, x_1, \ldots, x_r)$ is *good* for T in η if it satisfies the following two properties:

(a) $B^{\eta}_{t_j - t_{j-1}}(x_j)$ is a tree for $j \in [1,r]$ with $t_0 = 0$, and $\bar{B}^{\eta}_{t-t_r}(x_r)$ is a tree.

(b) The trees $B^{\eta}_{t_j - t_{j-1}}(x_j)$ for $j \in [1,r]$ and $\bar{B}^{\eta}_{t-t_r}(x_r)$ are all disjoint.

For a good $(r+1)$-tuple all the segmented paths, such that the ith segment starts from x_{i-1} and is of length $t_i - t_{i-1}$ for $i \in [1,r]$ and the $(r+1)$st segment ends at x_r and is of length $t - t_r$, are self-avoiding by the tree property. The next lemma states that whp in η almost all $(r+1)$-tuples are good. We denote by $N^{\eta}_t(T)$ the set of $(r+1)$-tuples that are good for T in η, and let $N^{\eta}_t(T)^c$ be the complement of $N^{\eta}_t(T)$. We have the following estimate on $|N^{\eta}_t(T)|$:

Lemma 2.4.3 (Estimate on good paths). *Suppose that $t = t(n) = o(\log n)$. Then there exist $\bar{\delta} > 0$ such that whp in η for all $T \subseteq [1,t]$,*

$$|N^{\eta}_t(T)| = (1 - n^{-\bar{\delta}})\ell^{|T|+1}. \tag{2.56}$$

Proof. Fix $\varepsilon > 0$ and $T \subseteq [1,t]$ with $|T| = r$. We want to show that whp $|N^{\eta}_t(T)^c| \leq \varepsilon \ell^{r+1}$. By the Markov inequality, we have

$$\mathbb{P}(|N^{\eta}_t(T)^c| > \varepsilon \ell^{r+1}) \leq \frac{\mathbb{E}(|N^{\eta}_t(T)^c|)}{\varepsilon \ell^{r+1}} = \frac{\mathbb{P}(Z_{[0,r]} \in N^{\eta}_t(T)^c)}{\varepsilon}, \tag{2.57}$$

where $Z_0, \dots, Z_r$ are i.i.d. uniform half-edges and we use that $1/\ell^{r+1}$ is the uniform probability over a collection of $r+1$ half-edges. Let $B_{i-1} = B^{\eta}_{t_i - t_{i-1}}(Z_{i-1})$ for $i \in [1,r]$ and $B_r = B^{\eta}_{t-t_r}(Z_r)$. By the union bound,

$$\mathbb{P}\big(Z_{[0,r]} \in N^{\eta}_t(T)^c\big) \leq \sum_{i=0}^{r} \mathbb{P}(B_i \text{ is not a tree}) + \sum_{i,j=0}^{r} \mathbb{P}(B_i \cap B_j \neq \varnothing). \tag{2.58}$$

By Lemma 2.4.1 and since $t = o(\log n)$, for any $0 < \delta < \frac{1}{2}$ we have $\mathbb{E}|B_i| = o(n^{\delta})$, and so by the Markov inequality $|B_i| = o(n^{(1-2\delta)/2})$ with probability $1 - o(n^{-\delta})$. Hence, by Lemma 2.4.2 and since $\ell = \Theta(n)$, for $i \in [1,r]$, we have

$$\mathbb{P}(B_{i-1} \text{ is not a tree}) = o(n^{-\delta}). \tag{2.59}$$

Again using Lemma 2.4.1, we see that for any $i, j \in [1,r]$,

$$\mathbb{P}(B_i \cap B_j \neq \varnothing) \leq \mathbb{P}(Z_j \in B^{\eta}_t(Z_i)) = \frac{\mathbb{E}(|B^{\eta}_t(Z_i)|)}{\ell} \leq o(n^{\delta - 1}). \tag{2.60}$$

Since $r \leq t = o(\log n)$, setting $\bar{\delta} = 2\delta/3$ and taking $\varepsilon = n^{-\delta}$, we get

$$\mathbb{P}(|N^{\eta}_t(T)^c| > \varepsilon \ell^{r+1}) \leq \frac{r n^{-\bar{\delta}} + r^2 n^{\bar{\delta} - 1}}{\varepsilon} = o(n^{-\delta/4}) \tag{2.61}$$

uniformly in $T \subseteq [1,t]$. Since there are 2^t different $T \subseteq [1,t]$ and $2^t = 2^{o(\log n)} = o(n^{\delta/8})$, taking the union bound we see that (2.56) holds for all $T \subseteq [1,t]$ with probability $1 - o(n^{-\delta/8})$. □

§2.5 Closeness to stationarity and tail behavior of stopping time

We are now ready to prove the lower bound in (2.26) and Proposition 2.2.1. Before giving these proofs, we need one more lemma, for which we introduce some new notation. For fixed $t \in \mathbb{N}$, $T \subseteq [1,t]$ with $|T| = r > 0$, $\eta \in \mathit{Conf}_H$ and $x, y \in H$, let $N_t^\eta(x,y;T)$ denote the set of $(r-1)$-tuples such that $(x, x_1, \dots, x_{r-1}, y)$ is good for T in η. Furthermore, for a given $(r+1)$-tuple $\mathbf{x} = (x, x_1, \dots, x_{r-1}, y)$ that is good for T in η, let $\mathsf{SP}_t^\eta(\mathbf{x};T)$ denote the set of all segmented paths in which the ith segment starts at x_{i-1} and is of length $t_i - t_{i-1}$ for $i \in [1,r]$ with $x_0 = x$ and $t_0 = 0$, and the $(r+1)$st segment ends at y and is of length $t - t_r$. By the definition of a good tuple, these paths are self-avoiding, and hence $\mathsf{SP}_t^\eta(\mathbf{x};T) \subset \mathsf{SP}_t^\eta(x,y;T)$.

Lemma 2.5.1 (Total mass of relevant paths). *Suppose that $t = t(n) = o(\log n)$. Then whp in η and x, y for all $T \subseteq [1,t]$,*

$$\sum_{x_{[0,t]} \in \mathsf{SP}_t^\eta(x,y;T)} \mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]} \mid A(x_{[0,t]};T)\big) \geq \frac{1-o(1)}{\ell}. \tag{2.62}$$

Proof. By Lemma 2.4.3, the number of pairs of half-edges x, y for which $|N_t^\eta(x,y;T)| \geq (1-n^{-\bar{\delta}})\ell^{|T|-1} = [1-o(1)]\,\ell^{|T|-1}$ for all $T \in [1,t]$ is at least $(1-2^t n^{-\bar{\delta}})\ell^2 = [1-o(1)]\,\ell^2$ whp in η. Take such a pair $x, y \in H$, and let $r = |T|$. By Lemma 2.3.2 and the last observation before the statement of Lemma 2.5.1, we have

$$\begin{aligned}
&\sum_{x_{[0,t]} \in \mathsf{SP}_t^\eta(x,y;T)} \mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]} \mid A(x_{[0,t]};T)\big) \\
&\qquad \geq \sum_{\mathbf{x} \in N_t^\eta(x,y;T)} \sum_{y_0 \dots y_t \in \mathsf{SP}_t^\eta(\mathbf{x},T)} \frac{1-o(1)}{\ell^r} \prod_{i \in [1,t] \setminus T} \frac{1}{\deg(y_i)}.
\end{aligned} \tag{2.63}$$

We analyze at the second sum by inspecting the contributions coming from each segment separately. For fixed $\mathbf{x} \in N_t^\eta(x,y;T)$, when we sum over the segmented paths in $\mathsf{SP}_t^\eta(\mathbf{x},T)$, we sum over all paths that go out of x_{i-1} of length $t_i - t_{i-1}$ for $i \in [1,r]$. Since $\prod_{j=t_{i-1}+1}^{t_i-1} \frac{1}{\deg(y_j)}$ is the probability that the random walk without backtracking follows this path on the static graph given by η starting from x_{i-1}, when we sum over all such paths the contribution from these terms sums up to 1 for each $i \in [1,r]$, i.e., the contributions of the first r segments coming from the products of inverse degrees sum up to 1. For the last segment we sum, over all paths going into y, the probability that the random walk without backtracking on the static graph given by η follows the path. Since the uniform distribution is stationary for this random walk, the sum over the last segment of the probabilities $\frac{1}{\ell}\prod_{j=t_r+1}^{t} \frac{1}{\deg(y_j)}$ gives us $1/\ell$. With this observation, using that $|N_t^\eta(x,y;T)| \geq (1-o(1))\ell^{r-1}$, we get

$$\begin{aligned}
&\sum_{x_{[0,t]} \in \mathsf{SP}_t^\eta(x,y;T)} \mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]} \mid A(x_{[0,t]};T)\big) \\
&\qquad \geq \frac{1-o(1)}{\ell} \sum_{\mathbf{x} \in N_t^\eta(x,y;T)} \frac{1-o(1)}{\ell^{r-1}} = \frac{1-o(1)}{\ell},
\end{aligned} \tag{2.64}$$

which is the desired result. □

• *Proof of* (2.21). For any self-avoiding segmented path $x_0 \cdots x_t$, we have $|x_{[s,s']}| = s' - s + 1$ for all $1 \le s < s' \le t$. By Lemma 2.3.1, the probability $\mathbb{P}_{\eta,x}(A(x_{[0,t]};T))$ depends on η and T only, and we can write $\mathbb{P}_{\eta,x}(A(x_{[0,t]};T)) = p_t^\eta(T)$ for any $xx_1 \cdots x_{t-1}y \in \mathsf{SP}_t^\eta(x,y;T)$. Applying Lemma 2.5.1, we get

$$\begin{aligned}
&\mathbb{P}_{\eta,x}(X_t = y, \tau \le t) \qquad (2.65)\\
&\ge \sum_{r=1}^{t} \sum_{\substack{T\subseteq[1,t]\\|T|=r}} \sum_{x_{[0,t]}\in\mathsf{SP}_t^\eta(x,y;T)} \mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]} \mid A(x_{[0,t]};T)\big)\, \mathbb{P}_{\eta,x}\big(A(x_{[0,t]};T)\big)\\
&\ge \frac{1-o(1)}{\ell} \sum_{r=1}^{t} \sum_{\substack{T\subseteq[1,t]\\|T|=r}} p_t^\eta(T).
\end{aligned}$$

If the t-neighborhood of x in η is a tree, then all t-step non-backtracking paths starting at x are self-avoiding. (Here is a place where the non-backtracking nature of our walk is crucially used!) In particular, for any such path $xx_1 \cdots x_t$ we have $\mathbb{P}_{\eta,x}(A(x_{[0,t]};\varnothing)) = p_t^\eta(\varnothing)$. Denoting by $\Gamma_t^\eta(x)$ the set of paths in η of length t that start from x, we also have

$$\begin{aligned}
\mathbb{P}_{\eta,x}(\tau > t) &= \sum_{x_0\cdots x_t\in\Gamma_t^\eta(x)} \mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]}, A(x_{[0,t]};\varnothing)\big)\\
&= \sum_{x_0\cdots x_t\in\Gamma_t^\eta(x)} \prod_{i=1}^{t} \frac{1}{\deg(x_i)}\, p_t^\eta(\varnothing) = p_t^\eta(\varnothing), \qquad (2.66)
\end{aligned}$$

since the product $\prod_{i=1}^t \frac{1}{\deg(x_i)}$ is the probability that a random walk without backtracking in the static η follows the path $x_0x_1\cdots x_t$, and we take the sum over all paths going out of x.

For a fixed path $x_0x_1\cdots x_t$, we have

$$\sum_{r=1}^{t} \sum_{\substack{T\subseteq[1,t]\\|T|=r}} \mathbb{P}_{\eta,x}\big(A(x_{[0,t]};T)\big) = 1 - \mathbb{P}_{\eta,x}\big(A(x_{[0,t]};\varnothing)\big). \qquad (2.67)$$

So, when the t-neighborhood of x in η is a tree, we have

$$\sum_{r=1}^{t} \sum_{\substack{T\subseteq[1,t]\\|T|=r}} p_t^\eta(T) = 1 - p_t^\eta(\varnothing) = 1 - \mathbb{P}_{\eta,x}(\tau > t) = \mathbb{P}_{\eta,x}(\tau \le t), \qquad (2.68)$$

which gives

$$\mathbb{P}_{\eta,x}(X_t = y, \tau \le t) \ge \frac{1-o(1)}{\ell}\, \mathbb{P}_{\eta,x}(\tau \le t) \qquad (2.69)$$

and settles the lower bound (2.26). Since the latter holds whp in η and x, y, we have that the number of y for which this holds is $[1-o(1)]\,\ell$ whp in η and x. Denoting the

set of $y \in H$ for which the lower bound in (2.26) holds by $N_t^\eta(x)$, we get that whp in η and x,

$$\begin{aligned}\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \le t) - U_H(\cdot)\|_{\text{TV}} &= \sum_{y\in H}\left[\frac{1}{\ell} - \mathbb{P}_{\eta,x}(X_t = y \mid \tau \le t)\right]^+ \\ &\le \sum_{y\in N_t^\eta(x)}\left[\frac{1}{\ell} - \frac{1-o(1)}{\ell}\right]^+ + \sum_{y\notin N_t^\eta(x)}\frac{1}{\ell} = o(1),\end{aligned} \tag{2.70}$$

which is (2.21). □

• *Proof of* (2.22). First note that $\mathbb{P}_{\eta,x}(X_t \in B_t^\eta(x) \mid \tau > t) = 1$. On the other hand, using Lemma 2.4.1 and the Markov inequality, we see that $U_H(B_t^\eta(x)) = |B_t^\eta(x)|/\ell = o(1)$ whp in η and x, and so we get

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H(\cdot)\|_{\text{TV}} \ge \mathbb{P}_{\eta,x}(X_t \in B_t^\eta(x) \mid \tau > t) - U_H(B_t^\eta(x)) = 1 - o(1). \tag{2.71}$$

□

• *Proof of* (2.23). Taking $T = \varnothing$ in Lemma 2.4.3, we see that $B_t^\eta(x)$ is a tree whp in η and x, so each path in η of length t that goes out of x is self-avoiding. By looking at pathwise probabilities, we see that

$$\mathbb{P}_{\eta,x}(\tau > t) = \sum_{x_0\cdots x_t \in \Gamma_t^\eta(x)} \mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]}, x_{i-1} \notin R_{\le i}\,\forall\, i \in [1,t]\big). \tag{2.72}$$

Since the event $\{x_{i-1} \notin R_{\le i}\,\forall\, i \in [1,t]\}$ implies that the edge involving x_{i-1} is open a time i,

$$\mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]} \mid x_{i-1} \notin R_{\le i}\,\forall\, i \in [1,t]\big) = \prod_{i=1}^t \frac{1}{\deg(x_i)}. \tag{2.73}$$

Next, let us look at the probability $\mathbb{P}_{\eta,x}(x_i \notin R_{\le i}\,\forall\, i \in [1,t])$. By rearranging and conditioning, we get

$$\begin{aligned}\mathbb{P}_{\eta,x}\big(x_{i-1} \notin R_{\le i}\,\forall\, i \in [1,t]\big) &= \mathbb{P}_{\eta,x}\big(x_j \notin R_i\,\forall\, j \in [i-1,t-1]\,\forall\, i \in [1,t]\big) \\ &= \prod_{i=1}^t \mathbb{P}_{\eta,x}\Big(x_j \notin R_i\,\forall j \in [i-1,t-1] \;\Big|\; x_k \notin R_j\,\forall\, k \in [j-1,t-1]\,\forall\, j \in [1,i-1]\Big).\end{aligned} \tag{2.74}$$

Observe that, on the event $\{x_k \notin R_j\,\forall\, k \in [j-1,t-1\,\forall\, j \in [1,i-1]\}$, the path $x_{i-1}\cdots x_{t-1}$ has not rewired until time $i-1$, and so the number of edges given by these half-edges is $t-i+1$, since it was originally a self-avoiding path. With this we see that for any $i \in [1,t]$,

$$\mathbb{P}_{\eta,x}\Big(x_j \notin R_i\,\forall j \in [i-1,t-1] \;\Big|\; x_k \notin R_j\,\forall\, k \in [j-1,t-1]\,\forall\, j \in [1,i-1]\Big) = \frac{\binom{m-t+i-1}{k}}{\binom{m}{k}}, \tag{2.75}$$

and hence

$$\mathbb{P}_{\eta,x}\big(x_{i-1} \notin R_{\le i}\,\forall\, i \in [1,t]\big) = \prod_{i=1}^{t} \frac{\binom{m-t+i-1}{k}}{\binom{m}{k}} = \prod_{i=1}^{t} \frac{\binom{m-i}{k}}{\binom{m}{k}}$$
$$= \prod_{i=1}^{t}\prod_{j=0}^{i-1}\left(1-\frac{k}{m-j}\right) = \prod_{j=1}^{t}\left(1-\frac{k}{m-j+1}\right)^{t-j+1}. \tag{2.76}$$

Since $j \le t = o(\log n)$, $m = \Theta(n)$ and $n/\log^2 n = o(k)$, we have

$$\mathbb{P}_{\eta,x}\big(x_{i-1} \notin R_{\le i} \text{ for all } i \in [1,t]\big) = [1+o(1)]\,(1-k/m)^{t(t+1)/2} = (1-\alpha_n)^{t(t+1)/2}+o(1). \tag{2.77}$$

Putting this together with (2.73) and inserting it into (2.72), we get

$$\mathbb{P}_{\eta,x}(\tau > t) = [(1-\alpha_n)^{t(t+1)/2} + o(1)] \sum_{x_0\cdots x_t \in \Gamma_t^{\eta}(x)} \prod_{i=1}^{t} \frac{1}{\deg(x_i)}$$
$$= (1-\alpha_n)^{t(t+1)/2} + o(1), \tag{2.78}$$

since, for each path $x_0 \cdots x_t$, the product $\prod_{i=1}^{t} \frac{1}{\deg(x_i)}$ is the probability that the random walk without backtracking on the static graph given by η follows the path, and we sum over all paths starting from x. □

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CHAPTER 3

Random walks on dynamic configuration models: a trichotomy

This chapter is based on a joint article with Luca Avena, Remco van der Hofstad and Frank den Hollander [13].

Abstract

We consider a dynamic random graph on n vertices that is obtained by starting from a random graph generated according to the configuration model with a prescribed degree sequence and at each unit of time randomly rewiring a fraction α_n of the edges. We are interested in the mixing time of a random walk without backtracking on this dynamic random graph in the limit as $n \to \infty$, when α_n is chosen such that $\lim_{n\to\infty} \alpha_n(\log n)^2 = \beta \in [0, \infty]$. In [12] we found that, under mild regularity conditions on the degree sequence, the mixing time is of order $1/\sqrt{\alpha_n}$ when $\beta = \infty$. In the present paper we investigate what happens when $\beta \in [0, \infty)$. It turns out that the mixing time is of order $\log n$, with the scaled mixing time exhibiting a one-sided cutoff when $\beta \in (0, \infty)$ and a two-sided cutoff when $\beta = 0$. The occurrence of a one-sided cutoff is a rare phenomenon. In our setting it comes from a competition between the time scales of mixing on the static graph, as identified by Ben-Hamou and Salez [16], and the regeneration time of first stepping across a rewired edge.

§3.1 Introduction

§3.1.1 Background

The goal of the present paper is to study the mixing time of a random walk *without backtracking* on a dynamic version of the configuration model. The *static* configuration model is a random graph with a prescribed degree sequence. For random walk on the static configuration model, with or without backtracking, the asymptotics of the associated mixing time, and related properties such as the presence of the so-called cutoff phenomenon, were derived recently by Berestycki, Lubetzky, Peres and Sly [21], and by Ben-Hamou and Salez [16]. In particular, under mild assumptions on the degree sequence, guaranteeing that the graph is an *expander* with high probability, the mixing time was shown to be of order $\log n$, with n the number of vertices.

In an earlier paper [12], we consider a *discrete-time dynamic* version of the configuration model, where at each unit of time a fraction α_n of the edges is sampled and rewired uniformly at random. Our dynamics *preserves the degrees* of the vertices. Consequently, when considering a random walk on this dynamic configuration model, its *stationary distribution remains constant over time* and the analysis of its mixing time is a well-posed question. It is natural to expect that, due to the graph dynamics, the random walk *mixes faster* than the $\log n$ order known for the static model. Under very mild assumptions on the prescribed degree sequence (Condition 3.1.2 below), we have shown that this is indeed the case when $(\alpha_n)_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} \alpha_n(\log n)^2 = \infty$, which corresponds to a regime of 'fast enough' graph dynamics. In particular, we have shown that for every $\varepsilon \in (0,1)$ the ε-mixing time grows like $\sqrt{2\log(1/\varepsilon)/\alpha_n}$ as $n \to \infty$ (when also $\lim_{n\to\infty} \alpha_n = 0$), with high probability (in the sense of Definition 3.1.1 below).

In the present paper we look at a slower dynamics, namely, $(\alpha_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} \alpha_n(\log n)^2 = \beta \in [0,\infty)$. Our main result (Theorem 3.1.4 below) states that, under somewhat stronger assumptions on the prescribed degree sequence (Condition 3.1.3 below), the mixing time is of order $\log n$, as for the static model, but that there is an interesting difference between the cases $\beta \in (0,\infty)$ and $\beta = 0$. Our proof builds on the strategy developed in [12] for the regime of fast dynamics. However, the argument in [12] establishing the almost self-avoiding nature of the random walk cannot be immediately extended to the regime of slow dynamics. This difficulty is overcome by using a different proof, in combination with an annealing argument (see Section 3.3).

The rest of the paper is organised as follows. In Section 3.1.2 we define the model. This is a verbatim repetition of what was written in [12, Section 1.2], in which we introduce notation and set the stage. In Section 3.1.3 we state our main theorem, which is a *trichotomy* for the cases $\beta = \infty$, $\beta \in (0,\infty)$ and $\beta = 0$. In Section 3.1.4 we place this theorem in its proper context.

Throughout the sequel we use standard notations for the asymptotic comparison of functions $f, g\colon \mathbb{N} \to [0,\infty)$: $f(n) = O(g(n))$ or $g(n) = \Omega(f(n))$ when $\limsup_{n\to\infty} f(n)/g(n) < \infty$; $f(n) = o(g(n))$ or $g(n) = \omega(f(n))$ when $\lim_{n\to\infty} f(n)/g(n) = 0$; $f(n) = \Theta(g(n))$ when both $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

§3.1.2 Model

Since this section was a verbatim repetition of Section 2.1.2, we remove it and refer to Section 2.1.2 whenever necessary.

§3.1.3 A trichotomy

We are interested in the behaviour of the total variation distance between the distribution of X_t and the uniform distribution

$$\mathcal{D}_{\eta,x}(t) := \|\mathbb{P}_{\eta,x}(X_t \in \cdot\,) - U_H(\cdot)\|_{\mathrm{TV}}. \tag{3.1}$$

Note that $\mathcal{D}_{\eta,x}(t)$ depends on the initial configuration η and half-edge x. We will prove statements that hold for *typical* choices of (η, x) under the uniform distribution μ_n (recall that H depends on the number of vertices n) given by

$$\mu_n := \mathrm{Conf}_H \times U_H \quad \text{ on } \mathit{Conf}_H \times H, \tag{3.2}$$

where *typical* is made precise through the following definition:

Definition 3.1.1 (**With high probability**). *A statement that depends on the initial configuration η and initial half-edge x is said to hold* with high probability (whp) *in η and x if the μ_n-measure of the set of pairs (η, x) for which the statement holds tends to 1 as $n \to \infty$.*

Regularity conditions

In Theorem 3.1.4 below we use two sets of regularity conditions on the degree sequence:

Condition 3.1.2. ***(Regularity of degrees)***

(R1) *ℓ is even and $\ell = \Theta(n)$ as $n \to \infty$.*

(R2) *$\limsup_{n\to\infty} \nu_n < \infty$, where*

$$\nu_n := \frac{\sum_{z\in H} \deg(z)}{\ell} = \frac{\sum_{v\in V} d(v)[d(v)-1]}{\sum_{v\in V} d(v)} \tag{3.3}$$

denotes the expected forward degree of a uniformly chosen half-edge.

(R3) *$d(v) \geq 2$ for all $v \in V$.*

Condition 3.1.3. ***(Regularity of degrees (Cont.))***

(R1*) *$d_{\max} = \ell^{o(1)}$ as $n \to \infty$, where*

$$d_{\max} := \max_{v\in V} d(v). \tag{3.4}$$

(R2*) *As* $n \to \infty$,

$$\frac{\lambda_2}{\lambda_1^3} = \omega\left(\frac{(\log\log \ell)^2}{\log \ell}\right), \qquad \frac{\lambda_2^{3/2}}{\lambda_3\sqrt{\lambda_1}} = \omega\left(\frac{1}{\sqrt{\log \ell}}\right), \tag{3.5}$$

where

$$\lambda_1 := \frac{1}{\ell}\sum_{z\in H} \log(\deg(z)), \qquad \lambda_m := \frac{1}{\ell}\sum_{z\in H} |\log(\deg(z)) - \lambda_1|^m, \quad m = 2, 3. \tag{3.6}$$

(R3*) $d(v) \geq 3$ *for all* $v \in V$.

Condition 3.1.2 was used in [12] to deal with the regime of 'fast graph dynamics'. Conditions (R1) and (R2) are minimal requirements to guarantee that the graph is locally tree-like. Condition (R3) ensures that the random walk without backtracking is well-defined. Condition 3.1.3 was used in [16] to deal with the regime of no graph dynamics, i.e., the static graph. Condition (R1*) provides control on the large degrees. Condition (R2*) is technical and states that the degrees vary neither too little nor too much. Condition (R3*) ensures that the graph is connected with high probability and that there are no nodes where the random walk without backtracking moves deterministically.

Below, we will work under the Conditions (R1)–(R3) as well as (R1*)–(R3*). If $D_n = d(V_n)$ denotes the degree of a random vertex, then Condition (R2*) is implied by the often used condition that $D_n \to D$ in distribution (when $\mathbb{P}(D \geq 3) > 0$), together with $\mathbb{E}[D_n] \to \mathbb{E}[D]$ (see e.g. van der Hofstad [93, Chapter 7]). Thus, Condition (R2*) is rather mild. Condition (R1*) excludes vertices with a degree that is a positive power of n, which is claimed to be realistic for real-world networks (see e.g. [93, Chapter 1] for an extensive introduction). We have a truncation argument, along the lines of the one in Berestycki, van der Hofstad and Salez [22], showing that the degrees can be truncated and the random walk is unlikely to notice this truncation. However, the truncated graph may have vertices of degree 2, so that it is not clear how to apply the results in Ben-Hamou and Salez [16]. Furthermore, we believe that Condition (R3*) is unnecessary for our results. We state it here because we rely on the work of [16], which considers random walk without backtracking started from the worst-possible starting point. When there is a positive proportion of vertices of degree 2, the configuration model is bound to contain a long path of such vertices. On such a stretch, the walk moves deterministically, but it slows down the mixing because it takes time $\omega(\log n)$ to leave the stretch. Thus, mixing would occur at a time that is $\omega(\log n)$ larger than that when the walk starts from a uniform vertex, which makes worst-case and average-case mixing different. Still, since our walk starts from the uniform measure on half-edges, it is unlikely to encounter such a stretch. We refrain from investigating this issue further.

Main theorem

Define the proportion of rewired edges per unit of time as

$$\alpha_n := k/m, \qquad n \in \mathbb{N}, \tag{3.7}$$

where $m = \ell/2$ is the total number of edges and k is the number of edges that get rewired per unit of time. For the static model ($\alpha_n \equiv 0$), under Condition 3.1.3, the ε-mixing time $\inf\{t \in \mathbb{N}_0 \colon \mathcal{D}_{\eta,x}(t) \le \varepsilon\}$ is known to scale like $[1 + o(1)]\, c_{n,\mathrm{stat}} \log n$ for all $\varepsilon \in (0,1)$, with $c_{n,\mathrm{stat}} = 1/\lambda_1 \in (0,\infty)$ (Ben-Hamou and Salez [16]). If Condition 3.1.2 holds too, then $n \mapsto c_{n,\mathrm{stat}}$ is bounded away from 0 and ∞. If also the degree distribution tends to a limit, then $\lim_{n\to\infty} c_{n,\mathrm{stat}} = c_{\mathrm{stat}} \in (0,\infty)$.

Our main theorem shows that the above behaviour turns into a *trichotomy* for the dynamic model:

Theorem 3.1.4 (Scaled mixing profiles). *Suppose that* $\lim_{n\to\infty} \alpha_n (\log n)^2 = \beta \in [0,\infty]$. *The following hold whp in* η *and* x*:*

(1) *Subject to Condition* 3.1.2*, if* $\beta = \infty$*, then*

$$\mathcal{D}_{\eta,x}\big(c\alpha_n^{-1/2}\big) = \mathrm{e}^{-c^2/2} + o(1), \quad c \in [0,\infty). \tag{3.8}$$

(2) *Subject to Condition* 3.1.2(R1) *and Condition* 3.1.3*, if* $\beta \in (0,\infty)$*, then*

$$\mathcal{D}_{\eta,x}\big(c \log n\big) = \begin{cases} \mathrm{e}^{-\beta c^2/2} + o(1), & c \in [0, c_{n,\mathrm{stat}}), \\ o(1), & c \in (c_{n,\mathrm{stat}}, \infty). \end{cases} \tag{3.9}$$

(3) *Subject to Condition* 3.1.2(R1) *and Condition* 3.1.3*, if* $\beta = 0$*, then*

$$\mathcal{D}_{\eta,x}\big(c \log n\big) = \begin{cases} 1 - o(1), & c \in [0, c_{n,\mathrm{stat}}), \\ o(1), & c \in (c_{n,\mathrm{stat}}, \infty). \end{cases} \tag{3.10}$$

The proof of Theorem 3.1.4 is organised as follows. Theorem 3.1.4(1) was already proved in [12]. In Section 3.2 we show that Theorems 3.1.4(2)–(3) follow from a key proposition (Proposition 3.2.1 below), which will be proved in Sections 3.3–3.4. In Section 3.3 we show that on scale $\log n$ with high probability the random walk is self-avoiding, i.e., does not visit the same vertex twice, and that the same holds for a version of the random walk with random resets. In Section 3.4 we compute probabilities of rewiring histories and of self-avoiding paths conditional on rewiring histories.

§3.1.4 Discussion

1. Theorem 3.1.4 gives the sharp asymptotics of the mixing profiles in three regimes, which we refer to as *supercritical* ($\beta = \infty$), *critical* ($\beta \in (0,\infty)$) and *subcritical* ($\beta = 0$). The latter includes the case of the static configuration model. While in the supercritical regime the mixing time is of order $1/\sqrt{\alpha_n} = o(\log n)$, in the critical and the subcritical regime it is of order $\log n$ (see Fig. 3.1). Note that for $\beta = \infty$ the scaling does not depend on the degrees, while for $\beta \in [0,\infty)$ it does via the constant $c_{n,\mathrm{stat}}$.

2. For the static model, because the scaling of the ε-mixing time does not depend on $\varepsilon \in (0,1)$ (Ben-Hamou and Salez [16]) there is *two-sided cutoff*, i.e., the total variation

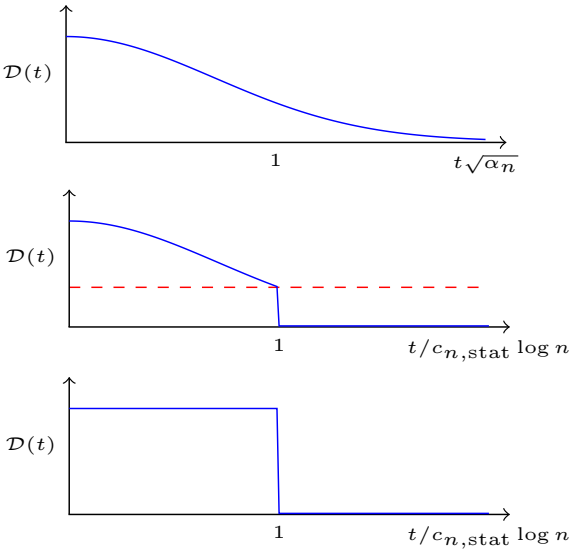

Figure 3.1: Plot of $\mathcal{D}(t)$ on time scale $1/\sqrt{\alpha_n}$ for $\beta = \infty$, respectively, on time scale $c_{n,\mathrm{stat}}\log n$ for $\beta \in (0,\infty)$ and $\beta = 0$. Because the scaling holds whp in η and x, we have suppressed these indices.

distance drops from 1 to 0 in a time window of width $o(\log n)$. Theorem 3.1.4 shows that this behaviour persists throughout the subcritical regime, but that in the critical regime the drop is not from height 1 but from height < 1, i.e., there is *one-sided cut-off*. In contrast, in the supercritical regime there is *no cutoff*, i.e., the total variation distance drops from 1 to 0 gradually on scale $1/\sqrt{\alpha_n}$.

3. We emphasize that we look at the mixing times for 'typical' initial conditions and at the distribution of the random walk averaged over the trajectories of the graph process: the 'annealed' model. It would be interesting to investigate different setups, such as 'worst-case' mixing, in which the maximum of the mixing time over all initial conditions is considered, or the 'quenched' model, in which the entire trajectory of the graph process is fixed instead of just the initial configuration. In such setups the results can be drastically different. For example, we might consider an initial configuration in which every vertex has a maximal number of self-loops, which would give a maximal component size of 2, and the initial position is a half-edge of an isolated vertex with small degree. In such a situation, we have to wait at least until one of the half-edges of the isolated vertex is rewired, and this time can be of order of $1/\alpha_n$, which is much larger than $1/\sqrt{\alpha_n}$. Another interesting example is to consider a uniformly sampled initial configuration, with a worst-case starting location for the random walk. We may expect our results to carry over because the mixing-time estimates of Ben-Hamou and Salez [16] hold for worst-case initial positions. However, to show that this is true we would require more sophisticated techniques, since the underlying graph changes at each step of the dynamics.

4. It would be of interest to extend our results to random walk *with backtracking*. This is much harder. Indeed, because the configuration model is locally tree-like and random walk without backtracking on a tree is the same as self-avoiding walk, in our proof we can exploit the fact that typical walk trajectories are self-avoiding. In contrast, for the random walk with backtracking, after it jumps over a rewired edge, which in our model serves as a randomized stopping time, it may jump back over the same edge, in which case it has not mixed. This problem remains to be resolved.

§3.2 Stopping time decomposition

As in [12], the proof is based on a randomized stopping time argument. Let

$$\tau \coloneqq \min\{t \in \mathbb{N}\colon\ X_{t-1} \in R_{\leq t}\}. \tag{3.11}$$

where $R_{\leq t} \coloneqq \cup_{s=1}^{t} R_s$ is the set of rewired edges up to time t. By the triangle inequality, we have

$$\begin{aligned}\mathcal{D}_{\eta,x}(t) \leq\ & \mathbb{P}_{\eta,x}(\tau > t)\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H\|_{\text{TV}} \\ & + \mathbb{P}_{\eta,x}(\tau \leq t)\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t) - U_H\|_{\text{TV}}\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\mathcal{D}_{\eta,x}(t) \geq\ & \mathbb{P}_{\eta,x}(\tau > t)\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H\|_{\text{TV}} \\ & - \mathbb{P}_{\eta,x}(\tau \leq t)\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t) - U_H\|_{\text{TV}}.\end{aligned} \tag{3.13}$$

Proposition 3.2.1 (Closeness to stationarity and stopping time tails). *Suppose that Condition* 3.1.2(R1) *and Condition* 3.1.3 *hold and that* $\beta \in [0,\infty)$. *If* $t = t(n) = [1 + o(1)]\, c \log n$ *for some* $c \in (0,\infty)$, *then whp in* η *and* x,

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H(\cdot)\|_{\text{TV}} = \begin{cases} 1 - o(1), & c \in [0, c_{n,\text{stat}}), \\ o(1), & c \in (c_{n,\text{stat}}, \infty), \end{cases} \tag{3.14}$$

$$\mathbb{P}_{\eta,x}(\tau > t) = (1 - \alpha_n)^{t(t+1)/2} + o(1). \tag{3.15}$$

If, in addition, $k = k(n) = \omega((\log n)^2)$, *then*

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t) - U_H(\cdot)\|_{\text{TV}} = o(1). \tag{3.16}$$

We show how Theorems 3.1.4(2)–(3) follow from Proposition 3.2.1:

Proof of Theorem 3.1.4(2)–(3). First we prove (3.9). Under the condition $\lim_{n\to\infty} \alpha_n (\log n)^2 = \beta \in (0,\infty)$, since $m = \Theta(n)$ we have $k = \omega((\log n)^2)$, and so we can use all three items of Proposition 3.2.1. From (3.12), (3.13) and (3.16) it follows that, for any $t = [1 + o(1)]\, c \log n$,

$$\mathcal{D}_{\eta,x}(t) = \mathbb{P}_{\eta,x}(\tau > t)\|\, \mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H\|_{\text{TV}} + o(1). \tag{3.17}$$

Since $\lim_{n\to\infty} \alpha_n = 0$ and $t\alpha_n = o(1)$, by (3.15) also

$$\mathbb{P}_{\eta,x}(\tau > t) = (1-\alpha_n)^{t(t+1)/2} + o(1) = \exp(-\alpha_n t^2/2) + o(1). \tag{3.18}$$

Since $\alpha_n = [1+o(1)]\,\beta/(\log n)^2$, (3.18) together with (3.14) gives us

$$\mathcal{D}_{\eta,x}(t) = \begin{cases} \exp(-\beta c^2/2) + o(1), & c \in [0, c_{n,\mathrm{stat}}), \\ o(1), & c \in (c_{n,\mathrm{stat}}, \infty). \end{cases} \tag{3.19}$$

Next, we prove (3.10). If $\lim_{n\to\infty} \alpha_n (\log n)^2 = \beta = 0$, then by (3.15), for any $t = [1+o(1)]\,c \log n$,

$$\mathbb{P}_{\eta,x}(\tau > t) = \exp(-\alpha_n t^2/2) + o(1) = 1 - o(1), \qquad \mathbb{P}_{\eta,x}(\tau \le t) = o(1). \tag{3.20}$$

Inserting (3.20) into (3.12) and (3.13), we get

$$\mathcal{D}_{\eta,x}(t) = [1-o(1)]\,\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H\|_{\mathrm{TV}} + o(1). \tag{3.21}$$

Using (3.14), we obtain

$$\mathcal{D}_{\eta,x}(t) = \begin{cases} 1 - o(1), & c \in [0, c_{n,\mathrm{stat}}), \\ o(1), & c \in (c_{n,\mathrm{stat}}, \infty). \end{cases} \tag{3.22}$$

□

§3.3 Self-avoiding trajectories

In this section, we show that the random walk trajectories are self-avoiding on the relevant time scales with high probability. We let SA_t denote the event $\{v(X_s) \neq v(X_{s'})$ for any $0 \le s < s' \le t\}$, i.e., no two half-edges are incident to the same vertex along the trajectory up to time t.

Along the way we need a random walk on the static model that is a slightly modified version of the random walk without backtracking. This version will be instrumental in the proof of our main theorem. For fixed $t \in \mathbb{N}$, let $[t] \coloneqq \{1, \dots, t\}$ and define the t-step *modified random walk* starting from configuration η and half-edge x as follows:

(a) Let $\mathcal{T}$ be a random subset of $[t]$ drawn according to a probability mass function $(p_t(T))_{T\subset[t]}$ with $p_t(\varnothing) \in (0,1)$ for all t (to be defined later on).

(b) At each time $s \in [t]$, if $s \notin \mathcal{T}$, then the random walk makes a non-backtracking move in configuration η, while if $s \in \mathcal{T}$, then it jumps to a uniformly chosen half-edge (possibly the half-edge it is on).

This is a random walk without backtracking that resets its position to a uniformly chosen half-edge at certain random times. We denote its law by $\mathbb{P}^{\mathrm{mod}}_{\eta,x}$, and put $\mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_0 = x) = 1$. Note that, although the distribution of this random walk depends on t and on the distribution of $\mathcal{T}$, we suppress these from the notation.

If we condition on the event that $\mathcal{T} \neq \varnothing$, then the modified random walk makes a uniform jump at some time in $[t]$ after which it becomes stationary, and so

$$\mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t \in \cdot \mid \mathcal{T} \neq \varnothing) = U_H(\cdot). \tag{3.23}$$

On the other hand, if we condition on the event that $\mathcal{T} = \varnothing$, then the modified random walk is the same as the random walk without backtracking on the static graph given by configuration η starting from x. Denoting the law of the latter by $\mathbb{P}^{\mathrm{stat}}_{\eta,x}$, we have

$$\mathbb{P}^{\mathrm{mod}}_{\eta,x}(\cdot \mid \mathcal{T} = \varnothing) = \mathbb{P}^{\mathrm{stat}}_{\eta,x}(\cdot). \tag{3.24}$$

The main result of this section is the following lemma:

Lemma 3.3.1. *Suppose that Condition* 3.1.2(R1) *and Condition* 3.1.3(R1*) *hold and that* $t = [1 + o(1)]\, c \log n$ *for some* $c \in (0, \infty)$. *Then whp in* η *and* x,

$$\mathbb{P}_{\eta,x}(\mathsf{SA}_t) = 1 - o(1), \qquad \mathbb{P}^{\mathrm{mod}}_{\eta,x}(\mathsf{SA}_t) = 1 - o(1). \tag{3.25}$$

Proof. The proof uses two exploration processes on the graph with the help of the two random walks in the annealed setting. Recall that $\mu_n = \mathit{Conf}_H \times U_H$. The annealed measures for the two random walks are defined as

$$\mathbb{P}(\cdot) := \sum_{\eta,x} \mu_n(\eta, x)\, \mathbb{P}_{\eta,x}(\cdot), \qquad \mathbb{P}^{\mathrm{mod}}(\cdot) := \sum_{\eta,x} \mu_n(\eta, x)\, \mathbb{P}^{\mathrm{mod}}_{\eta,x}(\cdot). \tag{3.26}$$

First, we describe the exploration process for the random walk on the dynamic configuration model. To compute the probability of a self-avoiding path, we keep track of already explored half-edges. The exploration process proceeds as follows:

(a) At time $s = 0$, choose x uniformly at random from H, set $X_0 = x$ and set A_0 to be the set containing x and all its siblings (the set of 'active' half-edges at time 0).

(b) At each time $s \in [t]$, reveal the pair of $X_{s-1} = x_{s-1}$ in C_s, say y_{s-1}. Denote the edge $\{x_{s-1}, y_{s-1}\}$ by e_s. Add y_{s-1} and all its siblings to A_{s-1} to obtain A_s (the set of 'active' half-edges at time s); some siblings may already have been added in a previous step.

(c) Choose one of the siblings of y_{s-1} uniformly at random, say x_s, and set $X_s = x_s$.

This procedure builds up the trajectory of the random walk while ignoring what happens in the rest of the graph. Note that we only pair the half-edges along the trajectory, while the siblings of the half-edges along the trajectory are not paired until they are visited by the random walk.

Under this construction, the first time the random walk is not self-avoiding is the first time the revealed pair at step 2 is in the set of active half-edges. Hence we want to bound the probability

$$\mathbb{P}\big(C_s(x_{s-1}) \in A_{s-1} \mid e_i \in C_i, i \in [s-1]\big), \tag{3.27}$$

where $e_1, \ldots, e_{s-1}$ form a self-avoiding path. For any $y \in H \setminus \{x_{s-1}\}$, if y is not paired up to time s, then it can be paired to x_{s-1} through the initial pairing at

time 0 or through rewiring at later times. Since the initial pairing is uniform and this distribution is stationary under the graph dynamics, for all such y the above conditional probability is the same, and so we have

$$\mathbb{P}\big(C_s(x_{s-1}) = y \mid e_i \in C_i, i \in [s-1]\big) \leq \frac{1}{\ell - 2s + 1}, \tag{3.28}$$

where we note that $2(s-1)$ half-edges are paired before time s. On the other hand, if $y \in H \setminus \{x_{s-1}\}$ is already paired before time s, then it can be paired to x_{s-1} only through rewiring. Hence the same probability is less than it is for an unpaired half-edge, and so we have the same upper bound. Summing over A_{s-1}, we get

$$\mathbb{P}\big(C_s(x_{s-1}) \in A_{s-1} \mid e_i \in C_i, i \in [s-1]\big) \leq \frac{|A_{s-1}| - 1}{\ell - 2s + 1} \leq \frac{s d_{\max}}{\ell - 2s + 1}, \tag{3.29}$$

where we use that at each time we activate at most $d_{\max} = \max_{v \in V} d(v)$ half-edges. Finally, since $d_{\max} = n^{o(1)}$ by Condition 3.1.3(R1*), $t = [1+o(1)]\, c \log n$ and $\ell = \Theta(n)$ by Condition 3.1.2(R1), via a union bound and summing over $s \in [t]$, we get

$$\mathbb{P}(\mathsf{SA}_t^c) \leq \frac{d_{\max} t(t+1)/2}{\ell - 2t + 1} = o(1), \tag{3.30}$$

which establishes the left-hand side of (3.25). Indeed, by the Markov inequality, for any $(w_n)_{n \in \mathbb{N}}$ that tends to zero arbitrarily slowly, we have

$$\mu_n(\mathbb{P}_{\eta,x}(\mathsf{SA}_t^c) > w_n) \leq \frac{\mathbb{P}(\mathsf{SA}_t^c)}{w_n}, \tag{3.31}$$

which implies that, with μ_n-probability at least $1 - o(1)$,

$$\mathbb{P}_{\eta,x}(\mathsf{SA}_t) = 1 - o(1). \tag{3.32}$$

Next, we describe the exploration process for the modified random walk. Again, we let A_t denote the set of active half-edges. Now, instead of random rewirings, we have a static configuration chosen randomly according to the configuration model, and we have a set of random times $\mathcal{T} \subset [t]$ at which the random walk makes uniform jumps. The exploration process proceeds as follows:

(a) At time $s = 0$, choose x uniformly at random from H, set $X_0 = x$ and let A_0 be the set containing x and all its siblings. Choose also $\mathcal{T} \subset [t]$ randomly with probabilities $(p_t(T))_{T \subset [t]}$.

(b) At each time $s \in [t]$, we do the following:

  (a) If $s \in [t] \setminus \mathcal{T}$, then reveal the pair of $X_{s-1} = x_{s-1}$ in η, say y_{s-1}. Add y_{s-1} and all its siblings to A_{s-1} to obtain A_s. Choose one of the siblings of y_{s-1} uniformly at random, say x_t, and set $X_s = x_s$.

  (b) If $s \in \mathcal{T}$, then choose x_s uniformly at random from H, set $X_s = x_s$, add x_s and all its siblings to A_{s-1} to obtain A_s.

Under this construction, the first time the random walk is not self-avoiding is the first time we either have that the revealed pair at step 2(a) is in the set of active half-edges or that the random walk jumps to an active half-edge at step 2(b). We look at the probability

$$\mathbb{P}^{\mathrm{mod}}(X_s \in A_{s-1} \mid X_{[0,s-1]} = x_{[0,s-1]}), \tag{3.33}$$

where $x_{[0,s-1]}$ is a self-avoiding segmented path. We see that if $s \in \mathcal{T}$, then this probability is $|A_{s-1}|/\ell$. Otherwise it is at most $(|A_{s-1}| - 1)/(\ell - 2s + 1)$, and so we get

$$\mathbb{P}^{\mathrm{mod}}\big(X_s \in A_{s-1} \mid X_{[0,s-1]} = x_{[0,s-1]}\big) \leq \frac{|A_{s-1}|}{\ell - 2s + 1} \leq \frac{sd_{\max}}{\ell - 2s + 1}. \tag{3.34}$$

This bound agrees with (3.30), so we get the same conclusion for $\mathbb{P}^{\mathrm{mod}}$. Hence, with μ_n-probability at least $1 - o(1)$,

$$\mathbb{P}^{\mathrm{mod}}_{\eta,x}(\mathsf{SA}_t) = 1 - o(1). \tag{3.35}$$

□

The proof for the modified random walk can easily be adapted to the random walk without backtracking on the static graph, simply by removing step 2(b) in the exploration process for the modified random walk. Hence we also have, whp in η and x,

$$\mathbb{P}^{\mathrm{stat}}_{\eta,x}(\mathsf{SA}_t) = 1 - o(1). \tag{3.36}$$

§3.4 Proof of the main proposition

In this section, we prove Proposition 3.2.1. We use the notation introduced in [12] and recall some of the definitions that are needed along the way.

For a fixed sequence of half-edges $x_{[0,t]}$ with $x_0 = x$ and a fixed set of times $T \subseteq [t]$, we use the short-hand notation

$$A(x_{[0,t]}; T) \coloneqq \big\{x_{i-1} \in R_{\leq i} \ \forall\, i \in T,\, x_{j-1} \notin R_{\leq j} \ \forall\, j \in [1,t] \setminus T\big\}, \tag{3.37}$$

where $R_{\leq i}$ denotes the set of half-edges that are rewired up to time i. This event gives us the rewiring history for the sequence of half-edges $x_{[0,t]}$. More precisely, it is the event that, for $i \in [t] \setminus T$, the half-edge x_{i-1} in not rewired until time i, and, for $i \in T$, the half-edge x_{i-1} is rewired at some time before or at time i.

We say that a sequence $x_{[0,t]}$ of half-edges of length t is a *self-avoiding segmented path* in the configuration η with respect to $T = \{t_1, \ldots, t_r\} \subset [t]$ if $x_{[0,t]}$ is self-avoiding, meaning that no two half-edges in $x_{[0,t]}$ are siblings, and each subsequence $x_{[t_{i-1},t_i-1]}$ induces a path in η for $i \in [r+1]$ with $t_0 = 0$ and $t_{r+1} = t + 1$. We denote by $\mathsf{SP}^{\eta}_t(x, y; T)$ the set of all self-avoiding segmented paths in η with respect to T with $x_0 = x$ and $x_t = y$ (see Fig. 3.2) and by $\mathsf{SP}^{\eta}_t(x; T)$ the set of all self-avoiding segmented paths in η with respect to T with $x_0 = x$. Note that for $T = \varnothing$ these are simply the sets of self-avoiding paths.

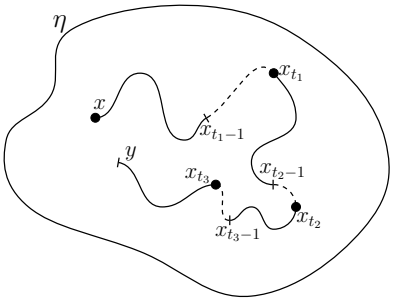

Figure 3.2: An element of $\mathsf{SP}_t^{\eta}(x, y; T)$ *with* $T = \{t_1, t_2, t_3\}$.

Lemmas 3.4.1 and 3.4.2 below are slight modifications of [12, Lemmas 3.1–3.2] and will be instrumental in the proof of Proposition 3.2.1. The first lemma is concerned with the probabilities of the rewiring histories of self-avoiding segmented paths:

Lemma 3.4.1 (Rewiring histories of self-avoiding segmented paths). *Fix* $t \in \mathbb{N}$, $T \subseteq [t]$ *and* $\eta, \zeta \in \mathrm{Conf}_H$. *Suppose that* $x_{[0,t]}$ *and* $y_{[0,t]}$ *are two self-avoiding segmented paths in* η *and* ζ, *respectively, of length* $t+1$. *Then*

$$\mathbb{P}_{\eta,x}\big(A(x_{[0,t]}; T)\big) = \mathbb{P}_{\zeta,y}\big(A(y_{[0,t]}; T)\big). \tag{3.38}$$

Proof. The proof follows the same line of argument as in the proof of [12, Lemma 3.1] and uses a coupling between two dynamic configuration models. Let f be a one-to-one map from H to itself with the property that it maps x_i to y_i for all $i \in [0, t]$, and preserves the edges between two configuration η and ζ, i.e., $f(\eta(x)) = \zeta(f(x))$ for all $x \in H$. The Markovian coupling $(C_t^x, C_t^y)_{t\in\mathbb{N}_0}$, where $C_0^x = \eta$ and $C_0^y = \zeta$, proceeds at every step $t \in \mathbb{N}$ as follows:

(a) Choose k edges from C_{t-1}^x uniformly at random without replacement, say $\{z_1, z_2\}$, $\ldots, \{z_{2k-1}, z_{2k}\}$. Choose the edges $\{f(z_1), f(z_2)\}, \ldots, \{f(z_{2k-1}), f(z_{2k})\}$ from C_{t-1}^y.

(b) Rewire the half-edges $z_1, \ldots, z_{2k}$ uniformly at random to obtain C_t^x and set $C_t^y(f(z_i)) = f(C_t^x(z_i))$.

Since under the coupling the event $A(x_{[0,t]}; T)$ on η is the same as the event $A(y_{[0,t]}; T)$ on ζ, we get the desired result. □

From this lemma we see that the probability of a specific rewiring history for a self-avoiding segmented path does not depend on the path itself nor on the configuration: it only depends on t and T. In what follows we set $p_t(T) = \mathbb{P}_{\eta,x}(A(x_{[0,t]}, T))$ for which it can be easily seen that

$$\mathbb{P}_{\eta,x}(A(x_{[0,t]}, T)) > 0 \text{ and } \sum_{T\subset[t]} \mathbb{P}_{\eta,x}(A(x_{[0,t]}, T)) = 1 \text{ for all } T \subset [t].$$

When we refer to the modified random walk we will use these probabilities as the distribution for the random times $\mathcal{T}$.

The second lemma is concerned with path probabilities for the random walk conditioned on the rewiring history:

Lemma 3.4.2 (Paths estimate given rewiring history). *Suppose that* $t = t(n) = [1 + o(1)]\, c \log n$ *for some* $c \in (0, \infty)$, $k = k(n) = \omega((\log n)^2)$ *and* $T = \{t_1, \dots, t_r\} \subseteq [t]$. *Let* $x_0 \cdots x_t \in \mathsf{SP}_t^\eta(x, y; T)$ *be a self-avoiding segmented path in* η *that starts at* x *and ends at* y. *Then*

$$\mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]} \mid A(x_{[0,t]}; T)\big) \geq \frac{1 - o(1)}{\ell^r} \prod_{i \in [1,t] \setminus T} \frac{1}{\deg(x_i)}. \tag{3.39}$$

Proof. The proof follows the same line of argument as the proof of [12, Lemma 3.2]. □

We continue with the proof of Proposition 3.2.1. We start by proving the result on the tail probabilities of τ, since this is easier.

▷ *Proof of* (3.15). Using (3.25), we see that whp in η and x

$$\mathbb{P}_{\eta,x}(\tau > t) - o(1) \leq \mathbb{P}_{\eta,x}(\mathsf{SA}_t, \tau > t) \leq \mathbb{P}_{\eta,x}(\tau > t). \tag{3.40}$$

On the other hand, by considering all possible self-avoiding paths,

$$\begin{aligned}
\mathbb{P}_{\eta,x}(\mathsf{SA}_t, \tau > t) &= \sum_{x_{[0,t]} \in \mathsf{SP}_t^\eta(x;\varnothing)} \mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]}, A(x_{[0,t]}; \varnothing)\big) \\
&= \sum_{x_{[0,t]} \in \mathsf{SP}_t^\eta(x;\varnothing)} \left(\prod_{i=1}^{t} \frac{1}{\deg(x_i)} \right) \mathbb{P}_{\eta,x}\big(A(x_{[0,t]}; \varnothing)\big) \\
&= p_t(\varnothing)\, \mathbb{P}_{\eta,x}^{\mathrm{stat}}(\mathsf{SA}_t),
\end{aligned} \tag{3.41}$$

where in the second line we use that

$$\mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]} \mid A(x_{[0,t]}; \varnothing)\big) = \prod_{i=1}^{t} \frac{1}{\deg(x_i)} \tag{3.42}$$

and in the third line that these are the path probabilities for the random walk without backtracking in the static model. By following the proof of [12, Eq. (2.6)], we also get

$$\mathbb{P}_{\eta,x}\big(A(x_{[0,t]}; \varnothing)\big) = (1 - \alpha_n)^{t(t+1)/2} + o(1). \tag{3.43}$$

Combining this with (3.36), we obtain

$$\mathbb{P}_{\eta,x}(\mathsf{SA}_t, \tau > t) = (1 - \alpha_n)^{t(t+1)/2} + o(1), \tag{3.44}$$

and the claim follows from (3.40). □

▷ *Proof of* (3.14). Fix $y \in H$. We have

$$\begin{aligned}\mathbb{P}_{\eta,x}(X_t = y, \mathsf{SA}_t, \tau > t) &= \sum_{x_{[0,t]} \in \mathsf{SP}_t^{\eta}(x,y;\varnothing)} \mathbb{P}_{\eta,x}\big(X_{[1,t]} = x_{[1,t]}, \tau > t\big) \\ &= \sum_{x_{[0,t]} \in \mathsf{SP}_t^{\eta}(x,y;\varnothing)} \left(\prod_{i=1}^{t} \frac{1}{\deg(x_i)}\right) \mathbb{P}_{\eta,x}\big(A(x_{[0,t]};\varnothing)\big) \qquad (3.45) \\ &= p_t(\varnothing)\, \mathbb{P}^{\mathrm{stat}}_{\eta,x}(X_t = y, \mathsf{SA}_t).\end{aligned}$$

Using the third line of (3.41), we obtain

$$\mathbb{P}_{\eta,x}(X_t = y \mid \mathsf{SA}_t, \tau > t) = \mathbb{P}^{\mathrm{stat}}_{\eta,x}(X_t = y \mid \mathsf{SA}_t). \qquad (3.46)$$

On the other hand, by partitioning according to SA_t and SA_t^c and using that $\mathbb{P}_{\eta,x}(\mathsf{SA}_t) = 1 - o(1)$, we obtain

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - \mathbb{P}_{\eta,x}(X_t \in \cdot \mid \mathsf{SA}_t, \tau > t)\|_{\mathrm{TV}} = o(1), \qquad (3.47)$$

and

$$\|\mathbb{P}^{\mathrm{stat}}_{\eta,x}(X_t \in \cdot\,) - \mathbb{P}^{\mathrm{stat}}_{\eta,x}(X_t \in \cdot \mid \mathsf{SA}_t)\|_{\mathrm{TV}} = o(1). \qquad (3.48)$$

Combining these relations with (3.46), we obtain

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - \mathbb{P}^{\mathrm{stat}}_{\eta,x}(X_t \in \cdot)\|_{\mathrm{TV}} = o(1). \qquad (3.49)$$

Using the results of [16] for the random walk without backtracking in the static configuration model, we see that if $t = [1 + o(1)]c \log n$ with $c \in (0, c_{n,\mathrm{stat}})$, then

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H\|_{\mathrm{TV}} = 1 - o(1), \qquad (3.50)$$

while if $t = [1 + o(1)]\, c \log n$ with $c \in (c_{n,\mathrm{stat}}, \infty)$, then

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau > t) - U_H\|_{\mathrm{TV}} = o(1). \qquad (3.51)$$

□

▷ *Proof of* (3.16). Fix $y \in H$ and suppose that $k = k(n) = \omega((\log n)^2)$. Using Lemmas 3.4.1 and 3.4.2,

$$\begin{aligned}&\mathbb{P}_{\eta,x}(X_t = y, \mathsf{SA}_t, \tau \le t) \\ &\quad = \sum_{r=1}^{t} \sum_{\substack{T \subseteq [1,t] \\ |T| = r}} \sum_{x_{[0,t]} \in \mathsf{SP}(x,y;T)} \mathbb{P}_{\eta,x}\big(X_{[0,t]} = x_{[0,t]} \mid A(x_{[0,t]};T)\big)\, \mathbb{P}_{\eta,x}\big(A(x_{[0,t]};T)\big) \\ &\quad \ge [1 - o(1)] \sum_{r=1}^{t} \sum_{\substack{T \subseteq [1,t] \\ |T| = r}} p_t(T) \sum_{x_{[0,t]} \in \mathsf{SP}(x,y;T)} \left(\prod_{i \in [t] \setminus T} \frac{1}{\deg(x_i)}\right) \frac{1}{\ell^r}. \qquad (3.52)\end{aligned}$$

We immediately note that

$$\mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t = y, \mathsf{SA}_t \mid \mathcal{T} = T) = \sum_{x_{[0,t]} \in \mathsf{SP}(x,y;T)} \left(\prod_{i \in [t] \setminus T} \frac{1}{\deg(x_i)} \right) \frac{1}{\ell^{|T|}}, \tag{3.53}$$

and so

$$\mathbb{P}_{\eta,x}(X_t = y, \mathsf{SA}_t, \tau \leq t) \geq [1 - o(1)]\, \mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t = y, \mathsf{SA}_t, \mathcal{T} \neq \varnothing). \tag{3.54}$$

Using (3.24), (3.25) and (3.41), whp in η and x, we also have

$$\begin{aligned} \mathbb{P}_{\eta,x}(\mathsf{SA}_t, \tau \leq t) &= \mathbb{P}_{\eta,x}(\mathsf{SA}_t) - \mathbb{P}_{\eta,x}(\mathsf{SA}_t, \tau > t) \\ &\leq \mathbb{P}^{\mathrm{mod}}_{\eta,x}(\mathsf{SA}_t) + o(1) - p_t(\varnothing)\, \mathbb{P}^{\mathrm{stat}}_{\eta,x}(\mathsf{SA}_t) \\ &= \mathbb{P}^{\mathrm{mod}}_{\eta,x}(\mathsf{SA}_t) + o(1) - \mathbb{P}^{\mathrm{mod}}_{\eta,x}(\mathsf{SA}_t, \mathcal{T} = \varnothing) \\ &= \mathbb{P}^{\mathrm{mod}}_{\eta,x}(\mathsf{SA}_t, \mathcal{T} \neq \varnothing) + o(1). \end{aligned} \tag{3.55}$$

Combining this with (3.54) we get, for any $y \in H$,

$$\mathbb{P}_{\eta,x}(X_t = y \mid \mathsf{SA}_t, \tau \leq t) \geq [1 - o(1)]\, \mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t = y \mid \mathsf{SA}_t, \mathcal{T} \neq \varnothing). \tag{3.56}$$

which in turn gives

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \mathsf{SA}_t, \tau \leq t) - \mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t \in \cdot \mid \mathsf{SA}_t, \mathcal{T} \neq \varnothing)\|_{\mathrm{TV}} = o(1). \tag{3.57}$$

On the other hand, (3.25) gives

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \mathsf{SA}_t, \tau \leq t) - \mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t)\|_{\mathrm{TV}} = o(1), \tag{3.58}$$

and

$$\|\mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t \in \cdot \mid \mathsf{SA}_t, \mathcal{T} \neq \varnothing) - \mathbb{P}^{\mathrm{mod}}_{\eta,x}(X_t \in \cdot \mid \mathcal{T} \neq \varnothing)\|_{\mathrm{TV}} = o(1). \tag{3.59}$$

Finally, from the latter two relations in combination with (3.23) and (3.57), we get

$$\|\mathbb{P}_{\eta,x}(X_t \in \cdot \mid \tau \leq t) - U_H(\cdot)\|_{\mathrm{TV}} = o(1), \tag{3.60}$$

which is the desired result. □

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CHAPTER 4

Mixing times of random walks with random rewirings

Abstract

We consider a random walk without backtracking on a general class of dynamic random graphs with n vertices, where the vertices and their degrees are fixed but the edges are rewired according to a prescribed rule. In previous works [12, 13], we considered the special case in which, at each unit of time, a certain fraction of the edges, chosen uniformly and independently of the random walk, are rewired uniformly. We showed that there are three different regimes, depending on how the fraction of edges to be rewired decays as a function of the number of vertices. In this paper, we show, for a general class of rewiring rules, how the mixing time of the random walk on the dynamically rewired random graph is linked to the mixing time of the random walk on static random graphs, drawn according to the configuration model. Furthermore, we give an explicit example, called *local rewiring*, in which the edges are rewired only along the random walk path, and using the above link, we show that, for this model, we have the same trichotomy as in [13] but on a different time scale. In our proof, we use a coupling argument where the random walk on the dynamically rewired random graph is coupled to a modified version of the random walk on the static random graph.

§4.1 Introduction

We consider a random walk on a dynamic random graph in which the vertices are fixed but the edges are randomly rewired at each unit of time according to a prescribed rule. By *rewiring* we mean an operation on the graph that changes the edges while keeping the degrees of the vertices fixed. This type of graph dynamics was considered in the context of uniform sampling of simple graphs with given degree sequences [34, 46, 54, 53, 69]. The main purpose of these works is to construct Markov chains on the set of simple graphs with a given degree sequence whose stationary distribution is uniform on this set. If the convergence to the stationary distribution of the Markov chain is sufficiently fast, i.e. *the mixing time* is sufficiently small, then it is possible to obtain approximately uniform samples in an efficient way, simply by simulating the Markov chain.

In [34, 54, 53], the authors consider a so-called *switch* chain in which, at each time unit, two edges (i, j) and (k, l), are chosen uniformly at random and their end-points are switched to obtain the edges (i, k) and (j, l), provided that the resulting graph is simple. In [34], the authors consider the switch chain in the context of simple regular graphs and show that the mixing time is polynomial in the size of the graph. Their results were later extended to the case of simple graphs with irregular degree sequences [54] and to directed graphs [53]. In [46, 69], the authors consider a so-called *flip* chain, which is a modified version of the switch chain in which a switch is performed if the two randomly chosen edges have a common neighbor, i.e., if (i, l) is an edge. In [46], the authors consider the flip chain in the context of simple regular graphs and they show that the mixing time is polynomial in the size of the graph by comparing the flip chain to a switch chain and using the results of [34].

In the present paper, we are interested in the behaviour of a random walk on a dynamically rewired random graph, rather than in the behaviour of the random graph dynamics itself. Namely, we study the mixing times of random walks on dynamically rewired random graphs, where the initial graph is drawn according to the configuration model. Our results are in the same spirit as those in [12, 13], in which random walks on a dynamic version of the configuration model with a specific rewiring mechanism were considered. In fact, we extend the results of [12, 13] to a more general class of dynamically rewired versions of the configuration model, which includes the dynamic configuration model of [12] as a special case.

The mixing times of random walks on static random graphs has been studied in the last few decades for a wide range of random graph models. For an overview, we refer to [12, 13] and references therein. In contrast, there are relatively few studies on the mixing times of random walks on dynamic random graphs. This line of research was started recently in [83], which considers random walks on dynamical percolation on $\mathbb{Z}^d$ in the subcritical regime. In [82], the results in [83] were extended to the supercritical regime. In [89], the authors consider random walks on a dynamic version of Erdős-Rényi random graph model and show that the joint chain of the random walk and the dynamic random graph exhibits cut-off phenomenon. Since there are two layers of randomness, the random walk and the graph dynamics, in all these works, several distinct notions of mixing times are considered, such as the annealed case vs. the

quenched case and the mixing time of the random walk vs. the mixing time of the joint chain. In our work, we consider the mixing time of the random walk component annealed over the graph dynamics, which will be made clear in the sequel.

The remainder of this paper is organised as follows. In Section 4.1.1, we introduce the model and set the notation. In Section 4.1.2, we state our main theorem (Theorem 4.1.5). Section 4.2 is devoted to the introduction of some core ingredients. In Section 4.3, we give the proof of the main theorem. In Section 4.4, we introduce a specific model within the framework of random walks on dynamically rewired random graph models and show that it exhibits the same trichotomy found in [13] but on a different time scale. In Section 4.5, we put our work in the proper context by discussing several issues in more detail and suggesting possible extensions.

Throughout the sequel we use standard notations for the asymptotic comparison of functions $f, g\colon \mathbb{N} \to [0,\infty)$: $f(n) = O(g(n))$ or $g(n) = \Omega(f(n))$ when $\limsup_{n\to\infty} f(n)/g(n) < \infty$; $f(n) = o(g(n))$ or $g(n) = \omega(f(n))$ when $\lim_{n\to\infty} f(n)/g(n) = 0$; $f(n) = \Theta(g(n))$ when both $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

§4.1.1 Model

We denote by V the set of vertices of the graph and by $\deg(v)$ the degree of vertex $v \in V$. To each vertex $v \in V$ we associate $\deg(v)$ half-edges and by H we denote the set of all half-edges, i.e.,

$$H = \{(v,i) : v \in V \text{ and } 1 \le i \le \deg(v)\}.$$

If a half-edge $x \in H$ is associated to a vertex $v \in V$, then we say that x *is incident to* v. We denote by $v(x) \in V$ the vertex to which $x \in H$ is incident and by $H(v) \coloneqq \{x \in H\colon\ v(x) = v\} \subset H$ the set of half-edges incident to $v \in V$. If $x, y \in H(v)$ with $x \neq y$, then we write $x \sim y$ and say that x and y are siblings of each other. The degree of a half-edge $x \in H$ is defined as

$$\deg(x) \coloneqq \deg(v(x)) - 1. \tag{4.1}$$

We consider graphs on n vertices, so that $|V| = n$, with m edges, so that $|H| = \sum_{v\in V} \deg(v) = 2m =: \ell$.

We view the set of edges as a pairing of half-edges. A pairing of half-edges ξ, called a *configuration*, is a bijection of H to itself without fixed points and with the property that $\xi(\xi(x)) = x$ for all $x \in H$. With a slight abuse of notation, we will use the same symbol ξ to denote the set of pairs of half-edges in ξ, so $\{x, y\} \in \xi$ means that $\xi(x) = y$ and $\xi(y) = x$. Each pair of half-edges in ξ will also be called an edge. The set of all configurations on H will be denoted by Conf_H, and the uniform distribution on Conf_H will be denoted by Conf_H.

We note that each configuration gives rise to a (multi-)graph that may contain self-loops (edges having the same vertex on both ends) or multiple edges (between the same pair of vertices). The distribution of the random graph corresponding to a uniformly distributed configuration is called the configuration model (see [93, Chapter 7]). On the other hand, a graph can be obtained via several distinct configurations.

We will consider asymptotic statements in the sense of $|V| = n \to \infty$. Quantities like $V, H, \deg, m$ and ℓ all depend on n. In order to lighten the notation, we often suppress n from the notation.

The central object of this study is a Markov chain $(X, C) = (X_t, C_t)_{t\in\mathbb{N}_0}$, where $X_t \in H$ and $C_t \in \mathit{Conf}_H$ for all $t \in \mathbb{N}_0$. Here, X denotes the random walk component and C denotes the random configuration component. The configuration component gives rise to a graph sequence in which each graph has the same degree sequence. At each time $t \in \mathbb{N}$, we first update the configuration and then let the walk move.

Random walk. We consider a random walk on a dynamic random graph in which some half-edges are rewired at each step. The random walk is not allowed to backtrack, in the sense that it cannot traverse the same edge twice in a row. Since in our model the underlying graph is dynamic and the edges change over time, it is more conveniently defined as a random walk on the set of half-edges H. Suppose that at time $t \in \mathbb{N}$ we updated the configuration to $C_t = \xi$. Then the random walk moves, according to the transition probabilities

$$P_\xi(x, y) \coloneqq \begin{cases} \frac{1}{\deg(y)} & \text{if } \xi(x) \sim y \text{ and } \xi(x) \neq y, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

In words, when the random walk is at half-edge x in configuration ξ, it jumps to one of the siblings of the half-edge it is paired to uniformly at random (see Fig. 4.1). The transition probabilities are symmetric with respect to the pairing given by ξ, i.e., $P_\xi(x, y) = P_\xi(\xi(y), \xi(x))$, in particular, the matrix of transition probabilities is doubly stochastic, and so the uniform distribution on H, denoted by U_H, is stationary for P_ξ for any $\xi \in \mathit{Conf}_H$. In the sequel, when we use the term *random walk* we always refer to this model.

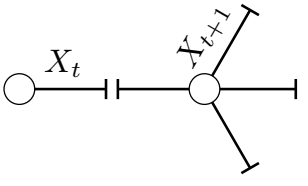

Figure 4.1: The random walk moves from half-edge X_t to half-edge X_{t+1}, one of the siblings of the half-edge that X_t is paired to.

Graph dynamics. We consider a general class of graph dynamics in which some edges are randomly rewired at each unit of time according to a presribed rule. A subset of edges to be rewired is chosen randomly, these edges are broken into half-edges and the resulting half-edges are paired randomly according to a prescribed distribution. The set of half-edges involved in the rewiring at time $t \in \mathbb{N}$ is denoted by R_t.

Suppose that $X_{t-1} = x$ and $C_{t-1} = \xi$. Then, at time t, the above dynamics gives rise to a distribution $Q_x(\xi, \cdot)$ on Conf_H. In [12, 13], a specific choice of dynamics was considered, in which $Q_x(\xi, \cdot)$ did not actually depend on x. In such a situation, the configuration component forms a Markov chain itself.

Joint chain. The law of the joint chain $(X, C) = (X_t, C_t)_{t\in\mathbb{N}_0}$, starting from initial half-edge x and initial configuration ξ, is given by the conditional probabilities

$$\begin{aligned}&\mathbb{P}_{x,\xi}(X_t = z, C_t = \zeta \mid X_{t-1} = y, C_{t-1} = \eta) \\ &= Q_y(\eta,\zeta)P_\zeta(y,z), \quad t \in \mathbb{N}\end{aligned} \tag{4.3}$$

with

$$\mathbb{P}_{x,\xi}(X_0 = x, C_0 = \xi) = 1. \tag{4.4}$$

While the joint chain is Markov, the marginal chains $X = (X_t)_{t\in\mathbb{N}}$ and $C = (C_t)_{t\in\mathbb{N}}$ are not necessarily Markov.

We note that when the graph dynamics does not depend on the random walk, i.e., $Q_x(\cdot,\cdot) = Q_y(\cdot,\cdot)$ for all $x, y \in H$, the uniform distribution U_H is a stationary distribution for the random walk, i.e., for all $\xi \in \mathit{Conf}_H$ and $t \in \mathbb{N}$,

$$\sum_{x\in H} \frac{1}{\ell}\mathbb{P}_{x,\xi}(X_t \in \cdot) = U_H(\cdot).$$

This can be easily seen by noting that the random walk conditioned on a realization of the graph dynamics is a time-inhomogeneous Markov chain for which U_H is a stationary distribution.

§4.1.2 Main theorem

We are interested in the behaviour of the total variation distance between the distribution of the random walk component and the uniform distribution on the set of half-edges U_H, i.e.,

$$\mathcal{D}_{x,\xi}(t) \coloneqq \|\mathbb{P}_{x,\xi}(X_t \in \cdot) - U_H(\cdot)\|_{\mathrm{TV}}. \tag{4.5}$$

The total variation distance between two probability measures μ and ν on the same finite state space S is defined by

$$\|\mu - \nu\|_{\mathrm{TV}} \coloneqq \sum_{x\in S} |\mu(x) - \nu(x)| = \sum_{x\in S} [\mu(x) - \nu(x)]_+ = \sup_{A\subseteq S} [\mu(A) - \nu(A)]. \tag{4.6}$$

We emphasize that the marginal chain $X = (X_t)_{t\in\mathbb{N}}$ is not Markov and the total variation distance $\|\mathbb{P}_{x,\xi}(X_t \in \cdot) - U_H(\cdot)\|_{\mathrm{TV}}$ is not guaranteed to be decreasing in t, even when it converges to 0.

Theorem 4.1.5 below concerns the behaviour of $\mathcal{D}_{x,\xi}(t)$ for *"typical"* choices of x and ξ. We formalize the notion of typicality now:

Definition 4.1.1 (With high probability). *Let $\mu = \mu_n \coloneqq U_H \times \mathrm{Conf}_H$. A statement that depends on the initial half-edge x and configuration ξ is said to hold* with high probability (whp) *in x and ξ if the μ-measure of the set of pairs (x, ξ) for which the statement holds tends to 1 as $n \to \infty$.*

One of the key objects of our study will be a randomized stopping time, namely, the first time the random walk steps along a previously rewired edge. Let $R_{\le t} \coloneqq \cup_{s=1}^{t} R_s$,

and let I_t denote the indicator of the event that the random walk steps along a previously rewired edge at time t, i.e., if $X_{t-1} \in R_{\leq t}$, then $I_t = 1$, and otherwise $I_t = 0$. We define the randomized stopping time τ as follows:

$$\tau \coloneqq \min\{t \in \mathbb{N} : I_t = 1\}. \tag{4.7}$$

Theorem 4.1.5 below will be stated in terms of the tail probabilities of τ, written $\mathbb{P}_{x,\xi}(\tau > t)$, and only holds under certain conditions. First, we give the conditions that concerns the degree sequences of the random graphs that we deal with:

Condition 4.1.2. *(Regularity of degrees)*

(R1) ℓ *is even and* $\ell = \Theta(n)$ *as* $n \to \infty$.

(R2) $\max_{v \in V} \deg(v) =: d_{\max} = o(n/(\log n)^2)$ *as* $n \to \infty$.

(R3) $\deg(v) \geq 2$ *for all* $v \in V$.

Condition 4.1.2(R1) ensures that the underlying graph is sparse, and together with Condition 4.1.2(R2) ensures that random walk paths are with high probability self-avoiding, as we will see in the proof of Lemma 4.2.2. Condition 4.1.2(R3) ensures that random walk is well-defined. These are the minimal conditions under which Theorem 4.1.5 is true. Next, we give additional conditions which allow us to use results of Ben-Hamou and Salez [16] on the mixing times of random walks on static configuration models:

Condition 4.1.3. *(Additional regularity of degrees)*

(R1*) $\max_{v \in V} \deg(v) =: d_{\max} = n^{o(1)}$ *as* $n \to \infty$.

(R2*)

$$\frac{\lambda_2}{\lambda_1^3} = \omega\left(\frac{(\log\log \ell)^2}{\log \ell}\right), \qquad \frac{\lambda_2^{3/2}}{\lambda_3 \sqrt{\lambda_1}} = \omega\left(\frac{1}{\sqrt{\log \ell}}\right), \quad n \to \infty,$$

where

$$\lambda_1 \coloneqq \frac{1}{\ell} \sum_{z \in H} \log(\deg(z)), \qquad \lambda_m \coloneqq \frac{1}{\ell} \sum_{z \in H} |\log(\deg(z)) - \lambda_1|^m, \quad m = 2, 3.$$

(R3*) $\deg(v) \geq 3$ *for all* $v \in V$.

Conditions 4.1.3(R1*) and (R2*) are technical and it might be possible to relax them via a truncation argument [22]. Condition 4.1.3(R3*) ensures that the random walk does not behave deterministically, and under this condition the configuration model is connected with high probability. Condition 4.1.3 will not be used in Theorem 4.1.5 below, but will be needed to use results of Ben-Hamou and Salez [16] to refine Theorem 4.1.5 in Corollary 4.1.6 below.

Next, we give the conditions that concern the graph dynamics. To do so we need more notation. We define the annealed distribution by

$$\mathbb{P} \coloneqq \sum_{\substack{x \in H, \\ \xi \in Conf}} \mu(x, \xi) \mathbb{P}_{x,\xi}, \tag{4.8}$$

which is the distribution of the random walk on the dynamically rewired graph annealed over the initial half-edge and the initial configuration. We will look at the annealed distribution of the graph dynamics at time t conditional on the walk before time t and on some partial information about the rewiring history before time t.

For $t \in \mathbb{N}$, let $[t] \coloneqq \{1, \dots, t\}$, and for $s \in \mathbb{N}$ with $s < t$, and let $[s, t] \coloneqq \{s, \dots, t\}$. Fix $t \in \mathbb{N}$, let $T = \{t_1, \dots, t_r\}$ be a subset of $[t-1]$. Consider four sequences of half-edges, $x_{[0,t-1]} = x_0 x_1 \dots x_{t-1}$, $\bar{x}_{[0,t-1]} = \bar{x}_0 \bar{x}_1 \dots \bar{x}_{t-1}$, $\hat{x}_{[r]} = \hat{x}_1 \hat{x}_2 \dots \hat{x}_r$ and $\tilde{x}_{[r]} = \tilde{x}_1 \tilde{x}_2 \dots \tilde{x}_r$, such that

- $\bar{x}_{s-1} \sim x_s$ for $s \in [t-1] \setminus T$,
- $\hat{x}_i \sim x_{t_i}$ for $i = 1, \dots, r$,
- the vertices $v(x_0), v(x_1), \dots, v(x_{t-1}), v(\bar{x}_{t_1-1}), \dots, v(\bar{x}_{t_r-1}), v(\bar{x}_{t-1}), v(\tilde{x}_1), \dots, v(\tilde{x}_r)$ are all distinct.

We call such sequences *dynamically self-avoiding with respect to* T. We will look at:

- the set T: the times up to time $t-1$ at which the random walk steps along a previously rewired edge,
- the sequence $x_0 \dots x_{t-1}$: the path of the random walk up to time $t-1$,
- the sequence $\bar{x}_0 \dots \bar{x}_{t-1}$: the pairs of the latter in the initial configuration,
- the sequence $\hat{x}_1 \dots \hat{x}_r$: the pairs of $x_{t_1-1} \dots x_{t_r-1}$ at the times $t_1, \dots, t_r$ respectively,
- the sequence $\tilde{x}_1, \dots \tilde{x}_r$: the pairs of $\hat{x}_1 \dots \hat{x}_r$ in the initial configuration.

For fixed $t \in \mathbb{N}$, $T = \{t_1, \dots, t_r\} \subset [t-1]$, and fixed sequences of half-edges $x_{[0,t-1]}$, $\bar{x}_{[0,t-1]}$, $\hat{x}_{[r]}$ and $\tilde{x}_{[r]}$, let $H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]})$ be the event that

- $I_s = 1$ for $s \in T$ and $I_s = 0$ for $s \in [t-1] \setminus T$,
- $C_0(x_s) = \bar{x}_s$ for $s = 0, \dots, t-1$,
- $C_{t_i}(x_{t_i-1}) = \hat{x}_i$ for $i = 1, \dots, r$,
- $C_0(\hat{x}_i) = \tilde{x}_i$ for $i = 1, \dots, r$,
- $X_s = x_s$ for $s = 0, \dots, t-1$.

When this event occurs we say that the *the history of the walk on the dynamically rewired graph up to time t is dynamically self-avoiding.*

With these definitions in hand, we can state the conditions on the graph dynamics:

Condition 4.1.4. *(Regularity of graph dynamics) For all $t = t(n) = O(\log n)$ and all $T = \{t_1, \dots, t_r\} \subset [t-1]$,*

(D1) $\mathbb{P}(I_t = 1 \mid H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]}))$ *is the same for all choices of $x_{[0,t-1]}$, $\bar{x}_{[0,t-1]}$, $\hat{x}_{[r]}$, $\tilde{x}_{[r]}$ that are dynamically self-avoiding with respect to T.*

(D2) $\|\mathbb{P}(C_t(x_{t-1}) \in \cdot \mid H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]}) \cap \{I_t = 1\}) - U_H(\cdot)\|_{\mathrm{TV}} = o(1/\log n)$ *for all choices of* $x_{[0,t-1]}$, $\bar{x}_{[0,t-1]}$, $\hat{x}_{[r]}$, $\tilde{x}_{[r]}$ *that are dynamically self-avoiding with respect to* T.

For $x \in H$ and $\xi \in Conf_H$, we denote by $\mathbb{P}^{\mathrm{stat}}_{x,\xi}(X_t \in \cdot)$ the law of the random walk on the static graph given by the configuration ξ, and by $\mathcal{D}^{\mathrm{stat}}_{x,\xi}(t)$ its total variation distance to the uniform distibution U_H at time t. Our main result reads as follows:

Theorem 4.1.5. *Suppose that* $t = t(n) = O(\log n)$. *Subject to Conditions 4.1.2 and 4.1.4, the following holds for the random walk on the dynamically rewired graph whp in* x *and* ξ*:*

$$\mathcal{D}_{x,\xi}\big(t\,\big) = \mathbb{P}_{x,\xi}(\tau > t)\mathcal{D}^{\mathrm{stat}}_{x,\xi}(t) + o(1). \tag{4.9}$$

For the static model, under Condition 4.1.3, the ε-mixing time $\inf\{t \in \mathbb{N}_0 \colon \mathcal{D}^{\mathrm{stat}}_{x,\xi}(t) \leq \varepsilon\}$ is known to scale like $t^{\mathrm{stat}}_{\mathrm{mix}} = t^{\mathrm{stat}}_{\mathrm{mix}}(n) \coloneqq [1 + o(1)]\, c_{n,\mathrm{stat}} \log n$ for all $\varepsilon \in (0,1)$, with $c_{n,\mathrm{stat}} = 1/\lambda_1 \in (0,\infty)$, where λ_1 is as defined in Condition 4.1.3(R2*)(Ben-Hamou and Salez [16]). This holds whp in ξ and uniformly in the starting position x. Using this relation we can refine Theorem 4.1.5:

Corollary 4.1.6. *Suppose* $t = t(n) = O(\log n)$. *Subject to Conditions 4.1.2(R1), 4.1.3 and 4.1.4, the following hold for the random walk on dynamically rewired graphs whp in* x *and* ξ*:*

$$\mathcal{D}_{x,\xi}\big(t\,\big) = \begin{cases} \mathbb{P}_{x,\xi}(\tau > t) + o(1) & \text{if } \limsup_{n\to\infty} t/t^{\mathrm{stat}}_{\mathrm{mix}} < 1, \\ o(1) & \text{if } \liminf_{n\to\infty} t/t^{\mathrm{stat}}_{\mathrm{mix}} > 1. \end{cases} \tag{4.10}$$

Proof. By the results in [16], whp in ξ we have

$$\mathcal{D}^{\mathrm{stat}}_{x,\xi}(t) = \begin{cases} 1 - o(1) & \text{if } \limsup_{n\to\infty} t/t^{\mathrm{stat}}_{\mathrm{mix}} < 1, \\ o(1) & \text{if } \liminf_{n\to\infty} t/t^{\mathrm{stat}}_{\mathrm{mix}} > 1. \end{cases}$$

Combining these with Theorem 4.1.5 we get the desired result. □

The proof of Theorem 4.1.5 will be given in Section 4.3. In the next section (Section 4.2), we introduce the key ingredients of the proof. After proving Theorem 4.1.5, we introduce a specific example of a random walk on dynamically rewired random graph, which we call *'random walk with local rewiring'* and prove a mixing time result for this model in Section 4.4, by using Corollary 4.1.6.

§4.2 Coupling to the modified random walk

We define *the modified random walk*, denoted by $(Y_t)_{t\in\mathbb{N}_0}$, as the random walk on the static graph that at certain random times makes uniform jumps. The distribution of the jump times does not depend on the random walk path. More formally, we have a sequence $(J_t)_{t\in\mathbb{N}}$ of random variables adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$, taking values in $\{0,1\}$ according to a given distribution on $\{0,1\}^{\mathbb{N}}$. For fixed $t \in \mathbb{N}$, J_t is seen as the indicator of the event that the modified random walk makes a uniform jump at

time t. The law of the modified random walk $(Y_t)_{t\in\mathbb{N}_0}$ on ξ that starts from the initial half-edge x, which is also adapted to $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$, is given by the conditional probabilities

$$\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t = z \mid Y_{t-1} = y, J_1 = j_1, \dots, J_t = j_t) \tag{4.11}$$

$$= \mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t = z \mid Y_{t-1} = y, J_t = j_t) = \begin{cases} P_\xi(y,z) & \text{if } j_t = 0, \\ \frac{1}{\ell} & \text{if } j_t = 1, \end{cases} \quad t \in \mathbb{N}, \tag{4.12}$$

with

$$\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_0 = x) = 1. \tag{4.13}$$

We note that, according to the definition, neither $(J_t)_{t\in\mathbb{N}}$ nor the pair $(Y_t, J_t)_{t\in\mathbb{N}}$ needs to be Markov but $(Y_t)_{t\in\mathbb{N}_0}$ is Markov conditional on a realisation of $(J_t)_{t\in\mathbb{N}}$.

Uniform jumps of the modified random walk can be rephrased in the following form. Let Y'_t be a uniformly chosen half-edge, independent of the random walk path and the jump times. If $J_t = 1$, then we choose a uniform sibling of Y'_t, say y, and set $Y_t = y$. Since Y'_t is uniform and one of its siblings is chosen uniformly at random, the resulting half-edge is distributed uniformly on H. In the following we use this formulation, since it makes the exposition more clear.

As an analogue of τ, we define σ to be the first time that the modified random walk makes a uniform jump, i.e.,

$$\sigma := \inf\{t \in \mathbb{N} : J_t = 1\} \tag{4.14}$$

Coupling of two random walks. We couple the law $\mathbb{P}_{x,\xi}(X_t \in \cdot)$ of the random walk on the dynamic random graph, with initial half-edge x and initial configuration ξ, to the law $\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot)$ of the modified random walk. We want the coupled random walks to stick together as much as possible. When the two random walks make different steps, we say that the coupling of the two random walks has *failed*, and we denote the first time that this happens by F. Until the coupling fails, the times at which the random walk on the dynamically rewired graph makes a step over a previously rewired edge correspond to the times at which the modified random walk makes a uniform jump.

We define an auxiliary random set A_t, called the set of *active* half-edges, which is constructed by adding half-edges at each unit of time. This set will keep track of the half-edges traversed by the two random walks, the half-edges that are rewired at the position of the random walk, and their pairs in the initial configuration. Note that A_0 consists of x and its siblings, i.e., $A_0 = H(v(x))$. The coupling is as follows:

(a) At time $t \in \mathbb{N}$, if the coupling has not failed yet and neither $\xi(X_{t-1})$ nor any of its siblings belongs to A_{t-1}, then maximally couple the distribution of I_t, conditional on the history of the random walk and the rewirings seen by the random walker, to the distribution of J_t, conditional on the values of indicators $J_1, \dots, J_{t-1}$:

   (a) If the coupling of the conditional distributions of I_t and J_t is successful and $I_t = J_t = 0$, then add $\xi(X_{t-1})$ and all of its siblings to A_{t-1} to obtain A_t, let X make a random walk move, and set $Y_t = X_t$.

(b) If the coupling of the conditional distributions of I_t and J_t is successful and $I_t = J_t = 1$, then maximally couple the distribution of the pair of X_{t-1} in C_t, $C_t(X_{t-1})$, conditional on the history of the random walk and the event that $I_t = 1$, to the distribution of Y'_t:

(a) If the coupling of $C_t(X_{t-1})$ and Y'_t is succesful, and neither $C_t(X_{t-1})$ nor any of its siblings is in A_{t-1}, then add $\xi(X_{t-1})$ and all of its siblings, $C_t(X_{t-1})$ and all of its siblings to A_{t-1} to obtain A_t, let X make a random walk move, and set $Y_t = X_t$.

(b) Otherwise, declare the coupling of the two random walks as failed.

(c) If the coupling of the conditional distributions of I_t and J_t is not succesful, i.e., $I_t \neq J_t$, then declare the coupling of the two random walks as failed.

(b) At time $t \in \mathbb{N}$, if the coupling has failed before, then let X and Y evolve independently. If the coupling has not failed yet but either $\xi(X_{t-1})$ or some of its siblings belong to A_{t-1}, then declare the coupling of the two random walks as failed, and let X and Y evolve independently.

Remark 4.2.1. *At each time $t \in \mathbb{N}$, the random walks try to avoid stepping on the active half-edges A_{t-1}. The coupling of the two random walks fails in three cases:*

(a) *if the coupling of $C_t(X_{t-1})$ and Y'_t fails, or the two random walks step over a half-edge in A_{t-1} in step* (b)*,*

(b) *if the coupling of I_t and J_t fails in step* (c)*,*

(c) *if the pair of X_{t-1} in the starting configuration is already in A_{t-1} as in step* (b)*.*

The second case in item 1, as well as item 3, correspond to the situation in which the random walks are not dynamically self-avoiding. We want to avoid this situation, since it might lead to a previously rewired half-edge that was stepped over previously. This implies that the random walks are dynamically self-avoiding before the coupling of the two random walks fail. The first case in item 1 corresponds to the situation in which the conditional distribution of $C_t(X_{t-1})$ is far from the uniform distribution in total variation distance. Item 3 corresponds to the situation in which the conditional distribution of the times at which the random walk on the dynamically rewired graph and the conditional distribution of the times at which the modified random walk makes uniform jumps are far from each other in total variation distance.

The next lemma states that these events are unlikely up to logarithmic times when Conditions 4.1.2 and 4.1.4 hold for the random walk on the dynamically rewired graph:

Lemma 4.2.2. *Suppose that $t = t(n) = O(\log n)$, and that Conditions 4.1.2 and 4.1.4 hold for the random walk on the dynamically rewired graph. For all $s \leq t$ and all $T = \{s_1, \dots, s_r\} \subset [s-1]$, fix a group of sequences $x^{s,T}_{[0,s-1]}, \bar{x}^{s,T}_{[0,t-1]}, \hat{x}^{s,T}_{[r]}, \tilde{x}^{s,T}_{[r]}$ that is dynamically self-avoiding with respect to T, and consider the modified random walk fpr which the jump distribution has conditional distribution*

$$\begin{aligned}&\mathbb{P}^{\mathrm{mod}}_{x,\xi}(J_s = 1 \mid J_{s'} = 0 \text{ for } s' \in [s-1] \setminus T, J_{s''} = 1 \text{ for } s'' \in T)\\ &= \mathbb{P}(I_s = 1 \mid H(T, x^{s,T}_{[0,s-1]}, \bar{x}^{s,T}_{[0,s-1]}, \hat{x}^{s,T}_{[r]}, \tilde{x}^{s,T}_{[r]})).\end{aligned} \tag{4.15}$$

Then, whp in x and ξ,

$$\|\mathbb{P}_{x,\xi}(X_t \in \cdot) - \mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot)\|_{\mathrm{TV}} = o(1) \tag{4.16}$$

and

$$\mathbb{P}_{x,\xi}(\tau > t) = \mathbb{P}^{\mathrm{mod}}_{x,\xi}(\sigma > t) + o(1). \tag{4.17}$$

Proof. Let $\mathbb{P}^{\mathrm{couple}}_{x,\xi}$ denote the law of the coupling of the two random walks with $X_0 = x$ and $C_0 = \xi$. Since the two random walks agree up to the time the coupling fails, we have

$$\|\mathbb{P}_{x,\xi}(X_t \in \cdot) - \mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot)\|_{\mathrm{TV}} \le \mathbb{P}^{\mathrm{couple}}_{x,\xi}(F \le t). \tag{4.18}$$

So, in order to prove our claim, it suffices to show that, whp in x and ξ,

$$\mathbb{P}^{\mathrm{couple}}_{x,\xi}(F \le t) = o(1). \tag{4.19}$$

To achieve this, we will use an annealing argument on the initial graph and the initial location. Recall that $\mu = U_H \times \mathit{Conf}_H$, and let

$$\mathbb{P}^{\mathrm{couple}} = \sum_{x,\xi} \mu(x,\xi)\mathbb{P}^{\mathrm{couple}}_{x,\xi}. \tag{4.20}$$

We will show that

$$\mathbb{P}^{\mathrm{couple}}(F \le t) = o(1) \tag{4.21}$$

by exploring the initial configuration through the coupled random walk paths until time F, the time at which the coupling fails. The exploration proceeds as follows:

(a) At time $s = 0$, choose a half-edge uniformly at random from H, say x, set $X_0 = Y_0 = x$ and $A_0 = H(v(x))$, the subset of H consisting of x and its siblings.

(b) At time $s \in \mathbb{N}$, first explore the pair of $X_{s-1} = Y_{s-1}$ in the initial configuration ξ, then make the coupled random walks move until the coupling fails, and update A_s accordingly.

According to this description, the exploration process explores the part of the graph seen by the random walks, as well as the parts changed by the rewiring at the positions of the random walks, and it stops as soon as the coupling of the two random walks fails. Suppose that the coupling of the two random walks has not failed before time s. Then it can fail at time s in the following three cases:

(a) if coupling of I_s and J_s fails in step (c),

(b) if coupling of $C_s(X_{s-1})$ and Y'_s fails in step (b),

(c) if the random walks step over a half-edge that is in A_{s-1} in step (b) or step (b).

By (4.15), I_s and J_s can be coupled perfectly, so the probability of the event in case 1 is 0.

For case 2 we note that, by Remark 4.2.1, before the coupling of the two random walks fails, the history of the random walk is dynamically self-avoiding. By Condition 4.1.4(D2), the total variation distance between the conditional distribution of

$C_s(X_{s-1})$ and the uniform distribution U_H is $o(1/\log n)$. Since Y'_s is also distributed uniformly on H, the probability of the event in case 2 is $o(1/\log n)$.

For case 3, we first need an upper bound on the size of A_{s-1}. Each time we explore the initial configuration, we add at most $d_{\max}$ half-edges to the set of active half-edges. If, in addition, a rewiring occurs, then we add at most $2d_{\max}$ half-edges to the set of active half-edges. This gives us

$$|A_{s-1}| \leq 3sd_{\max}. \tag{4.22}$$

For a fail event in step (b), we see that the probability that $C_s(X_{s-1}) \in A_{s-1}$ is smaller than

$$\frac{|A_{s-1}|}{\ell} + o(1/\log n) \leq \frac{3sd_{\max}}{\ell} + o(1/\log n), \tag{4.23}$$

since the random walk is dynamically self-avoiding before the coupling of the two random walks fails (see Remark 4.2.1), so the total variation distance between the conditional distribution of $C_s(X_{s-1})$ and the uniform distribution U_H is $o(1/\log n)$, by Condition 4.1.4(D2). For a fail event in step (b), we see that the probability that $C_0(X_{s-1}) \in A_{s-1}$ is smaller than

$$\frac{|A_{s-1}|}{\ell - 4s + 4} \leq \frac{3sd_{\max}}{\ell - 4s + 4}, \tag{4.24}$$

since up to time s we form at most $2s-2$ pairs in C_0, $s-1$ of them on the random walk path and an additional $s-1$ if rewiring occurs at each step up to time s.

The above estimates give us

$$\mathbb{P}^{\text{couple}}(F = s \mid F > s-1) \leq \frac{6sd_{\max}}{\ell - 4s + 4} + o(1/\log n). \tag{4.25}$$

Taking a union bound up to time t, and using that $t = O(\log n)$, $d_{\max} = o(n/(\log n)^2)$ and $\ell = \Theta(n)$, we get

$$\mathbb{P}^{\text{couple}}(F \leq t) \leq \frac{3t(t+1)d_{\max}}{\ell - 4t} + o(1) = o(1), \tag{4.26}$$

which in turn implies that, with μ-probability $1 - o(1)$,

$$\mathbb{P}^{\text{couple}}_{x,\xi}(F \leq t) = o(1). \tag{4.27}$$

Indeed, letting $\mathbb{P}^{\text{couple}}(F \leq t) = p_n$ and $B = \{(x,\xi) \in H \times Conf_H : \mathbb{P}^{\text{couple}}_{x,\xi}(F \leq t) > p_n^{1/2}\}$, we see that

$$\mathbb{P}^{\text{couple}}(F \leq t) = p_n > \mu(B) p_n^{1/2}, \tag{4.28}$$

and hence $\mu(B) < p_n^{1/2}$. So, with μ-probability at least $1 - p_n^{1/2}$, we have $\mathbb{P}^{\text{couple}}_{x,\xi}(F \leq t) \leq p_n^{1/2} = o(1)$.

Since the I_s's and J_s's are perfectly coupled until the coupling of the two random walks fails, we also have, whp in x and ξ,

$$|\mathbb{P}_{x,\xi}(\tau > t) - \mathbb{P}^{\text{mod}}_{x,\xi}(\sigma > t)| \leq \mathbb{P}^{\text{couple}}_{x,\xi}(F \leq t) = o(1). \tag{4.29}$$

□

§4.3 Link between the dynamic and the static models

In this section, we prove Theorem 4.1.5. Consider the modified random walk given in the statement of Lemma 4.2.2 and sample uniform jump times up to time t. For any fixed $T = \{t_1, \dots, t_r\} \subset [t]$, we see that the modified random walk conditional on the event $J(T) \coloneqq \{J_s = 0 \text{ for } s \in [t] \setminus T, J_s = 1 \text{ for } s \in T\}$ is a time-inhomogeneous Markov chain that makes random-walk moves at times $s \in [t] \setminus T$ and uniform jumps at times $s \in T$. Since this Markov chain becomes stationary when it makes a uniform jump, for any $\emptyset \neq T \subset [t]$, $x \in H$ and $\xi \in \mathit{Conf}_H$,

$$\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot \mid J(T)) = U_H(\cdot), \tag{4.30}$$

which gives us

$$\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot \mid \sigma \leq t) = \frac{\sum_{T \subset [t], T \neq \emptyset} \mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot \mid J(T))}{\sum_{T \subset [t], T \neq \emptyset} \mathbb{P}^{\mathrm{mod}}_{x,\xi}(J(T))} = U_H(\cdot). \tag{4.31}$$

On the other hand, since the modified random walk up to time t conditional on the event $\{\sigma > t\}$ is the same as the random walk on the static graph, for any $x \in H$ and $\xi \in \mathit{Conf}_H$ we have

$$\|\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot \mid \sigma > t) - U_H(\cdot)\|_{\mathrm{TV}} = \mathcal{D}^{\mathrm{stat}}_{x,\xi}(t). \tag{4.32}$$

Using the triangle inequality twice, we obtain

$$\begin{aligned} \|\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot) - U_H(\cdot)\|_{\mathrm{TV}} \leq & \mathbb{P}^{\mathrm{mod}}_{x,\xi}(\sigma > t)\|\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot \mid \sigma > t) - U_H(\cdot)\|_{\mathrm{TV}} \\ & + \mathbb{P}^{\mathrm{mod}}_{x,\xi}(\sigma \leq t)\|\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot \mid \sigma \leq t) - U_H(\cdot)\|_{\mathrm{TV}} \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \|\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot) - U_H(\cdot)\|_{\mathrm{TV}} \geq & \mathbb{P}^{\mathrm{mod}}_{x,\xi}(\sigma > t)\|\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot \mid \sigma > t) - U_H(\cdot)\|_{\mathrm{TV}} \\ & - \mathbb{P}^{\mathrm{mod}}_{x,\xi}(\sigma \leq t)\|\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot \mid \sigma \leq t) - U_H(\cdot)\|_{\mathrm{TV}}. \end{aligned} \tag{4.34}$$

Inserting (4.31) and (4.32), we obtain

$$\|\mathbb{P}^{\mathrm{mod}}_{x,\xi}(Y_t \in \cdot) - U_H(\cdot)\|_{\mathrm{TV}} = \mathbb{P}^{\mathrm{mod}}_{x,\xi}(\sigma > t)\mathcal{D}^{\mathrm{stat}}_{x,\xi}(t). \tag{4.35}$$

Now using Lemma 4.2.2, we see that, whp in x and ξ,

$$\mathcal{D}_{x,\xi}(t) = \mathbb{P}_{x,\xi}(\tau > t)\mathcal{D}^{\mathrm{stat}}_{x,\xi}(t) + o(1). \tag{4.36}$$

§4.4 Random walk with local rewiring

In this section, we consider a specific example of a random walk on a dynamically rewired graph in which the graph dynamics depends on the position of the random walk. We call this model the *random walk with local rewiring*. The rewiring mechanism works as follows:

(a) At each time $t \in \mathbb{N}$, we draw a Bernoulli random variable Z_t with parameter α, independent of each other and independent of the random walk and the configuration,

(b) If $Z_t = 0$, then the configuration does not change, $C_t = C_{t-1}$, and X_t makes a random-walk move,

(c) If $Z_t = 1$, then we draw a half-edge uniformly at random from $H \setminus \{X_{t-1}\}$, say y, we pair X_{t-1} to y and $C_{t-1}(X_{t-1})$ to $C_{t-1}(y)$ to obtain the new configuration C_t, and X_t makes a random walk move on C_t.

More fomally, let

$$Q_x^R(\xi,\eta) = Q_x^R(\eta,\xi) \coloneqq \begin{cases} \frac{1}{\ell-1} & \text{if } \xi(\eta(x)) = \eta(\xi(x)) \text{ and } |\xi \setminus \eta| \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.37}$$

Within the framework of Section 4.1.1, the above mechanism corresponds to the model in which

$$Q_x(\xi,\eta) = (1-\alpha)I(\xi,\eta) + \alpha Q_x^R(\xi,\eta), \tag{4.38}$$

where $I(\xi,\eta) = 1$ if $\eta = \xi$, and $I(\xi,\eta) = 0$ otherwise, i.e., I is the identity matrix. Since Q_x^R is symmetric for all $x \in H$, we see that the distribution Conf_H is a stationary distribution for Q_x^R for all $x \in H$. This implies that Conf_H is a stationary distribution for Q_x for all $x \in H$.

A direct calculation shows that $U_H \times \mathrm{Conf}_H$ is a stationary distribution of this dynamics:

Proposition 4.4.1. *$U_H \times Conf_H$ is a stationary distribution for the random walk with local rewiring with parameter α, for any $\alpha \in [0,1]$.*

Proof. Since U_H is stationary for P_η for any $\eta \in Conf_H$, and Conf_H is stationary for Q_x for any $x \in H$, for any $y \in H$ and $\eta \in Conf_H$,

$$\begin{aligned} &\sum_{x\in H}\sum_{\xi\in Conf_H} U_H(x)\mathrm{Conf}_H(\xi)\mathbb{P}_{x,\xi}(X_1 = y, C_1 = \eta) \\ &\quad= \sum_{x\in H}\sum_{\xi\in Conf_H} U_H(x)\mathrm{Conf}_H(\xi)Q_x(\xi,\eta)P_\eta(x,y) \\ &\quad= \sum_{x\in H} U_H(x)P_\eta(x,y)\sum_{\xi\in Conf_H}\mathrm{Conf}_H(\xi)Q_x(\xi,\eta) \\ &\quad= \mathrm{Conf}_H(\eta)\sum_{x\in H} U_H(x)P_\eta(x,y) = \mathrm{Conf}_H(\eta)U_H(y), \end{aligned}$$

which shows that $U_H \times \mathrm{Conf}_H$ is a stationary distribution for the random walk with local rewiring model. □

It is not easily seen that the Markov chain is irreducible and aperiodic. In Section 4.4.1 we show that this is indeed the case when $\alpha \in (0,1)$, and so the distribution of the joint chain converges to $U_H \times \mathrm{Conf}_H$ as $t \to \infty$. An important implication is

that the distribution of the random walk alone converges to U_H as $t \to \infty$. Indeed, for any $x \in H$, $\xi \in \mathit{Conf}_H$ and $t \in \mathbb{N}$ we have

$$\mathcal{D}_{x,\xi}(t) \leq \|\mathbb{P}_{x,\xi}((X_t, C_t) \in \cdot) - U_H \times \mathrm{Conf}_H(\cdot)\|_{\mathrm{TV}},$$

and since the right-hand side tends to 0 as $t \to \infty$, $\mathcal{D}_{x,\xi}(t)$ also tends to 0 as $t \to \infty$. On the other hand, this argument does not automatically imply that $\mathcal{D}_{x,\xi}(t)$ is non-increasing in t.

§4.4.1 Irreducibility and aperiodicity

In this section we show that the random walk with local rewiring model is irreducible and aperiodic, which ensures that the total variation distance $\mathcal{D}_{x,\xi}(t)$ converges to 0 as $t \to \infty$ for fixed $x \in H$, $\xi \in \mathit{Conf}_H$ and $\alpha \in (0,1)$. Our proof builds on the proof of irreducibility of the switch chain on multigraphs given by Eggleton and Holton [40].

Proposition 4.4.2. *The rewiring random walk $(X_t, C_t)_{t\in\mathbb{N}_0}$ is irreducible and aperiodic for any initial state $(x,\xi) \in H \times \mathrm{Conf}_H$ and any choice of $\alpha \in (0,1)$.*

Proof. Let $V = \{v_1, \ldots, v_n\}$ and assume that $\deg(v_1) \leq \deg(v_2) \leq \cdots \leq \deg(v_n)$. Identify the set of half-edges H with $[\ell] = \{1, \ldots, \ell\}$ such that the half-edges $1, \ldots, \deg(v_1)$ are associated to v_1, the half-edges $\deg(v_1)+1, \ldots, \deg(v_1)+\deg(v_2)$ to v_2, and so on. Let $v'_1, \ldots, v'_{2k} \in V$ be the odd-degree vertices. We fix a configuration $\xi_0 \in \mathit{Conf}_H$ such that each vertex has the maximum number of self-loops, i.e., each vertex $v \in V$ with even degree has $\deg(v)/2$ self-loops, each vertex $v \in V$ with odd degree has $(\deg(v)-1)/2$ self-loops, and there is exactly one edge between every pair of odd-degree vertices v'_{2i-1}, v'_{2i} for $i = 1, \ldots, k$ (see Figure 4.2). We will show that the pair $(1, \xi_0) \in H \times \mathit{Conf}_H$ is accessible from any pair $(x,\xi) \in H \times \mathit{Conf}_H$ by allowed moves in the random walk with local rewiring model.

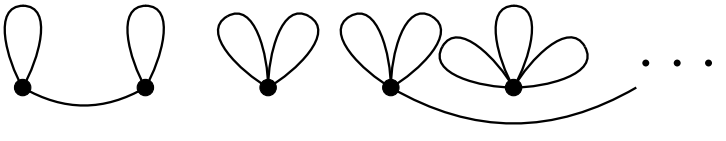

Figure 4.2: The configuration ξ_0.

First we show that, for any $x \in H$, $(1, \xi_0)$ is accessible from (x, ξ_0), by considering two different scenarios:

(a) Suppose that x is on a self-loop and $\xi_0(x) = x'$. We first move to $(1, \xi_1)$ from $(1, \xi_0)$ by rewiring the half-edges $x, x', 1$ and 2 where ξ_0 and ξ_1 agree on all the edges except that $\xi_1(1) = x'$ and $\xi_1(2) = x$. After that we again move to $(1, \xi_0)$ from $(1, \xi_1)$ by rewiring $1, 2, x$ and x' (see Figure 4.3).

(b) Suppose that x is not on a self-loop, i.e., it is on an edge between two odd-degree vertices. We first move to (x', ξ_0) without rewiring, where $x' \in H$ is on a self-loop. After that we apply the procedure in the item 1 to (x', ξ_0).

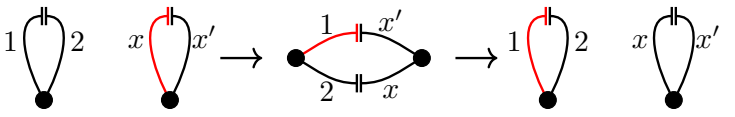

Figure 4.3: Move from half-edge x on a self-loop to half-edge 1 in ξ_0. The red color indicates the position of the walk.

Next, we show that for any $(x, \xi) \in H \times Conf_H$ with $\xi \neq \xi_0$ we have access from (x, ξ) to (y, ξ_0), for some $y \in H$. To do this, we show that we can move from (x, ξ) to some $(y, \eta) \in H \times Conf_H$ such that the configuration η has more edges in common with ξ_0 than ξ has, i.e., $|\xi \cap \xi_0| < |\eta \cap \xi_0|$, by considering the two scenarios:

(a) Suppose that x is on an edge that is not in ξ_0, i.e., $\xi(x) \neq \xi_0(x)$. Then we move to (y, η) by rewiring the half-edges $x, \xi(x), \xi_0(x)$ and $\xi(\xi_0(x))$, where ξ and η agree on all the edges except that $\eta(x) = \xi_0(x)$ and $\eta(\xi(x)) = \xi(\xi_0(x))$ and $y \sim \xi_0(x)$. Since $\eta(x) = \xi_0(x)$, we have that $|\xi \cap \xi_0| \leq |\eta \cap \xi_0| - 1$.

(b) Suppose that x is on an edge that is in ξ_0, i.e., $\xi(x) = \xi_0(x)$. Let $y \in H$ be a half-edge such that $\xi(y) \neq \xi_0(y)$, $\xi(x) = x'$ and $\xi(y) = y'$. Since $\deg(v) \geq 2$ for all $v \in V$, in the graph given by ξ there is a cycle of edges $\{y, y'\}, \{y_1, y_1'\}, \ldots, \{y_K, y_K'\}$ with $v(y') = v(y_1)$, $v(y_K') = v(y)$ and $v(y_i') = v(y_{i+1})$ for $i = 1, \ldots, K-1$. Let $\eta \in Conf_H$ be the configuration that agrees with ξ on all the edges except that $\eta(x) = y'$ and $\eta(y) = x'$, so that the edges $\{y_1, y_1'\}, \ldots, \{y_K, y_K'\}$ are present in η as well as in ξ. First we move from (x, ξ) to (y_1, η) by rewiring x, x', y and y'. Then we make K moves, from (y_i, η) to (y_{i+1}, η) for $i = 1, \ldots, K$, where $y_{K+1} = y$ without rewiring. After that we move from (y, η) to (y_1, ξ) by rewiring x, x', y and y', and finally we traverse the cycle again without rewiring to reach (y, ξ) from (y_1, ξ) (see Figure 4.4). Now y is on an edge that is not in ξ_0, so by applying the procedure in item 1 we can increase the number of edges we have in common with ξ_0.

By applying these procedures, we can reduce the number of edges that are not in ξ_0, so we can go from any $(x, \xi) \in H \times Conf_H$ to (y, ξ_0) for some $y \in H$, and then apply the above procedure to reach $(1, \xi_0)$.

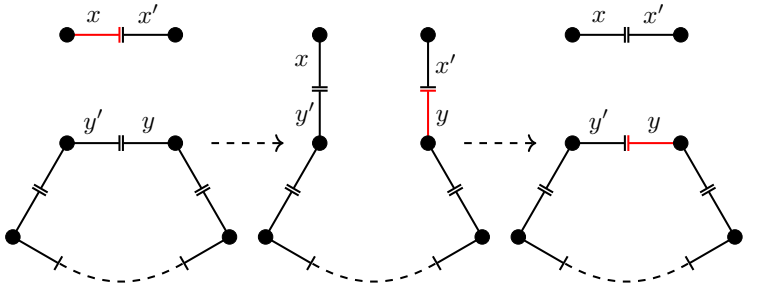

Figure 4.4: Moving from (x, ξ) to (y, η) by using a cycle. The red color indicates the position of the walk.

To show that we can access an arbitrary state (x, ξ) from $(1, \xi_0)$, we first note that we can access (y, ξ_0), for any y, from $(1, \xi_0)$ by relabelling the half-edges and using the first argument above. Then we see that we can access (x, ξ) from (y, ξ_0) for any y using the above strategy of reducing the edges and using the cycles to move around. Hence, the Markov chain is irreducible. Since, by traversing the self-loop without rewiring, we can reach $(1, \xi_0)$ from itself in one step, we see that the Markov chain is also aperiodic. □

§4.4.2 The mixing time of the random walk with local rewiring

In this section, we study the quantity $\mathcal{D}_{x,\xi}(t)$ for the random walk with local rewiring and show that we have the same trichotomy as for the random walk on the dynamic configuration model [13]:

Theorem 4.4.3 (**Scaled mixing profiles**). *Suppose that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \alpha_n \log n = \beta \in [0, \infty]$, *and consider the rewiring random walk with parameter* α_n. *Subject to Condition* 4.1.2(R1) *and Condition* 4.1.3, *the following hold whp in* x *and* ξ*:*

(1) *If* $\beta = \infty$, *then*

$$\mathcal{D}_{x,\xi}\big(\lfloor c\alpha_n^{-1} \rfloor\big) = \mathrm{e}^{-c} + o(1), \quad c \in [0, \infty). \tag{4.39}$$

(2) *If* $\beta \in (0, \infty)$, *then*

$$\mathcal{D}_{x,\xi}\big(\lfloor c \log n \rfloor\big) = \begin{cases} \mathrm{e}^{-\beta c} + o(1), & c \in [0, c_{n,\mathrm{stat}}), \\ o(1), & c \in (c_{n,\mathrm{stat}}, \infty). \end{cases} \tag{4.40}$$

(3) *If* $\beta = 0$, *then*

$$\mathcal{D}_{x,\xi}\big(\lfloor c \log n \rfloor\big) = \begin{cases} 1 - o(1), & c \in [0, c_{n,\mathrm{stat}}), \\ o(1), & c \in (c_{n,\mathrm{stat}}, \infty). \end{cases} \tag{4.41}$$

Proof. We show that Condition 4.1.4 holds and then use Corollary 4.1.6 to prove the claim. For fixed $t = O(\log n)$, fix some $T = \{t_1, \dots, t_r\} \subset [t-1]$ and some $x_{[0,t-1]}$, $\bar{x}_{[0,t-1]}$, $\hat{x}_{[r]}$ and $\tilde{x}_{[r]}$ that are dynamically self-avoiding with respect to T. Conditioned on the event $H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]})$, x_{t-1} cannot be rewired before time t. Indeed, by construction the half-edges that are rewired before time t are $x_{t_1-1}, \dots, x_{t_r-1}$, $\bar{x}_{t_1-1}, \dots, \bar{x}_{t_r-1}$, $\hat{x}_1, \dots, \hat{x}_r$ and $\tilde{x}_1, \dots, \tilde{x}_r$, and x_{t-1} is not equal to any of these. So we have

$$\begin{aligned} &\mathbb{P}(I_t = 1 \mid H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]})) \\ &= \mathbb{P}(Z_t = 1 \mid H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]})) = \alpha_n, \end{aligned} \tag{4.42}$$

and $\mathbb{P}(C_t(x_{t-1}) \in \cdot \mid H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]}) \cap \{I_t = 1\})$ is the uniform distribution on $H \setminus \{x_{t-1}\}$, which gives

$$\|\mathbb{P}(C_t(x_{t-1}) \in \cdot \mid H(T, x_{[0,t-1]}, \bar{x}_{[0,t-1]}, \hat{x}_{[r]}, \tilde{x}_{[r]}) \cap \{I_t = 1\}) - U_H(\cdot)\|_{\mathrm{TV}} = \frac{1}{\ell}. \tag{4.43}$$

Since this holds for any choice of $x_{[0,t-1]}$, $\bar{x}_{[0,t-1]}$, $\hat{x}_{[r]}$ and $\tilde{x}_{[r]}$, Condition 4.1.4 holds.

On the other hand, the event $\{\tau = t\}$ is the same as the event $\{\min\{s \in \mathbb{N} : R_s = 1\} = t\}$, since when a rewiring occurs the random walk steps over a rewired edge with probability 1. This implies that for any x and ξ, and since $\lim_{n\to\infty} \alpha_n = 0$,

$$\mathbb{P}_{x,\xi}(\tau > t) = (1 - \alpha_n)^t = \exp(-\alpha_n t) + o(1). \tag{4.44}$$

So we have

$$\mathbb{P}_{x,\xi}(\tau > t) = \exp(-c) + o(1) \text{ when } \lim_{n\to\infty} \alpha_n \log n = \infty \text{ and } t = \lfloor c\alpha_n^{-1} \rfloor, \tag{4.45}$$

$$\mathbb{P}_{x,\xi}(\tau > t) = \exp(-\beta c) + o(1) \text{ when } \lim_{n\to\infty} \alpha_n \log n = \beta \text{ and } t = \lfloor c \log n \rfloor, \tag{4.46}$$

$$\mathbb{P}_{x,\xi}(\tau > t) = 1 - o(1) \text{ when } \lim_{n\to\infty} \alpha_n \log n = 0 \text{ and } t = \lfloor c \log n \rfloor. \tag{4.47}$$

Combining these with Corollary 4.1.6, we obtain the desired result. □

§4.5 Discussion

1. Coupling between the two random walks: The core ingredient of the proof of the main result, which is the coupling between the random walk on the dynamically rewired graph and the modified random walk, is best visualised as follows: imagine we are looking at the random walk on the dynamically rewired graph from the point of view of the initial configuration. Then it looks as if the random walk performs an ordinary random walk on the static initial graph (when it walks on the parts that are not changed by the dynamics), with the exception that at some random times it makes uniform jumps (when it encounters a previously rewired edge). This suggests that the random walk on the dynamically rewired graph can be coupled to a random walk that exactly does this.

The framework of the coupling to a modified random walk introduced in this paper is based on the ideas developed in [13]. In fact, the coupling of the random walk on the dynamically rewired random graph and the modified random walk is implicit in the proof of the main theorem of [13]. There the main idea was that the path probabilities under the two random walk models coincide for self-avoiding paths, and it was shown that the random walk paths are with high probability self-avoiding.

The crucial observation is that the random walk paths on a typical configuration are self-avoiding with high probability under the law of the configuration model. The particular form of Condition 4.1.4 is motivated by this observation. This also suggests that the same results should hold when the distribution of the initial graph is replaced by some other distribution on graphs on which random walk paths are *'typically'* self-avoiding.

2. One-sided cut-off: It is easy to construct examples of one-sided cut-off in the more general framework of Markov chains. Suppose that P is the matrix of transition probabilities of an ergodic Markov chain on a state space $\mathcal{X}$ with a stationary distribution π, and let Π be the matrix whose rows are all equal to π. Fix $\alpha \in (0, 1]$ and consider the Markov chain where at each step transitions are made according to matrix P with probability $1 - \alpha$ and according to matrix Π with probability α, and these choices are made independently at each step. This corresponds to the Markov chain with transition probabilities given by $(1 - \alpha)P + \alpha\Pi$. Note that, as soon as Π is used for transition, the Markov chain becomes stationary. If we let σ be the first time Π is used for a transition then σ is a strong stationary time for the Markov chain, and hence the total variation distance can be bounded by tail probabilities of σ. In fact, for any $x \in \mathcal{X}$ and $t \in \mathbb{N}$ we have

$$\|Q^t(x, \cdot) - \pi\|_{\text{TV}} = (1 - \alpha)^t \|P^t(x, \cdot) - \pi\|_{\text{TV}}, \tag{4.48}$$

since the probability of the event $\{\sigma > t\}$ is $(1-\alpha)^t$ and the Markov chain is stationary at time t conditioned on the event $\{\sigma \leq t\}$.

Now, suppose $(P_n)_{n\in\mathbb{N}}$ is a sequence of ergodic Markov chains indexed by the size n of the state space, π_n is the stationary distribution and T_n is the mixing time of P_n with $T_n \to \infty$ as $n \to \infty$. Let Π_n be the matrix of transition probabilities whose rows are all equal to π_n, and consider the Markov chain whose transition probabilities are given by the matrix $Q_n = (1 - \alpha_n)P_n + \alpha_n\Pi_n$. If $(P_n)_{n\in\mathbb{N}}$ exhibits cut-off, then we have the same trichotomy as in Theorem 4.4.3:

- $\lim_{n\to\infty} \alpha_n T_n = \infty$: the mixing time is of order α_n^{-1} without cut-off,
- $\lim_{n\to\infty} \alpha_n T_n = \beta \in (0, \infty)$: the mixing time is of order T_n with one-sided cut-off,
- $\lim_{n\to\infty} \alpha_n T_n = 0$: the mixing time is of order T_n with two-sided cut-off (the same as for P_n).

3. Regularity of the graph dynamics: Simple modifications to the random walk with local rewiring model can lead to violations of Condition 4.1.4. Let us consider a modification in which the rewiring mechanism is slightly changed: When $Z_t = 1$ we choose an edge, say $\{y, z\}$, uniformly at random from the set of all edges of C_{t-1} except the edge $\{X_{t-1}, C_{t-1}(X_{t-1})\}$, and we pair the half-edges $X_{t-1}, C_{t-1}(X_{t-1}), y, z$ uniformly at random to obtain the new configuration C_t. In this case, the probability that X_{t-1} is paired to its previous pair $C_{t-1}(X_{t-1})$ is $1/3$, and hence Condition 4.1.4(D2) is not satisfied. Another possibility is to let α_n depend on X_{t-1}. Suppose that we are given a sequence $(\alpha_{n,x})_{x\in H}$, and $Z_t = 1$ with probability $\alpha_{n,x}$ conditioned on $X_{t-1} = x$. In this case Condition 4.1.4(D1) is violated.

4. Local vs. global rewiring mechanisms: The rewiring mechanism of the random walk with local rewiring model can be seen as a *'local-to-global'* rewiring mechanism: one end of the rewired edge is selected *'locally'* at the position of the random walk, while the other end is selected *'globally'* from the set of all possible half-edges. On the

other hand, the rewiring mechanism of the random walk on the dynamic configuration model introduced in [12], can be seen as a *'global-to-global'* rewiring mechanism, in the same sense. The effects of local versus global choices are best seen in the tail probabilities of the randomised stopping time τ. In the random walk on the dynamic configuration model, we had $\mathbb{P}_{x,\xi}(\tau > t) = (1-\alpha)^{t(t+1)/2} + o(1)$ whp in x and ξ, where the $t(t+1)/2$ term comes from the cumulative effect of doing a global rewiring at each step.

It would be interesting to study rewiring mechanisms that interpolate between these two examples. One possibility is to consider a model in which some of the half-edges in a neighborhood of the random walk are paired to randomly chosen half-edges. Formally, let $B^r_\xi(x)$ be the set of half-edges that can be reached from x by a random walk of at most r steps on the configuration ξ. Suppose that, at each time t, every half-edge in $B^r_{C_{t-1}}(X_{t-1})$ is rewired independently with probability α. The case $r = 0$ would correspond to the random walk with local rewiring model, while the case $r = \infty$ would correspond to a global-to-global rewiring mechanism similar to the rewiring mechanism of the dynamic configuration model. In between these two extremes, we expect to see that tail probabilities of τ interpolating between that of the random walk with local rewiring model and the random walk on the dynamic configuration model.

5. Comparison with the switch chain: The rewiring mechanism of the random walk with local rewiring model can be seen as a variation of the switch chain of [34]. There are two main differences:

- in the switch Markov chain, the switching edges are chosen uniformly at random from all possible pairs, while in the random walk with local rewiring model one of the switching edges is chosen according to the random walk,
- in the switch Markov chain, the underlying graph is forced to be simple, while in the random walk with local rewiring model, multiple edges and self-loops are allowed.

It would be interesting to study a variation of the random walk with local rewiring model in which the simplicity of the graph is preserved. The main challenge would be to deal with the combinatorial contraints that are imposed by the preservation of the simplicity.

PART II

UNION COMPLEXITY OF RANDOM DISK REGIONS

CHAPTER 5

Union complexity of random disk regions

This chapter is based on joint work with Mark de Berg.

Abstract

We study the union complexity of a set of n disks when disk centers are sampled uniformly and independently at random in a convex compact region S. We consider the case where all the disks have a common radius $R = \mathsf{diam}(S)$ and prove that if S is a square or a disk, then the expected union complexity is $\Theta(n^{1/3})$. Our proofs are based on the arguments used by Har-Peled [55] for the expected complexity of convex hulls of random points. We also show a connection between the union complexity of disk regions and the complexity of convex hull of a set of points.

§5.1 Introduction and main results

The introduction to this chapter was given in Section 1.2. Nevertheless, we repeat the setting and the definitions for ease of reading. Let S be a fixed convex compact region in $\mathbb{R}^2$, and $X = \{X_1, \ldots, X_n\}$ be a set of n points sampled independently and uniformly at random from S. Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a collection of n disks, where D_i is the closed disk centered at X_i with a fixed radius R such that $\mathsf{diam}(S) \leq R < \infty$, where $\mathsf{diam}(S)$ is the diameter of S, for $i = 1, \ldots, n$. By choosing the radius large enough such that any disk covers S completely, we make sure that the boundary of the disks always lie outside of S and this makes the analysis easier. The set of boundary disks of $\mathcal{D}$, denoted by $\mathsf{BD}(\mathcal{D})$, is the set of disks in $\mathcal{D}$ whose boundaries are not completely covered by other disks, i.e.,

$$\mathsf{BD}(\mathcal{D}) = \{D \in \mathcal{D} : \partial D \setminus \cup_{D' \in \mathcal{D} \setminus \{D\}} D' \neq \emptyset\},$$

where ∂D denotes the boundary of D. We are interested in union complexity of $\mathcal{D}$ which is the number of boundary arcs of $\mathcal{D}$. This number is linear in the number of disks in $\mathsf{BD}(\mathcal{D})$. Let B_n denote the number of boundary disks of $\mathcal{D}$ when $\mathcal{D}$ contains n disks. B_n is a random variable, since disk centers are random, and we are interested in the expected value of B_n as a function of n. We consider two cases: the case where S is a unit square and all the disks have radius $R = \sqrt{2}$, and the case where S is a unit disk and all the disks have radius $R = 2$.

In what follows, we use the notation for asymptotic comparison of functions $f, g : \mathbb{N} \to [0, \infty)$: $f(n) = O(g(n))$ or $g(n) = \Omega(f(n))$ when $\limsup_{n\to\infty} f(n)/g(n) < \infty$; $f(n) = o(g(n))$ or $g(n) = \omega(f(n))$ when $\lim_{n\to\infty} f(n)/g(n) = 0$; $f(n) = \Theta(g(n))$ when both $f(n) = O(g(n))$ and $g(n) = O(f(n))$. We denote by $d(x, y)$ the Euclidean distance between $x, y \in \mathbb{R}^2$, and with a slight abuse of notation we write $d(x, A) = \inf\{d(x, y) : y \in A\}$ for $x \in \mathbb{R}^2$ and $A \subset \mathbb{R}^2$. Our main result is given in the following theorem.

Theorem 5.1.1. *Suppose that*

(a) *either S is the unit square $[0, 1] \times [0, 1] \in \mathbb{R}^2$ and each disk has radius $R = \sqrt{2}$;*

(b) *or S is the unit disk $\{x \in \mathbb{R}^2 : d(x, o) \leq 1\}$, where o is the origin, and each disk has radius $R = 2$.*

Then

$$\mathbb{E}(B_n) = \Theta(n^{1/3}).$$

For the case of the unit square, the problem appears in the context of conflict-free colouring as discussed in Section 1.2.1. We present the unit-disk case as a generalisation. The union-complexity problem is related to the problem of the complexity of the convex hull, as we pointed out in Section 1.2.2. In fact, our proof follows some ideas developed for tackling convex-hull problems [39, 55]. In Section 5.2 we give the proof of Theorem 5.1.1 for the case of a unit square and in Section 5.3 for the case of a unit disk. In Section 5.4, we discuss several extensions of the boundary complexity problem.

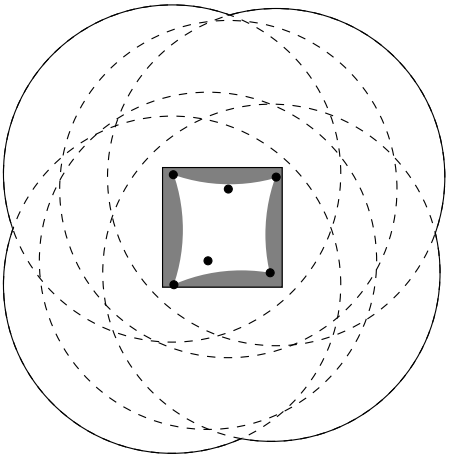

Figure 5.1: The halo for a set of 6 disks with centers inside the unit square is shown as the shaded region in the square.

Before proceeding with the proof of the Theorem 5.1.1, we introduce some further notation and state a lemma that will be crucial. Consider the general setting: S is a convex compact region and the disks have radius $\mathsf{diam}(S) \leq R < \infty$. Let $\mathsf{Cov}(\mathcal{D}) \coloneqq \cup_{D\in\mathcal{D}} D$ denote the coverage area of $\mathcal{D}$, i.e., the subset of $\mathbb{R}^2$ covered by the disks in $\mathcal{D}$. Let $\mathsf{Halo}(\mathcal{D}) = \{x \in S : d(x, \partial(\mathsf{Cov}(\mathcal{D})) \leq R\}$ be the set of points in S whose distance to the boundary of the coverage area is less than R (see Figure 5.1), and let $A_n = \mathbb{E}(\mathsf{Area}(\mathsf{Halo}(\mathcal{D})))$ be the expected area of the halo. To compute the expected number of boundary disks, we use the area of the halo. The two are related through the following lemma, which is analogous to Efron's Theorem for the convex hull [39].

Lemma 5.1.2. *Suppose that S is a convex compact region in $\mathbb{R}^2$ with unit area, and let $\mathcal{D}$ be a collection of n disks with a fixed radius R such that any single disk covers S completely and such that the centers are sampled uniformly and independently from S. Then $\mathbb{E}(B_n) = nA_{n-1}$, where A_n is the expected area of the halo of a set of n points sampled uniformly and independently at random from S.*

Proof. First we note that, for any $i = 1, \ldots, n$, the disk D_i is a boundary disk if and only if its center falls inside the halo of $\overline{\mathcal{D}}_i$, where $\overline{\mathcal{D}}_i \coloneqq \mathcal{D} \setminus \{D_i\}$. This gives

$$B_n = \sum_{i=1}^{n} \mathbb{1}_{\{D_i \in \mathsf{BD}(\mathcal{D})\}} = \sum_{i=1}^{n} \mathbb{1}_{\{X_i \in \mathsf{Halo}(\overline{\mathcal{D}}_i)\}},$$

so

$$\begin{aligned}\mathbb{E}(B_n) &= \sum_{i=1}^{n} \mathbb{P}(X_i \in \mathsf{Halo}(\overline{\mathcal{D}}_i)) \\ &= \sum_{i=1}^{n} \int_{S^{n-1}} \mathbb{P}(X_i \in \mathsf{Halo}(\overline{\mathcal{D}}_i) \mid X_j = x_j, j \in [n] \setminus \{i\}) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n.\end{aligned}$$

Note that the conditional probability $\mathbb{P}(X_i \in \mathsf{Halo}(\overline{\mathcal{D}}_i) \mid X_j = x_j, j \in [n] \setminus \{i\})$ is equal to $\mathsf{Area}(\mathsf{Halo}(\overline{\mathcal{D}}_i))$, and so by symmetry we have

$$\begin{aligned}\mathbb{E}(B_n) &= \sum_{i=1}^{n} \int_{S^{n-1}} \mathsf{Area}(\mathsf{Halo}(\overline{\mathcal{D}}_i)) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n \\ &= \sum_{i=1}^{n} \mathbb{E}(\mathsf{Area}(\mathsf{Halo}(\overline{\mathcal{D}}_i))) = nA_{n-1}.\end{aligned}$$

□

§5.2 Case of the unit square

In this section we give the proof of Theorem 5.1.1 for the case of the unit square. Thanks to Lemma 5.1.2, in order to compute the expected boundary complexity we only need to compute the expected area of the halo A_n. The next proposition gives an upper bound. The proof follows the arguments in [55] for the convex hull of a point set sampled in the unit square.

Proposition 5.2.1. *Suppose that S is the unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$. Then $A_n = O(n^{-2/3})$.*

Proof. We divide the unit square into n rows and n columns each of width $1/n$, which gives n^2 small squares of size $1/n \times 1/n$. We derive an upper bound for the number of squares that intersect the halo and multiply this by n^{-2} to get an upper bound for the area.

Let $S_{i,j} = [(i-1)/n, i/n] \times [(j-1)/n, j/n]$ be the jth square of the ith column, $C_i = \cup_{j=1}^{n} S_{i,j}$ be the ith column, and $C(k,l) = \cup_{i=k}^{l} C_i$. Let $X = \{X_1, \dots, X_n\}$ be the random set of disk centres. Let $m = \lfloor n^{2/3} \rfloor$ and for $j = m+1, \dots, n-m$ define $Y_j := \min\{k \in [n] : X \cap (\cup_{i=j-m}^{j-1} S_{i,k}) \neq \emptyset\}$, i.e., Y_j is the index of the lowest row that contains a point from X in $C(j-m, j-1)$. Define Y'_j analogously for $C(j+1, j+m)$ (see Figure 5.2a).

The squares at the bottom of the jth column that intersect the halo stay below or intersect the circle arc with radius $\sqrt{2}$ that passes through the lowest disk centers in $C(j-m, j-1)$ and $C(j+1, j+m)$. Furthermore, this arc stays below the arc that passes through the upper-left corner of the square $S_{j-m,\max\{Y_j,Y'_j\}}$ and the upper-right corner of the square $S_{j+m,\max\{Y_j,Y'_j\}}$. The latter arc has cord length $(2n^{2/3}+1)/n = 2n^{-1/3} + n^{-1}$, so the distance between the highest point of the arc and the cord is $\sqrt{2} - \sqrt{2 - \left(2n^{-1/3} + n^{-1}\right)^2} = O(n^{-2/3})$ as n tends to ∞. Let R_j denote the number of squares that stay between the chord and the highest point of the arc. Then R_j is of order $O(n^{1/3})$ (see Figure 5.2b).

Clearly, the number of small squares in C_j that intersects the halo is less than $\max\{Y_j, Y'_j\} + R_j < Y_j + Y'_j + R_j$. Next we compute the $\mathbb{E}(Y_j)$ and $\mathbb{E}(Y'_j)$. For Y_j, we divide the area $C(j-m, j-1)$ into rectangles of area $1/n$, so that each rectangle is m squares wide and n/m squares high. Let Z_j be the index of the lowest

rectangle that contains a point from X. Then $Y_j \leq n^{1/3} Z_j$. Now, observe that $\mathbb{P}(Z_j \geq k) \leq (1-(k-1)/n)^n \leq e^{-(k-1)}$. Hence

$$\mathbb{E}(Z_j) \leq \sum_{k=1}^{n^{2/3}} k\mathbb{P}(Z_j = k) \leq \sum_{k=1}^{\infty} \mathbb{P}(Z_j \geq k) \leq \sum_{k=1}^{\infty} k e^{-(k-1)} = O(1). \tag{5.1}$$

From this we get $\mathbb{E}(Y_j) = O(n^{1/3})$ and similarly $\mathbb{E}(Y_j') = O(n^{1/3})$. Summing over $j = m, \ldots, n-m$, we see that the expected number of small squares that fall into the halo at the bottom of the square between the columns $m+1$ and $n-m$ is $O(n^{4/3})$. Doing the same for the upper, left and right sides, we get a total number of $O(n^{4/3})$ small squares contributing to the halo. We have not accounted for the four squares with side length $m = n^{2/3}$ at the corners, but these contain a total number of $O(n^{2/3})$ small squares. So in total the halo has $O(n^{4/3})$ small squares. Since each small square has area n^{-2}, we get $A_n = O(n^{-2/3})$. □

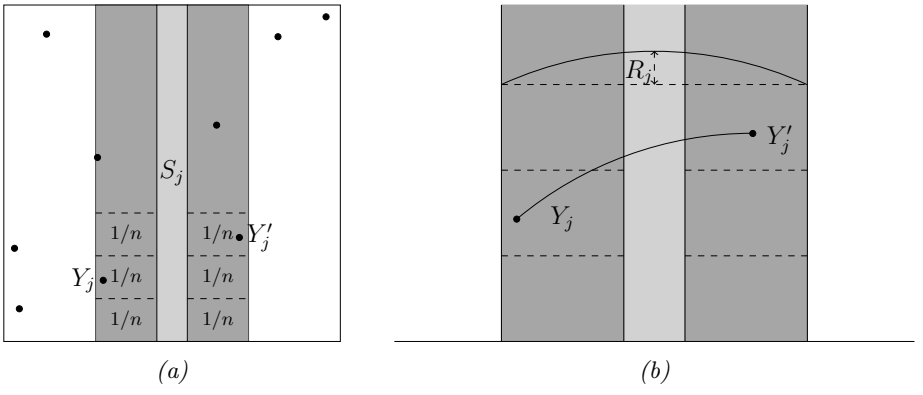

Figure 5.2: Illustration of proof of Proposition 5.2.1

Using similar arguments, we next prove that $n^{2/3}$ is the correct order for the expected area of the halo.

Proposition 5.2.2. *Suppose that S is the unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$. Then $A_n = \Omega(n^{-2/3})$.*

Proof. As in the proof of Proposition 5.2.1, consider the window of width $2m + 1 = 2\lfloor n^{2/3} \rfloor + 1$ around the jth column. Consider the arc whose endpoints are $((j-m-1)/n, 0)$ and $((j+m)/n, 0)$ and whose center lies below the unit square. The cord length of this arc is $(2m+1)/n$, so y-coordinate of the highest point of this arc is $\sqrt{2} - \sqrt{2 - ((2m+1)/n)^2} = \Omega(n^{-2/3})$ and hence the latter point lies in a row with index $\Omega(n^{1/3})$. The expected number of small squares on jth column that stay in the halo is bounded from below by the minimum of the row index of the highest point of the latter arc and $\mathbb{E}(\min\{Y_j, Y_j'\}) - 1$. Dividing $C(j-m, j-1)$ and $C(j+1, j+m)$ into rectangles of area $1/n$, and defining Z_j and Z_j' as in the proof of Proposition 5.2.1, we see that $Y_j - 1 \geq n^{1/3}(Z_j - 1)$ and $Y_j' - 1 \geq n^{1/3}(Z_j' - 1)$,

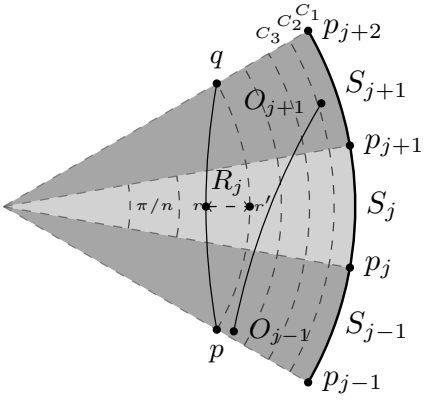

Figure 5.3: Illustration of proof of Proposition 5.3.1.

so $\mathbb{E}(\min\{Y_j, Y_j'\}) - 1 \geq n^{1/3}\mathbb{E}(\min\{Z_j, Z_j'\} - 1)$. Note that $\mathbb{E}(\min\{Z_j, Z_j'\} - 1) \geq \mathbb{P}(Z_j > 1, Z_j' > 1) = (1 - 2/n)^n \geq e^{-3}$ for large enough n. From this we conclude that $\mathbb{E}(\min\{Y_j, Y_j'\}) - 1 = \Omega(n^{1/3})$, so the expected number of squares on jth column that stay in the halo is $\Omega(n^{1/3})$, which gives us the desired result. □

Combining the last two propositions with Lemma 5.1.2, we obtain the result of Theorem 5.1.1 for the unit square.

§5.3 Case of the unit disk

In this section we give the proof of Theorem 5.1.1 for the case of the unit disk. As in the case of the unit square, we obtain upper and lower bounds for the expected area of the halo A_n, then we combine these bounds with Lemma 5.1.2 to obtain the result of Theorem 5.1.1. The next proposition gives an upper bound. Again, the proof follows the arguments in [55] for the convex hull.

Proposition 5.3.1. *Suppose that S is the unit disk $\{x \in \mathbb{R}^2 : d(x, o) \leq 1\} \subset \mathbb{R}^2$. Then $A_n = O(n^{-2/3})$.*

Proof. Assuming without loss of generality that $n = m^3$ for some $m \in \mathbb{N}$, we divide the unit disk S into n tiles of equal area as follows: divide S into m slices, $S_1, \ldots, S_m$, by drawing m lines from the center to m equally spaced points $p_1, \ldots, p_m$ on ∂S. Then divide each slice into m^2 tiles of equal area as follows: consider m^2 concentric rings given by m^2 concentric circles $C_1 = \partial S, C_2, \ldots, C_{m^2}$, with radii $r_1 = 1, r_2, \ldots, r_{m^2}$ respectively, such that the intersection of each slice and ring gives a tile of area π/n (see Figure 5.3). Let $S_{i,j}$ be the ith outermost tile in S_j for $i = 1, \ldots, m^2$ and $j = 1, \ldots, m$, i.e., $S_{i,j}$ is the intersection of S_j and the ring between the circles C_i and C_{i+1}. We compute the expected number of tiles that intersects the halo and multiply the result by $1/n$ to get an upper bound for the expected area of the halo. We do this by computing the expected number of tiles that intersect the halo for each slice.

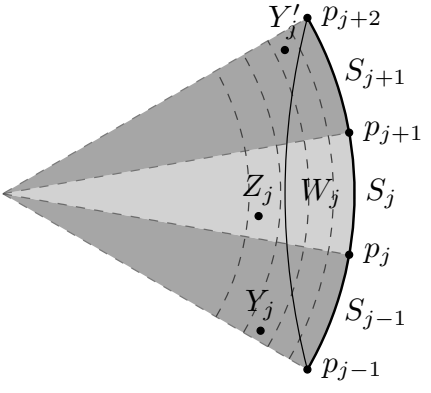

Figure 5.4: Illustration of proof of Proposition 5.3.2.

Let Y_j be the index of the outermost tile of the slice S_{j-1} that contains a disk center, i.e., $Y_j = \min\{i : S_{i,j-1} \cap X \neq \emptyset\}$, and analogously define Y'_j for the slice S_{j+1}. Let O_{j-1} and O_{j+1} be the outermost disk centers, that are the disk centers furthest away from the origin, in S_{j-1} and S_{j+1} respectively. Consider the arc with radius 2 that passes through O_{j-1} and O_{j+1} whose center lies away from the origin relative to the line passing through O_{j-1} and O_{j+1}. The tiles of S_j that intersect the halo stay outside this arc. Furthermore, the latter arc stays outside the arc a with radius 2 that passes through points p and q and whose center lies away from the origin relative to the line passing through p and q, where p and q are the extreme points of the arc $a' = C_{Z_j+1} \cap (S_{j-1} \cup S_j \cup S_{j+1})$ and $Z_j = \max\{Y_j, Y'_j\}$. Let r and r' be the midpoints of the arcs a and a', respectively, and let R_j be the number of tiles between r and r' (see Figure 5.3). Then the number of tiles in S_j that intersect the halo is bounded from above by $R_j + Z_j \leq R_j + Y_j + Y'_j$.

The length of the line segment connecting r and r' is

$$d(o,p) - d(o,p)\sin\left(\frac{3\pi}{m}\right) + 2\left(1 - \cos\left(\arcsin\left(\frac{|op|}{2}\sin\left(\frac{3\pi}{m}\right)\right)\right)\right) = O(m^{-2}),$$

On the other hand, the radial length of every tile is greater than or equal to $r_1 - r_2 \geq 1/(2m^2)$, so we have $R_j = O(1)$. To compute $\mathbb{E}(Y_j)$, we note that $\mathbb{P}(Y_j \geq k) = (1-(k-1)/n)^n \leq \exp(-(k-1))$. This gives

$$\mathbb{E}(Y_j) = \sum_{k=1}^{\infty} \mathbb{P}(Y_j \geq k) \leq \sum_{k=1}^{\infty} e^{-(k-1)} = O(1). \tag{5.2}$$

Thus, the expected number of tiles in S_j that intersect the halo is $O(1)$ and the expected total number of tiles that intersect the halo is $O(m)$, which gives $A_n = O(m^{-2}) = O(n^{-2/3})$. □

The next proposition gives a lower bound for the expected area of the halo A_n:

Proposition 5.3.2. *Suppose S is the unit disk $\{x \in \mathbb{R}^2 : d(x,o) \leq 1\} \subset \mathbb{R}^2$. Then $A_n = \Omega(n^{-2/3})$.*

Proof. As in the proof of Proposition 5.3.1, we divide the disk into tiles and obtain a lower bound for the number of tiles of S_j that stay in the halo for $j = 1, \dots, m$. Let Y_j and Y_j' be as defined in the proof of Proposition 5.3.1, Z_j be the index of the outermost tile of the slice S_j, and W_j be the index of the innermost tile of S_j that does not intersect the arc with radius 2 that passes through the points p_{j-1} and p_{j+2} (see Figure 5.4). Then a lower bound for the number of tiles of S_j that stay in the halo is $\min\{Y_j - 1, Y_j' - 1, Z_j - 1, W_j\}$. We note that W_j is not random, and a calculation similar to that of R_j in the proof of Proposition 5.3.1 gives that $W_j > 0$. We also note that $\mathbb{E}(\min\{Y_j, Y_j', Z_j\} - 1) \geq \mathbb{P}(Y_j > 1, Y_j' > 1, Z_j > 1) = (1 - 3/n)^n \geq e^{-4} = \Omega(1)$ for large enough n. So the expected number of tiles of S_j that stay in the halo is $\Omega(1)$. Taking the sum over $j = 1, \dots, m = n^{1/3}$, we see that the expected number of tiles that intersect the halo is $\Omega(n^{1/3})$ and multiplying by the area of each tile, which is $1/n$, we obtain the desired result.

□

§5.4 Discussion

In this section, we briefly discuss several extensions of the boundary complexity problem. One possible extension is where the radius of the random disks depends on the number of disks n. For instance the random disks have radius r_n with $\lim_{n\to\infty} r_n = 0$. In this case, the expected number of boundary disks is a function of n and r_n, and its behaviour depends on how fast r_n tends to 0. For example, when $r_n = O(n^{-2})$, the number of isolated disks, i.e., the disks that have no intersection with any other disk, is of order n, which tells us that the number of boundary disks is of order n as well. When $r_n = \Omega(n^{-2})$, however, the problem becomes more complicated and we will address it in future work. Another interesting regime is the case where $\lim_{n\to\infty} r_n = \infty$. For example, in this case Proposition 1.2.3 suggests that if r_n tends to infinity fast enough, then the boundary complexity is the same as the complexity of the convex hull. Also this will be the subject of future work.

Another possible extension is to replace S by an arbitrary convex polygon or convex compact region with smooth boundary. By following the proofs in [55] for the convex hull, the proofs for the unit square and the unit disk can be adapted to arbitrary polygons and regions with smooth boundary, to show that the order of the number of boundary disks is again $n^{1/3}$. The leading order coefficient can be different for different shapes and its computation requires a more detailed analysis.

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Samenvatting

Dit proefschrift bestaat uit twee delen, met als gemeenschappelijk thema 'Exploratie op en van Netwerken'. In Deel I onderzoeken we het thema 'exploratie op netwerken' door random wandelingen op dynamische random grafen te bestuderen. In Deel II onderzoeken we het thema 'exploratie van netwerken', door het probleem van complexiteit van random schijfgebieden te bestuderen.

Deel I: Random wandelingen op dynamische random grafen

Random grafen zijn wiskundige objecten die worden gebruikt voor het bestuderen van netwerken. Random wandelingen op random grafen kunnen worden gezien als een model voor exploratie op netwerken. Een bepaalde functie van een random wandeling, de mengtijd genoemd, kan worden beschouwd als een maat voor hoe snel het netwerk wordt geëxploreerd door een random wandeling. De mengtijden van random wandelingen op random grafen zijn de afgelopen decennia uitgebreid bestudeerd. Het grootste deel van het onderzoek rond dit onderwerp heeft zich echter geconcentreerd op statische grafen, terwijl de meeste netwerken juist in de tijd veranderen. In Deel I van dit proefschrift bestuderen we mengtijden van random wandelingen op random grafen wanneer de random graaf zelf ook in de tijd verandert. We concentreren ons op een bepaald type dynamica, de zogenaamde herbedradingsdynamiek, waarin de graden worden vastgelegd terwijl de verbindingen willekeurig worden gereorganiseerd.

In hoofdstuk 2 van het proefschrift introduceren we eerst het dynamische configuratiemodel en bestuderen we vervolgens de mengtijd van random wandelingen zonder backtracking. In het dynamische configuratiemodel wordt op elke tijdseenheid een vast deel van de lijnen uniform willekeurig gekozen en vervolgens willekeurig herbedraad. We laten zien, onder bepaalde regelmatigheidscondities op de gradenreeks, dat als de fractie van de opnieuw bedrade lijnen groot genoeg is, de mengtijd veel kleiner is dan de mengtijd op de statische graaf, en dat er geen zogenaamde cutoff is. Onze resultaten gelden met een kans die naar 1 convergeert als het aantal knopen naar oneindig gaat, en wel met betrekking tot een uniforme keuze van de initiële knoop en de initiële configuratie. In onze bewijzen laten we zien dat de mengtijd gerelateerd is aan een regeneratietijd, namelijk, de tijd dat de random wandeling langs een eerder bedrade lijn beweegt, waarbij we de boomachtige structuur van het configuratiemodel gebruiken.

In hoofdstuk 3 van het proefschrift breiden we de resultaten van hoofdstuk 2 uit naar complementaire regimes. In het bijzonder identificeren we drie verschillende regimes die superkritiek, kritiek en subkritiek worden genoemd. Het superkritieke

regime komt overeen met de resultaten van hoofdstuk 2. In het kritieke regime, waar de proportie van de opnieuw bedrade lijen niet al te groot en niet al te klein is, vinden we dat de mengtijd logaritmisch groeit in het aantal knopen, zoals voor de statische graaf, en er is eenzijdige cutoff, in tegenstelling tot wat we zien voor de mengtijd op de statische graaf. In het subkritieke regime, waar de proportie van de opnieuw bedrade lijnen klein is, zien we dat de mengtijd hetzelfde is als de mengtijd in de statische graaf: die groeit logaritmisch in het aantal knopen, en er is een tweezijdige cutoff. In onze bewijzen gebruiken we opnieuw het argument voor de regeneratietijd, maar dit keer gebruiken we impliciet een koppelingsargument, waarbij de random wandeling op het dynamische configuratiemodel wordt gekoppeld aan een gewijzigde random wandeling op de statische graaf. Het bestaan van deze koppeling hangt af van de boomachtige structuur van het configuratiemodel.

In hoofdstuk 4 van het proefschrift beschouwen we een meer algemene klasse van dynamische random graafmodellen, die het dynamische configuratiemodel als een speciaal geval omvat. Voortbouwend op de ideeën die in hoofdstuk 3 zijn ontwikkeld, laten we zien dat, onder bepaalde regulariteitscondities op de gradenrij en de graafdynamiek, de random wandeling zonder backtracking op de dynamische random graaf kan worden gekoppeld aan een gewijzigde random wandeling op de statische graaf. Met behulp van deze koppeling laten we een verband zien tussen de mengtijd van de random wandeling op de dynamische graaf en de mengtijd van de random wandeling op de statische graaf, en relateren we de mengtijd aan de hierboven beschreven regeneratietijd. Verder geven we een voorbeeld van een model binnen deze klasse van dynamische random graafmodellen, genaamd random wandeling met lokale herbedrading, waarbij het opnieuw bedraden plaatsvindt langs de random wandeling zelf. Met behulp van de bovenstaande link verkrijgen we dezelfde trichotomie als in hoofdstuk 3 voor de random wandeling met lokale herbedrading, maar op een andere tijdschaal.

Deel II: Complexiteit van random schijfgebieden

De complexiteit van random schijfgebieden hangt nauw samen met het probleem van conflictvrije kleuringen van schijfgebieden. Het concept van conflictvrije kleuringen is een generalisatie van het concept van graafkleuringen en ontstaat in de context van frequentie-toekenning in een netwerk van draadloze zenders. Elke draadloze zender wordt verondersteld een dekkingsgebied in de vorm van een schijf te hebben en wordt een zodanige frequentie toegekend dat een ontvanger niet wordt beïnvloed door interferentie van signalen van verschillende zenders. In dit kader kan het minimale aantal kleuren dat vereist is voor een conflictvrije kleuring van schijfgebieden worden beschouwd als een structurele parameter van het onderliggende draadloze netwerk. In Deel II van het proefschrift onderzoeken we de typische structuur van de netwerken die ontstaan als een resultaat van een random proces. Om dit te doen, voeren we een 'mean-case' analyse uit voor de complexiteit van random schijfgebieden met behulp van ideeën uit de geometrische waarschijnlijkheidstheorie.

In hoofdstuk 5 van het proefschrift bestuderen we de gemiddelde complexiteit van random schijfgebieden in twee verschillende situaties, een waarin de schijfcentra uni-

form random worden gekozen in een vierkant en een waarin de schijfcentra uniform worden gekozen in een schijf. We laten zien dat, onder bepaalde condities op de straal van de schijfgebieden, de gemiddelde complexiteit in beide gevallen van orde $n^{1/3}$ is, waarbij n het aantal schijfgebieden is. Onze bewijzen zijn geïnspireerd door de argumenten die worden gebruikt voor de gemiddelde complexiteit van convexe omhulsels van random punten. Meer specifiek laten we zien dat de gemiddelde complexiteit van de unie gerelateerd is aan het gemiddelde oppervlak van een zogenaamde 'halo' van de schijfcentra, wat analoog is aan het complement van het convexe omhulsel van random punten, en berekenen we het gemiddelde oppervlak van de halo via een discretiseringsargument. We laten ook een verband zien tussen de complexiteit van schijfgebieden en de complexiteit van convexe omhulsels van een reeks punten.

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Curriculum Vitae

Hakan Güldaş was born in Istanbul in 1988. He graduated from Izmir Fen Lisesi (Izmir Science High School) in 2006. Afterwards, he studied at Bilkent University, Ankara, graduating with a B.Sc. degree in Mathematics in 2011. The same year, he enrolled in the M.Sc. program in the Department of Computer Engineering, Boğaziçi University, Istanbul. During his master's studies he carried out research on low-dimensional matrix completion and parallelization of particle filters under the supervision of Prof. A. Taylan Cemgil. He defended his master's thesis, entitled 'Acceleration of sequential Monte Carlo methods via parallelization of resampling algorithms', in February 2015. In March 2015, he moved to The Netherlands, where he started his PhD research in Leiden University under the supervision of Prof. Frank den Hollander and Prof. Remco van der Hofstad. During his PhD, he carried out research in random graph theory, in collaboration with Prof. Frank den Hollander, Prof. Remco van der Hofstad and Dr. Luca Avena, and in geometric probability, in collaboration with Prof. Mark de Berg. On April 29, 2019, he started working as Data Scientist at ABN AMRO Bank, Amsterdam. He is married to Müzeyyen, since 2015, and together they have a son, Kayra, born in 2017.