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Travelling Patterns on Discrete Media

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Chapter 2

Bichromatic travelling waves

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Abstract. We discuss bichromatic (two-color) front solutions to the bistable Nagumo lattice differential equation. Such fronts connect the stable spatially homogeneous equilibria with spatially heterogeneous 2-periodic equilibria and hence are not monotonic like the standard monochromatic fronts. We provide explicit criteria that can determine whether or not these fronts are stationary and show that the bichromatic fronts can travel in parameter regimes where the monochromatic fronts are pinned. The presence of these bichromatic waves allows the two stable homogeneous equilibria to both spread out through the spatial domain towards each other, buffered by a shrinking intermediate zone in which the periodic pattern is visible.

2.1 Introduction

In this chapter we consider the Nagumo lattice differential equation (LDE)

$$\dot{u}_j(t) = d[u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)] + g(u_j(t); a), \quad (2.1.1)$$

posed on the spatial lattice $j \in \mathbb{Z}$, with $t \in \mathbb{R}$. We assume $d > 0$ and use the standard cubic bistable nonlinearity $g(u; a) = u(1 - u)(u - a)$ with $a \in (0, 1)$. This LDE is well-known as a prototypical model that describes the competition between two stable states $u = 0$ and $u = 1$ in a discrete spatial environment. A crucial role is reserved for so-called travelling front solutions, which have the form

$$u_j(t) = \Phi(j - ct), \quad \Phi(-\infty) = 0, \quad \Phi(+\infty) = 1. \quad (2.1.2)$$

Such solutions are often referred to as invasion waves, as they provide a mechanism by which the energetically preferred state can invade the spatial domain.

Our work focuses on the case where $c = 0$ holds for these primary invasion waves, indicating a delicate balance between the two competing states. In this case (2.1.1) can admit stable spatially periodic rest-states. Numerical results indicate that these states can act as a buffer between regions of space where $u = 0$ and $u = 1$ dominate the dynamics. This buffer shrinks as these two stable states appear to move towards each other. This latter process is governed by secondary two-component invasion waves that we analyze in detail in this chapter.

Nagumo PDE The LDE (2.1.1) can be seen as the nearest-neighbour discretization of the Nagumo reaction-diffusion PDE [99]

$$u_t = u_{xx} + g(u; a), \quad x \in \mathbb{R} \quad (2.1.3)$$

on a spatial grid with size $h = d^{-1/2}$. This PDE has been used as a highly simplified model for the spread of genetic traits [3] and the propagation of electrical signals through nerve fibres [9]. In higher space dimensions it also serves as a desingularization of the standard mean-curvature flow that is often used to describe the evolution of interfaces [45].

Fife and McLeod [51] used phase plane analysis to show that (2.1.3) admits a front solution for each $a \in [0, 1]$. Such solutions have the form

$$u(x, t) = \Phi(x - ct), \quad \Phi(-\infty) = 0, \quad \Phi(+\infty) = 1, \quad (2.1.4)$$

for some smooth waveprofile Φ and wavespeed c that has the same sign as $a - \frac{1}{2}$. These fronts hence connect the two stable spatially homogeneous equilibria $u(x, t) \equiv 0$ and $u(x, t) \equiv 1$.

Exploiting the comparison principle, Fife and McLeod were able to show that these front solutions have a surprisingly large basin of attraction. Indeed, any solution to (2.1.3) with an initial condition $u(x, 0) = u_0(x)$ that has $u_0(x) \approx 0$ for $x \ll -1$ and $u_0(x) \approx 1$ for $x \gg +1$ will converge to a shifted version of this front as $t \rightarrow \infty$.

These front solutions can be used as building blocks to capture the behaviour of a more general class of solutions to (2.1.3). Consider for example the two-parameter family of functions

$$u_{\text{plt}; \alpha_0, \alpha_1}(x, t) = \Phi(x - ct + \alpha_0) + \Phi(-x - ct + \alpha_1) - 1, \quad (2.1.5)$$

with $\alpha_1 \geq \alpha_0$. Each of these functions can be interpreted as a shifted version of the front solution (2.1.4) that is reflected in a vertical line to form a plateau.

If $c < 0$, then any initial configuration that has $u_0(x) \approx 0$ for $|x| \gg L$ and $u_0(x) \approx 1$ for $|x| \leq L$ will converge to a member of the family (2.1.5) as $t \rightarrow \infty$. This provides a mechanism by which compact regions where $u \sim 1$ can spread out to fill the entire domain.

On the other hand, when $c > 0$ one can construct entire solutions that converge to an element of (2.1.5) as $t \rightarrow -\infty$ and tend to zero as $t \rightarrow +\infty$. These solutions are stable under small perturbations [137]. In particular, they can be viewed as a robust elimination process whereby compact regions that have $u \sim 1$ are annihilated by two incoming travelling fronts that collide as $t \rightarrow \infty$.

Nagumo LDE For many physical phenomena such as crystal growth in materials [20], the formation of fractures in elastic bodies [119] and the motion of dislocations [25] and domain walls [37] through crystals, the discreteness and topology of the underlying spatial domain have a major impact on the dynamical behaviour. It is hence important to develop mathematical modelling tools that can incorporate such structures effectively. Indeed, by now it is well known that discrete models can capture dynamical behaviour that their continuous counterparts can not.

The LDE (2.1.1) has served as a prototype system in which such effects can be explored. It arises as a highly simplified model for the propagation of action potentials through nerve fibers that have regularly spaced gaps in their myeline coating [9]. Two-dimensional versions have been used to describe phase transitions in Ising models [7], to analyze predator-prey interactions [118] and to develop pattern recognition algorithms in image processing [31, 32]. Recently, an interest has also arisen in Nagumo equations posed on graphs [121], motivated by the network structure present in many biological systems [115].

Many authors have studied the LDE (2.1.1), focusing primarily on the richness of the set of equilibria [89] and the existence of travelling and standing front solutions [91, 139]. Such solutions have the form (2.1.2), which leads naturally to the waveprofile equation

$$-c\Phi'(\xi) = d[\Phi(\xi - 1) - 2\Phi(\xi) + \Phi(\xi + 1)] + g(\Phi(\xi); a). \quad (2.1.6)$$

Since the behaviour of every lattice point is governed by the same profile Φ , we refer to these front solutions as monochromatic waves in this chapter (in order to distinguish them from the bichromatic waves we discuss in the sequel). The seminal results by Mallet-Paret [91] show that for each $a \in [0, 1]$ and $d > 0$ there exists a unique $c = c_{\text{mc}}(a, d)$ for which such monochromatic (mc) solutions exist.

Pinning Upon fixing $a \in (0, 1) \setminus \{\frac{1}{2}\}$, Zinner [139] established that $c_{\text{mc}}(a, d) \neq 0$ for $d \gg 1$, while Keener [83] showed that $c_{\text{mc}}(a, d) = 0$ for $0 < d \ll 1$. Upon fixing $d > 0$, Mallet-Paret established [91] that $c_{\text{mc}}(a, d) \neq 0$ for $a \approx 1$ and $a \approx 0$. In addition, again for fixed $d > 0$, the results in [67, 92] strongly suggest that there exists $\delta > 0$ so that $c_{\text{mc}}(a, d) = 0$ whenever $|a - \frac{1}{2}| \leq \delta$; see Figure 2.1.

This last phenomenon is called pinning and distinguishes the LDE (2.1.1) from the PDE (2.1.3). It is a direct consequence of the fact that we have broken the translational invariance of space. Indeed, (2.1.6) becomes singular in the limit $c \rightarrow 0$ and the corresponding waveprofiles indeed typically lose their smoothness. Many results suggest that this phenomenon is generic for discrete systems [7, 21, 41, 43, 75]. However, by carefully tuning the nonlinearity g it is possible to design systems for which this pinning is absent [40, 70]. Understanding the pinning phenomenon is an important and challenging mathematical problem that also has practical ramifications.

Periodicity In this chapter we study waves that connect spatially homogeneous stationary solutions of (2.1.1) with spatially heterogeneous 2-periodic stationary solutions. It is well known that many physical systems exhibit spatially periodic features

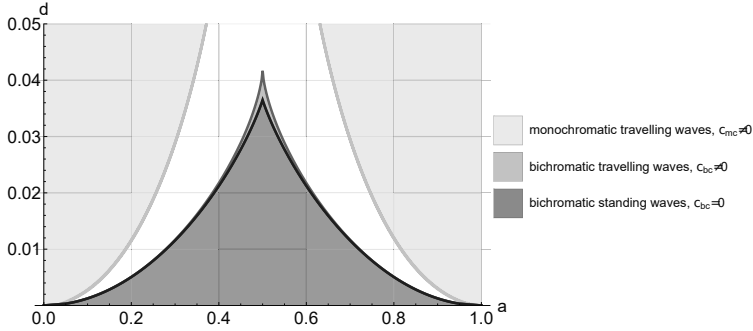


Figure 2.1: Existence regions for monochromatic and bichromatic wave solutions to (2.1.1).

[53, 57, 117]. Examples that also feature spatial discreteness include the presence of twinning microstructures in shape memory alloys [14] and the formation of domain-wall microstructures in dielectric crystals [123].

In many cases the underlying periodicity comes from the spatial system itself. For example, in [46, 47, 69] the authors consider chains of alternating masses connected by identical springs (and vice versa). The dynamical behaviour of such systems can be easily modelled by LDEs with periodic coefficients. In certain limiting cases the authors were able to construct so-called nanopterons, which are multi-component wave solutions that have low-amplitude oscillations in their tails.

However, periodic patterns also arise naturally as solutions to spatially homogeneous discrete systems. Indeed, we shall see in §2.3 that the LDE (2.1.1) with $d > 0$ admits many periodic equilibria. In addition, the results in [89] explore the periodicity and chaos present in the set of equilibria to homogeneous LDEs with simplified nonlinearities.

It is also possible to introduce a natural periodicity into the structure of (2.1.1) by taking $d < 0$. This can be seen by introducing new variables $v_j = (-1)^j u_j$, which restores the applicability of the comparison principle. This choice essentially decomposes the lattice sites $\mathbb{Z}$ into two groups $\mathbb{Z}_{\text{odd}}$ and $\mathbb{Z}_{\text{even}}$ that each have their own characteristic behaviour.

Such anti-diffusion models have been used to describe phase transitions for grids of particles that have visco-elastic interactions [22, 23, 128]. In [19] this problem has been analyzed in considerable detail. The authors show that the resulting two component system admits co-existing patterns that can be both monostable and bistable in nature. Similar results with piecewise linear nonlinearities but more general couplings between neighbours can be found in [129].

Bichromatic waves In this chapter we are interested in the parameter region where $c_{\text{mc}}(a, d) = 0$. In a subset of this region it is possible to show that (2.1.1) has spatially

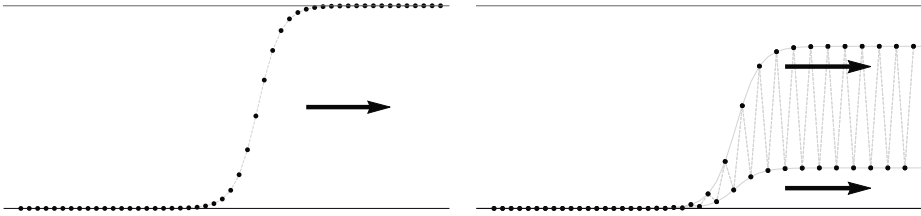


Figure 2.2: A monochromatic travelling wave of (2.1.1) (left panel) connects two spatially homogeneous stationary solutions. A bichromatic travelling wave of (2.1.1) (right panel) connects a spatially homogeneous stationary solution with a spatially heterogeneous one.

heterogeneous stable equilibria. We focus on the simplest case and consider so-called bichromatic (two-color) equilibria, which are spatially periodic with period two. As such, they are closely connected to solutions of the Nagumo equation posed on a graph with two vertices. We set out to construct bichromatic front-solutions to (2.1.1), which can be seen as waves that connect the spatially homogeneous equilibrium $u \equiv 0$ with such a 2-periodic state. We emphasize that these differ from the traditional front-solutions (2.1.2) in the sense that the odd and even lattice sites each have their own waveprofile, as illustrated in Figure 2.2. Consequently, the bichromatic front-solutions are not monotone.

Our first main contribution is contained in §2.3, where we give a detailed description of the set of parameters (a, d) where such 2-periodic equilibria exist and where they are stable. In contrast to the setting encountered in [19], the relevant bifurcation curves cannot all be described explicitly. Besides a global result stating that the number of such equilibria decreases as d is increased, we also obtain precise asymptotics that describe the boundaries near the three corners $(a, d) \in \{(0, 0), (1/2, 1/24), (1, 0)\}$ in Figure 2.1.

As in [19], these preparations allow the existence of bichromatic fronts to be established in a straightforward fashion. Indeed, one can apply the general theory developed by Chen, Guo and Wu in [27] for discrete periodic systems that admit a comparison-principle. These results imply that there exists a unique wavespeed $c_{bc}(a, d)$ for which such bichromatic (bc) fronts exist. If $c_{bc}(a, d) \neq 0$, the Fredholm theory developed in [74] together with the techniques from [73, §3] can be used to show that these travelling fronts depend smoothly on (a, d) and are nonlinearly stable.

However, these general results cannot distinguish between the cases $c_{bc}(a, d) = 0$ and $c_{bc}(a, d) \neq 0$ where we have standing respectively travelling fronts. This should be contrasted to the situation for the PDE (2.1.3), where the sign of the wavespeed is given by the sign of a simple integral [51]. Indeed, there is a large set of parameters (a, d) for which the discrete bichromatic fronts fail to travel, even though the analogous integral does not vanish.

Our second main contribution is that we provide explicit criteria in §2.4 that can guarantee $c_{bc} = 0$ or $c_{bc} > 0$. Together these results cover most of the parameter region where bichromatic fronts exist. In any case, they provide a two-component generalization of the coercivity conditions introduced in [91], which ensure $c_{mc}(a, d) \neq 0$ for the boundary regions $a \approx 1$ and $a \approx 0$.

Our arguments to guarantee $c_{bc} = 0$ are closely related to the setup used by Keener [83] to establish that monochromatic waves are pinned for $0 < d \ll 1$. In particular, for small values of d one can neglect the diffusion term in §2.1.1 and use properties of the cubic to show that the derivative of the waveprofile must change signs. This contradicts the fact that waveprofiles must be strictly monotonic if they travel.

On the other hand, in §2.4 we develop an intuitive geometric construction involving reflections to describe a planar recurrence relation that standing bichromatic fronts must satisfy. This allows us to rule out the presence of such fronts when a scalar inequality is violated. This consequently implies the presence of travelling bichromatic fronts.

The parameter regimes where these two arguments apply both converge towards the corner points $(0, 0)$ and $(1, 0)$. Near these corners we need the delicate asymptotics described above to distinguish between the two cases.

Colliding fronts One of the main reasons for our interest in these bichromatic fronts is that they present mechanisms via which the stable homogeneous states $u = 0$ and $u = 1$ can spread throughout the domain, even though the primary invasion waves are blocked from propagation. By using similar techniques as in [137], we believe it should be possible to construct entire solutions consisting of a right-travelling bichromatic front connection between the homogeneous equilibrium $u \equiv 0$ and a 2-periodic intermediate state that collides with a left-travelling bichromatic connection between 2-periodic intermediate state and the homogeneous equilibrium $u \equiv 1$; see Figure 2.3. The resulting state after the collision is then a pinned monochromatic front that connects 0 with 1. We have been able to numerically verify the existence of these solutions in the parameter regions predicted by the theory developed in this chapter.

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2.2 Main results

Our interest here is in the lattice differential equation

$$\dot{x}_j(t) = d[x_{j-1}(t) - 2x_j(t) + x_{j+1}(t)] + g(x_j(t); a) \quad (2.2.1)$$

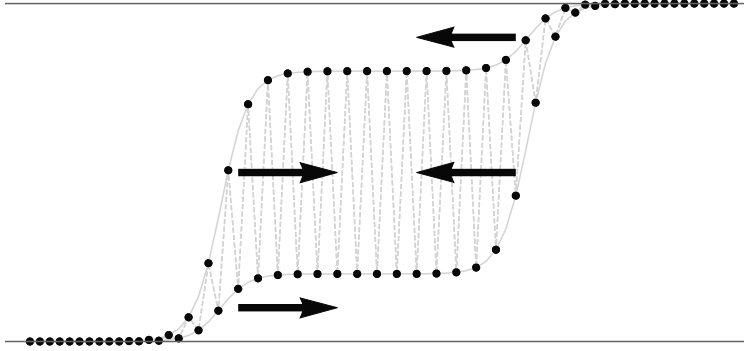


Figure 2.3: Colliding front of (2.1.1) consisting of a right-travelling bichromatic front connection between the homogeneous equilibrium $u \equiv 0$ and a 2-periodic intermediate state that collides with a left-travelling bichromatic connection between the 2-periodic state and the homogeneous equilibrium $u \equiv 1$.

posed on the one-dimensional lattice, i.e., $j \in \mathbb{Z}$. The bistable nonlinearity is explicitly given by

$$g(u; a) = u(1 - u)(u - a), \quad (2.2.2)$$

with $a \in (0, 1)$. Our results concern so-called bichromatic (two-colour) travelling wave solutions to the LDE (2.2.1). Such solutions can be written in the form

$$x_j(t) = \begin{cases} \Phi_u(j - ct) & \text{if } j \text{ is even,} \\ \Phi_v(j - ct) & \text{if } j \text{ is odd,} \end{cases} \quad (2.2.3)$$

for some wavespeed $c \in \mathbb{R}$ and $\mathbb{R}^2$ -valued waveprofile

$$\Phi = (\Phi_u, \Phi_v) : \mathbb{R} \rightarrow \mathbb{R}^2. \quad (2.2.4)$$

Substituting this Ansatz into (2.2.1) we obtain the travelling wave system

$$\begin{aligned} -c\Phi'_u(\xi) &= d[\Phi_v(\xi - 1) - 2\Phi_u(\xi) + \Phi_v(\xi + 1)] + g(\Phi_u(\xi); a), \\ -c\Phi'_v(\xi) &= d[\Phi_u(\xi - 1) - 2\Phi_v(\xi) + \Phi_u(\xi + 1)] + g(\Phi_v(\xi); a). \end{aligned} \quad (2.2.5)$$

Upon introducing the functions

$$G(u, v; a, d) = \begin{pmatrix} G_1(u, v; a, d) \\ G_2(u, v; a, d) \end{pmatrix} = \begin{pmatrix} 2d(v - u) + g(u; a) \\ 2d(u - v) + g(v; a) \end{pmatrix}, \quad (2.2.6)$$

we see that any stationary solution

$$(\Phi_u, \Phi_v)(\xi) = (\bar{u}, \bar{v}) \quad (2.2.7)$$

to (2.2.5) must satisfy the nonlinear algebraic equation

$$G(\bar{u}, \bar{v}; a, d) = 0. \quad (2.2.8)$$

The full bifurcation diagram for this equation is described in §2.3. For our purposes here however it suffices to summarize a subset of the conclusions from this analysis, which we do in our first result below. In particular, there exists a region Ω_{bc} in the (a, d) -plane for which the spatially homogeneous system $(\dot{u}, \dot{v}) = G(u, v; a, d)$ has a stable equilibrium $(\bar{u}_{bc}(a, d), \bar{v}_{bc}(a, d))$ that can be interpreted as a bichromatic equilibrium state for the LDE (2.2.1).

Proposition 2.2.1 (see §2.3). There exists a continuous curve $d_{bc} : [0, 1] \rightarrow [0, \frac{1}{24}]$ with $d_{bc}(\frac{1}{2}) = \frac{1}{24}$ and $d_{bc}(1-a) = d_{bc}(a)$ so that for every $0 \leq d < d_{bc}$ and $0 < a < 1$ the system (2.2.8) has nine distinct equilibria $(\bar{u}, \bar{v}) \in [0, 1]^2$. Upon writing

$$\Omega_{bc} = \{0 < d < d_{bc}(a) \text{ and } 0 < a < 1\}, \quad (2.2.9)$$

there exist C^∞ -smooth maps

$$(\bar{u}_{bc}, \bar{v}_{bc}) : \Omega_{bc} \rightarrow (0, 1)^2 \quad (2.2.10)$$

with $\bar{u}_{bc} < a < \bar{v}_{bc}$ so that for every $(a, d) \in \Omega_{bc}$ we have

$$G(\bar{u}_{bc}(a, d), \bar{v}_{bc}(a, d); a, d) = 0 \quad (2.2.11)$$

together with

$$\det D_{1,2}G(\bar{u}_{bc}(a, d), \bar{v}_{bc}(a, d); a, d) > 0, \quad \text{Tr } D_{1,2}G(\bar{u}_{bc}(a, d), \bar{v}_{bc}(a, d); a, d) < 0. \quad (2.2.12)$$

We note that the statements (2.2.11)-(2.2.12) are also valid upon replacing the bichromatic rest-state $(\bar{u}_{bc}, \bar{v}_{bc})$ by the monochromatic equilibria $(0, 0)$ and $(1, 1)$. We will be interested in waves that connect these mono- and bichromatic equilibria together. More precisely, we set out to find solutions to (2.2.5) that satisfy either the 'lower' boundary conditions

$$\lim_{\xi \rightarrow -\infty} \Phi(\xi) = (0, 0), \quad \lim_{\xi \rightarrow +\infty} \Phi(\xi) = (\bar{u}_{bc}, \bar{v}_{bc}), \quad (2.2.13)$$

or the 'upper' boundary conditions

$$\lim_{\xi \rightarrow -\infty} \Phi(\xi) = (\bar{u}_{bc}, \bar{v}_{bc}) \quad \lim_{\xi \rightarrow +\infty} \Phi(\xi) = (1, 1). \quad (2.2.14)$$

The result below summarizes several key facts concerning the existence and uniqueness of such waves. It introduces subregions of Ω_{bc} denoted by $\mathcal{T}_{low}$ and $\mathcal{T}_{up}$ where the bichromatic travelling waves (2.2.3) exist with nonzero speeds $c_{low} > 0$ and $c_{up} < 0$; see Figure 2.4. With the exception of the inequalities $c_{low} \geq 0$ and $c_{up} \leq 0$, these properties follow directly from the theory developed in [27, 74].

Theorem 2.2.2 (see §2.4.1). There exist continuous maps

$$c_{\text{low}} : \Omega_{\text{bc}} \rightarrow [0, \infty), \quad c_{\text{up}} : \Omega_{\text{bc}} \rightarrow (-\infty, 0] \quad (2.2.15)$$

that satisfy the following properties.

- (i) Upon introducing the open sets

$$\begin{aligned} \mathcal{T}_{\text{low}} &= \{(a, d) \in \Omega_{\text{bc}} : c_{\text{low}} > 0\}, \\ \mathcal{T}_{\text{up}} &= \{(a, d) \in \Omega_{\text{bc}} : c_{\text{up}} < 0\}, \end{aligned} \quad (2.2.16)$$

the functions c_{low} and c_{up} are C^∞ -smooth on $\mathcal{T}_{\text{low}}$ respectively $\mathcal{T}_{\text{up}}$.

- (ii) There exist C^∞ -smooth functions

$$\Phi_{\text{low}} : \mathcal{T}_{\text{low}} \rightarrow W^{1,\infty}(\mathbb{R}; \mathbb{R}^2), \quad \Phi_{\text{up}} : \mathcal{T}_{\text{up}} \rightarrow W^{1,\infty}(\mathbb{R}; \mathbb{R}^2), \quad (2.2.17)$$

such that for any $\# \in \{\text{low}, \text{up}\}$ and any $(a, d) \in \mathcal{T}_\#$, the pair

$$(c, \Phi) = (c_\#(a, d), \Phi_\#(a, d)) \quad (2.2.18)$$

satisfies (2.2.5) together with the boundary condition (2.2.13) if $\# = \text{low}$ or (2.2.14) if $\# = \text{up}$. In addition, we have the componentwise inequality $\Phi'_\# > (0, 0)$.

- (iii) For any $\# \in \{\text{low}, \text{up}\}$ and any $(a, d) \in \Omega_{\text{bc}} \setminus \mathcal{T}_\#$, there exists a non-decreasing function $\Phi : \mathbb{R} \rightarrow \mathbb{R}^2$ that satisfies (2.2.5) with $c = 0$ together with the boundary condition (2.2.13) if $\# = \text{low}$ or (2.2.14) if $\# = \text{up}$.
- (iv) Pick $\# \in \{\text{low}, \text{up}\}$ and $(a, d) \in \Omega_{\text{bc}}$ and consider any $c \neq 0$ together with a function $\Phi \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^2)$ that satisfies (2.2.5) together with the boundary condition (2.2.13) if $\# = \text{low}$ or (2.2.14) if $\# = \text{up}$. Then we must have $c = c_\#(a, d)$ and $\Phi = \Phi_\#(a, d)(\cdot - \vartheta)$ for some $\vartheta > 0$. In particular, we have $(a, d) \in \mathcal{T}_\#$.
- (v) Pick $\# \in \{\text{low}, \text{up}\}$ and $(a, d) \in \Omega_{\text{bc}}$ and consider any non-decreasing function $\Phi : \mathbb{R} \rightarrow \mathbb{R}^2$ that satisfies (2.2.5) with $c = 0$ together with the boundary condition (2.2.13) if $\# = \text{low}$ or (2.2.14) if $\# = \text{up}$. Then we must have $(a, d) \in \Omega_{\text{bc}} \setminus \mathcal{T}_\#$.

We numerically determined the locations of the sets $\mathcal{T}_{\text{low}}$ and $\mathcal{T}_{\text{up}}$ in Figure 2.4. In particular, we simulated (2.2.1) with an initial condition that consists of the stable periodic pattern multiplied by a hyperbolic tangent. By checking if this solution converges to a travelling or stationary pattern one can decide whether $(a, d) \in \mathcal{T}_{\text{low}}$.

This procedure hinges on the crucial fact that the waves in item (ii) of Theorem 2.2.2 are nonlinearly stable under the dynamics of (2.1.1); see [27, Thm. 4].

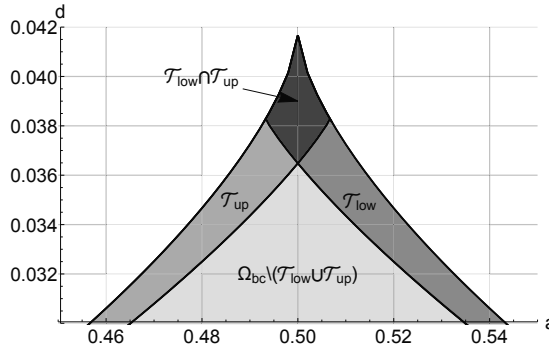


Figure 2.4: Numerical bounds for the parameter sets Ω_{bc} , $\mathcal{T}_{low}$ and $\mathcal{T}_{up}$ introduced in Theorem 2.2.2, in the neighbourhood of the cusp $(\frac{1}{2}, \frac{1}{24})$.

The main contribution of this chapter is that we provide an explicit and verifiable inequality to rigorously confirm that the sets $\mathcal{T}_{low}$ and $\mathcal{T}_{up}$ touch the boundary curve $d = d_{bc}(a)$. In order to formulate this inequality, we first extend the bichromatic equilibria (2.2.10) to this boundary by writing

$$(\bar{u}_{bc}, \bar{v}_{bc})(a, d_{bc}(a)) = \lim_{d \uparrow d_{bc}(a)} (\bar{u}_{bc}, \bar{v}_{bc})(a, d). \quad (2.2.19)$$

Inspection of (2.2.8) readily shows that this pair (2.2.19) is a solution to the fixed-point problem

$$(u, v) = (u_a(v), v_a(u)), \quad (2.2.20)$$

in which we have introduced the functions

$$u_a(v) = v - \frac{g(v; a)}{2d_{bc}(a)}, \quad v_a(u) = u - \frac{g(u; a)}{2d_{bc}(a)}. \quad (2.2.21)$$

The two critical points of these cubic functions are given by

$$\gamma_{\pm}(a) = \frac{1}{3} \left[a + 1 \pm \sqrt{a^2 - a + 1 - 6d_{bc}(a)} \right]. \quad (2.2.22)$$

Our approach relies on a geometric construction that rules out the existence of solutions to the recurrence relation that standing waves must satisfy. Indeed, in §2.4.2 we will recast (2.2.5) with $c = 0$ as a discrete map that involves horizontal and vertical reflections through the curves $u = u_a(v)$ and $v = v_a(u)$. Due to the monotonicity of the waveprofiles, several key features of the stable manifold for the pair (2.2.19) can therefore be captured by reflecting the lines $u = \bar{u}(a, d_{bc}(a))$ and $v = \bar{v}(a, d_{bc}(a))$ through these curves.

The resulting reflected functions can be written as

$$u_{a,r}(v) = 2u_a(v) - \bar{u}_{bc}(a, d_{bc}(a)), \quad v_{a,r}(u) = 2v_a(u) - \bar{v}_{bc}(a, d_{bc}(a)). \quad (2.2.23)$$

For any $a \in (0, 1)$, we now define the set of intersections

$$\mathcal{I}_a = \{(u, v) : u = u_a(v) \text{ and } v = v_{a,r}(u)\}, \quad (2.2.24)$$

which are all mapped onto the horizontal line $v = \bar{v}(a, d_{bc}(a))$. Assuming that this set is non-empty, we pick the specific representative

$$u_*(a) = \max\{u : \text{there exists } (u, v) \in \mathcal{I}_a \text{ with } 0 \leq u \leq \bar{u}_{bc}(a, d_{bc}(a)) \text{ and } 0 \leq v \leq \gamma_-(a)\}, \quad (2.2.25)$$

which allows us to define the scalar function

$$\Gamma(a) = u_{a,r}(\gamma_+(a)) - u_*(a). \quad (2.2.26)$$

The full geometric interpretation of this definition will be clarified and illustrated in §2.4.2. For now, we simply remark that it measures the difference between two critical points associated to the unstable manifold of $(0, 0)$ and the stable manifold of the pair (2.2.19). When $\Gamma(a) > 0$ holds, the critical points are ordered in such a way that connections between these two manifolds are blocked. This forces the inequality $c \neq 0$, implying that $\mathcal{T}_{low}$ indeed touches the boundary curve $d = d_{bc}(a)$.

We wish to emphasize that there is an essential difference between computing $\Gamma(a)$ and simulating the full LDE (2.2.1) to check whether $c \neq 0$. Indeed, the first problem only requires knowledge of the curves (2.2.21)-(2.2.23) and is hence two dimensional. On the other hand, the second problem is infinite dimensional and therefore necessarily involves truncations.

We are now ready to formulate our final result, which provides analytical bounds for the parameter regions $\mathcal{T}_{low}$ and $\mathcal{T}_{up}$ where the bichromatic waves actually travel (i.e., where $c_{low} > 0$ respectively $c_{up} < 0$). Both regions contain a neighbourhood of the cusp $(a, d) = (\frac{1}{2}, \frac{1}{24})$. In addition, the corners $(0, 0)$ and $(1, 0)$ are accumulation points for the sets $\mathcal{T}_{up}$ respectively $\mathcal{T}_{low}$.

Theorem 2.2.3 (see §2.4.3). The sets $\mathcal{T}_{low}$ and $\mathcal{T}_{up}$ satisfy the following properties.

- (i) For each $(a, d) \in \mathcal{T}_{up}$ we have $d > \frac{1}{8}a^2$, while for each $(a, d) \in \mathcal{T}_{low}$ we have $d > \frac{1}{8}(1-a)^2$.
- (ii) If $(a, d) \in \mathcal{T}_{low}$ then also $(a', d) \in \mathcal{T}_{low}$ for all $(a', d) \in \Omega_{bc}$ that have $a' \geq a$. On the other hand, if $(a, d) \in \mathcal{T}_{up}$ then also $(a', d) \in \mathcal{T}_{up}$ for all $(a', d) \in \Omega_{bc}$ that have $a' \leq a$.
- (iii) There exists $\epsilon > 0$ so that we have the inclusions

$$(a, d) \in \mathcal{T}_{low} \cap \mathcal{T}_{up} \quad (2.2.27)$$

for all $(a, d) \in \Omega_{bc}$ that have

$$0 < \left|a - \frac{1}{2}\right| + \left|d - \frac{1}{24}\right| < \epsilon. \quad (2.2.28)$$

- (iv) The expression (2.2.26) is well-defined for all $0 < a < 1$. If $\Gamma(a_*) > 0$ for some $0 < a_* < 1$, then there exists $\epsilon > 0$ so that $(a, d) \in \mathcal{T}_{\text{low}}$ for all $(a, d) \in \Omega_{\text{bc}}$ that have

$$0 < |a - a_*| + |d - d_{\text{bc}}(a_*)| < \epsilon. \quad (2.2.29)$$

- (v) The inequality $\Gamma(a) > 0$ holds whenever $1 - a > 0$ is sufficiently small. In particular, we have $(0, 0) \in \overline{\mathcal{T}}_{\text{up}}$ and $(1, 0) \in \overline{\mathcal{T}}_{\text{low}}$.

Using numerics we have verified that $\Gamma(a) > 0$ holds for $a \in [.498, .999]$; see §2.4.3. Together with (ii) and (v) above, this strongly suggests that $\mathcal{T}_{\text{low}}$ is a connected set that extends towards the right boundary of Ω_{bc} . By symmetry, this is equivalent to the statement that $\mathcal{T}_{\text{up}}$ is a connected set that extends towards the left boundary of Ω_{bc} .

2.3 Bichromatic stationary solutions

In this section we uncover the structure of the solution set to $G(u, v; a, d) = 0$ as a function of the parameters (a, d) . Our first result shows that for $d \gg 1$ this equation only has the monochromatic roots $(0, 0)$, (a, a) and $(1, 1)$. The threshold $d_+(a)$ between this region and the region with five distinct roots can be explicitly computed. However, we only have qualitative and asymptotic results for the boundary $d_-(a)$ where the root-count increases to the maximal value of nine. In Figure 2.5 we compare these asymptotics to numerically computed values for $d_-(a)$. We remark here that the monotonicity of the root count with respect to d does not hold for general bistable nonlinearities g .

Proposition 2.3.1 (see §2.3.5). There exist two continuous functions

$$d_{\pm} : [0, 1] \rightarrow [0, \infty) \quad (2.3.1)$$

that satisfy the following properties.

- (i) For any $0 < a < 1$ we have the explicit expression

$$d_+(a) = \frac{g'(a; a)}{4}, \quad (2.3.2)$$

together with the identities

$$d_-(a) = d_-(1 - a), \quad d_+(a) = d_+(1 - a) \quad (2.3.3)$$

and the inequality $d_-(a) < d_+(a)$. In addition, we have

$$d_-(0) = d_+(0) = d_-(1) = d_+(1) = 0 \quad (2.3.4)$$

together with $d_-(\frac{1}{2}) = \frac{1}{24}$.

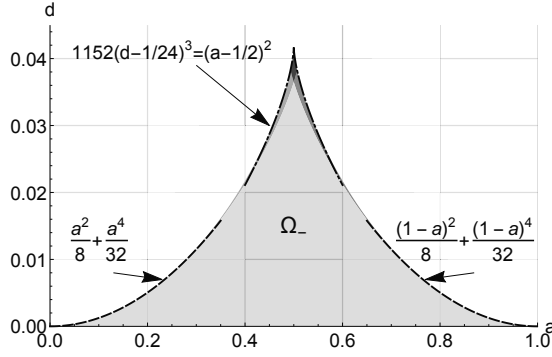


Figure 2.5: Comparison of the asymptotics for d_- described in Proposition 2.3.1 (v) with the numerically computed border of the set Ω_- .

- (ii) We have $d_- \in C^\infty([0, \frac{1}{2})) \cap C^\infty((\frac{1}{2}, 1])$. In addition, d_- is strictly increasing on $[0, \frac{1}{2}]$ and strictly decreasing on $[\frac{1}{2}, 1]$.
- (iii) Pick any $a \in (0, 1)$. The equation $G(u, v; a, d) = 0$ has nine distinct roots for $0 \leq d < d_-(a)$, five distinct roots for $d_- < d < d_+(a)$ and three distinct real roots for $d \geq d_+(a)$.
- (iv) Pick any $a \in (0, 1)$. The equation $G(u, v; a, d) = 0$ has seven distinct roots for $d = d_-(a)$ if $a \neq \frac{1}{2}$ and five if $a = \frac{1}{2}$.
- (v) We have the expansion $d_-(a) = \frac{1}{8}a^2 + \frac{1}{32}a^4 + O(a^5)$ for $a \downarrow 0$. In addition, writing $a_- : [0, \frac{1}{24}] \rightarrow [0, \frac{1}{2}]$ for the inverse function of d_- on $[0, \frac{1}{2}]$, we have the expansion

$$a_-(d) = \frac{1}{2} - \sqrt{-1152(d - \frac{1}{24})^3} + O((d - \frac{1}{24})^2) \quad (2.3.5)$$

as $d \uparrow \frac{1}{24}$.

In order to break the symmetry caused by the swap $u \leftrightarrow v$, we set out to describe the roots of $G(u, v; a, d) = 0$ that have $v > u$. To this end, we introduce two regions

$$\begin{aligned} \Omega_- &= \{(a, d) : 0 < a < 1 \text{ and } 0 < d < d_-(a)\}, \\ \Omega_+ &= \{(a, d) : 0 < a < 1 \text{ and } d_-(a) < d < d_+(a)\} \end{aligned} \quad (2.3.6)$$

that are studied separately in the two results below. In Ω_- there are three such bichromatic equilibria with $v > u$. These equilibria can be ordered and the middle one is the only stable one. Two (or three) of these equilibria collide at $d = d_-(a)$ in a saddle node (or pitchfork) bifurcation, leaving a single unstable bichromatic equilibrium in Ω_+ . This equilibrium in turns collides with its swapped counterpart and the monochromatic equilibrium (a, a) on the boundary $d_+(a)$. These processes

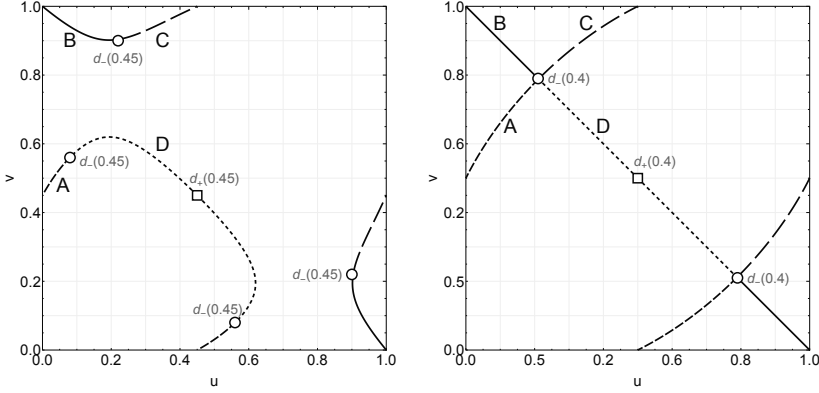


Figure 2.6: Illustration of the functions $(\bar{u}_A, \bar{v}_A)$, $(\bar{u}_B, \bar{v}_B)$, $(\bar{u}_C, \bar{v}_C)$ from Proposition 2.3.2 and the function $(\bar{u}_D, \bar{v}_D)$ from Proposition 2.3.3 for $a = .45$ (left panel) and $a = .5$ (right panel). The bifurcations occurring at $d_+(a)$ and $d_-(a)$ are indicated by squares and circles; see Proposition 2.3.1.

are illustrated in Figure 2.6. In particular, we see that $\bar{\Omega}_-$ coincides with the set Ω_{bc} introduced in §2.2; cf. Figures 2.4 and 2.5.

Proposition 2.3.2 (see §2.3.5). There exist continuous functions

$$(\bar{u}_A, \bar{v}_A) : \bar{\Omega}_- \rightarrow [0, 1]^2 \quad (\bar{u}_B, \bar{v}_B) : \bar{\Omega}_- \rightarrow [0, 1]^2, \quad (\bar{u}_C, \bar{v}_C) : \bar{\Omega}_- \rightarrow [0, 1]^2 \quad (2.3.7)$$

that satisfy the following properties.

- (i) Pick any $(a, d) \in \bar{\Omega}_-$. Then we have

$$G(\bar{u}_\#(a, d), \bar{v}_\#(a, d); a, d) = 0 \quad (2.3.8)$$

for all $\# \in \{A, B, C\}$. If also $(a, d) \in \Omega_-$, then the matrix

$$D_{1,2}G(\bar{u}_\#(a, d), \bar{v}_\#(a, d); a, d) \quad (2.3.9)$$

has two strictly negative eigenvalues if $\# = B$ or one strictly positive and one strictly negative eigenvalue if $\# \in \{A, C\}$.

- (ii) For any $0 \leq a \leq 1$ we have the identities

$$(\bar{u}_A, \bar{v}_A)(a, 0) = (0, a), \quad (\bar{u}_B, \bar{v}_B)(a, 0) = (0, 1), \quad (\bar{u}_C, \bar{v}_C)(a, 0) = (a, 1). \quad (2.3.10)$$

- (iii) For any $(a, d) \in \Omega_-$ we have the ordering

$$0 < \bar{u}_A(a, d) < \bar{u}_B(a, d) < \bar{u}_C(a, d) < a < \bar{v}_A(a, d) < \bar{v}_B(a, d) < \bar{v}_C(a, d). \quad (2.3.11)$$

(iv) For any $a \in [0, \frac{1}{2}]$ we have

$$(\bar{u}_B, \bar{v}_B)(a, d_-(a)) = (\bar{u}_C, \bar{v}_C)(a, d_-(a)), \quad (2.3.12)$$

while for any $a \in [\frac{1}{2}, 1]$ we have

$$(\bar{u}_A, \bar{v}_A)(a, d_-(a)) = (\bar{u}_B, \bar{v}_B)(a, d_-(a)). \quad (2.3.13)$$

Proposition 2.3.3 (see §2.3.5). There exist continuous functions

$$(\bar{u}_D, \bar{v}_D) : \bar{\Omega}_+ \rightarrow [0, 1]^2 \quad (2.3.14)$$

that satisfy the following properties.

(i) Pick any $(a, d) \in \bar{\Omega}_+$. Then we have

$$G(\bar{u}_D(a, d), \bar{v}_D(a, d); a, d) = 0. \quad (2.3.15)$$

If also $(a, d) \in \Omega_+$, then the matrix

$$D_{1,2}G(\bar{u}_D(a, d), \bar{v}_D(a, d); a, d) \quad (2.3.16)$$

has one strictly positive and one strictly negative eigenvalue.

(ii) For any $0 \leq a \leq 1$ we have the identity

$$(\bar{u}_D, \bar{v}_D)(a, d_+(a)) = (a, a). \quad (2.3.17)$$

(iii) For any $(a, d) \in \Omega_+$ we have the ordering

$$0 < \bar{u}_D(a, d) < a < \bar{v}_D(a, d) < 1. \quad (2.3.18)$$

(iv) For any $a \in [0, \frac{1}{2}]$ we have the identity

$$(\bar{u}_D, \bar{v}_D)(a, d_-(a)) = (\bar{u}_A, \bar{v}_A)(a, d_-(a)), \quad (2.3.19)$$

while for any $a \in [\frac{1}{2}, 1]$ we have

$$(\bar{u}_D, \bar{v}_D)(a, d_-(a)) = (\bar{u}_C, \bar{v}_C)(a, d_-(a)). \quad (2.3.20)$$

Corollary 2.3.4. For any $(a, d) \in \bar{\Omega}_-$, we have the identities

$$\begin{aligned} (\bar{u}_A, \bar{v}_A)(a, d) &= (1 - \bar{v}_C, 1 - \bar{u}_C)(1 - a, d), \\ (\bar{u}_B, \bar{v}_B)(a, d) &= (1 - \bar{v}_B, 1 - \bar{u}_B)(1 - a, d), \\ (\bar{u}_C, \bar{v}_C)(a, d) &= (1 - \bar{v}_A, 1 - \bar{u}_A)(1 - a, d). \end{aligned} \quad (2.3.21)$$

In addition, for any $(a, d) \in \bar{\Omega}_+$ we have the identity

$$(\bar{u}_D, \bar{v}_D)(a, d) = (1 - \bar{v}_D, 1 - \bar{u}_D)(1 - a, d). \quad (2.3.22)$$

Proof. The symmetry $g(1-u, 1-a) = -g(u, a)$ implies that

$$G(1-u, 1-v; 1-a, d) = -G(u, v; a, d). \quad (2.3.23)$$

In addition, we have $G(u, v; a, d) = 0$ if and only if $G(v, u; a, d) = 0$. The statements hence follow from the ordering (2.3.11). $\square$

Our final result concerns the special case $a = \frac{1}{2}$, in which case it is possible to be more explicit. In particular, the bichromatic roots $(\bar{u}_B, \bar{v}_B)$ and $(\bar{u}_D, \bar{v}_D)$ lie on the line $u + v = 1$ and collide precisely when $g'(u; \frac{1}{2}) = g'(v; \frac{1}{2}) = 0$.

Corollary 2.3.5. For any $0 \leq d \leq \frac{1}{24}$ we have

$$\bar{u}_B(\frac{1}{2}, d) = 1 - \bar{v}_B(\frac{1}{2}, d), \quad (2.3.24)$$

while for any $\frac{1}{24} \leq d \leq \frac{1}{16} = d_+(1/2)$ we have

$$\bar{u}_D(\frac{1}{2}, d) = 1 - \bar{v}_D(\frac{1}{2}, d). \quad (2.3.25)$$

In addition, we have the identities

$$\begin{aligned} \bar{u}_A(\frac{1}{2}, \frac{1}{24}) &= \bar{u}_B(\frac{1}{2}, \frac{1}{24}) = \bar{u}_C(\frac{1}{2}, \frac{1}{24}) = \bar{u}_D(\frac{1}{2}, \frac{1}{24}) &= \frac{1}{2} - \frac{1}{6}\sqrt{3}, \\ \bar{v}_A(\frac{1}{2}, \frac{1}{24}) &= \bar{v}_B(\frac{1}{2}, \frac{1}{24}) = \bar{v}_C(\frac{1}{2}, \frac{1}{24}) = \bar{v}_D(\frac{1}{2}, \frac{1}{24}) &= \frac{1}{2} + \frac{1}{6}\sqrt{3}. \end{aligned} \quad (2.3.26)$$

2.3.1 Saddle-nodes around $a = 0$

In this section we construct the branches $(\bar{u}_B, \bar{v}_B)$ and $(\bar{u}_C, \bar{v}_C)$ of solutions to $G(u, v; a, d) = 0$ in the regime where $(a, d) \approx (0, 0)$. In particular, we define

$$H(u, v; a, d) = G(u, 1+v; a, d) \quad (2.3.27)$$

and determine the zeroes of H for which (u, v, a, d) are small.

Proposition 2.3.6. There exist constants $\delta_a > 0$, $\delta_d > 0$ and $\epsilon > 0$ together with a function $d_c : (0, \delta_a) \rightarrow (0, \delta_d)$ and a constant $K \geq 1$ so that the following holds true.

- (i) For every $0 < a < \delta_a$ and $0 < d < d_c(a)$ the equation $H(u, v; a, d) = 0$ has precisely two solutions on the set $\{|u| + |v| < \epsilon\}$.
- (ii) For every $0 < a < \delta_a$ and $d_c(a) < d < \delta_d$ the equation $H(u, v; a, d) = 0$ has no solutions on the set $\{|u| + |v| < \epsilon\}$.
- (iii) For every $0 < a < \delta_a$ we have the estimate

$$\left| d_c(a) - \frac{1}{8}a^2 - \frac{1}{32}a^4 \right| \leq Ka^5. \quad (2.3.28)$$

- (iv) For every $0 < a < \delta_a$ the equation $H(u, v; d_c(a), a) = 0$ has precisely one solution $(u_c(a), v_c(a))$ on the set $\{|u| + |v| < \epsilon\}$. We have the estimates

$$\begin{aligned} |u_c(a) - \tfrac{1}{2}a| &\leq Ka^4, \\ |v_c(a) + \tfrac{1}{4}a^2 + \tfrac{1}{8}a^3| &\leq Ka^4. \end{aligned} \quad (2.3.29)$$

Writing $H(u, v; a, d) = (H_1(u, v; a, d), H_2(u, v; a, d))^T$ we can compute

$$\begin{aligned} H_1(u, v; a, d) &= u^2 - ua - u^3 + u^2a + 2d + 2dv - 2du, \\ H_2(u, v; a, d) &= -v - 2v^2 + va - v^3 + v^2a + 2du - 2d - 2dv. \end{aligned} \quad (2.3.30)$$

Our strategy is to use the identity $H_2 = 0$ to eliminate v and then recast $H_1 = 0$ into the normal form of a saddle-node bifurcation.

Lemma 2.3.7. Pick $\delta > 0$ sufficiently small. Then there exist constants $\epsilon > 0$ and $K \geq 1$ together with functions

$$\alpha_0 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \alpha_1 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad R_{\alpha;2} : (-\delta, \delta)^3 \rightarrow \mathbb{R} \quad (2.3.31)$$

that satisfy the following properties.

- (i) For every $(u, a, d) \in (-\delta, \delta)^3$ the equation $H_2(u, v; a, d) = 0$ has a unique solution $v = v_*$ in the set $\{|v| < \epsilon\}$. This solution is given by

$$v_*(u; a, d) = \alpha_0(a, d) + \alpha_1(a, d)u + u^2 R_{\alpha;2}(u; a, d). \quad (2.3.32)$$

- (ii) Upon writing

$$\begin{aligned} \alpha_0(a, d) &= -2d - 2ad + S_{\alpha_0}(a, d), \\ \alpha_1(a, d) &= 2d + S_{\alpha_1}(a, d), \end{aligned} \quad (2.3.33)$$

the bounds

$$\begin{aligned} |S_{\alpha_0}(a, d)| &\leq K(d^2 + |d|a^2), \\ |S_{\alpha_1}(a, d)| &\leq K|d|(|a| + |d|), \end{aligned} \quad (2.3.34)$$

together with

$$\begin{aligned} |\partial_d S_{\alpha_0}(a, d)| &\leq K(|d| + a^2), \\ |\partial_d S_{\alpha_1}(a, d)| &\leq K(|a| + |d|) \end{aligned} \quad (2.3.35)$$

hold for all $(a, d) \in (-\delta, \delta)^2$.

- (iii) For every $(u, a, d) \in (-\delta, \delta)^3$ we have the bounds

$$|R_{\alpha;2}(u; a, d)| + |R'_{\alpha;2}(u; a, d)| + |R''_{\alpha;2}(u; a, d)| + |R'''_{\alpha;2}(u; a, d)| \leq Kd^2, \quad (2.3.36)$$

together with

$$\begin{aligned} |\partial_d R_{\alpha;2}(u; a, d)| + |\partial_d R'_{\alpha;2}(u; a, d)| \\ + |\partial_d R''_{\alpha;2}(u; a, d)| + |\partial_d R'''_{\alpha;2}(u; a, d)| \leq K|d|. \end{aligned} \quad (2.3.37)$$

Proof. Substituting the Ansatz (2.3.32) into H_2 , we obtain the fixed point problems

$$\begin{aligned}\alpha_0 &= -2d - 2d\alpha_0 + \alpha_0 a + \alpha_0^2 a - 2\alpha_0^2 - \alpha_0^3, \\ \alpha_1 &= 2d - 4\alpha_0\alpha_1 + 2\alpha_0\alpha_1 a - 2d\alpha_1 + \alpha_1 a - 3\alpha_0^2\alpha_1,\end{aligned}\tag{2.3.38}$$

together with

$$\begin{aligned}R_2 &= 2\alpha_0 R_2 a + R_2 a + \alpha_1^2 a - 3\alpha_0\alpha_1^2 - 2dR_2 - 2\alpha_1^2 - 3\alpha_0^2 R_2 - 4\alpha_0 R_2 \\ &\quad + (2\alpha_1 R_2 a - 6\alpha_0\alpha_1 R_2 - 4\alpha_1 R_2 - \alpha_1^3)u \\ &\quad + (R_2^2 a - 3\alpha_0 R_2^2 - 2R_2^2 - 3\alpha_1^2 R_2)u^2 \\ &\quad - 3\alpha_1 R_2^2 u^3 - R_2^3 u^4.\end{aligned}\tag{2.3.39}$$

These fixed point problems can be successively solved for small a , d and u , along with their differentiated counterparts. The desired estimates can subsequently be obtained in a standard fashion by computing Taylor expansions. $\square$

In order to eliminate v from H_1 , we define the function

$$\begin{aligned}\mathcal{J}(u; a, d) &= H_1(u, v_*(u; a, d); a, d) \\ &= \beta_0(a, d) + \beta_1(a, d)u + u^2(1 + a - u + R_{\beta;2}(u; a, d)),\end{aligned}\tag{2.3.40}$$

which forces us to write

$$\begin{aligned}\beta_0(a, d) &= 2d + 2d\alpha_0(a, d), \\ \beta_1(a, d) &= -a - 2d + 2d\alpha_1(a, d), \\ R_{\beta;2}(u; a, d) &= 2dR_{\alpha;2}(u; a, d).\end{aligned}\tag{2.3.41}$$

By applying a shift to u the linear term in (2.3.40) can be removed, transforming (2.3.40) into a normal form for saddle node bifurcations.

Lemma 2.3.8. Pick $\delta > 0$ sufficiently small. Then there exist constants $\epsilon > 0$ and $K \geq 1$ together with functions

$$u_* : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \zeta_0 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad R_{\zeta;2} : (-\delta, \delta)^3 \rightarrow \mathbb{R}\tag{2.3.42}$$

that satisfy the following properties.

- (i) For every $(\tilde{u}, a, d) \in (-\delta, \delta)^3$, we have the identity

$$\mathcal{J}(u_*(a, d) + \tilde{u}; a, d) = \zeta_0(a, d) + \tilde{u}^2[1 + R_{\zeta;2}(\tilde{u}; a, d)].\tag{2.3.43}$$

- (ii) Upon writing

$$\begin{aligned}u_*(a, d) &= \frac{1}{2}a + d - \frac{1}{8}a^2 + \frac{1}{2}ad - \frac{1}{16}a^3 + S_{u_*}(a, d), \\ \zeta_0(a, d) &= 2d - \frac{1}{4}a^2 - da - 5d^2 + \frac{1}{8}a^3 + \frac{1}{4}da^2 - \frac{1}{64}a^4 + S_{\zeta_0}(a, d),\end{aligned}\tag{2.3.44}$$

the bounds

$$\begin{aligned} |S_{u_*}(a, d)| &\leq K[d^2 + a^2|d| + a^4], \\ |S_{\zeta_0}(a, d)| &\leq K(a^5 + |d|^3 + d^2|a| + |a|^3|d|), \end{aligned} \quad (2.3.45)$$

together with

$$\begin{aligned} |\partial_d S_{u_*}(a, d)| &\leq K[|d| + a^2], \\ |\partial_d S_{\zeta_0}(a, d)| &\leq K(d^2 + |d||a| + |a|^3) \end{aligned} \quad (2.3.46)$$

hold for all $(a, d) \in (-\delta, \delta)^2$.

(iii) For every $(\tilde{u}, a, d) \in (-\delta, \delta)^3$ we have the bounds

$$\begin{aligned} |R_{\zeta;2}(\tilde{u}; a, d)| &\leq K(|a| + |d|), \\ |R'_{\zeta;2}(\tilde{u}; a, d)| &\leq K|d|^3. \end{aligned} \quad (2.3.47)$$

Proof. We first introduce the notation

$$\mathcal{N}_{\beta;2;u_*}(\tilde{u}; a, d) = \tilde{u}^{-2} \left[R_{\beta;2}(u_* + \tilde{u}) - R_{\beta;2}(u_*; a, d) - R'_{\beta;2}(u_*; a, d)\tilde{u} \right] \quad (2.3.48)$$

for $\tilde{u} \neq 0$, together with $\mathcal{N}_{\beta;2;u_*}(0; a, d) = \frac{1}{2}R''_{\beta;2}(u_*; a, d)$. This allows us to compute

$$\mathcal{J}(u_* + \tilde{u}, a, d) = \gamma_0(a, d, u_*) + \gamma_1(a, d, u_*)\tilde{u} + (1 + a - 3u_* + R_{\gamma;2}(\tilde{u}; a, d, u_*))\tilde{u}^2, \quad (2.3.49)$$

in which

$$\begin{aligned} \gamma_0(a, d, u_*) &= \beta_0 + \beta_1 u_* + u_*^2[1 + a - u_* + R_{\beta;2}(u_*)], \\ \gamma_1(a, d, u_*) &= \beta_1 + u_*^2[-1 + R'_{\beta;2}(u_*; a, d)] + 2u_*[1 + a - u_* + R_{\beta;2}(u_*)], \\ R_{\gamma;2}(\tilde{u}; a, d, u_*) &= (u_* + \tilde{u})^2 \mathcal{N}_{\beta;2;u_*}(\tilde{u}; a, d) + R'_{\beta;2}(u_*; a, d)(2u_*\tilde{u} + \tilde{u}^2) \\ &\quad + R_{\beta;2}(u_*; a, d). \end{aligned} \quad (2.3.50)$$

On account of (2.3.36) we have¹

$$|\mathcal{N}_{\beta;2;u_*}(\tilde{u}; a, d)| + |\mathcal{N}'_{\beta;2;u_*}(\tilde{u}; a, d)| \leq C'_1 \quad (2.3.51)$$

and hence also

$$|\mathcal{R}_{\gamma;2}(\tilde{u}; a, d, u_*)| + |\mathcal{R}'_{\gamma;2}(\tilde{u}; a, d, u_*)| \leq C'_2. \quad (2.3.52)$$

Setting $\gamma_1 = 0$ leads to the fixed point problem

$$u_* = -\frac{1}{2}\beta_1 - au_* + \frac{3}{2}u_*^2 - \frac{1}{2}u_*^2 R'_{\beta;2}(u_*; a, d) - u_* R_{\beta;2}(u_*; a, d), \quad (2.3.53)$$

¹All primed constants in this section are strictly positive and do not depend on the variables appearing on the left hand side of the inequalities where they appear.

which has a unique small solution that we write for the moment as

$$u_*(a, d, \beta_1) = -\frac{1}{2}\beta_1 + \frac{1}{2}a\beta_1 + \frac{3}{8}\beta_1^2 - \frac{1}{2}a^2\beta_1 - \frac{9}{8}a\beta_1^2 - \frac{9}{16}\beta_1^3 + \tilde{S}_{u_*}(a, d, \beta_1). \quad (2.3.54)$$

In a standard fashion one obtains the bound

$$\left| \tilde{S}_{u_*}(a, d, \beta_1) \right| \leq C'_3 [\beta_1^4 + |a| |\beta_1|^3 + (a^2 + d^2)\beta_1^2 + (|a|^3 + |d|^3) |\beta_1|]. \quad (2.3.55)$$

Differentiating (2.3.53) we obtain

$$\begin{aligned} D_d u_* &= -\frac{1}{2}\partial_d \beta_1 - \frac{1}{2}u_*^2 \partial_d R'_{\beta;2}(u_*; a, d) - u_* \partial_d R_{\beta;2}(u_*; a, d) \\ &\quad - a D_d u_* + 3u_* D_d u_* - u_* R'_{\beta;2}(u_*; a, d) D_d u_* - R_{\beta;2}(u_*; a, d) D_d u_*, \end{aligned} \quad (2.3.56)$$

which yields the estimate

$$\left| D_d u_* - \left[-\frac{1}{2} + \frac{1}{2}a + \frac{3}{4}\beta_1 \right] \partial_d \beta_1 \right| \leq C'_4 [d^2 |\beta_1| + a^2 + |a| |\beta_1| + \beta_1^2] |\partial_d \beta_1|. \quad (2.3.57)$$

Using the bounds

$$\begin{aligned} |\beta_0(a, d) - [2d - 4d^2]| &\leq C'_5 d^2 (|a| + |d|), \\ |\beta_1(a, d) + (a + 2d)| &\leq C'_5 d^2, \end{aligned} \quad (2.3.58)$$

together with

$$\begin{aligned} |\partial_d \beta_0(a, d) - [2 - 8d]| &\leq C'_5 |d| (|a| + |d|), \\ |\partial_d \beta_1(a, d) + 2| &\leq C'_5 |d| \end{aligned} \quad (2.3.59)$$

and finally

$$|R_{\beta;2}(u_*; a, d)| + |R'_{\beta;2}(u_*; a, d)| + |R''_{\beta;2}(u_*; a, d)| + |R'''_{\beta;2}(u_*; a, d)| \leq C'_6 |d|^3, \quad (2.3.60)$$

the desired estimates follow by writing

$$\begin{aligned} \zeta_0(a, d) &= \gamma_0(a, d, u_*(a, d)), \\ R_{\zeta;2}(\tilde{u}; a, d) &= a - 3u_*(a, d) + R_{\gamma;2}(\tilde{u}, a, d, u_*(a, d)) \end{aligned} \quad (2.3.61)$$

and computing $\partial_d \zeta_0$ directly. $\square$

Proof of Proposition 2.3.6. We note first that the map

$$\tilde{u} \mapsto \tilde{u} \sqrt{1 + R_{\zeta;2}(\tilde{u}; a, d)} \quad (2.3.62)$$

is invertible for $(\tilde{u}, a, d) \in (-\delta, \delta)^3$. In order to find d_c it hence suffices to solve $\zeta_0(a, d_c) = 0$, which gives the fixed-point problem

$$d_c = \frac{1}{8}a^2 - \frac{1}{16}a^3 + \frac{1}{128}a^4 + \frac{1}{2}ad_c - \frac{1}{8}d_c a^2 + \frac{5}{2}d_c^2 - \frac{1}{2}S_{\zeta_0}(a, d_c). \quad (2.3.63)$$

Our estimate on $\partial_d S_{\zeta_0}$ guarantees the existence of a unique small solution for small a . The estimate (iii) now follows in a standard fashion. Writing $u_c = u_*(a, d_c(a))$ and $v_c = v_*(u_*(a, d_c(a)); a, d_c(a))$ the expansions in (iv) follow directly by substitution. $\square$

2.3.2 The cusp bifurcation around $a = 1/2$

Our goal here is to unfold the structure of the solution-set to $G(u, v; a, d)$ near the critical point $(a, d) = (1/2, 1/24)$ where the branches $(\bar{u}_\#, \bar{v}_\#)$ with $\# \in \{A, B, C, D\}$ all collide in a cusp bifurcation; see Figures 2.1 and 2.6. In particular, we introduce the cusp location

$$\begin{aligned} (u_{\text{cp}}, v_{\text{cp}}, a_{\text{cp}}, d_{\text{cp}}) &= (u_{\min}(\tfrac{1}{2}), u_{\max}(\tfrac{1}{2}), \tfrac{1}{12}, \tfrac{1}{24}) \\ &= (\tfrac{1}{2} - \tfrac{1}{6}\sqrt{3}, \tfrac{1}{2} + \tfrac{1}{6}\sqrt{3}, \tfrac{1}{2}, \tfrac{1}{24}) \end{aligned} \quad (2.3.64)$$

together with the function

$$H_{\text{cp}}(u, v; a, d) = G(u_{\text{cp}} + u, v_{\text{cp}} + v; a_{\text{cp}} + a, d_{\text{cp}} + d) \quad (2.3.65)$$

and set out to determine the zeroes of H_{cp} for which (u, v, a, d) are small.

Proposition 2.3.9. There exist constants $\delta_a > 0$, $\delta_d > 0$ and $\epsilon > 0$ together with a function $a_c : (-\delta_d, 0) \rightarrow (0, \delta_a)$ and a constant $K \geq 1$ so that the following holds true.

- (i) For every $0 \leq d < \delta_d$ and any $a \in (-\delta_a, \delta_a)$, the equation $H_{\text{cp}}(u, v; a, d) = 0$ has precisely one solution on the set $\{|u| + |v| < \epsilon\}$.
- (ii) For every $-\delta_d < d < 0$ and $a \in (-\delta_a, \delta_a)$ with $|a| > a_c(d)$, the equation $H_{\text{cp}}(u, v; a, d) = 0$ has precisely one solution on the set $\{|u| + |v| < \epsilon\}$.
- (iii) For every $-\delta_d < d < 0$ and $a \in (-a_c(d), a_c(d))$, the equation $H_{\text{cp}}(u, v; a, d) = 0$ has precisely three solutions on the set $\{|u| + |v| < \epsilon\}$.
- (iv) For any $-\delta_d < d < 0$, the equation $H_{\text{cp}}(u, v; a_c(d), d) = 0$ has precisely two solutions on the set $\{|u| + |v| < \epsilon\}$.
- (v) For any $d \in (-\delta_d, 0)$ we have the estimate

$$\left| a_c(d) - \sqrt{-1152d^3} \right| \leq Kd^2. \quad (2.3.66)$$

In order to recast our equation into an efficient form, we introduce two functions (h_1, h_2) by writing

$$\begin{pmatrix} h_1(p, q; a, d) \\ h_2(p, q; a, d) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} H_{\text{cp}}(p + q, p - q; a, d), \quad (2.3.67)$$

which can be evaluated as

$$\begin{aligned} h_1(p, q; a, d) &= -\frac{1}{3}a + 2ap^2 - 6pq^2 + 2aq^2 - \frac{2}{3}\sqrt{3}aq - 2p^3 + 2\sqrt{3}pq, \\ h_2(p, q; a, d) &= -\frac{2}{3}\sqrt{3}ap - \frac{1}{3}q + \frac{4}{3}\sqrt{3}d + \sqrt{3}p^2 + \sqrt{3}q^2 - 6p^2q - 8dq + 4apq - 2q^3. \end{aligned} \quad (2.3.68)$$

Since h_2 contains a term that is linear in q , we set out to eliminate this variable by demanding $h_2 = 0$.

Lemma 2.3.10. Pick $\delta > 0$ sufficiently small. Then there exist constants $\epsilon > 0$ and $K \geq 1$, together with functions

$$\alpha_0 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \alpha_1 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \alpha_2 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad R_{\alpha;3} : (-\delta, \delta)^3 \rightarrow \mathbb{R} \quad (2.3.69)$$

that satisfy the following properties.

- (i) For every $(p, a, d) \in (-\delta, \delta)^3$ the equation $h_2(p, q; a, d) = 0$ has a unique solution $q = q_*$ in the set $\{|q| < \epsilon\}$. This solution is given by

$$q_*(p; a, d) = \alpha_0(a, d) + \alpha_1(a, d)p + \alpha_2(a, d)p^2 + p^3 R_{\alpha;3}(p; a, d). \quad (2.3.70)$$

- (ii) Upon writing

$$\begin{aligned} \alpha_0(a, d) &= 4\sqrt{3}d + S_{\alpha_0}(a, d), \\ \alpha_1(a, d) &= -2\sqrt{3}a + S_{\alpha_1}(a, d), \\ \alpha_2(a, d) &= 3\sqrt{3} + S_{\alpha_2}(a, d), \end{aligned} \quad (2.3.71)$$

the bounds

$$\begin{aligned} |S_{\alpha_0}(a, d)| &\leq Kd^2, & |DS_{\alpha_0}(a, d)| &\leq K|d|, \\ |S_{\alpha_1}(a, d)| &\leq K|a||d|, & |DS_{\alpha_1}(a, d)| &\leq K(|a| + |d|), \\ |S_{\alpha_2}(a, d)| &\leq K(|d| + a^2), & |DS_{\alpha_2}(a, d)| &\leq K \end{aligned} \quad (2.3.72)$$

hold for all $(a, d) \in (-\delta, \delta)^2$.

- (iii) For every $(p, a, d) \in (-\delta, \delta)^3$ we have the bounds

$$|R_{\alpha;3}(p; a, d)| \leq K(|a| + |p|), \quad |D_{a,d}R_{\alpha;3}(p; a, d)| \leq K, \quad (2.3.73)$$

together with

$$\left| R_{\alpha;3}^{(i)}(p; a, d) \right| + \left| D_{a,d}R_{\alpha;3}^{(i)}(p; a, d) \right| \leq K \quad (2.3.74)$$

for all $1 \leq i \leq 6$.

Proof. Substituting the Ansatz (2.3.70) into h_2 , we obtain the fixed point problems

$$\begin{aligned}\frac{1}{3}\alpha_0 &= \frac{4}{3}\sqrt{3}d - 8d\alpha_0 + \sqrt{3}\alpha_0^2 - 2\alpha_0^3, \\ \frac{1}{3}\alpha_1 &= -\frac{2}{3}\sqrt{3}a + 4\alpha_0a + 2\sqrt{3}\alpha_0\alpha_1 - 6\alpha_0^2\alpha_1 - 8\alpha_1d, \\ \frac{1}{3}\alpha_2 &= \sqrt{3} - 6\alpha_0 + \sqrt{3}\alpha_1^2 + 4a\alpha_1 - 6\alpha_1^2\alpha_0 + 2\sqrt{3}\alpha_0\alpha_2 - 6\alpha_0^2\alpha_2 - 8d\alpha_2,\end{aligned}\tag{2.3.75}$$

together with

$$\begin{aligned}\frac{R_{\alpha;3}}{3} &= -2\alpha_1^3 - 6\alpha_1 + 2\sqrt{3}\alpha_1\alpha_2 - 12\alpha_2\alpha_1\alpha_0 + 4a\alpha_2 + 2\sqrt{3}\alpha_0R_{\alpha;3} - 6\alpha_0^2R_{\alpha;3} - 8dR_{\alpha;3} \\ &\quad + \left(\sqrt{3}\alpha_2^2 - 6\alpha_2 - 6\alpha_1^2\alpha_2 - 6\alpha_0\alpha_2^2 + 2\sqrt{3}\alpha_1R_{\alpha;3} - 12\alpha_0\alpha_1R_{\alpha;3} + 4aR_{\alpha;3}\right)p \\ &\quad + \left(2\sqrt{3}\alpha_2R_{\alpha;3} - 6\alpha_1\alpha_2^2 - 6\alpha_1^2R_{\alpha;3} - 6R_3 - 12\alpha_0\alpha_2R_{\alpha;3}\right)p^2 \\ &\quad + \left(\sqrt{3}R_{\alpha;3}^2 - 2\alpha_2^3 - 12\alpha_2\alpha_1R_{\alpha;3} - 6\alpha_0R_{\alpha;3}^2\right)p^3 \\ &\quad + \left(-6\alpha_2^2R_{\alpha;3} - 6\alpha_1R_{\alpha;3}^2\right)p^4 - 6\alpha_2R_{\alpha;3}^2p^5 - 2R_{\alpha;3}^3p^6.\end{aligned}\tag{2.3.76}$$

These fixed point problems can be successively solved for small a , d and p , which yields the desired estimates. $\square$

In order to eliminate q from h_1 , we define the function

$$\mathcal{J}(p; a, d) = \tilde{h}_1(p, q_*(p; a, d), a, d)\tag{2.3.77}$$

and obtain the following representation.

Corollary 2.3.11. Pick $\delta > 0$ sufficiently small. Then there exist a constant $K \geq 1$, together with functions

$$\beta_0 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \beta_1 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \beta_2 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad R_{\beta;3} : (-\delta, \delta)^3 \rightarrow \mathbb{R}\tag{2.3.78}$$

that satisfy the following properties.

(i) For every $(p, a, d) \in (-\delta, \delta)^3$ we have

$$\mathcal{J}(p; a, d) = \beta_0(a, d) + \beta_1(a, d)p + \beta_2(a, d)p^2 + p^3[16 + R_{\beta;3}(p; a, d)].\tag{2.3.79}$$

(ii) Upon writing

$$\begin{aligned}\beta_0(a, d) &= -\frac{1}{3}a + S_{\beta_0}(a, d), \\ \beta_1(a, d) &= 24d + S_{\beta_1}(a, d), \\ \beta_2(a, d) &= -16a + S_{\beta_2}(a, d),\end{aligned}\tag{2.3.80}$$

the bounds

$$\begin{aligned}
|S_{\beta_0}(a, d)| &\leq K |a| |d|, & |DS_{\alpha_0}(a, d)| &\leq K(|a| + |d|), \\
|S_{\beta_1}(a, d)| &\leq K(a^2 + d^2), & |DS_{\alpha_1}(a, d)| &\leq K(|a| + |d|), \\
|S_{\beta_2}(a, d)| &\leq K |a| (|d| + a^2), & |DS_{\alpha_2}(a, d)| &\leq K(|a| + |d|)
\end{aligned} \tag{2.3.81}$$

hold for all $(a, d) \in (-\delta, \delta)^2$.

(iii) For every $(p, a, d) \in (-\delta, \delta)^3$ we have the bounds

$$\begin{aligned}
|R_{\beta;3}(p; a, d)| &\leq K(|d| + a^2 + |a| |p| + p^2), \\
|R'_{\beta;3}(p; a, d)| &\leq K(|a| + |p|), \\
|D_{a,d}R_{\beta;3}(p; a, d)| + |D_{a,d}R'_{\beta;3}(p; a, d)| &\leq K,
\end{aligned} \tag{2.3.82}$$

together with

$$|R_{\beta;3}^{(i)}(p; a, d)| + |D_{a,d}R_{\beta;3}^{(i)}(p; a, d)| \leq K \tag{2.3.83}$$

for all $2 \leq i \leq 6$.

Proof. Substitution yields

$$\begin{aligned}
\beta_0 &= -\frac{1}{3}a - \frac{2}{3}\sqrt{3}a\alpha_0 + 2a\alpha_0^2, \\
\beta_1 &= 4a\alpha_0\alpha_1 - \frac{2}{3}\sqrt{3}a\alpha_1 - 6\alpha_0^2 + 2\sqrt{3}\alpha_0, \\
\beta_2 &= 4a\alpha_0\alpha_2 - \frac{2}{3}\sqrt{3}a\alpha_2 + 2a - 12\alpha_0\alpha_1 + 2\sqrt{3}\alpha_1 + 2a\alpha_1^2,
\end{aligned} \tag{2.3.84}$$

together with

$$\begin{aligned}
R_{\beta;3} &= 4a\alpha_1\alpha_2 - 6\alpha_1^2 - 12\alpha_0\alpha_2 + 2\sqrt{3}(\alpha_2 - 3\sqrt{3}) - \frac{2}{3}\sqrt{3}aR_{\alpha;3} + 4a\alpha_0R_{\alpha;3} \\
&\quad + \left(2a\alpha_2^2 - 12\alpha_1\alpha_2 + 2\sqrt{3}R_{\alpha;3} + 4a\alpha_1R_{\alpha;3} - 12\alpha_0R_{\alpha;3}\right)p \\
&\quad + \left(4a\alpha_2R_{\alpha;3} - 6\alpha_2^2 - 12\alpha_1R_{\alpha;3}\right)p^2 \\
&\quad + \left(2aR_{\alpha;3}^2 - 12\alpha_2R_{\alpha;3}\right)p^3 - 6R_{\alpha;3}^2p^4.
\end{aligned} \tag{2.3.85}$$

The desired estimates can be determined directly from these expressions. $\square$

By applying a (small) shift to p the undesired quadratic term in (2.3.79) can be eliminated. The bifurcation curve in (a, d) space can subsequently be found by determining when the remaining equation has roots of order two or higher.

Lemma 2.3.12. Pick $\delta > 0$ sufficiently small. Then there exists a constant $K \geq 1$ together with functions

$$\zeta_0 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \zeta_1 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \zeta_3 : (-\delta, \delta)^2 \rightarrow \mathbb{R} \quad (2.3.86)$$

and

$$p_* : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad R_{\zeta;4} : (-\delta, \delta)^3 \rightarrow \mathbb{R} \quad (2.3.87)$$

that satisfy the following properties.

(i) For every $(\tilde{p}, a, d) \in (-\delta, \delta)^3$, we have the identity

$$\mathcal{J}(p_*(a, d) + \tilde{p}; a, d) = \zeta_0(a, d) + \zeta_1(a, d)\tilde{p} + \zeta_3(a, d)\tilde{p}^3 + \tilde{p}^4 R_{\zeta;4}(\tilde{p}; a, d). \quad (2.3.88)$$

(ii) Upon writing

$$\begin{aligned} \zeta_0(a, d) &= -\frac{1}{3}a + S_{\zeta_0}(a, d), \\ \zeta_1(a, d) &= 24d + S_{\zeta_1}(a, d), \\ \zeta_3(a, d) &= 16 + S_{\zeta_3}(a, d), \end{aligned} \quad (2.3.89)$$

the bounds

$$\begin{aligned} |S_{\zeta_0}(a, d)| &\leq K(|a||d| + |a|^3), & |DS_{\zeta_0}(a, d)| &\leq K(|a| + |d|), \\ |S_{\zeta_1}(a, d)| &\leq K(a^2 + d^2), & |DS_{\zeta_1}(a, d)| &\leq K(|a| + |d|), \\ |S_{\zeta_3}(a, d)| &\leq K(|a| + |d|), & |DS_{\zeta_3}(a, d)| &\leq K \end{aligned} \quad (2.3.90)$$

hold for all $(a, d) \in (-\delta, \delta)^2$.

(iii) For every $(\tilde{p}, a, d) \in (-\delta, \delta)^3$ we have the bounds

$$\begin{aligned} |R_{\zeta;4}(\tilde{p}; a, d)| + |R'_{\zeta;4}(\tilde{p}; a, d)| + |R''_{\zeta;4}(\tilde{p}; a, d)| &\leq K, \\ |D_{a,d}R_{\zeta;4}(\tilde{p}; a, d)| + |D_{a,d}R'_{\zeta;4}(\tilde{p}; a, d)| &\leq K. \end{aligned} \quad (2.3.91)$$

Proof. We first introduce the notation

$$\mathcal{N}_{\beta;3;p_*}(\tilde{p}; a, d) = \tilde{p}^{-3} \left[R_{\beta;3}(p_* + \tilde{p}; a, d) - R_{\beta;3}(p_*; a, d) - R'_{\beta;3}(p_*; a, d)\tilde{p} - \frac{1}{2}R''_{\beta;3}(p_*; a, d)\tilde{p}^2 \right] \quad (2.3.92)$$

for $\tilde{p} \neq 0$, with $\mathcal{N}_{\beta;3;p_*}(0; a, d) = \frac{1}{6}R'''_{\beta;3}(p_*; a, d)$. This allows us to compute

$$\begin{aligned} \mathcal{J}(p_* + \tilde{p}; a, d) &= \gamma_0(a, d, p_*) + \gamma_1(a, d, p_*)\tilde{p} + \gamma_2(a, d, p_*)\tilde{p}^2 \\ &\quad + [16 + R_{\gamma;3}(\tilde{p}; a, d, p_*)]\tilde{p}^3, \end{aligned} \quad (2.3.93)$$

in which

$$\begin{aligned}
\gamma_0(a, d, p_*) &= \beta_0(a, d) + \beta_1(a, d)p_*(a, d) + \beta_2(a, d)p_*(a, d)^2 \\
&\quad + p_*^3 [16 + R_{\beta;3}(p_*; a, d)], \\
\gamma_1(a, d, p_*) &= \beta_1(a, d) + 2\beta_2(a, d)p_* \\
&\quad + p_*^3 R'_{\beta;3}(p_*; a, d) + 3p_*^2 [16 + R_{\beta;3}(p_*; a, d)], \\
\gamma_2(a, d, p_*) &= \beta_2(a, d) + \frac{1}{2}p_*^3 R''_{\beta;3}(p_*; a, d) + 3p_*^2 R'_{\beta;3}(p_*; a, d) + 3p_* [16 + R_{\beta;3}(p_*; a, d)],
\end{aligned} \tag{2.3.94}$$

together with

$$\begin{aligned}
R_{\gamma;3}(\tilde{p}; a, d, p_*) &= (p_* + \tilde{p})^3 \mathcal{N}_{\beta;3;p_*}(\tilde{p}; a, d) + \frac{1}{2}(3p_*^2 + 3p_*\tilde{p} + \tilde{p}^2) R''_{\beta;3}(p_*; a, d) \\
&\quad + (3p_* + \tilde{p}) R'_{\beta;3}(p_*; a, d) + R_{\beta;3}(p_*; a, d).
\end{aligned} \tag{2.3.95}$$

On account of (2.3.83) we have

$$|\mathcal{N}_{\beta;3;p_*}(\tilde{p}; a, d)| + |\mathcal{N}'_{\beta;3;p_*}(\tilde{p}; a, d)| + |\mathcal{N}''_{\beta;3;p_*}(\tilde{p}; a, d)| + |\mathcal{N}'''_{\beta;3;p_*}(\tilde{p}; a, d)| \leq C'_1 \tag{2.3.96}$$

and hence also

$$|\mathcal{R}_{\gamma;3}(\tilde{p}; a, d, p_*)| + |\mathcal{R}'_{\gamma;3}(\tilde{p}; a, d, p_*)| + |\mathcal{R}''_{\gamma;3}(\tilde{p}; a, d, p_*)| + |\mathcal{R}'''_{\gamma;3}(\tilde{p}; a, d, p_*)| \leq C'_2 \tag{2.3.97}$$

for small $|\tilde{p}|$.

Setting $\gamma_2(a, d, p_*) = 0$ leads to a fixed-point equation for p_* , which has a unique small solution $p_*(a, d)$ that admits the bounds

$$|p_*(a, d)| \leq C'_3 |\beta_2(a, d)| \leq C'_4 a, \quad |D_{a,d} p_*(a, d)| \leq C'_4. \tag{2.3.98}$$

The results hence follow by writing

$$\begin{aligned}
\zeta_0(a, d) &= \gamma_0(a, d, p_*(a, d)), \\
\zeta_1(a, d) &= \gamma_1(a, d, p_*(a, d)), \\
\zeta_3(a, d) &= 16 + R_{\gamma;3}(0; a, d, p_*(a, d)),
\end{aligned} \tag{2.3.99}$$

together with

$$R_{\zeta;4}(\tilde{p}; a, d) = \tilde{p}^{-1} [R_{\gamma;3}(\tilde{p}; a, d, p_*(a, d)) - R_{\gamma;3}(0; a, d, p_*(a, d))] \tag{2.3.100}$$

for $\tilde{p} \neq 0$ and

$$R_{\zeta;4}(0; a, d) = R'_{\gamma;3}(0; a, d, p_*(a, d)). \tag{2.3.101}$$

Here we use (2.3.97) to get bounds on $R_{\zeta;4}$ and its derivatives. $\square$

Proof of Proposition 2.3.9. In order to find roots of order two or higher we need to solve the system

$$\begin{aligned}\zeta_0(a, d) + \zeta_1(a, d)\tilde{p} + \zeta_3(a, d)\tilde{p}^3 + \tilde{p}^4 R_{\zeta;4}(\tilde{p}, a, d) &= 0, \\ \zeta_1(a, d) + 3\zeta_3(a, d)\tilde{p}^2 + 4\tilde{p}^3 R_{\zeta;4}(\tilde{p}, a, d) + \tilde{p}^4 R'_{\zeta;4}(\tilde{p}, a, d) &= 0.\end{aligned}\tag{2.3.102}$$

Solving the second equation we find two branches

$$\tilde{p}_{\pm}(a, d) = \pm\sqrt{-d/2} + S_{\tilde{p}_{\pm}}(a, d),\tag{2.3.103}$$

in which we have

$$\begin{aligned}S_{\tilde{p}_{\pm}}(a, d) &\leq C'_1(|a| + |d|), \\ D_{a,d}S_{\tilde{p}_{\pm}}(a, d) &\leq C'_1.\end{aligned}\tag{2.3.104}$$

Plugging this into the first line of (2.3.102) we obtain

$$\frac{1}{3}a = \pm 16d\sqrt{-d/2} + O(d^2 + |ad| + |a|^3 + a^2\sqrt{|d|}),\tag{2.3.105}$$

from which the desired expression for a_c follows. $\square$

2.3.3 Geometry of the cubic

Our strategy to establish Propositions 2.3.1-2.3.3 hinges upon geometric properties of the cubic $g(u; a)$. As a preparation, we introduce the notation

$$u_{\text{infl}}(a) = \frac{1}{3}(a + 1)\tag{2.3.106}$$

together with

$$u_{\min}(a) = u_{\text{infl}}(a) - \frac{1}{3}\sqrt{1 - a(1 - a)}, \quad u_{\max}(a) = u_{\text{infl}}(a) + \frac{1}{3}\sqrt{1 - a(1 - a)}\tag{2.3.107}$$

and note that

$$g'(u_{\min}(a); a) = 0, \quad g''(u_{\text{infl}}(a); a) = 0, \quad g'(u_{\max}(a); a) = 0.\tag{2.3.108}$$

In addition, we note that $g''(u; a) > 0$ for $u < u_{\text{infl}}(a)$ and $g''(u; a) < 0$ for $u > u_{\text{infl}}(a)$. When $0 < a < \frac{1}{2}$ we have the ordering

$$0 < u_{\min}(a) < a < u_{\text{infl}}(a) < u_{\max}(a) < 1.\tag{2.3.109}$$

Recalling the definition (2.2.6), one readily sees that

$$G_1(u, v; a, d) + G_2(u, v; a, d) = g(u; a) + g(v; a).\tag{2.3.110}$$

In order to exploit the fact that this identity does not depend on d , we first use basic properties of the cubic to parametrize solutions to $g(u; a) + g(v; a) = 0$. Restricting ourselves to $a \in [0, \frac{1}{2}]$, it is possible to construct two solution curves $v_{\pm}(u)$ that are defined for $u \in [0, a]$; see Figure 2.7.

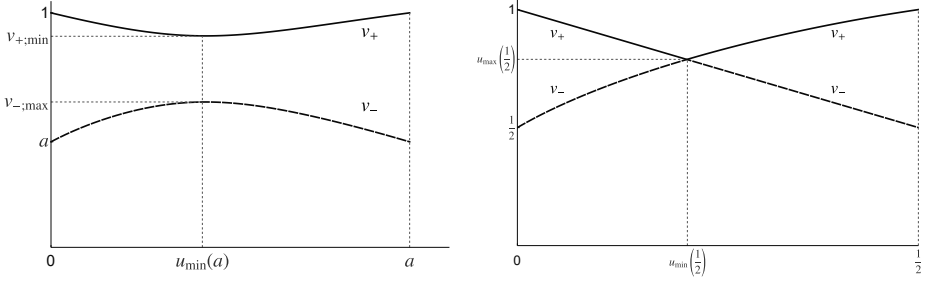


Figure 2.7: The functions $v_{\pm}$ defined in Lemmata 2.3.13-2.3.14 for $a = 0.45$ (left) and $a = 0.5$ (right). In the latter case the derivatives have a discontinuity at $u_{\min}(\frac{1}{2})$.

Lemma 2.3.13. Fix $0 < a < \frac{1}{2}$. Then there are two constants

$$a < v_{-;\max} < v_{+;\min} < 1 \quad (2.3.111)$$

together with two C^∞ -smooth functions

$$v_- : [0, a] \rightarrow [a, v_{-;\max}], \quad v_+ : [0, a] \rightarrow [v_{+;\min}, 1] \quad (2.3.112)$$

that satisfy the following properties.

- (i) We have $g(u; a) = -g(v_-(u); a) = -g(v_+(u); a)$ for all $0 \leq u \leq a$.
- (ii) If $g(u; a) = -g(v; a)$ for some pair $u \in (0, a)$ and $v \in [0, 1]$, then $v = v_-(u)$ or $v = v_+(u)$.
- (iii) We have $v_-(0) = v_-(a) = a$ and $v_+(0) = v_+(a) = 1$.
- (iv) We have the identities

$$v'_\pm(u) = -[g'(v_\pm(u); a)]^{-1} g'(u; a) \quad (2.3.113)$$

for all $0 \leq u \leq a$.

- (v) We have $v'_-(u) > -1$ for all $0 \leq u < a$, together with $v'_-(a) = -1$.

Proof. Since $0 < a < \frac{1}{2}$ we have $g(u_{\max}(a); a) > -g(u_{\min}(a); a)$, which implies that $v_\pm(u) \neq u_{\max}(a)$. Properties (i)-(iv) hence follow immediately from the implicit function theorem.

To obtain (v), we take $u_{\min}(a) < u < a$ and recall $a < v_-(u) < u_{\max}$. If $v_-(u) \leq u_{\text{infl}}(a)$ then clearly $g'(v_-(u); a) > g'(u; a) > 0$, as desired. In order to handle the remaining case $v_-(u) > u_{\text{infl}}(a)$, we introduce the reflection $u_{\text{refl}} = 2u_{\text{infl}}(a) - u$. Exploiting the point symmetry of the graph of $g(\cdot; a)$ around its inflection point $(u_{\text{infl}}(a), g(u_{\text{infl}}(a); a))$, the inequality $g(u_{\text{infl}}(a); a) > 0$ implies that

$$u_{\text{infl}}(a) < v_-(u) < u_{\text{refl}}. \quad (2.3.114)$$

Since $g''(\tilde{u}; a) < 0$ for $\tilde{u} > u_{\inf}(a)$, we obtain

$$g'(v_-(u); a) > g'(u_{\text{ref}}; a) = g'(u; a) > 0, \quad (2.3.115)$$

which implies $v'_-(u) > -1$. $\square$

Lemma 2.3.14. Fix $a = \frac{1}{2}$. Then there are two functions

$$v_- : [0, a] \rightarrow [a, u_{\max}(a)], \quad v_+ : [0, a] \rightarrow [u_{\max}(a), 1] \quad (2.3.116)$$

that satisfy items (i) - (iii) from Lemma 2.3.13 together with the following additional properties.

(i) We have

$$v_-(u_{\min}(a)) = v_+(u_{\min}(a)) = u_{\max}(a). \quad (2.3.117)$$

(ii) For any $u \in [0, a] \setminus \{u_{\min}(a)\}$ we have the identities

$$v'_\pm(u) = -[g'(v_\pm(u); a)]^{-1} g'(u; a). \quad (2.3.118)$$

(iii) We have $v_-(u) = 1 - u$ for all $u_{\min}(a) \leq u \leq 1$, while $v_+(u) = 1 - u$ for all $0 \leq u \leq u_{\min}(a)$.

(iv) We have the limits

$$\lim_{u \uparrow u_{\min}(a)} v'_-(u) = \lim_{u \downarrow u_{\min}(a)} v'_+(u) = 1, \quad (2.3.119)$$

together with

$$\lim_{u \uparrow u_{\min}(a)} v''_-(u) = \lim_{u \downarrow u_{\min}(a)} v''_+(u) = -\frac{4}{3}\sqrt{3}. \quad (2.3.120)$$

Proof. Items (i) and (iii) follow directly from the symmetry of $g(\cdot; a)$, while (ii) follows from the implicit function theorem. To obtain (iv), we first compute

$$g(u_{\min} + u; a) = g(u_{\min}; a) + \frac{1}{2}\sqrt{3}u^2 - u^3, \quad g(u_{\max} + v; a) = -g(u_{\min}; a) - \frac{1}{2}\sqrt{3}v^2 - v^3. \quad (2.3.121)$$

In particular, the identity

$$g(u_{\min} + u; a) = -g(u_{\max} + v; a) \quad (2.3.122)$$

can be rewritten as

$$v^2 = u^2 - \frac{2}{3}\sqrt{3}[u^3 + v^3], \quad (2.3.123)$$

which can be interpreted as a fixed point problem for v^2 upon assuming that v and u have the same sign. For small $|u|$ this problem has a solution that can be expanded as

$$v^2 = u^2 - \frac{4}{3}\sqrt{3}u^3 + O(u^4), \quad (2.3.124)$$

which yields

$$v = u - \frac{2}{3}\sqrt{3}u^2 + O(u^3). \quad (2.3.125)$$

The desired limits follow directly from this expansion. $\square$

2.3.4 Tangencies

Let us now fix $a \in [0, \frac{1}{2}]$. In order to find solutions to $G(u, v; a, d) = 0$ with $d > 0$, we introduce the function

$$v_d(u) = u - \frac{g(u; a)}{2d} \quad (2.3.126)$$

and note that the results above show that it suffices to find $u \in [0, a]$ for which one of the equations

$$v_{\pm}(u) = v_d(u) \quad (2.3.127)$$

holds.

Our goal here is to show that non-transverse intersections of this type can only occur at local minima of $v_{\pm} - v_d$. Together with the strict monotonicity

$$\partial_d v_d(u) < 0 \quad (2.3.128)$$

that holds for $u \in (0, a)$, this will allow us to obtain global results in §2.3.5.

For the moment, we note that

$$v_-(a) = v_d(a) = a \quad (2.3.129)$$

for every $d > 0$. In addition, we may compute

$$v'_d(a) = 1 - \frac{g'(a; a)}{2d}, \quad (2.3.130)$$

together with

$$v'_-(a) = -[g'(a; a)]^{-1}g'(a; a) = -1. \quad (2.3.131)$$

In particular, when $g'(a) = 4d$ the intersection (2.3.129) is tangential. In the sequel we show that in fact $v_- > v_d$ on $[0, a)$ for this critical value of d .

For intersections with $u \in (0, a)$ such explicit computations are significantly harder to carry out, which is why we pursue a more indirect approach here. As a preparation, we compute

$$v''_{\pm}(u) = -\frac{g''(u)}{g'(v_{\pm}(u))} - \frac{g'(u)^2}{g'(v_{\pm}(u))^3}g''(v_{\pm}(u)) \quad (2.3.132)$$

together with

$$v''_d(u) = -\frac{g''(u)}{2d}. \quad (2.3.133)$$

In addition, for any $\kappa \leq g'(u_{\text{infl}}(a); a) = \frac{1}{3}(a^2 - a + 1)$, we introduce the expressions

$$\begin{aligned} u_l(\kappa) &= u_{\text{infl}}(a) - \sqrt{\frac{1}{3}(g'(u_{\text{infl}}(a); a) - \kappa)}, \\ u_r(\kappa) &= u_{\text{infl}}(a) + \sqrt{\frac{1}{3}(g'(u_{\text{infl}}(a); a) - \kappa)}. \end{aligned} \quad (2.3.134)$$

It is easy to verify that $u_l(\kappa)$ and $u_r(\kappa)$ are the two solutions to the quadratic equation $g'(u; a) = \kappa$.

Lemma 2.3.15. Fix $0 < a \leq \frac{1}{2}$ and $d > 0$ and suppose that

$$v'_{\#}(u) = v'_d(u) = \beta \quad (2.3.135)$$

for some $\# \in \{-, +\}$ and $0 \leq u \leq a$, with $u \neq u_{\min}(a)$ in case $a = \frac{1}{2}$. Then the following statements hold.

(i) We have

$$2d[v''_{\#}(u) - v''_d(u)] = \frac{1}{1-\beta}g''(u; a) + \frac{\beta^3}{1-\beta}g''(v_{\#}(u); a). \quad (2.3.136)$$

(ii) If $\# = -$ then we have $\beta \in [-1, 0) \cup (1, \infty)$. On the other hand, the inclusion $\beta \in (0, 1)$ holds if $\# = +$.

(iii) We have $u = u_l(2d(1 - \beta))$.

(iv) Suppose that $\# = -$. Then the identity

$$v_{\#}(u) = u_l(2d(1 - \beta^{-1})) \quad (2.3.137)$$

holds if $v_{-}(u) \leq u_{\inf}(a)$. On the other hand, we have

$$v_{\#}(u) = u_r(2d(1 - \beta^{-1})) \quad (2.3.138)$$

if $v_{-}(u) > u_{\inf}(a)$.

(v) The identity

$$v_{\#}(u) = u_r(2d(1 - \beta^{-1})) \quad (2.3.139)$$

holds if $\# = +$.

Proof. We first consider the case $0 < a < \frac{1}{2}$. For any $0 \leq \tilde{u} \leq a$, one sees that $v'_d(\tilde{u}) = 1$ holds if and only if $g'(\tilde{u}; a) = v'_{\pm}(\tilde{u}) = 0$, which shows that $\beta \notin \{0, 1\}$.

For any $0 \leq \tilde{u} \leq a$ we have $g'(v_{+}(\tilde{u}); a) < 0$ and hence

$$\text{sign}[v'_{+}(\tilde{u})] = \text{sign}[g'(\tilde{u}; a)]. \quad (2.3.140)$$

If $\# = +$, this shows that $g'(u; a) > 0$ and $v'_d(u) > 0$, hence $\beta \in (0, 1)$.

On the other hand, we have $g'(v_{-}(\tilde{u}); a) > 0$ for all $0 \leq \tilde{u} \leq a$. If $g'(u; a) < 0$ then we have $v'_d(u) = \beta > 1$, while if $g'(u; a) > 0$ and $\# = -$ we may use item (v) from Lemma 2.3.13 to conclude

$$-1 \leq v'_-(u) = \beta < 0, \quad (2.3.141)$$

which establishes (ii).

In order to obtain (iii), it suffices to recall the bound $u \leq a \leq u_{\inf}(a)$ and note that the identity $v'_*(u) = \beta$ implies that

$$g'(u) = 2d(1 - \beta). \quad (2.3.142)$$

On the other hand, the identity $v'_\pm(u) = \beta$ implies

$$g'(v_\pm(u)) = -2d\beta^{-1}(1 - \beta). \quad (2.3.143)$$

Items (iv) and (v) now follow directly, remembering that $v_+(u) \geq u_{\max}(a) \geq u_{\inf}(a)$.

Exploiting (2.3.132) we may now compute

$$\begin{aligned} v''_\pm(u) &= \frac{\beta}{2d(1-\beta)}g''(u; a) + \frac{(2d)^2(1-\beta)^2\beta^3}{(2d)^3(1-\beta)^3}g''(v_\pm(u); a) \\ &= \frac{\beta}{2d(1-\beta)}g''(u; a) + \frac{\beta^3}{2d(1-\beta)}g''(v_\pm(u); a). \end{aligned} \quad (2.3.144)$$

The desired identity in (i) hence follows directly from (2.3.133). In order to conclude the proof, it suffices to note that the arguments above remain valid when $a = \frac{1}{2}$. Indeed, the critical cases $g'(u; a) = 0$ and $g'(v_\pm; a) = 0$ are excluded by the requirement that $u \neq u_{\min}(a)$. □

Lemma 2.3.16. Fix $0 < a \leq \frac{1}{2}$ and suppose that

$$v'_-(u) = v'_d(u) \quad (2.3.145)$$

for some $u_{\min}(a) \leq u < a$ and $d > 0$, with $u \neq u_{\min}(a)$ if $a = \frac{1}{2}$. Then we have

$$v''_-(u) > v''_d(u). \quad (2.3.146)$$

Proof. Exploiting (v) of Lemma 2.3.13 and (ii) of Lemma 2.3.15 we have $v'_-(u) \in (-1, 0)$. In addition, the inequalities $v_-(u) > u$ and $g''' < 0$ imply that $g''(v_-(u); a) \leq g''(u; a)$. Writing $\beta = v'_-(u)$ we may hence estimate

$$\begin{aligned} 2d[v''_-(u) - v''_d(u)] &\geq \frac{1}{1-\beta}g''(u; a) + \frac{\beta^3}{1-\beta}g''(u; a) \\ &= \frac{\beta^3+1}{1-\beta}g''(u; a) \\ &> 0, \end{aligned} \quad (2.3.147)$$

in which we used $g''(u; a) > 0$. □

Intersections with $v'_\pm(u) > 0$ are more delicate to analyze. Items (i), (iii) and (iv) of Lemma 2.3.15 suggest that it is worthwhile to consider the two functions

$$\begin{aligned} h_l(\beta) &= \frac{1}{2d(1-\beta)} \left[g''(u_l(2d(1-\beta)); a) + \beta^3 g''(u_l(2d(1-\beta^{-1})); a) \right], \\ h_r(\beta) &= \frac{1}{2d(1-\beta)} \left[g''(u_l(2d(1-\beta)); a) + \beta^3 g''(u_r(2d(1-\beta^{-1})); a) \right]. \end{aligned} \quad (2.3.148)$$

Lemma 2.3.17. Pick $0 < a \leq \frac{1}{2}$ and $0 < d \leq \frac{g'(a; a)}{4}$. Then for any $\beta > 1$ the inequality

$$h_l(\beta) < 0 \quad (2.3.149)$$

holds, while for any $\beta \in (0, \infty) \setminus \{1\}$ we have

$$h_r(\beta) > 0. \quad (2.3.150)$$

Proof. Observe first that for $\beta > 0$ we have

$$\max\{2d(1 - \beta), 2d(1 - \beta^{-1})\} \leq 2d \leq \frac{g'(a; a)}{2} \leq \frac{g'(u_{\inf}(a); a)}{2}, \quad (2.3.151)$$

which implies that $h_l(\beta)$ and $h_r(\beta)$ are well-defined.

A little algebra yields

$$\begin{aligned} h_l(\beta) &= \frac{\sqrt{3}}{d(1-\beta)} \left[\sqrt{g'(u_{\inf}(a); a) + 2d(\beta - 1)} + \beta^3 \sqrt{g'(u_{\inf}(a); a) + 2d\beta^{-1}(1 - \beta)} \right], \\ h_r(\beta) &= \frac{\sqrt{3}}{d(1-\beta)} \left[\sqrt{g'(u_{\inf}(a); a) + 2d(\beta - 1)} - \beta^3 \sqrt{g'(u_{\inf}(a); a) + 2d\beta^{-1}(1 - \beta)} \right]. \end{aligned} \quad (2.3.152)$$

It is clear that $h_l(\beta) < 0$ for $\beta > 1$. Upon writing

$$\begin{aligned} \Delta(\beta) &= g'(u_{\inf}(a); a) + 2d(\beta - 1) - (\beta^6 g'(u_{\inf}(a); a) + 2d\beta^5(1 - \beta)) \\ &= (1 - \beta^6)(g'(u_{\inf}(a); a) - 2d) + 2d\beta(1 - \beta^4), \end{aligned} \quad (2.3.153)$$

it is easy to verify that $\Delta(\beta) < 0$ for $\beta > 1$ and $\Delta(\beta) > 0$ for $0 < \beta < 1$. This yields the final inequality (2.3.150). $\square$

Lemma 2.3.18. Pick $0 < a \leq \frac{1}{2}$ and $0 < d \leq \frac{g'(a; a)}{4}$. Then we have

$$v'_-(0) < v'_d(0). \quad (2.3.154)$$

In addition, suppose that

$$v'_-(u) = v'_*(u) \quad (2.3.155)$$

for some $0 \leq u \leq a$, with $u \neq u_{\min}(a)$ if $a = \frac{1}{2}$. Then one of the following two statements must hold.

(a) We have the inequality

$$v''_-(u) > v''_d(u). \quad (2.3.156)$$

(b) We have the identities

$$u = a, \quad d = \frac{g'(a)}{4}, \quad v''_-(u) = v''_d(u). \quad (2.3.157)$$

Proof. An easy computation yields

$$v'_d(0) = 1 - \frac{g'(0; a)}{2d} \geq 1 - \frac{2g'(0; a)}{g'(a; a)} > -\frac{g'(0; a)}{g'(a; a)} = v'_-(0). \quad (2.3.158)$$

We introduce the critical value

$$u_c = \sup\{0 \leq u \leq a : v_-(\tilde{u}) \leq u_{\inf}(a) \text{ for all } 0 \leq \tilde{u} \leq u\} \quad (2.3.159)$$

and remark that $u_c = 0$ when $a = \frac{1}{2}$. This allows us to define the value

$$u_I = \sup\{0 \leq u \leq \min\{u_c, u_{\min}(a)\} : v'_d(u) > v'_-(u)\}, \quad (2.3.160)$$

which again satisfies $u_I = 0$ when $a = \frac{1}{2}$.

We claim that also $v'_d(u_I) > v'_-(u_I)$. Indeed, assuming this is false we can define $\beta = v'_d(u_I) = v'_-(u_I) \geq 0$. Item (ii) of Lemma 2.3.15 then implies $\beta > 1$. Since $h_l(\beta) < 0$ we must have

$$v''_-(u_I) < v''_d(u_I), \quad (2.3.161)$$

which yields a contradiction.

In particular, if (2.3.155) holds then we must have $u \geq \min\{u_{\min}(a), u_c\}$. If $u_{\min}(a) \leq u < a$, Lemma 2.3.16 shows that (a) must hold. On the other hand, if $u_c \leq u < u_{\min}(a)$, then we can define $\beta = v'_d(u) = v'_-(u) \geq 0$ and conclude as above that $\beta > 1$. In addition, we have $v_-(u) \geq u_{\inf}(a)$, which allows us to use $h_r(\beta) > 0$ and item (iv) of Lemma 2.3.15 to show that (a) must hold. In the final case $u = a$, the remarks at the start of this section together with a direct computation of $v''_d(a)$ and $v''_-(a)$ imply the identities in (b). $\square$

In the remainder of this section we collect several consequences of these computations. In each case, we either rule out non-transverse intersections of $v_{\pm}$ with v_d or show that they must occur at local minima of $v_{\pm} - v_d$.

Corollary 2.3.19. Fix $0 < a \leq \frac{1}{2}$ together with $0 < d \leq \frac{g'(a;a)}{4}$ and suppose that

$$v'_+(u) = v'_d(u) \quad (2.3.162)$$

for some $0 \leq u \leq a$, with $u \neq u_{\min}(a)$ if $a = \frac{1}{2}$. Then we have

$$v''_+(u) > v''_d(u). \quad (2.3.163)$$

Proof. Using the fact that $h_r(\beta) > 0$ for $\beta \in (0, 1)$, this follows directly from items (i), (ii) and (v) of Lemma 2.3.15. $\square$

Corollary 2.3.20. Fix $0 < a \leq \frac{1}{2}$ together with $0 < d < \frac{g'(a;a)}{4}$ and suppose that

$$v'_-(u) = v'_*(u) \quad (2.3.164)$$

for some $0 \leq u \leq a$, with $u \neq u_{\min}(a)$ if $a = \frac{1}{2}$. Then we have

$$v''_-(u) > v''_d(u). \quad (2.3.165)$$

In addition, we have

$$v'_-(a) > v'_d(a). \quad (2.3.166)$$

Proof. The first inequality follows directly from the fact that option (b) in Lemma 2.3.18 cannot hold because of the restriction on d . The final inequality can be verified directly by noting that

$$v'_d(a) = 1 - \frac{g'(a; a)}{2d} < 1 - \frac{2g'(a; a)}{g'(a; a)} = -1 = v'_-(a). \quad (2.3.167)$$

□

Corollary 2.3.21. Fix $0 < a \leq \frac{1}{2}$ together with $d = \frac{g'(a; a)}{4}$. Then we have

$$v'_-(u) < v'_d(u) \quad (2.3.168)$$

for all $0 \leq u < a$, with the exception of $u = u_{\min}(a)$ in the special case $a = \frac{1}{2}$.

Proof. It is easy to verify that $v'_-(a) = v'_d(a)$ and $v''_-(a) = v''_d(a)$. We also compute

$$v'''_d(a) = -2 \frac{g'''(a; a)}{g'(a; a)} = \frac{12}{a(1-a)} > 0 \quad (2.3.169)$$

together with

$$\begin{aligned} v'''_-(u) &= -\frac{g'''(u; a)}{g'(v_-(u); a)} - 3 \frac{g''(u; a)g'(u; a)}{g'(v_-(u); a)^3} g''(v_-(u); a) \\ &\quad - 3 \frac{g'(u; a)^3}{g'(v_-(u); a)^5} g''(v_-(u); a)^2 + \frac{g'(u; a)^3}{g'(v_-(u); a)^4} g'''(v_-(u); a), \end{aligned} \quad (2.3.170)$$

which gives

$$v'''_-(a) = -6 \frac{g''(a; a)^2}{g'(a; a)^2} \leq 0. \quad (2.3.171)$$

In particular, we see that

$$v'_-(a - \epsilon) < v'_d(a - \epsilon) \quad (2.3.172)$$

for all sufficiently small $\epsilon > 0$. If $a \neq \frac{1}{2}$, the conclusion now follows from (2.3.154) together with (a) from Lemma 2.3.18.

For $a = \frac{1}{2}$, one also needs to use the identities

$$v'_d(u_{\min}(a)) = 1, \quad v''_d(u_{\min}(a)) = -12\sqrt{3} \quad (2.3.173)$$

together with the limits in item (iv) of Lemma 2.3.14 to conclude that

$$v'_-(u_{\min}(a) - \epsilon) < v'_d(u_{\min}(a) - \epsilon) \quad (2.3.174)$$

for all sufficiently small $\epsilon > 0$. The arguments above allow us to extend this to $\epsilon \in (0, u_{\min}(a)]$. In addition, we have $v'_-(\tilde{u}) = -1 < v'_d(\tilde{u})$ for $u_{\min}(a) < \tilde{u} < a$. □

Corollary 2.3.22. Pick $0 < a \leq \frac{1}{2}$ and $0 < d \leq \frac{g'(a; a)}{4}$. Then we have

$$v'_-(u) < v'_d(u) \quad (2.3.175)$$

for all $0 \leq u < u_{\min}(a)$.

Proof. Writing $d_c = \frac{g'(a;a)}{4}$, we may use Corollary 2.3.21 to compute

$$v'_d(u) = 1 - \frac{1}{2d}g'(u;a) \geq 1 - \frac{2}{g'(a;a)}g'(u;a) = v'_{d_c}(u) > v'_-(u) \quad (2.3.176)$$

for $u \in [0, u_{\min}(a)]$. $\square$

Corollary 2.3.23. Fix $a = \frac{1}{2}$ and $0 < d \leq \frac{g'(a;a)}{4}$. Then we have

$$v'_+(u) > v'_d(u) \quad (2.3.177)$$

for all $u_{\min}(a) < u \leq a$.

Proof. Item (iv) of Lemma 2.3.14 allow us to compute

$$v'_d(u_{\min}(a)) = 1 = \lim_{u \downarrow u_{\min}(a)} v'_+(u) \quad (2.3.178)$$

together with

$$v''_d(u_{\min}(a)) < \lim_{u \downarrow u_{\min}(a)} v''_+(u), \quad (2.3.179)$$

which allows us to conclude that

$$v'_+(u_{\min}(a) + \epsilon) > v'_d(u_{\min}(a) + \epsilon) \quad (2.3.180)$$

for all sufficiently small $\epsilon > 0$. Corollary 2.3.19 allows us to extend this conclusion to the desired interval $\epsilon \in (0, a - u_{\min}(a))$. $\square$

2.3.5 Structure

We are now ready to analyze the global structure of the solution set to $G(u, v; a, d) = 0$. Our first two results fix $a \in (0, \frac{1}{2}]$ and track the intersections of the curves $v_{\pm}$ that were introduced in §2.3.3 with the curve v_d introduced in §2.3.4. These intersections disappear as the parameter d is increased; see Figure 2.8.

Lemma 2.3.24. Fix $0 < a \leq \frac{1}{2}$. Then there exists a continuous strictly increasing function

$$u_{AD} : [0, \frac{g'(a;a)}{4}] \rightarrow [0, a] \quad (2.3.181)$$

that satisfies the following properties.

- (i) We have $u_{AD}(0) = 0$ and $u_{AD}(\frac{g'(a;a)}{4}) = a$.
- (ii) The identity $v_-(u_{AD}(d)) = v_d(u_{AD}(d))$ holds for any $0 < d \leq \frac{g'(a;a)}{4}$.
- (iii) Suppose that $v_-(u) = v_d(u)$ for some $0 < d \leq \frac{g'(a;a)}{4}$ and $0 \leq u \leq a$. Then in fact $u \in \{u_{AD}(d), a\}$.

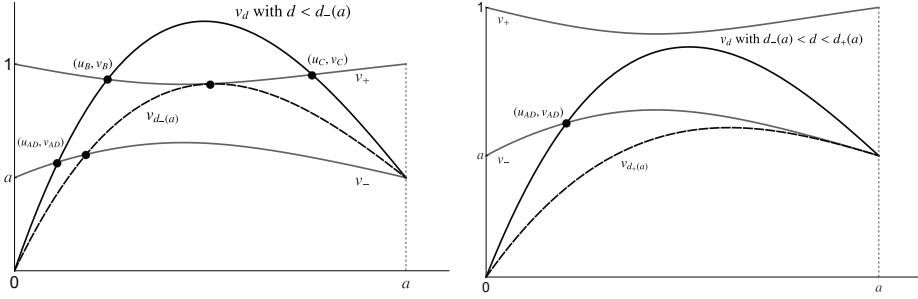


Figure 2.8: Fix $0 < a < \frac{1}{2}$. The branches u_B and u_C described in Lemma 2.3.25 arise as the two intersections of v_+ and v_d on $[0, a]$, which collide as $d \uparrow d_-(a)$ (left). On the other hand, the branch u_{AD} described in Lemma 2.3.24 arises as the unique intersection of the curves v_- and v_d on $[0, a)$, which converges to a as $d \uparrow d_+(a)$ (right).

- (iv) Consider any $0 < d < \frac{g'(a;a)}{4}$ for which $(a, d) \neq (\frac{1}{2}, \frac{1}{24})$. Then we have the inequality

$$v'_-(u_{AD}(d)) < v'_d(u_{AD}(d)). \quad (2.3.182)$$

- (v) For any $d > \frac{g'(a;a)}{4}$ and $0 \leq u < a$ we have $v_-(u) > v_d(u)$.

Proof. For convenience, we introduce the function $h_d(u) = v_-(u) - v_d(u)$ and set out to count the zeroes of h_d on the interval $[0, a]$. We first note that $h_d(a) = 0$ for all $d > 0$. When $d = \frac{g'(a;a)}{4}$ this is in fact the only zero, which can be seen by using (2.3.168) and explicitly verifying the inequality

$$v_-(u_{\min}(a)) > v_d(u_{\min}(a)) \quad (2.3.183)$$

for $(a, d) = (\frac{1}{2}, \frac{g'(\frac{1}{2}; \frac{1}{2})}{4})$.

On the other hand, (2.3.166) implies that h_d has at least two zeroes for $0 < d < \frac{g'(a;a)}{4}$. Furthermore, we claim that h_d has precisely two zeroes for $0 < d \leq d_1$ upon choosing $d_1 > 0$ to be sufficiently small. Indeed, for any $u \in (0, a)$ we have $v''_d(u) < 0$ and we can enforce

$$|v'_d(u)| + v_d(u) \geq 1 + \max_{0 \leq \tilde{u} \leq a} \{|v'_-(\tilde{u})|\} \quad (2.3.184)$$

by restricting the size of $d > 0$.

We now pick $\epsilon > 0$ in such a way that the function h_d is strictly decreasing on $[0, u_{\min}(a) + \epsilon]$ for all $d_1 \leq d \leq \frac{g'(a;a)}{4}$. This is possible because Corollary 2.3.22 allows us to enforce $h'_d < 0$ on this interval, with the exception of the single point $u = u_{\min}(a)$ when $a = \frac{1}{2}$.

Let us now define the critical value

$$d_c = \sup\{d_1 \leq d \leq \frac{g'(a; a)}{4} : h_d = 0 \text{ has two distinct solutions on } [0, a]\} \quad (2.3.185)$$

and assume for the moment that $d_c < g'(a; a)/4$. The preparations above show that there exists $u_{\min}(a) < u < a$ with $h_{d_c}(u) = 0$ and $h'_{d_c}(u) = 0$. Corollary 2.3.20 now implies $h''_{d_c}(u) > 0$. As a consequence of the monotonicity $\partial_d v_d(u) < 0$, this means that for all sufficiently small $\delta > 0$, the function h_d with $d = d_c - \delta$ must have at least three zeroes. This yields a contradiction, which implies that $d_c = g'(a; a)/4$.

We may hence define $u_{AD}(d) \in [0, a]$ to be the left-most root of $h_d(u) = 0$ for $0 < d \leq d_c$. The statements (i)-(v) follow readily from the observations above together with the monotonicity $\partial_d v_d(u) < 0$ that holds for $u \in (0, a)$. $\square$

Lemma 2.3.25. Fix $0 < a \leq \frac{1}{2}$. Then there exists a constant $0 < d_- < \frac{g'(a; a)}{4}$ together with two continuous functions

$$(u_B, u_C) : [0, d_-] \rightarrow [0, a] \times [0, a] \quad (2.3.186)$$

that satisfy the following properties.

- (i) We have $(u_B, u_C)(0) = (0, a)$ and $u_B(d_-) = u_C(d_-)$.
- (ii) The function u_B is strictly increasing, while the function u_C is strictly decreasing.
- (iii) For any $0 < d \leq d_-$ the identity $v_+(u_{\#}) = v_d(u_{\#})$ holds for $\# \in \{B, C\}$.
- (iv) If $v_+(u) = v_d(u)$ for some $0 < d \leq d_-$ and $0 \leq u \leq a$ then in fact $u \in \{u_B(d), u_C(d)\}$.
- (v) For any $0 < d < d_-$ we have

$$v'_+(u_B(d)) < v'_d(u_B(d)), \quad v'_+(u_C(d)) > v'_d(u_C(d)). \quad (2.3.187)$$

If $a \neq \frac{1}{2}$, then we also have

$$v'_+(u_C(d_-)) = v'_+(u_B(d_-)) = v'_d(u_B(d_-)) = v'_d(u_C(d_-)). \quad (2.3.188)$$

- (vi) For any $d > d_-$ the inequality $v_+(u) > v_d(u)$ holds for all $0 \leq u \leq a$.
- (vii) If $a = \frac{1}{2}$ then we have $d_-(\frac{1}{2}) = \frac{1}{24}$ together with $u_B(\frac{1}{24}) = u_C(\frac{1}{24}) = u_{\min}(a)$.

Proof. Writing $h_d(u) = v_+(u) - v_d(u)$, we observe first that h_d is strictly decreasing on $[0, u_{\min}(a)]$ because $v'_d > 0$ and $v'_+ < 0$ on the interior of this interval.

In addition, in the special case $a = \frac{1}{2}$ we can use Corollary 2.3.23 to conclude that h_d is strictly increasing on $[u_{\min}(a), a]$. Since $h_d(u_{\min}(a)) = 0$ occurs precisely when $d = \frac{1}{24}$, all the desired statements can be easily verified.

Throughout the remainder of this proof we therefore assume that $0 < a < \frac{1}{2}$. Arguing as in the proof of Lemma 2.3.24, we may pick $d_1 > 0$ in such a way that h_d has precisely two zeroes on $[0, a]$ for every $0 < d \leq d_1$. This allows us to define the critical value

$$d_- = \sup\{d_1 \leq d : h_d = 0 \text{ has two distinct solutions on } [0, a]\}. \quad (2.3.189)$$

Since $v_- < v_+$ it is clear that $d_- < \frac{g'(a; a)}{4}$.

Let us assume for the moment that h_{d_-} has two or more zeroes on $[0, a]$. This implies that there exists at least one $u \in (0, a)$ for which $h_{d_-}(u) = 0$ and $h'_{d_-}(u) = 0$. Using Corollary 2.3.20 it follows that $h''_{d_-}(u) > 0$ must hold for all such zeroes. As a consequence of the monotonicity $\partial_d v_d(u) < 0$, this means that for all sufficiently small $\delta > 0$, the function h_d with $d = d_- - \delta$ must have at least three zeroes. This yields a contradiction, which by continuity shows that $h_{d_-}(u) = 0$ has precisely one root on $[0, a]$. Upon defining $u_B(d)$ and $u_C(d)$ to be the left-most respectively right-most root of $h_d(u) = 0$, the desired properties (i) - (vii) can be easily verified. $\square$

Recalling the function G introduced in (2.2.6), we see that

$$D_{1,2}G(u, v; a, d) = \begin{pmatrix} g'(u) - 2d & 2d \\ 2d & g'(v) - 2d \end{pmatrix}. \quad (2.3.190)$$

In order to study the stability and parameter-dependence of the roots constructed in Lemmata 2.3.24-2.3.25, it is crucial to understand when the determinant of $D_{1,2}G$ vanishes. The result below states that this happens at tangential intersections of $v_{\pm}$ and v_d .

Lemma 2.3.26. Fix $0 < a < \frac{1}{2}$ together with $d > 0$ and suppose that

$$v_{\#}(u) = v_d(u) \quad (2.3.191)$$

for some $\# \in \{-, +\}$ and $u \in [0, a]$. Then we have $\det D_{1,2}G(u, v_d(u); a, d) = 0$ if and only if $v'_{\#}(u) = v'_d(u)$.

Proof. We first consider the case $v'_{\#}(u) = 0$, which occurs when $g'(u; a) = 0$ and hence $u = u_{\min}(a)$. Using $0 < a < \frac{1}{2}$ we see that

$$v_-(u) < u_{\max}(a) < v_+(u). \quad (2.3.192)$$

Writing $v = v_d(u) = v_{\pm}(u)$, we hence obtain $g'(v; a) \neq 0$ and hence

$$\det D_{1,2}G(u, v; a, d) = -2dg'(v; a) \neq 0. \quad (2.3.193)$$

Since $v'_d(u) = 1$, the desired equivalence indeed holds for this case.

Assuming now that $v'_{\#}(u) \neq 0$ and hence $g'(u; a) \neq 0$, we again write $v = v_d(u) = v_{\#}(u)$ and use (2.3.113) to compute

$$g'(v; a) = -[v'_{\#}(u)]^{-1}g'(u; a). \quad (2.3.194)$$

In particular, we find

$$\begin{aligned} \det D_{1,2}G(u, v; a, d) &= (g'(u; a) - 2d)(-[v'_{\#}(u)]^{-1}g'(u; a) - 2d) - 4d^2 \\ &= -[v'_{\#}(u)]^{-1}g'(u; a)^2 - 2dg'(u; a) + 2d[v'_{\#}(u)]^{-1}g'(u; a) \\ &= 2dg'(u; a)[v'_{\#}(u)]^{-1}\left[1 - \frac{g'(u; a)}{2d} - v'_{\#}(u)\right] \\ &= 2dg'(u; a)[v'_{\#}(u)]^{-1}\left[v'_d(u) - v'_{\#}(u)\right], \end{aligned} \quad (2.3.195)$$

from which the statement follows. $\square$

In order to characterize the dependence of $d_-(a)$ on a , we introduce the function

$$G_{\text{sn}}(u, v, d; a) = (G_1(u, v; a, d), G_2(u, v; a, d), \det D_{1,2}G(u, v; a, d))^T \quad (2.3.196)$$

and symbolically write

$$\begin{aligned} [D_{1,2,3}G_{\text{sn}}(u, v, d; a)]^{-1} &= [\det D_{1,2,3}G_{\text{sn}}(u, v, d; a)]^{-1} \\ &\quad \times \begin{pmatrix} * & * & * \\ * & * & * \\ \gamma_1(u, v, d; a) & \gamma_2(u, v, d; a) & \gamma_3(u, v, d; a) \end{pmatrix}. \end{aligned} \quad (2.3.197)$$

For any $0 < a \leq \frac{1}{2}$, we use the functions defined in Lemma 2.3.25 to introduce the notation

$$\omega(a) = (u_B(d_-(a)), v_+(u_B(d_-(a))), d_-(a)) = (u_C(d_-(a)), v_+(u_C(d_-(a))), d_-(a)), \quad (2.3.198)$$

which corresponds to the critical point where the branches u_B and u_C collide.

Corollary 2.3.27. Upon fixing $0 < a < \frac{1}{2}$, the following two statements are equivalent.

- (a) The identity $G_{\text{sn}}(u, v, d; a) = 0$ holds for some $d > 0$ and some pair $(u, v) \in [0, 1]^2$ that has $v \geq u$.
- (b) We have $(u, v, d) = \omega(a)$ or $(u, v, d) = (a, a, \frac{g'(a; a)}{4})$.

Proof. As a preparation, we note that $\det D_{1,2}G(a, a; a, d) = 0$ if and only if $d = \frac{g'(a; a)}{4}$. In addition, it is easy to check that $\det D_{1,2}G(0, 0; a, d) > 0$ and $\det D_{1,2}G(1, 1; a, d) > 0$ for all $d \geq 0$.

The implication (b) $\rightarrow$ (a) can now be verified directly using Lemma 2.3.26 and (2.3.188). In addition, we only need to establish the reverse implication under the additional assumption that $v > u$.

Let us therefore assume that (a) holds with $g(u) = -g(v) \neq 0$, which allows us to write

$$0 < u < a < v < 1. \quad (2.3.199)$$

In particular, Lemma 2.3.13 implies that $v = v_-(u)$ or $v = v_+(u)$. Lemma 2.3.26 together with (2.3.187) and (2.3.182) now imply that (b) must hold. $\square$

Lemma 2.3.28. We have the identities

$$\begin{aligned} \gamma_1(0, 1, 0; 0) &= -2, \\ \gamma_2(0, 1, 0; 0) &= 0. \end{aligned} \quad (2.3.200)$$

In addition, the identity

$$\gamma_3(\omega(a); a) = 0 \quad (2.3.201)$$

holds for any $0 < a < \frac{1}{2}$, together with the inequalities

$$\begin{aligned} \det D_{1,2,3}G_{\text{sn}}(\omega(a); a) &< 0, \\ \gamma_1(\omega(a); a) - \gamma_2(\omega(a); a) &< 0, \\ \gamma_1(\omega(a); a)\gamma_2(\omega(a); a) &\geq 0. \end{aligned} \quad (2.3.202)$$

Proof. Let us assume for the moment that $G_{\text{sn}}(u, v, d; a) = 0$, which directly implies

$$(g'(u; a) - 2d)g'(v; a) = 2dg'(u; a). \quad (2.3.203)$$

In view of the identity

$$\begin{aligned} D_{1,2,3}G_{\text{sn}}(u, v, d; a) &= \\ &= \begin{pmatrix} g'(u; a) - 2d & 2d & 2(v - u) \\ 2d & g'(v; a) - 2d & 2(u - v) \\ (g'(v; a) - 2d)g''(u; a) & (g'(u; a) - 2d)g''(v; a) & -2(g'(u; a) + g'(v; a)) \end{pmatrix}, \end{aligned} \quad (2.3.204)$$

we can use (2.3.203) to compute

$$\begin{aligned} \det D_{1,2,3}G_{\text{sn}}(u, v, d; a) &= \\ &= 2(v - u) \left[g'(u; a)(g'(u; a) - 2d)g''(v; a) - g'(v; a)(g'(v; a) - 2d)g''(u; a) \right] \\ &= 2(v - u) \left[2dg'(u; a)^2 \frac{1}{g'(v; a)} g''(v; a) - g'(v; a)(g'(v; a) - 2d)g''(u; a) \right]. \end{aligned} \quad (2.3.205)$$

Expanding the subdeterminants of (2.3.204) and reusing (2.3.203), we also find

$$\begin{aligned}\gamma_1(u, v, d; a) &= 2d(g'(u; a) - 2d)g''(v; a) - (g'(v; a) - 2d)^2g''(u; a), \\ \gamma_2(u, v, d; a) &= 2d(g'(v; a) - 2d)g''(u; a) - (g'(u; a) - 2d)^2g''(v; a), \\ \gamma_3(u, v, d; a) &= 0,\end{aligned}\tag{2.3.206}$$

which allows us to explicitly verify (2.3.200).

Let us now fix $0 < a < \frac{1}{2}$. For any $(u, v) \in [0, 1]^2$ and $d > 0$ we introduce the function

$$\begin{aligned}h(u, v, d) &= \frac{1}{2d}g''(u; a) - \frac{g''(u; a)}{g'(v; a)} - \frac{g'(u; a)^2}{g'(v; a)^3}g''(v; a) \\ &= \frac{1}{2dg'(v; a)^2} \left[g'(v; a)(g'(v; a) - 2d)g''(u; a) - 2d \frac{g'(u; a)^2}{g'(v; a)} g''(v; a) \right],\end{aligned}\tag{2.3.207}$$

which allows us to rewrite (2.3.205) in the form

$$\det D_{1,2,3}G_{\text{sn}}(u, v, d; a) = -4d(v - u)g'(v; a)^2h(u, v, d).\tag{2.3.208}$$

Using (2.3.132) and (2.3.133) we observe that

$$v_+''(u) - v_d''(u) = h(u, v_+(u), d).\tag{2.3.209}$$

In particular, Corollary 2.3.19 and (2.3.188) imply that

$$h(\omega(a)) > 0,\tag{2.3.210}$$

which shows that

$$\det D_{1,2,3}G_{\text{sn}}(\omega(a); a) < 0,\tag{2.3.211}$$

as desired.

Let us now write $(u, v, d) = \omega(a)$ and compute

$$\begin{aligned}\gamma_1(\omega(a); a) - \gamma_2(\omega(a); a) &= g'(u)(g'(u) - 2d)g''(v) - g'(v)(g'(v) - 2d)g''(u) \\ &= \frac{1}{2}(v - u)^{-1} \det DG_{\text{sn}}(u, v, d; a) \\ &\leq 0.\end{aligned}\tag{2.3.212}$$

In addition, since $u > u_{\min}(a)$ and hence $v_d'(u) = v_+'(u) > 0$, we have $g'(u) < 2d$. This allows us to define

$$\begin{aligned}\alpha &= \sqrt{2d}(2d - g'(u; a))^{3/2}g''(v; a), \\ \beta &= \sqrt{2d}(2d - g'(v; a))^{3/2}g''(u; a)\end{aligned}\tag{2.3.213}$$

and compute

$$\begin{aligned}
 \gamma_1(\omega(a); a)\gamma_2(\omega(a); a) &= (4d^2)(4d^2)g''(u; a)g''(v; a) + (4d^2)^2g''(u; a)g''(v; a) \\
 &\quad - 2d(g'(u; a) - 2d)^3g''(v; a)^2 - 2d(g'(v; a) - 2d)^3g''(u; a)^2 \\
 &= 2\alpha\beta + \alpha^2 + \beta^2 \\
 &\geq 0,
 \end{aligned} \tag{2.3.214}$$

as desired. $\square$

Corollary 2.3.29. The map $a \mapsto \omega(a)$ is C^∞ -smooth on $(0, \frac{1}{2})$ and we have $d'_-(a) > 0$.

Proof. Since $d_- < \frac{g'(a; a)}{4}$, we may use Corollary 2.3.27, together with the first inequality in (2.3.202) to apply the implicit function and establish the C^∞ -smoothness of ω .

Since $g'(a; a)/4 \downarrow 0$ as $a \downarrow 0$, a squeezing argument shows that $\lim_{a \downarrow 0} \omega(a) = (0, 1, 0)$. Exploiting (2.3.200), and (2.3.202), the continuity of γ_1 and γ_2 implies that

$$\gamma_1(\omega(a); a) < \gamma_2(\omega(a); a) \leq 0 \tag{2.3.215}$$

for all $0 < a < \frac{1}{2}$. In particular, writing $(u, v, d) = \omega(a)$, we may use the inequalities

$$D_2g(u; a) < 0, \quad D_2g(v; a) < 0 \tag{2.3.216}$$

to compute

$$d'_-(a) = -[\det D_{1,2,3}G_{\text{sn}}(\omega(a); a)]^{-1} [\gamma_1(\omega(a); a)D_2g(u; a) + \gamma_2(\omega(a); a)D_2g(v; a)] > 0. \tag{2.3.217}$$

$\square$

Proof of Proposition 2.3.1. Items (i), (iii), (iv) follow from Lemmata 2.3.24-2.3.25 and the observations in the proof of Corollary 2.3.27. Item (ii) follows from Corollary 2.3.29, while the expansions in (v) follow from Propositions 2.3.6 and 2.3.9. $\square$

Proof of Proposition 2.3.2. The identity $g(1-u, 1-a) = -g(u, a)$ implies that

$$G(1-u, 1-v; 1-a, d) = -G(u, v; a, d). \tag{2.3.218}$$

This allows us to extend the solutions constructed in Lemmata 2.3.24-2.3.25 to the entire interval $0 \leq a \leq 1$ in the fashion outlined in Corollary 2.3.4, which yields (ii) and (iv).

To establish (i), we note that for each $\# \in \{A, B, C\}$ the sign of $\det D_{1,2}G(\bar{u}_\#, \bar{v}_\#; a, d)$ is constant on Ω_- on account of Corollary 2.3.27. Using (ii) the eigenvalues of these matrices can be explicitly computed at $d = 0$, which yields the desired stability properties.

To obtain the strict ordering in (iii), we fix $0 < a \leq \frac{1}{2}$ and $0 < d < d_-(a)$. We observe that $v_+(u) \geq v_-(u)$ for $u \in [0, a]$, with equality only at $u = u_{\min}(a)$ for $a = \frac{1}{2}$. Since $v_d(0) = 0$ this implies that $\bar{u}_A(a, d) < \bar{u}_B(a, d)$ and $\bar{v}_A(a, d) < \bar{v}_B(a, d)$. By construction, we also have $\bar{u}_B(a, d) < \bar{u}_C(a, d)$. In addition, it cannot be the case that $\bar{v}_B(a, \tilde{d}) = \bar{v}_C(a, \tilde{d})$ for any $0 < \tilde{d} < d_-(a)$ since then $G_2 = 0$ implies that also $\bar{u}_B(a, \tilde{d}) = \bar{u}_C(a, \tilde{d})$. Using the general identity

$$D_{a,d}(\bar{u}_\#, \bar{v}_\#) = -[D_{1,2}G(\bar{u}_\#, \bar{v}_\#; a, d)]^{-1} D_{3,4}G(\bar{u}_\#, \bar{v}_\#; a, d) \quad (2.3.219)$$

we see that

$$\partial_d \bar{v}_\#(a, 0) = -\frac{1}{g'(\bar{v}_\#(a, 0); a)} (\bar{u}_\#(a, 0) - \bar{v}_\#(a, 0)), \quad (2.3.220)$$

which yields

$$\partial_d \bar{v}_B(a, 0) = \frac{1}{g'(1; a)} < \frac{1-a}{g'(1; a)} = \partial_d \bar{v}_C(a, 0). \quad (2.3.221)$$

These observations allow us to conclude $\bar{v}_B(a, d) < \bar{v}_C(a, d)$, as desired. $\square$

Proof of Proposition 2.3.3. Arguing in a similar fashion as the proof of Proposition 2.3.2, the statements follow from Lemma 2.3.24 and Corollary 2.3.27. $\square$

2.4 Travelling waves

Our goal here is to establish the existence of bichromatic wave solutions to the Nagumo LDE (2.2.1) and to obtain detailed results concerning their speed. In particular, we establish Theorems 2.2.2 and 2.2.3, which are the main results of this chapter.

Upon introducing the standard discrete Laplacian

$$[\Delta^+ u](\xi) = u(\xi + 1) + u(\xi - 1) - 2u(\xi) \quad (2.4.1)$$

together with the off-diagonal matrix

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (2.4.2)$$

the travelling wave system (2.2.5) can be rewritten as

$$-c\Phi' = d\mathcal{J}\Delta^+\Phi + G(\Phi; a, d). \quad (2.4.3)$$

For any $(a, d) \in \Omega_-$, we set out to seek solutions to (2.4.3) that satisfy the boundary conditions

$$\lim_{\xi \rightarrow -\infty} \Phi(\xi) = (0, 0), \quad \lim_{\xi \rightarrow +\infty} \Phi(\xi) = (\bar{u}_B(a, d), \bar{v}_B(a, d)). \quad (2.4.4)$$

The existence of such solutions will be established in §2.4.1, where we also show $c \geq 0$ and establish Theorem 2.2.2. In §2.4.2 we subsequently set out to derive criteria that distinguish between the two cases $c = 0$ and $c > 0$. We verify these criteria in §2.4.3 for parameters $(a, d) \in \Omega_-$ that are close to the cusp $(\frac{1}{2}, \frac{1}{24})$ and the corner $(1, 0)$. This allows us to establish our main results contained in Theorem 2.2.3.

2.4.1 Existence of waves

Our preparatory work in §2.3 allows us to invoke the theory developed in [27] to establish a general existence result for bichromatic waves. Indeed, the equilibria $(0, 0)$ and $(\bar{u}_B, \bar{v}_B)$ are both stable under the dynamics of $(\dot{u}, \dot{v}) = G(u, v; a, d)$ and all intermediate equilibria are unstable. Using a straightforward estimate based on the ordering (2.3.11), the wavespeeds can be shown to be non-negative.

Lemma 2.4.1. Pick $(a, d) \in \Omega_-$. Then there exists a constant $c \in \mathbb{R}$ and a non-decreasing function $\Phi : \mathbb{R} \rightarrow \mathbb{R}^2$ that satisfy (2.4.3)-(2.4.4). This constant c is unique, while the function Φ is unique up to translation if $c \neq 0$. In the latter case we also have $\Phi'(\xi) > (0, 0)$ for all $\xi \in \mathbb{R}$.

Proof. These statements follow directly from the main results in [27]. $\square$

Lemma 2.4.2. Pick $(a, d) \in \Omega_-$. Then the constant c defined in Lemma 2.4.1 satisfies $c \geq 0$.

Proof. Suppose that $c \neq 0$. We estimate

$$\begin{aligned} -c\Phi'_v(\xi) &= d[\Phi_u(\xi + 1) + \Phi_u(\xi - 1) - 2\Phi_v(\xi)] + g(\Phi_v(\xi); a) \\ &\leq 2d[\bar{u} - v(\xi)] + g(\Phi_v(\xi); a). \end{aligned} \quad (2.4.5)$$

Since $0 < \bar{u}_B(a, d) < a < \bar{v}_B(a, d)$, there exists $\xi_* \in \mathbb{R}$ for which $\Phi_v(\xi_*) = \bar{u}_B(a, d)$. This yields

$$-c\Phi'_v(\xi_*) \leq g(\bar{u}; a) < 0, \quad (2.4.6)$$

from which we conclude $c > 0$. $\square$

Writing $c(a, d)$ and $\Phi(a, d)$ for the wavespeed and waveprofile defined in Lemma 2.4.1, we introduce the set

$$\mathcal{T} = \{(a, d) \in \Omega_- : c(a, d) > 0\}, \quad (2.4.7)$$

which corresponds to the set $\mathcal{T}_{\text{low}}$ used in §2.2. In addition, for any $(a, d) \in \mathcal{T}$ we introduce the linear operators

$$\mathcal{L}_{a,d} : W^{1,\infty}(\mathbb{R}; \mathbb{R}^2) \rightarrow L^\infty(\mathbb{R}; \mathbb{R}^2), \quad \mathcal{L}_{a,d}^{\text{adj}} : W^{1,\infty}(\mathbb{R}; \mathbb{R}^2) \rightarrow L^\infty(\mathbb{R}; \mathbb{R}^2), \quad (2.4.8)$$

that act as

$$\begin{aligned} \mathcal{L}_{a,d}\phi &= -c(a, d)\phi' - d\mathcal{J}\Delta^+\phi - DG(\Phi(a, d); a, d)\phi, \\ \mathcal{L}_{a,d}^{\text{adj}}\psi &= c(a, d)\psi' - d\mathcal{J}\Delta^+\psi - DG(\Phi(a, d); a, d)\psi. \end{aligned} \quad (2.4.9)$$

The results in [74, §8] imply that there exists $\Psi = \Psi(a, d) \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^2)$ with $\Psi > (0, 0)$ for which we have the identities

$$\text{Ker } \mathcal{L}_{a,d} = \text{span}\{\Phi'(a, d)\}, \quad \text{Ker } \mathcal{L}_{a,d}^{\text{adj}} = \text{span}\{\Psi(a, d)\} \quad (2.4.10)$$

together with the normalization

$$\int_{-\infty}^{\infty} \langle \Psi(\xi), \Phi'(\xi) \rangle d\xi = 1. \quad (2.4.11)$$

In particular, [90, Thm. A] implies that

$$\text{Range } \mathcal{L}_{a,d} = \{f \in L^\infty(\mathbb{R}; \mathbb{R}^2) : \int_{-\infty}^{\infty} \langle \Psi(\xi), f(\xi) \rangle d\xi = 0\}. \quad (2.4.12)$$

These ingredients allow us to use the implicit function theorem to show that the pair (c, Φ) depends smoothly on the parameters $(a, d) \in \mathcal{T}$. In addition, we obtain a sign on $\partial_a c$.

Lemma 2.4.3. The maps

$$\begin{aligned} \mathcal{T} \ni (a, d) &\mapsto c(a, d) \in (0, \infty), \\ \mathcal{T} \ni (a, d) &\mapsto \Phi(a, d) \in BC^1(\mathbb{R}; \mathbb{R}^2) \end{aligned} \quad (2.4.13)$$

are C^∞ -smooth. In addition, we have

$$\partial_a c(a, d) > 0 \quad (2.4.14)$$

for all $(a, d) \in \mathcal{T}$.

Proof. The C^1 -smoothness of the maps (c, Φ) follows from [74, Thm. 2.3]. On account of the smoothness of the function g , this can readily be extended to the desired C^∞ -smoothness by using the ideas in [74, §8] to set up an implicit function argument along the lines of [91, Prop. 6.5].

Differentiating (2.4.3) with respect to a , we compute

$$-[\partial_a c]\Phi' - c[\partial_a \Phi'] = d\mathcal{J}\Delta^+[\partial_a \Phi] + DG(\Phi; a, d)\partial_a \Phi + \partial_a G(\Phi; a, d), \quad (2.4.15)$$

which gives

$$-[\partial_a c]\Phi' + \mathcal{L}_{a,d}\partial_a \Phi = \partial_a G(\Phi; a, d). \quad (2.4.16)$$

Applying (2.4.12) and noting that $\partial_a g(u; a) = -u(1 - u) < 0$ for all $u \in (0, 1)$, we obtain

$$-\partial_a c = \int_{-\infty}^{\infty} \langle \partial_a G(\Phi(\xi); a, d), \Psi(\xi) \rangle d\xi < 0, \quad (2.4.17)$$

as desired. $\square$

Corollary 2.4.4. If $(a, d) \in \mathcal{T}$, then also $(a', d) \in \mathcal{T}$ for all $(a', d) \in \Omega_-$ with $a' \geq a$.

Proof of Theorem 2.2.2. The statements follow directly from Lemmata 2.4.1-2.4.3. $\square$

2.4.2 Characterization of waves

We now set out to derive conditions that guarantee either $c(a, d) = 0$ or $c(a, d) > 0$. To this end, we introduce the functions

$$u_{a,d}(v) = v - \frac{g(v; a)}{2d}, \quad v_{a,d}(u) = u - \frac{g(u; a)}{2d} \quad (2.4.18)$$

and note that

$$\bar{u}_{\#}(a, d) = u_{a,d}(\bar{v}_{\#}(a, d)), \quad \bar{v}_{\#}(a, d) = v_{a,d}(\bar{u}_{\#}(a, d)) \quad (2.4.19)$$

for each $\# \in \{A, B, C\}$ and $(a, d) \in \Omega_-$. The local extrema of these functions are located at the critical points

$$\begin{aligned} \gamma_{c;\pm}(a, d) &= \frac{1}{3}(a+1) \pm \frac{1}{3}\sqrt{a^2 - a + 1 - 6d} \\ &= u_{\text{infl}}(a) \pm \sqrt{\frac{1}{3}(g'(u_{\text{infl}}(a); a) - 2d)}. \end{aligned} \quad (2.4.20)$$

One can verify that the functions $u_{a,d}$ and $v_{a,d}$ are strictly decreasing on $[\gamma_{c;-}(a, d), \gamma_{c;+}(a, d)]$ and strictly increasing outside this interval. The following ordering result exploits this characterization and will allow us to establish $c(a, d) = 0$ for a significant portion of the parameter set Ω_- .

Lemma 2.4.5. For any $(a, d) \in \bar{\Omega}_-$ we have the ordering

$$0 \leq \gamma_{c;-}(a, d) \leq a \leq \gamma_{c;+}(a, d) \leq \bar{v}_B(a, d). \quad (2.4.21)$$

If $(a, d) \in \Omega_-$ then the inequalities in (2.4.21) are all strict.

Proof. Let us first fix $(a, d) \in \Omega_-$. We note that $2d < 2d_+(a) = \frac{g'(a; a)}{2} \leq \frac{g'(u_{\text{infl}}(a); a)}{2}$, which implies that $\gamma_{c;\pm}(a, d)$ are well-defined. In addition, this allows us to compute

$$u'_{a,d}(a) = 1 - \frac{g'(a; a)}{2d} \leq -1, \quad (2.4.22)$$

which means that $a \in (\gamma_{c;-}(a, d), \gamma_{c;+}(a, d))$.

In particular, the function $v \mapsto u_{a,d}(v)$ is strictly decreasing on $[a, \gamma_{c;+}(a, d)]$. On account of the orderings

$$u_{a,d}(\bar{v}_A(a, d)) = \bar{u}_A(a, d) < \bar{u}_B(a, d) = u_{a,d}(\bar{v}_B(a, d)), \quad a < \bar{v}_A(a, d) < \bar{v}_B(a, d) \quad (2.4.23)$$

we hence cannot have $\bar{v}_B(a, d) \leq \gamma_{c;+}(a)$. The results for the general case $(a, d) \in \bar{\Omega}_-$ now follow by continuity. $\square$

Lemma 2.4.6. Consider any $(a, d) \in \Omega_-$ for which $d \leq \frac{1}{8}(1-a)^2$. Then we have $c(a, d) = 0$.

Proof. Fix any $0 < a < 1$, write $d_* = \frac{1}{8}(1-a)^2$ and suppose that $(a, d_*) \in \Omega_-$. On account of Corollary 2.4.4 it suffices to show that $c(a, d_*) = 0$. Assuming to the contrary that $c = c(a, d_*) > 0$, we compute

$$\begin{aligned}
 0 &> -c\Phi'_v(\xi) \\
 &= d_*[\Phi_u(\xi+1) + \Phi_u(\xi-1) - 2\Phi_v(\xi)] + g(\Phi_v(\xi); a) \\
 &> -2d_*\Phi_v(\xi) + g(\Phi_v(\xi); a) \\
 &= -2d_*u_{a,d_*}(\Phi_v(\xi)).
 \end{aligned} \tag{2.4.24}$$

Since $0 < \gamma_{c,+}(a, d_*) < \bar{v}_B(a, d_*)$, there exists $\xi_* \in \mathbb{R}$ for which $\Phi_v(\xi_*) = \gamma_{c,+}(a, d_*)$. The key point is that

$$u_{a,d_*}(\gamma_{c,+}(a, d_*)) = 0, \tag{2.4.25}$$

which allows us to obtain a contradiction by picking $\xi = \xi_*$ in (2.4.24). $\square$

We now set out to derive a (non-sharp) condition that guarantees $c(a, d) > 0$. The strategy will be to rule out the existence of standing bichromatic waves. Let us therefore pick any $(a, d) \in \Omega_- \setminus \mathcal{T}$, which implies $c(a, d) = 0$. Writing $(\Phi_u, \Phi_v) = \Phi(a, d)$ for the corresponding profile, we introduce the sequence

$$(u_i, v_i) = (\Phi_u(2i), \Phi_v(2i+1)), \tag{2.4.26}$$

which satisfies the limits

$$\lim_{i \rightarrow -\infty} (u_i, v_i) = (0, 0) \quad \lim_{i \rightarrow +\infty} (u_i, v_i) = (\bar{u}_B(a, d), \bar{v}_B(a, d)), \tag{2.4.27}$$

together with the difference equation

$$\begin{aligned}
 0 &= d[v_i + v_{i-1} - 2u_i] + g(u_i; a), \\
 0 &= d[u_{i+1} + u_i - 2v_i] + g(v_i; a).
 \end{aligned} \tag{2.4.28}$$

Applying a shift to the first equation, we obtain the implicit system

$$\begin{aligned}
 v_{i+1} &= 2\left[u_{i+1} - \frac{g(u_{i+1}; a)}{2d}\right] - v_i, \\
 u_{i+1} &= 2\left[v_i - \frac{g(v_i; a)}{2d}\right] - u_i,
 \end{aligned} \tag{2.4.29}$$

which can be written as

$$\begin{aligned}
 v_{i+1} &= 2v_{a,d}(u_{i+1}) - v_i, \\
 u_{i+1} &= 2u_{a,d}(v_i) - u_i.
 \end{aligned} \tag{2.4.30}$$

In particular, we can obtain (u_{i+1}, v_{i+1}) by first reflecting (u_i, v_i) horizontally through the curve $u = u_{a,d}(v)$ and then vertically through the curve $v = v_{a,d}(u)$. Based on this geometric intuition, we set out to construct a rectangle

$$[\bar{u}_{\text{top}}(a, d), \bar{u}_B(a, d)] \times [\bar{v}_{\text{top}}(a, d), \bar{v}_B(a, d)] \tag{2.4.31}$$

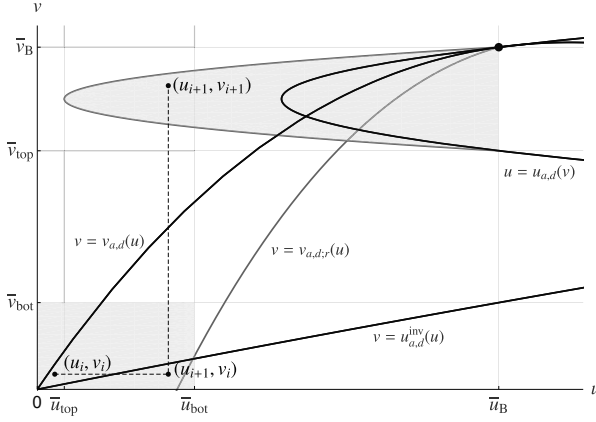


Figure 2.9: Illustration of the two reflections described by the system (2.4.30). The values have been modified for illustrative purposes, since the real regions are minuscule.

that must contain (u_{i_0+1}, v_{i_0+1}) for some critical i_0 , together with a rectangle

$$[0, \bar{u}_{\text{bot}}(a, d)] \times [0, \bar{v}_{\text{bot}}(a, d)] \quad (2.4.32)$$

that must contain the intermediate point (u_{i_0+1}, v_{i_0}) ; see Figure 2.9.

Lemma 2.4.7. There exist continuous functions

$$(\bar{v}_{\text{bot}}, \bar{v}_{\text{top}}) : \Omega_- \rightarrow [0, 1]^2 \quad (2.4.33)$$

that satisfy the inequalities

$$0 < \bar{v}_{\text{bot}}(a, d) < \gamma_{c;-}(a, d) < \bar{v}_{\text{top}}(a, d) < \gamma_{c;+}(a, d) < \bar{v}_B(a, d) \quad (2.4.34)$$

together with the identities

$$u_{a,d}(\bar{v}_{\text{bot}}(a, d)) = u_{a,d}(\bar{v}_{\text{top}}(a, d)) = \bar{u}_B(a, d) \quad (2.4.35)$$

for each $(a, d) \in \Omega_-$. Furthermore, these functions can be continuously extended to $\bar{\Omega}_-$ in such a way that (2.4.35) holds whenever $d > 0$.

Proof. Pick any $(a, d) \in \Omega_-$. On account of the identity $u_{a,d}(0) = 0$ and the inequalities

$$u_{a,d}(\gamma_{c;+}(a, d)) < u_{a,d}(\bar{v}_B(a, d)) = \bar{u}_B(a, d) < a = u_{a,d}(a), \quad (2.4.36)$$

the equation $u_{a,d}(v) = \bar{u}_B(a, d)$ has three distinct solutions on $(0, \bar{v}_B(a, d)]$. Using the fact that $\bar{u}_B(a, 0) = 0$, these solutions can be extended continuously to $d = 0$ by writing $\bar{v}_{\text{bot}}(a, 0) = 0$ and $\bar{v}_{\text{top}}(a, 0) = a$. The extension to $d = d_-(a) > 0$ can be achieved by standard continuity arguments. $\square$

For any $(a, d) \in \overline{\Omega}_-$ with $d > 0$ we now define the constant

$$\bar{u}_{\text{top}}(a, d) = 2u_{a,d}(\gamma_{c;+}(a, d)) - \bar{u}_B(a, d). \quad (2.4.37)$$

We note that $(\bar{u}_{\text{top}}(a, d), \gamma_{c;+}(a, d))$ can be seen as the horizontal reflection of $(\bar{u}_B(a, d), \gamma_{c;+}(a, d))$ through the curve $u = u_{a,d}(v)$.

Lemma 2.4.8. Pick any $(a, d) \in \Omega_-$ for which $c(a, d) = 0$ and consider the sequence $\{(u_i, v_i)\}$ defined in (2.4.26). Then there exists $i_0 \in \mathbb{Z}$ for which

$$(\bar{u}_{\text{top}}(a, d), \bar{v}_{\text{top}}(a, d)) \leq (u_{i_0+1}, v_{i_0+1}) \leq (\bar{u}_B(a, d), \bar{v}_B(a, d)), \quad (2.4.38)$$

while

$$(0, 0) \leq (u_{i_0}, v_{i_0}) \leq (\bar{u}_B(a, d), \bar{v}_{\text{bot}}(a, d)). \quad (2.4.39)$$

Proof. For any $i \in \mathbb{Z}$ we have the inequalities

$$u_i \leq u_{i+1} \leq \bar{u}_B(a, d), \quad (2.4.40)$$

which implies that we must have $u_i \leq u_{a,d}(v_i) \leq u_{i+1}$. In particular, we see that

$$u_{a,d}(v_i) \leq \bar{u}_B(a, d), \quad (2.4.41)$$

which implies that

$$v_i \notin (\bar{v}_{\text{bot}}(a, d), \bar{v}_{\text{top}}(a, d)). \quad (2.4.42)$$

In addition, we have

$$u_i = 2u_{a,d}(v_i) - u_{i+1} \geq 2u_{a,d}(v_i) - \bar{u}_B(a, d). \quad (2.4.43)$$

In particular, for every i we either have

$$(0, 0) \leq (u_i, v_i) \leq (\bar{u}_B(a, d), \bar{v}_{\text{bot}}(a, d)) \quad (2.4.44)$$

or

$$(\bar{u}_{\text{top}}(a, d), \bar{v}_{\text{top}}(a, d)) \leq (u_i, v_i) \leq (\bar{u}_B(a, d), \bar{v}_B(a, d)). \quad (2.4.45)$$

The limits (2.4.27) imply that there exists $M \gg 1$ so that (2.4.44) holds for all $i \leq -M$ and (2.4.45) holds for all $i \geq M$. In particular, there must be jump between these two sets. $\square$

For any $(a, d) \in \overline{\Omega}_-$ with $d > 0$, the inequalities (2.4.34) imply that the function $u_{a,d}$ is strictly increasing on $[0, \bar{v}_{\text{bot}}(a, d)]$. This allows us to define an inverse

$$u_{a,d}^{\text{inv}} : [0, \bar{u}_B(a, d)] \rightarrow [0, \bar{v}_{\text{bot}}(a, d)]. \quad (2.4.46)$$

In addition, we introduce the function

$$v_{a,d;r}(u) = 2v_{a,d}(u) - \bar{v}_B(a, d), \quad (2.4.47)$$

which can be interpreted as the vertical reflection of the line $v = \bar{v}_B(a, d)$ through the curve $v = v_{a,d}(u)$.

For any $(a, d) \in \bar{\Omega}_-$ with $d > 0$, these definitions allow us to introduce the notation

$$\bar{u}_{\text{bot}}(a, d) = \max\{u \in [0, \bar{u}_B(a, d)] : v_{a,d;r}(u) = u_{a,d}^{\text{inv}}(u)\}. \quad (2.4.48)$$

On account of the inequalities

$$v_{a,d;r}(0) < 0 = u_{a,d}^{\text{inv}}(0) < u_{a,d}^{\text{inv}}(\bar{u}_B(a, d)) = \bar{v}_{\text{bot}}(a, d) < \bar{v}_B(a, d) = v_{a,d;r}(\bar{u}_B(a, d)), \quad (2.4.49)$$

one can verify that $\bar{u}_{\text{bot}}$ is well-defined and continuous in (a, d) .

Lemma 2.4.9. Pick $(a, d) \in \Omega_-$ and assume that

$$\bar{u}_{\text{bot}}(a, d) < \bar{u}_{\text{top}}(a, d) \quad (2.4.50)$$

holds. Then we have $c(a, d) > 0$.

Proof. Suppose the contrary that $c(a, d) = 0$ and consider the sequence $\{(u_i, v_i)\}$ defined by (2.4.26), together with the critical value $i_0 \in \mathbb{Z}$ that appears in Lemma 2.4.8. Since the sequence $\{u_i\}$ is non-decreasing, we have $u_{i_0+1} \geq u_{i_0}$ which implies that

$$u_{i_0+1} \geq u_{a,d}(v_{i_0}). \quad (2.4.51)$$

Exploiting $0 \leq v_{i_0} \leq \bar{v}_{\text{bot}}(a, d)$ this gives

$$0 \leq v_{i_0} \leq u_{a,d}^{\text{inv}}(u_{i_0+1}). \quad (2.4.52)$$

On the other hand, the inequality $v_{i_0+1} \leq \bar{v}_B(a, d)$ yields

$$v_{i_0} \geq v_{a,d;r}(u_{i_0+1}). \quad (2.4.53)$$

Combining this with (2.4.52), we see that $u_{i_0+1} \leq \bar{u}_{\text{bot}}(a, d)$. However, (2.4.38) implies that $u_{i_0+1} \geq \bar{u}_{\text{top}}(a, d)$, which yields the desired contradiction. $\square$

2.4.3 Verification of $\bar{u}_{\text{top}} > \bar{u}_{\text{bot}}$

In Figure 2.10 we show where one may numerically verify that the scalar inequality (2.4.50) holds, which ensures that $c(a, d) > 0$. We also plot the curve $d = \frac{(1-a)^2}{8}$, below which we have established that $c(a, d) = 0$. Taken together, we feel that these results cover a reasonable portion of the parameter space Ω_- .

Our final task is to analytically verify (without resorting to any numerics) that $c(a, d) > 0$ near the cusp $(a, d) = (\frac{1}{2}, \frac{1}{24})$ and the corner $(a, d) = (1, 0)$. As a preparation, we construct a simplified but weaker version of the condition $\bar{u}_{\text{top}} > \bar{u}_{\text{bot}}$ by exploiting the monotonicity of $v_{a,d;r}$.

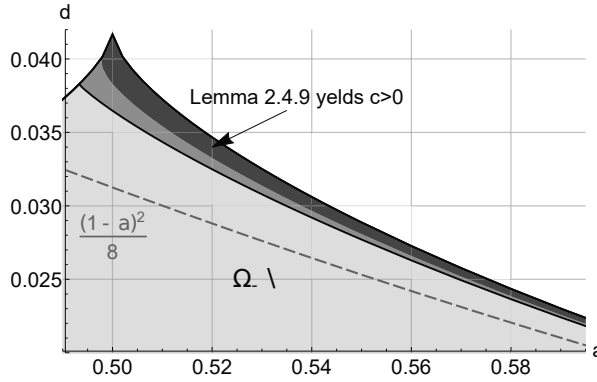


Figure 2.10: The darkest region contains all pairs (a, d) where the assumption $\bar{u}_{\text{bot}}(a, d) < \bar{u}_{\text{top}}(a, d)$ was verified numerically. We also include the boundaries of the sets $\mathcal{T}$ and Ω_- as computed numerically by the procedure described in §2.2.

Lemma 2.4.10. Pick $(a, d) \in \bar{\Omega}_-$ with $d > 0$. Then the function $v_{a,d;r}$ is strictly increasing on $[0, \bar{u}_B(a, d)]$.

Proof. We note first that $v'_{a,d}(u) = 1 - \frac{g'(u;a)}{2d} > 0$ for $u \in [0, u_{\min}(a)]$. In particular, we only have to consider the case $\bar{u}_B(a, d) > u_{\min}(a)$, which cannot occur for $a = \frac{1}{2}$.

Let us first assume that $0 < a < \frac{1}{2}$. Using (2.3.187) or (2.3.188) we may conclude that

$$v'_{a,d}(\bar{u}_B(a, d)) \geq v'_+(\bar{u}_B(a, d)) > 0. \quad (2.4.54)$$

Since $\bar{u}_B(a, d) \leq a \leq \gamma_{c;+}(a, d)$ this implies that $\bar{u}_B(a, d) \leq \gamma_{c;-}(a, d)$. In particular, $v_{a,d}$ and hence $v_{a,d;r}$ are strictly increasing on $[0, \bar{u}_B(a, d)]$.

It remains to consider $a \in (\frac{1}{2}, 1)$. Since $\bar{v}_B(1-a, d) > u_{\max}(1-a)$, we may use Corollary 2.3.4 to conclude

$$\bar{u}_B(a, d) = 1 - \bar{v}_B(1-a, d) < 1 - u_{\max}(1-a) = u_{\min}(a). \quad (2.4.55)$$

□

Corollary 2.4.11. Consider any $(a, d) \in \bar{\Omega}_-$ with $d > 0$ and suppose that

$$v_{a,d;r}(\bar{u}_{\text{top}}(a, d)) > \bar{v}_{\text{bot}}(a, d). \quad (2.4.56)$$

Then we have $\bar{u}_{\text{top}}(a, d) > \bar{u}_{\text{bot}}(a, d)$.

Proof. This follows from the uniform bound $u_{a,d}^{\text{inv}} \leq \bar{v}_{\text{bot}}(a, d)$ and the fact that $v_{a,d;r}$ is strictly increasing. □

We now set out to verify the explicit condition (2.4.56) for the boundary points $(a, d_-(a))$ with $a \sim 1$ and $a = \frac{1}{2}$. Using the continuity of $\bar{u}_{\text{top}}$ and $\bar{u}_{\text{bot}}$, this means that $c(a, d) > 0$ for all $(a, d) \in \Omega_-$ that are sufficiently close to these critical boundary points.

Lemma 2.4.12. We have the expansions

$$\begin{aligned}\bar{u}_B(a, d_-(a)) &= \frac{1}{4}(1-a)^2 + \frac{1}{8}(1-a)^3 + O((1-a)^4), \\ \bar{v}_B(a, d_-(a)) &= 1 - \frac{1}{2}(1-a) + O((1-a)^4)\end{aligned}\tag{2.4.57}$$

as $a \uparrow 1$.

Proof. Exploiting Corollary 2.3.4 together with the symmetry $d(a) = d(1-a)$, we obtain

$$\bar{u}_B(a, d_-(a)) = 1 - \bar{v}_B(1-a, d_-(1-a)), \quad \bar{v}_B(a, d_-(a)) = 1 - \bar{u}_B(1-a, d_-(1-a)).\tag{2.4.58}$$

The desired expansions hence follow from Proposition 2.3.6. $\square$

Lemma 2.4.13. We have the expansions

$$\begin{aligned}\bar{u}_{\text{top}}(a, d_-(a)) &= \frac{1}{4}(a-1)^2 + O((1-a)^3), \\ v_{a, d_-(a); r}(\bar{u}_{\text{top}}(a, d_-(a))) &= 1 + O(1-a)\end{aligned}\tag{2.4.59}$$

as $a \uparrow 1$.

Proof. These expansions can be found by substitution of (2.4.57) into the definitions (2.4.37) and (2.4.47). $\square$

On account of the identity $\gamma_{c;-}(1, 0) = \frac{1}{3}$ and the inequality $\bar{v}_{\text{bot}}(a, d) \leq \gamma_{c;-}(a, d)$, we see that Corollary 2.4.11 implies that

$$\bar{u}_{\text{top}}(a, d_-(a)) > \bar{u}_{\text{bot}}(a, d_-(a))\tag{2.4.60}$$

whenever $1-a > 0$ is sufficiently small.

Lemma 2.4.14. The inequality (2.4.56) holds for $(a, d) = (\frac{1}{2}, \frac{1}{24})$.

Proof. Writing $(a_{\text{cp}}, d_{\text{cp}}) = (\frac{1}{2}, \frac{1}{24})$, we can explicitly compute

$$\gamma_{c;+}(a_{\text{cp}}, d_{\text{cp}}) = \frac{1}{2} + \frac{1}{6}\sqrt{2},\tag{2.4.61}$$

which together with the expressions (2.3.26) and (2.4.37) yields

$$\bar{u}_{\text{top}}(a_{\text{cp}}, d_{\text{cp}}) = \frac{1}{2} - \frac{4}{9}\sqrt{2} + \frac{1}{6}\sqrt{3}.\tag{2.4.62}$$

Using (2.4.47) we obtain

$$v_{a_{\text{cp}}, d_{\text{cp}}; r}(\bar{u}_{\text{top}}(a_{\text{cp}}, d_{\text{cp}})) = \frac{1}{2} - \frac{1240}{243}\sqrt{2} + \frac{229}{54}\sqrt{3} \sim 0.6286.\tag{2.4.63}$$

In particular, we have

$$v_{a_{\text{cp}}, d_{\text{cp}}; r}(\bar{u}_{\text{top}}(a_{\text{cp}}, d_{\text{cp}})) > a_{\text{cp}} \geq \gamma_{c;-}(a_{\text{cp}}, d_{\text{cp}}) \geq \bar{v}_{\text{bot}}(a_{\text{cp}}, d_{\text{cp}}),\tag{2.4.64}$$

as desired. $\square$

Proof of Theorem 2.2.3. Items (i), (ii) and (iii) follow from Lemma 2.4.6, Corollary 2.4.4 and Lemma 2.4.14 respectively. Item (iv) follows from Lemma 2.4.5, together with Lemma 2.4.9 and the continuity of the functions $\bar{u}_{\text{bot}}$ and $\bar{u}_{\text{top}}$. Indeed, the expression (2.2.26) can be rewritten as

$$\Gamma(a) = \bar{u}_{\text{top}}(a, d_-(a)) - \bar{u}_{\text{bot}}(a, d_-(a)). \quad (2.4.65)$$

Finally, item (v) follows directly from (2.4.60). $\square$