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Travelling Patterns on Discrete Media

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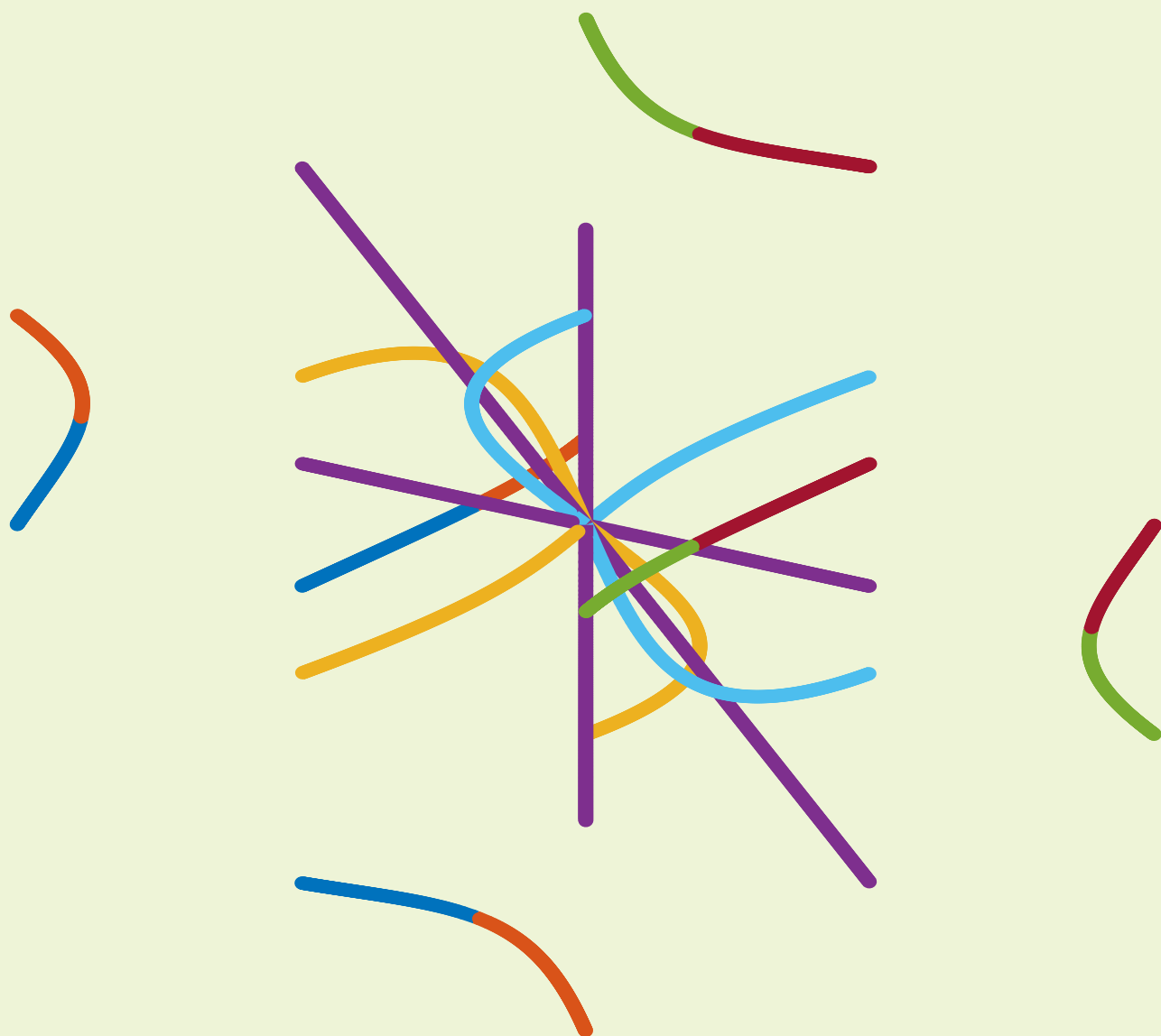
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Travelling Patterns on Discrete Media



Leonardo Morelli

Travelling Patterns on Discrete Media

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Chapter 1

Introduction

The main theme of this thesis is the study of travelling patterns that arise as solutions of the Nagumo problem on the regular lattices $\mathbb{Z}$ and $\mathbb{Z}^2$. As the field of Lattice Differential Equations (LDEs) is relatively young compared to the classic Partial Differential Equations (PDEs) field, our knowledge of the former is pretty limited in comparison to the latter. We can divide the results for LDEs into two types.

On the one hand, we can try generalize ideas and techniques developed for PDEs, often running into technical issues. In Chapter 4 we will present some results of this type, where corner defect solutions found in the PDE setting by Haragus and Scheel [61, 60] can be constructed for LDEs as well.

On the other hand, we can think of phenomena that only occur in the spatially discrete setting but that do not find a direct translation, or at least not a well defined one, in the world of PDEs. As we will see, the results obtained in Chapters 2 and 3 regarding solutions that are asymptotically spatially periodic belong to the lattice setting and to the lattice setting only. Indeed, the structures we consider there cannot be transferred to a spatially continuous domain.

1.1 Modelling with LDEs

There are many systems in the real world where it seems more appropriate to use a spatially discrete model rather than a continuous one. Indeed, lattice equations are used in many different scientific areas [84] ranging from physics [30, 64] to dynamics of DNA strands [85, 103]. Furthermore, it is very common for numerical simulations to replace continuous variables by their discretized approximations.

Let us imagine a chain of elements in which every particle u_j is influenced by the behaviour of the two adjacent ones, u_{j-1} and u_{j+1} . We can think of modelling such behaviour with a system of the form

$$\dot{u}_j(t) = G(u_{j-1}(t), u_j(t), u_{j+1}(t)). \quad (1.1.1)$$

This is the general form of a LDE with nearest-neighbour interactions. It consists of a countable number of Ordinary Differential Equations (ODEs) that all depend on the two adjacent elements. However, we can also think of longer or even infinite range interactions [113, 124].

We are interested in solutions that take the form of a fixed shape that moves in time with some constant speed c , that is, a solution of the form $u_j(t) = \varphi(j - ct)$. We immediately see that φ would need to satisfy a particular equation, namely

$$-c\varphi'(\xi) = G(\varphi(\xi - 1), \varphi(\xi), \varphi(\xi + 1)). \quad (1.1.2)$$

Such problems are called Mixed-type Functional Differential Equations (MFDEs), due to the presence of advanced and delayed terms. Unfortunately, these problems are in general not well posed. Indeed, equation (1.1.2) needs an initial condition in the infinite dimensional space $C^0([-1, 1], \mathbb{R})$, which is not necessarily compatible with the equation itself.

Although the study of MFDEs is a fairly recent endeavour, a reasonably solid basic framework has been developed. See for example [109, 110, 93, 62] for general theory and [21, 91, 139, 140] for more focused results on travelling solutions.

FPUT equation The Fermi-Pasta-Ulam-Tsingou problem, first introduced and studied by Enrico Fermi, John Pasta, Stan Ulam and Mary Tsingou [49, 34], was designed to model a nonlinear wave in a discrete spatial setting where single atoms are chained one next to the other. This was the first attempt to model and numerically study a crystal in a one-dimensional setting. The springs between two consecutive atoms are assumed to follow Hooke's law, thus creating a linear interaction with the neighbours. Furthermore, nonlinear correction terms are taken into account, typically of quadratic (FPUT- α)

$$\ddot{u}_j = (u_{j+1} - 2u_j + u_{j-1}) + \alpha ((u_{j+1} - u_j)^2 - (u_j - u_{j-1})^2), \quad (1.1.3)$$

or cubic (FPUT- β) type

$$\ddot{u}_j = (u_{j+1} - 2u_j + u_{j-1}) + \beta ((u_{j+1} - u_j)^3 - (u_j - u_{j-1})^3). \quad (1.1.4)$$

While the authors conjectured that an initial localized impulse would eventually spread uniformly through the whole system, they later numerically found that the initial configuration was a recurring state of the system. This discovery came as a surprise, since the common idea at the time was that nonlinearity is a sufficient condition for energy diffusion. Years later, Zabusky and Kruskal [138] further numerically investigated the FPUT equation, finding a close relation to the Korteweg-de Vries (KdV) PDE

$$u_t + 6uu_x + u_{xxx} = 0, \quad (1.1.5)$$

a problem known to admit explicit soliton solutions. It took more than thirty additional years, however, to analytically establish the existence of soliton solutions to the FPUT problem [54].

Toda lattice The Toda lattice [125, 126] is a classical example of a fully integrable discrete system that also admits solitary waves as solution. It can be written as

$$\ddot{u}_j = e^{-(u_j - u_{j-1})} - e^{-(u_{j+1} - u_j)}. \quad (1.1.6)$$

Here $u_j(t)$ represents the horizontal displacement of the j th element at time t . The Toda lattice system has been used to model certain mechanics of DNA strands at body temperature [97, 98].

It was shown by Chen and Liu [26] that a discrete Bäcklund Transform can be used to construct more complex solutions to (1.1.6) by combining multiple solitary waves. The stability of such n -soliton solutions was proved only recently by Benes, Hoffman and Wayne [11] through the use of so-called Bäcklund Transform ladders, where the Bäcklund Transform is applied iteratively to reduce the complexity of the problem.

FitzHugh-Nagumo The FitzHugh-Nagumo model is a two component system devised by FitzHugh [52] and Nagumo [99] in order to recreate the dynamics of signal transmissions along the axon of a neuron. The axon is a nerve fiber which is covered by a myelin coating. However, this coating admits gaps at regular distances (about every 1mm) along the fiber. These gaps are about $1\mu\text{m}$ in length and are called Nodes of Ranvier.

The model tries to encompass the most essential characteristics of a ion flow in a sodium-potassium media via two equations. The first one is related to the voltage, while the second one describes a recovery variable. The strength of this approach is that it is able to capture the recessive phase that appears right after the passage of a strong enough signal.

The standard formulation in continuous spatial coordinates is

$$\begin{cases} \dot{v} &= v_{xx} + g(v, a) - w, \\ \dot{w} &= \rho(v - \gamma w), \end{cases} \quad (1.1.7)$$

in which $g(v, a) = v(1 - v)(v - a)$ is a bistable cubic nonlinearity and ρ and γ are positive parameters. A great deal of attention has been spent on this system: the singular limit $\rho \downarrow 0$ regime has for example strongly influenced the development of geometric singular perturbation theory; see [24, 63, 78, 79, 80, 86].

It turns out to be advantageous to consider how signals jump from one Node of Ranvier to the next, while ignoring the dynamics through the myelin coating between the two nodes [9, 72]. In particular, let us switch to a spatially discrete setting by replacing $v(x, t)$ and $w(x, t)$ with $v_j(t)$ and $w_j(t)$ respectively. In addition we replace the diffusion v_{xx} by its nearest neighbour counterpart $v_{j-1} - 2v_j + v_{j+1}$, thus obtaining

$$\begin{cases} \dot{v}_j &= (v_{j-1} - 2v_j + v_{j+1}) + g(v_j, a) - w_j, \\ \dot{w}_j &= \rho(v_j - \gamma w_j). \end{cases} \quad (1.1.8)$$

In recent years there have been substantial developments regarding the existence [72, 48, 114] and stability [73, 113] of travelling pulses for (1.1.8) and some of its extensions.

Nagumo In 1960 John W. Cahn devised a model to describe the growth of crystals on atomic scales [20]. The main feature of this model is that the boundary of the crystalline formation can be represented by a front that moves slowly through the domain. The PDE that he considered is

$$u_t = u_{xx} + g(u, a), \quad (1.1.9)$$

with $g(u, a) = u(1 - u)(u - a)$.

A few years later Mitio Nagumo [94] independently used the same equation to describe similar dynamics among populations invading spatial domains. For this reason equation (1.1.9) is known both as the Allen-Cahn [2] and the Nagumo equation. It is well known [50] that equation (1.1.9) admits a solution in the form of a travelling front $u(x, t) = \varphi(x - ct)$, for some $c \in \mathbb{R}$ and wave profile φ that satisfies

$$\lim_{\xi \rightarrow -\infty} \varphi(\xi) = 0, \quad \lim_{\xi \rightarrow \infty} \varphi(\xi) = 1. \quad (1.1.10)$$

If we plug this Ansatz into (1.1.9) we obtain

$$-c\varphi'(\xi) = \varphi''(\xi) + g(\varphi(\xi), a), \quad (1.1.11)$$

a fairly standard second order ODE.

The natural way to transfer this model to a spatially discrete setting is to write

$$\dot{u}_j = d(u_{j-1} - 2u_j + u_{j+1}) + g(u_j, a), \quad (1.1.12)$$

with $j \in \mathbb{Z}$. Here the diffusion coefficient d is directly related to the discretization distance h . Indeed, using the approximation $u_{xx}(x) = (u(x - h) - 2u(x) + u(x + h))/h^2 + O(h^3)$ and assuming $u_j = u(jh)$, we see that is natural to take $d \sim h^{-2}$ to bring us from the continuous to the discrete formulation.

If we now look for a travelling front of the form $u_j(t) = \varphi(j - ct)$, we find that such a solution must satisfy

$$-c\varphi'(\xi) = d(\varphi(\xi - 1) - 2\varphi(\xi) + \varphi(\xi + 1)) + g(\varphi(\xi), a). \quad (1.1.13)$$

In the PDE setting it is easy to show that the speed coefficient c of the travelling solution is negative when $a < \frac{1}{2}$ and positive if $a > \frac{1}{2}$. This also implies that the wavespeed vanishes¹ whenever $a = \frac{1}{2}$. When we move to the LDE setting however, one encounters a new phenomenon, called *propagation failure*. More specifically, there exists a small buffer interval $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$ for a inside of which we have pinned solutions (i.e. $c = 0$). This phenomenon has been studied extensively, both analytically [21, 67,

¹due to the point symmetry that $g(u, \frac{1}{2}) = u(\frac{1}{2} - u)(1 - u)$ possesses with respect to $u = \frac{1}{2}$.

70, 83, 92, 40] and numerically [1, 42, 44, 55]. Propagation failure is guaranteed to occur when the diffusion coefficient d is sufficiently small, but in general it is highly dependent on the shape of the nonlinearity g . In Chapter 2 we will see some numerical simulations for the null speed threshold in the parameter region $(a, d) \in [0, 1] \times \mathbb{R}^+$.

Throughout this thesis we will analyze the Lattice Nagumo equation (1.1.12) together with its planar counterpart. In particular, we show that it is possible to obtain a range of solutions that can be seen as variations of the classical travelling waves.

1.2 Multichromatic travelling waves

Many examples in nature, ranging from polymer dynamics [122] to DNA strings [35, 135], feature spatial structures that are not constant but rather periodically distributed. This tells us that besides considering globally constant steady states and parameters, it might be worthwhile to investigate systems that exhibit periodic behaviour or possess periodic parameters.

While [46] and [114] analyze LDEs with periodic parameters, in this thesis we will investigate periodic steady states. More in detail, we will consider the Nagumo problem on regular lattices and look for waves that have n -periodical asymptotic values.

1.2.1 Bichromatic waves

As a first step, let us separate the equations for the even and odd nodes in the lattice. This can be achieved by looking for solutions to (1.1.12) of the form

$$u_j(t) = \begin{cases} \Phi_e(j - ct) & \text{if } j \text{ is even,} \\ \Phi_o(j - ct) & \text{if } j \text{ is odd,} \end{cases} \quad (1.2.1)$$

for some speed c and profile waves $(\Phi_o, \Phi_e) : \mathbb{R} \rightarrow \mathbb{R}^2$. If we plug this Ansatz into (1.1.12) we obtain

$$\begin{cases} -c\Phi_o'(\xi) = d(\Phi_e(\xi - 1) + \Phi_e(\xi + 1) - 2\Phi_o(\xi)) + g(\Phi_o(\xi), a), \\ -c\Phi_e'(\xi) = d(\Phi_o(\xi - 1) + \Phi_o(\xi + 1) - 2\Phi_e(\xi)) + g(\Phi_e(\xi), a). \end{cases} \quad (1.2.2)$$

Rather than considering asymptotic conditions such as (1.1.10), we can now impose the two-periodic limit values

$$\lim_{\xi \rightarrow -\infty} (\Phi_o(\xi), \Phi_e(\xi)) = (0, 0), \quad \lim_{\xi \rightarrow \infty} (\Phi_o(\xi), \Phi_e(\xi)) = (\bar{u}_o, \bar{u}_e), \quad (1.2.3)$$

for some $\bar{u}_o, \bar{u}_e \in (0, 1)$. The couple $(\bar{u}_o, \bar{u}_e)$ in (1.2.3) cannot be chosen arbitrarily, but it must satisfy the stationary conditions

$$\begin{cases} 0 = 2d(\bar{u}_e - \bar{u}_o) + g(\bar{u}_o, a), \\ 0 = 2d(\bar{u}_o - \bar{u}_e) + g(\bar{u}_e, a). \end{cases} \quad (1.2.4)$$

From (1.2.4) we see that $\bar{u}_o$ and $\bar{u}_e$ must satisfy

$$g(\bar{u}_o, a) + g(\bar{u}_e, a) = 0, \quad (1.2.5)$$

independently from the value of d . Assuming $a \leq 1/2$, we can define for $u \in (0, a)$ the functions

$$\begin{aligned} u_e^-(u; a) &= \min\{g^{-1}(-g(u, a), a)\}, \\ u_e^+(u; a) &= \max\{g^{-1}(-g(u, a), a)\}. \end{aligned} \quad (1.2.6)$$

If $u \in \{0, a\}$, then we can use continuity to define

$$u_e^-(0; a) = u_e^-(a; a) = a, \quad u_e^+(0; a) = u_e^+(a; a) = 1. \quad (1.2.7)$$

Geometrically, for any given $u \in [0, a]$ these functions represent the two solutions that (1.2.5) possesses.

Inspired by the first equation of (1.2.4), let us furthermore define the function

$$u_e^d(u; d, a) := u - g(u, a)/2d. \quad (1.2.8)$$

All that is needed for a couple $(\bar{u}_o, \bar{u}_e)$ to be a solution of (1.2.4) is that $u_e^\pm(\bar{u}_o; a) = u_e^d(\bar{u}_o; d, a) = \bar{u}_e$, see the left graph in Figure 1.1. Motivated by this, let us introduce the set of couples $(a, d) \in [0, 1] \times \mathbb{R}^+$ for which we have two distinct solutions to $u_e^+(\bar{u}_o) = u_e^d(\bar{u}_o; d, a)$ with $\bar{u}_o \in (0, 1)$, and let us call this set Ω_{bc} , see the right graph in Figure 1.1.

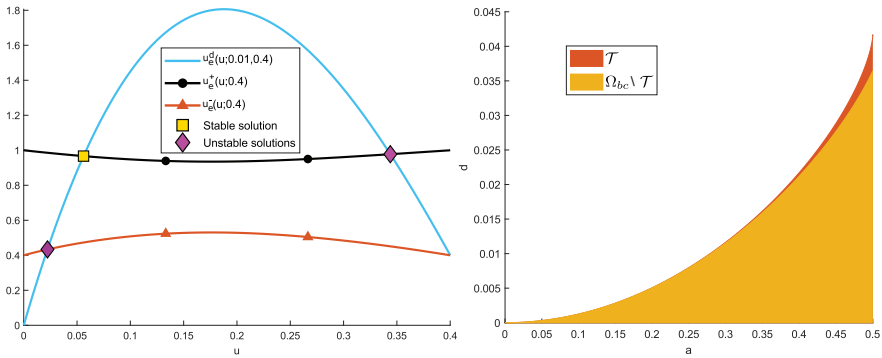


Figure 1.1: On the left, the graphs of u_e^d and $u_e^\pm$ for $(a, d) = (0.4, 0.01) \in \Omega_{bc}$. On the right, the region Ω_{bc} and its subregion $\mathcal{T}$ where we find a nonzero speed for the front connecting the heterogeneous stable steady state to the homogeneous state $(1, 1)$. For $a \in [0.5, 1]$ these regions can be reflected through the line $a = \frac{1}{2}$.

Travelling fronts The general theory in [27] allows us to find wave solutions connecting the stable heterogeneous steady states with the homogeneous ones $(0, 0)$ and $(1, 1)$. A natural question is, thus, whether such solutions actually travel or stand still. Inside of Ω_{bc} we can find a region $\mathcal{T}$, see the right graph in Figure 1.1, where the wave profile connecting the stable heterogeneous root to the homogeneous state $(1, 1)$ has a nonzero speed.

In Chapter 2 we will analyze this region in depth. We will manage to find analytic expansions for Ω_{bc} 's upper boundary around $a = 0$ (and by symmetry $a = 1$) and around its cusp in $(a, d) = (\frac{1}{2}, \frac{1}{24})$. We also derive a geometric condition to conclude $(a, d) \in \mathcal{T}$, which guarantees that $\mathcal{T}$ is not empty. In Chapter 3 we will investigate trichromatic and quadrichromatic dynamics and we will show that even richer results can be obtained.

1.2.2 Custom nonlinearity

Although in this thesis we will only consider the polynomial nonlinearities $g(u, a) = u(1 - u)(u - a)$ or $g(u, a) = (1 - u^2)(u - a)$, it is interesting to think about what happens with "slightly" different bistable functions.

In Section 2.3 we will see that some computations heavily rely on the geometry of the cubic. Even though such an approach does not appear to be as general as one would hope for, a quick way to recognize its value is to consider a different custom-made bistable nonlinearity, namely

$$g_{cus}(u, a) = \begin{cases} \frac{2(u - a)u(81a^2 - 208au + 176u^2)}{a^3} & u \leq a, \\ \frac{(1-u)(u-a)}{20(a-1)^4} [5292u^3 + u^2(-10116a - 5760) \\ + u(3993a^2 + 12246a - 363) - 1129a^3 - 606a^2 - 5517a + 1960] & u > a. \end{cases} \quad (1.2.9)$$

In accordance with the third order polynomial g , we see that g_{cus} has only one local minimum in $[0, a]$ and one local maximum in $[a, 1]$. Moreover, we have $g_{cus}(\cdot, a) \in C^1([0, 1])$, $\forall a \in [0, 1]$. In Figure 1.2 we can see the graph of $g_{cus}(\cdot, 0.4)$ together with a multiple of the standard cubic $g(u, a) = u(1 - u)(u - a)$.

Due to the shape that g_{cus} has in $[0, a]$, we can observe from Figure 1.3 that the function $u_e^d(u; d, a)$ can have two local maximum points. In particular, we can define a region $\Omega_{bc}^{(1)}$ where u_e^d and u_e^+ have two intersection points, together with a region $\Omega_{bc}^{(2)}$ where $u_e^+(u; a)$ and $u_e^d(u; d, a)$ cross in four different points.

Travelling fronts It is possible to numerically check if there exist moving connections between the various steady states. As before, in $\Omega_{bc}^{(1)}$ there is a region $\mathcal{T}$ where the heterogeneous stable steady state is connected to 1 via a moving front. In $\Omega_{bc}^{(2)}$, however, we do not find any wave solution that possesses a nonzero speed and

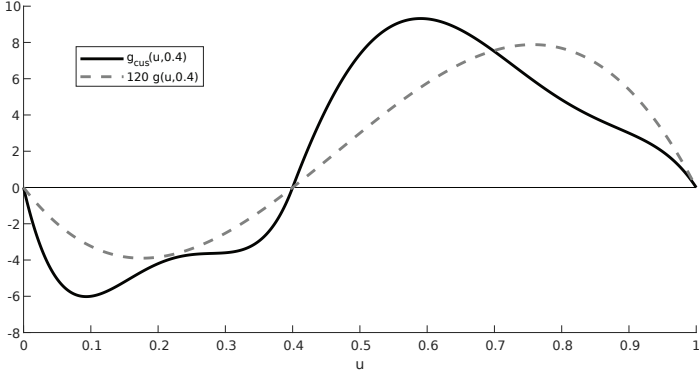


Figure 1.2: Graph of $g_{cus}(\cdot, 0.4)$ compared to a multiple of the standard cubic $g(\cdot, 0.4)$. Through (1.2.4) and (1.2.8) we see how a multiplication of the nonlinearity g or g_{cus} can be dealt with by rescaling d in the same way.

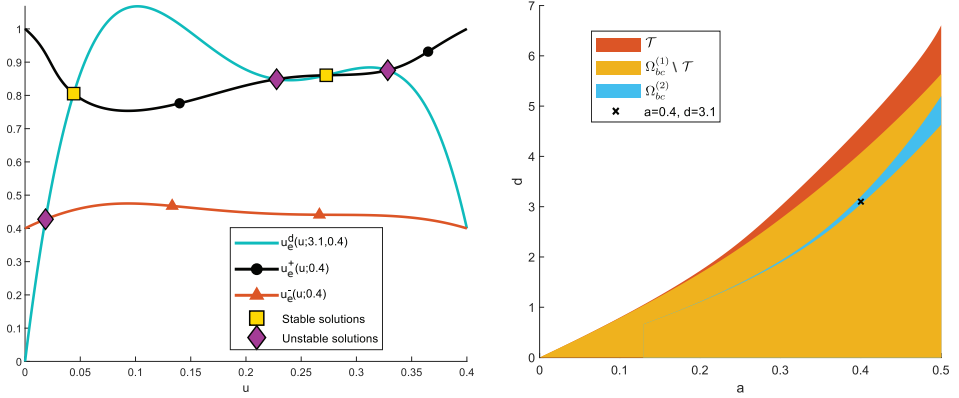


Figure 1.3: On the left, we can see the intersections between u_e^d with u_e^+ and u_e^- for $a = 0.4$ and $d = 3.1$ for g_{cus} . On the right, the numerically computed regions $\Omega_{bc}^{(1)}$ and $\Omega_{bc}^{(2)}$, together with the $\mathcal{T}$ region.

connects any two of the four stable steady states (namely 0, 1 and the two stable heterogeneous solutions depicted in Figure 1.3).

1.3 Planar systems

Let us now explore systems with two spatial dimensions. What might first come to mind when thinking about dynamics evolving on bidimensional surfaces, especially if you happen to live in the Netherlands, is probably the weather forecast [107, 88]. There is, however, an incredible richness of other phenomena that can be efficiently studied with planar models, for example in biology [127, 116, 39], chemistry [133] or signal processing [33].

When increasing the number of spatial dimensions, we usually find two types of theoretical challenges. The first one is how to transfer results from lower dimensional frames into the higher dimensional setting, which often brings new technical obstacles. In the next pages we will shortly introduce some possible planar extensions to mono- and multichromatic dynamics. Even though we only analyze unidimensional lattices in Chapters 2 and 3, we will see in the following pages that most of the theory can be directly exported to bidimensional lattice dynamics.

The second challenge revolves around the identification of new phenomena that can arise. In Chapter 4 we will discuss a class of bidimensional travelling patterns that does not possess any lower dimensional counterpart.

1.3.1 Bidimensional Nagumo PDE

Let us consider the extension of the standard Nagumo PDE (1.1.9) onto the two-dimensional plane $\mathbb{R}^2$. We now have to take care of the diffusion operator and make sure that it encompasses both spatial directions, so let us write

$$u_t = (\partial_{xx} + \partial_{yy})u + g(u, a). \quad (1.3.1)$$

Looking for a solution that travels along a direction $\theta \in [0, 2\pi]$, we write

$$u(x, y, t) = \varphi(\cos(\theta)x + \sin(\theta)y - ct). \quad (1.3.2)$$

The equation that needs to be solved by φ is now

$$-c\varphi'(\xi) = (\cos^2(\theta) + \sin^2(\theta))\varphi''(\xi) + g(\varphi(\xi), a) = \varphi''(\xi) + g(\varphi(\xi), a), \quad (1.3.3)$$

which coincides perfectly with (1.1.11). Note that this ODE does not depend on the angle θ , due to the radial symmetry that $\mathbb{R}^2$ possesses.

Nevertheless, the planar setting does pose some challenges. We must remark now that a front travelling on $\mathbb{R}$ only possesses a bounded area where it takes values u with $0 < \varepsilon < u < 1 - \varepsilon < 1$, for any $\varepsilon > 0$. In higher dimensions however we usually find an infinite region where the solution takes such values. One must therefore take care when choosing an appropriate class of admissible perturbations.

Nevertheless in [82] Kapitula manages to prove the asymptotic stability of multi-dimensional travelling fronts. In addition, in [13] the authors establish the asymptotic stability of bidimensional fronts when encountering bounded obstacles, while in [4] localized initial profiles are shown to be able to expand through the whole spatial domain.

1.3.2 Square lattices

The most natural way to formulate the Nagumo equation on $\mathbb{Z}^2$ is to include the four nearest neighbours into the action of the diffusion operator, so that (1.1.12) becomes

$$\dot{u}_{i,j}(t) = d(u_{i,j-1}(t) + u_{i,j+1}(t) + u_{i-1,j}(t) + u_{i+1,j}(t) - 4u_{i,j}(t)) + g(u_{i,j}(t), a). \quad (1.3.4)$$

Let us consider again travelling solutions moving along the direction θ , by writing

$$u_{i,j}(t) = \varphi(\cos(\theta)i + \sin(\theta)j - ct). \quad (1.3.5)$$

In order to solve (1.3.4), we see that the pair (c, Φ) needs to satisfy

$$-c\varphi'(\xi) = d(\varphi(\xi + \cos(\theta)) + \varphi(\xi - \cos(\theta)) + \varphi(\xi + \sin(\theta)) + \varphi(\xi - \sin(\theta)) - 4\varphi(\xi)) + g(\varphi(\xi), a), \quad (1.3.6)$$

where we can clearly see a dependency on the angle θ . Solutions of this type were constructed in [91].

As in the unidimensional case, the propagation failure phenomenon plays an important role in the study of (1.3.4). Due to the presence of the new angular parameter θ , the speed now depends greatly on the direction the travelling front is facing. This dependence can be rather intricate, up to the point of waves being pinned in certain directions but travelling in others [21, 66, 67].

1.3.2.1 Bichromatic Nagumo LDE on square lattices

Let us consider the bidimensional system (1.3.4), and let us assume that for some angle $\theta \in [0, 2\pi)$ there exists a travelling solution of the form

$$u_{i,j}(t; \theta) = \begin{cases} \Phi_e(\cos(\theta)i + \sin(\theta)j - ct) & \text{if } i + j \text{ is even,} \\ \Phi_o(\cos(\theta)i + \sin(\theta)j - ct) & \text{if } i + j \text{ is odd.} \end{cases} \quad (1.3.7)$$

If we plug the Ansatz (1.3.7) into (1.3.4), we obtain

$$\begin{cases} -c\Phi_e'(\xi) = d(\Phi_o(\xi + \cos(\theta)) + \Phi_o(\xi - \cos(\theta)) + \Phi_o(\xi + \sin(\theta)) + \Phi_o(\xi - \sin(\theta)) \\ \quad - 4\Phi_e(\xi)) + g(\Phi_e(\xi), a), \\ -c\Phi_o'(\xi) = d(\Phi_e(\xi + \cos(\theta)) + \Phi_e(\xi - \cos(\theta)) + \Phi_e(\xi + \sin(\theta)) + \Phi_e(\xi - \sin(\theta)) \\ \quad - 4\Phi_o(\xi)) + g(\Phi_o(\xi), a), \end{cases} \quad (1.3.8)$$

which closely resembles the MFDE (1.1.13). In the same fashion as in the unidimensional case we see that a steady state couple $(\bar{u}_e, \bar{u}_o)$ would have to satisfy

$$\begin{cases} 0 = 4d(\bar{u}_o - \bar{u}_e) + g(\bar{u}_e, a), \\ 0 = 4d(\bar{u}_e - \bar{u}_o) + g(\bar{u}_o, a), \end{cases} \quad (1.3.9)$$

the solutions of which can be obtained by the ones for (1.2.4) by simply rescaling the diffusion parameter to $\tilde{d} = 2d$.

We can find some additional similarities with the unidimensional dynamic. If we set $\theta = \pi/4 + k\pi/2$, the system (1.3.8) takes the form

$$\begin{cases} -c\Phi'_e(\xi) = 2d(\Phi_o(\xi + \sqrt{2}/2) + \Phi_o(\xi - \sqrt{2}/2) - 2\Phi_e(\xi)) + g(\Phi_e(\xi), a), \\ -c\Phi'_o(\xi) = 2d(\Phi_e(\xi + \sqrt{2}/2) + \Phi_e(\xi - \sqrt{2}/2) - 2\Phi_o(\xi)) + g(\Phi_o(\xi), a), \end{cases} \quad (1.3.10)$$

which can be seen as the unidimensional bichromatic travelling wave problem (1.2.2) on $\frac{\sqrt{2}}{2}\mathbb{Z}$ with a new diffusion parameter $\tilde{d} = 2d$. In Section 4.2 we will numerically examine the full dependence of c on θ .

1.3.3 Hexagonal lattices

Another common type of regular planar lattice that is encountered in real-world applications is the unitary hexagonal lattice. Since the theoretical discovery of graphene [133] as a single layer of a graphite structure in 1947, and especially since its actual isolation [101] in 2004, the hexagonal lattice has risen in popularity.

In order to best approach the study of planar travelling waves on such lattices it is convenient to make a distinction between *top* and *bottom* vertices (as seen from the perspective of the embedded hexagons). In Figure 1.4 we can see how the *top* vertices are coloured in green and the *bottom* ones are coloured in red.

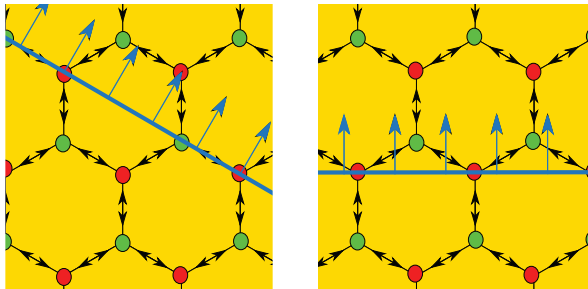


Figure 1.4: Two examples of travelling fronts that are aligned with the hexagonal structure, with $\theta = \pi/3$ on the left and $\theta = \pi/2$ on the right. Here we used red dots to represent the *bottom* edges and green dots for the *top* ones.

Since we will be dealing with planar waves, the most natural way to address the vertices is by their cartesian coordinates. We can choose an arbitrary *bottom* edge as the origin. Given the asymmetry between *top* and *bottom* vertices, let us split our system in two separate components:

$$\begin{cases} \dot{u}_t = d\Delta^\wedge(u_b, u_t) + g(u_t, a), \\ \dot{u}_b = d\Delta^\vee(u_t, u_b) + g(u_b, a). \end{cases} \quad (1.3.11)$$

Here the top- and bottom-diffusion operators $\Delta^\wedge$ respectively $\Delta^\vee$ are defined as

$$\begin{aligned}\Delta^\wedge(u_b, u_t)(x, y) &= u_b(x, y+1) + u_b\left(x - \frac{\sqrt{3}}{2}, y - \frac{1}{2}\right) + u_b\left(x + \frac{\sqrt{3}}{2}, y - \frac{1}{2}\right) - 3u_t(x, y), \\ \Delta^\vee(u_t, u_b)(x, y) &= u_t\left(x + \frac{\sqrt{3}}{2}, y + \frac{1}{2}\right) + u_t\left(x - \frac{\sqrt{3}}{2}, y + \frac{1}{2}\right) + u_t(x, y-1) - 3u_b(x, y).\end{aligned}\tag{1.3.12}$$

Let us remark that every *top* edge is connected uniquely with three *bottom* edges and conversely, every *bottom* edge is connected uniquely to three *top* edges.

Let us assume that for some $\theta, c \in \mathbb{R}$, we have found a travelling solution

$$(u_t, u_b)(x, y, t) = (\phi_t, \phi_b)(\cos(\theta)x + \sin(\theta)y - ct) \tag{1.3.13}$$

that satisfies (1.3.11). By plugging (ϕ_t, ϕ_b) into (1.3.11), we obtain the new system

$$\begin{cases} -c\phi_t'(\xi) = d(\phi_b(\xi + r_1) + \phi_b(\xi + r_2) + \phi_b(\xi + r_3) - 3\phi_t(\xi)) + g(\phi_t, a), \\ -c\phi_b'(\xi) = d(\phi_t(\xi + r_4) + \phi_t(\xi + r_5) + \phi_t(\xi + r_6) - 3\phi_b(\xi)) + g(\phi_b, a), \end{cases} \tag{1.3.14}$$

with

$$\begin{aligned}r_1 &= \cos(\pi/2 - \theta), & r_4 &= \cos(\pi/6 - \theta), \\ r_2 &= \cos(7\pi/6 - \theta), & r_5 &= \cos(5\pi/6 - \theta), \\ r_3 &= \cos(11\pi/6 - \theta), & r_6 &= \cos(3\pi/2 - \theta).\end{aligned} \tag{1.3.15}$$

Due to the bichromatic splitting we had to enforce, it seems natural to consider asymptotic conditions such as

$$\begin{aligned}\lim_{\xi \rightarrow -\infty} (\phi_t, \phi_b)(\xi) &= (0, 0), \\ \lim_{\xi \rightarrow \infty} (\phi_t, \phi_b)(\xi) &= (\bar{u}_t, \bar{u}_b).\end{aligned} \tag{1.3.16}$$

By inspecting the system (1.3.11), we see that $\bar{u}_t$ and $\bar{u}_b$ this time must satisfy the steady state conditions

$$\begin{cases} 0 = 3d(\bar{u}_b - \bar{u}_t) + g(\bar{u}_t, a), \\ 0 = 3d(\bar{u}_t - \bar{u}_b) + g(\bar{u}_b, a). \end{cases} \tag{1.3.17}$$

We can hence again borrow existence results for the heterogeneous solutions from the unidimensional setting after rewriting $\tilde{d} = \frac{3}{2}d$.

It is furthermore possible to choose $\bar{u}_t = \bar{u}_b = 1$ and obtain a front (still composed of two different functions ϕ_t and ϕ_b) that connects the two stable homogeneous states.

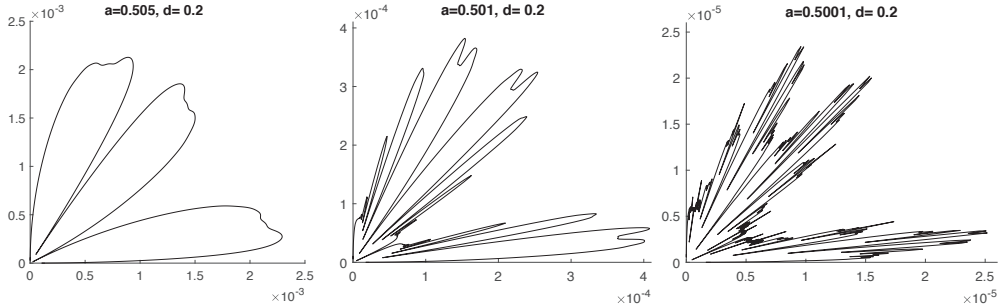


Figure 1.5: Radial speed for the travelling wave connecting $(0,0)$ to $(1,1)$ on the regular hexagonal lattice for different values of a . These curves correspond to $(\cos(\theta)c(\theta), \sin(\theta)c(\theta))$.

Travelling fronts Similarly as for the square lattice, we can numerically compute the speed of the bichromatic fronts on hexagonal lattices. In Figure 1.5 we see how the speed of the fronts connecting $(0,0)$ and $(1,1)$ varies when changing the direction θ , for three values of a close to the critical case $a = \frac{1}{2}$.

On the other hand, in Figure 1.6 we can observe how the radial speed changes when reducing the diffusion parameter d for the travelling front connecting $(0,0)$ to the stable heterogeneous rest state $(\bar{u}_t, \bar{u}_b)$. Let us remark that these plots are computed for $a = 0.5$, where we cannot find travelling fronts connecting the homogeneous states. Even though the speed values are fairly uniform for higher diffusion coefficients, we see that when d decreases the main lattice directions $\theta = k\pi/6$ are subject to more variations than the other non-aligned directions.

1.3.3.1 Multichromatic fronts on hexagonal lattices

It is no longer a straightforward exercise to come up with periodic colouring schemes for which a corresponding wave equation can be formulated. We will now shortly introduce the two most immediate examples.

The first one uses a 6-colours pattern as illustrated in Figure 1.7. The values B , D and F always appear on the *top* vertices while A , C and E correspond with the

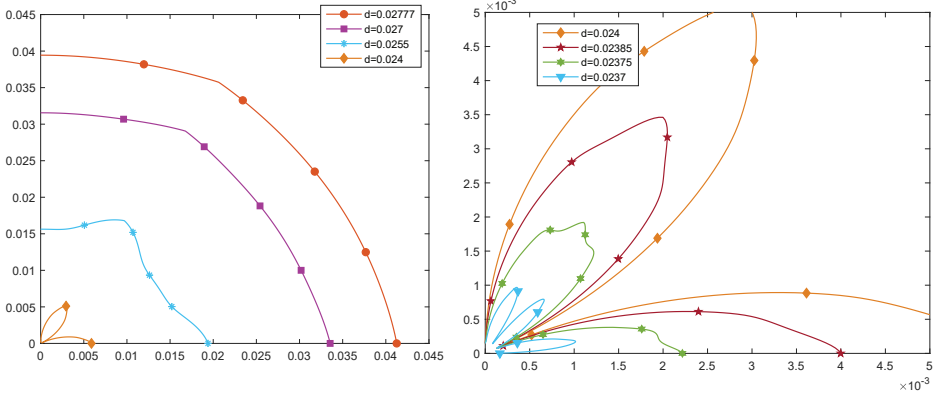


Figure 1.6: Radial speed for the bichromatic wave posed on the regular hexagonal lattice for $a = 0.5$. Notice the quicker speed reduction along the directions corresponding to the lattice orientation.

bottom ones. The wave equation associated to this system is

$$\begin{cases} -cA'(\xi) = d(B(\xi + r_4) + D(\xi + r_5) + F(\xi + r_6) - 3A(\xi)) + g(A(\xi), a), \\ -cB'(\xi) = d(E(\xi + r_1) + A(\xi + r_2) + C(\xi + r_3) - 3B(\xi)) + g(B(\xi), a), \\ -cC'(\xi) = d(F(\xi + r_4) + B(\xi + r_5) + D(\xi + r_6) - 3C(\xi)) + g(C(\xi), a), \\ -cD'(\xi) = d(C(\xi + r_1) + E(\xi + r_2) + A(\xi + r_3) - 3D(\xi)) + g(D(\xi), a), \\ -cE'(\xi) = d(D(\xi + r_4) + F(\xi + r_5) + B(\xi + r_6) - 3E(\xi)) + g(E(\xi), a), \\ -cF'(\xi) = d(A(\xi + r_1) + C(\xi + r_2) + E(\xi + r_3) - 3F(\xi)) + g(F(\xi), a). \end{cases} \quad (1.3.18)$$

It is easy to see that any 6-valued heterogeneous rest state $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{E}, \bar{F}) \in [0, 1]^6$ must satisfy the system

$$\begin{cases} 0 = d(\bar{B} + \bar{D} + \bar{F} - 3\bar{A}) + g(\bar{A}, a), \\ 0 = d(\bar{A} + \bar{C} + \bar{E} - 3\bar{B}) + g(\bar{B}, a), \\ 0 = d(\bar{B} + \bar{D} + \bar{F} - 3\bar{C}) + g(\bar{C}, a), \\ 0 = d(\bar{A} + \bar{C} + \bar{E} - 3\bar{D}) + g(\bar{D}, a), \\ 0 = d(\bar{B} + \bar{D} + \bar{F} - 3\bar{E}) + g(\bar{E}, a), \\ 0 = d(\bar{A} + \bar{C} + \bar{E} - 3\bar{F}) + g(\bar{F}, a). \end{cases} \quad (1.3.19)$$

It is interesting to note that this algebraic system also arises when considering the equilibria of the Nagumo equation

$$\dot{u}_v = d \sum_{w \sim v} (u_w - u_v) + g(u_v, a) \quad (1.3.20)$$

posed on the graph $v \in K_{3,3}$ depicted on the left of Figure 1.9. Here the relation $w \sim v$ means that the vertices v and w are connected. In particular, the one dimensional theory developed in Chapters 2 and 3 can no longer be applied to this situation. However, the algebraic system (1.3.19) can be simplified by taking $\overline{B} = \overline{D} = \overline{F} = \overline{u}_t$ and $\overline{A} = \overline{C} = \overline{E} = \overline{u}_b$, in which case (1.3.19) and (1.3.18) get simplified to (1.3.17) and (1.3.14) respectively.

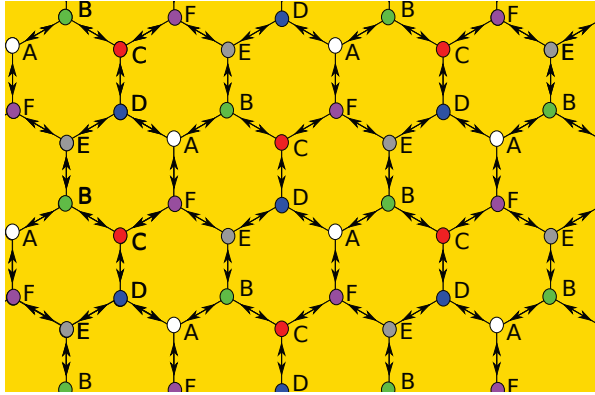


Figure 1.7: The 6-colour pattern corresponding with (1.3.18) and (1.3.19).

The final multichromatic pattern that we are going to consider on the hexagonal lattice is the 8-coloured one depicted in Figure 1.8. The corresponding wave system

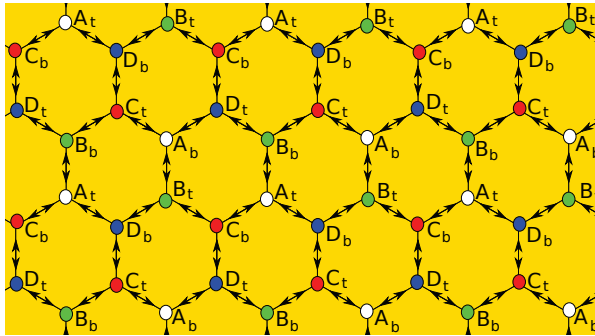


Figure 1.8: The 8-colour pattern corresponding with (1.3.21) and (1.3.23).

for the eight variables is

$$\begin{cases} -cA'_t(\xi) = d(B_b(\xi + r_1) + C_b(\xi + r_2) + D_b(\xi + r_3) - 3A_t(\xi)) + g(A_t(\xi), a), \\ -cA'_b(\xi) = d(D_t(\xi + r_4) + C_t(\xi + r_5) + B_t(\xi + r_6) - 3A_b(\xi)) + g(A_b(\xi), a), \\ -cB'_t(\xi) = d(A_b(\xi + r_1) + D_b(\xi + r_2) + C_b(\xi + r_3) - 3B_t(\xi)) + g(B_t(\xi), a), \\ -cB'_b(\xi) = d(C_t(\xi + r_4) + D_t(\xi + r_5) + A_t(\xi + r_6) - 3B_b(\xi)) + g(B_b(\xi), a), \\ -cC'_t(\xi) = d(D_b(\xi + r_1) + B_b(\xi + r_2) + A_b(\xi + r_3) - 3C_t(\xi)) + g(C_t(\xi), a), \\ -cC'_b(\xi) = d(A_t(\xi + r_4) + B_t(\xi + r_5) + D_t(\xi + r_6) - 3C_b(\xi)) + g(C_b(\xi), a), \\ -cD'_t(\xi) = d(C_b(\xi + r_1) + A_b(\xi + r_2) + B_b(\xi + r_3) - 3D_t(\xi)) + g(D_t(\xi), a), \\ -cD'_b(\xi) = d(B_t(\xi + r_4) + A_t(\xi + r_5) + C_t(\xi + r_6) - 3D_b(\xi)) + g(D_b(\xi), a). \end{cases} \quad (1.3.21)$$

We can in general consider eight different asymptotic values for the variables $(A_t, A_b, B_t, B_b, C_t, C_b, D_t, D_b)$. The corresponding steady-state system is then equivalent to that of the Nagumo equation on the cubic graph, see Figure 1.9.

As the naming and colouring suggests, however, we are interested in the case where we only have to provide the four asymptotic values

$$\begin{aligned} \lim_{\xi \rightarrow \infty} A_t(\xi) &= \lim_{\xi \rightarrow \infty} A_b(\xi) = \overline{A}, \\ \lim_{\xi \rightarrow \infty} B_t(\xi) &= \lim_{\xi \rightarrow \infty} B_b(\xi) = \overline{B}, \\ \lim_{\xi \rightarrow \infty} C_t(\xi) &= \lim_{\xi \rightarrow \infty} C_b(\xi) = \overline{C}, \\ \lim_{\xi \rightarrow \infty} D_t(\xi) &= \lim_{\xi \rightarrow \infty} D_b(\xi) = \overline{D}. \end{aligned} \quad (1.3.22)$$

In order to find the heterogeneous steady state quadruplets $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ we would then need to solve

$$\begin{cases} 0 = d(\overline{B} + \overline{C} + \overline{D} - 3\overline{A}) + g(\overline{A}, a), \\ 0 = d(\overline{A} + \overline{C} + \overline{D} - 3\overline{B}) + g(\overline{B}, a), \\ 0 = d(\overline{A} + \overline{B} + \overline{D} - 3\overline{C}) + g(\overline{C}, a), \\ 0 = d(\overline{A} + \overline{B} + \overline{C} - 3\overline{D}) + g(\overline{D}, a), \end{cases} \quad (1.3.23)$$

which can only be further reduced by taking $\overline{A} = \overline{B} = \overline{C} = \overline{D}$.

The interesting feature of the steady state system (1.3.23) is that it corresponds to the complete graph K_4 ; see Figure 1.9. Following the steps of [121], we will see in Chapter 3 that the steady state equation for the unidimensional trichromatic waves can be related to the Nagumo system on K_3 , while K_2 is associated to the bichromatic problem² discussed in Chapter 2 and Section 1.2.1 above.

For completeness let us mention that the periodic extension of the Nagumo system on K_5 must be posed on $\mathbb{Z}^2$. On the other hand, K_6 does not admit a regular planar

²With a proper readjustment of the diffusion parameter d .

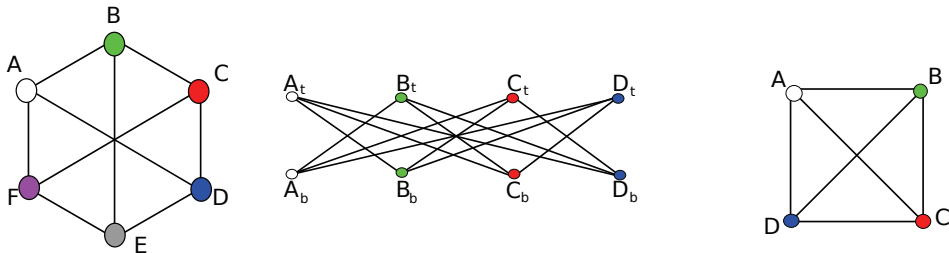


Figure 1.9: On the left, the bipartite graph $K_{3,3}$ that generates (1.3.19), also known as the utility graph. In the centre, the crown S_4^0 graph that generates the 8-coloured steady states system for $(A_t, A_b, B_t, B_b, C_t, C_b, D_t, D_b)$, also known as the cubic graph. On the right, the complete graph K_4 that generates the reduced system (1.3.23).

extension, but K_7 can be recovered from a triangular planar lattice (or $\mathbb{Z}^3$). The graph K_5 does not seem to admit any novel behaviour, but the wave equation for K_7 turned out to be computationally too cumbersome for a detailed analysis.

1.3.4 Beyond waves: travelling corners

Upon looking at the system (1.3.4), one might wonder about the behaviour of travelling profiles whose initial shape is close to a perfectly planar front except for some type of defect. The subject is a good starting point if we are interested in studying the microscopic dynamics of crystal growth, which is closely related to Wulff shapes [108, 136].

The results in [61] show that only interior corners can be found for the standard Nagumo PDE, while in [60] a whole range of defects (interior corners, exterior corners, steps or holes) can arise if a (small) amount of spatial heterogeneity is added to the system.

In Chapter 4 we will establish the existence of interior and exterior corners in the bidimensional LDE setting, with the presence of one excluding the other. We will also briefly investigate such defects in the context of bidimensional bichromatic travelling waves. This study is a perfect example of how it is possible to construct more complex moving structures starting from simple travelling waves.

Chapter 2

Bichromatic travelling waves

This paper will appear in SIAM J. Applied Dynamical Systems as: H. J. Hupkes, L. Morelli, P. Stehlík, "Bichromatic Travelling Waves for Lattice Nagumo Equations"

Abstract. We discuss bichromatic (two-color) front solutions to the bistable Nagumo lattice differential equation. Such fronts connect the stable spatially homogeneous equilibria with spatially heterogeneous 2-periodic equilibria and hence are not monotonic like the standard monochromatic fronts. We provide explicit criteria that can determine whether or not these fronts are stationary and show that the bichromatic fronts can travel in parameter regimes where the monochromatic fronts are pinned. The presence of these bichromatic waves allows the two stable homogeneous equilibria to both spread out through the spatial domain towards each other, buffered by a shrinking intermediate zone in which the periodic pattern is visible.

2.1 Introduction

In this chapter we consider the Nagumo lattice differential equation (LDE)

$$\dot{u}_j(t) = d[u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)] + g(u_j(t); a), \quad (2.1.1)$$

posed on the spatial lattice $j \in \mathbb{Z}$, with $t \in \mathbb{R}$. We assume $d > 0$ and use the standard cubic bistable nonlinearity $g(u; a) = u(1 - u)(u - a)$ with $a \in (0, 1)$. This LDE is well-known as a prototypical model that describes the competition between two stable states $u = 0$ and $u = 1$ in a discrete spatial environment. A crucial role is reserved for so-called travelling front solutions, which have the form

$$u_j(t) = \Phi(j - ct), \quad \Phi(-\infty) = 0, \quad \Phi(+\infty) = 1. \quad (2.1.2)$$

Such solutions are often referred to as invasion waves, as they provide a mechanism by which the energetically preferred state can invade the spatial domain.

Our work focuses on the case where $c = 0$ holds for these primary invasion waves, indicating a delicate balance between the two competing states. In this case (2.1.1) can admit stable spatially periodic rest-states. Numerical results indicate that these states can act as a buffer between regions of space where $u = 0$ and $u = 1$ dominate the dynamics. This buffer shrinks as these two stable states appear to move towards each other. This latter process is governed by secondary two-component invasion waves that we analyze in detail in this chapter.

Nagumo PDE The LDE (2.1.1) can be seen as the nearest-neighbour discretization of the Nagumo reaction-diffusion PDE [99]

$$u_t = u_{xx} + g(u; a), \quad x \in \mathbb{R} \quad (2.1.3)$$

on a spatial grid with size $h = d^{-1/2}$. This PDE has been used as a highly simplified model for the spread of genetic traits [3] and the propagation of electrical signals through nerve fibres [9]. In higher space dimensions it also serves as a desingularization of the standard mean-curvature flow that is often used to describe the evolution of interfaces [45].

Fife and McLeod [51] used phase plane analysis to show that (2.1.3) admits a front solution for each $a \in [0, 1]$. Such solutions have the form

$$u(x, t) = \Phi(x - ct), \quad \Phi(-\infty) = 0, \quad \Phi(+\infty) = 1, \quad (2.1.4)$$

for some smooth waveprofile Φ and wavespeed c that has the same sign as $a - \frac{1}{2}$. These fronts hence connect the two stable spatially homogeneous equilibria $u(x, t) \equiv 0$ and $u(x, t) \equiv 1$.

Exploiting the comparison principle, Fife and McLeod were able to show that these front solutions have a surprisingly large basin of attraction. Indeed, any solution to (2.1.3) with an initial condition $u(x, 0) = u_0(x)$ that has $u_0(x) \approx 0$ for $x \ll -1$ and $u_0(x) \approx 1$ for $x \gg +1$ will converge to a shifted version of this front as $t \rightarrow \infty$.

These front solutions can be used as building blocks to capture the behaviour of a more general class of solutions to (2.1.3). Consider for example the two-parameter family of functions

$$u_{\text{plt}; \alpha_0, \alpha_1}(x, t) = \Phi(x - ct + \alpha_0) + \Phi(-x - ct + \alpha_1) - 1, \quad (2.1.5)$$

with $\alpha_1 \geq \alpha_0$. Each of these functions can be interpreted as a shifted version of the front solution (2.1.4) that is reflected in a vertical line to form a plateau.

If $c < 0$, then any initial configuration that has $u_0(x) \approx 0$ for $|x| \gg L$ and $u_0(x) \approx 1$ for $|x| \leq L$ will converge to a member of the family (2.1.5) as $t \rightarrow \infty$. This provides a mechanism by which compact regions where $u \sim 1$ can spread out to fill the entire domain.

On the other hand, when $c > 0$ one can construct entire solutions that converge to an element of (2.1.5) as $t \rightarrow -\infty$ and tend to zero as $t \rightarrow +\infty$. These solutions are stable under small perturbations [137]. In particular, they can be viewed as a robust elimination process whereby compact regions that have $u \sim 1$ are annihilated by two incoming travelling fronts that collide as $t \rightarrow \infty$.

Nagumo LDE For many physical phenomena such as crystal growth in materials [20], the formation of fractures in elastic bodies [119] and the motion of dislocations [25] and domain walls [37] through crystals, the discreteness and topology of the underlying spatial domain have a major impact on the dynamical behaviour. It is hence important to develop mathematical modelling tools that can incorporate such structures effectively. Indeed, by now it is well known that discrete models can capture dynamical behaviour that their continuous counterparts can not.

The LDE (2.1.1) has served as a prototype system in which such effects can be explored. It arises as a highly simplified model for the propagation of action potentials through nerve fibers that have regularly spaced gaps in their myeline coating [9]. Two-dimensional versions have been used to describe phase transitions in Ising models [7], to analyze predator-prey interactions [118] and to develop pattern recognition algorithms in image processing [31, 32]. Recently, an interest has also arisen in Nagumo equations posed on graphs [121], motivated by the network structure present in many biological systems [115].

Many authors have studied the LDE (2.1.1), focusing primarily on the richness of the set of equilibria [89] and the existence of travelling and standing front solutions [91, 139]. Such solutions have the form (2.1.2), which leads naturally to the waveprofile equation

$$-c\Phi'(\xi) = d[\Phi(\xi - 1) - 2\Phi(\xi) + \Phi(\xi + 1)] + g(\Phi(\xi); a). \quad (2.1.6)$$

Since the behaviour of every lattice point is governed by the same profile Φ , we refer to these front solutions as monochromatic waves in this chapter (in order to distinguish them from the bichromatic waves we discuss in the sequel). The seminal results by Mallet-Paret [91] show that for each $a \in [0, 1]$ and $d > 0$ there exists a unique $c = c_{\text{mc}}(a, d)$ for which such monochromatic (mc) solutions exist.

Pinning Upon fixing $a \in (0, 1) \setminus \{\frac{1}{2}\}$, Zinner [139] established that $c_{\text{mc}}(a, d) \neq 0$ for $d \gg 1$, while Keener [83] showed that $c_{\text{mc}}(a, d) = 0$ for $0 < d \ll 1$. Upon fixing $d > 0$, Mallet-Paret established [91] that $c_{\text{mc}}(a, d) \neq 0$ for $a \approx 1$ and $a \approx 0$. In addition, again for fixed $d > 0$, the results in [67, 92] strongly suggest that there exists $\delta > 0$ so that $c_{\text{mc}}(a, d) = 0$ whenever $|a - \frac{1}{2}| \leq \delta$; see Figure 2.1.

This last phenomenon is called pinning and distinguishes the LDE (2.1.1) from the PDE (2.1.3). It is a direct consequence of the fact that we have broken the translational invariance of space. Indeed, (2.1.6) becomes singular in the limit $c \rightarrow 0$ and the corresponding waveprofiles indeed typically lose their smoothness. Many results suggest that this phenomenon is generic for discrete systems [7, 21, 41, 43, 75]. However, by carefully tuning the nonlinearity g it is possible to design systems for which this pinning is absent [40, 70]. Understanding the pinning phenomenon is an important and challenging mathematical problem that also has practical ramifications.

Periodicity In this chapter we study waves that connect spatially homogeneous stationary solutions of (2.1.1) with spatially heterogeneous 2-periodic stationary solutions. It is well known that many physical systems exhibit spatially periodic features

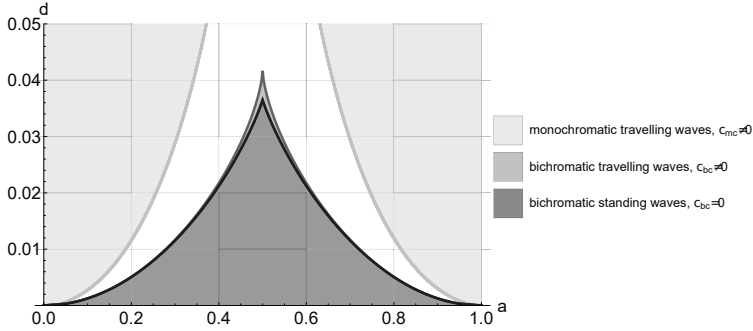


Figure 2.1: Existence regions for monochromatic and bichromatic wave solutions to (2.1.1).

[53, 57, 117]. Examples that also feature spatial discreteness include the presence of twinning microstructures in shape memory alloys [14] and the formation of domain-wall microstructures in dielectric crystals [123].

In many cases the underlying periodicity comes from the spatial system itself. For example, in [46, 47, 69] the authors consider chains of alternating masses connected by identical springs (and vice versa). The dynamical behaviour of such systems can be easily modelled by LDEs with periodic coefficients. In certain limiting cases the authors were able to construct so-called nanopterons, which are multi-component wave solutions that have low-amplitude oscillations in their tails.

However, periodic patterns also arise naturally as solutions to spatially homogeneous discrete systems. Indeed, we shall see in §2.3 that the LDE (2.1.1) with $d > 0$ admits many periodic equilibria. In addition, the results in [89] explore the periodicity and chaos present in the set of equilibria to homogeneous LDEs with simplified nonlinearities.

It is also possible to introduce a natural periodicity into the structure of (2.1.1) by taking $d < 0$. This can be seen by introducing new variables $v_j = (-1)^j u_j$, which restores the applicability of the comparison principle. This choice essentially decomposes the lattice sites $\mathbb{Z}$ into two groups $\mathbb{Z}_{\text{odd}}$ and $\mathbb{Z}_{\text{even}}$ that each have their own characteristic behaviour.

Such anti-diffusion models have been used to describe phase transitions for grids of particles that have visco-elastic interactions [22, 23, 128]. In [19] this problem has been analyzed in considerable detail. The authors show that the resulting two component system admits co-existing patterns that can be both monostable and bistable in nature. Similar results with piecewise linear nonlinearities but more general couplings between neighbours can be found in [129].

Bichromatic waves In this chapter we are interested in the parameter region where $c_{\text{mc}}(a, d) = 0$. In a subset of this region it is possible to show that (2.1.1) has spatially

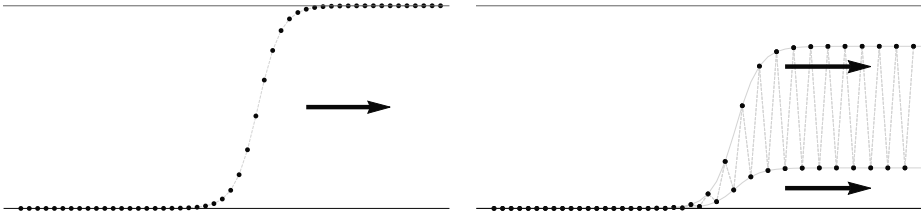


Figure 2.2: A monochromatic travelling wave of (2.1.1) (left panel) connects two spatially homogeneous stationary solutions. A bichromatic travelling wave of (2.1.1) (right panel) connects a spatially homogeneous stationary solution with a spatially heterogeneous one.

heterogeneous stable equilibria. We focus on the simplest case and consider so-called bichromatic (two-color) equilibria, which are spatially periodic with period two. As such, they are closely connected to solutions of the Nagumo equation posed on a graph with two vertices. We set out to construct bichromatic front-solutions to (2.1.1), which can be seen as waves that connect the spatially homogeneous equilibrium $u \equiv 0$ with such a 2-periodic state. We emphasize that these differ from the traditional front-solutions (2.1.2) in the sense that the odd and even lattice sites each have their own waveprofile, as illustrated in Figure 2.2. Consequently, the bichromatic front-solutions are not monotone.

Our first main contribution is contained in §2.3, where we give a detailed description of the set of parameters (a, d) where such 2-periodic equilibria exist and where they are stable. In contrast to the setting encountered in [19], the relevant bifurcation curves cannot all be described explicitly. Besides a global result stating that the number of such equilibria decreases as d is increased, we also obtain precise asymptotics that describe the boundaries near the three corners $(a, d) \in \{(0, 0), (1/2, 1/24), (1, 0)\}$ in Figure 2.1.

As in [19], these preparations allow the existence of bichromatic fronts to be established in a straightforward fashion. Indeed, one can apply the general theory developed by Chen, Guo and Wu in [27] for discrete periodic systems that admit a comparison-principle. These results imply that there exists a unique wavespeed $c_{bc}(a, d)$ for which such bichromatic (bc) fronts exist. If $c_{bc}(a, d) \neq 0$, the Fredholm theory developed in [74] together with the techniques from [73, §3] can be used to show that these travelling fronts depend smoothly on (a, d) and are nonlinearly stable.

However, these general results cannot distinguish between the cases $c_{bc}(a, d) = 0$ and $c_{bc}(a, d) \neq 0$ where we have standing respectively travelling fronts. This should be contrasted to the situation for the PDE (2.1.3), where the sign of the wavespeed is given by the sign of a simple integral [51]. Indeed, there is a large set of parameters (a, d) for which the discrete bichromatic fronts fail to travel, even though the analogous integral does not vanish.

Our second main contribution is that we provide explicit criteria in §2.4 that can guarantee $c_{bc} = 0$ or $c_{bc} > 0$. Together these results cover most of the parameter region where bichromatic fronts exist. In any case, they provide a two-component generalization of the coercivity conditions introduced in [91], which ensure $c_{mc}(a, d) \neq 0$ for the boundary regions $a \approx 1$ and $a \approx 0$.

Our arguments to guarantee $c_{bc} = 0$ are closely related to the setup used by Keener [83] to establish that monochromatic waves are pinned for $0 < d \ll 1$. In particular, for small values of d one can neglect the diffusion term in §2.1.1 and use properties of the cubic to show that the derivative of the waveprofile must change signs. This contradicts the fact that waveprofiles must be strictly monotonic if they travel.

On the other hand, in §2.4 we develop an intuitive geometric construction involving reflections to describe a planar recurrence relation that standing bichromatic fronts must satisfy. This allows us to rule out the presence of such fronts when a scalar inequality is violated. This consequently implies the presence of travelling bichromatic fronts.

The parameter regimes where these two arguments apply both converge towards the corner points $(0, 0)$ and $(1, 0)$. Near these corners we need the delicate asymptotics described above to distinguish between the two cases.

Colliding fronts One of the main reasons for our interest in these bichromatic fronts is that they present mechanisms via which the stable homogeneous states $u = 0$ and $u = 1$ can spread throughout the domain, even though the primary invasion waves are blocked from propagation. By using similar techniques as in [137], we believe it should be possible to construct entire solutions consisting of a right-travelling bichromatic front connection between the homogeneous equilibrium $u \equiv 0$ and a 2-periodic intermediate state that collides with a left-travelling bichromatic connection between 2-periodic intermediate state and the homogeneous equilibrium $u \equiv 1$; see Figure 2.3. The resulting state after the collision is then a pinned monochromatic front that connects 0 with 1. We have been able to numerically verify the existence of these solutions in the parameter regions predicted by the theory developed in this chapter.

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2.2 Main results

Our interest here is in the lattice differential equation

$$\dot{x}_j(t) = d[x_{j-1}(t) - 2x_j(t) + x_{j+1}(t)] + g(x_j(t); a) \quad (2.2.1)$$

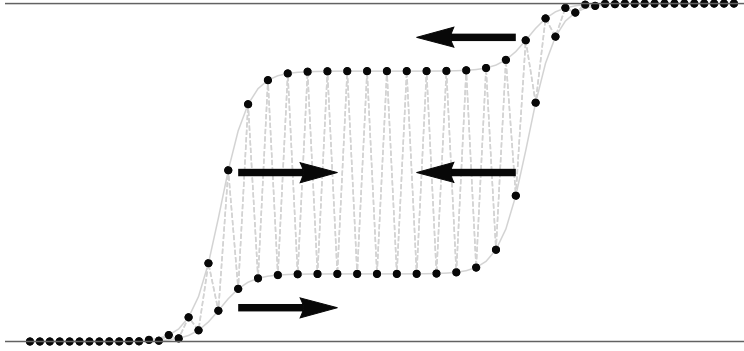


Figure 2.3: Colliding front of (2.1.1) consisting of a right-travelling bichromatic front connection between the homogeneous equilibrium $u \equiv 0$ and a 2-periodic intermediate state that collides with a left-travelling bichromatic connection between the 2-periodic state and the homogeneous equilibrium $u \equiv 1$.

posed on the one-dimensional lattice, i.e., $j \in \mathbb{Z}$. The bistable nonlinearity is explicitly given by

$$g(u; a) = u(1 - u)(u - a), \quad (2.2.2)$$

with $a \in (0, 1)$. Our results concern so-called bichromatic (two-colour) travelling wave solutions to the LDE (2.2.1). Such solutions can be written in the form

$$x_j(t) = \begin{cases} \Phi_u(j - ct) & \text{if } j \text{ is even,} \\ \Phi_v(j - ct) & \text{if } j \text{ is odd,} \end{cases} \quad (2.2.3)$$

for some wavespeed $c \in \mathbb{R}$ and $\mathbb{R}^2$ -valued waveprofile

$$\Phi = (\Phi_u, \Phi_v) : \mathbb{R} \rightarrow \mathbb{R}^2. \quad (2.2.4)$$

Substituting this Ansatz into (2.2.1) we obtain the travelling wave system

$$\begin{aligned} -c\Phi'_u(\xi) &= d[\Phi_v(\xi - 1) - 2\Phi_u(\xi) + \Phi_v(\xi + 1)] + g(\Phi_u(\xi); a), \\ -c\Phi'_v(\xi) &= d[\Phi_u(\xi - 1) - 2\Phi_v(\xi) + \Phi_u(\xi + 1)] + g(\Phi_v(\xi); a). \end{aligned} \quad (2.2.5)$$

Upon introducing the functions

$$G(u, v; a, d) = \begin{pmatrix} G_1(u, v; a, d) \\ G_2(u, v; a, d) \end{pmatrix} = \begin{pmatrix} 2d(v - u) + g(u; a) \\ 2d(u - v) + g(v; a) \end{pmatrix}, \quad (2.2.6)$$

we see that any stationary solution

$$(\Phi_u, \Phi_v)(\xi) = (\bar{u}, \bar{v}) \quad (2.2.7)$$

to (2.2.5) must satisfy the nonlinear algebraic equation

$$G(\bar{u}, \bar{v}; a, d) = 0. \quad (2.2.8)$$

The full bifurcation diagram for this equation is described in §2.3. For our purposes here however it suffices to summarize a subset of the conclusions from this analysis, which we do in our first result below. In particular, there exists a region Ω_{bc} in the (a, d) -plane for which the spatially homogeneous system $(\dot{u}, \dot{v}) = G(u, v; a, d)$ has a stable equilibrium $(\bar{u}_{bc}(a, d), \bar{v}_{bc}(a, d))$ that can be interpreted as a bichromatic equilibrium state for the LDE (2.2.1).

Proposition 2.2.1 (see §2.3). There exists a continuous curve $d_{bc} : [0, 1] \rightarrow [0, \frac{1}{24}]$ with $d_{bc}(\frac{1}{2}) = \frac{1}{24}$ and $d_{bc}(1-a) = d_{bc}(a)$ so that for every $0 \leq d < d_{bc}$ and $0 < a < 1$ the system (2.2.8) has nine distinct equilibria $(\bar{u}, \bar{v}) \in [0, 1]^2$. Upon writing

$$\Omega_{bc} = \{0 < d < d_{bc}(a) \text{ and } 0 < a < 1\}, \quad (2.2.9)$$

there exist C^∞ -smooth maps

$$(\bar{u}_{bc}, \bar{v}_{bc}) : \Omega_{bc} \rightarrow (0, 1)^2 \quad (2.2.10)$$

with $\bar{u}_{bc} < a < \bar{v}_{bc}$ so that for every $(a, d) \in \Omega_{bc}$ we have

$$G(\bar{u}_{bc}(a, d), \bar{v}_{bc}(a, d); a, d) = 0 \quad (2.2.11)$$

together with

$$\det D_{1,2}G(\bar{u}_{bc}(a, d), \bar{v}_{bc}(a, d); a, d) > 0, \quad \text{Tr } D_{1,2}G(\bar{u}_{bc}(a, d), \bar{v}_{bc}(a, d); a, d) < 0. \quad (2.2.12)$$

We note that the statements (2.2.11)-(2.2.12) are also valid upon replacing the bichromatic rest-state $(\bar{u}_{bc}, \bar{v}_{bc})$ by the monochromatic equilibria $(0, 0)$ and $(1, 1)$. We will be interested in waves that connect these mono- and bichromatic equilibria together. More precisely, we set out to find solutions to (2.2.5) that satisfy either the 'lower' boundary conditions

$$\lim_{\xi \rightarrow -\infty} \Phi(\xi) = (0, 0), \quad \lim_{\xi \rightarrow +\infty} \Phi(\xi) = (\bar{u}_{bc}, \bar{v}_{bc}), \quad (2.2.13)$$

or the 'upper' boundary conditions

$$\lim_{\xi \rightarrow -\infty} \Phi(\xi) = (\bar{u}_{bc}, \bar{v}_{bc}) \quad \lim_{\xi \rightarrow +\infty} \Phi(\xi) = (1, 1). \quad (2.2.14)$$

The result below summarizes several key facts concerning the existence and uniqueness of such waves. It introduces subregions of Ω_{bc} denoted by $\mathcal{T}_{\text{low}}$ and $\mathcal{T}_{\text{up}}$ where the bichromatic travelling waves (2.2.3) exist with nonzero speeds $c_{\text{low}} > 0$ and $c_{\text{up}} < 0$; see Figure 2.4. With the exception of the inequalities $c_{\text{low}} \geq 0$ and $c_{\text{up}} \leq 0$, these properties follow directly from the theory developed in [27, 74].

Theorem 2.2.2 (see §2.4.1). There exist continuous maps

$$c_{\text{low}} : \Omega_{\text{bc}} \rightarrow [0, \infty), \quad c_{\text{up}} : \Omega_{\text{bc}} \rightarrow (-\infty, 0] \quad (2.2.15)$$

that satisfy the following properties.

- (i) Upon introducing the open sets

$$\begin{aligned} \mathcal{T}_{\text{low}} &= \{(a, d) \in \Omega_{\text{bc}} : c_{\text{low}} > 0\}, \\ \mathcal{T}_{\text{up}} &= \{(a, d) \in \Omega_{\text{bc}} : c_{\text{up}} < 0\}, \end{aligned} \quad (2.2.16)$$

the functions c_{low} and c_{up} are C^∞ -smooth on $\mathcal{T}_{\text{low}}$ respectively $\mathcal{T}_{\text{up}}$.

- (ii) There exist C^∞ -smooth functions

$$\Phi_{\text{low}} : \mathcal{T}_{\text{low}} \rightarrow W^{1,\infty}(\mathbb{R}; \mathbb{R}^2), \quad \Phi_{\text{up}} : \mathcal{T}_{\text{up}} \rightarrow W^{1,\infty}(\mathbb{R}; \mathbb{R}^2), \quad (2.2.17)$$

such that for any $\# \in \{\text{low}, \text{up}\}$ and any $(a, d) \in \mathcal{T}_\#$, the pair

$$(c, \Phi) = (c_\#(a, d), \Phi_\#(a, d)) \quad (2.2.18)$$

satisfies (2.2.5) together with the boundary condition (2.2.13) if $\# = \text{low}$ or (2.2.14) if $\# = \text{up}$. In addition, we have the componentwise inequality $\Phi'_\# > (0, 0)$.

- (iii) For any $\# \in \{\text{low}, \text{up}\}$ and any $(a, d) \in \Omega_{\text{bc}} \setminus \mathcal{T}_\#$, there exists a non-decreasing function $\Phi : \mathbb{R} \rightarrow \mathbb{R}^2$ that satisfies (2.2.5) with $c = 0$ together with the boundary condition (2.2.13) if $\# = \text{low}$ or (2.2.14) if $\# = \text{up}$.
- (iv) Pick $\# \in \{\text{low}, \text{up}\}$ and $(a, d) \in \Omega_{\text{bc}}$ and consider any $c \neq 0$ together with a function $\Phi \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^2)$ that satisfies (2.2.5) together with the boundary condition (2.2.13) if $\# = \text{low}$ or (2.2.14) if $\# = \text{up}$. Then we must have $c = c_\#(a, d)$ and $\Phi = \Phi_\#(a, d)(\cdot - \vartheta)$ for some $\vartheta > 0$. In particular, we have $(a, d) \in \mathcal{T}_\#$.
- (v) Pick $\# \in \{\text{low}, \text{up}\}$ and $(a, d) \in \Omega_{\text{bc}}$ and consider any non-decreasing function $\Phi : \mathbb{R} \rightarrow \mathbb{R}^2$ that satisfies (2.2.5) with $c = 0$ together with the boundary condition (2.2.13) if $\# = \text{low}$ or (2.2.14) if $\# = \text{up}$. Then we must have $(a, d) \in \Omega_{\text{bc}} \setminus \mathcal{T}_\#$.

We numerically determined the locations of the sets $\mathcal{T}_{\text{low}}$ and $\mathcal{T}_{\text{up}}$ in Figure 2.4. In particular, we simulated (2.2.1) with an initial condition that consists of the stable periodic pattern multiplied by a hyperbolic tangent. By checking if this solution converges to a travelling or stationary pattern one can decide whether $(a, d) \in \mathcal{T}_{\text{low}}$.

This procedure hinges on the crucial fact that the waves in item (ii) of Theorem 2.2.2 are nonlinearly stable under the dynamics of (2.1.1); see [27, Thm. 4].

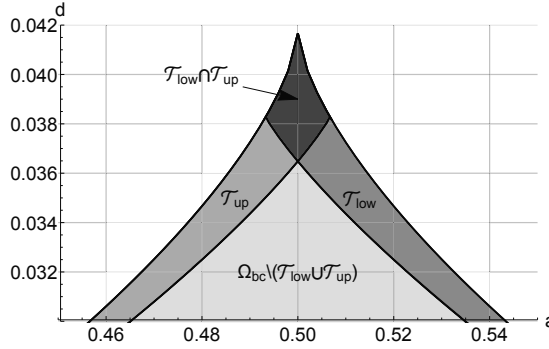


Figure 2.4: Numerical bounds for the parameter sets Ω_{bc} , $\mathcal{T}_{low}$ and $\mathcal{T}_{up}$ introduced in Theorem 2.2.2, in the neighbourhood of the cusp $(\frac{1}{2}, \frac{1}{24})$.

The main contribution of this chapter is that we provide an explicit and verifiable inequality to rigorously confirm that the sets $\mathcal{T}_{low}$ and $\mathcal{T}_{up}$ touch the boundary curve $d = d_{bc}(a)$. In order to formulate this inequality, we first extend the bichromatic equilibria (2.2.10) to this boundary by writing

$$(\bar{u}_{bc}, \bar{v}_{bc})(a, d_{bc}(a)) = \lim_{d \uparrow d_{bc}(a)} (\bar{u}_{bc}, \bar{v}_{bc})(a, d). \quad (2.2.19)$$

Inspection of (2.2.8) readily shows that this pair (2.2.19) is a solution to the fixed-point problem

$$(u, v) = (u_a(v), v_a(u)), \quad (2.2.20)$$

in which we have introduced the functions

$$u_a(v) = v - \frac{g(v; a)}{2d_{bc}(a)}, \quad v_a(u) = u - \frac{g(u; a)}{2d_{bc}(a)}. \quad (2.2.21)$$

The two critical points of these cubic functions are given by

$$\gamma_{\pm}(a) = \frac{1}{3} \left[a + 1 \pm \sqrt{a^2 - a + 1 - 6d_{bc}(a)} \right]. \quad (2.2.22)$$

Our approach relies on a geometric construction that rules out the existence of solutions to the recurrence relation that standing waves must satisfy. Indeed, in §2.4.2 we will recast (2.2.5) with $c = 0$ as a discrete map that involves horizontal and vertical reflections through the curves $u = u_a(v)$ and $v = v_a(u)$. Due to the monotonicity of the waveprofiles, several key features of the stable manifold for the pair (2.2.19) can therefore be captured by reflecting the lines $u = \bar{u}(a, d_{bc}(a))$ and $v = \bar{v}(a, d_{bc}(a))$ through these curves.

The resulting reflected functions can be written as

$$u_{a,r}(v) = 2u_a(v) - \bar{u}_{bc}(a, d_{bc}(a)), \quad v_{a,r}(u) = 2v_a(u) - \bar{v}_{bc}(a, d_{bc}(a)). \quad (2.2.23)$$

For any $a \in (0, 1)$, we now define the set of intersections

$$\mathcal{I}_a = \{(u, v) : u = u_a(v) \text{ and } v = v_{a,r}(u)\}, \quad (2.2.24)$$

which are all mapped onto the horizontal line $v = \bar{v}(a, d_{bc}(a))$. Assuming that this set is non-empty, we pick the specific representative

$$u_*(a) = \max\{u : \text{there exists } (u, v) \in \mathcal{I}_a \text{ with } 0 \leq u \leq \bar{u}_{bc}(a, d_{bc}(a)) \text{ and } 0 \leq v \leq \gamma_-(a)\}, \quad (2.2.25)$$

which allows us to define the scalar function

$$\Gamma(a) = u_{a,r}(\gamma_+(a)) - u_*(a). \quad (2.2.26)$$

The full geometric interpretation of this definition will be clarified and illustrated in §2.4.2. For now, we simply remark that it measures the difference between two critical points associated to the unstable manifold of $(0, 0)$ and the stable manifold of the pair (2.2.19). When $\Gamma(a) > 0$ holds, the critical points are ordered in such a way that connections between these two manifolds are blocked. This forces the inequality $c \neq 0$, implying that $\mathcal{T}_{low}$ indeed touches the boundary curve $d = d_{bc}(a)$.

We wish to emphasize that there is an essential difference between computing $\Gamma(a)$ and simulating the full LDE (2.2.1) to check whether $c \neq 0$. Indeed, the first problem only requires knowledge of the curves (2.2.21)-(2.2.23) and is hence two dimensional. On the other hand, the second problem is infinite dimensional and therefore necessarily involves truncations.

We are now ready to formulate our final result, which provides analytical bounds for the parameter regions $\mathcal{T}_{low}$ and $\mathcal{T}_{up}$ where the bichromatic waves actually travel (i.e., where $c_{low} > 0$ respectively $c_{up} < 0$). Both regions contain a neighbourhood of the cusp $(a, d) = (\frac{1}{2}, \frac{1}{24})$. In addition, the corners $(0, 0)$ and $(1, 0)$ are accumulation points for the sets $\mathcal{T}_{up}$ respectively $\mathcal{T}_{low}$.

Theorem 2.2.3 (see §2.4.3). The sets $\mathcal{T}_{low}$ and $\mathcal{T}_{up}$ satisfy the following properties.

- (i) For each $(a, d) \in \mathcal{T}_{up}$ we have $d > \frac{1}{8}a^2$, while for each $(a, d) \in \mathcal{T}_{low}$ we have $d > \frac{1}{8}(1-a)^2$.
- (ii) If $(a, d) \in \mathcal{T}_{low}$ then also $(a', d) \in \mathcal{T}_{low}$ for all $(a', d) \in \Omega_{bc}$ that have $a' \geq a$. On the other hand, if $(a, d) \in \mathcal{T}_{up}$ then also $(a', d) \in \mathcal{T}_{up}$ for all $(a', d) \in \Omega_{bc}$ that have $a' \leq a$.
- (iii) There exists $\epsilon > 0$ so that we have the inclusions

$$(a, d) \in \mathcal{T}_{low} \cap \mathcal{T}_{up} \quad (2.2.27)$$

for all $(a, d) \in \Omega_{bc}$ that have

$$0 < \left|a - \frac{1}{2}\right| + \left|d - \frac{1}{24}\right| < \epsilon. \quad (2.2.28)$$

- (iv) The expression (2.2.26) is well-defined for all $0 < a < 1$. If $\Gamma(a_*) > 0$ for some $0 < a_* < 1$, then there exists $\epsilon > 0$ so that $(a, d) \in \mathcal{T}_{\text{low}}$ for all $(a, d) \in \Omega_{\text{bc}}$ that have

$$0 < |a - a_*| + |d - d_{\text{bc}}(a_*)| < \epsilon. \quad (2.2.29)$$

- (v) The inequality $\Gamma(a) > 0$ holds whenever $1 - a > 0$ is sufficiently small. In particular, we have $(0, 0) \in \overline{\mathcal{T}}_{\text{up}}$ and $(1, 0) \in \overline{\mathcal{T}}_{\text{low}}$.

Using numerics we have verified that $\Gamma(a) > 0$ holds for $a \in [.498, .999]$; see §2.4.3. Together with (ii) and (v) above, this strongly suggests that $\mathcal{T}_{\text{low}}$ is a connected set that extends towards the right boundary of Ω_{bc} . By symmetry, this is equivalent to the statement that $\mathcal{T}_{\text{up}}$ is a connected set that extends towards the left boundary of Ω_{bc} .

2.3 Bichromatic stationary solutions

In this section we uncover the structure of the solution set to $G(u, v; a, d) = 0$ as a function of the parameters (a, d) . Our first result shows that for $d \gg 1$ this equation only has the monochromatic roots $(0, 0)$, (a, a) and $(1, 1)$. The threshold $d_+(a)$ between this region and the region with five distinct roots can be explicitly computed. However, we only have qualitative and asymptotic results for the boundary $d_-(a)$ where the root-count increases to the maximal value of nine. In Figure 2.5 we compare these asymptotics to numerically computed values for $d_-(a)$. We remark here that the monotonicity of the root count with respect to d does not hold for general bistable nonlinearities g .

Proposition 2.3.1 (see §2.3.5). There exist two continuous functions

$$d_{\pm} : [0, 1] \rightarrow [0, \infty) \quad (2.3.1)$$

that satisfy the following properties.

- (i) For any $0 < a < 1$ we have the explicit expression

$$d_+(a) = \frac{g'(a; a)}{4}, \quad (2.3.2)$$

together with the identities

$$d_-(a) = d_-(1 - a), \quad d_+(a) = d_+(1 - a) \quad (2.3.3)$$

and the inequality $d_-(a) < d_+(a)$. In addition, we have

$$d_-(0) = d_+(0) = d_-(1) = d_+(1) = 0 \quad (2.3.4)$$

together with $d_-(\frac{1}{2}) = \frac{1}{24}$.

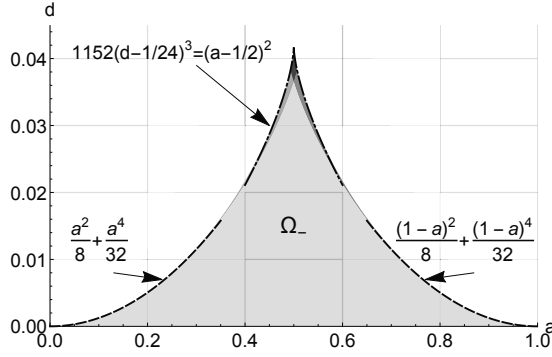


Figure 2.5: Comparison of the asymptotics for d_- described in Proposition 2.3.1 (v) with the numerically computed border of the set Ω_- .

- (ii) We have $d_- \in C^\infty([0, \frac{1}{2})) \cap C^\infty((\frac{1}{2}, 1])$. In addition, d_- is strictly increasing on $[0, \frac{1}{2}]$ and strictly decreasing on $[\frac{1}{2}, 1]$.
- (iii) Pick any $a \in (0, 1)$. The equation $G(u, v; a, d) = 0$ has nine distinct roots for $0 \leq d < d_-(a)$, five distinct roots for $d_- < d < d_+(a)$ and three distinct real roots for $d \geq d_+(a)$.
- (iv) Pick any $a \in (0, 1)$. The equation $G(u, v; a, d) = 0$ has seven distinct roots for $d = d_-(a)$ if $a \neq \frac{1}{2}$ and five if $a = \frac{1}{2}$.
- (v) We have the expansion $d_-(a) = \frac{1}{8}a^2 + \frac{1}{32}a^4 + O(a^5)$ for $a \downarrow 0$. In addition, writing $a_- : [0, \frac{1}{24}] \rightarrow [0, \frac{1}{2}]$ for the inverse function of d_- on $[0, \frac{1}{2}]$, we have the expansion

$$a_-(d) = \frac{1}{2} - \sqrt{-1152(d - \frac{1}{24})^3} + O((d - \frac{1}{24})^2) \quad (2.3.5)$$

as $d \uparrow \frac{1}{24}$.

In order to break the symmetry caused by the swap $u \leftrightarrow v$, we set out to describe the roots of $G(u, v; a, d) = 0$ that have $v > u$. To this end, we introduce two regions

$$\begin{aligned} \Omega_- &= \{(a, d) : 0 < a < 1 \text{ and } 0 < d < d_-(a)\}, \\ \Omega_+ &= \{(a, d) : 0 < a < 1 \text{ and } d_-(a) < d < d_+(a)\} \end{aligned} \quad (2.3.6)$$

that are studied separately in the two results below. In Ω_- there are three such bichromatic equilibria with $v > u$. These equilibria can be ordered and the middle one is the only stable one. Two (or three) of these equilibria collide at $d = d_-(a)$ in a saddle node (or pitchfork) bifurcation, leaving a single unstable bichromatic equilibrium in Ω_+ . This equilibrium in turns collides with its swapped counterpart and the monochromatic equilibrium (a, a) on the boundary $d_+(a)$. These processes

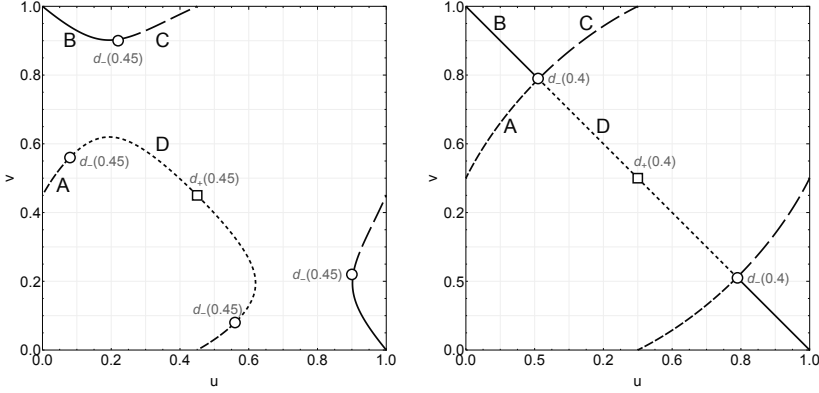


Figure 2.6: Illustration of the functions $(\bar{u}_A, \bar{v}_A)$, $(\bar{u}_B, \bar{v}_B)$, $(\bar{u}_C, \bar{v}_C)$ from Proposition 2.3.2 and the function $(\bar{u}_D, \bar{v}_D)$ from Proposition 2.3.3 for $a = .45$ (left panel) and $a = .5$ (right panel). The bifurcations occurring at $d_+(a)$ and $d_-(a)$ are indicated by squares and circles; see Proposition 2.3.1.

are illustrated in Figure 2.6. In particular, we see that $\bar{\Omega}_-$ coincides with the set Ω_{bc} introduced in §2.2; cf. Figures 2.4 and 2.5.

Proposition 2.3.2 (see §2.3.5). There exist continuous functions

$$(\bar{u}_A, \bar{v}_A) : \bar{\Omega}_- \rightarrow [0, 1]^2 \quad (\bar{u}_B, \bar{v}_B) : \bar{\Omega}_- \rightarrow [0, 1]^2, \quad (\bar{u}_C, \bar{v}_C) : \bar{\Omega}_- \rightarrow [0, 1]^2 \quad (2.3.7)$$

that satisfy the following properties.

- (i) Pick any $(a, d) \in \bar{\Omega}_-$. Then we have

$$G(\bar{u}_\#(a, d), \bar{v}_\#(a, d); a, d) = 0 \quad (2.3.8)$$

for all $\# \in \{A, B, C\}$. If also $(a, d) \in \Omega_-$, then the matrix

$$D_{1,2}G(\bar{u}_\#(a, d), \bar{v}_\#(a, d); a, d) \quad (2.3.9)$$

has two strictly negative eigenvalues if $\# = B$ or one strictly positive and one strictly negative eigenvalue if $\# \in \{A, C\}$.

- (ii) For any $0 \leq a \leq 1$ we have the identities

$$(\bar{u}_A, \bar{v}_A)(a, 0) = (0, a), \quad (\bar{u}_B, \bar{v}_B)(a, 0) = (0, 1), \quad (\bar{u}_C, \bar{v}_C)(a, 0) = (a, 1). \quad (2.3.10)$$

- (iii) For any $(a, d) \in \Omega_-$ we have the ordering

$$0 < \bar{u}_A(a, d) < \bar{u}_B(a, d) < \bar{u}_C(a, d) < a < \bar{v}_A(a, d) < \bar{v}_B(a, d) < \bar{v}_C(a, d). \quad (2.3.11)$$

(iv) For any $a \in [0, \frac{1}{2}]$ we have

$$(\bar{u}_B, \bar{v}_B)(a, d_-(a)) = (\bar{u}_C, \bar{v}_C)(a, d_-(a)), \quad (2.3.12)$$

while for any $a \in [\frac{1}{2}, 1]$ we have

$$(\bar{u}_A, \bar{v}_A)(a, d_-(a)) = (\bar{u}_B, \bar{v}_B)(a, d_-(a)). \quad (2.3.13)$$

Proposition 2.3.3 (see §2.3.5). There exist continuous functions

$$(\bar{u}_D, \bar{v}_D) : \bar{\Omega}_+ \rightarrow [0, 1]^2 \quad (2.3.14)$$

that satisfy the following properties.

(i) Pick any $(a, d) \in \bar{\Omega}_+$. Then we have

$$G(\bar{u}_D(a, d), \bar{v}_D(a, d); a, d) = 0. \quad (2.3.15)$$

If also $(a, d) \in \Omega_+$, then the matrix

$$D_{1,2}G(\bar{u}_D(a, d), \bar{v}_D(a, d); a, d) \quad (2.3.16)$$

has one strictly positive and one strictly negative eigenvalue.

(ii) For any $0 \leq a \leq 1$ we have the identity

$$(\bar{u}_D, \bar{v}_D)(a, d_+(a)) = (a, a). \quad (2.3.17)$$

(iii) For any $(a, d) \in \Omega_+$ we have the ordering

$$0 < \bar{u}_D(a, d) < a < \bar{v}_D(a, d) < 1. \quad (2.3.18)$$

(iv) For any $a \in [0, \frac{1}{2}]$ we have the identity

$$(\bar{u}_D, \bar{v}_D)(a, d_-(a)) = (\bar{u}_A, \bar{v}_A)(a, d_-(a)), \quad (2.3.19)$$

while for any $a \in [\frac{1}{2}, 1]$ we have

$$(\bar{u}_D, \bar{v}_D)(a, d_-(a)) = (\bar{u}_C, \bar{v}_C)(a, d_-(a)). \quad (2.3.20)$$

Corollary 2.3.4. For any $(a, d) \in \bar{\Omega}_-$, we have the identities

$$\begin{aligned} (\bar{u}_A, \bar{v}_A)(a, d) &= (1 - \bar{v}_C, 1 - \bar{u}_C)(1 - a, d), \\ (\bar{u}_B, \bar{v}_B)(a, d) &= (1 - \bar{v}_B, 1 - \bar{u}_B)(1 - a, d), \\ (\bar{u}_C, \bar{v}_C)(a, d) &= (1 - \bar{v}_A, 1 - \bar{u}_A)(1 - a, d). \end{aligned} \quad (2.3.21)$$

In addition, for any $(a, d) \in \bar{\Omega}_+$ we have the identity

$$(\bar{u}_D, \bar{v}_D)(a, d) = (1 - \bar{v}_D, 1 - \bar{u}_D)(1 - a, d). \quad (2.3.22)$$

Proof. The symmetry $g(1-u, 1-a) = -g(u, a)$ implies that

$$G(1-u, 1-v; 1-a, d) = -G(u, v; a, d). \quad (2.3.23)$$

In addition, we have $G(u, v; a, d) = 0$ if and only if $G(v, u; a, d) = 0$. The statements hence follow from the ordering (2.3.11). $\square$

Our final result concerns the special case $a = \frac{1}{2}$, in which case it is possible to be more explicit. In particular, the bichromatic roots $(\bar{u}_B, \bar{v}_B)$ and $(\bar{u}_D, \bar{v}_D)$ lie on the line $u + v = 1$ and collide precisely when $g'(u; \frac{1}{2}) = g'(v; \frac{1}{2}) = 0$.

Corollary 2.3.5. For any $0 \leq d \leq \frac{1}{24}$ we have

$$\bar{u}_B(\frac{1}{2}, d) = 1 - \bar{v}_B(\frac{1}{2}, d), \quad (2.3.24)$$

while for any $\frac{1}{24} \leq d \leq \frac{1}{16} = d_+(1/2)$ we have

$$\bar{u}_D(\frac{1}{2}, d) = 1 - \bar{v}_D(\frac{1}{2}, d). \quad (2.3.25)$$

In addition, we have the identities

$$\begin{aligned} \bar{u}_A(\frac{1}{2}, \frac{1}{24}) = \bar{u}_B(\frac{1}{2}, \frac{1}{24}) = \bar{u}_C(\frac{1}{2}, \frac{1}{24}) = \bar{u}_D(\frac{1}{2}, \frac{1}{24}) &= \frac{1}{2} - \frac{1}{6}\sqrt{3}, \\ \bar{v}_A(\frac{1}{2}, \frac{1}{24}) = \bar{v}_B(\frac{1}{2}, \frac{1}{24}) = \bar{v}_C(\frac{1}{2}, \frac{1}{24}) = \bar{v}_D(\frac{1}{2}, \frac{1}{24}) &= \frac{1}{2} + \frac{1}{6}\sqrt{3}. \end{aligned} \quad (2.3.26)$$

2.3.1 Saddle-nodes around $a = 0$

In this section we construct the branches $(\bar{u}_B, \bar{v}_B)$ and $(\bar{u}_C, \bar{v}_C)$ of solutions to $G(u, v; a, d) = 0$ in the regime where $(a, d) \approx (0, 0)$. In particular, we define

$$H(u, v; a, d) = G(u, 1+v; a, d) \quad (2.3.27)$$

and determine the zeroes of H for which (u, v, a, d) are small.

Proposition 2.3.6. There exist constants $\delta_a > 0$, $\delta_d > 0$ and $\epsilon > 0$ together with a function $d_c : (0, \delta_a) \rightarrow (0, \delta_d)$ and a constant $K \geq 1$ so that the following holds true.

- (i) For every $0 < a < \delta_a$ and $0 < d < d_c(a)$ the equation $H(u, v; a, d) = 0$ has precisely two solutions on the set $\{|u| + |v| < \epsilon\}$.
- (ii) For every $0 < a < \delta_a$ and $d_c(a) < d < \delta_d$ the equation $H(u, v; a, d) = 0$ has no solutions on the set $\{|u| + |v| < \epsilon\}$.
- (iii) For every $0 < a < \delta_a$ we have the estimate

$$\left| d_c(a) - \frac{1}{8}a^2 - \frac{1}{32}a^4 \right| \leq Ka^5. \quad (2.3.28)$$

- (iv) For every $0 < a < \delta_a$ the equation $H(u, v; d_c(a), a) = 0$ has precisely one solution $(u_c(a), v_c(a))$ on the set $\{|u| + |v| < \epsilon\}$. We have the estimates

$$\begin{aligned} |u_c(a) - \tfrac{1}{2}a| &\leq Ka^4, \\ |v_c(a) + \tfrac{1}{4}a^2 + \tfrac{1}{8}a^3| &\leq Ka^4. \end{aligned} \quad (2.3.29)$$

Writing $H(u, v; a, d) = (H_1(u, v; a, d), H_2(u, v; a, d))^T$ we can compute

$$\begin{aligned} H_1(u, v; a, d) &= u^2 - ua - u^3 + u^2a + 2d + 2dv - 2du, \\ H_2(u, v; a, d) &= -v - 2v^2 + va - v^3 + v^2a + 2du - 2d - 2dv. \end{aligned} \quad (2.3.30)$$

Our strategy is to use the identity $H_2 = 0$ to eliminate v and then recast $H_1 = 0$ into the normal form of a saddle-node bifurcation.

Lemma 2.3.7. Pick $\delta > 0$ sufficiently small. Then there exist constants $\epsilon > 0$ and $K \geq 1$ together with functions

$$\alpha_0 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \alpha_1 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad R_{\alpha;2} : (-\delta, \delta)^3 \rightarrow \mathbb{R} \quad (2.3.31)$$

that satisfy the following properties.

- (i) For every $(u, a, d) \in (-\delta, \delta)^3$ the equation $H_2(u, v; a, d) = 0$ has a unique solution $v = v_*$ in the set $\{|v| < \epsilon\}$. This solution is given by

$$v_*(u; a, d) = \alpha_0(a, d) + \alpha_1(a, d)u + u^2 R_{\alpha;2}(u; a, d). \quad (2.3.32)$$

- (ii) Upon writing

$$\begin{aligned} \alpha_0(a, d) &= -2d - 2ad + S_{\alpha_0}(a, d), \\ \alpha_1(a, d) &= 2d + S_{\alpha_1}(a, d), \end{aligned} \quad (2.3.33)$$

the bounds

$$\begin{aligned} |S_{\alpha_0}(a, d)| &\leq K(d^2 + |d|a^2), \\ |S_{\alpha_1}(a, d)| &\leq K|d|(|a| + |d|), \end{aligned} \quad (2.3.34)$$

together with

$$\begin{aligned} |\partial_d S_{\alpha_0}(a, d)| &\leq K(|d| + a^2), \\ |\partial_d S_{\alpha_1}(a, d)| &\leq K(|a| + |d|) \end{aligned} \quad (2.3.35)$$

hold for all $(a, d) \in (-\delta, \delta)^2$.

- (iii) For every $(u, a, d) \in (-\delta, \delta)^3$ we have the bounds

$$|R_{\alpha;2}(u; a, d)| + |R'_{\alpha;2}(u; a, d)| + |R''_{\alpha;2}(u; a, d)| + |R'''_{\alpha;2}(u; a, d)| \leq Kd^2, \quad (2.3.36)$$

together with

$$\begin{aligned} |\partial_d R_{\alpha;2}(u; a, d)| + |\partial_d R'_{\alpha;2}(u; a, d)| \\ + |\partial_d R''_{\alpha;2}(u; a, d)| + |\partial_d R'''_{\alpha;2}(u; a, d)| \leq K|d|. \end{aligned} \quad (2.3.37)$$

Proof. Substituting the Ansatz (2.3.32) into H_2 , we obtain the fixed point problems

$$\begin{aligned}\alpha_0 &= -2d - 2d\alpha_0 + \alpha_0 a + \alpha_0^2 a - 2\alpha_0^2 - \alpha_0^3, \\ \alpha_1 &= 2d - 4\alpha_0\alpha_1 + 2\alpha_0\alpha_1 a - 2d\alpha_1 + \alpha_1 a - 3\alpha_0^2\alpha_1,\end{aligned}\tag{2.3.38}$$

together with

$$\begin{aligned}R_2 &= 2\alpha_0 R_2 a + R_2 a + \alpha_1^2 a - 3\alpha_0\alpha_1^2 - 2dR_2 - 2\alpha_1^2 - 3\alpha_0^2 R_2 - 4\alpha_0 R_2 \\ &\quad + (2\alpha_1 R_2 a - 6\alpha_0\alpha_1 R_2 - 4\alpha_1 R_2 - \alpha_1^3)u \\ &\quad + (R_2^2 a - 3\alpha_0 R_2^2 - 2R_2^2 - 3\alpha_1^2 R_2)u^2 \\ &\quad - 3\alpha_1 R_2^2 u^3 - R_2^3 u^4.\end{aligned}\tag{2.3.39}$$

These fixed point problems can be successively solved for small a , d and u , along with their differentiated counterparts. The desired estimates can subsequently be obtained in a standard fashion by computing Taylor expansions. $\square$

In order to eliminate v from H_1 , we define the function

$$\begin{aligned}\mathcal{J}(u; a, d) &= H_1(u, v_*(u; a, d); a, d) \\ &= \beta_0(a, d) + \beta_1(a, d)u + u^2(1 + a - u + R_{\beta;2}(u; a, d)),\end{aligned}\tag{2.3.40}$$

which forces us to write

$$\begin{aligned}\beta_0(a, d) &= 2d + 2d\alpha_0(a, d), \\ \beta_1(a, d) &= -a - 2d + 2d\alpha_1(a, d), \\ R_{\beta;2}(u; a, d) &= 2dR_{\alpha;2}(u; a, d).\end{aligned}\tag{2.3.41}$$

By applying a shift to u the linear term in (2.3.40) can be removed, transforming (2.3.40) into a normal form for saddle node bifurcations.

Lemma 2.3.8. Pick $\delta > 0$ sufficiently small. Then there exist constants $\epsilon > 0$ and $K \geq 1$ together with functions

$$u_* : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \zeta_0 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad R_{\zeta;2} : (-\delta, \delta)^3 \rightarrow \mathbb{R}\tag{2.3.42}$$

that satisfy the following properties.

- (i) For every $(\tilde{u}, a, d) \in (-\delta, \delta)^3$, we have the identity

$$\mathcal{J}(u_*(a, d) + \tilde{u}; a, d) = \zeta_0(a, d) + \tilde{u}^2[1 + R_{\zeta;2}(\tilde{u}; a, d)].\tag{2.3.43}$$

- (ii) Upon writing

$$\begin{aligned}u_*(a, d) &= \frac{1}{2}a + d - \frac{1}{8}a^2 + \frac{1}{2}ad - \frac{1}{16}a^3 + S_{u_*}(a, d), \\ \zeta_0(a, d) &= 2d - \frac{1}{4}a^2 - da - 5d^2 + \frac{1}{8}a^3 + \frac{1}{4}da^2 - \frac{1}{64}a^4 + S_{\zeta_0}(a, d),\end{aligned}\tag{2.3.44}$$

the bounds

$$\begin{aligned} |S_{u_*}(a, d)| &\leq K[d^2 + a^2|d| + a^4], \\ |S_{\zeta_0}(a, d)| &\leq K(a^5 + |d|^3 + d^2|a| + |a|^3|d|), \end{aligned} \quad (2.3.45)$$

together with

$$\begin{aligned} |\partial_d S_{u_*}(a, d)| &\leq K[|d| + a^2], \\ |\partial_d S_{\zeta_0}(a, d)| &\leq K(d^2 + |d||a| + |a|^3) \end{aligned} \quad (2.3.46)$$

hold for all $(a, d) \in (-\delta, \delta)^2$.

(iii) For every $(\tilde{u}, a, d) \in (-\delta, \delta)^3$ we have the bounds

$$\begin{aligned} |R_{\zeta;2}(\tilde{u}; a, d)| &\leq K(|a| + |d|), \\ |R'_{\zeta;2}(\tilde{u}; a, d)| &\leq K|d|^3. \end{aligned} \quad (2.3.47)$$

Proof. We first introduce the notation

$$\mathcal{N}_{\beta;2;u_*}(\tilde{u}; a, d) = \tilde{u}^{-2} \left[R_{\beta;2}(u_* + \tilde{u}) - R_{\beta;2}(u_*; a, d) - R'_{\beta;2}(u_*; a, d)\tilde{u} \right] \quad (2.3.48)$$

for $\tilde{u} \neq 0$, together with $\mathcal{N}_{\beta;2;u_*}(0; a, d) = \frac{1}{2}R''_{\beta;2}(u_*; a, d)$. This allows us to compute

$$\mathcal{J}(u_* + \tilde{u}, a, d) = \gamma_0(a, d, u_*) + \gamma_1(a, d, u_*)\tilde{u} + (1 + a - 3u_* + R_{\gamma;2}(\tilde{u}; a, d, u_*))\tilde{u}^2, \quad (2.3.49)$$

in which

$$\begin{aligned} \gamma_0(a, d, u_*) &= \beta_0 + \beta_1 u_* + u_*^2[1 + a - u_* + R_{\beta;2}(u_*)], \\ \gamma_1(a, d, u_*) &= \beta_1 + u_*^2[-1 + R'_{\beta;2}(u_*; a, d)] + 2u_*[1 + a - u_* + R_{\beta;2}(u_*)], \\ R_{\gamma;2}(\tilde{u}; a, d, u_*) &= (u_* + \tilde{u})^2 \mathcal{N}_{\beta;2;u_*}(\tilde{u}; a, d) + R'_{\beta;2}(u_*; a, d)(2u_*\tilde{u} + \tilde{u}^2) \\ &\quad + R_{\beta;2}(u_*; a, d). \end{aligned} \quad (2.3.50)$$

On account of (2.3.36) we have¹

$$|\mathcal{N}_{\beta;2;u_*}(\tilde{u}; a, d)| + |\mathcal{N}'_{\beta;2;u_*}(\tilde{u}; a, d)| \leq C'_1 \quad (2.3.51)$$

and hence also

$$|\mathcal{R}_{\gamma;2}(\tilde{u}; a, d, u_*)| + |\mathcal{R}'_{\gamma;2}(\tilde{u}; a, d, u_*)| \leq C'_2. \quad (2.3.52)$$

Setting $\gamma_1 = 0$ leads to the fixed point problem

$$u_* = -\frac{1}{2}\beta_1 - au_* + \frac{3}{2}u_*^2 - \frac{1}{2}u_*^2 R'_{\beta;2}(u_*; a, d) - u_* R_{\beta;2}(u_*; a, d), \quad (2.3.53)$$

¹All primed constants in this section are strictly positive and do not depend on the variables appearing on the left hand side of the inequalities where they appear.

which has a unique small solution that we write for the moment as

$$u_*(a, d, \beta_1) = -\frac{1}{2}\beta_1 + \frac{1}{2}a\beta_1 + \frac{3}{8}\beta_1^2 - \frac{1}{2}a^2\beta_1 - \frac{9}{8}a\beta_1^2 - \frac{9}{16}\beta_1^3 + \tilde{S}_{u_*}(a, d, \beta_1). \quad (2.3.54)$$

In a standard fashion one obtains the bound

$$\left| \tilde{S}_{u_*}(a, d, \beta_1) \right| \leq C'_3 [\beta_1^4 + |a| |\beta_1|^3 + (a^2 + d^2)\beta_1^2 + (|a|^3 + |d|^3) |\beta_1|]. \quad (2.3.55)$$

Differentiating (2.3.53) we obtain

$$\begin{aligned} D_d u_* &= -\frac{1}{2}\partial_d \beta_1 - \frac{1}{2}u_*^2 \partial_d R'_{\beta;2}(u_*; a, d) - u_* \partial_d R_{\beta;2}(u_*; a, d) \\ &\quad - a D_d u_* + 3u_* D_d u_* - u_* R'_{\beta;2}(u_*; a, d) D_d u_* - R_{\beta;2}(u_*; a, d) D_d u_*, \end{aligned} \quad (2.3.56)$$

which yields the estimate

$$\left| D_d u_* - \left[-\frac{1}{2} + \frac{1}{2}a + \frac{3}{4}\beta_1 \right] \partial_d \beta_1 \right| \leq C'_4 [d^2 |\beta_1| + a^2 + |a| |\beta_1| + \beta_1^2] |\partial_d \beta_1|. \quad (2.3.57)$$

Using the bounds

$$\begin{aligned} |\beta_0(a, d) - [2d - 4d^2]| &\leq C'_5 d^2 (|a| + |d|), \\ |\beta_1(a, d) + (a + 2d)| &\leq C'_5 d^2, \end{aligned} \quad (2.3.58)$$

together with

$$\begin{aligned} |\partial_d \beta_0(a, d) - [2 - 8d]| &\leq C'_5 |d| (|a| + |d|), \\ |\partial_d \beta_1(a, d) + 2| &\leq C'_5 |d| \end{aligned} \quad (2.3.59)$$

and finally

$$|R_{\beta;2}(u_*; a, d)| + |R'_{\beta;2}(u_*; a, d)| + |R''_{\beta;2}(u_*; a, d)| + |R'''_{\beta;2}(u_*; a, d)| \leq C'_6 |d|^3, \quad (2.3.60)$$

the desired estimates follow by writing

$$\begin{aligned} \zeta_0(a, d) &= \gamma_0(a, d, u_*(a, d)), \\ R_{\zeta;2}(\tilde{u}; a, d) &= a - 3u_*(a, d) + R_{\gamma;2}(\tilde{u}, a, d, u_*(a, d)) \end{aligned} \quad (2.3.61)$$

and computing $\partial_d \zeta_0$ directly. □

Proof of Proposition 2.3.6. We note first that the map

$$\tilde{u} \mapsto \tilde{u} \sqrt{1 + R_{\zeta;2}(\tilde{u}; a, d)} \quad (2.3.62)$$

is invertible for $(\tilde{u}, a, d) \in (-\delta, \delta)^3$. In order to find d_c it hence suffices to solve $\zeta_0(a, d_c) = 0$, which gives the fixed-point problem

$$d_c = \frac{1}{8}a^2 - \frac{1}{16}a^3 + \frac{1}{128}a^4 + \frac{1}{2}ad_c - \frac{1}{8}d_c a^2 + \frac{5}{2}d_c^2 - \frac{1}{2}S_{\zeta_0}(a, d_c). \quad (2.3.63)$$

Our estimate on $\partial_d S_{\zeta_0}$ guarantees the existence of a unique small solution for small a . The estimate (iii) now follows in a standard fashion. Writing $u_c = u_*(a, d_c(a))$ and $v_c = v_*(u_*(a, d_c(a)); a, d_c(a))$ the expansions in (iv) follow directly by substitution. $\square$

2.3.2 The cusp bifurcation around $a = 1/2$

Our goal here is to unfold the structure of the solution-set to $G(u, v; a, d)$ near the critical point $(a, d) = (1/2, 1/24)$ where the branches $(\bar{u}_\#, \bar{v}_\#)$ with $\# \in \{A, B, C, D\}$ all collide in a cusp bifurcation; see Figures 2.1 and 2.6. In particular, we introduce the cusp location

$$\begin{aligned} (u_{\text{cp}}, v_{\text{cp}}, a_{\text{cp}}, d_{\text{cp}}) &= (u_{\min}(\tfrac{1}{2}), u_{\max}(\tfrac{1}{2}), \tfrac{1}{12}, \tfrac{1}{24}) \\ &= (\tfrac{1}{2} - \tfrac{1}{6}\sqrt{3}, \tfrac{1}{2} + \tfrac{1}{6}\sqrt{3}, \tfrac{1}{2}, \tfrac{1}{24}) \end{aligned} \quad (2.3.64)$$

together with the function

$$H_{\text{cp}}(u, v; a, d) = G(u_{\text{cp}} + u, v_{\text{cp}} + v; a_{\text{cp}} + a, d_{\text{cp}} + d) \quad (2.3.65)$$

and set out to determine the zeroes of H_{cp} for which (u, v, a, d) are small.

Proposition 2.3.9. There exist constants $\delta_a > 0$, $\delta_d > 0$ and $\epsilon > 0$ together with a function $a_c : (-\delta_d, 0) \rightarrow (0, \delta_a)$ and a constant $K \geq 1$ so that the following holds true.

- (i) For every $0 \leq d < \delta_d$ and any $a \in (-\delta_a, \delta_a)$, the equation $H_{\text{cp}}(u, v; a, d) = 0$ has precisely one solution on the set $\{|u| + |v| < \epsilon\}$.
- (ii) For every $-\delta_d < d < 0$ and $a \in (-\delta_a, \delta_a)$ with $|a| > a_c(d)$, the equation $H_{\text{cp}}(u, v; a, d) = 0$ has precisely one solution on the set $\{|u| + |v| < \epsilon\}$.
- (iii) For every $-\delta_d < d < 0$ and $a \in (-a_c(d), a_c(d))$, the equation $H_{\text{cp}}(u, v; a, d) = 0$ has precisely three solutions on the set $\{|u| + |v| < \epsilon\}$.
- (iv) For any $-\delta_d < d < 0$, the equation $H_{\text{cp}}(u, v; a_c(d), d) = 0$ has precisely two solutions on the set $\{|u| + |v| < \epsilon\}$.
- (v) For any $d \in (-\delta_d, 0)$ we have the estimate

$$\left| a_c(d) - \sqrt{-1152d^3} \right| \leq Kd^2. \quad (2.3.66)$$

In order to recast our equation into an efficient form, we introduce two functions (h_1, h_2) by writing

$$\begin{pmatrix} h_1(p, q; a, d) \\ h_2(p, q; a, d) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} H_{\text{cp}}(p + q, p - q; a, d), \quad (2.3.67)$$

which can be evaluated as

$$\begin{aligned} h_1(p, q; a, d) &= -\frac{1}{3}a + 2ap^2 - 6pq^2 + 2aq^2 - \frac{2}{3}\sqrt{3}aq - 2p^3 + 2\sqrt{3}pq, \\ h_2(p, q; a, d) &= -\frac{2}{3}\sqrt{3}ap - \frac{1}{3}q + \frac{4}{3}\sqrt{3}d + \sqrt{3}p^2 + \sqrt{3}q^2 - 6p^2q - 8dq + 4apq - 2q^3. \end{aligned} \quad (2.3.68)$$

Since h_2 contains a term that is linear in q , we set out to eliminate this variable by demanding $h_2 = 0$.

Lemma 2.3.10. Pick $\delta > 0$ sufficiently small. Then there exist constants $\epsilon > 0$ and $K \geq 1$, together with functions

$$\alpha_0 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \alpha_1 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \alpha_2 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad R_{\alpha;3} : (-\delta, \delta)^3 \rightarrow \mathbb{R} \quad (2.3.69)$$

that satisfy the following properties.

- (i) For every $(p, a, d) \in (-\delta, \delta)^3$ the equation $h_2(p, q; a, d) = 0$ has a unique solution $q = q_*$ in the set $\{|q| < \epsilon\}$. This solution is given by

$$q_*(p; a, d) = \alpha_0(a, d) + \alpha_1(a, d)p + \alpha_2(a, d)p^2 + p^3 R_{\alpha;3}(p; a, d). \quad (2.3.70)$$

- (ii) Upon writing

$$\begin{aligned} \alpha_0(a, d) &= 4\sqrt{3}d + S_{\alpha_0}(a, d), \\ \alpha_1(a, d) &= -2\sqrt{3}a + S_{\alpha_1}(a, d), \\ \alpha_2(a, d) &= 3\sqrt{3} + S_{\alpha_2}(a, d), \end{aligned} \quad (2.3.71)$$

the bounds

$$\begin{aligned} |S_{\alpha_0}(a, d)| &\leq Kd^2, & |DS_{\alpha_0}(a, d)| &\leq K|d|, \\ |S_{\alpha_1}(a, d)| &\leq K|a||d|, & |DS_{\alpha_1}(a, d)| &\leq K(|a| + |d|), \\ |S_{\alpha_2}(a, d)| &\leq K(|d| + a^2), & |DS_{\alpha_2}(a, d)| &\leq K \end{aligned} \quad (2.3.72)$$

hold for all $(a, d) \in (-\delta, \delta)^2$.

- (iii) For every $(p, a, d) \in (-\delta, \delta)^3$ we have the bounds

$$|R_{\alpha;3}(p; a, d)| \leq K(|a| + |p|), \quad |D_{a,d}R_{\alpha;3}(p; a, d)| \leq K, \quad (2.3.73)$$

together with

$$\left| R_{\alpha;3}^{(i)}(p; a, d) \right| + \left| D_{a,d}R_{\alpha;3}^{(i)}(p; a, d) \right| \leq K \quad (2.3.74)$$

for all $1 \leq i \leq 6$.

Proof. Substituting the Ansatz (2.3.70) into h_2 , we obtain the fixed point problems

$$\begin{aligned}\frac{1}{3}\alpha_0 &= \frac{4}{3}\sqrt{3}d - 8d\alpha_0 + \sqrt{3}\alpha_0^2 - 2\alpha_0^3, \\ \frac{1}{3}\alpha_1 &= -\frac{2}{3}\sqrt{3}a + 4\alpha_0a + 2\sqrt{3}\alpha_0\alpha_1 - 6\alpha_0^2\alpha_1 - 8\alpha_1d, \\ \frac{1}{3}\alpha_2 &= \sqrt{3} - 6\alpha_0 + \sqrt{3}\alpha_1^2 + 4a\alpha_1 - 6\alpha_1^2\alpha_0 + 2\sqrt{3}\alpha_0\alpha_2 - 6\alpha_0^2\alpha_2 - 8d\alpha_2,\end{aligned}\tag{2.3.75}$$

together with

$$\begin{aligned}\frac{R_{\alpha;3}}{3} &= -2\alpha_1^3 - 6\alpha_1 + 2\sqrt{3}\alpha_1\alpha_2 - 12\alpha_2\alpha_1\alpha_0 + 4a\alpha_2 + 2\sqrt{3}\alpha_0R_{\alpha;3} - 6\alpha_0^2R_{\alpha;3} - 8dR_{\alpha;3} \\ &\quad + \left(\sqrt{3}\alpha_2^2 - 6\alpha_2 - 6\alpha_1^2\alpha_2 - 6\alpha_0\alpha_2^2 + 2\sqrt{3}\alpha_1R_{\alpha;3} - 12\alpha_0\alpha_1R_{\alpha;3} + 4aR_{\alpha;3}\right)p \\ &\quad + \left(2\sqrt{3}\alpha_2R_{\alpha;3} - 6\alpha_1\alpha_2^2 - 6\alpha_1^2R_{\alpha;3} - 6R_3 - 12\alpha_0\alpha_2R_{\alpha;3}\right)p^2 \\ &\quad + \left(\sqrt{3}R_{\alpha;3}^2 - 2\alpha_2^3 - 12\alpha_2\alpha_1R_{\alpha;3} - 6\alpha_0R_{\alpha;3}^2\right)p^3 \\ &\quad + \left(-6\alpha_2^2R_{\alpha;3} - 6\alpha_1R_{\alpha;3}^2\right)p^4 - 6\alpha_2R_{\alpha;3}^2p^5 - 2R_{\alpha;3}^3p^6.\end{aligned}\tag{2.3.76}$$

These fixed point problems can be successively solved for small a , d and p , which yields the desired estimates. $\square$

In order to eliminate q from h_1 , we define the function

$$\mathcal{J}(p; a, d) = \tilde{h}_1(p, q_*(p; a, d), a, d)\tag{2.3.77}$$

and obtain the following representation.

Corollary 2.3.11. Pick $\delta > 0$ sufficiently small. Then there exist a constant $K \geq 1$, together with functions

$$\beta_0 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \beta_1 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \beta_2 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad R_{\beta;3} : (-\delta, \delta)^3 \rightarrow \mathbb{R}\tag{2.3.78}$$

that satisfy the following properties.

(i) For every $(p, a, d) \in (-\delta, \delta)^3$ we have

$$\mathcal{J}(p; a, d) = \beta_0(a, d) + \beta_1(a, d)p + \beta_2(a, d)p^2 + p^3[16 + R_{\beta;3}(p; a, d)].\tag{2.3.79}$$

(ii) Upon writing

$$\begin{aligned}\beta_0(a, d) &= -\frac{1}{3}a + S_{\beta_0}(a, d), \\ \beta_1(a, d) &= 24d + S_{\beta_1}(a, d), \\ \beta_2(a, d) &= -16a + S_{\beta_2}(a, d),\end{aligned}\tag{2.3.80}$$

the bounds

$$\begin{aligned}
|S_{\beta_0}(a, d)| &\leq K |a| |d|, & |DS_{\alpha_0}(a, d)| &\leq K(|a| + |d|), \\
|S_{\beta_1}(a, d)| &\leq K(a^2 + d^2), & |DS_{\alpha_1}(a, d)| &\leq K(|a| + |d|), \\
|S_{\beta_2}(a, d)| &\leq K |a| (|d| + a^2), & |DS_{\alpha_2}(a, d)| &\leq K(|a| + |d|)
\end{aligned} \tag{2.3.81}$$

hold for all $(a, d) \in (-\delta, \delta)^2$.

(iii) For every $(p, a, d) \in (-\delta, \delta)^3$ we have the bounds

$$\begin{aligned}
|R_{\beta;3}(p; a, d)| &\leq K(|d| + a^2 + |a| |p| + p^2), \\
|R'_{\beta;3}(p; a, d)| &\leq K(|a| + |p|), \\
|D_{a,d}R_{\beta;3}(p; a, d)| + |D_{a,d}R'_{\beta;3}(p; a, d)| &\leq K,
\end{aligned} \tag{2.3.82}$$

together with

$$|R_{\beta;3}^{(i)}(p; a, d)| + |D_{a,d}R_{\beta;3}^{(i)}(p; a, d)| \leq K \tag{2.3.83}$$

for all $2 \leq i \leq 6$.

Proof. Substitution yields

$$\begin{aligned}
\beta_0 &= -\frac{1}{3}a - \frac{2}{3}\sqrt{3}a\alpha_0 + 2a\alpha_0^2, \\
\beta_1 &= 4a\alpha_0\alpha_1 - \frac{2}{3}\sqrt{3}a\alpha_1 - 6\alpha_0^2 + 2\sqrt{3}\alpha_0, \\
\beta_2 &= 4a\alpha_0\alpha_2 - \frac{2}{3}\sqrt{3}a\alpha_2 + 2a - 12\alpha_0\alpha_1 + 2\sqrt{3}\alpha_1 + 2a\alpha_1^2,
\end{aligned} \tag{2.3.84}$$

together with

$$\begin{aligned}
R_{\beta;3} &= 4a\alpha_1\alpha_2 - 6\alpha_1^2 - 12\alpha_0\alpha_2 + 2\sqrt{3}(\alpha_2 - 3\sqrt{3}) - \frac{2}{3}\sqrt{3}aR_{\alpha;3} + 4a\alpha_0R_{\alpha;3} \\
&\quad + \left(2a\alpha_2^2 - 12\alpha_1\alpha_2 + 2\sqrt{3}R_{\alpha;3} + 4a\alpha_1R_{\alpha;3} - 12\alpha_0R_{\alpha;3}\right)p \\
&\quad + \left(4a\alpha_2R_{\alpha;3} - 6\alpha_2^2 - 12\alpha_1R_{\alpha;3}\right)p^2 \\
&\quad + \left(2aR_{\alpha;3}^2 - 12\alpha_2R_{\alpha;3}\right)p^3 - 6R_{\alpha;3}^2p^4.
\end{aligned} \tag{2.3.85}$$

The desired estimates can be determined directly from these expressions. $\square$

By applying a (small) shift to p the undesired quadratic term in (2.3.79) can be eliminated. The bifurcation curve in (a, d) space can subsequently be found by determining when the remaining equation has roots of order two or higher.

Lemma 2.3.12. Pick $\delta > 0$ sufficiently small. Then there exists a constant $K \geq 1$ together with functions

$$\zeta_0 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \zeta_1 : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad \zeta_3 : (-\delta, \delta)^2 \rightarrow \mathbb{R} \quad (2.3.86)$$

and

$$p_* : (-\delta, \delta)^2 \rightarrow \mathbb{R}, \quad R_{\zeta;4} : (-\delta, \delta)^3 \rightarrow \mathbb{R} \quad (2.3.87)$$

that satisfy the following properties.

(i) For every $(\tilde{p}, a, d) \in (-\delta, \delta)^3$, we have the identity

$$\mathcal{J}(p_*(a, d) + \tilde{p}; a, d) = \zeta_0(a, d) + \zeta_1(a, d)\tilde{p} + \zeta_3(a, d)\tilde{p}^3 + \tilde{p}^4 R_{\zeta;4}(\tilde{p}; a, d). \quad (2.3.88)$$

(ii) Upon writing

$$\begin{aligned} \zeta_0(a, d) &= -\frac{1}{3}a + S_{\zeta_0}(a, d), \\ \zeta_1(a, d) &= 24d + S_{\zeta_1}(a, d), \\ \zeta_3(a, d) &= 16 + S_{\zeta_3}(a, d), \end{aligned} \quad (2.3.89)$$

the bounds

$$\begin{aligned} |S_{\zeta_0}(a, d)| &\leq K(|a||d| + |a|^3), & |DS_{\zeta_0}(a, d)| &\leq K(|a| + |d|), \\ |S_{\zeta_1}(a, d)| &\leq K(a^2 + d^2), & |DS_{\zeta_1}(a, d)| &\leq K(|a| + |d|), \\ |S_{\zeta_3}(a, d)| &\leq K(|a| + |d|), & |DS_{\zeta_3}(a, d)| &\leq K \end{aligned} \quad (2.3.90)$$

hold for all $(a, d) \in (-\delta, \delta)^2$.

(iii) For every $(\tilde{p}, a, d) \in (-\delta, \delta)^3$ we have the bounds

$$\begin{aligned} |R_{\zeta;4}(\tilde{p}; a, d)| + |R'_{\zeta;4}(\tilde{p}; a, d)| + |R''_{\zeta;4}(\tilde{p}; a, d)| &\leq K, \\ |D_{a,d}R_{\zeta;4}(\tilde{p}; a, d)| + |D_{a,d}R'_{\zeta;4}(\tilde{p}; a, d)| &\leq K. \end{aligned} \quad (2.3.91)$$

Proof. We first introduce the notation

$$\mathcal{N}_{\beta;3;p_*}(\tilde{p}; a, d) = \tilde{p}^{-3} \left[R_{\beta;3}(p_* + \tilde{p}; a, d) - R_{\beta;3}(p_*; a, d) - R'_{\beta;3}(p_*; a, d)\tilde{p} - \frac{1}{2}R''_{\beta;3}(p_*; a, d)\tilde{p}^2 \right] \quad (2.3.92)$$

for $\tilde{p} \neq 0$, with $\mathcal{N}_{\beta;3;p_*}(0; a, d) = \frac{1}{6}R'''_{\beta;3}(p_*; a, d)$. This allows us to compute

$$\begin{aligned} \mathcal{J}(p_* + \tilde{p}; a, d) &= \gamma_0(a, d, p_*) + \gamma_1(a, d, p_*)\tilde{p} + \gamma_2(a, d, p_*)\tilde{p}^2 \\ &\quad + [16 + R_{\gamma;3}(\tilde{p}; a, d, p_*)]\tilde{p}^3, \end{aligned} \quad (2.3.93)$$

in which

$$\begin{aligned}
\gamma_0(a, d, p_*) &= \beta_0(a, d) + \beta_1(a, d)p_*(a, d) + \beta_2(a, d)p_*(a, d)^2 \\
&\quad + p_*^3 [16 + R_{\beta;3}(p_*; a, d)], \\
\gamma_1(a, d, p_*) &= \beta_1(a, d) + 2\beta_2(a, d)p_* \\
&\quad + p_*^3 R'_{\beta;3}(p_*; a, d) + 3p_*^2 [16 + R_{\beta;3}(p_*; a, d)], \\
\gamma_2(a, d, p_*) &= \beta_2(a, d) + \frac{1}{2}p_*^3 R''_{\beta;3}(p_*; a, d) + 3p_*^2 R'_{\beta;3}(p_*; a, d) + 3p_* [16 + R_{\beta;3}(p_*; a, d)],
\end{aligned} \tag{2.3.94}$$

together with

$$\begin{aligned}
R_{\gamma;3}(\tilde{p}; a, d, p_*) &= (p_* + \tilde{p})^3 \mathcal{N}_{\beta;3;p_*}(\tilde{p}; a, d) + \frac{1}{2}(3p_*^2 + 3p_*\tilde{p} + \tilde{p}^2) R''_{\beta;3}(p_*; a, d) \\
&\quad + (3p_* + \tilde{p}) R'_{\beta;3}(p_*; a, d) + R_{\beta;3}(p_*; a, d).
\end{aligned} \tag{2.3.95}$$

On account of (2.3.83) we have

$$|\mathcal{N}_{\beta;3;p_*}(\tilde{p}; a, d)| + |\mathcal{N}'_{\beta;3;p_*}(\tilde{p}; a, d)| + |\mathcal{N}''_{\beta;3;p_*}(\tilde{p}; a, d)| + |\mathcal{N}'''_{\beta;3;p_*}(\tilde{p}; a, d)| \leq C'_1 \tag{2.3.96}$$

and hence also

$$|\mathcal{R}_{\gamma;3}(\tilde{p}; a, d, p_*)| + |\mathcal{R}'_{\gamma;3}(\tilde{p}; a, d, p_*)| + |\mathcal{R}''_{\gamma;3}(\tilde{p}; a, d, p_*)| + |\mathcal{R}'''_{\gamma;3}(\tilde{p}; a, d, p_*)| \leq C'_2 \tag{2.3.97}$$

for small $|\tilde{p}|$.

Setting $\gamma_2(a, d, p_*) = 0$ leads to a fixed-point equation for p_* , which has a unique small solution $p_*(a, d)$ that admits the bounds

$$|p_*(a, d)| \leq C'_3 |\beta_2(a, d)| \leq C'_4 a, \quad |D_{a,d} p_*(a, d)| \leq C'_4. \tag{2.3.98}$$

The results hence follow by writing

$$\begin{aligned}
\zeta_0(a, d) &= \gamma_0(a, d, p_*(a, d)), \\
\zeta_1(a, d) &= \gamma_1(a, d, p_*(a, d)), \\
\zeta_3(a, d) &= 16 + R_{\gamma;3}(0; a, d, p_*(a, d)),
\end{aligned} \tag{2.3.99}$$

together with

$$R_{\zeta;4}(\tilde{p}; a, d) = \tilde{p}^{-1} [R_{\gamma;3}(\tilde{p}; a, d, p_*(a, d)) - R_{\gamma;3}(0; a, d, p_*(a, d))] \tag{2.3.100}$$

for $\tilde{p} \neq 0$ and

$$R_{\zeta;4}(0; a, d) = R'_{\gamma;3}(0; a, d, p_*(a, d)). \tag{2.3.101}$$

Here we use (2.3.97) to get bounds on $R_{\zeta;4}$ and its derivatives. $\square$

Proof of Proposition 2.3.9. In order to find roots of order two or higher we need to solve the system

$$\begin{aligned}\zeta_0(a, d) + \zeta_1(a, d)\tilde{p} + \zeta_3(a, d)\tilde{p}^3 + \tilde{p}^4 R_{\zeta;4}(\tilde{p}, a, d) &= 0, \\ \zeta_1(a, d) + 3\zeta_3(a, d)\tilde{p}^2 + 4\tilde{p}^3 R_{\zeta;4}(\tilde{p}, a, d) + \tilde{p}^4 R'_{\zeta;4}(\tilde{p}, a, d) &= 0.\end{aligned}\tag{2.3.102}$$

Solving the second equation we find two branches

$$\tilde{p}_{\pm}(a, d) = \pm\sqrt{-d/2} + S_{\tilde{p}_{\pm}}(a, d),\tag{2.3.103}$$

in which we have

$$\begin{aligned}S_{\tilde{p}_{\pm}}(a, d) &\leq C'_1(|a| + |d|), \\ D_{a,d}S_{\tilde{p}_{\pm}}(a, d) &\leq C'_1.\end{aligned}\tag{2.3.104}$$

Plugging this into the first line of (2.3.102) we obtain

$$\frac{1}{3}a = \pm 16d\sqrt{-d/2} + O(d^2 + |ad| + |a|^3 + a^2\sqrt{|d|}),\tag{2.3.105}$$

from which the desired expression for a_c follows. $\square$

2.3.3 Geometry of the cubic

Our strategy to establish Propositions 2.3.1-2.3.3 hinges upon geometric properties of the cubic $g(u; a)$. As a preparation, we introduce the notation

$$u_{\text{infl}}(a) = \frac{1}{3}(a + 1)\tag{2.3.106}$$

together with

$$u_{\min}(a) = u_{\text{infl}}(a) - \frac{1}{3}\sqrt{1 - a(1 - a)}, \quad u_{\max}(a) = u_{\text{infl}}(a) + \frac{1}{3}\sqrt{1 - a(1 - a)}\tag{2.3.107}$$

and note that

$$g'(u_{\min}(a); a) = 0, \quad g''(u_{\text{infl}}(a); a) = 0, \quad g'(u_{\max}(a); a) = 0.\tag{2.3.108}$$

In addition, we note that $g''(u; a) > 0$ for $u < u_{\text{infl}}(a)$ and $g''(u; a) < 0$ for $u > u_{\text{infl}}(a)$. When $0 < a < \frac{1}{2}$ we have the ordering

$$0 < u_{\min}(a) < a < u_{\text{infl}}(a) < u_{\max}(a) < 1.\tag{2.3.109}$$

Recalling the definition (2.2.6), one readily sees that

$$G_1(u, v; a, d) + G_2(u, v; a, d) = g(u; a) + g(v; a).\tag{2.3.110}$$

In order to exploit the fact that this identity does not depend on d , we first use basic properties of the cubic to parametrize solutions to $g(u; a) + g(v; a) = 0$. Restricting ourselves to $a \in [0, \frac{1}{2}]$, it is possible to construct two solution curves $v_{\pm}(u)$ that are defined for $u \in [0, a]$; see Figure 2.7.

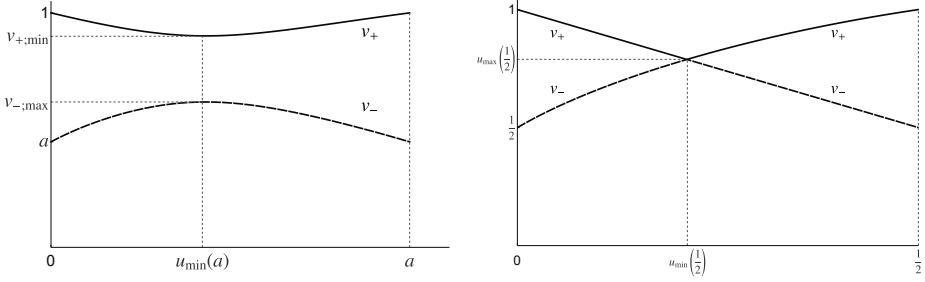


Figure 2.7: The functions $v_{\pm}$ defined in Lemmata 2.3.13-2.3.14 for $a = 0.45$ (left) and $a = 0.5$ (right). In the latter case the derivatives have a discontinuity at $u_{\min}(\frac{1}{2})$.

Lemma 2.3.13. Fix $0 < a < \frac{1}{2}$. Then there are two constants

$$a < v_{-;\max} < v_{+;\min} < 1 \quad (2.3.111)$$

together with two C^∞ -smooth functions

$$v_- : [0, a] \rightarrow [a, v_{-;\max}], \quad v_+ : [0, a] \rightarrow [v_{+;\min}, 1] \quad (2.3.112)$$

that satisfy the following properties.

- (i) We have $g(u; a) = -g(v_-(u); a) = -g(v_+(u); a)$ for all $0 \leq u \leq a$.
- (ii) If $g(u; a) = -g(v; a)$ for some pair $u \in (0, a)$ and $v \in [0, 1]$, then $v = v_-(u)$ or $v = v_+(u)$.
- (iii) We have $v_-(0) = v_-(a) = a$ and $v_+(0) = v_+(a) = 1$.
- (iv) We have the identities

$$v'_\pm(u) = -[g'(v_\pm(u); a)]^{-1} g'(u; a) \quad (2.3.113)$$

for all $0 \leq u \leq a$.

- (v) We have $v'_-(u) > -1$ for all $0 \leq u < a$, together with $v'_-(a) = -1$.

Proof. Since $0 < a < \frac{1}{2}$ we have $g(u_{\max}(a); a) > -g(u_{\min}(a); a)$, which implies that $v_\pm(u) \neq u_{\max}(a)$. Properties (i)-(iv) hence follow immediately from the implicit function theorem.

To obtain (v), we take $u_{\min}(a) < u < a$ and recall $a < v_-(u) < u_{\max}$. If $v_-(u) \leq u_{\text{infl}}(a)$ then clearly $g'(v_-(u); a) > g'(u; a) > 0$, as desired. In order to handle the remaining case $v_-(u) > u_{\text{infl}}(a)$, we introduce the reflection $u_{\text{refl}} = 2u_{\text{infl}}(a) - u$. Exploiting the point symmetry of the graph of $g(\cdot; a)$ around its inflection point $(u_{\text{infl}}(a), g(u_{\text{infl}}(a); a))$, the inequality $g(u_{\text{infl}}(a); a) > 0$ implies that

$$u_{\text{infl}}(a) < v_-(u) < u_{\text{refl}}. \quad (2.3.114)$$

Since $g''(\tilde{u}; a) < 0$ for $\tilde{u} > u_{\inf}(a)$, we obtain

$$g'(v_-(u); a) > g'(u_{\text{ref}}; a) = g'(u; a) > 0, \quad (2.3.115)$$

which implies $v'_-(u) > -1$. $\square$

Lemma 2.3.14. Fix $a = \frac{1}{2}$. Then there are two functions

$$v_- : [0, a] \rightarrow [a, u_{\max}(a)], \quad v_+ : [0, a] \rightarrow [u_{\max}(a), 1] \quad (2.3.116)$$

that satisfy items (i) - (iii) from Lemma 2.3.13 together with the following additional properties.

(i) We have

$$v_-(u_{\min}(a)) = v_+(u_{\min}(a)) = u_{\max}(a). \quad (2.3.117)$$

(ii) For any $u \in [0, a] \setminus \{u_{\min}(a)\}$ we have the identities

$$v'_\pm(u) = -[g'(v_\pm(u); a)]^{-1} g'(u; a). \quad (2.3.118)$$

(iii) We have $v_-(u) = 1 - u$ for all $u_{\min}(a) \leq u \leq 1$, while $v_+(u) = 1 - u$ for all $0 \leq u \leq u_{\min}(a)$.

(iv) We have the limits

$$\lim_{u \uparrow u_{\min}(a)} v'_-(u) = \lim_{u \downarrow u_{\min}(a)} v'_+(u) = 1, \quad (2.3.119)$$

together with

$$\lim_{u \uparrow u_{\min}(a)} v''_-(u) = \lim_{u \downarrow u_{\min}(a)} v''_+(u) = -\frac{4}{3}\sqrt{3}. \quad (2.3.120)$$

Proof. Items (i) and (iii) follow directly from the symmetry of $g(\cdot; a)$, while (ii) follows from the implicit function theorem. To obtain (iv), we first compute

$$g(u_{\min} + u; a) = g(u_{\min}; a) + \frac{1}{2}\sqrt{3}u^2 - u^3, \quad g(u_{\max} + v; a) = -g(u_{\min}; a) - \frac{1}{2}\sqrt{3}v^2 - v^3. \quad (2.3.121)$$

In particular, the identity

$$g(u_{\min} + u; a) = -g(u_{\max} + v; a) \quad (2.3.122)$$

can be rewritten as

$$v^2 = u^2 - \frac{2}{3}\sqrt{3}[u^3 + v^3], \quad (2.3.123)$$

which can be interpreted as a fixed point problem for v^2 upon assuming that v and u have the same sign. For small $|u|$ this problem has a solution that can be expanded as

$$v^2 = u^2 - \frac{4}{3}\sqrt{3}u^3 + O(u^4), \quad (2.3.124)$$

which yields

$$v = u - \frac{2}{3}\sqrt{3}u^2 + O(u^3). \quad (2.3.125)$$

The desired limits follow directly from this expansion. $\square$

2.3.4 Tangencies

Let us now fix $a \in [0, \frac{1}{2}]$. In order to find solutions to $G(u, v; a, d) = 0$ with $d > 0$, we introduce the function

$$v_d(u) = u - \frac{g(u; a)}{2d} \quad (2.3.126)$$

and note that the results above show that it suffices to find $u \in [0, a]$ for which one of the equations

$$v_{\pm}(u) = v_d(u) \quad (2.3.127)$$

holds.

Our goal here is to show that non-transverse intersections of this type can only occur at local minima of $v_{\pm} - v_d$. Together with the strict monotonicity

$$\partial_d v_d(u) < 0 \quad (2.3.128)$$

that holds for $u \in (0, a)$, this will allow us to obtain global results in §2.3.5.

For the moment, we note that

$$v_-(a) = v_d(a) = a \quad (2.3.129)$$

for every $d > 0$. In addition, we may compute

$$v'_d(a) = 1 - \frac{g'(a; a)}{2d}, \quad (2.3.130)$$

together with

$$v'_-(a) = -[g'(a; a)]^{-1}g'(a; a) = -1. \quad (2.3.131)$$

In particular, when $g'(a) = 4d$ the intersection (2.3.129) is tangential. In the sequel we show that in fact $v_- > v_d$ on $[0, a)$ for this critical value of d .

For intersections with $u \in (0, a)$ such explicit computations are significantly harder to carry out, which is why we pursue a more indirect approach here. As a preparation, we compute

$$v''_{\pm}(u) = -\frac{g''(u)}{g'(v_{\pm}(u))} - \frac{g'(u)^2}{g'(v_{\pm}(u))^3}g''(v_{\pm}(u)) \quad (2.3.132)$$

together with

$$v''_d(u) = -\frac{g''(u)}{2d}. \quad (2.3.133)$$

In addition, for any $\kappa \leq g'(u_{\text{infl}}(a); a) = \frac{1}{3}(a^2 - a + 1)$, we introduce the expressions

$$\begin{aligned} u_l(\kappa) &= u_{\text{infl}}(a) - \sqrt{\frac{1}{3}(g'(u_{\text{infl}}(a); a) - \kappa)}, \\ u_r(\kappa) &= u_{\text{infl}}(a) + \sqrt{\frac{1}{3}(g'(u_{\text{infl}}(a); a) - \kappa)}. \end{aligned} \quad (2.3.134)$$

It is easy to verify that $u_l(\kappa)$ and $u_r(\kappa)$ are the two solutions to the quadratic equation $g'(u; a) = \kappa$.

Lemma 2.3.15. Fix $0 < a \leq \frac{1}{2}$ and $d > 0$ and suppose that

$$v'_{\#}(u) = v'_d(u) = \beta \quad (2.3.135)$$

for some $\# \in \{-, +\}$ and $0 \leq u \leq a$, with $u \neq u_{\min}(a)$ in case $a = \frac{1}{2}$. Then the following statements hold.

(i) We have

$$2d[v''_{\#}(u) - v''_d(u)] = \frac{1}{1-\beta}g''(u; a) + \frac{\beta^3}{1-\beta}g''(v_{\#}(u); a). \quad (2.3.136)$$

(ii) If $\# = -$ then we have $\beta \in [-1, 0) \cup (1, \infty)$. On the other hand, the inclusion $\beta \in (0, 1)$ holds if $\# = +$.

(iii) We have $u = u_l(2d(1 - \beta))$.

(iv) Suppose that $\# = -$. Then the identity

$$v_{\#}(u) = u_l(2d(1 - \beta^{-1})) \quad (2.3.137)$$

holds if $v_{-}(u) \leq u_{\inf}(a)$. On the other hand, we have

$$v_{\#}(u) = u_r(2d(1 - \beta^{-1})) \quad (2.3.138)$$

if $v_{-}(u) > u_{\inf}(a)$.

(v) The identity

$$v_{\#}(u) = u_r(2d(1 - \beta^{-1})) \quad (2.3.139)$$

holds if $\# = +$.

Proof. We first consider the case $0 < a < \frac{1}{2}$. For any $0 \leq \tilde{u} \leq a$, one sees that $v'_d(\tilde{u}) = 1$ holds if and only if $g'(\tilde{u}; a) = v'_{\pm}(\tilde{u}) = 0$, which shows that $\beta \notin \{0, 1\}$.

For any $0 \leq \tilde{u} \leq a$ we have $g'(v_{+}(\tilde{u}); a) < 0$ and hence

$$\text{sign}[v'_{+}(\tilde{u})] = \text{sign}[g'(\tilde{u}; a)]. \quad (2.3.140)$$

If $\# = +$, this shows that $g'(u; a) > 0$ and $v'_d(u) > 0$, hence $\beta \in (0, 1)$.

On the other hand, we have $g'(v_{-}(\tilde{u}); a) > 0$ for all $0 \leq \tilde{u} \leq a$. If $g'(u; a) < 0$ then we have $v'_d(u) = \beta > 1$, while if $g'(u; a) > 0$ and $\# = -$ we may use item (v) from Lemma 2.3.13 to conclude

$$-1 \leq v'_-(u) = \beta < 0, \quad (2.3.141)$$

which establishes (ii).

In order to obtain (iii), it suffices to recall the bound $u \leq a \leq u_{\inf}(a)$ and note that the identity $v'_*(u) = \beta$ implies that

$$g'(u) = 2d(1 - \beta). \quad (2.3.142)$$

On the other hand, the identity $v'_\pm(u) = \beta$ implies

$$g'(v_\pm(u)) = -2d\beta^{-1}(1 - \beta). \quad (2.3.143)$$

Items (iv) and (v) now follow directly, remembering that $v_+(u) \geq u_{\max}(a) \geq u_{\inf}(a)$.

Exploiting (2.3.132) we may now compute

$$\begin{aligned} v''_\pm(u) &= \frac{\beta}{2d(1-\beta)}g''(u; a) + \frac{(2d)^2(1-\beta)^2\beta^3}{(2d)^3(1-\beta)^3}g''(v_\pm(u); a) \\ &= \frac{\beta}{2d(1-\beta)}g''(u; a) + \frac{\beta^3}{2d(1-\beta)}g''(v_\pm(u); a). \end{aligned} \quad (2.3.144)$$

The desired identity in (i) hence follows directly from (2.3.133). In order to conclude the proof, it suffices to note that the arguments above remain valid when $a = \frac{1}{2}$. Indeed, the critical cases $g'(u; a) = 0$ and $g'(v_\pm; a) = 0$ are excluded by the requirement that $u \neq u_{\min}(a)$. □

Lemma 2.3.16. Fix $0 < a \leq \frac{1}{2}$ and suppose that

$$v'_-(u) = v'_d(u) \quad (2.3.145)$$

for some $u_{\min}(a) \leq u < a$ and $d > 0$, with $u \neq u_{\min}(a)$ if $a = \frac{1}{2}$. Then we have

$$v''_-(u) > v''_d(u). \quad (2.3.146)$$

Proof. Exploiting (v) of Lemma 2.3.13 and (ii) of Lemma 2.3.15 we have $v'_-(u) \in (-1, 0)$. In addition, the inequalities $v_-(u) > u$ and $g''' < 0$ imply that $g''(v_-(u); a) \leq g''(u; a)$. Writing $\beta = v'_-(u)$ we may hence estimate

$$\begin{aligned} 2d[v''_-(u) - v''_d(u)] &\geq \frac{1}{1-\beta}g''(u; a) + \frac{\beta^3}{1-\beta}g''(u; a) \\ &= \frac{\beta^3+1}{1-\beta}g''(u; a) \\ &> 0, \end{aligned} \quad (2.3.147)$$

in which we used $g''(u; a) > 0$. □

Intersections with $v'_\pm(u) > 0$ are more delicate to analyze. Items (i), (iii) and (iv) of Lemma 2.3.15 suggest that it is worthwhile to consider the two functions

$$\begin{aligned} h_l(\beta) &= \frac{1}{2d(1-\beta)} \left[g''(u_l(2d(1-\beta)); a) + \beta^3 g''(u_l(2d(1-\beta^{-1})); a) \right], \\ h_r(\beta) &= \frac{1}{2d(1-\beta)} \left[g''(u_l(2d(1-\beta)); a) + \beta^3 g''(u_r(2d(1-\beta^{-1})); a) \right]. \end{aligned} \quad (2.3.148)$$

Lemma 2.3.17. Pick $0 < a \leq \frac{1}{2}$ and $0 < d \leq \frac{g'(a; a)}{4}$. Then for any $\beta > 1$ the inequality

$$h_l(\beta) < 0 \quad (2.3.149)$$

holds, while for any $\beta \in (0, \infty) \setminus \{1\}$ we have

$$h_r(\beta) > 0. \quad (2.3.150)$$

Proof. Observe first that for $\beta > 0$ we have

$$\max\{2d(1 - \beta), 2d(1 - \beta^{-1})\} \leq 2d \leq \frac{g'(a; a)}{2} \leq \frac{g'(u_{\inf}(a); a)}{2}, \quad (2.3.151)$$

which implies that $h_l(\beta)$ and $h_r(\beta)$ are well-defined.

A little algebra yields

$$\begin{aligned} h_l(\beta) &= \frac{\sqrt{3}}{d(1-\beta)} \left[\sqrt{g'(u_{\inf}(a); a) + 2d(\beta - 1)} + \beta^3 \sqrt{g'(u_{\inf}(a); a) + 2d\beta^{-1}(1 - \beta)} \right], \\ h_r(\beta) &= \frac{\sqrt{3}}{d(1-\beta)} \left[\sqrt{g'(u_{\inf}(a); a) + 2d(\beta - 1)} - \beta^3 \sqrt{g'(u_{\inf}(a); a) + 2d\beta^{-1}(1 - \beta)} \right]. \end{aligned} \quad (2.3.152)$$

It is clear that $h_l(\beta) < 0$ for $\beta > 1$. Upon writing

$$\begin{aligned} \Delta(\beta) &= g'(u_{\inf}(a); a) + 2d(\beta - 1) - (\beta^6 g'(u_{\inf}(a); a) + 2d\beta^5(1 - \beta)) \\ &= (1 - \beta^6)(g'(u_{\inf}(a); a) - 2d) + 2d\beta(1 - \beta^4), \end{aligned} \quad (2.3.153)$$

it is easy to verify that $\Delta(\beta) < 0$ for $\beta > 1$ and $\Delta(\beta) > 0$ for $0 < \beta < 1$. This yields the final inequality (2.3.150). $\square$

Lemma 2.3.18. Pick $0 < a \leq \frac{1}{2}$ and $0 < d \leq \frac{g'(a; a)}{4}$. Then we have

$$v'_-(0) < v'_d(0). \quad (2.3.154)$$

In addition, suppose that

$$v'_-(u) = v'_*(u) \quad (2.3.155)$$

for some $0 \leq u \leq a$, with $u \neq u_{\min}(a)$ if $a = \frac{1}{2}$. Then one of the following two statements must hold.

(a) We have the inequality

$$v''_-(u) > v''_d(u). \quad (2.3.156)$$

(b) We have the identities

$$u = a, \quad d = \frac{g'(a)}{4}, \quad v''_-(u) = v''_d(u). \quad (2.3.157)$$

Proof. An easy computation yields

$$v'_d(0) = 1 - \frac{g'(0; a)}{2d} \geq 1 - \frac{2g'(0; a)}{g'(a; a)} > -\frac{g'(0; a)}{g'(a; a)} = v'_-(0). \quad (2.3.158)$$

We introduce the critical value

$$u_c = \sup\{0 \leq u \leq a : v_-(\tilde{u}) \leq u_{\inf}(a) \text{ for all } 0 \leq \tilde{u} \leq u\} \quad (2.3.159)$$

and remark that $u_c = 0$ when $a = \frac{1}{2}$. This allows us to define the value

$$u_I = \sup\{0 \leq u \leq \min\{u_c, u_{\min}(a)\} : v'_d(u) > v'_-(u)\}, \quad (2.3.160)$$

which again satisfies $u_I = 0$ when $a = \frac{1}{2}$.

We claim that also $v'_d(u_I) > v'_-(u_I)$. Indeed, assuming this is false we can define $\beta = v'_d(u_I) = v'_-(u_I) \geq 0$. Item (ii) of Lemma 2.3.15 then implies $\beta > 1$. Since $h_l(\beta) < 0$ we must have

$$v''_-(u_I) < v''_d(u_I), \quad (2.3.161)$$

which yields a contradiction.

In particular, if (2.3.155) holds then we must have $u \geq \min\{u_{\min}(a), u_c\}$. If $u_{\min}(a) \leq u < a$, Lemma 2.3.16 shows that (a) must hold. On the other hand, if $u_c \leq u < u_{\min}(a)$, then we can define $\beta = v'_d(u) = v'_-(u) \geq 0$ and conclude as above that $\beta > 1$. In addition, we have $v_-(u) \geq u_{\inf}(a)$, which allows us to use $h_r(\beta) > 0$ and item (iv) of Lemma 2.3.15 to show that (a) must hold. In the final case $u = a$, the remarks at the start of this section together with a direct computation of $v''_d(a)$ and $v''_-(a)$ imply the identities in (b). $\square$

In the remainder of this section we collect several consequences of these computations. In each case, we either rule out non-transverse intersections of $v_{\pm}$ with v_d or show that they must occur at local minima of $v_{\pm} - v_d$.

Corollary 2.3.19. Fix $0 < a \leq \frac{1}{2}$ together with $0 < d \leq \frac{g'(a;a)}{4}$ and suppose that

$$v'_+(u) = v'_d(u) \quad (2.3.162)$$

for some $0 \leq u \leq a$, with $u \neq u_{\min}(a)$ if $a = \frac{1}{2}$. Then we have

$$v''_+(u) > v''_d(u). \quad (2.3.163)$$

Proof. Using the fact that $h_r(\beta) > 0$ for $\beta \in (0, 1)$, this follows directly from items (i), (ii) and (v) of Lemma 2.3.15. $\square$

Corollary 2.3.20. Fix $0 < a \leq \frac{1}{2}$ together with $0 < d < \frac{g'(a;a)}{4}$ and suppose that

$$v'_-(u) = v'_*(u) \quad (2.3.164)$$

for some $0 \leq u \leq a$, with $u \neq u_{\min}(a)$ if $a = \frac{1}{2}$. Then we have

$$v''_-(u) > v''_d(u). \quad (2.3.165)$$

In addition, we have

$$v'_-(a) > v'_d(a). \quad (2.3.166)$$

Proof. The first inequality follows directly from the fact that option (b) in Lemma 2.3.18 cannot hold because of the restriction on d . The final inequality can be verified directly by noting that

$$v'_d(a) = 1 - \frac{g'(a; a)}{2d} < 1 - \frac{2g'(a; a)}{g'(a; a)} = -1 = v'_-(a). \quad (2.3.167)$$

□

Corollary 2.3.21. Fix $0 < a \leq \frac{1}{2}$ together with $d = \frac{g'(a; a)}{4}$. Then we have

$$v'_-(u) < v'_d(u) \quad (2.3.168)$$

for all $0 \leq u < a$, with the exception of $u = u_{\min}(a)$ in the special case $a = \frac{1}{2}$.

Proof. It is easy to verify that $v'_-(a) = v'_d(a)$ and $v''_-(a) = v''_d(a)$. We also compute

$$v'''_d(a) = -2 \frac{g'''(a; a)}{g'(a; a)} = \frac{12}{a(1-a)} > 0 \quad (2.3.169)$$

together with

$$\begin{aligned} v'''_-(u) &= -\frac{g'''(u; a)}{g'(v_-(u); a)} - 3 \frac{g''(u; a)g'(u; a)}{g'(v_-(u); a)^3} g''(v_-(u); a) \\ &\quad - 3 \frac{g'(u; a)^3}{g'(v_-(u); a)^5} g''(v_-(u); a)^2 + \frac{g'(u; a)^3}{g'(v_-(u); a)^4} g'''(v_-(u); a), \end{aligned} \quad (2.3.170)$$

which gives

$$v'''_-(a) = -6 \frac{g''(a; a)^2}{g'(a; a)^2} \leq 0. \quad (2.3.171)$$

In particular, we see that

$$v'_-(a - \epsilon) < v'_d(a - \epsilon) \quad (2.3.172)$$

for all sufficiently small $\epsilon > 0$. If $a \neq \frac{1}{2}$, the conclusion now follows from (2.3.154) together with (a) from Lemma 2.3.18.

For $a = \frac{1}{2}$, one also needs to use the identities

$$v'_d(u_{\min}(a)) = 1, \quad v''_d(u_{\min}(a)) = -12\sqrt{3} \quad (2.3.173)$$

together with the limits in item (iv) of Lemma 2.3.14 to conclude that

$$v'_-(u_{\min}(a) - \epsilon) < v'_d(u_{\min}(a) - \epsilon) \quad (2.3.174)$$

for all sufficiently small $\epsilon > 0$. The arguments above allow us to extend this to $\epsilon \in (0, u_{\min}(a)]$. In addition, we have $v'_-(\tilde{u}) = -1 < v'_d(\tilde{u})$ for $u_{\min}(a) < \tilde{u} < a$. □

Corollary 2.3.22. Pick $0 < a \leq \frac{1}{2}$ and $0 < d \leq \frac{g'(a; a)}{4}$. Then we have

$$v'_-(u) < v'_d(u) \quad (2.3.175)$$

for all $0 \leq u < u_{\min}(a)$.

Proof. Writing $d_c = \frac{g'(a;a)}{4}$, we may use Corollary 2.3.21 to compute

$$v'_d(u) = 1 - \frac{1}{2d}g'(u;a) \geq 1 - \frac{2}{g'(a;a)}g'(u;a) = v'_{d_c}(u) > v'_-(u) \quad (2.3.176)$$

for $u \in [0, u_{\min}(a)]$. $\square$

Corollary 2.3.23. Fix $a = \frac{1}{2}$ and $0 < d \leq \frac{g'(a;a)}{4}$. Then we have

$$v'_+(u) > v'_d(u) \quad (2.3.177)$$

for all $u_{\min}(a) < u \leq a$.

Proof. Item (iv) of Lemma 2.3.14 allow us to compute

$$v'_d(u_{\min}(a)) = 1 = \lim_{u \downarrow u_{\min}(a)} v'_+(u) \quad (2.3.178)$$

together with

$$v''_d(u_{\min}(a)) < \lim_{u \downarrow u_{\min}(a)} v''_+(u), \quad (2.3.179)$$

which allows us to conclude that

$$v'_+(u_{\min}(a) + \epsilon) > v'_d(u_{\min}(a) + \epsilon) \quad (2.3.180)$$

for all sufficiently small $\epsilon > 0$. Corollary 2.3.19 allows us to extend this conclusion to the desired interval $\epsilon \in (0, a - u_{\min}(a))$. $\square$

2.3.5 Structure

We are now ready to analyze the global structure of the solution set to $G(u, v; a, d) = 0$. Our first two results fix $a \in (0, \frac{1}{2}]$ and track the intersections of the curves $v_{\pm}$ that were introduced in §2.3.3 with the curve v_d introduced in §2.3.4. These intersections disappear as the parameter d is increased; see Figure 2.8.

Lemma 2.3.24. Fix $0 < a \leq \frac{1}{2}$. Then there exists a continuous strictly increasing function

$$u_{AD} : [0, \frac{g'(a;a)}{4}] \rightarrow [0, a] \quad (2.3.181)$$

that satisfies the following properties.

- (i) We have $u_{AD}(0) = 0$ and $u_{AD}(\frac{g'(a;a)}{4}) = a$.
- (ii) The identity $v_-(u_{AD}(d)) = v_d(u_{AD}(d))$ holds for any $0 < d \leq \frac{g'(a;a)}{4}$.
- (iii) Suppose that $v_-(u) = v_d(u)$ for some $0 < d \leq \frac{g'(a;a)}{4}$ and $0 \leq u \leq a$. Then in fact $u \in \{u_{AD}(d), a\}$.

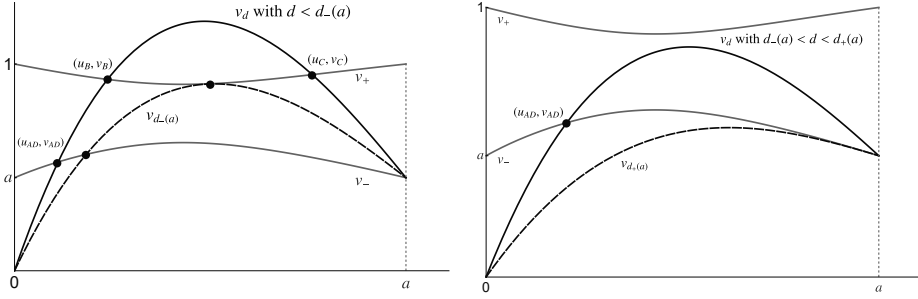


Figure 2.8: Fix $0 < a < \frac{1}{2}$. The branches u_B and u_C described in Lemma 2.3.25 arise as the two intersections of v_+ and v_d on $[0, a]$, which collide as $d \uparrow d_-(a)$ (left). On the other hand, the branch u_{AD} described in Lemma 2.3.24 arises as the unique intersection of the curves v_- and v_d on $[0, a)$, which converges to a as $d \uparrow d_+(a)$ (right).

- (iv) Consider any $0 < d < \frac{g'(a;a)}{4}$ for which $(a, d) \neq (\frac{1}{2}, \frac{1}{24})$. Then we have the inequality

$$v'_-(u_{AD}(d)) < v'_d(u_{AD}(d)). \quad (2.3.182)$$

- (v) For any $d > \frac{g'(a;a)}{4}$ and $0 \leq u < a$ we have $v_-(u) > v_d(u)$.

Proof. For convenience, we introduce the function $h_d(u) = v_-(u) - v_d(u)$ and set out to count the zeroes of h_d on the interval $[0, a]$. We first note that $h_d(a) = 0$ for all $d > 0$. When $d = \frac{g'(a;a)}{4}$ this is in fact the only zero, which can be seen by using (2.3.168) and explicitly verifying the inequality

$$v_-(u_{\min}(a)) > v_d(u_{\min}(a)) \quad (2.3.183)$$

for $(a, d) = (\frac{1}{2}, \frac{g'(\frac{1}{2}; \frac{1}{2})}{4})$.

On the other hand, (2.3.166) implies that h_d has at least two zeroes for $0 < d < \frac{g'(a;a)}{4}$. Furthermore, we claim that h_d has precisely two zeroes for $0 < d \leq d_1$ upon choosing $d_1 > 0$ to be sufficiently small. Indeed, for any $u \in (0, a)$ we have $v''_d(u) < 0$ and we can enforce

$$|v'_d(u)| + v_d(u) \geq 1 + \max_{0 \leq \tilde{u} \leq a} \{|v'_-(\tilde{u})|\} \quad (2.3.184)$$

by restricting the size of $d > 0$.

We now pick $\epsilon > 0$ in such a way that the function h_d is strictly decreasing on $[0, u_{\min}(a) + \epsilon]$ for all $d_1 \leq d \leq \frac{g'(a;a)}{4}$. This is possible because Corollary 2.3.22 allows us to enforce $h'_d < 0$ on this interval, with the exception of the single point $u = u_{\min}(a)$ when $a = \frac{1}{2}$.

Let us now define the critical value

$$d_c = \sup\{d_1 \leq d \leq \frac{g'(a; a)}{4} : h_d = 0 \text{ has two distinct solutions on } [0, a]\} \quad (2.3.185)$$

and assume for the moment that $d_c < g'(a; a)/4$. The preparations above show that there exists $u_{\min}(a) < u < a$ with $h_{d_c}(u) = 0$ and $h'_{d_c}(u) = 0$. Corollary 2.3.20 now implies $h''_{d_c}(u) > 0$. As a consequence of the monotonicity $\partial_d v_d(u) < 0$, this means that for all sufficiently small $\delta > 0$, the function h_d with $d = d_c - \delta$ must have at least three zeroes. This yields a contradiction, which implies that $d_c = g'(a; a)/4$.

We may hence define $u_{AD}(d) \in [0, a]$ to be the left-most root of $h_d(u) = 0$ for $0 < d \leq d_c$. The statements (i)-(v) follow readily from the observations above together with the monotonicity $\partial_d v_d(u) < 0$ that holds for $u \in (0, a)$. $\square$

Lemma 2.3.25. Fix $0 < a \leq \frac{1}{2}$. Then there exists a constant $0 < d_- < \frac{g'(a; a)}{4}$ together with two continuous functions

$$(u_B, u_C) : [0, d_-] \rightarrow [0, a] \times [0, a] \quad (2.3.186)$$

that satisfy the following properties.

- (i) We have $(u_B, u_C)(0) = (0, a)$ and $u_B(d_-) = u_C(d_-)$.
- (ii) The function u_B is strictly increasing, while the function u_C is strictly decreasing.
- (iii) For any $0 < d \leq d_-$ the identity $v_+(u_{\#}) = v_d(u_{\#})$ holds for $\# \in \{B, C\}$.
- (iv) If $v_+(u) = v_d(u)$ for some $0 < d \leq d_-$ and $0 \leq u \leq a$ then in fact $u \in \{u_B(d), u_C(d)\}$.
- (v) For any $0 < d < d_-$ we have

$$v'_+(u_B(d)) < v'_d(u_B(d)), \quad v'_+(u_C(d)) > v'_d(u_C(d)). \quad (2.3.187)$$

If $a \neq \frac{1}{2}$, then we also have

$$v'_+(u_C(d_-)) = v'_+(u_B(d_-)) = v'_d(u_B(d_-)) = v'_d(u_C(d_-)). \quad (2.3.188)$$

- (vi) For any $d > d_-$ the inequality $v_+(u) > v_d(u)$ holds for all $0 \leq u \leq a$.
- (vii) If $a = \frac{1}{2}$ then we have $d_-(\frac{1}{2}) = \frac{1}{24}$ together with $u_B(\frac{1}{24}) = u_C(\frac{1}{24}) = u_{\min}(a)$.

Proof. Writing $h_d(u) = v_+(u) - v_d(u)$, we observe first that h_d is strictly decreasing on $[0, u_{\min}(a)]$ because $v'_d > 0$ and $v'_+ < 0$ on the interior of this interval.

In addition, in the special case $a = \frac{1}{2}$ we can use Corollary 2.3.23 to conclude that h_d is strictly increasing on $[u_{\min}(a), a]$. Since $h_d(u_{\min}(a)) = 0$ occurs precisely when $d = \frac{1}{24}$, all the desired statements can be easily verified.

Throughout the remainder of this proof we therefore assume that $0 < a < \frac{1}{2}$. Arguing as in the proof of Lemma 2.3.24, we may pick $d_1 > 0$ in such a way that h_d has precisely two zeroes on $[0, a]$ for every $0 < d \leq d_1$. This allows us to define the critical value

$$d_- = \sup\{d_1 \leq d : h_d = 0 \text{ has two distinct solutions on } [0, a]\}. \quad (2.3.189)$$

Since $v_- < v_+$ it is clear that $d_- < \frac{g'(a; a)}{4}$.

Let us assume for the moment that h_{d_-} has two or more zeroes on $[0, a]$. This implies that there exists at least one $u \in (0, a)$ for which $h_{d_-}(u) = 0$ and $h'_{d_-}(u) = 0$. Using Corollary 2.3.20 it follows that $h''_{d_-}(u) > 0$ must hold for all such zeroes. As a consequence of the monotonicity $\partial_d v_d(u) < 0$, this means that for all sufficiently small $\delta > 0$, the function h_d with $d = d_- - \delta$ must have at least three zeroes. This yields a contradiction, which by continuity shows that $h_{d_-}(u) = 0$ has precisely one root on $[0, a]$. Upon defining $u_B(d)$ and $u_C(d)$ to be the left-most respectively right-most root of $h_d(u) = 0$, the desired properties (i) - (vii) can be easily verified. $\square$

Recalling the function G introduced in (2.2.6), we see that

$$D_{1,2}G(u, v; a, d) = \begin{pmatrix} g'(u) - 2d & 2d \\ 2d & g'(v) - 2d \end{pmatrix}. \quad (2.3.190)$$

In order to study the stability and parameter-dependence of the roots constructed in Lemmata 2.3.24-2.3.25, it is crucial to understand when the determinant of $D_{1,2}G$ vanishes. The result below states that this happens at tangential intersections of $v_\pm$ and v_d .

Lemma 2.3.26. Fix $0 < a < \frac{1}{2}$ together with $d > 0$ and suppose that

$$v_\#(u) = v_d(u) \quad (2.3.191)$$

for some $\# \in \{-, +\}$ and $u \in [0, a]$. Then we have $\det D_{1,2}G(u, v_d(u); a, d) = 0$ if and only if $v'_\#(u) = v'_d(u)$.

Proof. We first consider the case $v'_\#(u) = 0$, which occurs when $g'(u; a) = 0$ and hence $u = u_{\min}(a)$. Using $0 < a < \frac{1}{2}$ we see that

$$v_-(u) < u_{\max}(a) < v_+(u). \quad (2.3.192)$$

Writing $v = v_d(u) = v_\pm(u)$, we hence obtain $g'(v; a) \neq 0$ and hence

$$\det D_{1,2}G(u, v; a, d) = -2dg'(v; a) \neq 0. \quad (2.3.193)$$

Since $v'_d(u) = 1$, the desired equivalence indeed holds for this case.

Assuming now that $v'_{\#}(u) \neq 0$ and hence $g'(u; a) \neq 0$, we again write $v = v_d(u) = v_{\#}(u)$ and use (2.3.113) to compute

$$g'(v; a) = -[v'_{\#}(u)]^{-1}g'(u; a). \quad (2.3.194)$$

In particular, we find

$$\begin{aligned} \det D_{1,2}G(u, v; a, d) &= (g'(u; a) - 2d)(-[v'_{\#}(u)]^{-1}g'(u; a) - 2d) - 4d^2 \\ &= -[v'_{\#}(u)]^{-1}g'(u; a)^2 - 2dg'(u; a) + 2d[v'_{\#}(u)]^{-1}g'(u; a) \\ &= 2dg'(u; a)[v'_{\#}(u)]^{-1}\left[1 - \frac{g'(u; a)}{2d} - v'_{\#}(u)\right] \\ &= 2dg'(u; a)[v'_{\#}(u)]^{-1}\left[v'_d(u) - v'_{\#}(u)\right], \end{aligned} \quad (2.3.195)$$

from which the statement follows. $\square$

In order to characterize the dependence of $d_-(a)$ on a , we introduce the function

$$G_{\text{sn}}(u, v, d; a) = (G_1(u, v; a, d), G_2(u, v; a, d), \det D_{1,2}G(u, v; a, d))^T \quad (2.3.196)$$

and symbolically write

$$\begin{aligned} [D_{1,2,3}G_{\text{sn}}(u, v, d; a)]^{-1} &= [\det D_{1,2,3}G_{\text{sn}}(u, v, d; a)]^{-1} \\ &\quad \times \begin{pmatrix} * & * & * \\ * & * & * \\ \gamma_1(u, v, d; a) & \gamma_2(u, v, d; a) & \gamma_3(u, v, d; a) \end{pmatrix}. \end{aligned} \quad (2.3.197)$$

For any $0 < a \leq \frac{1}{2}$, we use the functions defined in Lemma 2.3.25 to introduce the notation

$$\omega(a) = (u_B(d_-(a)), v_+(u_B(d_-(a))), d_-(a)) = (u_C(d_-(a)), v_+(u_C(d_-(a))), d_-(a)), \quad (2.3.198)$$

which corresponds to the critical point where the branches u_B and u_C collide.

Corollary 2.3.27. Upon fixing $0 < a < \frac{1}{2}$, the following two statements are equivalent.

- (a) The identity $G_{\text{sn}}(u, v, d; a) = 0$ holds for some $d > 0$ and some pair $(u, v) \in [0, 1]^2$ that has $v \geq u$.
- (b) We have $(u, v, d) = \omega(a)$ or $(u, v, d) = (a, a, \frac{g'(a; a)}{4})$.

Proof. As a preparation, we note that $\det D_{1,2}G(a, a; a, d) = 0$ if and only if $d = \frac{g'(a; a)}{4}$. In addition, it is easy to check that $\det D_{1,2}G(0, 0; a, d) > 0$ and $\det D_{1,2}G(1, 1; a, d) > 0$ for all $d \geq 0$.

The implication (b) $\rightarrow$ (a) can now be verified directly using Lemma 2.3.26 and (2.3.188). In addition, we only need to establish the reverse implication under the additional assumption that $v > u$.

Let us therefore assume that (a) holds with $g(u) = -g(v) \neq 0$, which allows us to write

$$0 < u < a < v < 1. \quad (2.3.199)$$

In particular, Lemma 2.3.13 implies that $v = v_-(u)$ or $v = v_+(u)$. Lemma 2.3.26 together with (2.3.187) and (2.3.182) now imply that (b) must hold. $\square$

Lemma 2.3.28. We have the identities

$$\begin{aligned} \gamma_1(0, 1, 0; 0) &= -2, \\ \gamma_2(0, 1, 0; 0) &= 0. \end{aligned} \quad (2.3.200)$$

In addition, the identity

$$\gamma_3(\omega(a); a) = 0 \quad (2.3.201)$$

holds for any $0 < a < \frac{1}{2}$, together with the inequalities

$$\begin{aligned} \det D_{1,2,3}G_{\text{sn}}(\omega(a); a) &< 0, \\ \gamma_1(\omega(a); a) - \gamma_2(\omega(a); a) &< 0, \\ \gamma_1(\omega(a); a)\gamma_2(\omega(a); a) &\geq 0. \end{aligned} \quad (2.3.202)$$

Proof. Let us assume for the moment that $G_{\text{sn}}(u, v, d; a) = 0$, which directly implies

$$(g'(u; a) - 2d)g'(v; a) = 2dg'(u; a). \quad (2.3.203)$$

In view of the identity

$$\begin{aligned} D_{1,2,3}G_{\text{sn}}(u, v, d; a) &= \\ &= \begin{pmatrix} g'(u; a) - 2d & 2d & 2(v - u) \\ 2d & g'(v; a) - 2d & 2(u - v) \\ (g'(v; a) - 2d)g''(u; a) & (g'(u; a) - 2d)g''(v; a) & -2(g'(u; a) + g'(v; a)) \end{pmatrix}, \end{aligned} \quad (2.3.204)$$

we can use (2.3.203) to compute

$$\begin{aligned} \det D_{1,2,3}G_{\text{sn}}(u, v, d; a) &= \\ &= 2(v - u) \left[g'(u; a)(g'(u; a) - 2d)g''(v; a) - g'(v; a)(g'(v; a) - 2d)g''(u; a) \right] \\ &= 2(v - u) \left[2dg'(u; a)^2 \frac{1}{g'(v; a)} g''(v; a) - g'(v; a)(g'(v; a) - 2d)g''(u; a) \right]. \end{aligned} \quad (2.3.205)$$

Expanding the subdeterminants of (2.3.204) and reusing (2.3.203), we also find

$$\begin{aligned}\gamma_1(u, v, d; a) &= 2d(g'(u; a) - 2d)g''(v; a) - (g'(v; a) - 2d)^2g''(u; a), \\ \gamma_2(u, v, d; a) &= 2d(g'(v; a) - 2d)g''(u; a) - (g'(u; a) - 2d)^2g''(v; a), \\ \gamma_3(u, v, d; a) &= 0,\end{aligned}\tag{2.3.206}$$

which allows us to explicitly verify (2.3.200).

Let us now fix $0 < a < \frac{1}{2}$. For any $(u, v) \in [0, 1]^2$ and $d > 0$ we introduce the function

$$\begin{aligned}h(u, v, d) &= \frac{1}{2d}g''(u; a) - \frac{g''(u; a)}{g'(v; a)} - \frac{g'(u; a)^2}{g'(v; a)^3}g''(v; a) \\ &= \frac{1}{2dg'(v; a)^2} \left[g'(v; a)(g'(v; a) - 2d)g''(u; a) - 2d \frac{g'(u; a)^2}{g'(v; a)} g''(v; a) \right],\end{aligned}\tag{2.3.207}$$

which allows us to rewrite (2.3.205) in the form

$$\det D_{1,2,3}G_{\text{sn}}(u, v, d; a) = -4d(v - u)g'(v; a)^2h(u, v, d).\tag{2.3.208}$$

Using (2.3.132) and (2.3.133) we observe that

$$v_+''(u) - v_d''(u) = h(u, v_+(u), d).\tag{2.3.209}$$

In particular, Corollary 2.3.19 and (2.3.188) imply that

$$h(\omega(a)) > 0,\tag{2.3.210}$$

which shows that

$$\det D_{1,2,3}G_{\text{sn}}(\omega(a); a) < 0,\tag{2.3.211}$$

as desired.

Let us now write $(u, v, d) = \omega(a)$ and compute

$$\begin{aligned}\gamma_1(\omega(a); a) - \gamma_2(\omega(a); a) &= g'(u)(g'(u) - 2d)g''(v) - g'(v)(g'(v) - 2d)g''(u) \\ &= \frac{1}{2}(v - u)^{-1} \det DG_{\text{sn}}(u, v, d; a) \\ &\leq 0.\end{aligned}\tag{2.3.212}$$

In addition, since $u > u_{\min}(a)$ and hence $v_d'(u) = v_+'(u) > 0$, we have $g'(u) < 2d$. This allows us to define

$$\begin{aligned}\alpha &= \sqrt{2d}(2d - g'(u; a))^{3/2}g''(v; a), \\ \beta &= \sqrt{2d}(2d - g'(v; a))^{3/2}g''(u; a)\end{aligned}\tag{2.3.213}$$

and compute

$$\begin{aligned}
 \gamma_1(\omega(a); a)\gamma_2(\omega(a); a) &= (4d^2)(4d^2)g''(u; a)g''(v; a) + (4d^2)^2g''(u; a)g''(v; a) \\
 &\quad - 2d(g'(u; a) - 2d)^3g''(v; a)^2 - 2d(g'(v; a) - 2d)^3g''(u; a)^2 \\
 &= 2\alpha\beta + \alpha^2 + \beta^2 \\
 &\geq 0,
 \end{aligned} \tag{2.3.214}$$

as desired. $\square$

Corollary 2.3.29. The map $a \mapsto \omega(a)$ is C^∞ -smooth on $(0, \frac{1}{2})$ and we have $d'_-(a) > 0$.

Proof. Since $d_- < \frac{g'(a; a)}{4}$, we may use Corollary 2.3.27, together with the first inequality in (2.3.202) to apply the implicit function and establish the C^∞ -smoothness of ω .

Since $g'(a; a)/4 \downarrow 0$ as $a \downarrow 0$, a squeezing argument shows that $\lim_{a \downarrow 0} \omega(a) = (0, 1, 0)$. Exploiting (2.3.200), and (2.3.202), the continuity of γ_1 and γ_2 implies that

$$\gamma_1(\omega(a); a) < \gamma_2(\omega(a); a) \leq 0 \tag{2.3.215}$$

for all $0 < a < \frac{1}{2}$. In particular, writing $(u, v, d) = \omega(a)$, we may use the inequalities

$$D_2g(u; a) < 0, \quad D_2g(v; a) < 0 \tag{2.3.216}$$

to compute

$$d'_-(a) = -[\det D_{1,2,3}G_{\text{sn}}(\omega(a); a)]^{-1} [\gamma_1(\omega(a); a)D_2g(u; a) + \gamma_2(\omega(a); a)D_2g(v; a)] > 0. \tag{2.3.217}$$

$\square$

Proof of Proposition 2.3.1. Items (i), (iii), (iv) follow from Lemmata 2.3.24-2.3.25 and the observations in the proof of Corollary 2.3.27. Item (ii) follows from Corollary 2.3.29, while the expansions in (v) follow from Propositions 2.3.6 and 2.3.9. $\square$

Proof of Proposition 2.3.2. The identity $g(1-u, 1-a) = -g(u, a)$ implies that

$$G(1-u, 1-v; 1-a, d) = -G(u, v; a, d). \tag{2.3.218}$$

This allows us to extend the solutions constructed in Lemmata 2.3.24-2.3.25 to the entire interval $0 \leq a \leq 1$ in the fashion outlined in Corollary 2.3.4, which yields (ii) and (iv).

To establish (i), we note that for each $\# \in \{A, B, C\}$ the sign of $\det D_{1,2}G(\bar{u}_\#, \bar{v}_\#; a, d)$ is constant on Ω_- on account of Corollary 2.3.27. Using (ii) the eigenvalues of these matrices can be explicitly computed at $d = 0$, which yields the desired stability properties.

To obtain the strict ordering in (iii), we fix $0 < a \leq \frac{1}{2}$ and $0 < d < d_-(a)$. We observe that $v_+(u) \geq v_-(u)$ for $u \in [0, a]$, with equality only at $u = u_{\min}(a)$ for $a = \frac{1}{2}$. Since $v_d(0) = 0$ this implies that $\bar{u}_A(a, d) < \bar{u}_B(a, d)$ and $\bar{v}_A(a, d) < \bar{v}_B(a, d)$. By construction, we also have $\bar{u}_B(a, d) < \bar{u}_C(a, d)$. In addition, it cannot be the case that $\bar{v}_B(a, \tilde{d}) = \bar{v}_C(a, \tilde{d})$ for any $0 < \tilde{d} < d_-(a)$ since then $G_2 = 0$ implies that also $\bar{u}_B(a, \tilde{d}) = \bar{u}_C(a, \tilde{d})$. Using the general identity

$$D_{a,d}(\bar{u}_\#, \bar{v}_\#) = -[D_{1,2}G(\bar{u}_\#, \bar{v}_\#; a, d)]^{-1} D_{3,4}G(\bar{u}_\#, \bar{v}_\#; a, d) \quad (2.3.219)$$

we see that

$$\partial_d \bar{v}_\#(a, 0) = -\frac{1}{g'(\bar{v}_\#(a, 0); a)} (\bar{u}_\#(a, 0) - \bar{v}_\#(a, 0)), \quad (2.3.220)$$

which yields

$$\partial_d \bar{v}_B(a, 0) = \frac{1}{g'(1; a)} < \frac{1-a}{g'(1; a)} = \partial_d \bar{v}_C(a, 0). \quad (2.3.221)$$

These observations allow us to conclude $\bar{v}_B(a, d) < \bar{v}_C(a, d)$, as desired. $\square$

Proof of Proposition 2.3.3. Arguing in a similar fashion as the proof of Proposition 2.3.2, the statements follow from Lemma 2.3.24 and Corollary 2.3.27. $\square$

2.4 Travelling waves

Our goal here is to establish the existence of bichromatic wave solutions to the Nagumo LDE (2.2.1) and to obtain detailed results concerning their speed. In particular, we establish Theorems 2.2.2 and 2.2.3, which are the main results of this chapter.

Upon introducing the standard discrete Laplacian

$$[\Delta^+ u](\xi) = u(\xi + 1) + u(\xi - 1) - 2u(\xi) \quad (2.4.1)$$

together with the off-diagonal matrix

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (2.4.2)$$

the travelling wave system (2.2.5) can be rewritten as

$$-c\Phi' = d\mathcal{J}\Delta^+\Phi + G(\Phi; a, d). \quad (2.4.3)$$

For any $(a, d) \in \Omega_-$, we set out to seek solutions to (2.4.3) that satisfy the boundary conditions

$$\lim_{\xi \rightarrow -\infty} \Phi(\xi) = (0, 0), \quad \lim_{\xi \rightarrow +\infty} \Phi(\xi) = (\bar{u}_B(a, d), \bar{v}_B(a, d)). \quad (2.4.4)$$

The existence of such solutions will be established in §2.4.1, where we also show $c \geq 0$ and establish Theorem 2.2.2. In §2.4.2 we subsequently set out to derive criteria that distinguish between the two cases $c = 0$ and $c > 0$. We verify these criteria in §2.4.3 for parameters $(a, d) \in \Omega_-$ that are close to the cusp $(\frac{1}{2}, \frac{1}{24})$ and the corner $(1, 0)$. This allows us to establish our main results contained in Theorem 2.2.3.

2.4.1 Existence of waves

Our preparatory work in §2.3 allows us to invoke the theory developed in [27] to establish a general existence result for bichromatic waves. Indeed, the equilibria $(0, 0)$ and $(\bar{u}_B, \bar{v}_B)$ are both stable under the dynamics of $(\dot{u}, \dot{v}) = G(u, v; a, d)$ and all intermediate equilibria are unstable. Using a straightforward estimate based on the ordering (2.3.11), the wavespeeds can be shown to be non-negative.

Lemma 2.4.1. Pick $(a, d) \in \Omega_-$. Then there exists a constant $c \in \mathbb{R}$ and a non-decreasing function $\Phi : \mathbb{R} \rightarrow \mathbb{R}^2$ that satisfy (2.4.3)-(2.4.4). This constant c is unique, while the function Φ is unique up to translation if $c \neq 0$. In the latter case we also have $\Phi'(\xi) > (0, 0)$ for all $\xi \in \mathbb{R}$.

Proof. These statements follow directly from the main results in [27]. $\square$

Lemma 2.4.2. Pick $(a, d) \in \Omega_-$. Then the constant c defined in Lemma 2.4.1 satisfies $c \geq 0$.

Proof. Suppose that $c \neq 0$. We estimate

$$\begin{aligned} -c\Phi'_v(\xi) &= d[\Phi_u(\xi + 1) + \Phi_u(\xi - 1) - 2\Phi_v(\xi)] + g(\Phi_v(\xi); a) \\ &\leq 2d[\bar{u} - v(\xi)] + g(\Phi_v(\xi); a). \end{aligned} \quad (2.4.5)$$

Since $0 < \bar{u}_B(a, d) < a < \bar{v}_B(a, d)$, there exists $\xi_* \in \mathbb{R}$ for which $\Phi_v(\xi_*) = \bar{u}_B(a, d)$. This yields

$$-c\Phi'_v(\xi_*) \leq g(\bar{u}; a) < 0, \quad (2.4.6)$$

from which we conclude $c > 0$. $\square$

Writing $c(a, d)$ and $\Phi(a, d)$ for the wavespeed and waveprofile defined in Lemma 2.4.1, we introduce the set

$$\mathcal{T} = \{(a, d) \in \Omega_- : c(a, d) > 0\}, \quad (2.4.7)$$

which corresponds to the set $\mathcal{T}_{\text{low}}$ used in §2.2. In addition, for any $(a, d) \in \mathcal{T}$ we introduce the linear operators

$$\mathcal{L}_{a,d} : W^{1,\infty}(\mathbb{R}; \mathbb{R}^2) \rightarrow L^\infty(\mathbb{R}; \mathbb{R}^2), \quad \mathcal{L}_{a,d}^{\text{adj}} : W^{1,\infty}(\mathbb{R}; \mathbb{R}^2) \rightarrow L^\infty(\mathbb{R}; \mathbb{R}^2), \quad (2.4.8)$$

that act as

$$\begin{aligned} \mathcal{L}_{a,d}\phi &= -c(a, d)\phi' - d\mathcal{J}\Delta^+\phi - DG(\Phi(a, d); a, d)\phi, \\ \mathcal{L}_{a,d}^{\text{adj}}\psi &= c(a, d)\psi' - d\mathcal{J}\Delta^+\psi - DG(\Phi(a, d); a, d)\psi. \end{aligned} \quad (2.4.9)$$

The results in [74, §8] imply that there exists $\Psi = \Psi(a, d) \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^2)$ with $\Psi > (0, 0)$ for which we have the identities

$$\text{Ker } \mathcal{L}_{a,d} = \text{span}\{\Phi'(a, d)\}, \quad \text{Ker } \mathcal{L}_{a,d}^{\text{adj}} = \text{span}\{\Psi(a, d)\} \quad (2.4.10)$$

together with the normalization

$$\int_{-\infty}^{\infty} \langle \Psi(\xi), \Phi'(\xi) \rangle d\xi = 1. \quad (2.4.11)$$

In particular, [90, Thm. A] implies that

$$\text{Range } \mathcal{L}_{a,d} = \{f \in L^\infty(\mathbb{R}; \mathbb{R}^2) : \int_{-\infty}^{\infty} \langle \Psi(\xi), f(\xi) \rangle d\xi = 0\}. \quad (2.4.12)$$

These ingredients allow us to use the implicit function theorem to show that the pair (c, Φ) depends smoothly on the parameters $(a, d) \in \mathcal{T}$. In addition, we obtain a sign on $\partial_a c$.

Lemma 2.4.3. The maps

$$\begin{aligned} \mathcal{T} \ni (a, d) &\mapsto c(a, d) \in (0, \infty), \\ \mathcal{T} \ni (a, d) &\mapsto \Phi(a, d) \in BC^1(\mathbb{R}; \mathbb{R}^2) \end{aligned} \quad (2.4.13)$$

are C^∞ -smooth. In addition, we have

$$\partial_a c(a, d) > 0 \quad (2.4.14)$$

for all $(a, d) \in \mathcal{T}$.

Proof. The C^1 -smoothness of the maps (c, Φ) follows from [74, Thm. 2.3]. On account of the smoothness of the function g , this can readily be extended to the desired C^∞ -smoothness by using the ideas in [74, §8] to set up an implicit function argument along the lines of [91, Prop. 6.5].

Differentiating (2.4.3) with respect to a , we compute

$$-[\partial_a c]\Phi' - c[\partial_a \Phi'] = d\mathcal{J}\Delta^+[\partial_a \Phi] + DG(\Phi; a, d)\partial_a \Phi + \partial_a G(\Phi; a, d), \quad (2.4.15)$$

which gives

$$-[\partial_a c]\Phi' + \mathcal{L}_{a,d}\partial_a \Phi = \partial_a G(\Phi; a, d). \quad (2.4.16)$$

Applying (2.4.12) and noting that $\partial_a g(u; a) = -u(1 - u) < 0$ for all $u \in (0, 1)$, we obtain

$$-\partial_a c = \int_{-\infty}^{\infty} \langle \partial_a G(\Phi(\xi); a, d), \Psi(\xi) \rangle d\xi < 0, \quad (2.4.17)$$

as desired. $\square$

Corollary 2.4.4. If $(a, d) \in \mathcal{T}$, then also $(a', d) \in \mathcal{T}$ for all $(a', d) \in \Omega_-$ with $a' \geq a$.

Proof of Theorem 2.2.2. The statements follow directly from Lemmata 2.4.1-2.4.3. $\square$

2.4.2 Characterization of waves

We now set out to derive conditions that guarantee either $c(a, d) = 0$ or $c(a, d) > 0$. To this end, we introduce the functions

$$u_{a,d}(v) = v - \frac{g(v; a)}{2d}, \quad v_{a,d}(u) = u - \frac{g(u; a)}{2d} \quad (2.4.18)$$

and note that

$$\bar{u}_{\#}(a, d) = u_{a,d}(\bar{v}_{\#}(a, d)), \quad \bar{v}_{\#}(a, d) = v_{a,d}(\bar{u}_{\#}(a, d)) \quad (2.4.19)$$

for each $\# \in \{A, B, C\}$ and $(a, d) \in \Omega_-$. The local extrema of these functions are located at the critical points

$$\begin{aligned} \gamma_{c;\pm}(a, d) &= \frac{1}{3}(a+1) \pm \frac{1}{3}\sqrt{a^2 - a + 1 - 6d} \\ &= u_{\inf}(a) \pm \sqrt{\frac{1}{3}(g'(u_{\inf}(a); a) - 2d)}. \end{aligned} \quad (2.4.20)$$

One can verify that the functions $u_{a,d}$ and $v_{a,d}$ are strictly decreasing on $[\gamma_{c;-}(a, d), \gamma_{c;+}(a, d)]$ and strictly increasing outside this interval. The following ordering result exploits this characterization and will allow us to establish $c(a, d) = 0$ for a significant portion of the parameter set Ω_- .

Lemma 2.4.5. For any $(a, d) \in \bar{\Omega}_-$ we have the ordering

$$0 \leq \gamma_{c;-}(a, d) \leq a \leq \gamma_{c;+}(a, d) \leq \bar{v}_B(a, d). \quad (2.4.21)$$

If $(a, d) \in \Omega_-$ then the inequalities in (2.4.21) are all strict.

Proof. Let us first fix $(a, d) \in \Omega_-$. We note that $2d < 2d_+(a) = \frac{g'(a; a)}{2} \leq \frac{g'(u_{\inf}(a); a)}{2}$, which implies that $\gamma_{c;\pm}(a, d)$ are well-defined. In addition, this allows us to compute

$$u'_{a,d}(a) = 1 - \frac{g'(a; a)}{2d} \leq -1, \quad (2.4.22)$$

which means that $a \in (\gamma_{c;-}(a, d), \gamma_{c;+}(a, d))$.

In particular, the function $v \mapsto u_{a,d}(v)$ is strictly decreasing on $[a, \gamma_{c;+}(a, d)]$. On account of the orderings

$$u_{a,d}(\bar{v}_A(a, d)) = \bar{u}_A(a, d) < \bar{u}_B(a, d) = u_{a,d}(\bar{v}_B(a, d)), \quad a < \bar{v}_A(a, d) < \bar{v}_B(a, d) \quad (2.4.23)$$

we hence cannot have $\bar{v}_B(a, d) \leq \gamma_{c;+}(a)$. The results for the general case $(a, d) \in \bar{\Omega}_-$ now follow by continuity. $\square$

Lemma 2.4.6. Consider any $(a, d) \in \Omega_-$ for which $d \leq \frac{1}{8}(1-a)^2$. Then we have $c(a, d) = 0$.

Proof. Fix any $0 < a < 1$, write $d_* = \frac{1}{8}(1-a)^2$ and suppose that $(a, d_*) \in \Omega_-$. On account of Corollary 2.4.4 it suffices to show that $c(a, d_*) = 0$. Assuming to the contrary that $c = c(a, d_*) > 0$, we compute

$$\begin{aligned}
 0 &> -c\Phi'_v(\xi) \\
 &= d_*[\Phi_u(\xi+1) + \Phi_u(\xi-1) - 2\Phi_v(\xi)] + g(\Phi_v(\xi); a) \\
 &> -2d_*\Phi_v(\xi) + g(\Phi_v(\xi); a) \\
 &= -2d_*u_{a,d_*}(\Phi_v(\xi)).
 \end{aligned} \tag{2.4.24}$$

Since $0 < \gamma_{c,+}(a, d_*) < \bar{v}_B(a, d_*)$, there exists $\xi_* \in \mathbb{R}$ for which $\Phi_v(\xi_*) = \gamma_{c,+}(a, d_*)$. The key point is that

$$u_{a,d_*}(\gamma_{c,+}(a, d_*)) = 0, \tag{2.4.25}$$

which allows us to obtain a contradiction by picking $\xi = \xi_*$ in (2.4.24). $\square$

We now set out to derive a (non-sharp) condition that guarantees $c(a, d) > 0$. The strategy will be to rule out the existence of standing bichromatic waves. Let us therefore pick any $(a, d) \in \Omega_- \setminus \mathcal{T}$, which implies $c(a, d) = 0$. Writing $(\Phi_u, \Phi_v) = \Phi(a, d)$ for the corresponding profile, we introduce the sequence

$$(u_i, v_i) = (\Phi_u(2i), \Phi_v(2i+1)), \tag{2.4.26}$$

which satisfies the limits

$$\lim_{i \rightarrow -\infty} (u_i, v_i) = (0, 0) \quad \lim_{i \rightarrow +\infty} (u_i, v_i) = (\bar{u}_B(a, d), \bar{v}_B(a, d)), \tag{2.4.27}$$

together with the difference equation

$$\begin{aligned}
 0 &= d[v_i + v_{i-1} - 2u_i] + g(u_i; a), \\
 0 &= d[u_{i+1} + u_i - 2v_i] + g(v_i; a).
 \end{aligned} \tag{2.4.28}$$

Applying a shift to the first equation, we obtain the implicit system

$$\begin{aligned}
 v_{i+1} &= 2\left[u_{i+1} - \frac{g(u_{i+1}; a)}{2d}\right] - v_i, \\
 u_{i+1} &= 2\left[v_i - \frac{g(v_i; a)}{2d}\right] - u_i,
 \end{aligned} \tag{2.4.29}$$

which can be written as

$$\begin{aligned}
 v_{i+1} &= 2v_{a,d}(u_{i+1}) - v_i, \\
 u_{i+1} &= 2u_{a,d}(v_i) - u_i.
 \end{aligned} \tag{2.4.30}$$

In particular, we can obtain (u_{i+1}, v_{i+1}) by first reflecting (u_i, v_i) horizontally through the curve $u = u_{a,d}(v)$ and then vertically through the curve $v = v_{a,d}(u)$. Based on this geometric intuition, we set out to construct a rectangle

$$[\bar{u}_{\text{top}}(a, d), \bar{u}_B(a, d)] \times [\bar{v}_{\text{top}}(a, d), \bar{v}_B(a, d)] \tag{2.4.31}$$

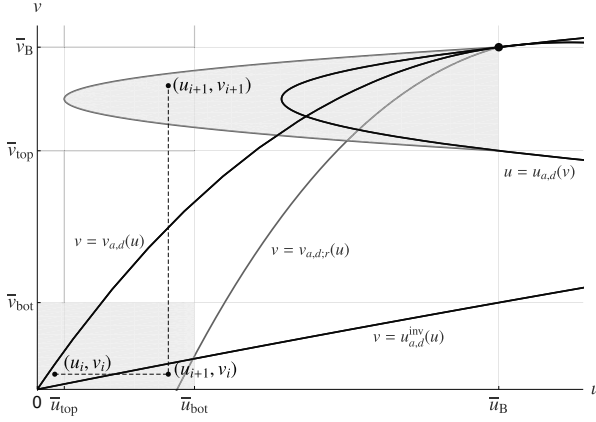


Figure 2.9: Illustration of the two reflections described by the system (2.4.30). The values have been modified for illustrative purposes, since the real regions are minuscule.

that must contain (u_{i_0+1}, v_{i_0+1}) for some critical i_0 , together with a rectangle

$$[0, \bar{u}_{\text{bot}}(a, d)] \times [0, \bar{v}_{\text{bot}}(a, d)] \quad (2.4.32)$$

that must contain the intermediate point (u_{i_0+1}, v_{i_0}) ; see Figure 2.9.

Lemma 2.4.7. There exist continuous functions

$$(\bar{v}_{\text{bot}}, \bar{v}_{\text{top}}) : \Omega_- \rightarrow [0, 1]^2 \quad (2.4.33)$$

that satisfy the inequalities

$$0 < \bar{v}_{\text{bot}}(a, d) < \gamma_{c;-}(a, d) < \bar{v}_{\text{top}}(a, d) < \gamma_{c;+}(a, d) < \bar{v}_B(a, d) \quad (2.4.34)$$

together with the identities

$$u_{a,d}(\bar{v}_{\text{bot}}(a, d)) = u_{a,d}(\bar{v}_{\text{top}}(a, d)) = \bar{u}_B(a, d) \quad (2.4.35)$$

for each $(a, d) \in \Omega_-$. Furthermore, these functions can be continuously extended to $\bar{\Omega}_-$ in such a way that (2.4.35) holds whenever $d > 0$.

Proof. Pick any $(a, d) \in \Omega_-$. On account of the identity $u_{a,d}(0) = 0$ and the inequalities

$$u_{a,d}(\gamma_{c;+}(a, d)) < u_{a,d}(\bar{v}_B(a, d)) = \bar{u}_B(a, d) < a = u_{a,d}(a), \quad (2.4.36)$$

the equation $u_{a,d}(v) = \bar{u}_B(a, d)$ has three distinct solutions on $(0, \bar{v}_B(a, d)]$. Using the fact that $\bar{u}_B(a, 0) = 0$, these solutions can be extended continuously to $d = 0$ by writing $\bar{v}_{\text{bot}}(a, 0) = 0$ and $\bar{v}_{\text{top}}(a, 0) = a$. The extension to $d = d_-(a) > 0$ can be achieved by standard continuity arguments. $\square$

For any $(a, d) \in \overline{\Omega}_-$ with $d > 0$ we now define the constant

$$\bar{u}_{\text{top}}(a, d) = 2u_{a,d}(\gamma_{c;+}(a, d)) - \bar{u}_B(a, d). \quad (2.4.37)$$

We note that $(\bar{u}_{\text{top}}(a, d), \gamma_{c;+}(a, d))$ can be seen as the horizontal reflection of $(\bar{u}_B(a, d), \gamma_{c;+}(a, d))$ through the curve $u = u_{a,d}(v)$.

Lemma 2.4.8. Pick any $(a, d) \in \Omega_-$ for which $c(a, d) = 0$ and consider the sequence $\{(u_i, v_i)\}$ defined in (2.4.26). Then there exists $i_0 \in \mathbb{Z}$ for which

$$(\bar{u}_{\text{top}}(a, d), \bar{v}_{\text{top}}(a, d)) \leq (u_{i_0+1}, v_{i_0+1}) \leq (\bar{u}_B(a, d), \bar{v}_B(a, d)), \quad (2.4.38)$$

while

$$(0, 0) \leq (u_{i_0}, v_{i_0}) \leq (\bar{u}_B(a, d), \bar{v}_{\text{bot}}(a, d)). \quad (2.4.39)$$

Proof. For any $i \in \mathbb{Z}$ we have the inequalities

$$u_i \leq u_{i+1} \leq \bar{u}_B(a, d), \quad (2.4.40)$$

which implies that we must have $u_i \leq u_{a,d}(v_i) \leq u_{i+1}$. In particular, we see that

$$u_{a,d}(v_i) \leq \bar{u}_B(a, d), \quad (2.4.41)$$

which implies that

$$v_i \notin (\bar{v}_{\text{bot}}(a, d), \bar{v}_{\text{top}}(a, d)). \quad (2.4.42)$$

In addition, we have

$$u_i = 2u_{a,d}(v_i) - u_{i+1} \geq 2u_{a,d}(v_i) - \bar{u}_B(a, d). \quad (2.4.43)$$

In particular, for every i we either have

$$(0, 0) \leq (u_i, v_i) \leq (\bar{u}_B(a, d), \bar{v}_{\text{bot}}(a, d)) \quad (2.4.44)$$

or

$$(\bar{u}_{\text{top}}(a, d), \bar{v}_{\text{top}}(a, d)) \leq (u_i, v_i) \leq (\bar{u}_B(a, d), \bar{v}_B(a, d)). \quad (2.4.45)$$

The limits (2.4.27) imply that there exists $M \gg 1$ so that (2.4.44) holds for all $i \leq -M$ and (2.4.45) holds for all $i \geq M$. In particular, there must be jump between these two sets. $\square$

For any $(a, d) \in \overline{\Omega}_-$ with $d > 0$, the inequalities (2.4.34) imply that the function $u_{a,d}$ is strictly increasing on $[0, \bar{v}_{\text{bot}}(a, d)]$. This allows us to define an inverse

$$u_{a,d}^{\text{inv}} : [0, \bar{u}_B(a, d)] \rightarrow [0, \bar{v}_{\text{bot}}(a, d)]. \quad (2.4.46)$$

In addition, we introduce the function

$$v_{a,d;r}(u) = 2v_{a,d}(u) - \bar{v}_B(a, d), \quad (2.4.47)$$

which can be interpreted as the vertical reflection of the line $v = \bar{v}_B(a, d)$ through the curve $v = v_{a,d}(u)$.

For any $(a, d) \in \bar{\Omega}_-$ with $d > 0$, these definitions allow us to introduce the notation

$$\bar{u}_{\text{bot}}(a, d) = \max\{u \in [0, \bar{u}_B(a, d)] : v_{a,d;r}(u) = u_{a,d}^{\text{inv}}(u)\}. \quad (2.4.48)$$

On account of the inequalities

$$v_{a,d;r}(0) < 0 = u_{a,d}^{\text{inv}}(0) < u_{a,d}^{\text{inv}}(\bar{u}_B(a, d)) = \bar{v}_{\text{bot}}(a, d) < \bar{v}_B(a, d) = v_{a,d;r}(\bar{u}_B(a, d)), \quad (2.4.49)$$

one can verify that $\bar{u}_{\text{bot}}$ is well-defined and continuous in (a, d) .

Lemma 2.4.9. Pick $(a, d) \in \Omega_-$ and assume that

$$\bar{u}_{\text{bot}}(a, d) < \bar{u}_{\text{top}}(a, d) \quad (2.4.50)$$

holds. Then we have $c(a, d) > 0$.

Proof. Suppose the contrary that $c(a, d) = 0$ and consider the sequence $\{(u_i, v_i)\}$ defined by (2.4.26), together with the critical value $i_0 \in \mathbb{Z}$ that appears in Lemma 2.4.8. Since the sequence $\{u_i\}$ is non-decreasing, we have $u_{i_0+1} \geq u_{i_0}$ which implies that

$$u_{i_0+1} \geq u_{a,d}(v_{i_0}). \quad (2.4.51)$$

Exploiting $0 \leq v_{i_0} \leq \bar{v}_{\text{bot}}(a, d)$ this gives

$$0 \leq v_{i_0} \leq u_{a,d}^{\text{inv}}(u_{i_0+1}). \quad (2.4.52)$$

On the other hand, the inequality $v_{i_0+1} \leq \bar{v}_B(a, d)$ yields

$$v_{i_0} \geq v_{a,d;r}(u_{i_0+1}). \quad (2.4.53)$$

Combining this with (2.4.52), we see that $u_{i_0+1} \leq \bar{u}_{\text{bot}}(a, d)$. However, (2.4.38) implies that $u_{i_0+1} \geq \bar{u}_{\text{top}}(a, d)$, which yields the desired contradiction. $\square$

2.4.3 Verification of $\bar{u}_{\text{top}} > \bar{u}_{\text{bot}}$

In Figure 2.10 we show where one may numerically verify that the scalar inequality (2.4.50) holds, which ensures that $c(a, d) > 0$. We also plot the curve $d = \frac{(1-a)^2}{8}$, below which we have established that $c(a, d) = 0$. Taken together, we feel that these results cover a reasonable portion of the parameter space Ω_- .

Our final task is to analytically verify (without resorting to any numerics) that $c(a, d) > 0$ near the cusp $(a, d) = (\frac{1}{2}, \frac{1}{24})$ and the corner $(a, d) = (1, 0)$. As a preparation, we construct a simplified but weaker version of the condition $\bar{u}_{\text{top}} > \bar{u}_{\text{bot}}$ by exploiting the monotonicity of $v_{a,d;r}$.

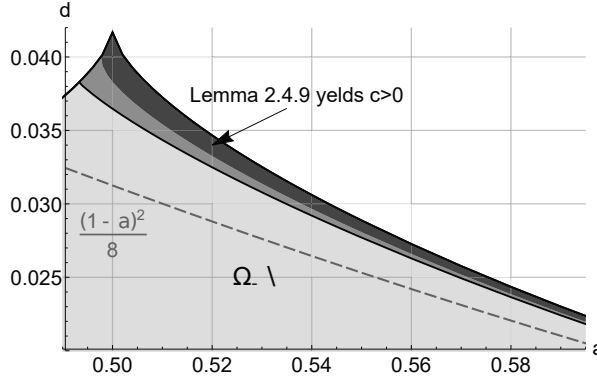


Figure 2.10: The darkest region contains all pairs (a, d) where the assumption $\bar{u}_{\text{bot}}(a, d) < \bar{u}_{\text{top}}(a, d)$ was verified numerically. We also include the boundaries of the sets $\mathcal{T}$ and Ω_- as computed numerically by the procedure described in §2.2.

Lemma 2.4.10. Pick $(a, d) \in \bar{\Omega}_-$ with $d > 0$. Then the function $v_{a,d;r}$ is strictly increasing on $[0, \bar{u}_B(a, d)]$.

Proof. We note first that $v'_{a,d}(u) = 1 - \frac{g'(u;a)}{2d} > 0$ for $u \in [0, u_{\min}(a)]$. In particular, we only have to consider the case $\bar{u}_B(a, d) > u_{\min}(a)$, which cannot occur for $a = \frac{1}{2}$.

Let us first assume that $0 < a < \frac{1}{2}$. Using (2.3.187) or (2.3.188) we may conclude that

$$v'_{a,d}(\bar{u}_B(a, d)) \geq v'_+(\bar{u}_B(a, d)) > 0. \quad (2.4.54)$$

Since $\bar{u}_B(a, d) \leq a \leq \gamma_{c;+}(a, d)$ this implies that $\bar{u}_B(a, d) \leq \gamma_{c;-}(a, d)$. In particular, $v_{a,d}$ and hence $v_{a,d;r}$ are strictly increasing on $[0, \bar{u}_B(a, d)]$.

It remains to consider $a \in (\frac{1}{2}, 1)$. Since $\bar{v}_B(1-a, d) > u_{\max}(1-a)$, we may use Corollary 2.3.4 to conclude

$$\bar{u}_B(a, d) = 1 - \bar{v}_B(1-a, d) < 1 - u_{\max}(1-a) = u_{\min}(a). \quad (2.4.55)$$

□

Corollary 2.4.11. Consider any $(a, d) \in \bar{\Omega}_-$ with $d > 0$ and suppose that

$$v_{a,d;r}(\bar{u}_{\text{top}}(a, d)) > \bar{v}_{\text{bot}}(a, d). \quad (2.4.56)$$

Then we have $\bar{u}_{\text{top}}(a, d) > \bar{u}_{\text{bot}}(a, d)$.

Proof. This follows from the uniform bound $u_{a,d}^{\text{inv}} \leq \bar{v}_{\text{bot}}(a, d)$ and the fact that $v_{a,d;r}$ is strictly increasing. □

We now set out to verify the explicit condition (2.4.56) for the boundary points $(a, d_-(a))$ with $a \sim 1$ and $a = \frac{1}{2}$. Using the continuity of $\bar{u}_{\text{top}}$ and $\bar{u}_{\text{bot}}$, this means that $c(a, d) > 0$ for all $(a, d) \in \Omega_-$ that are sufficiently close to these critical boundary points.

Lemma 2.4.12. We have the expansions

$$\begin{aligned}\bar{u}_B(a, d_-(a)) &= \frac{1}{4}(1-a)^2 + \frac{1}{8}(1-a)^3 + O((1-a)^4), \\ \bar{v}_B(a, d_-(a)) &= 1 - \frac{1}{2}(1-a) + O((1-a)^4)\end{aligned}\tag{2.4.57}$$

as $a \uparrow 1$.

Proof. Exploiting Corollary 2.3.4 together with the symmetry $d(a) = d(1-a)$, we obtain

$$\bar{u}_B(a, d_-(a)) = 1 - \bar{v}_B(1-a, d_-(1-a)), \quad \bar{v}_B(a, d_-(a)) = 1 - \bar{u}_B(1-a, d_-(1-a)).\tag{2.4.58}$$

The desired expansions hence follow from Proposition 2.3.6. $\square$

Lemma 2.4.13. We have the expansions

$$\begin{aligned}\bar{u}_{\text{top}}(a, d_-(a)) &= \frac{1}{4}(a-1)^2 + O((1-a)^3), \\ v_{a, d_-(a); r}(\bar{u}_{\text{top}}(a, d_-(a))) &= 1 + O(1-a)\end{aligned}\tag{2.4.59}$$

as $a \uparrow 1$.

Proof. These expansions can be found by substitution of (2.4.57) into the definitions (2.4.37) and (2.4.47). $\square$

On account of the identity $\gamma_{c;-}(1, 0) = \frac{1}{3}$ and the inequality $\bar{v}_{\text{bot}}(a, d) \leq \gamma_{c;-}(a, d)$, we see that Corollary 2.4.11 implies that

$$\bar{u}_{\text{top}}(a, d_-(a)) > \bar{u}_{\text{bot}}(a, d_-(a))\tag{2.4.60}$$

whenever $1-a > 0$ is sufficiently small.

Lemma 2.4.14. The inequality (2.4.56) holds for $(a, d) = (\frac{1}{2}, \frac{1}{24})$.

Proof. Writing $(a_{\text{cp}}, d_{\text{cp}}) = (\frac{1}{2}, \frac{1}{24})$, we can explicitly compute

$$\gamma_{c;+}(a_{\text{cp}}, d_{\text{cp}}) = \frac{1}{2} + \frac{1}{6}\sqrt{2},\tag{2.4.61}$$

which together with the expressions (2.3.26) and (2.4.37) yields

$$\bar{u}_{\text{top}}(a_{\text{cp}}, d_{\text{cp}}) = \frac{1}{2} - \frac{4}{9}\sqrt{2} + \frac{1}{6}\sqrt{3}.\tag{2.4.62}$$

Using (2.4.47) we obtain

$$v_{a_{\text{cp}}, d_{\text{cp}}; r}(\bar{u}_{\text{top}}(a_{\text{cp}}, d_{\text{cp}})) = \frac{1}{2} - \frac{1240}{243}\sqrt{2} + \frac{229}{54}\sqrt{3} \sim 0.6286.\tag{2.4.63}$$

In particular, we have

$$v_{a_{\text{cp}}, d_{\text{cp}}; r}(\bar{u}_{\text{top}}(a_{\text{cp}}, d_{\text{cp}})) > a_{\text{cp}} \geq \gamma_{c;-}(a_{\text{cp}}, d_{\text{cp}}) \geq \bar{v}_{\text{bot}}(a_{\text{cp}}, d_{\text{cp}}),\tag{2.4.64}$$

as desired. $\square$

Proof of Theorem 2.2.3. Items (i), (ii) and (iii) follow from Lemma 2.4.6, Corollary 2.4.4 and Lemma 2.4.14 respectively. Item (iv) follows from Lemma 2.4.5, together with Lemma 2.4.9 and the continuity of the functions $\bar{u}_{\text{bot}}$ and $\bar{u}_{\text{top}}$. Indeed, the expression (2.2.26) can be rewritten as

$$\Gamma(a) = \bar{u}_{\text{top}}(a, d_-(a)) - \bar{u}_{\text{bot}}(a, d_-(a)). \quad (2.4.65)$$

Finally, item (v) follows directly from (2.4.60). $\square$

Chapter 3

Multichromatic travelling waves

This chapter has been submitted as: H. J. Hupkes, L. Morelli, P. Stehlík, V. Švígler, "Multichromatic Travelling Waves for Lattice Nagumo Equations"

Abstract. We discuss multichromatic front solutions to the bistable Nagumo lattice differential equation. Such fronts connect the stable spatially homogeneous equilibria with spatially heterogeneous n -periodic equilibria and hence are not monotonic like the standard monochromatic fronts. In contrast to the bichromatic case, our results show that these multichromatic fronts can disappear and reappear as the diffusion coefficient is increased. In addition, these multichromatic waves can travel in parameter regimes where the monochromatic fronts are also free to travel. This leads to intricate collision processes where an incoming multichromatic wave can reverse its direction and turn into a monochromatic wave.

3.1 Introduction

In this chapter we are interested in the Nagumo lattice differential equation (LDE)

$$\dot{u}_j(t) = d[u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)] + u_j(t)(1 - u_j(t))(u_j(t) - a), \quad (3.1.1)$$

posed on the one-dimensional lattice $j \in \mathbb{Z}$, for small values of the diffusion-coefficient $d > 0$. This so-called anti-continuum regime features spatially periodic equilibria for (3.1.1) that can serve as buffer zones between regions of space where the homogeneous stable equilibria $u \equiv 0$ and $u \equiv 1$ dominate the dynamics. Our goal here is to continue the program initiated in Chapter 2 where two-periodic patterns and their connection to bichromatic waves were rigorously analyzed.

In particular, we build a classification framework that also allows larger periods to be considered. We also construct so-called multichromatic waves, which connect heterogeneous n -periodic rest-states to each other or their homogeneous counterparts

$u \equiv 0$ and $u \equiv 1$. We numerically analyze these multichromatic waves for $n \in \{3, 4\}$ and show that they exhibit richer behaviour than the bichromatic versions. Indeed, these multichromatic waves can disappear and reappear as the diffusion d is increased. In addition, open sets of parameters (a, d) exist where monochromatic and multichromatic waves can travel simultaneously. This allows us to explore various new types of wave collisions.

Reaction-diffusion systems The LDE (3.1.1) can be seen as the spatial discretization of the Nagumo PDE

$$u_t = u_{xx} + u(1 - u)(u - a) \quad (3.1.2)$$

onto a uniform grid with node-spacing $h = d^{-1/2}$. This scalar reaction-diffusion PDE can serve as a highly simplified model to describe the interaction between two species or states (described by $u = 0$ and $u = 1$) that compete for dominance in a spatial domain [3]. It admits a comparison principle and can be equipped with a variational structure [56], but it also has a rich global attractor. As such, it has served as a prototype system to investigate many of the key concepts in the field of pattern formation, such as spreading speeds for compact disturbances [134], the existence and stability of travelling waves [51, 111] and other non-trivial entire solutions [95, 137].

The semi-discrete version (3.1.1) has served as a playground to investigate the impact of the transition from a spatially continuous to a spatially discrete domain. From a mathematical point of view, interesting questions and complications arise due to the broken translational invariance [89]. From the practical point of view, it is highly desirable to be able to incorporate the natural spatial discreteness present in many physical systems such as myelinated nerve fibres [106], meta-materials [22, 23, 128] and crystals [25, 37].

Monochromatic waves Substitution of the travelling wave Ansatz $u(x, t) = \Phi(x - ct)$ into the PDE (3.1.2) yields the travelling wave ODE

$$-c\Phi' = \Phi'' + \Phi(1 - \Phi)(\Phi - a). \quad (3.1.3)$$

On the other hand, substitution of the discrete analog $u_j(t) = \Phi(j - ct)$ into the LDE (3.1.1) yields the monochromatic wave equation

$$-c\Phi'(\xi) = d[\Phi(\xi - 1) - 2\Phi(\xi) + \Phi(\xi + 1)] + \Phi(\xi)(1 - \Phi(\xi))(\Phi(\xi) - a), \quad (3.1.4)$$

which is a functional differential equation of mixed type (MFDE). We use the term monochromatic here to refer to the fact that each spatial index j follows the same waveprofile Φ . We are specially interested in waves that connect the two stable equilibria $u = 0$ and $u = 1$. In particular, we impose the boundary conditions

$$\Phi(-\infty) = 0, \quad \Phi(+\infty) = 1. \quad (3.1.5)$$

The ODE (3.1.3) with (3.1.5) can be analyzed by phase-plane analysis [51] (and even solved explicitly) to yield the existence of solutions that increase monotonically and have $\text{sign}(c) = \text{sign}(a - \frac{1}{2})$. These waves have a large basin of attraction [51] and can be used as building blocks to construct and analyze more complicated solutions [134].

More advanced techniques are required to analyze (3.1.4), but again it is possible to show that non-decreasing solutions exist [91]. However, it is now a very delicate question to determine whether the uniquely determined wavespeed $c = c_{\text{mc}}(a, d)$ satisfies $c_{\text{mc}}(a, d) = 0$ or $c_{\text{mc}}(a, d) \neq 0$. Indeed, the broken translational invariance causes an energy-barrier that must be overcome before waves are able to travel. As such, there is an open region in the (a, d) -plane for which $c_{\text{mc}}(a, d) = 0$ holds; see Fig. 3.10. This pinning phenomenon is generic [7, 21, 41, 43, 67, 75, 83, 92] but not omnipresent [40, 70] in discrete systems and has received considerable attention.

Bichromatic waves The discrete second derivative allows (3.1.1) to have a much larger class of equilibrium solutions than the PDE (3.1.2). For example, two-periodic equilibria of the form

$$u_j = \begin{cases} v & j \text{ is even,} \\ w & j \text{ is odd.} \end{cases} \quad (3.1.6)$$

can be found by solving the two-component system $G(v, w; a, d) = 0$ given by

$$G(v, w; a, d) = \begin{pmatrix} 2d(w - v) + v(1 - v)(v - a) \\ 2d(v - w) + w(1 - w)(w - a) \end{pmatrix}. \quad (3.1.7)$$

The variable w can be readily eliminated, leading to a ninth-order polynomial equation for the remaining component v . For $0 < d \ll 1$ this polynomial has nine roots, leading to two stable and four unstable two-periodic equilibria for (3.1.1) besides the three spatially homogeneous equilibria $\{0, a, 1\}$.

In Chapter 2 we performed a full rigorous analysis of this system, which shows that the number of these two-periodic equilibria decreases as $d > 0$ is increased. In particular, there exist two functions $0 < d_s(a) < d_u(a)$ defined for $a \in (0, 1)$ so that the two stable patterns $(v_{\text{bc}}, w_{\text{bc}})$ and $(w_{\text{bc}}, v_{\text{bc}})$ collide with two unstable patterns and disappear as d crosses $d_s(a)$. The remaining two unstable patterns subsequently collide with (a, a) as d crosses $d_u(a)$, leaving only the three spatially homogeneous equilibria. We emphasize that all these two-periodic equilibria only exist in the region where monochromatic waves are pinned, i.e. $c_{\text{mc}}(a, d) = 0$.

Based on general results in [27] we showed that for $0 < d < d_s(a)$ the system (3.1.1) admits two types of bichromatic waves

$$u_j(t) = \begin{cases} \Phi_e(j - ct) & j \text{ is even,} \\ \Phi_o(j - ct) & j \text{ is odd.} \end{cases} \quad (3.1.8)$$

The first class satisfies the lower limits

$$\lim_{\xi \rightarrow -\infty} (\Phi_e(\xi), \Phi_o(\xi)) = (0, 0), \quad \lim_{\xi \rightarrow +\infty} (\Phi_e(\xi), \Phi_o(\xi)) = (v_{\text{bc}}, w_{\text{bc}}), \quad (3.1.9)$$

and has wavespeed $c_{0 \rightarrow bc} \geq 0$, while the second class satisfies the upper limits

$$\lim_{\xi \rightarrow -\infty} (\Phi_e(\xi), \Phi_o(\xi)) = (v_{bc}, w_{bc}), \quad \lim_{\xi \rightarrow +\infty} (\Phi_e(\xi), \Phi_o(\xi)) = (1, 1), \quad (3.1.10)$$

and has $c_{bc \rightarrow 1} \leq 0$. In Chapter 2 we showed that there exist two thresholds

$$0 < d_{0 \rightarrow bc}(a) \leq d_s(a), \quad 0 < d_{bc \rightarrow 1}(a) \leq d_s(a), \quad (3.1.11)$$

so that in fact $c_{0 \rightarrow bc} > 0$ respectively $c_{bc \rightarrow 1} < 0$ holds as d is increased above these thresholds. In addition, for all $a \in (0, 1)$ one or both of the inequalities in (3.1.11) is strict, indicating the presence of one or more *travelling* bichromatic waves for d sufficiently close to $d_s(a)$.

Numerical results indicate that these two types of bichromatic waves can be glued together via an intermediate buffer zone that displays the two-periodic pattern (v_{bc}, w_{bc}) . This buffer zone is consumed as the waves move towards each other and eventually collide to form a trapped monochromatic wave; see Figure 3.8 for the trichromatic analogue.

Bichromatic waves have also been found in several other spatially discrete settings. The results in [19, 129] apply to an anti-diffusion version of (3.1.1) where $d < 0$. This can be reformulated as a two-component problem with positive alternating diffusion coefficients, allowing the general results in [27] to be applied. Several versions of the two-periodic FPU problem are considered in [46, 47, 69]. Using a different palette of techniques, the authors obtain so-called nanopterons, which can have small high-frequency oscillations in their tails. Finally, the two-periodic FitzHugh-Nagumo problem was considered in [114] using a modified spectral-convergence argument.

Multichromatic waves The main purpose of this chapter is to illustrate the novel behaviour that arises for (3.1.1) when considering wave connections to/from stable n -periodic patterns with $n \geq 3$. Our two main conclusions are that the monotonicity properties described above are no longer valid and that travelling multichromatic waves can co-exist with travelling monochromatic waves. In particular, travelling multichromatic waves can appear, disappear and reappear as $d > 0$ is increased and can collide with other multichromatic waves to form travelling monochromatic waves.

Since the degree of the polynomial that governs the n -periodic equilibria is given by 3^n , it is essential to develop an appropriate classification system to keep track of all the roots and their ordering properties. We develop such a system in this chapter, using words from the set $\{\mathbf{o}, \mathbf{a}, \mathbf{1}\}^n$ to track roots that bifurcate off the corresponding sequence of zeroes of the cubic at $d = 0$. The lack of monotonicity with respect to d leads to complications and forces us to allow both parameters (a, d) to vary when tracking roots of a related algebraic problem, unlike in Chapter 2. Although the general theory in [27] also applies to our setting, it is still a challenge to check the conditions in a systematic fashion.

Wave collisions Understanding the interaction between waves is an important topic that is attracting considerable attention, primarily in the spatially continuous setting at present. The so-called weak interaction regime where the waves are far apart is relatively well-understood; see e.g. the exit manifold developed by Wright and Hoffman [68] for the discrete setting and the numerous studies on renormalization techniques for the continuous setting [10, 38, 131].

However, at present there is no general theory to understand strong interactions, where the core of the waves approach each other and deform significantly. Early numerical results by Nishiura and coworkers [100] for the Gray-Scott and a three-component FitzHugh-Nagumo system suggest that the fate of colliding waves (annihilation, combination or scattering) is related to the properties of a special class of unstable solutions called separators. Even the internal dynamics of a single pulse under the influence of essential spectrum (a proxy for the advance of a second wave) can be highly complicated, see e.g. [29] for the (partial) unfolding of a butterfly catastrophe.

Naturally, more information can be obtained in the presence of a comparison principle. Indeed, for the PDE (3.1.2) one can show that monostable waves can merge to form a bistable wave [95] and that counterpropagating waves can annihilate [96, 137]. If one modifies the nonlinearity to allow more zero-crossings, one can stack waves that connect a chain of equilibria to form so-called propagating terraces [105].

We emphasize that the collisions described in this chapter are far richer than those described above for (3.1.2). This is a direct consequence of the delicate structure of the set of equilibria for (3.1.1). By exploiting the comparison principle, our hope is that this system can serve as a playground for generating and understanding complicated collision processes.

Organization In §3.2 we discuss the algebraic problem that n -periodic equilibria must satisfy, develop a naming scheme for its roots and formulate a result concerning the existence of travelling waves. In §3.3 we discuss trichromatic waves and focus on the fact that for certain values of a three-periodic stable equilibria can disappear and reappear as the diffusion parameter d is increased. We move on to quadrichromatic waves in §3.4, highlighting the fact that quadrichromatic and monochromatic waves can travel simultaneously in certain parameter regions. This allows us to study various types of wave collisions. Finally, in §3.5 we prove our main result Theorem 3.2.2, which establishes the existence of multichromatic waves.

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3.2 Multichromatic root naming and ordering

The main focus of this chapter is the Nagumo lattice differential equation (LDE)

$$\dot{u}_j(t) = d[u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)] + g(u_j(t); a), \quad j \in \mathbb{Z}, \quad (3.2.1)$$

in which the parameters (a, d) are taken from the half-strip

$$\mathcal{H} = [0, 1] \times [0, \infty) \quad (3.2.2)$$

and the nonlinearity is given by the cubic

$$g(u; a) = u(1 - u)(u - a). \quad (3.2.3)$$

Our results focus on n -periodic stationary solutions to (3.2.1) and the waves that connect them.

In §3.2.1 we develop a naming system that allows us to partially classify these stationary solutions in an intuitive fashion. We proceed in §3.2.2 by formulating a result for the existence of waves that uses our naming system to decide which equilibria can be connected. Finally, equivalence classes for these waves are introduced in §3.2.3 by exploiting the symmetries present in (3.2.1).

3.2.1 Equilibrium types

We will write n -periodic equilibria for the LDE (3.2.1) in the form

$$u_i = u_{\text{mod}(i, n)} \quad (3.2.4)$$

for some vector $u \in \mathbb{R}^n$, where we let the modulo operator take values in

$$\text{mod}(i, n) \in \{1, \dots, n\}. \quad (3.2.5)$$

We remark that u can be interpreted as a solution of the Nagumo equation posed on a cyclic graph of length n ; see [121]. Taking $n \geq 3$ and introducing the nonlinear mapping

$$G(u; a, d) := \begin{pmatrix} d(u_n - 2u_1 + u_2) + g(u_1; a) \\ d(u_1 - 2u_2 + u_3) + g(u_2; a) \\ \vdots \\ d(u_{n-1} - 2u_n + u_1) + g(u_n; a) \end{pmatrix} \in \mathbb{R}^n, \quad (3.2.6)$$

we see that any such equilibrium must satisfy $G(u; a, d) = 0$.

For any $a \in (0, 1)$ and $u \in \{0, a, 1\}^n$, it is easy to see that $G(u; a, 0) = 0$ and to confirm that the diagonal matrix

$$D_1 G(u; a, 0) = \text{diag}(g'(u_1; a), \dots, g'(u_n; a)) \quad (3.2.7)$$

has non-zero entries. In particular, the implicit function theorem implies that each of these 3^n roots is part of a smooth one-parameter family of roots that exists whenever $|d|$ is small. In fact, one can track the location of each of these roots as d is increased, up until the point where the root in question disappears by colliding with another root. This procedure forms the heart of the naming scheme that we develop here, which will allow us to refer to different types of roots in an efficient manner.

In particular, we set out to label solutions of the equation $G(\cdot; a, d) = 0$ with words w taken from the set $\{\mathbf{o}, \mathbf{a}, \mathbf{1}\}^n$. We emphasize that we are using the fixed symbol $\mathbf{a}$ as placeholder for the parameter $a \in (0, 1)$, which is allowed to vary. Indeed, for any $w \in \{\mathbf{o}, \mathbf{a}, \mathbf{1}\}^n$ we introduce the vector $w|_a \in \mathbb{R}^n$ by writing

$$(w|_a)_i = \begin{cases} 0 & \text{if } w_i = \mathbf{o}, \\ a & \text{if } w_i = \mathbf{a}, \\ 1 & \text{if } w_i = \mathbf{1}. \end{cases} \quad (3.2.8)$$

Definition 3.2.1. Consider a word $w \in \{\mathbf{o}, \mathbf{a}, \mathbf{1}\}^n$ together with a triplet

$$(u, a, d) \in [0, 1]^n \times (0, 1) \times [0, \infty). \quad (3.2.9)$$

Then we say that u is an equilibrium of type w if there exists a C^1 -smooth curve

$$[0, 1] \ni t \mapsto (v(t), \alpha(t), \delta(t)) \in [0, 1]^n \times (0, 1) \times [0, \infty) \quad (3.2.10)$$

so that we have

$$\begin{aligned} (v, \alpha, \delta)(0) &= (w|_a, a, 0), \\ (v, \alpha, \delta)(1) &= (u, a, d), \end{aligned} \quad (3.2.11)$$

together with

$$G(v(t); \alpha(t), \delta(t)) = 0, \quad \det D_1 G(v(t); \alpha(t), \delta(t)) \neq 0 \quad (3.2.12)$$

for all $0 \leq t \leq 1$.

We note that substituting $t = 1$ in (3.2.12) shows that indeed $G(u; a, d) = 0$, justifying the terminology of an equilibrium. In addition, the second requirement in (3.2.12) allows us to apply the implicit function theorem to conclude that $G(\cdot; \tilde{a}, \tilde{d}) = 0$ also has equilibria of type w for all pairs $(\tilde{a}, \tilde{d})$ sufficiently close to (a, d) . In particular, these observations allow us to introduce the pathwise connected set

$$\Omega_w = \{(a, d) \in \mathcal{H} : \text{the system } G(\cdot; a, d) = 0 \text{ admits an equilibrium of type } w\}, \quad (3.2.13)$$

which is open in the half-strip $\mathcal{H} = [0, 1] \times [0, \infty)$.

We now impose the following conditions on the structure of these sets Ω_w . The second of these basically states that the interior of the curve (3.2.10) can be perturbed freely within Ω_w without changing the value of the equilibrium u .

(H01) For any two words $w_A, w_B \in \{\mathbf{o}, \mathbf{a}, \mathbf{1}\}^n$ the intersection $\Omega_{w_A} \cap \Omega_{w_B}$ is connected.

(HΩ2) Consider any word $w \in \{0, a, 1\}^n$ and any $(a, d) \in \Omega_w$, together with a pair of curves

$$[0, 1] \ni t \mapsto (\alpha_A, \delta_A)(t) \in \Omega_w, \quad [0, 1] \ni t \mapsto (\alpha_B, \delta_B)(t) \in \Omega_w \quad (3.2.14)$$

that have

$$(\alpha_A, \delta_A)(0) = (\alpha_B, \delta_B)(0) = (a, 0), \quad (\alpha_A, \delta_A)(1) = (\alpha_B, \delta_B)(1) = (a, d). \quad (3.2.15)$$

Then there exist unique functions

$$[0, 1] \ni t \mapsto (v_A, v_B)(t) \in \mathbb{R}^n \times \mathbb{R}^n \quad (3.2.16)$$

so that the triplets $(v_A, \alpha_A, \delta_A)$ and $(v_B, \alpha_B, \delta_B)$ both satisfy (3.2.12) for all $0 \leq t \leq 1$, together with the identities

$$v_A(0) = v_B(0) = w|_a, \quad v_A(1) = v_B(1). \quad (3.2.17)$$

At first glance, the condition (HΩ2) appears to be rather cumbersome to verify in practice. To make this more feasible, it is useful to introduce the set

$$\begin{aligned} \Gamma = \{ (a, d) \in \mathcal{H} : \text{there exists } u \in \mathbb{R}^n \text{ for which } G(u; a, d) = 0 \\ \text{and } \det D_1 G(u; a, d) = 0 \}, \end{aligned} \quad (3.2.18)$$

which by continuity is a closed subset of $\mathcal{H} = [0, 1] \times [0, \infty)$. Let us assume that $\mathcal{H} \setminus \Gamma$ consists of a finite number N_Γ of components $\{V_i\}_{i=1}^{N_\Gamma}$ that are open in $\mathcal{H}$ and *simply connected*. On account of a global implicit function theorem [15, Thm. 3], there exist non-negative integers $\{m_i\}_{i=1}^{N_\Gamma}$ together with smooth functions

$$u_{i,j} : V_i \rightarrow \mathbb{R}^n, \quad 1 \leq i \leq N_\Gamma, \quad 1 \leq j \leq m_i \quad (3.2.19)$$

so that $G(u_{i,j}(a, d); a, d) = 0$ for each $(a, d) \in V_i$. In addition, $u_{i,j_1}(a, d) \neq u_{i,j_2}(a, d)$ whenever $j_1 \neq j_2$ and every solution $G(\cdot; a, d) = 0$ with $(a, d) \in V_i$ can be written in this way.

In this setting, (HΩ2) can be verified by checking which of these functions can be connected continuously through the boundaries of adjacent components. Indeed, (HΩ2) is satisfied if there is no sequence of connections that starts in u_{i,j_1} and ends in u_{i,j_2} for some $1 \leq i \leq N_\Gamma$ and some pair $1 \leq j_1 \neq j_2 \leq m_i$.

In any case, writing V_* for the component of $\mathcal{H} \setminus \Gamma$ that contains the horizontal segment $(0, 1) \times \{0\}$, (HΩ2) can always be achieved if one replaces Ω_w by subsets of the form

$$\Omega_w^* = \{ (a, d) \in V_* : \text{the system } G(\cdot; a, d) = 0 \text{ admits an equilibrium of type } w \}. \quad (3.2.20)$$

However, we emphasize that (HΩ2) appears to be valid without this artificial restriction for the regions that we have numerically computed in this chapter.

Corollary 3.2.1. Fix an integer $n \geq 2$ and suppose that (HΩ1) and (HΩ2) both hold. Then for any word $w \in \{\mathbf{o}, \mathbf{a}, \mathbf{1}\}^n$ there is a smooth function $u_w : \Omega_w \rightarrow \mathbb{R}^n$ so that $u_w(a, d)$ is the unique equilibrium of type w for the system $G(\cdot; a, d) = 0$ for all $(a, d) \in \Omega_w$. In addition, whenever $(a, d) \in \Omega_{w_A} \cap \Omega_{w_B}$ for two distinct words $w_A, w_B \in \{\mathbf{o}, \mathbf{a}, \mathbf{1}\}^n$ we have

$$u_{w_A}(a, d) \neq u_{w_B}(a, d). \quad (3.2.21)$$

Proof. The first statement is a consequence of (HΩ2) and the implicit function theorem. For the second statement, let us argue by contradiction and assume that both vectors in (3.2.21) are equal to $u \in \mathbb{R}^n$. Condition (HΩ1) allows us to pick a path from $(a, 0)$ to (a, d) that lies entirely within $\Omega_{w_A} \cap \Omega_{w_B}$. Applying (HΩ2) shows that u can be continued as an equilibrium along this path back to both $(w_A)|_a$ and $(w_B)|_a$. However, the implicit function theorem implies that this continuation should be unique. $\square$

We emphasize that our classification scheme only track roots until the first time the associated Jacobian becomes singular and the implicit function theorem can no longer be applied. We will encounter several different types of behaviour at such points. It is possible for two roots to collide and become complex and sometimes even recombine at ‘later’ parameter values. We will also encounter multi-root collisions where one or more roots survive the collision process. In such cases, we often use an ad-hoc naming system, where we label the emerging branch by reusing or combining the types of the original colliding branches.

3.2.2 Waves

In this section we focus on wave solutions to (3.2.1) that connect the n -periodic stationary solutions investigated in the previous section. These so-called multichromatic waves can be written as

$$u_j(t) = \Phi_{\text{mod}(j,n)}(j - ct) \quad (3.2.22)$$

for some wavespeed $c \in \mathbb{R}$ and $\mathbb{R}^n$ -valued waveprofile

$$\Phi = (\Phi_1, \Phi_2, \dots, \Phi_n) : \mathbb{R} \rightarrow \mathbb{R}^n \quad (3.2.23)$$

that satisfies the boundary conditions

$$\lim_{\xi \rightarrow -\infty} \Phi(\xi) = u_{w_-}, \quad \lim_{\xi \rightarrow +\infty} \Phi(\xi) = u_{w_+} \quad (3.2.24)$$

for some pair of words $w_{\pm} \in \{\mathbf{o}, \mathbf{1}\}^n$; see Corollary 3.2.1.

Substituting this Ansatz into (3.2.1) yields the traveling wave functional differential equation

$$\begin{aligned} -c\Phi'_1(\xi) &= d[\Phi_n(\xi - 1) - 2\Phi_1(\xi) + \Phi_2(\xi + 1)] + g(\Phi_1(\xi); a), \\ -c\Phi'_2(\xi) &= d[\Phi_1(\xi - 1) - 2\Phi_2(\xi) + \Phi_3(\xi + 1)] + g(\Phi_2(\xi); a), \\ &\vdots \\ -c\Phi'_n(\xi) &= d[\Phi_{n-1}(\xi - 1) - 2\Phi_n(\xi) + \Phi_1(\xi + 1)] + g(\Phi_n(\xi); a), \end{aligned} \quad (3.2.25)$$

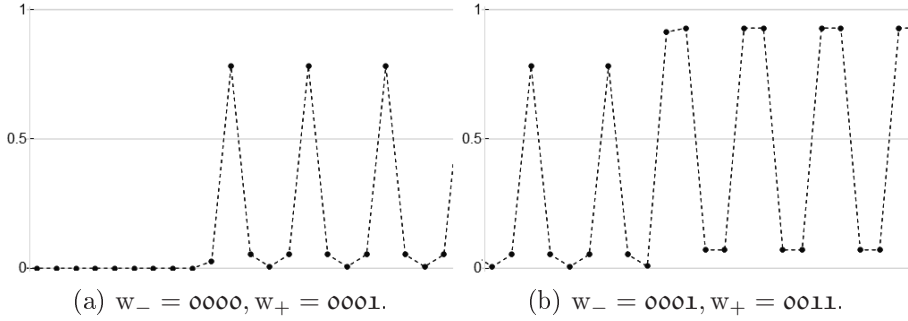


Figure 3.1: Two examples of 4-chromatic waves as described in Theorem 3.2.2.

which has positive coefficients on all shifted terms and also has diagonal nonlinearities. This system hence fits into the framework developed in [27], provided that the assumptions pertaining to the boundary conditions (3.2.24) can also be validated.

This is in fact the key question that we address in our main result below. This result requires the following assumption, which states that our root tracking scheme captures all (marginally) stable¹ n -periodic equilibria of the LDE. In particular, the types of these equilibria correspond with words from the *stable* subset $\{\mathbf{o}, \mathbf{1}\}^n$.

(HS) Recall the definitions (3.2.2) and (3.2.18) and suppose that $G(\mathbf{u}; a, d) = 0$ for some $\mathbf{u} \in \mathbb{R}^n$ and $(a, d) \in \mathcal{H}$. Suppose furthermore that all eigenvalues λ of $D_1 G(\mathbf{u}; a, d)$ satisfy $\lambda \leq 0$. Then we have² $\mathbf{u} = \mathbf{u}_w(a, d)$ for some $w \in \{\mathbf{o}, \mathbf{1}\}^n$ and $(a, d) \in \overline{\Omega}_w$.

Theorem 3.2.2 (see §3.5). *Fix an integer $n \geq 2$ and assume that (HΩ1), (HΩ2) and (HS) all hold. Consider two distinct words $w_-, w_+ \in \{\mathbf{o}, \mathbf{1}\}^n$ with $w_- \leq w_+$ and pick $(a, d) \in \Omega_{w_-} \cap \Omega_{w_+}$ with $d > 0$. Suppose furthermore that one of the following conditions holds.*

- (a) *The words w_- and w_+ differ at precisely one location.*
- (b) *For each $w \in \{\mathbf{o}, \mathbf{1}\}^n \setminus \{w_-, w_+\}$ that satisfies $w_- \leq w \leq w_+$ we have $(a, d) \notin \overline{\Omega}_w$.*

Then there exists a unique $c \in \mathbb{R}$ for which the travelling system (3.2.25) admits a solution $\Phi : \mathbb{R} \rightarrow \mathbb{R}^n$ that satisfies the boundary conditions

$$\lim_{\xi \rightarrow -\infty} \Phi(\xi) = \mathbf{u}_{w_-}(a, d), \quad \lim_{\xi \rightarrow +\infty} \Phi(\xi) = \mathbf{u}_{w_+}(a, d). \quad (3.2.26)$$

If $c \neq 0$, then Φ is unique up to translation and each component is strictly increasing.

¹We use the eigenvalues of the matrix $D_1 G(\mathbf{u}; a, d)$ to characterize the stability of $\mathbf{u}$. Using the comparison principle this can be easily transferred to the full LDE (3.1.1).

²If $(a, d) \in \partial\Omega_w$ this identity should be interpreted as a limit.

3.2.3 Symmetries

There are a number of useful symmetries present in the equilibrium equation $G(u; a, d) = 0$ and the travelling wave MFDE (3.2.25). We explore three important transformations here that significantly reduce the number of cases that need to be considered.

We first note that the identity $g(1 - u; a) = -g(u; 1 - a)$ implies that

$$G(\mathbf{1} - \mathbf{u}; a, d) = -G(\mathbf{u}; 1 - a, d) \quad (3.2.27)$$

holds for all $\mathbf{u} \in \mathbb{R}^n$, with $\mathbf{1} = (1, \dots, 1)^\top \in \mathbb{R}^n$. In order to exploit this, we pick $\mathbf{w} \in \{\mathbf{o}, \mathbf{a}, \mathbf{1}\}^n$ and write $\mathbf{w}_{\mathbf{o} \leftrightarrow \mathbf{1}} \in \{\mathbf{o}, \mathbf{a}, \mathbf{1}\}^n$ for the ‘inverted’ word

$$(\mathbf{w}_{\mathbf{o} \leftrightarrow \mathbf{1}})_i = \begin{cases} \mathbf{1} & \text{if } \mathbf{w}_i = \mathbf{o}, \\ \mathbf{a} & \text{if } \mathbf{w}_i = \mathbf{a}, \\ \mathbf{o} & \text{if } \mathbf{w}_i = \mathbf{1}. \end{cases} \quad (3.2.28)$$

The identity (3.2.27) directly implies the reflection relation

$$\Omega_{\mathbf{w}_{\mathbf{o} \leftrightarrow \mathbf{1}}} = \{(a, d) : (1 - a, d) \in \Omega_{\mathbf{w}}\}, \quad (3.2.29)$$

together with

$$\mathbf{u}_{\mathbf{w}_{\mathbf{o} \leftrightarrow \mathbf{1}}}(a, d) = \mathbf{1} - \mathbf{u}_{\mathbf{w}}(1 - a, d). \quad (3.2.30)$$

We proceed by introducing the coordinate-shifts $\{T_k\}_{k=1}^n : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and reflection $R : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that act as

$$(T_k \mathbf{u})_i = \mathbf{u}_{\text{mod}(i+k, n)}, \quad (R \mathbf{u})_i = \mathbf{u}_{\text{mod}(1-i, n)}. \quad (3.2.31)$$

It is easy to verify that the identity $G(\mathbf{u}; a, d) = 0$ implies that also

$$G(T_k \mathbf{u}; a, d) = G(R \mathbf{u}; a, d) = 0. \quad (3.2.32)$$

In addition, when (c, Φ) is a solution to the travelling wave MFDE (3.2.25), the same holds for the coordinate-shifted pair $(c, T_k \Phi)$ and the reflected version $(-c, \tilde{\Phi})$ with

$$\tilde{\Phi}(\xi) = R\Phi(-\xi). \quad (3.2.33)$$

These identities are all consequence of the invariance of the Nagumo LDE (3.2.1) under the transformations $j \mapsto j + k$ and $j \mapsto -j$.

Our choice to only consider waves where $\Phi(-\infty) \leq \Phi(+\infty)$ breaks the reflection invariance (3.2.33), which allows us to focus solely on the symmetry caused by the coordinate shifts T_k . In particular, for any word $\mathbf{w} \in \{\mathbf{o}, \mathbf{a}, \mathbf{1}\}^n$, we write

$$\Omega_{[\mathbf{w}]} = \Omega_{\mathbf{w}} = \Omega_{T_1 \mathbf{w}} = \dots = \Omega_{T_{n-1} \mathbf{w}}. \quad (3.2.34)$$

In addition, we introduce the shorthand notation

$$[\mathbf{w}] = \{\mathbf{u}_{\mathbf{w}}, T_1 \mathbf{u}_{\mathbf{w}}, \dots, T_{n-1} \mathbf{u}_{\mathbf{w}}\} \quad (3.2.35)$$

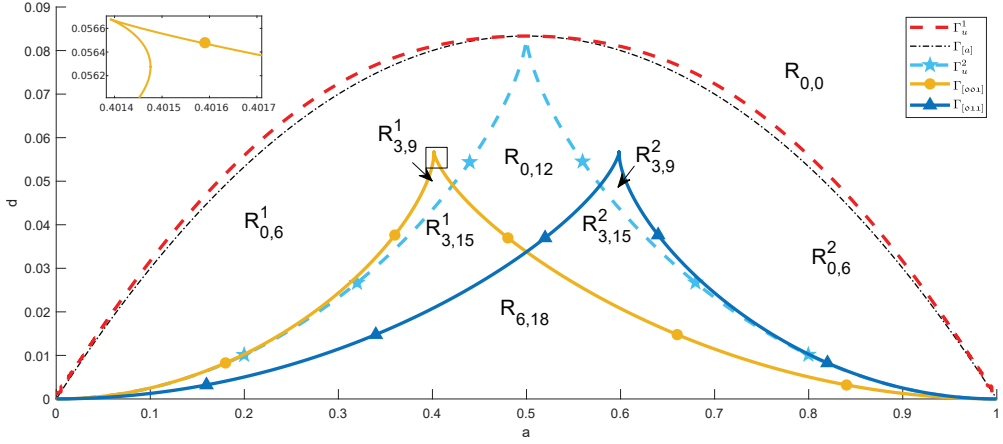


Figure 3.2: Bifurcation thresholds for the number of roots of (3.3.1). The subscripts in the regions R indicate the number of stable and unstable roots present in each region, counting their multiplicities but excluding the homogeneous roots $[o]$, $[a]$ and $[1]$.

to refer to all the roots in the corresponding equivalence class, which are defined for $(a, d) \in \Omega_{[w]}$.

Note that we are hence treating $[0a1]$ and $[01a]$ as separate classes, even though they correspond with equivalent equilibria of $G(\cdot; a, d) = 0$. Interestingly enough, when restricting attention to the alphabet $\{o, 1\}^n$, this distinction only plays a role for $n \geq 6$. For example, $0o1o11$ is not shift-related to its reflection $11o1oo$, but all shorter binary sequences are.

Whenever we are referring to an equivalence class of roots, we will use the word that is the smallest in the lexicographical³ sense (the so-called Lyndon word) as a class representative. For example, $0a11$ is the Lyndon word for the class

$$[0a11] = \{u_{0a11}, u_{10a1}, u_{110a}, u_{a110}\}. \quad (3.2.36)$$

We note that shorter words with a length that divides n can also be interpreted as a word of length n by periodic extension. For example, we write

$$[o1] = [o1o1] = \{u_{o1o1}, u_{1o1o}\}. \quad (3.2.37)$$

For any pair $w_{\pm} \in \{o, 1\}^n$ with $w_- \leq w_+$, we will use the shorthand notation $u_{w_-} \rightarrow u_{w_+}$ to refer to any pair (c, Φ) that satisfies the travelling wave MFDE (3.2.25) together with the boundary conditions (3.2.26). The observations above show that $T_k u_{w_-} \rightarrow T_k u_{w_+}$ then corresponds with the pair $(c, T_k \Phi)$.

³For ordering purposes we assume that $o < a < 1$ holds.

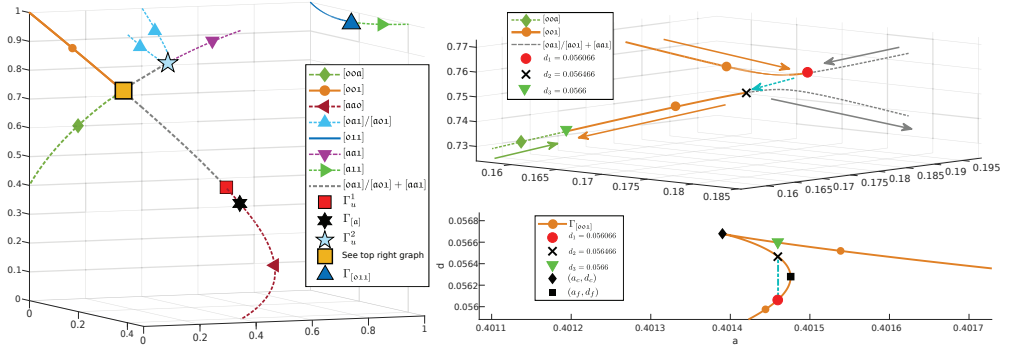


Figure 3.4: Behaviour of the root branches near the cusp of the yellow $\Gamma_{[001]}$ curve from Figure 3.2, which is magnified in the bottom right panel. In particular, we fix $a = 0.40146 \in (a_c, a_f)$ and note that $\Gamma_{[001]}$ is crossed three times as d is increased. The top right panel contains a zoom of the yellow square from the left panel, which is impacted by these crossings. The blue dashed line between the red circle and black cross represents the region where two of the roots are temporarily complex. These plots illustrate the mechanism by which the stable branch $[001]$ switches its unstable connecting branch as a is increased through the cusp and fold points a_c and a_f .

intervals of the parameter a in which the number of (stable) equilibria can actually increase as d is increased (e.g., for $a = 0.40146$, see Figures 3.2 and 3.4). In addition, for values of a in these intervals, there are two disjoint intervals of parameters d for which *travelling* trichromatic waves exist that connect the homogeneous states $[0]$ and $[1]$ to a stable 3-periodic equilibrium. Similarly, for a fixed d , the number of (stable) equilibria can increase as $|a - \frac{1}{2}|$ is increased (e.g., for $d = 0.04$, see Figure 3.2).

3.3.1 Equilibria

Trichromatic equilibria $u \in \mathbb{R}^3$ for (3.2.1) correspond to roots of the nonlinear function

$$G(u; a, d) = \begin{pmatrix} d(u_3 - 2u_1 + u_2) + g(u_1; a) \\ d(u_1 - 2u_2 + u_3) + g(u_2; a) \\ d(u_2 - 2u_3 + u_1) + g(u_3; a) \end{pmatrix}. \quad (3.3.1)$$

Inspection of this system shows that one component can be removed if one enforces either $u_1 = u_2$, $u_2 = u_3$ or $u_1 = u_3$.

In Figure 3.2 we numerically computed the critical set Γ defined in (3.2.18) by searching for roots of the augmented system

$$G_*(u; a, d) = \begin{pmatrix} G(u; a, d) \\ \det D_1 G(u; a, d) \end{pmatrix}. \quad (3.3.2)$$

The results show that Γ can be decomposed into 5 piecewise-smooth curves that we label as $\Gamma_{[\mathbf{a}]}$, $\Gamma_{[\mathbf{o}\mathbf{o}\mathbf{1}]}$, $\Gamma_{[\mathbf{o}\mathbf{1}\mathbf{1}]}$, Γ_u^1 and Γ_u^2 . The labels of the form $\Gamma_{[\mathbf{w}]}$ imply that $\Gamma_{[\mathbf{w}]} \subset \partial\Omega_{[\mathbf{w}]}$. We emphasize that this naming scheme is ambiguous by its very nature. Indeed, collisions between roots occur precisely on these curves, which also intersect each other.

These 5 curves divide the remaining parameter space $\mathcal{H} \setminus \Gamma$ into 11 open and simply connected components. Ignoring the three homogeneous roots⁴ $[\mathbf{o}]$, $[\mathbf{a}]$ and $[\mathbf{1}]$, the bottom region $R_{6,18}$ contains 6 stable roots and 18 unstable roots. The stable roots are represented by the equivalence classes $[\mathbf{o}\mathbf{o}\mathbf{1}]$ and $[\mathbf{o}\mathbf{1}\mathbf{1}]$, while all the remaining equivalence classes generate the unstable roots.

In the discussion below we indicate how this configuration changes as each of the critical curves is crossed. For now, we recall the identity (3.2.27), which explains the reflection symmetry through the line $a = \frac{1}{2}$ and allows us to focus on the case $a \in (0, \frac{1}{2}]$.

The $\Gamma_{[\mathbf{o}\mathbf{1}\mathbf{1}]}$ threshold The first threshold that is encountered when increasing d for $a \in (0, \frac{1}{2})$ is the curve $\Gamma_{[\mathbf{o}\mathbf{1}\mathbf{1}]}$. Here the stable roots $[\mathbf{o}\mathbf{1}\mathbf{1}]$ collide with the unstable roots $[\mathbf{a}\mathbf{1}\mathbf{1}]$, after which both branches disappear. This collision is visible in Figures 3.3, 3.4 and 3.5.

In order to find an expansion for this threshold near the corner $(a, d) = (0, 0)$, we exploit the observations above which allow us to consider equilibria close to $(0, 1, 1)$ for which the second and third components are equal. In particular, we construct solutions to the problem

$$G((x, 1 + y, 1 + y); a, d) = 0 \quad (3.3.3)$$

for which $|x| + |y| + |a| + |d|$ is small. The resulting system has a structure that is very similar to that encountered in Chapter 2, allowing us to follow the exact same procedure to unfold the saddle node bifurcations. In particular, viewing $\Gamma_{[\mathbf{o}\mathbf{1}\mathbf{1}]}$ locally as the graph of the function $d_{[\mathbf{o}\mathbf{1}\mathbf{1}]}$, we obtain the expansion

$$d_{[\mathbf{o}\mathbf{1}\mathbf{1}]}(a) = \frac{a^2}{8} + \frac{a^4}{64} + O(a^5), \quad (3.3.4)$$

together with

$$u_{011}(a; d_{[\mathbf{o}\mathbf{1}\mathbf{1}]}(a)) = \left(\frac{a}{2}, 1 - \frac{a^2}{8} - \frac{a^3}{16} - \frac{a^4}{64}, 1 - \frac{a^2}{8} - \frac{a^3}{16} - \frac{a^4}{64}\right) + O(a^5). \quad (3.3.5)$$

The Γ_u^2 threshold This curve features the triple collision of the unstable branches $[\mathbf{o}\mathbf{a}\mathbf{1}]$, $[\mathbf{a}\mathbf{o}\mathbf{1}]$ and $[\mathbf{a}\mathbf{a}\mathbf{1}]$ when $a \in (0, \frac{1}{2})$. One unstable branch of roots emerges from this collision. This collision is visible in Figures 3.3, 3.4 and 3.5.

⁴In the current trichromatic context, these roots are given by $(0, 0, 0)$, (a, a, a) and $(1, 1, 1)$.

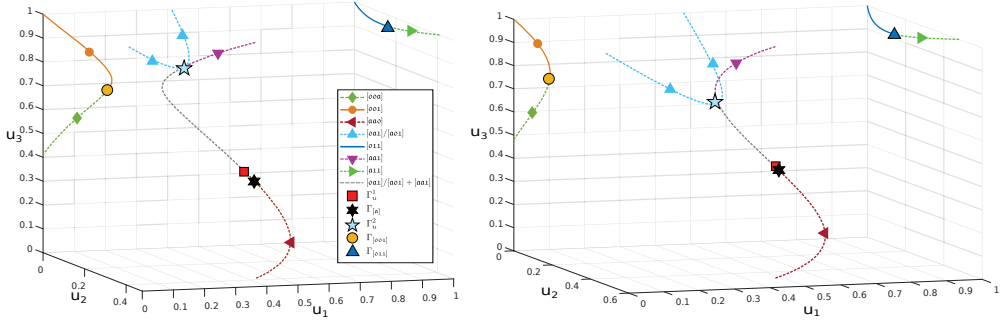


Figure 3.5: Root branches for $a = 0.41 > a_f$ (left) and $a = 0.48 > a_f$ (right). The branch $[001]$ now collides with $[00a]$, while the unstable branch spawned by the Γ_u^2 collision collides with $[00a]$ at Γ_u^1 . The collision points Γ_u^1 and Γ_u^2 slide through (a, a, a) as a is increased beyond $a = \frac{1}{2}$.

The $\Gamma_{[001]}$ threshold The important feature of this curve is that d cannot be expressed as a function of a . Indeed, the function

$$G_{**}(u; a, d) = \begin{pmatrix} G_*(u; a, d) \\ \det D_{1,3} G_*(u; a, d) \end{pmatrix} \quad (3.3.6)$$

has the (numerically computed) roots

$$(a_c, d_c) \approx (0.4013889, 0.05668), \quad (a_f, d_f) \approx (0.401476, 0.056275), \quad (3.3.7)$$

which correspond with a cusp respectively fold point for the curve $\Gamma_{[001]}$; see Figures 3.2 and 3.4.

When $a \in (0, a_c)$, the root $[001]$ hits the branch spawned by the Γ_u^2 collision and disappears; see Figure 3.3. On the other hand, for $a \in (a_f, 1)$ the root $[001]$ hits $[00a]$; see Figure 3.5. This corresponds with the scenario described above for $\Gamma_{[011]}$ after applying the $0 \leftrightarrow 1$ swap.

The intermediate case $a \in (a_c, a_f)$ is illustrated in Figure 3.4. In this case the root $[001]$ again hits the branch spawned by the Γ_u^2 collision and disappears, but this pair reappears and splits off from each other after d crosses the $\Gamma_{[001]}$ threshold a second time. The stable branch of this pair collides with $[00a]$ and disappears when d crosses $\Gamma_{[001]}$ for the third and final time, while the unstable branch emerges as the survivor of the full triple crossing process.

The black cross and green triangle in Figure 3.4 overlap when $a = a_c$, in which case the $[00a]$ branch has a triple root at the critical value $d = d_c$. On the other hand, the black cross and red circle in this figure overlap when $a = a_f$, in which case the branch $[001]$ and the branch spawned by the Γ_u^2 collision can be said to bounce off each other at $d = d_f$.

In order to find an expansion for this threshold near the corner $(a, d) = (0, 0)$, we now look for solutions to

$$G((x, x, 1 + y); a, d) = 0 \quad (3.3.8)$$

for which $|x| + |y| + |a| + |d|$ is small. Viewing $\Gamma_{[\circ\circ 1]}$ locally as the graph of the function $d_{[\circ\circ 1]}$, we may again use the same procedure as in Chapter 2 to obtain the expansion

$$d_{[\circ\circ 1]}(a) = \frac{a^2}{4} + \frac{a^4}{8} + O(a^5), \quad (3.3.9)$$

together with

$$u_{001}(a; d_{[\circ\circ 1]}(a)) = \left(\frac{a}{2}, \frac{a}{2}, 1 - \frac{a^2}{8} - \frac{a^3}{16} - \frac{a^4}{64}\right) + O(a^5). \quad (3.3.10)$$

The $\Gamma_{[a]}$ threshold This curve is characterized by the relation $\det D_1 G([a]; a, d) = 0$, which can be explicitly solved to yield $d_{[a]}(a) = a(1 - a)/3$. As shown in Figure 3.3, the three branches of roots contained in the equivalence class $[\circ aa]$ all pass through (a, a, a) at $d = d_{[a]}$ and survive the collision.

The Γ_u^1 threshold On this threshold the unstable branch that survived the Γ_u^2 and $\Gamma_{[\circ\circ 1]}$ collisions hits the branch $[\circ aa]$ that passed through (a, a, a) ; see Figures 3.3, 3.4, 3.5. Above this threshold the only remaining equilibria are the homogeneous states $[o]$ $[a]$ and $[1]$.

The critical case $a = \frac{1}{2}$ The right panel in Figure 3.5 describes the situation just before a reaches the critical value $a = \frac{1}{2}$. Upon increasing a through this value, the collisions at Γ_u^2 and Γ_u^1 both cross through the center $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ at $a = \frac{1}{2}$ and $d = d_{[a]}(\frac{1}{2}) = \frac{1}{12}$, transitioning to occur on the $[\circ aa]$ branch.

3.3.2 Wave connections

Our numerical results strongly suggest that $(H\Omega 1)$, $(H\Omega 2)$ and (HS) are satisfied, allowing us to apply the results in §3.2. Figure 3.7 represents the equivalence classes of wave connections between neighbouring words. We note that we are not drawing an edge from 101 because this is not a Lyndon word.

We recall that Theorem 3.2.2 does not provide any information about the speed c of the travelling wave. We therefore resorted to numerics to find parameter values where $c \neq 0$ for the waves discussed in the diagram above. This was done by connecting the two endstates with a tanh profile and letting this initial profile evolve under the flow of (3.2.1). Exploiting the stability of the moving waves, one can test whether $c = 0$ by determining whether movement ceases after an initial transient period.

For the $[\circ\circ 1 \rightarrow \circ 11]$ and $[\circ\circ 1 \rightarrow 101]$ connections we were not able to find any regions where $c \neq 0$. However, in Figure 3.6 we can observe the numerically computed minimum threshold for d where we in fact have $c \neq 0$ for the waves that connect to

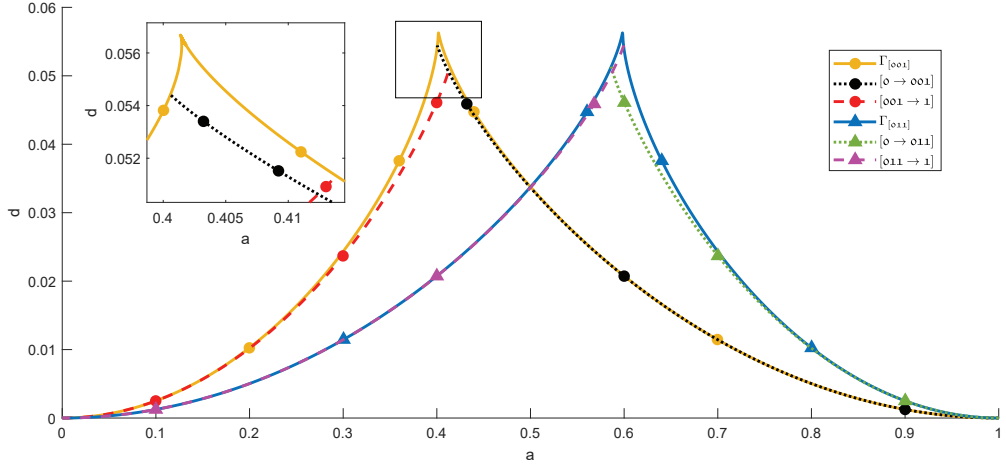


Figure 3.6: Thresholds where standing waves ($c = 0$) transition to travelling waves ($c \neq 0$) for the connections to and from the homogeneous equilibria $[0]$ and $[1]$.

and from the homogeneous states $[0]$ and $[1]$. Notice that both types of waves have non-zero speed in the region around the fold and cusp points of $\Gamma_{[001]}$. This indicates that travelling waves can appear and disappear twice as the diffusion coefficient is increased, which does not happen in the bichromatic case.

In Figure 3.8 we give snapshots of the collision process that occurs as a $[0 \rightarrow 001]$ wave collides with a $[001 \rightarrow 1]$ wave. In both cases an intermediate buffer-zone consisting of the trichromatic state $[001]$ is consumed by one (or both) of the incoming trichromatic waves, leading eventually to a pinned monochromatic wave. This type of collision can also be observed in the bichromatic setting.

3.4 Quadrichromatic waves

In this section we discuss the quadrichromatic case $n = 4$. As in the trichromatic case, stable 4-periodic equilibria can disappear and reappear as d is increased for a fixed a . The novel behaviour in this setting is that *travelling* quadrichromatic waves can co-exist with *travelling* monochromatic waves for an open set of parameters (a, d) . This allows several new types of collisions to occur. For example, two incoming connections with intermediate quadrichromatic states can collide to form a monochromatic travelling wave.

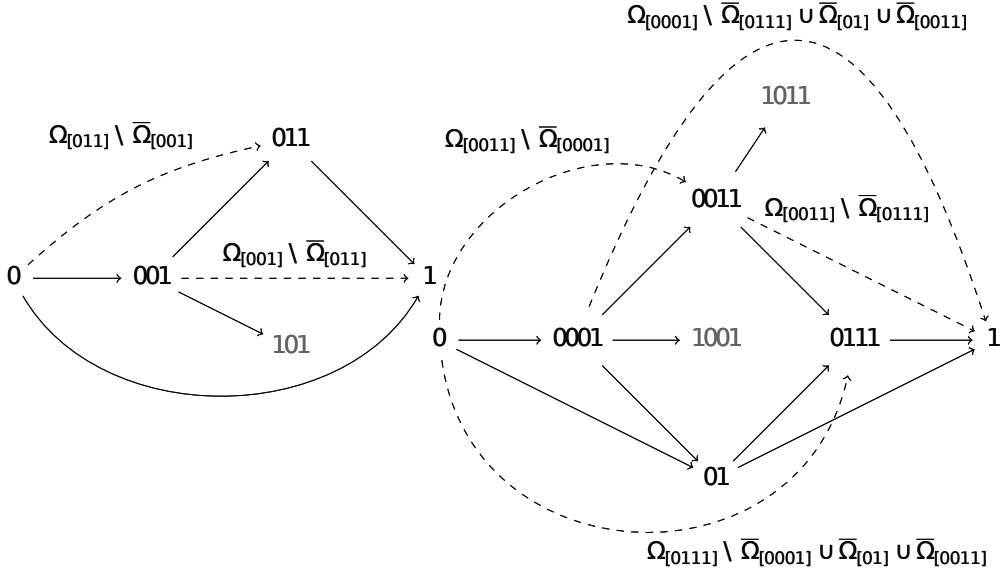


Figure 3.7: This diagram depicts the trichromatic (left) and quadrichromatic (right) wave connections predicted by the theory in §3.2. A solid edge from w_- to w_+ indicates that waves of type $[w_- \rightarrow w_+]$ exist for $(a, d) \in \Omega_{[w_+]} \cap \Omega_{[w_-]}$ with $d > 0$ by applying Theorem 3.2.2 with option (a). A dashed edge indicates that this connection is covered by option (b). In this case, the connection is only guaranteed to exist in the parameter region displayed next to the edge, because intermediate stable roots have to be ruled out. The asymptotics in §3.3.1 and §3.4.1 guarantee that all the relevant parameter regions are non-empty. In addition, the connections $[0001 \rightarrow 1101]$, $[0001 \rightarrow 0111]$ and $[0001 \rightarrow 1011]$ are not listed because $\Omega_{[0001]} \cap \Omega_{[0111]}$ is contained in $\Omega_{[0011]}$. To prevent clutter, we have also left out the monochromatic $[0 \rightarrow 1]$ connection in the diagram on the right. The words that are gray are *not* Lyndon words and as such do not have any outgoing arrows; see the discussion in §3.2.3.

3.4.1 Equilibria

The relevant nonlinearity that governs quadrichromatic equilibria to (3.2.1) is now given by

$$G(u; a, d) := \begin{pmatrix} d(u_4 - 2u_1 + u_2) + g(u_1; a) \\ d(u_1 - 2u_2 + u_3) + g(u_2; a) \\ d(u_2 - 2u_3 + u_4) + g(u_3; a) \\ d(u_3 - 2u_4 + u_1) + g(u_4; a) \end{pmatrix}. \quad (3.4.1)$$

Inspecting this system shows that one component can be removed by enforcing either $u_1 = u_3$ or $u_2 = u_4$. Of course, this problem reduces to the bichromatic case $n = 2$ if

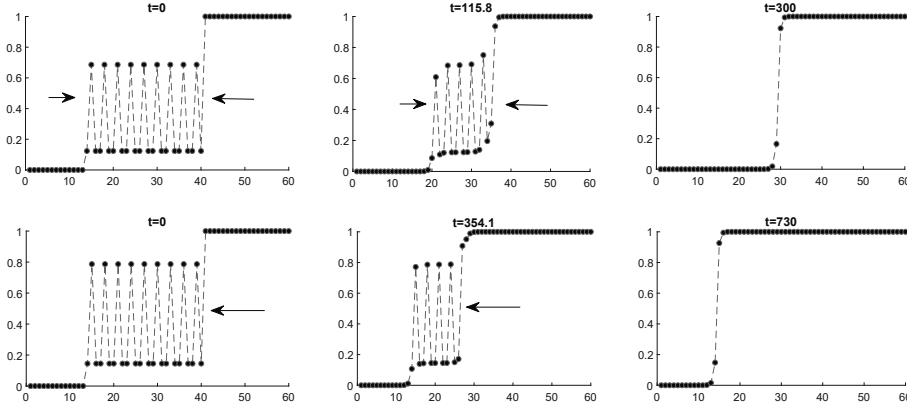


Figure 3.8: These panels describe two simulations of the LDE (3.2.1) that feature a collision between a $[0 \rightarrow 001]$ wave and a $[001 \rightarrow 1]$ wave. For the top three panels we have $a = 0.404$ and $d = 0.054$, which is above both trichromatic speed thresholds. However, it is still below the monochromatic threshold. In particular, the buffer zone is now consumed from both sides, but the end result is again a pinned monochromatic front. For the bottom three panels we have $a = 0.404$ and $d = 0.05$, which is now below the $[0 \rightarrow 001]$ but above the $[001 \rightarrow 1]$ speed thresholds. Here the $[001]$ buffer zone is consumed from the right, resulting in a pinned monochromatic front.

both these identities are enforced. On the other hand, if one takes

$$u_1 = u_2 = u_A, \quad u_3 = u_4 = u_B, \quad (3.4.2)$$

the system $G(u; a, d) = 0$ reduces to

$$d(u_B - u_A) + g(u_A; a) = d(u_A - u_B) + g(u_B; a) = 0. \quad (3.4.3)$$

This again corresponds to the bichromatic case $n = 2$ but now with the halved diffusion coefficient $d_{bc} = \frac{1}{2}d$.

In Figure 3.9 we display several numerically computed curves $\Gamma_{[w]}$ in the critical set Γ that correspond with the upper boundaries of the sets $\Omega_{[w]}$. For visual clarity, we only consider the words

$$w \in \{0001, 0011, 01, 0111\}, \quad (3.4.4)$$

which correspond with the stable non-homogeneous equilibria for $G(u; a, d) = 0$.

The curves $\Gamma_{[0001]}$ and $\Gamma_{[0111]}$ again contain cusp and fold points. However, all four curves can *locally* be described as a graph $d = d_{[w]}(a)$ near the corner $(a, d) = (0, 0)$. We now set out to compute the first two terms in the asymptotic expansion of each of these curves.

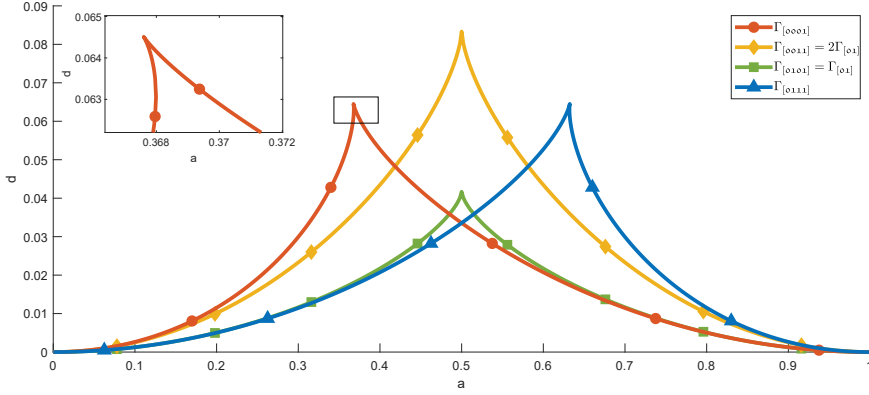


Figure 3.9: Bifurcation thresholds for the four stable quadrichromatic root classes. The $\Gamma_{[0001]}$ and $\Gamma_{[0111]}$ curves again feature slanted cusps; see the inset.

The $\Gamma_{[0111]}$ threshold In view of the symmetries discussed above we consider solutions where the second and fourth component are equal. In particular, we consider the problem

$$G((x, 1 + y, 1 + z, 1 + y); a, d) = 0, \quad (3.4.5)$$

which can be written as

$$(H_1, H_2, H_3)(x, y, z; a, d) = 0 \quad (3.4.6)$$

with

$$\begin{aligned} H_1(x, y, z; a, d) &= 2d(1 + y - x) + x(1 - x)(x - a), \\ H_2(x, y, z; a, d) &= d(x + z - 2y - 1) - y(1 + y)(y + 1 - a), \\ H_3(x, y, z; a, d) &= 2d(y - z) - z(1 + z)(z + 1 - a). \end{aligned} \quad (3.4.7)$$

Notice that H_2 and H_3 feature terms of order $O(y)$ respectively $O(z)$, which corresponds with the fact that the root $g(1; a) = 0$ is simple when $a = 0$. In addition, H_3 is independent of x . Setting $H_3 = 0$ hence allows us to write $z = z_*(y; a, d)$, which can be substituted into $H_2 = 0$ to yield $y = y_*(x; a, d)$. Plugging these expressions into H_1 by writing

$$\tilde{H}_1(x; a, d) = H_1(x, y_*(x; a, d), z_*(y; a, d); a, d), \quad (3.4.8)$$

we find

$$\tilde{H}_1(x; a, d) = x^2 + O(x^3 + ax + d). \quad (3.4.9)$$

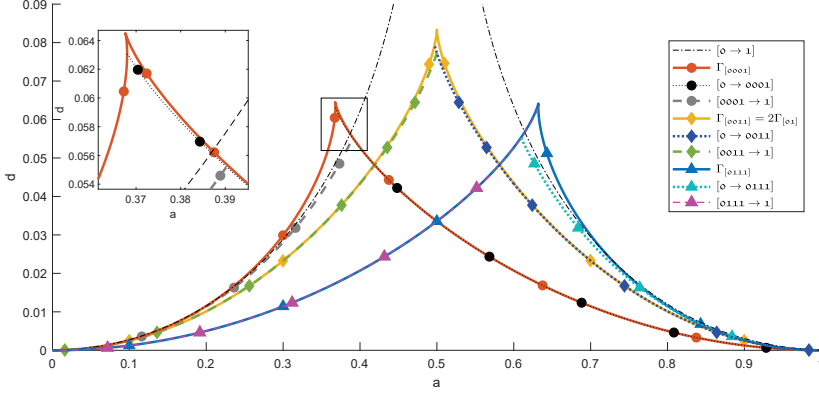


Figure 3.10: Speed thresholds for the wave connections to and from the stable roots $[0001]$, $[0011]$ and $[0111]$. The thresholds for $[0101]$ are not shown, as they coincide with the bichromatic results obtained in Chapter 2.

This allows us to uncover the saddle-node bifurcations in a fashion analogous to Chapter 2. In particular, we obtain the expansion

$$d_{[0111]}(a) = \frac{a^2}{8} + \frac{a^4}{64} + O(a^5), \quad (3.4.10)$$

together with

$$u_{0111}(a; d_{[0111]}(a)) = \left(\frac{a}{2}, 1 - \frac{a^2}{8} - \frac{a^3}{16}, 1, 1 - \frac{a^2}{8} - \frac{a^3}{16} \right) + O(a^4). \quad (3.4.11)$$

The $\Gamma_{[0101]} = \Gamma_{[01]}$ threshold The discussion above implies that this threshold is identical to the corresponding threshold for the bichromatic case $n = 2$. We can hence copy the results from Proposition 2.3.6 and write

$$d_{[01]}(a) = \frac{a^2}{8} + \frac{a^4}{32} + O(a^5), \quad (3.4.12)$$

together with

$$u_{01}(a; d_{[01]}(a)) = \left(\frac{a}{2}, 1 - \frac{a^2}{4} - \frac{a^3}{8} \right) + O(a^4). \quad (3.4.13)$$

The $\Gamma_{[0011]}$ threshold The identity (3.4.3) allows us to write

$$d_{[0011]}(a) = 2d_{[01]}(a) = \frac{a^2}{4} + \frac{a^4}{16} + O(a^5). \quad (3.4.14)$$

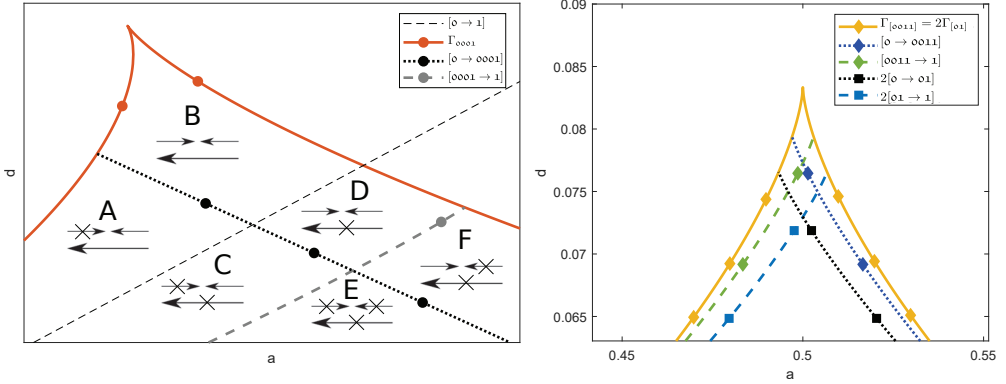


Figure 3.11: The left panel contains a schematic exaggeration of the area around the cusp of $\Gamma_{[0001]}$, together with a description of the collision type that can be expected in each region. The top two arrows indicate the nature of the $[0 \rightarrow 0001]$ and $[0001 \rightarrow 1]$ connections that move towards each other. The bottom arrow describes whether or not the resulting monochromatic wave is pinned. The right panel compares the speed threshold for the connections involving $[0011]$ with the doubled bichromatic thresholds. One can see that the reduction (3.4.2) respects the equilibrium structure but not the wave structure of the system.

In addition, we can reuse the expressions (3.4.13) to find

$$u_{0011}(a; d_{[0011]}(a)) = \left(\frac{a}{2}, \frac{a}{2}, 1 - \frac{a^2}{4} - \frac{a^3}{8}, 1 - \frac{a^2}{4} - \frac{a^3}{8} \right) + O(a^4). \quad (3.4.15)$$

The $\Gamma_{[0001]}$ threshold The symmetries discussed above allow us to consider solutions where the first and third component are equal. In particular, we consider the problem

$$G((x, y, x, 1 + z); a, d) = 0, \quad (3.4.16)$$

which can be written as

$$(H_1, H_2, H_3)(x, y, z; a, d) = 0 \quad (3.4.17)$$

with

$$\begin{aligned} H_1(x, y, z; a, d) &= d(y + z + 1 - 2x) + x(1 - x)(x - a), \\ H_2(x, y, z; a, d) &= 2d(x - y) + y(1 - y)(y - a), \\ H_3(x, y, z; a, d) &= 2d(x - z - 1) - z(1 + z)(z + 1 - a). \end{aligned} \quad (3.4.18)$$

Notice that H_3 features a term of order $O(z)$ and is independent of y . In fact, it is very similar to the equation of (2.3.30), which allows us to write

$$z = z_*(x; a, d) = -2d - 2ad + 2dx + O(d^2 + da^2). \quad (3.4.19)$$

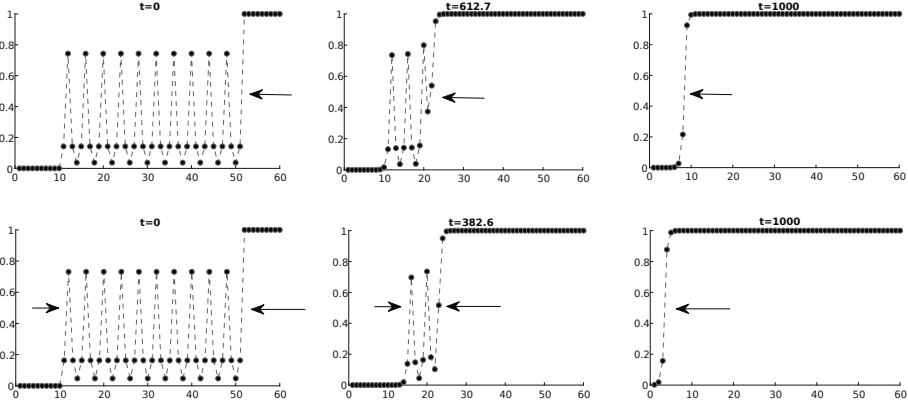


Figure 3.12: These panels describe two simulations of the LDE (3.2.1) that feature a collision between a $[0 \rightarrow 0001]$ wave and a $[0001 \rightarrow 1]$ wave. The top three panels feature a collision of type *A*, with $a = 0.378$ and $d = 0.058$. The quadrichromatic buffer zone is invaded from the right, forming a travelling monochromatic wave once it is extinguished. The bottom three panels feature a collision of type *B*, with $a = 0.37$ and $d = 0.0625$. Here the left part of the buffer zone is first pulled towards zero by the incoming wave on the left, but then gets pulled towards one by the final travelling monochromatic wave.

However H_2 features $O((a+d)y)$ terms, which prevents us from expressing y in terms of x as before. This corresponds with the fact that the root $g(0; a) = 0$ is double at $a = 0$. On the other hand, setting $H_2 = 0$ and introducing the scalings

$$y = a\tilde{y}, \quad d = d_*(a, \tilde{d}) = \frac{1}{4}a^2(1 + \tilde{d}) \quad (3.4.20)$$

does allow us to write

$$x = x_*(\tilde{y}; a, \tilde{d}) = a\tilde{y} - \frac{2\tilde{y}}{1 + 4\tilde{d}}(1 - a\tilde{y})(\tilde{y} - 1). \quad (3.4.21)$$

Substituting these expressions into H_1 , we write

$$\tilde{H}_1(\tilde{y}; a, \tilde{d}) = H_1\left(x_*(\tilde{y}; a, \tilde{d}), a\tilde{y}, z_*(x_*(\tilde{y}; a, \tilde{d}); a, d_*(a, \tilde{d})); a, d_*(a, \tilde{d})\right) \quad (3.4.22)$$

and find

$$\tilde{H}_1(\tilde{y}; a, \tilde{d}) = 4\tilde{y}^2 + O(\tilde{y}^3 + (a + \tilde{d})\tilde{y}^2 + a\tilde{y} + a^2). \quad (3.4.23)$$

The saddle-node bifurcations can now be unfolded by examining the terms in this equation using the procedure in Chapter 2. In particular, we find

$$d_{[0001]}(a) = \frac{a^2}{4} + \frac{a^3}{8} + O(a^4), \quad (3.4.24)$$

together with

$$u_{\text{oooo1}}(a; d_{[\text{oooo1}]}(a)) = \left(\frac{a}{2} + \frac{a^2}{8} + \frac{3a^3}{16}, \frac{a^2}{4} + \frac{a^3}{8}, \frac{a}{2} + \frac{a^2}{8} + \frac{3a^3}{16}, 1 - \frac{a^2}{2} - \frac{a^3}{2} \right) + O(a^4). \quad (3.4.25)$$

The main point of interest here is that $d_{[\text{oooo1}]}(a)$ contains a cubic term. In fact, our expansion here agrees with the expansion of the formula [83, Eq. (5.1)], which provides a (non-sharp) upper bound for the values of d where monochromatic waves are pinned. As indicated in Figure 3.10, our numerical results confirm that $\Omega_{[\text{oooo1}]}$ intersects the region in (a, d) space where monochromatic waves can travel.

3.4.2 Wave connections

As in the trichromatic case, a visual inspection confirms that assumptions (H Ω 1), (H Ω 2) and (HS) are satisfied. The connections predicted by Theorem 3.2.2 are depicted in Figure 3.7. In Figure 3.10 we provide the numerically computed minimal values for d for which the waves connecting to and from the spatially homogeneous equilibria [0] and [1] have a non-zero speed. The novel feature here is that there is overlap with the region where the monochromatic [0 $\rightarrow$ 1] wave has non-zero speed. This allows for situations where the end-product of a collision between two quadrichromatic wave is no longer a pinned monochromatic wave but in fact a travelling monochromatic wave.

The parameter regions where various types of collisions can occur are described in Figure 3.11. Types *C-F* closely resemble those encountered in the bichromatic and trichromatic cases. Types *A* and *B* are new and indeed feature travelling monochromatic end-products. Two examples with snapshots of such collisions are provided in Figure 3.12.

3.5 Proof of Theorem 3.2.2

Here we provide the proof of our main result, which allows us to establish the existence of wave connections between equilibria by simply comparing their types.

We first show that a pair of distinct ordered stationary solutions cannot have two equal components. Throughout this section we assume that expressions such as $i - 1$ or $i + 1$ should be evaluated within the modulo arithmetic on indices $\{1, 2, \dots, n\}$.

Lemma 3.5.1. *Assume that $u, v \in \mathbb{R}^n$ satisfy $G(u; a, d) = G(v; a, d) = 0$ for some pair $a \in (0, 1)$ and $d > 0$. Assume furthermore that $u \leq v$ and that $u_i = v_i$ for some $i \in \{1, \dots, n\}$. Then in fact $u = v$.*

Proof. Since $u_i = v_i$ we have

$$\begin{aligned} 0 &= d(u_{i-1} - 2u_i + u_{i+1}) + g(u_i; a) = d(u_{i-1} - 2v_i + u_{i+1}) + g(v_i; a) \\ &= d(v_{i-1} - 2v_i + v_{i+1}) + g(v_i; a), \end{aligned} \quad (3.5.1)$$

which implies that

$$u_{i-1} + u_{i+1} = v_{i-1} + v_{i+1}. \quad (3.5.2)$$

Since $u_{i-1} \leq v_{i-1}$ and $u_{i+1} \leq v_{i+1}$ we obtain $u_{i-1} = v_{i-1}$ and $u_{i+1} = v_{i+1}$. This argument can subsequently be repeated a number of times to yield $u = v$. $\square$

The main ingredient in our proof of Theorem 3.2.2 is that the ordering of any pair of words from the full set $\{\mathbf{o}, \mathbf{a}, \mathbf{1}\}^n$ and the stable subset $\{\mathbf{o}, \mathbf{1}\}^n$ is preserved for the equilibria that have the corresponding types. For example, for $n = 4$ we have the partial ordering

$$\begin{array}{ccccc}
 & & [\mathbf{o}\mathbf{o}\mathbf{a}\mathbf{1}] < [\mathbf{o}\mathbf{o}\mathbf{1}\mathbf{1}] < [\mathbf{o}\mathbf{a}\mathbf{1}\mathbf{1}] & & \\
 & \triangle & & \triangleright & \\
 [\mathbf{o}] < [\mathbf{o}\mathbf{o}\mathbf{o}\mathbf{a}] < [\mathbf{o}\mathbf{o}\mathbf{o}\mathbf{1}] & & & & [\mathbf{o}\mathbf{1}\mathbf{1}\mathbf{1}] < [\mathbf{a}\mathbf{1}\mathbf{1}\mathbf{1}] < [\mathbf{1}] \\
 & \triangleright & & \triangle & \\
 & & [\mathbf{o}\mathbf{a}\mathbf{o}\mathbf{1}] < [\mathbf{o}\mathbf{1}] < [\mathbf{o}\mathbf{1}\mathbf{a}\mathbf{1}] & &
 \end{array}$$

whereby each $[w_A] < [w_B]$ connection in this diagram indicates that $u_{w_A}(a, d) < u_{w_B}(a, d)$ whenever $(a, d) \in \Omega_{w_A} \cap \Omega_{w_B}$ with $d > 0$.

Lemma 3.5.2. *Assume that (H Ω 1) and (H Ω 2) are satisfied and consider a distinct pair $w_A, w_B \in \{\mathbf{o}, \mathbf{a}, \mathbf{1}\}^n$ that admits the ordering $w_A \leq w_B$. Suppose furthermore that at least one of these two words is contained in $\{\mathbf{o}, \mathbf{1}\}^n$. Then for any $(a, d) \in \Omega_{w_A} \cap \Omega_{w_B}$ we have the strict component-wise inequality*

$$u_{w_A}(a, d) < u_{w_B}(a, d). \quad (3.5.3)$$

Proof. Fixing $(a, d) \in \Omega_{w_A} \cap \Omega_{w_B}$, we note that (H Ω 1) and (H Ω 2) allow us to pick a curve

$$[0, 1] \ni t \mapsto (v_A(t), v_B(t), \alpha(t), \delta(t)) \in [0, 1]^n \times [0, 1]^n \times (0, 1) \times [0, \infty) \quad (3.5.4)$$

so that we have

$$\begin{aligned}
 (v_A, v_B, \alpha, \delta)(0) &= ([w_A]_{|a}, [w_B]_{|a}, a, 0), \\
 (v_A, v_B, \alpha, \delta)(1) &= (u_{w_A}(a, d), u_{w_B}(a, d), a, d),
 \end{aligned} \quad (3.5.5)$$

while the inclusion

$$(\alpha(t), \delta(t)) \in \Omega_{w_A} \cap \Omega_{w_B} \quad (3.5.6)$$

and the identities

$$G(v_A(t); \alpha(t), \delta(t)) = G(v_B(t); \alpha(t), \delta(t)) = 0 \quad (3.5.7)$$

all hold for $0 \leq t \leq 1$. By slightly modifying the path and picking a small $\epsilon > 0$, we can also ensure that $\alpha(t) = a$ and $\delta(t) = t$ for all $t \in [0, \epsilon]$.

Upon introducing the graph Laplacian $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and the nonlinearity $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that act as

$$(Bu)_i = u_{i-1} - 2u_i + u_{i+1}, \quad \Psi(u)_i = g(u_i; a), \quad (3.5.8)$$

we see that

$$0 = G(v_\#(t); a, t) = tBv_\#(t) + \Psi(v_\#(t)) \quad (3.5.9)$$

for t small and $\# \in \{A, B\}$. Taking implicit derivatives of (3.5.9), we find

$$tB \frac{d^k}{dt^k} v_\#(t) + kB \frac{d^{k-1}}{dt^{k-1}} v_\#(t) + \mathcal{R}_\#^{(k)}(t) = -D\Psi(v_\#(t)) \frac{d^k}{dt^k} v_\#(t), \quad (3.5.10)$$

in which we have defined

$$\mathcal{R}_\#^{(k)}(t) = \sum_{j=1}^{k-1} \binom{k-1}{j-1} \left[\frac{d^{k-j}}{dt^{k-j}} D\Psi(v_\#(t)) \right] \frac{d^j}{dt^j} v_\#(t) \quad (3.5.11)$$

for $k \geq 2$, setting this expression to zero for $k = 1$.

For any index i we define the quantity

$$\ell_i = \min\{j' \geq 0 : (w_A)_{i+j'} < (w_B)_{i+j'} \text{ or } (w_A)_{i-j'} < (w_B)_{i-j'}\}, \quad (3.5.12)$$

which measures the distance to the closest index where w_A and w_B are unequal. We now claim that for any $k \geq 0$ we have

$$\frac{d^k}{dt^k} (v_A)_i(0) \leq \frac{d^k}{dt^k} (v_B)_i(0) \quad (3.5.13)$$

if $\ell_i \geq k$, with the inequality being strict if and only if $\ell_i = k$. For $k = 0$ this is obvious. Assuming this holds for $k - 1$, consider any index with $\ell_i \geq k \geq 1$. Our alphabet assumption implies that

$$(w_A)_i = (w_B)_i \neq a, \quad (3.5.14)$$

which implies that the i -component of the two diagonal matrices $D\Psi(v_A(0))$ and $D\Psi(v_B(0))$ are strictly negative; see (3.2.7). In addition, our induction hypothesis implies that

$$\mathcal{R}_A^{(k)}(0) = \mathcal{R}_B^{(k)}(0). \quad (3.5.15)$$

By definition, we have

$$\ell_{i \pm 1} \geq \ell_i - 1 \geq k - 1. \quad (3.5.16)$$

In addition, we have $\ell_i = k$ if and only if $\ell_{i+1} = k - 1$ or $\ell_{i-1} = k - 1$ holds. Our induction hypothesis hence implies

$$\left(B \frac{d^{k-1}}{dt^{k-1}} v_A(0) \right)_i \leq \left(B \frac{d^{k-1}}{dt^{k-1}} v_B(0) \right)_i, \quad (3.5.17)$$

with strict inequality if and only if $\ell_i = k$. Our claim now follows immediately from (3.5.10).

The argument above shows that $v_A(t) < v_B(t)$ for all $t \in (0, \epsilon)$. If (3.5.3) fails to hold, this hence means that there exists $t_* \in [\epsilon, 1]$ for which $v_A(t_*) \leq v_B(t_*)$, with also $(v_A(t_*))_i = (v_B(t_*))_i$ for some $i \in \{1, \dots, n\}$. Lemma 3.5.1 now implies $v_A(t_*) = v_B(t_*)$ and hence

$$u_{w_A}(\alpha(t_*), \delta(t_*)) = u_{w_B}(\alpha(t_*), \delta(t_*)), \quad (3.5.18)$$

which violates Corollary 3.2.1. $\square$

By combining Lemma's 3.5.1 and 3.5.2 we can control all the (marginally) stable equilibria in the box $[u_{w_-}, u_{w_+}]$. This allows us to finally prove our main result.

Proof of Theorem 3.2.2. Pick any distinct pair $w_{\pm} \in \{\mathbf{o}, \mathbf{1}\}^n$ with $w_- \leq w_+$ and any $(a, d) \in \Omega_{w_-} \cap \Omega_{w_+}$ with $d > 0$. Lemma 3.5.2 implies that the cuboid

$$\mathcal{K} = \{u \in \mathbb{R}^n : u_{w_-}(a, d) \leq u \leq u_{w_+}(a, d)\} \subset [0, 1]^n \quad (3.5.19)$$

has non-empty volume. In addition, for any $w \in \{\mathbf{o}, \mathbf{1}\}^n$ that does not satisfy $w_- \leq w \leq w_+$ and for which $(a, d) \in \overline{\Omega}_w$, we have $u_w(a, d) \notin \mathcal{K}$ on account of Lemma 3.5.1, the connectedness of $\Omega_{w_{\pm}} \cap \Omega_w$ and continuity considerations.

In view of (HS), all equilibria inside the cube $\mathcal{K}$ besides the two corner points $u_{w_{\pm}}(a, d)$ hence have a strictly positive eigenvalue. The existence of the wave (c, Φ) now follows from [27, Thm. 6]. $\square$

Chapter 4

Travelling corner defects

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Abstract. We consider reaction-diffusion equations on the planar square lattice that admit spectrally stable planar travelling wave solutions. We show that these solutions can be continued into a branch of travelling corners. As an example, we consider the monochromatic and bichromatic Nagumo lattice differential equation and show that both systems exhibit interior and exterior corners.

Our result is valid in the setting where the group velocity is zero. In this case, the equations for the corner can be written as a difference equation posed on an appropriate Hilbert space. Using a non-standard global center manifold reduction, we recover a two-component difference equation that describes the behaviour of solutions that bifurcate off the planar travelling wave. The main technical complication is the lack of regularity caused by the spatial discreteness, which prevents the symmetry group from being factored out in a standard fashion.

4.1 Introduction

In this Chapter we construct travelling corner solutions to a class of planar lattice differential equations (LDEs) that includes the Nagumo LDE

$$\dot{u}_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} + g_{\text{cub}}(u; \rho) \quad (4.1.1)$$

posed on the two-dimensional square lattice $(i, j) \in \mathbb{Z}^2$, in which the nonlinearity is given by the bistable cubic

$$g_{\text{cub}}(u; \rho) = (u^2 - 1)(\rho - u), \quad -1 < \rho < 1. \quad (4.1.2)$$

Such corners can be seen as interfaces that connect planar waves travelling in slightly different directions. In particular, our analysis does not require the use of the comparison principle, but merely requires a number of spectral and geometric conditions to hold for the underlying planar travelling waves. This allows our results to be applied to a wide range of LDEs, highlighting the important role that anisotropy and topology play in spatially discrete settings.

Reaction-diffusion systems The LDE (4.1.1) can be seen as a nearest-neighbour spatial discretization of the Nagumo PDE

$$u_t = u_{xx} + u_{yy} + g_{\text{cub}}(u; \rho). \quad (4.1.3)$$

In modelling contexts one often uses the two stable equilibria of the nonlinearity g to represent material phases or biological species that compete for dominance in a spatial domain. Indeed, the diffusion term tends to attenuate high frequency oscillations, while the bistable nonlinearity promotes these. The balance between these two dynamical features leads to interesting pattern forming behaviour.

As a consequence, the PDE (4.1.3) has served as a prototype system for the understanding of many basic concepts at the heart of dynamical systems theory, including the existence and stability of planar travelling waves, the expansion of localized structures and the study of obstacles. Multi-component versions of (4.1.3) such as the Gray-Scott model [58] play an important role in the formation of patterns, generating spatially periodic structures from equilibria that destabilize through Turing bifurcations. Memory devices have been designed using FitzHugh-Nagumo-type systems with two components [81], which support stable stationary radially symmetric spot patterns. Similarly, one can find stable travelling spots [132] for three-component FitzHugh-Nagumo systems, which have been used to describe gas discharges [102, 112].

At present, a major effort is underway to understand the impact that non-local effects can have on reaction-diffusion systems. For example, many neural field models include infinite-range convolution terms to describe the dynamics of large networks of neurons [17, 18, 104, 120], which interact with each other over long distances. The description of phase transitions in Ising models [6, 7] features non-local interactions that can be both attractive and repulsive depending on the length scale involved.

It is well-known by now that the topology of the underlying spatial domain can have a major impact on the dynamical behaviour exhibited by such non-local systems. For example, nerve fibers have a myeline coating that admits gaps at regular intervals [106], which can block signals from propagating through the fiber [28, 77, 87]. In order to study the growth of plants, one must take into account that cells divide and grow in a fashion that is influenced heavily by the spatial configuration of their neighbours [59]. Finally, the periodic structure inherent in many meta-materials strongly influences the phase transitions that can occur [22, 23, 128] as a consequence of the visco-elastic interactions between their building blocks.

We view the planar LDE (4.1.1) as a prototype model that allows the impact of such non-local spatially-discrete effects to be explored. Indeed, the spatial $\mathbb{R}^2 \rightarrow \mathbb{Z}^2$

transition breaks the locality but also the translational and rotational symmetry of (4.1.3), leading to several interesting phenomena and mathematical challenges.

Existence of planar waves It is well-known that the balance between the diffusion and reaction terms in the PDE (4.1.3) is resolved through the formation of planar travelling wave solutions

$$u(x, y, t) = \Phi(x \cos \zeta + y \sin \zeta + ct); \quad \Phi(-\infty) = -1, \quad \Phi(\infty) = 1, \quad (4.1.4)$$

which connect the two stable equilibria $u = \pm 1$. When $c \neq 0$, these waves can be thought of as a mechanism by which the fitter species or more energetically favourable phase invades the spatial domain. The existence of these waves can be obtained by applying a phase-plane analysis [51] to the travelling wave ODE

$$c\Phi' = \Phi'' + g_{\text{cub}}(\Phi; \rho), \quad \Phi(\pm\infty) = \pm 1, \quad (4.1.5)$$

which results after substituting (4.1.4) into (4.1.3).

On the other hand, substitution of the analogous Ansatz

$$u_{ij}(t) = \Phi(i \cos \zeta + j \sin \zeta + ct); \quad \Phi(-\infty) = -1, \quad \Phi(\infty) = 1, \quad (4.1.6)$$

into the LDE (4.1.1) leads to the mixed-type functional differential equation (MFDE)

$$c\Phi'(\xi) = \Phi(\xi + \cos \zeta) + \Phi(\xi + \sin \zeta) + \Phi(\xi - \cos \zeta) + \Phi(\xi - \sin \zeta) - 4\Phi(\xi) + g_{\text{cub}}(\Phi(\xi); \rho). \quad (4.1.7)$$

The broken rotational invariance in the transition from (4.1.3) to (4.1.1) is manifested by the explicit presence of the propagation direction in (4.1.7). The broken translational invariance causes the wavespeed c to appear in (4.1.7) as a singular parameter.

A comprehensive existence theory for solutions to (4.1.7) was obtained in [91]. In particular, for every $\zeta \in [0, 2\pi]$ and $\rho \in [-1, 1]$ there exists a unique wavespeed $c = c_{\rho, \zeta}$ for which (4.1.7) admits a solution. However, it is a delicate question to decide whether $c \neq 0$ or $c = 0$. Indeed, a sufficient energy difference between the two stable equilibrium states is needed for the propagation of waves [7, 9, 43, 83], a phenomenon referred to as propagation failure. In fact, due to the angular dependence in (4.1.7), planar waves can fail to propagate in certain directions that resonate with the lattice, whilst travelling freely in others [21, 67, 92].

Linearization It is well-known that planar travelling waves can be used as a skeleton to describe the global dynamics of the PDE (4.1.4) [4]. In particular, they have been used as building blocks to construct other more complicated types of solutions. A key ingredient in such constructions is to understand the dynamics of the system that arises after linearizing (4.1.3) around the planar waves (4.1.4).

Performing this linearization for $\zeta = 0$, we obtain the system

$$\partial_t v(x, y, t) = \partial_{xx} v(x, y, t) + \partial_{yy} v(x, y, t) + g'_{\text{cub}}(\Phi(x + ct); \rho) v(x, y, t), \quad (4.1.8)$$

which can be transformed to the temporally autonomous system

$$\partial_t v(x, y, t) = \partial_{xx} v(x, y, t) + \partial_{yy} v(x, y, t) - c \partial_x v(x, y, t) + g'_{\text{cub}}(\Phi(x); \rho) v(x, y, t) \quad (4.1.9)$$

by the variable transformation $x' = x + ct$. Since this system is also autonomous with respect to the y -coordinate, which is transverse to the motion of the wave, it is convenient to apply a Fourier transform in this direction. Upon introducing the symbol

$$[\mathcal{L}_z p](x) = \partial_{xx} p(x) + z^2 p(x) - c \partial_x p(x) + g'_{\text{cub}}(\Phi(x); \rho) p(x), \quad (4.1.10)$$

we easily find

$$\partial_t \hat{v}_\omega(x, t) = [\mathcal{L}_{i\omega} \hat{v}_\omega(\cdot, t)](x). \quad (4.1.11)$$

Inspecting (4.1.10), we readily see that the spectrum of $\mathcal{L}_z$ can be obtained by rigidly shifting the spectrum of $\mathcal{L}_0$ by z^2 . In particular, writing $\lambda_z = z^2$, we find that

$$\mathcal{L}_z \Phi' = \lambda_z \Phi'. \quad (4.1.12)$$

Noting that $\lambda_{i\omega} = -\omega^2$, we hence see that perturbations of the form $v(x, y, 0) = \theta(y, 0) \Phi'(x)$ evolve under (4.1.9) according to the heat semiflow $\theta_t = \theta_{yy}$. These perturbations are important because they correspond at the linear level with transverse deformations of the planar wave interface.

On the other hand, linearizing the LDE (4.1.1) around the spatially-discrete wave (4.1.6) that travels in the horizontal direction $\zeta = 0$, we obtain the temporally non-autonomous system

$$\dot{v}_{ij}(t) = v_{i+1,j}(t) + v_{i,j+1}(t) + v_{i-1,j}(t) + v_{i,j-1}(t) - 4v_{ij}(t) + g'_{\text{cub}}(\Phi(i + ct); \rho) v_{ij}(t). \quad (4.1.13)$$

Although the time dependence cannot be readily transformed away, it is still possible to take a discrete Fourier transform in the j -direction. This leads to the system

$$\begin{aligned} \frac{d}{dt} [\hat{v}_\omega]_i(t) &= [\hat{v}_\omega]_{i+1}(t) + e^{i\omega} [\hat{v}_\omega]_i(t) + [\hat{v}_\omega]_{i-1}(t) + e^{-i\omega} [\hat{v}_\omega]_i(t) - 4[\hat{v}_\omega]_i(t) \\ &\quad + g'_{\text{cub}}(\Phi(i + ct); \rho) [\hat{v}_\omega]_i(t). \end{aligned} \quad (4.1.14)$$

Looking for solutions of the form

$$[\hat{v}_\omega]_i = e^{\lambda t} w_\omega(i + ct), \quad (4.1.15)$$

we arrive the eigenvalue problem

$$\mathcal{L}_{i\omega}^{\text{hor}} w_\omega = \lambda w_\omega \quad (4.1.16)$$

for the linear operator

$$[\mathcal{L}_z^{\text{hor}} p](\xi) = -c p'(\xi) + 2 \cosh(z) p(\xi) + p(\xi + 1) + p(\xi - 1) - 4p(\xi) + g'_{\text{cub}}(\Phi(\xi); \rho) p(\xi). \quad (4.1.17)$$

The theory developed in [8, 12] essentially justifies this formal calculation and confirms that the spectral properties of $\mathcal{L}_z$ can be used to understand the dynamics of the time-dependent problem (4.1.14). Upon writing $\lambda_z = 2(\cosh(z) - 1)$, we again have

$$\mathcal{L}_z^{\text{hor}} \Phi' = \lambda_z \Phi'. \quad (4.1.18)$$

To find the evolution of perturbations of the form $v_{ij}(0) = \theta_j \Phi'$ under (4.1.14), we must now solve the discrete heat equation

$$\dot{\theta}_j = \theta_{j+1} + \theta_{j-1} - 2\theta_j. \quad (4.1.19)$$

The situation is hence similar to that encountered for the PDE (4.1.3).

More material changes arise however when considering the diagonal direction $\zeta = \frac{\pi}{4}$. Following a similar procedure as above, one arrives at the linear operator

$$[\mathcal{L}_z^{\text{diag}} p](\xi) = -cp'(\xi) + 2 \cosh(z)[p(\xi + 1) + p(\xi - 1)] - 4p(\xi) + g'_{\text{cub}}(\Phi_*(\xi); \rho)p(\xi), \quad (4.1.20)$$

which has a spectrum that can no longer be directly related to that of $\mathcal{L}_0^{\text{diag}}$. It is hence no longer clear how to formulate an analogue for (4.1.19) to describe the linear evolution of interface deformations. However, it is still the case that $\lambda_z = O(z^2)$ as $z \rightarrow 0$ for the curve of eigenvalues that bifurcates from the zero eigenvalue $\mathcal{L}_0^{\text{diag}} \Phi' = 0$.

For general rational angles ζ this quadratic behaviour need no longer be true. In fact, we obtain the relation

$$[\partial_z \lambda_z]_{z=0} = \partial_\zeta c_{\rho, \zeta} \quad (4.1.21)$$

for the quantity that is often referred to as the group velocity. A similar relation was found in [60] for planar PDEs with direction-dependent diffusion coefficients. However, in this case it is always possible to change the coordinate system in such a way that $\lambda_z = O(z^2)$ holds again.

Such a transformation is not possible in the spatially discrete setting (4.1.1), since this would require the transverse spatial coordinate to become continuous. However, we do remark here that the function $\phi \mapsto c_{\rho, \zeta}$ can behave rather wildly in the critical regime where ρ is small, allowing the group velocity to vanish at specific values for ρ even if $\zeta \notin \frac{\pi}{4}\mathbb{Z}$.

Stability of planar waves The realization that transverse interfacial deformations are governed by a heat equation led to the development of two main approaches to establish the nonlinear stability of the planar waves (4.1.4). Both approaches exploit the coordinate system

$$u(x, y, t) = \Phi(x + ct + \theta(y, t)) + v(x, y, t) \quad (4.1.22)$$

in the neighbourhood of the planar travelling wave and require the initial perturbations $\theta(y, 0)$ and $v(x, y, 0)$ to be localized in a suitable sense.

The first approach was pioneered by Kapitula in [82], where he used semigroup methods and fixed-point arguments to show that θ tends algebraically to zero, while v

decays exponentially fast. The advantage of this approach is that only weak spectral assumptions need to be imposed on the underlying system. However, the crude estimates on the nonlinear terms lead to rather weak estimates for the basin of attraction.

The second approach leverages the comparison principle to obtain stability for a much larger class of initial perturbations. By slowing down the natural decay-rate of the fundamental solution of the heat equation, the authors of the landmark paper [13] were able to construct explicit super and sub-solutions to (4.1.3) that trap perturbations that can be arbitrarily large (but localized). In fact, the authors use their construction to show that these planar waves can pass around large compact obstacles and still eventually recover their shape.

In [65, 66] these approaches were generalized to the discrete setting of (4.1.1), thereby continuing the early work by Bates and Chen [5] featuring a related four-dimensional non-local problem. In both cases the key technical challenge was the analysis of troublesome non-selfadjoint terms spawned by the anisotropy of the lattice, especially in situation where the group velocity does not vanish. These terms have slower decay rates than their PDE counterparts and hence require special care to close the nonlinear bootstrapping procedure. For example, the sub-solutions in [13] consist of only two terms, while 33 terms were required in [65] to correct for the slower decay.

Spreading phenomena The classic result [4, Thm. 5.3] obtained by Weinberger for the PDE (4.1.3) states that large compact blobs with $u \approx 1$ inside and $u \approx -1$ outside can expand throughout the plane. The proof of this result relies on the construction of radially expanding sub- and super-solutions by glueing together planar travelling waves.

In [65] a weak version of this expansion result was established for the LDE (4.1.1) in the special case that no direction is pinned. However, the underlying sub- and super-solutions expand at the speeds $\min_{0 \leq \zeta \leq 2\pi} c_{\rho, \zeta}$ and $\max_{0 \leq \zeta \leq 2\pi} c_{\rho, \zeta}$ respectively, which still leaves a considerable hole in our knowledge of the expansion process. Indeed, the numerical results in [130] provide strong evidence that the limiting shape of the expanding blob can be found by applying the Wulff construction [108] to the polar plot of the $\zeta \mapsto c_{\rho, \zeta}$ relation. For a large subset of parameters ρ this limiting shape resembles a polygon.

The main motivation behind this chapter is to take a step towards understanding this expansion process by looking at the evolution of a single corner. Indeed, when the expanding blob is sufficiently large, it would seem to be very reasonable to assume that the corners of the polygon behave in an almost independent fashion.

Corners for PDEs Assuming for concreteness that $\rho < 0$, the horizontal planar wave (c, Φ) given by (4.1.4) with $\phi = 0$ satisfies $c > 0$, which means that it travels towards the left. In [61] Haragus and Scheel construct travelling corner solutions to (4.1.3) by ‘bending’ this planar wave to the left in the spatial limits $y \rightarrow \pm\infty$, so that the interface resembles a $>$ sign.

In particular, for any small opening angle $\varphi > 0$, the authors establish the existence

and stability of solutions of the form

$$u(x, y, t) = \Phi(x + \theta(y) + \frac{c}{\cos \varphi} t) + v(x + \frac{c}{\cos \varphi} t, y). \quad (4.1.23)$$

Here $\|v(\cdot, y)\|_{H^2} = O(\varphi^2)$ uniformly in y , while the phase θ satisfies the limits

$$\lim_{y \rightarrow \pm\infty} \theta'(y) = \pm \tan \varphi. \quad (4.1.24)$$

Notice that the horizontal speed $\frac{c}{\cos \varphi}$ of these corners is faster than the original speed of the planar wave.

The result is obtained by using the change of variable $x' = x + \tilde{c}t$ to recast (4.1.3) as

$$u_t = u_{xx} - \tilde{c}u_x + u_{yy} + g_{\text{cub}}(u; \rho) \quad (4.1.25)$$

and subsequently demanding $u_t = 0$. The resulting system can be written in the first-order form

$$\begin{aligned} u_y &= v, \\ v_y &= \tilde{c}u_x - u_{xx} - g_{\text{cub}}(u; \rho), \end{aligned} \quad (4.1.26)$$

which admits the family of y -independent equilibria

$$\tilde{c} = c, \quad (u, v)(x, y) = (\Phi(x + \vartheta), 0), \quad \vartheta \in \mathbb{R}. \quad (4.1.27)$$

The linearization of (4.1.26) around $(\tilde{c}, u) = (c, \Phi)$ can be written as

$$\begin{aligned} p_y &= q, \\ q_y &= \tilde{c}p_x - p_{xx} - g'_{\text{cub}}(\Phi(\cdot); \rho)p. \end{aligned} \quad (4.1.28)$$

This system admits the y -independent solutions $(\Phi', 0)$ caused by the translational invariance, together with the linearly growing solution $(y\Phi', \Phi')$. In particular, the desired corner (4.1.23) lives on the two-dimensional global center manifold associated to the family (4.1.27). The solutions on this manifold can be represented in the form

$$(u, v) = \left(\Phi(\theta(y) + x), 0 \right) + \kappa(y) \left(0, \Phi'(\theta(y) + x) \right) + h^*(\kappa(y), \tilde{c})(\theta(y) + x) \quad (4.1.29)$$

for some function

$$h^* : (-\epsilon, \epsilon) \times (c - \epsilon, c + \epsilon) \rightarrow H^2 \times H^1. \quad (4.1.30)$$

One can subsequently obtain two skew-coupled ODEs to describe the dynamics of the scalar functions κ and θ . A relatively straightforward analysis shows that these ODEs have solutions for which θ satisfies the limits (4.1.24), while κ remains small. This suffices to establish the existence of the corners (4.1.23).

In addition, in [60] anisotropic effects were introduced into the problem by allowing the nonlinearity g to depend on the gradient of u and considering non-diagonal diffusion coefficients. In such cases the group velocity c_g defined by the quantities

(4.1.21) need not vanish, but it can be removed by applying a coordinate transformation $y' = y - c_g t$ in the transverse direction.

By restricting their attention to small opening angles φ and using center manifold arguments, Scheel and Haragus were also able to apply their techniques to multi-component reaction-diffusion PDEs such as the FitzHugh-Nagumo and Gray-Scott equations [60, 61]. However, it is also possible to consider large opening angles when considering equations that admit a comparison principle. Indeed, in [16] explicit sub- and super-solutions are used to construct corners for the Nagumo PDE (4.1.3) that can be arbitrarily sharp.

Corners for LDEs The crucial point in the analysis outlined above for the corners (4.1.23) is that the phase shift $\theta(y)$ can be completely factored out from the system. This implies that the ODE for κ does not depend on θ . In addition, it allows the center manifold to be constructed by a standard fixed point argument analogous to the local case.

This is possible because the right-hand side of (4.1.28) maps $H^2 \times H^1$ into $H^1 \times L^2$, which roughly means that its inverse gains an order of regularity in both components. This precisely compensates for the loss of regularity that arises by factoring out the phase-shift.

However, when attempting to mimic this procedure for the LDE (4.1.1) one runs into a fundamental difficulty. Indeed, the analog of (4.1.28) has a right-hand side that now maps $H^1 \times H^1$ into $H^1 \times L^2$ due to the lack of second derivatives in the equation. This forces us to construct a full two-dimensional global center manifold that takes into the account the dynamics of θ and v simultaneously.

A similar situation was encountered by one of the authors in [71], where modulated travelling wave solutions were constructed to a class of non-local systems. However, the analog of (4.1.28) is a difference equation rather than a differential equation. The order of this difference equation can become arbitrarily large depending on the height of the fraction $\tan \zeta$, which we require to be rational. Nevertheless, the final step in our analysis requires us to uncover a first-order difference equation for the center variables.

The main technical contribution in this chapter is that we adapt the spirit of the approach in [71] to construct global center manifolds for the differential-difference systems that we encounter here. This approach uses two intertwined fixed point procedures to separate the flow problem for the two center variables from the task of capturing the shape of the remainder function h^* . The underlying linear problems have non-autonomous slowly-varying coefficients, for which we develop appropriate solution operators.

In this chapter we do require the group velocity (4.1.21) to vanish. Unlike in the spatially continuous setting, this cannot always be arranged by a simple variable transformation. Indeed, such a transformation would force the spatial variable transverse to the propagation direction to become continuous, destroying the difference structure of the system. This prevents us from exploiting the $\omega \mapsto \omega + 2\pi$ periodicity

in the Fourier variable. As a result, resonances start to appear in the spectrum that are very hard to control. A similar situation was encountered in [71], which forced the authors to add a smoothening term to the underlying system.

We emphasize that the group velocity for (4.1.1) vanishes automatically in the directions $\zeta = 0$ and $\zeta = \frac{\pi}{4}$. In addition, directions where the wavespeed is minimal (and hence the group velocity is zero) play an important role in the Wulff construction, which is the primary motivation for our analysis here. In any case, the delicate behaviour of the $c_{\rho,\zeta}$ map for the Nagumo LDE (4.1.1) leads to a much richer class of behaviour than that displayed by its continuous counterpart (4.1.3). For example, the latter only features interior corners, while the former can also admit exterior corners. The former also allows for so-called bichromatic corners, which connect spatially homogeneous equilibria to checkerboard patterns.

While we are confident that our center manifold construction will also allow us to establish the (linearized) stability of the corners constructed here along the lines of the approach in [60], we do not pursue this in the present chapter. The main reason is that there is no coordinate transformation that can freeze our corners and also leave the discrete structure of the equation intact. One would need to generalize the approach developed in [12, 8, 66] to accommodate solutions that vary in two directions instead of just one, which we expect to be a tedious task.

Organization Our main results are formulated in §4.2 and applied to the Nagumo LDE (4.1.1) in §4.2.1-4.2.2. In §4.3 we derive the differential-difference system that the pair (θ, v) must satisfy and formulate the global center manifold result. We proceed in §4.4 by deriving a representation formula for solutions to the linearized problem with constant phase. This requires us to compute a convoluted spectral projection operator that arises from the second order pole that the operator $\mathcal{L}_z^{-1}$ has in $z = 0$. In §4.5-§4.6 we combine this representation formula with Fourier analysis to construct a solution operator for the linearized problem where the phase is allowed to vary slowly. Finally, we setup the fixed point problems required to build the global center manifold in §4.7, appealing at times to the results in [71] for overlapping parts of the program.

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4.2 Main results

In this chapter we consider the nearest-neighbour lattice differential equation

$$\dot{u}_{ij}(t) = f(u_{i+1,j}(t), u_{i,j+1}(t), u_{i-1,j}(t), u_{i,j-1}(t), u_{ij}(t)), \quad (4.2.1)$$

posed on the planar lattice $(i, j) \in \mathbb{Z}^2$, in which u takes values in $\mathbb{R}^d$. For convenience, we introduce the operator $\pi_{ij}^+ : \ell^\infty(\mathbb{Z}^2; \mathbb{R}^d) \rightarrow (\mathbb{R}^d)^5$ that acts as

$$\pi_{ij}^+ u = (u_{i+1,j}, u_{i,j+1}, u_{i-1,j}, u_{i,j-1}, u_{ij}) \in (\mathbb{R}^d)^5 \quad (4.2.2)$$

for any $(i, j) \in \mathbb{Z}^2$, which allows us to rewrite (4.2.1) in the condensed form

$$\dot{u}_{ij}(t) = f(\pi_{ij}^+ u(t)). \quad (4.2.3)$$

The plus sign corresponds with the fact that a ”+”-shaped stencil is used to sample u .

The conditions we impose on the nonlinearity f are summarized in the following assumption.

(Hf) The nonlinearity $f : (\mathbb{R}^d)^5 \rightarrow \mathbb{R}^d$ is C^r -smooth for some $r \geq 2$ and there exist two points $u_\pm \in \mathbb{R}^d$ with

$$f(u_\pm, u_\pm, u_\pm, u_\pm, u_\pm) = 0. \quad (4.2.4)$$

We emphasize that the two points $u_\pm$ are allowed to be equal. These two equilibria are required to be connected by a planar travelling wave solution to (4.2.1). In particular, we pick an arbitrary rational direction $(\sigma_A, \sigma_B) \in \mathbb{Z}^2$ with $\gcd(\sigma_A, \sigma_B) = 1$ and impose the following condition.

(HΦ) There exists a wave speed $c_* \neq 0$ and a wave profile $\Phi_* \in C^{r+1}(\mathbb{R}, \mathbb{R}^d)$ so that the function

$$u_{ij}(t) = \Phi_*(i\sigma_A + j\sigma_B + c_*t) \quad (4.2.5)$$

satisfies (4.2.1) for all $t \in \mathbb{R}$. In addition, we have the limits

$$\lim_{\xi \rightarrow \pm\infty} \Phi_*(\xi) = u_\pm. \quad (4.2.6)$$

Upon introducing the operator $\tau : C(\mathbb{R}; \mathbb{R}^d) \rightarrow C(\mathbb{R}; (\mathbb{R}^d)^5)$ that acts as

$$[\tau p](\xi) = (p(\xi + \sigma_A), p(\xi + \sigma_B), p(\xi - \sigma_A), p(\xi - \sigma_B), p(\xi)) \in (\mathbb{R}^d)^5, \quad (4.2.7)$$

we note that the pair (c_*, Φ_*) must satisfy the functional differential equation of mixed type (MFDE)

$$\begin{aligned} c_* \Phi'_*(\xi) &= f(\Phi_*(\xi + \sigma_A), \Phi_*(\xi + \sigma_B), \Phi_*(\xi - \sigma_A), \Phi_*(\xi - \sigma_B), \Phi_*(\xi)) \\ &= f([\tau \Phi_*](\xi)). \end{aligned} \quad (4.2.8)$$

In particular, the C^{r+1} -continuity mentioned in (HΦ) is automatic upon assuming that Φ_* is merely continuous.

For convenience, we now introduce the new coordinates

$$\begin{aligned} n &= \sigma_A i + \sigma_B j, \\ l &= \sigma_A j - \sigma_B i, \end{aligned} \quad (4.2.9)$$

which are parallel respectively orthogonal to the direction of motion of the wave (4.2.5). Upon introducing the notation

$$\pi_{nl}^\times u = (u_{n+\sigma_A, l-\sigma_B}, u_{n+\sigma_B, l+\sigma_A}, u_{n-\sigma_A, l+\sigma_B}, u_{n-\sigma_B, l-\sigma_A}, u_{nl}), \quad (4.2.10)$$

the LDE (4.2.3) transforms into the equivalent problem

$$\dot{u}_{nl}(t) = f(\pi_{nl}^\times u(t)), \quad (4.2.11)$$

which admits the travelling wave solution

$$u_{nl}(t) = \Phi_*(n + c_* t). \quad (4.2.12)$$

A standard approach towards establishing the stability of the wave (4.2.12) under the nonlinear dynamics of the LDE (4.2.11) is to consider the linear variational problem

$$\dot{v}_{nl}(t) = Df([\tau\Phi_*](n + c_* t))\pi_{nl}^\times v(t). \quad (4.2.13)$$

Looking for a solution of the form

$$v_{nl}(t) = e^{\lambda t} e^{z l} p(n + c_* t), \quad (4.2.14)$$

we readily find that p must satisfy the eigenvalue problem

$$\mathcal{L}_z p = \lambda p. \quad (4.2.15)$$

Here the linear operator $\mathcal{L}_z : W^{1,\infty}(\mathbb{R}; \mathbb{C}^d) \rightarrow L^\infty(\mathbb{R}; \mathbb{C}^d)$ acts as

$$[\mathcal{L}_z p](\xi) = -c_* p'(\xi) + \sum_{j=1}^5 A_{z,j}(\xi) p(\xi + r_j), \quad (4.2.16)$$

with shifts r_j and functions $A_{z,j}$ that are given by

$$\begin{aligned} r_1 &= \sigma_A, & A_{z,1}(\xi) &= e^{-\sigma_B z} D_1 f([\tau\Phi_*](\xi)), \\ r_2 &= \sigma_B, & A_{z,2}(\xi) &= e^{+\sigma_A z} D_2 f([\tau\Phi_*](\xi)), \\ r_3 &= -\sigma_A, & A_{z,3}(\xi) &= e^{+\sigma_B z} D_3 f([\tau\Phi_*](\xi)), \\ r_4 &= -\sigma_B, & A_{z,4}(\xi) &= e^{-\sigma_A z} D_4 f([\tau\Phi_*](\xi)), \\ r_5 &= 0, & A_{z,5}(\xi) &= D_5 f([\tau\Phi_*](\xi)). \end{aligned} \quad (4.2.17)$$

Since $\Phi_*(\xi)$ approaches $u_\pm$ as $\xi \rightarrow \pm\infty$, it is possible to define the characteristic $\mathbb{C}^{d \times d}$ -valued functions

$$\Delta_z^\pm(s) = -c_* s I + \sum_{j=1}^5 \left[\lim_{\xi \rightarrow \pm\infty} A_{z,j}(\xi) \right] e^{sr_j}. \quad (4.2.18)$$

Our first spectral assumption states that these characteristic functions cannot have roots on the imaginary axis whenever z is purely imaginary.

(HS1) For all $\omega \in [-\pi, \pi]$ and $\nu \in \mathbb{R}$ we have

$$\det [\Delta_{i\omega}^\pm(i\nu)] \neq 0. \quad (4.2.19)$$

We note that (4.2.19) can be used to rule out kernel elements of $\mathcal{L}_{i\omega}$ that behave as $e^{i\nu\xi}$ as $\xi \rightarrow \pm\infty$. In fact, using [90, Thm. A] we see that (HS1) implies that $\mathcal{L}_{i\omega}$ is a Fredholm operator for all $\omega \in [-\pi, \pi]$. Our next condition demands that these operators are actually invertible for $\omega \neq 0$.

(HS2) For any $\omega \neq 0$ the operator $\mathcal{L}_{i\omega}$ is invertible as a map from $W^{1,\infty}(\mathbb{R}, \mathbb{C}^d)$ into $L^\infty(\mathbb{R}, \mathbb{C}^d)$.

Since the Fredholm index varies continuously, (HS1) and (HS2) together imply that the Fredholm index of $\mathcal{L}_0$ is zero. The translational invariance of the problem implies that $\mathcal{L}_0 \Phi_*' = 0$, which means that zero is an eigenvalue for $\mathcal{L}_0$. Our next assumption states that this eigenvalue is in fact algebraically simple.

(HS3) We have the characterization

$$\text{Ker}(\mathcal{L}_0) = \text{span}\{\Phi_*'\} \quad (4.2.20)$$

and the algebraic simplicity condition

$$\Phi_*' \notin \text{Range}(\mathcal{L}_0). \quad (4.2.21)$$

For any $z \in \mathbb{C}$ we now introduce the linear operator

$$\mathcal{L}_z^{\text{adj}} : W^{1,\infty}(\mathbb{R}, \mathbb{C}^d) \rightarrow L^\infty(\mathbb{R}, \mathbb{C}^d) \quad (4.2.22)$$

that acts as

$$[\mathcal{L}_z^{\text{adj}} q](\xi) = c_* q'(\xi) + \sum_{j=1}^5 A_{z,j}^*(\xi - r_j) q(\xi - r_j). \quad (4.2.23)$$

An easy computation shows that

$$\int_{-\infty}^{\infty} \langle q(\xi), [\mathcal{L}_z p](\xi) \rangle_{\mathbb{C}^d} d\xi = \int_{-\infty}^{\infty} \langle [\mathcal{L}_z^{\text{adj}} q](\xi), p(\xi) \rangle_{\mathbb{C}^d} d\xi \quad (4.2.24)$$

holds for all pairs $p, q \in W^{1,\infty}(\mathbb{R}, \mathbb{C}^d)$. For this reason, we refer to this operator as the formal adjoint of $\mathcal{L}_z$.

Using [90, Thm. A] together with (HS3), one sees that the kernel of $\mathcal{L}_0^{\text{adj}}$ must also be one-dimensional. In particular, it is spanned by a function $\psi_* \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^d)$ that can be uniquely fixed by the identity

$$\int_{-\infty}^{\infty} \langle \psi_*(\xi), \Phi'_*(\xi) \rangle d\xi = 1 \quad (4.2.25)$$

on account of (4.2.21). We note that (HS1) implies that both $\Phi'_*(\xi)$ and $\psi_*(\xi)$ decay exponentially as $\xi \rightarrow \pm\infty$.

We now explore two important consequences of the algebraic simplicity condition (HS3). The first of these states that the zero eigenvalue can be extended to a branch of eigenvalues λ_z for $\mathcal{L}_z$ when $|z|$ is small.

Lemma 4.2.1 (see [66, Prop. 2.2]). Assume that (Hf), $(H\Phi)$ and (HS1)-(HS3) are all satisfied. Then there exists a constant $\delta_z > 0$ together with pairs

$$(\lambda_z, \phi_z) \in \mathbb{C} \times W^{1,\infty}(\mathbb{R}, \mathbb{C}^d), \quad (4.2.26)$$

defined for each $z \in \mathbb{C}$ with $|z| < \delta_z$, such that the following hold true.

(i) The characterization

$$\text{Ker}(\mathcal{L}_z - \lambda_z) = \text{span}\{\phi_z\} \quad (4.2.27)$$

together with the algebraic simplicity condition

$$\phi_z \notin \text{Range}(\mathcal{L}_z - \lambda_z) \quad (4.2.28)$$

hold for each $z \in \mathbb{C}$ with $|z| < \delta_z$.

(ii) We have $\lambda_0 = 0$, $\phi_0 = \Phi'_*$ and the maps $z \mapsto \lambda_z$ and $z \mapsto \phi_z$ are analytic.

(iii) The normalization condition

$$\langle \psi_*, \phi_z \rangle_{L^2} = 1 \quad (4.2.29)$$

holds for every $z \in \mathbb{C}$ with $|z| < \delta_z$.

The second consequence is that wave (c_*, Φ_*) travelling in the rational direction (σ_A, σ_B) can be perturbed to yield waves travelling in nearby directions. In particular, we introduce the constants (ζ_*, σ_*) by writing

$$\sigma_* = \sqrt{\sigma_A^2 + \sigma_B^2}, \quad (\sigma_A, \sigma_B) = \sigma_*(\cos \zeta_*, \sin \zeta_*). \quad (4.2.30)$$

Looking for solutions to the LDE (4.2.11) of the form

$$u_{nl}(t) = \Phi_\varphi(n \cos \varphi + l \sin \varphi + c_\varphi t), \quad (4.2.31)$$

a short computation shows that the pair $(c_\varphi, \Phi_\varphi)$ must satisfy the MFDE

$$c_\varphi \Phi'_\varphi(\xi) = f(\tau_\varphi \Phi_\varphi)(\xi), \quad (4.2.32)$$

in which we have introduced the notation

$$\begin{aligned} [\tau_\varphi p](\xi) &= \left(p(\xi + \sigma_* \cos(\varphi + \zeta_*)), p(\xi + \sigma_* \sin(\varphi + \zeta_*)), \right. \\ &\quad \left. p(\xi - \sigma_* \cos(\varphi + \zeta_*)), p(\xi - \sigma_* \sin(\varphi + \zeta_*)), p(\xi) \right). \end{aligned} \quad (4.2.33)$$

In order to translate these waves back to the original coordinates, we remark that any solution to (4.2.32) yields a solution to the original LDE (4.2.1) by writing

$$u_{ij}(t) = \tilde{\Phi}_\zeta(i \cos \tilde{\zeta} + j \sin \tilde{\zeta} + \tilde{c}_\zeta t) \quad (4.2.34)$$

with the rescaled quantities

$$\tilde{\zeta} = \zeta_* + \varphi, \quad \tilde{c}_\zeta = \sigma_*^{-1} c_\varphi, \quad \tilde{\Phi}_\zeta(\xi) = \Phi_\varphi(\sigma_* \xi). \quad (4.2.35)$$

Lemma 4.2.2 (see §4.4). Assume that (Hf), $(H\Phi)$ and (HS1)-(HS3) are all satisfied. Then there exists a constant $\delta_\varphi > 0$ together with pairs

$$(c_\varphi, \Phi_\varphi) \in \mathbb{R} \times W^{1,\infty}(\mathbb{R}, \mathbb{C}^d), \quad (4.2.36)$$

defined for each $\varphi \in (-\delta_\varphi, \delta_\varphi)$, such that the following hold true.

- (i) For every $\varphi \in (-\delta_\varphi, \delta_\varphi)$, the pair $(c_\varphi, \Phi_\varphi)$ satisfies the MFDE (4.2.32), while the function

$$u_{nl}(t) = \Phi_\varphi(n \cos \varphi + l \sin \varphi + c_\varphi t) \quad (4.2.37)$$

satisfies the LDE (4.2.11) for all $t \in \mathbb{R}$.

- (ii) The maps $\varphi \mapsto c_\varphi$ and $\varphi \mapsto \Phi_\varphi$ are C^{r-1} -smooth. In addition, we have $c_0 = c_*$, together with $\Phi_0 = \Phi_*$.
- (iii) The normalization condition

$$\langle \psi_*, \Phi_\varphi \rangle = \langle \psi_*, \Phi_* \rangle \quad (4.2.38)$$

holds for every $\varphi \in (-\delta_\varphi, \delta_\varphi)$.

- (iv) We have the identities

$$[\partial_\varphi c_\varphi]_{\varphi=0} = [\partial_z \lambda_z]_{z=0}, \quad [\partial_\varphi \Phi_\varphi]_{\varphi=0} = [\partial_z \phi_z]_{z=0}. \quad (4.2.39)$$

We remark that the first quantities in (4.2.39) can be interpreted as a so-called *group velocity*, which represents the speed at which long-amplitude perturbations travel in the transverse direction. Indeed, expanding (4.2.14) with $p = \phi_z$ and $\lambda = \lambda_z$ we find

$$v_{nl}(t) = \exp[z(l + [\partial_z \lambda_z]_{z=0}t) + O(z^2t)] \phi_z(n + c_*t). \quad (4.2.40)$$

Our final condition requires λ_z to depend quadratically on z , which means that this group velocity has to vanish. We emphasize that the inequality $[\partial_z^2 \lambda_z]_{z=0} > 0$ was required in [66] to obtain the nonlinear stability of the planar wave (c_*, Φ_*) .

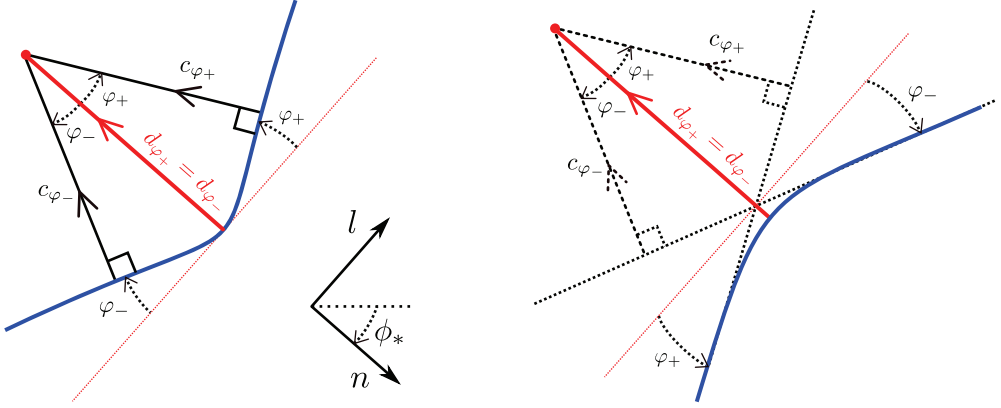


Figure 4.1: The blue curves in the left and right panels depict the interface of an interior respectively exterior corner. Both corners travel at the speed $d_{\varphi-} = d_{\varphi+}$ and share the coordinate system (n, l) depicted in the center. Angles are positive when oriented counter-clockwise and negative otherwise. All speeds are positive.

(HM) We have the identities

$$[\partial_{\varphi} c_{\varphi}]_{\varphi=0} = [\partial_z \lambda_z]_{z=0} = 0, \quad (4.2.41)$$

together with the inequality

$$[\partial_z^2 \lambda_z]_{z=0} \neq 0. \quad (4.2.42)$$

As a final preparation, we introduce the directional dispersion

$$d_{\varphi} = \frac{c_{\varphi}}{\cos \varphi}. \quad (4.2.43)$$

Assuming that the original wave travels in the horizontal direction $\zeta_* = 0$, the quantity d_{φ} represents the speed at which level-sets of the wave $(c_{\varphi}, \Phi_{\varphi})$ travel along the horizontal axis; see Figure 4.1. An easy calculation using (4.2.41) shows that

$$d_0 = c_*, \quad [\partial_{\varphi} d_{\varphi}]_{\varphi=0} = 0, \quad [\partial_{\varphi}^2 d_{\varphi}]_{\varphi=0} = [\partial_{\varphi}^2 c_{\varphi}]_{\varphi=0} + c_*. \quad (4.2.44)$$

Our main result establishes the existence of travelling corners in the setting where $[\partial_{\varphi}^2 d_{\varphi}]_{\varphi=0} \neq 0$. Assuming again that $\phi_* = 0$ and that $[\partial_z^2 \lambda_z]_{z=0}$ and $[\partial_{\varphi}^2 d_{\varphi}]_{\varphi=0}$ are both strictly positive, the level-sets resemble a $>$ sign. In particular, when $c_* > 0$ this resembles an interior corner travelling to the left.

Theorem 4.2.3 (see §4.3). Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are all satisfied. Assume furthermore that $[\partial_{\varphi}^2 d_{\varphi}]_{\varphi=0} \neq 0$ and pick a sufficiently large $K > 0$. Then for any $c \neq c_*$ sufficiently close to c_* with

$$\text{sign}(c - c_*) = \text{sign}\left([\partial_{\varphi}^2 d_{\varphi}]_{\varphi=0}\right), \quad (4.2.45)$$

there exist two sequences

$$(\theta, v) : \mathbb{Z} \rightarrow \mathbb{R} \times H^1(\mathbb{R}; \mathbb{R}^d) \quad (4.2.46)$$

together with two angles $\varphi_- < 0 < \varphi_+$ that satisfy the following properties.

(i) The function

$$u_{nl}(t) = \Phi_*(n + ct + \theta_l) + v_l(n + ct) \quad (4.2.47)$$

satisfies the LDE (4.2.11) for all $t \in \mathbb{R}$.

(ii) We have $\langle \psi_*(\cdot + \theta_l), v_l \rangle_{L^2} = 0$ for all $l \in \mathbb{Z}$.

(iii) We have the identities

$$d(\varphi_{\pm}) = c. \quad (4.2.48)$$

(iv) If $[\partial_{\varphi}^2 d_{\varphi}]_{\varphi=0}$ and $[\partial_z^2 \lambda_z]_{z=0}$ have the same sign, then we have the limits

$$\theta_{l+1} - \theta_l \rightarrow \tan(\varphi_{\pm}), \quad l \rightarrow \pm\infty. \quad (4.2.49)$$

On the other hand, if these quantities have opposing signs, then we have the limits

$$\theta_{l+1} - \theta_l \rightarrow \tan(\varphi_{\mp}), \quad l \rightarrow \pm\infty. \quad (4.2.50)$$

(v) For every $l \in \mathbb{Z}$ we have the bound

$$|\varphi_{\pm}|^2 + \|v_l\|_{H^1} \leq K |c - c_*|. \quad (4.2.51)$$

4.2.1 The Nagumo LDE

As an example, we return to the Nagumo LDE

$$\dot{u}_{ij} = u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} + g(u_{ij}; \rho) \quad (4.2.52)$$

in which the nonlinearity is given by the scaled cubic

$$g(u; \rho) = \frac{5}{2} g_{\text{cub}}(u) = \frac{5}{2} (u^2 - 1)(\rho - u) \quad (4.2.53)$$

for some detuning parameter $\rho \in (-1, 1)$. In the terminology of (4.2.1), we hence have

$$f(v_1, \dots, v_5) = v_1 + v_2 + v_3 + v_4 - 4v_5 + g(v_5; \rho), \quad (4.2.54)$$

which shows that (Hf) is satisfied upon picking $u_{\pm} = \pm 1$.

Turning to $(H\Phi)$, we note that the results in [91] show that for each $\zeta \in [0, 2\pi]$ and $\rho \in (-1, 1)$ there is a unique wavespeed $c = c_{\rho, \zeta}$ for which the system

$$c\Phi'(\xi) = \Phi(\xi + \cos \zeta) + \Phi(\xi - \cos \zeta) + \Phi(\xi + \sin \zeta) + \Phi(\xi - \sin \zeta) - 4\Phi(\xi) + g(\Phi(\xi); \rho) \quad (4.2.55)$$

admits a monotonic solution $\Phi = \Phi_{\rho, \zeta}$ that also satisfies the limits (4.2.6). Figure 4.2 contains polar plots of the $\zeta \mapsto c_{\rho, \zeta}$ relation, which can be very delicate whenever $|\rho|$ is small.

By symmetry, we have $c_{\rho, \zeta} = -c_{-\rho, \zeta}$ and hence $c_{0, \zeta} = 0$ for all angles $\zeta \in [0, 2\pi]$. Upon writing

$$\rho_*(\zeta) = \sup\{\rho : c_{\rho, \zeta} = 0\}, \quad (4.2.56)$$

the results in [91] show that $0 \leq \rho_*(\zeta) < 1$ for all $\zeta \in [0, 2\pi]$. In particular, this means that $(H\Phi)$ is satisfied whenever $\tan \zeta$ is rational (or infinite) and $\rho_*(\zeta) < |\rho| < 1$. Under the same conditions, the discussion in [66, §6] uses arguments based on the comparison principle to show that also (HS1)-(HS3) are valid.

The verification of the conditions in (HM) is much more subtle. In order to make the angular dependence fully explicit, we first pick

$$(\tilde{\sigma}_A, \tilde{\sigma}_B) = (\cos \zeta, \sin \zeta) \quad (4.2.57)$$

and consider the operators

$$\tilde{\mathcal{L}}_z : H^1(\mathbb{R}; \mathbb{R}) \rightarrow L^2(\mathbb{R}; \mathbb{R}) \quad (4.2.58)$$

that act as

$$\begin{aligned} [\tilde{\mathcal{L}}_z p](\xi) &= -c_* p'(\xi) + e^{-\tilde{\sigma}_B z} p(\xi + \tilde{\sigma}_A) + e^{\tilde{\sigma}_A z} p(\xi + \tilde{\sigma}_B) \\ &\quad + e^{\tilde{\sigma}_B z} p(\xi - \tilde{\sigma}_A) + e^{-\tilde{\sigma}_A z} p(\xi - \tilde{\sigma}_B) - 4p(\xi) + g'(\Phi_*(\xi); \rho) p(\xi). \end{aligned} \quad (4.2.59)$$

Writing $\tilde{\lambda}_z$ for the branch of eigenvalues for $\tilde{\mathcal{L}}_z$ bifurcating from $\tilde{\lambda}_0 = 0$ and comparing this to the branch λ_z defined in Lemma 4.2.1, we have

$$\tilde{\lambda}_z = \lambda_{\sigma_*^{-1} z}, \quad (4.2.60)$$

with $\sigma_* > 0$ the smallest number so that

$$\sigma_*(\tilde{\sigma}_A, \tilde{\sigma}_B) = (\sigma_A, \sigma_B) \in \mathbb{Z}^2. \quad (4.2.61)$$

In particular, we have

$$[\partial_z \tilde{\lambda}_z]_{z=0} = \sigma_*^{-1} [\partial_z \lambda_z]_{z=0}. \quad (4.2.62)$$

In view of the similar rescalings (4.2.35) and the fact that the statements in Theorem 4.2.3 merely depend on the signs of the quantities $[\partial_\varphi^2 d_\varphi]_{\varphi=0}$ and $[\partial_z^2 \lambda_z]_{z=0}$, we simply write¹

$$\begin{aligned} [\mathcal{L}_{z; \rho, \zeta} p](\xi) &= -c_{\rho, \zeta} p'(\xi) + e^{\mp z \sin \zeta} p(\xi \pm \cos \zeta) + e^{\pm z \cos \zeta} p(\xi \pm \sin \zeta) - 4p(\xi) \\ &\quad + g'(\Phi_{\rho, \zeta}(\xi); \rho) p(\xi) \end{aligned} \quad (4.2.63)$$

¹The terms involving $\pm$ and $\mp$ should be evaluated twice, once with the top signs and once with the bottom signs. For example, $\pm p(\xi \mp 1) = p(\xi - 1) - p(\xi + 1)$.

and focus on the eigenvalues $\lambda_{z;\rho,\zeta}$ and eigenfunctions $\phi_{z;\rho,\zeta}$ bifurcating from $(0, \Phi'_{\rho,\zeta})$ for $\rho_*(\zeta) < |\rho| < 1$. We write $\psi_{\rho,\zeta}$ for the solution to the adjoint equation

$$-c_{\rho,\zeta} \psi'(\xi) = \psi(\xi \pm \cos \zeta) + \psi(\xi \pm \sin \zeta) - 4\psi(\xi) + g'(\Phi_{\rho,\zeta}(\xi); \rho) \psi(\xi) \quad (4.2.64)$$

that is normalized to have $\langle \psi_{\rho,\zeta}, \Phi'_{\rho,\zeta} \rangle = 1$.

In our context, the operators defined in (4.4.13) and (4.4.40) act as

$$\begin{aligned} [A_1 p](\xi) &= \mp \sin(\zeta) p(\xi \pm \cos \zeta) \pm \cos(\zeta) p(\xi \pm \sin \zeta), \\ [A_2 p](\xi) &= \sin(\zeta)^2 p(\xi \pm \cos \zeta) + \cos(\zeta)^2 p(\xi \pm \sin \zeta), \\ [B_1 p](\xi) &= \mp \cos(\zeta) p(\xi \pm \cos \zeta) \mp \sin(\zeta) p(\xi \pm \sin \zeta). \end{aligned} \quad (4.2.65)$$

In particular, Lemma 4.4.2 allows us to compute

$$[\partial_z \lambda_{z;\rho,\zeta}]_{z=0} = \mp \sin \zeta \langle \psi_{\rho,\zeta}, \Phi'_{\rho,\zeta}(\cdot \pm \cos \zeta) \rangle \pm \cos \zeta \langle \psi_{\rho,\zeta}, \Phi'_{\rho,\zeta}(\cdot \pm \sin \zeta) \rangle, \quad (4.2.66)$$

which in turn allows us to find $[\partial_z \phi_{z;\rho,\zeta}]_{z=0}$ by solving the MFDE

$$\mathcal{L}_{0;\rho,\zeta} [\partial_z \phi_{z;\rho,\zeta}]_{z=0} = \pm \sin(\zeta) \Phi'_{\rho,\zeta}(\cdot \pm \cos \zeta) \mp \cos(\zeta) \Phi'_{\rho,\zeta}(\cdot \pm \sin \zeta) + [\partial_z \lambda_{z;\rho,\zeta}]_{z=0} \Phi'_{\rho,\zeta}. \quad (4.2.67)$$

In addition, item (iv) of Lemma 4.2.2 shows that

$$[\partial_\zeta c_{\rho,\zeta}] = [\partial_z \lambda_{z;\rho,\zeta}]_{z=0}, \quad [\partial_\zeta \Phi_{\rho,\zeta}] = [\partial_z \phi_{z;\rho,\zeta}]_{z=0}. \quad (4.2.68)$$

Turning to the second derivatives, we again use Lemma 4.4.2 to compute

$$\begin{aligned} [\partial_z^2 \lambda_{z;\rho,\zeta}]_{z=0} &= \sin(\zeta)^2 \langle \psi_{\rho,\zeta}, \Phi'_{\rho,\zeta}(\cdot \pm \cos \zeta) \rangle + \cos(\zeta)^2 \langle \psi_{\rho,\zeta}, \Phi'_{\rho,\zeta}(\cdot \pm \sin \zeta) \rangle \\ &\quad \mp 2 \sin \zeta \langle \psi_{\rho,\zeta}, [\partial_z \phi_{z;\rho,\zeta}]_{z=0}(\cdot \pm \cos \zeta) \rangle \\ &\quad \pm 2 \cos \zeta \langle \psi_{\rho,\zeta}, [\partial_z \phi_{z;\rho,\zeta}]_{z=0}(\cdot \pm \sin \zeta) \rangle \\ &\quad - 2 [\partial_z \lambda_{z;\rho,\zeta}]_{z=0} \langle \psi_{\rho,\zeta}, [\partial_z \phi_{z;\rho,\zeta}]_{z=0} \rangle. \end{aligned} \quad (4.2.69)$$

We remark that the last line vanishes in principle if the normalization (4.2.29) is imposed. However, numerically it is convenient to be free to utilize a different normalization, in which case this term should be included.

Finally, in view of the fact that $D^2 f$ acts only on its fifth argument, we can use Lemma 4.4.5 to obtain

$$\begin{aligned} [\partial_\zeta^2 c_{\rho,\zeta}] &= \langle \psi_{\rho,\zeta}, g''(\Phi_{\rho,\zeta}; \rho) [\partial_\zeta \Phi_{\rho,\zeta}]^2 \rangle \\ &\quad + \sin(\zeta)^2 \langle \psi_{\rho,\zeta}, \Phi''_{\rho,\zeta}(\cdot \pm \cos \zeta) \rangle + \cos(\zeta)^2 \langle \psi_{\rho,\zeta}, \Phi''_{\rho,\zeta}(\cdot \pm \sin \zeta) \rangle \\ &\quad \mp \cos \zeta \langle \psi_{\rho,\zeta}, \Phi'_{\rho,\zeta}(\cdot \pm \cos \zeta) \rangle \mp \sin \zeta \langle \psi_{\rho,\zeta}, \Phi'_{\rho,\zeta}(\cdot \pm \sin \zeta) \rangle \\ &\quad \mp 2 \sin \zeta \langle \psi_{\rho,\zeta}, [\partial_\zeta \Phi'_{\rho,\zeta}](\cdot \pm \cos \zeta) \rangle \pm 2 \cos \zeta \langle \psi_{\rho,\zeta}, [\partial_\zeta \Phi'_{\rho,\zeta}](\cdot \pm \sin \zeta) \rangle \\ &\quad - 2 [\partial_\zeta c_{\rho,\zeta}] \langle \psi_{\rho,\zeta}, [\partial_\zeta \Phi'_{\rho,\zeta}] \rangle. \end{aligned} \quad (4.2.70)$$

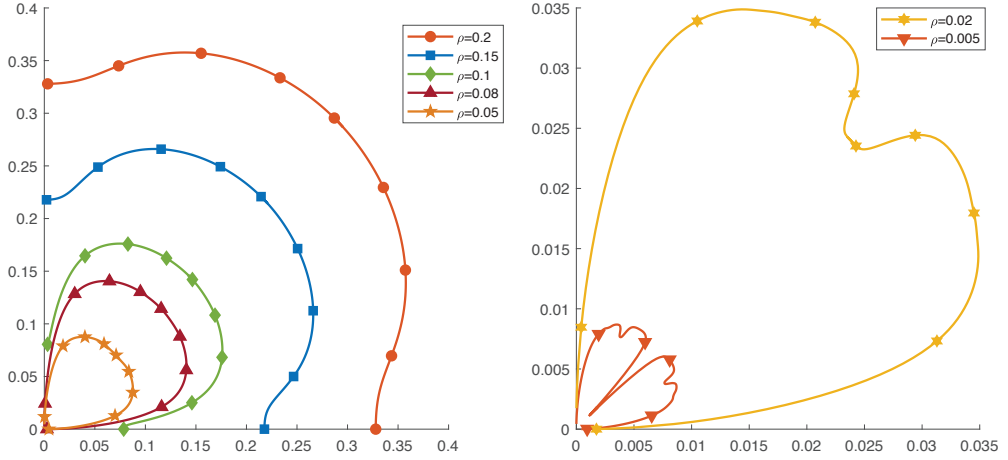


Figure 4.2: Both panels contain polar plots of the $\zeta \mapsto c_{\rho, \zeta}$ relation, for various values of $\rho > 0$. Since $c \leq 0$ in this setting, we have actually plotted the points $-c_{\rho, \zeta}(\cos \zeta, \sin \zeta)$ for $0 \leq \zeta \leq \frac{\pi}{2}$. Notice the extra minima that start to form in the directions $\tan \zeta = 1$ and subsequently $\tan \zeta = \frac{2}{3}$ as ρ is decreased.

The last line can be ignored if indeed $[\partial_\zeta c_{\rho, \zeta}] = 0$.

In the special cases where $\zeta = k\frac{\pi}{4}$ for some $k \in \mathbb{Z}$, we have $A_1 = 0$ and hence

$$[\partial_z \lambda_{z; \rho, k\frac{\pi}{4}}]_{z=0} = [\partial_\zeta c_{\rho, \zeta}]_{\zeta=k\frac{\pi}{4}} = 0, \quad [\partial_z \phi_{z; \rho, k\frac{\pi}{4}}]_{z=0} = [\partial_\zeta \Phi_{\rho, \zeta}]_{\zeta=k\frac{\pi}{4}} = 0. \quad (4.2.71)$$

For $\zeta = 0$ this allows us to write

$$\begin{aligned} [\partial_z^2 \lambda_{z; \rho, 0}]_{z=0} &= 2\langle \psi_{\rho, \zeta}, \Phi'_{\rho, \zeta} \rangle = 2, \\ [\partial_\zeta^2 c_{\rho, \zeta}]_{\zeta=0} &= \langle \psi_{\rho, \zeta}, 2\Phi''_{\rho, \zeta} + \Phi'_{\rho, \zeta}(\cdot - 1) - \Phi'_{\rho, \zeta}(\cdot + 1) \rangle. \end{aligned} \quad (4.2.72)$$

On the other hand, for $\zeta = \frac{\pi}{4}$ we have

$$\begin{aligned} [\partial_z^2 \lambda_{z; \rho, \frac{\pi}{4}}]_{z=0} &= \langle \psi_{\rho, \zeta}, \Phi'_{\rho, \zeta}(\cdot + \tfrac{1}{2}\sqrt{2}) + \Phi'_{\rho, \zeta}(\cdot - \tfrac{1}{2}\sqrt{2}) \rangle, \\ [\partial_\zeta^2 c_{\rho, \zeta}]_{\zeta=\frac{\pi}{4}} &= \langle \psi_{\rho, \zeta}, \Phi''_{\rho, \zeta}(\cdot + \tfrac{1}{2}\sqrt{2}) + \Phi''_{\rho, \zeta}(\cdot - \tfrac{1}{2}\sqrt{2}) \\ &\quad + \sqrt{2}\Phi'_{\rho, \zeta}(\cdot - \tfrac{1}{2}\sqrt{2}) - \sqrt{2}\Phi'_{\rho, \zeta}(\cdot + \tfrac{1}{2}\sqrt{2}) \rangle. \end{aligned} \quad (4.2.73)$$

Since $\psi_{\rho, \zeta}$ and $\Phi'_{\rho, \zeta}$ are strictly positive we hence see that

$$[\partial_z^2 \lambda_{z; \rho, k\frac{\pi}{4}}]_{z=0} > 0 \quad (4.2.74)$$

for all $k \in \mathbb{Z}$. The numerical results in [66, §6] suggest that this inequality extends to a wide range of (ρ, ζ) and we take this for granted for the remainder of our discussion. However, even for the straightforward expressions (4.2.72)-(4.2.73), it is not clear whether the quantity $c_{\rho, \zeta} + \partial_\zeta^2 c_{\rho, \zeta}$ has a sign.

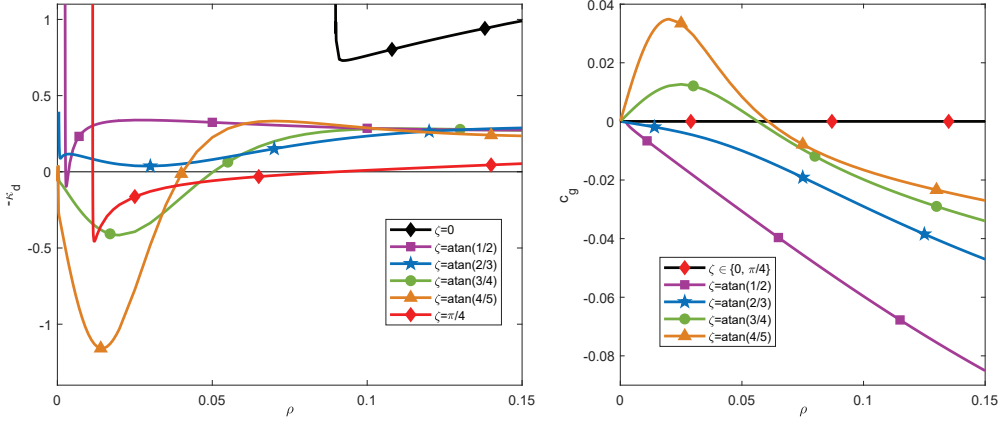


Figure 4.3: The left panel contains numerically computed values for $-\kappa_d(\rho)$. The sharp spikes occur at the critical value $\rho_*(\zeta)$ where pinning sets in. We note that sign changes appear for $\zeta = \frac{\pi}{2}$ but not for $\zeta = 0$. In particular, the identity $c_g \equiv 0$ for these directions implies that interior and exterior corners can both occur for $\zeta = \frac{\pi}{2}$, while the horizontal direction $\zeta = 0$ features interior corners only. The right panel contains numerically computed values for $c_g(\rho)$. Notice the zero-crossings for $\tan \zeta = \frac{3}{4}$ and $\tan \zeta = \frac{4}{5}$, which indicates the presence of interior corners at these two critical values for ρ .

For any fixed ζ_* , we now introduce the notation

$$c_g(\rho) = [\partial_z \lambda_{z;\rho,\zeta}]_{z=0}, \quad \kappa_d(\rho) = c_{\rho,\zeta_*} + [\partial_\zeta^2 c_{\rho,\zeta}]_{\zeta=\zeta_*} \quad (4.2.75)$$

for the group velocity and second derivative of the directional dispersion that play a role in Theorem 4.2.3. In particular, to apply this result we need $c_g(\rho) = 0$ and $\kappa_d(\rho) \neq 0$. Since $c_{\rho,\zeta} \leq 0$ whenever $\rho \geq 0$, we have an interior corner for $\kappa_d(\rho) < 0$ and an exterior corner for $\kappa_d(\rho) > 0$. In both cases the corner travels in the rightward direction (provided $|\zeta| < \frac{\pi}{2}$).

In Figure 4.3 we have numerically computed the quantities (4.2.75) for a range of rational directions. In all cases, we also confirmed numerically that $[\partial_z^2 \lambda_{z;\rho,\zeta}]_{z=0} > 0$. In particular, the results predict interior corners travelling in the horizontal direction $\zeta_* = 0$, while both types of corners can travel in the diagonal direction $\zeta_* = \frac{\pi}{4}$. In addition, for two critical values of $\rho > 0$ there are interior corners that travel in the direction $\zeta_* = \arctan(3/4)$ respectively $\zeta_* = \arctan(4/5)$. To obtain these results, we simultaneously solved the systems (4.2.55), (4.2.64) and (4.2.67). For well-posedness reasons, we added the extra terms $\gamma \Phi''$, $\gamma \psi''$ respectively $\gamma [\partial_z \phi_{z;\rho,\zeta}]''$ to the right-hand side of each equation, taking $\gamma = 10^{-6}$. For the precise procedure, we refer to [66, §6].

4.2.2 Bichromatic Nagumo LDE

We here reconsider the Nagumo LDE

$$\dot{u}_{ij} = \alpha [u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij}] + g_{\text{cub}}(u_{ij}; \rho), \quad (4.2.76)$$

but are now interested in so-called bichromatic planar travelling wave solutions. Such solutions can be written in the form

$$u_{ij}(t) = \begin{cases} \Phi^{(u)}(i \cos \phi + j \sin \phi + ct) & \text{if } i + j \text{ is even,} \\ \Phi^{(v)}(i \cos \phi + j \sin \phi + ct) & \text{if } i + j \text{ is odd,} \end{cases} \quad (4.2.77)$$

for some wavespeed $c \in \mathbb{R}$ and $\mathbb{R}^2$ -valued waveprofile

$$\Phi = (\Phi^{(u)}, \Phi^{(v)}) : \mathbb{R} \rightarrow \mathbb{R}^2. \quad (4.2.78)$$

These waves fit into the framework of this chapter, since they can be seen as travelling wave solutions for the ‘doubled’ LDE

$$\begin{aligned} \dot{u}_{ij} &= \alpha [v_{i+1,j} + v_{i,j+1} + v_{i-1,j} + v_{i,j-1} - 4u_{ij}] + g_{\text{cub}}(u_{ij}; \rho), \\ \dot{v}_{ij} &= \alpha [u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4v_{ij}] + g_{\text{cub}}(v_{ij}; \rho). \end{aligned} \quad (4.2.79)$$

We now introduce the notation

$$G(u, v; \rho, \alpha) = \begin{pmatrix} 4\alpha(v - u) + g_{\text{cub}}(u; \rho) \\ 4\alpha(u - v) + g_{\text{cub}}(v; \rho) \end{pmatrix}, \quad (4.2.80)$$

together with the matrix

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (4.2.81)$$

and the operator

$$\Delta_{z,\zeta} p = e^{\mp z \sin \zeta} p(\xi \pm \cos \zeta) + e^{\pm z \cos \zeta} p(\xi \pm \sin \zeta) - 4p(\xi). \quad (4.2.82)$$

Substitution of the bichromatic wave Ansatz (4.2.77) into the LDE (4.2.76) leads to the travelling wave MFDE

$$c\Phi' = \alpha \mathcal{J} \Delta_{0,\zeta} \Phi + G(\Phi; \rho, \alpha), \quad (4.2.83)$$

while the spatially homogeneous equilibria $(u_{ij}, v_{ij}) = (u, v)$ to the doubled LDE (4.2.79) satisfy the system

$$G(u, v; \rho, \alpha) = 0. \quad (4.2.84)$$

The results in Chapter 2 show that there exists an open set of values (ρ, α) with $\rho \in (-1, 1)$ and $\alpha > 0$ for which (4.2.84) admits solutions $(u_{\text{bc}}, v_{\text{bc}}) \in (0, 1)^2$ that are

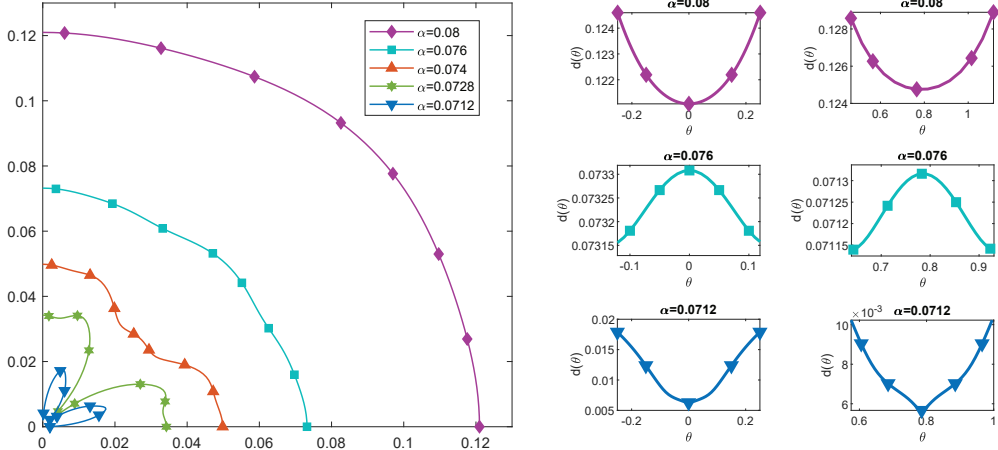


Figure 4.4: The left panel contains polar plots of the $\zeta \mapsto c_{\rho, \alpha, \zeta}$ relation, with fixed $\rho = 0$. In particular, the curves consist of the points $c_{\rho, \alpha, \zeta}(\cos \zeta, \sin \zeta)$. The right panel depicts the directional dispersion $d(\zeta) = \frac{c_{\rho, \alpha, \zeta}}{\cos(\zeta - \zeta_*)}$, with $\zeta_* = 0$ for the left column and $\zeta_* = \frac{\pi}{4}$ for the right column, again with $\rho = 0$. These results strongly suggest that $[\partial_\zeta^2 d(\zeta)]_{\zeta=\zeta_*}$ can take both signs as the diffusion coefficient α is varied. In particular, both the horizontal and diagonal directions can have interior and exterior corners.

stable spatially homogeneous equilibria for (4.2.79). By applying the theory in [74], we hence obtain the existence of solutions to (4.2.83) that satisfy the limits

$$\lim_{\xi \rightarrow -\infty} (\Phi^{(u)}, \Phi^{(v)}) = (0, 0), \quad \lim_{\xi \rightarrow +\infty} (\Phi^{(u)}, \Phi^{(v)}) = (u_{bc}, v_{bc}), \quad (4.2.85)$$

together with similar solutions that connect (u_{bc}, v_{bc}) with $(1, 1)$.

We remark that the existence theory in [74] does not prescribe whether $c \neq 0$ or $c = 0$. However, in the special case $\zeta = \frac{\pi}{4}$ the travelling wave system can be written as

$$\begin{aligned} c[\Phi^{(u)}]'(\xi) &= 2\alpha[\Phi^{(v)}(\xi + \tfrac{1}{2}\sqrt{2}) + \Phi^{(v)}(\xi - \tfrac{1}{2}\sqrt{2}) - 2\Phi^{(u)}(\xi)] + g_{\text{cub}}(\Phi^{(u)}(\xi); \rho), \\ c[\Phi^{(v)}]'(\xi) &= 2\alpha[\Phi^{(u)}(\xi + \tfrac{1}{2}\sqrt{2}) + \Phi^{(u)}(\xi - \tfrac{1}{2}\sqrt{2}) - 2\Phi^{(v)}(\xi)] + g_{\text{cub}}(\Phi^{(v)}(\xi); \rho). \end{aligned} \quad (4.2.86)$$

After a spatial rescaling, this corresponds precisely with the bichromatic travelling wave MFDE equation (2.2.5) encountered in the one-dimensional spatial setting. We remark here that this does **not** hold for the horizontal direction $\zeta = 0$.

In any case, this observation allows us to apply Theorem 2.2.3. As a consequence, for $\zeta = \frac{\pi}{4}$ there exists an open set of values (ρ, α) for which the wavespeed does not vanish, allowing us to verify $(H\Phi)$. By continuity in ζ , this hence also holds for nearby angles. In addition, our numerical results suggest that the diagonal direction

is the first to become pinned as α is decreased; see Figure 4.4. This contrasts the situation encountered in the monochromatic case, where the diagonal waves satisfy the same travelling wave MFDE as the horizontal waves, but with a doubled diffusion coefficient. As a result, the monochromatic horizontal waves pin earlier than their diagonal counterparts.

Since the spectral conditions (HS1)-(HS3) can be verified with techniques similar to those used for the monochromatic case, we now turn our attention to (HM). In particular, writing $(c_{\rho,\alpha,\zeta}, \Phi_{\rho,\alpha,\zeta})$ for the solution to (4.2.83) that satisfies the limits (4.2.85), we introduce the operator

$$\mathcal{L}_{z;\rho,\alpha,\zeta}p = -c_{\rho,\alpha,\zeta}p' + \alpha\mathcal{J}\Delta_{z,\zeta}p + DG(\Phi_{\rho,\alpha,\zeta}; \rho, \alpha)p \quad (4.2.87)$$

for any $p = (p^{(u)}, p^{(v)}) \in H^1(\mathbb{R}; \mathbb{R}) \times H^1(\mathbb{R}; \mathbb{R})$. We now introduce the notation A_1^{mc} , A_2^{mc} and B_1^{mc} for the operators (4.2.65) defined for the monochromatic equation. In addition, we write A_1^{bc} , A_2^{bc} and B_1^{bc} for the operators defined in (4.4.13) and (4.4.40) associated to the bichromatic problem (4.2.83). It is not hard to verify the relations

$$A_1^{\text{bc}} = \alpha\mathcal{J}A_1^{\text{mc}}, \quad A_2^{\text{bc}} = \alpha\mathcal{J}A_2^{\text{mc}}, \quad B_1^{\text{bc}} = \alpha\mathcal{J}B_1^{\text{mc}}. \quad (4.2.88)$$

We write $(\lambda_{z;\rho,\alpha,\zeta}, \phi_{z;\rho,\alpha,\zeta})$ and $\psi_{\rho,\alpha,\zeta}$ for the analogs of the similar named expressions defined in §4.2.1. Since $A_1^{\text{bc}} = 0$ for $\zeta = 0$ and $\zeta = \frac{\pi}{4}$, we again have

$$[\partial_z \lambda_{z;\rho,\alpha,k\frac{\pi}{4}}]_{z=0} = [\partial_\zeta c_{\rho,\alpha,\zeta}]_{\zeta=k\frac{\pi}{4}} = 0, \quad [\partial_z \phi_{z;\rho,\alpha,k\frac{\pi}{4}}]_{z=0} = [\partial_\zeta \Phi_{\rho,\alpha,\zeta}]_{\zeta=k\frac{\pi}{4}} = 0 \quad (4.2.89)$$

for all $k \in \mathbb{Z}$.

For $\zeta = 0$ this allows us to write

$$\begin{aligned} [\partial_z^2 \lambda_{z;\rho,\alpha,0}]_{z=0} &= 2\alpha \langle \psi_{\rho,\alpha,\zeta}, \mathcal{J}\Phi'_{\rho,\alpha,\zeta} \rangle, \\ [\partial_\zeta^2 c_{\rho,\alpha,\zeta}]_{\zeta=0} &= \alpha \langle \psi_{\rho,\alpha,\zeta}, \mathcal{J}[2\Phi''_{\rho,\alpha,\zeta} + \Phi'_{\rho,\alpha,\zeta}(\cdot - 1) - \Phi'_{\rho,\alpha,\zeta}(\cdot + 1)] \rangle. \end{aligned} \quad (4.2.90)$$

On the other hand, for $\zeta = \frac{\pi}{4}$ we have

$$\begin{aligned} [\partial_z^2 \lambda_{z;\rho,\alpha,\frac{\pi}{4}}]_{z=0} &= \alpha \langle \psi_{\rho,\alpha,\zeta}, \mathcal{J}[\Phi'_{\rho,\alpha,\zeta}(\cdot + \tfrac{1}{2}\sqrt{2}) + \Phi'_{\rho,\alpha,\zeta}(\cdot - \tfrac{1}{2}\sqrt{2})] \rangle, \\ [\partial_\zeta^2 c_{\rho,\alpha,\zeta}]_{\zeta=\frac{\pi}{4}} &= \alpha \langle \psi_{\rho,\alpha,\zeta}, \mathcal{J}[\Phi''_{\rho,\alpha,\zeta}(\cdot + \tfrac{1}{2}\sqrt{2}) + \Phi''_{\rho,\alpha,\zeta}(\cdot - \tfrac{1}{2}\sqrt{2})] \\ &\quad + \sqrt{2}\alpha \langle \psi_{\rho,\alpha,\zeta}, \mathcal{J}[\Phi'_{\rho,\alpha,\zeta}(\cdot - \tfrac{1}{2}\sqrt{2}) - \Phi'_{\rho,\alpha,\zeta}(\cdot + \tfrac{1}{2}\sqrt{2})] \rangle \rangle. \end{aligned} \quad (4.2.91)$$

Since both components of $\psi_{\rho,\alpha,\zeta}$ and $\Phi'_{\rho,\alpha,\zeta}$ are strictly positive, we again see that

$$[\partial_z^2 \lambda_{z;\rho,\alpha,k\frac{\pi}{4}}]_{z=0} > 0 \quad (4.2.92)$$

for all $k \in \mathbb{Z}$.

As before, it is unclear if the derivatives of c have a sign. As a consequence of the increasing number of components in the MFDEs, our numerical method is at present unable to compute the desired derivatives in the same fashion as above.

Instead, we simply compute the directional dispersion relation directly and determine by inspection whether it is concave or convex; see Figure 4.4. Interestingly enough, we find that this characterization flips at least twice as α is decreased, both for the horizontal direction $\zeta = 0$ and the diagonal direction $\zeta = \frac{\pi}{4}$. In contrast to the monochromatic case, we hence see that interior and exterior bichromatic corners can both travel in the horizontal direction.

4.3 Problem setup

In this section we setup a differential-algebraic equation to describe solutions to the LDE

$$\dot{u}_{nl}(t) = f(\pi_{nl}^\times u(t)) \quad (4.3.1)$$

that can be written in the form

$$u_{nl}(t) = \Xi_l(n + ct) \quad (4.3.2)$$

for some sequence

$$\Xi : \mathbb{Z} \rightarrow W^{1,\infty}(\mathbb{R}; \mathbb{R}^d). \quad (4.3.3)$$

The elements Ξ_l will be required to lie in the orbital vicinity of the waveprofile Φ_* . In particular, we formulate a global center manifold reduction that allows us to find an equivalent two component difference equation of skew-product form.

For any sequence Ξ of the form (4.3.3), we introduce the notation

$$\Xi^\diamond : \mathbb{Z} \rightarrow W^{1,\infty}(\mathbb{R}; (\mathbb{R}^d)^5) \quad (4.3.4)$$

to refer to the expanded sequence

$$\Xi_l^\diamond = \left(\Xi_{l-\sigma_B} - \Xi_l, \Xi_{l+\sigma_A} - \Xi_l, \Xi_{l+\sigma_B} - \Xi_l, \Xi_{l-\sigma_A} - \Xi_l, 0 \right). \quad (4.3.5)$$

In addition, for any

$$\bar{p} = (p_1, \dots, p_5) \in C(\mathbb{R}; (\mathbb{R}^d)^5) \quad (4.3.6)$$

we introduce the function $\bar{\tau}\bar{p}$ that is given by

$$[\bar{\tau}\bar{p}](\xi) = (p_1(\xi + \sigma_A), p_2(\xi + \sigma_B), p_3(\xi - \sigma_A), p_4(\xi - \sigma_B), p_5(\xi)) \in (\mathbb{R}^d)^5. \quad (4.3.7)$$

This allows us to write

$$\pi_{nl}^\times u(t) = \tau \Xi_l(n + ct) + \bar{\tau} \Xi_l^\diamond(n + ct) \quad (4.3.8)$$

for the function u defined in (4.3.2). In particular, this function u satisfies the LDE (4.3.1) if and only if

$$c \Xi'_l(\xi) = f([\tau \Xi_l](\xi) + [\bar{\tau} \Xi_l^\diamond](\xi)) \quad (4.3.9)$$

holds for all $l \in \mathbb{Z}$ and $\xi \in \mathbb{R}$.

From now on, we often drop the explicit ξ -dependence. For instance, we simply write

$$c\Xi'_l = f([\tau\Xi_l] + [\bar{\tau}\Xi_l^\diamond]) \quad (4.3.10)$$

instead of the longer form (4.3.9).

For technical reasons it is advantageous to recast (4.3.10) as a $2d$ -component system. To this end, we introduce the first differences

$$\Upsilon_l = \Xi_l - \Xi_{l-1}. \quad (4.3.11)$$

In addition, we introduce the notation

$$s^\diamond[\Upsilon] : \mathbb{Z} \rightarrow C(\mathbb{R}; (\mathbb{R}^d)^5) \quad (4.3.12)$$

for the summed sequence

$$s^\diamond[\Upsilon]_l = \left(-\sum_{j=0}^{\sigma_B-1} \Upsilon_{l-j}, \sum_{j=1}^{\sigma_A} \Upsilon_{l+j}, \sum_{j=1}^{\sigma_B} \Upsilon_{l+j}, -\sum_{j=0}^{\sigma_A-1} \Upsilon_{l-j}, 0 \right), \quad (4.3.13)$$

in which we make the convention that sums where the lower index is strictly larger than the upper index are set to zero. For example, in the special case $(\sigma_A, \sigma_B) = (1, 0)$ we have

$$s^\diamond[\Upsilon]_l = (0, \Upsilon_{l+1}, 0, -\Upsilon_l, 0). \quad (4.3.14)$$

These definitions allow us to observe that

$$\Xi^\diamond = s^\diamond[\Upsilon]. \quad (4.3.15)$$

In particular, the system (4.3.10) can now be rewritten in the equivalent form

$$\begin{aligned} \Xi_{l+1} - \Xi_l &= \Upsilon_{l+1}, \\ c\Xi'_l &= f(\tau\Xi_l + \bar{\tau}s^\diamond[\Upsilon]_l). \end{aligned} \quad (4.3.16)$$

In the special case $c = c_*$, the travelling wave solution (4.2.12) gives rise to l -independent solutions to (4.3.16) of the form

$$(\Xi_l, \Upsilon_l) = (T_\vartheta \Phi_*, 0), \quad (4.3.17)$$

in which $\vartheta \in \mathbb{R}$ can be chosen arbitrarily. Here we have introduced the left-shift operator

$$[T_\vartheta p](\xi) = p(\xi + \vartheta) \quad (4.3.18)$$

for any $p \in C(\mathbb{R}; \mathbb{R}^d)$.

We now look for a branch of solutions to (4.3.16) that bifurcates from the travelling waves (4.3.17) for $c \neq c_*$. In particular, we consider the Ansatz

$$\begin{aligned} \Xi_l &= T_{\theta_l} \Phi_* + v_l, \\ \Upsilon_l &= w_l, \end{aligned} \quad (4.3.19)$$

for three sequences

$$(\theta, v, w) : \mathbb{Z} \rightarrow \mathbb{R} \times H^1(\mathbb{R}; \mathbb{R}^d) \times H^1(\mathbb{R}; \mathbb{R}^d) \quad (4.3.20)$$

that much hence satisfy

$$\begin{aligned} v_{j+1} - v_j &= w_{j+1} - T_{\theta_{j+1}} \Phi_* + T_{\theta_j} \Phi_*, \\ c[T_{\theta_j} \Phi' + v'] &= f(T_{\theta_j} \tau \Phi_* + \tau v + \bar{\tau} s^\diamond[w]_l). \end{aligned} \quad (4.3.21)$$

In order to close the system, we augment (4.3.21) by demanding that

$$\langle T_{\theta_l} \psi_*, v_l \rangle = 0 \quad (4.3.22)$$

for all $l \in \mathbb{Z}$.

We now set out to isolate the linear and nonlinear parts of the system (4.3.21). For any $\tilde{v} \in C(\mathbb{R}; (\mathbb{R}^d)^5)$ and $\vartheta \in \mathbb{R}$ we therefore introduce the function $\mathcal{N}_{f;\vartheta}(\tilde{v}) \in C(\mathbb{R}; \mathbb{R}^d)$ that is given by

$$\mathcal{N}_{f;\vartheta}(\tilde{v}) = f(\tau T_\vartheta \Phi_* + \tilde{v}) - f(\tau T_\vartheta \Phi_*) - Df(\tau T_\vartheta \Phi_*) \tilde{v}. \quad (4.3.23)$$

In addition, for any phase $\vartheta \in \mathbb{R}$ and difference $\theta_d \in \mathbb{R}$ we introduce the function $\mathcal{N}_{\Phi_*;\vartheta}(\theta_d) \in C(\mathbb{R}; \mathbb{R}^d)$ by writing

$$\mathcal{N}_{\Phi_*;\vartheta}(\theta_d) = T_{\vartheta+\theta_d} \Phi_* - T_\vartheta \Phi_* - T_\vartheta \Phi'_* \theta_d. \quad (4.3.24)$$

Using these functions, the system (4.3.21) can be recast as

$$\begin{aligned} v_{j+1} - v_j &= w_{j+1} - T_{\theta_j} \Phi'_*(\theta_{j+1} - \theta_j) - \mathcal{N}_{\Phi;\theta_j}(\theta_{j+1} - \theta_j), \\ (c - c_*)[T_{\theta_j} \Phi' + v'] &= \mathcal{L}_*^{(\theta_j)} v_j + Df(T_{\theta_j} \tau \Phi_*) \bar{\tau} s^\diamond[w]_j + \mathcal{N}_{f;\theta_j}(\tau v + \bar{\tau} s^\diamond[w]_j), \end{aligned} \quad (4.3.25)$$

in which we have

$$\mathcal{L}_*^{(\vartheta)} v = -c_* v' + Df(T_\vartheta \tau \Phi_*) \tau v. \quad (4.3.26)$$

For any $v \in L^2(\mathbb{R}; \mathbb{R}^d)$, we now introduce the notation

$$Q_\vartheta v = \langle T_\vartheta \psi_*, v \rangle_{L^2}, \quad P_\vartheta v = [Q_\vartheta v] T_\vartheta \Phi'_*. \quad (4.3.27)$$

Applying a difference to (4.3.22), we obtain

$$\begin{aligned} 0 &= Q_{\theta_{j+1}} v_{j+1} - Q_{\theta_j} v_j \\ &= Q_{\theta_j} (v_{j+1} - v_j) + [Q_{\theta_{j+1}} - Q_{\theta_j}] v_{j+1}. \end{aligned} \quad (4.3.28)$$

Substituting the equation for v , we arrive at

$$[Q_{\theta_j} - Q_{\theta_{j+1}}] v_{j+1} = Q_{\theta_j} w_{j+1} - (\theta_{j+1} - \theta_j) - Q_{\theta_j} \mathcal{N}_{\Phi;\theta_j}(\theta_{j+1} - \theta_j). \quad (4.3.29)$$

In order to formulate this in a more compact fashion, we write

$$\sigma_* = \max\{|\sigma_A|, |\sigma_B|\} \quad (4.3.30)$$

together with

$$\text{ev}_l p = \left(p_{l-\sigma_*+1}, \dots, p_{l+\sigma_*-1}, p_{l+\sigma_*} \right) \quad (4.3.31)$$

for any sequence p . In addition, we introduce the shorthand notation

$$L^2 = L^2(\mathbb{R}; \mathbb{R}^d), \quad H^1 = H^1(\mathbb{R}; \mathbb{R}^d), \quad \mathbf{L}^2 = (L^2)^{2\sigma_*}, \quad \mathbf{H}^1 = (H^1)^{2\sigma_*}. \quad (4.3.32)$$

With a slight abuse of notation, we introduce the function $\mathcal{S} : \mathbb{R}^{2\sigma_*} \times \mathbf{H}^1 \times \mathbf{H}^1 \rightarrow \mathbb{R}$ that acts as

$$\mathcal{S}(\text{ev}_l(\theta, v, w)) = [Q_{\theta_{l+1}} - Q_{\theta_l}]v_{l+1} - Q_{\theta_l}\mathcal{N}_{\Phi; \theta_l}(\theta_{l+1} - \theta_l). \quad (4.3.33)$$

This allows us to rewrite (4.3.29) in the form

$$\theta_{l+1} - \theta_l = Q_{\theta_l}w_{l+1} + \mathcal{S}(\text{ev}_l(\theta, v, w)). \quad (4.3.34)$$

Substituting this back into (4.3.25), we obtain

$$\begin{aligned} v_{l+1} - v_l &= (1 - P_{\theta_l})w_{l+1} - \mathcal{S}(\text{ev}_l(\theta, v, w))T_{\theta_l}\Phi'_* \\ &\quad - \mathcal{N}_{\Phi; \theta_l}(\theta_{l+1} - \theta_l). \end{aligned} \quad (4.3.35)$$

For any triplet $(\theta, v, w) : \mathbb{Z} \rightarrow \mathbb{R} \times H^1 \times H^1$, we now introduce the sequences

$$\mathcal{T}(\theta)[v, w] : \mathbb{Z} \rightarrow H^1 \times L^2, \quad \mathcal{D}(\theta)[v, w] : \mathbb{Z} \rightarrow H^1 \times H^1, \quad (4.3.36)$$

defined by

$$\begin{aligned} \mathcal{T}(\theta)[v, w]_l &= (w_{l+1}, -\mathcal{L}_*^{(\theta_l)}v_l), \\ \mathcal{D}(\theta)[v, w]_l &= (v_{l+1} - v_l, Df(T_{\theta_l}\tau\Phi_*)\bar{\tau}s^\diamond[w]_l). \end{aligned} \quad (4.3.37)$$

In addition, we introduce the nonlinear function

$$\mathcal{R} : \mathbb{R}^{2\sigma_*} \times \mathbf{H}^1 \times \mathbf{H}^1 \rightarrow H^1 \times L^2 \quad (4.3.38)$$

that acts as

$$\begin{aligned} \mathcal{R}(\text{ev}_l(\theta, v, w)) &= (c - c_*)(0, v'_l) - \mathcal{S}(\text{ev}_l(\theta, v, w))(T_{\theta_l}\Phi'_*, 0) \\ &\quad - \left(\mathcal{N}_{\Phi; \theta_l}(\theta_{l+1} - \theta_l), \mathcal{N}_{f; \theta_l}(\tau v_l + \bar{\tau}s^\diamond[w]_l) \right). \end{aligned} \quad (4.3.39)$$

Finally, we introduce the operators

$$Q_\vartheta^{(1)}[v, w] = Q_\vartheta v, \quad Q_\vartheta^{(2)}[v, w] = Q_\vartheta w \quad (4.3.40)$$

and the associated projections

$$P_{\vartheta}^{(1)}[v, w] = (P_{\vartheta}v, 0), \quad P_{\vartheta}^{(2)}(v, w) = (0, P_{\vartheta}w). \quad (4.3.41)$$

This allows us to represent the full problem as

$$\begin{aligned} \theta_{l+1} - \theta_l &= Q_{\theta_l} w_{l+1} + \mathcal{S}(\text{ev}_l(\theta, v, w)), \\ \text{pev}_l \mathcal{D}(\theta)[v, w] &= (I - P_{\theta_l}^{(1)}) \text{pev}_l \mathcal{T}(\theta)[v, w] + (c - c_*)(0, T_{\theta_l} \Phi'_*) + \mathcal{R}(\text{ev}_l(\theta, v, w)), \end{aligned} \quad (4.3.42)$$

in which we have introduced the pointwise evaluation operator

$$\text{pev}_l H = H_l \quad (4.3.43)$$

that acts on sequences H .

Lemma 4.3.1. Assume that (Hf), $(H\Phi)$ and $(HS1)$ – $(HS3)$ are satisfied. Pick a constant $c \in \mathbb{R}$ together with three sequences

$$(\theta, v, w) : \mathbb{Z} \rightarrow \mathbb{R} \times H^1 \times H^1 \quad (4.3.44)$$

and consider the pair (Ξ, Υ) defined by (4.3.19). Then the differential-algebraic system (4.3.16) and the identity

$$Q_{\theta_l} v_l = Q_{\theta_0} v_0 \quad (4.3.45)$$

are both satisfied for all $l \in \mathbb{Z}$, if and only if (4.3.42) holds for all $l \in \mathbb{Z}$.

Proof. The computations above shows that indeed (4.3.42) holds whenever (4.3.16) and (4.3.45) are satisfied. The converse implication can be checked by using (4.3.35) to compute

$$\begin{aligned} Q_l[v_{l+1} - v_l] &= -\mathcal{S}(\text{ev}_l(\theta, v, w)) - Q_{\theta_l} \mathcal{N}_{\Phi, \theta_l}(\theta_{l+1} - \theta_l) \\ &= -[Q_{\theta_{l+1}} - Q_{\theta_l}]v_{l+1}, \end{aligned} \quad (4.3.46)$$

which gives (4.3.45). The identity (4.3.16) then follows readily. $\square$

We now proceed to obtain estimates on the nonlinear terms. These are mostly standard bounds for quadratic nonlinearities that will be used in §4.7 for the center manifold construction. Notice however that any dependencies on the phase θ always involve differences in θ .

Lemma 4.3.2. Assume that (Hf) and $(H\Phi)$ hold and pick a sufficiently large constant $K > 0$. Then for any $\tilde{v} \in H^1(\mathbb{R}; (\mathbb{R}^d)^5)$ with $\|\tilde{v}\|_{H^1} \leq 1$ we have the bound

$$\|\mathcal{N}_{f; \vartheta}(\tilde{v})\|_{L^2} \leq K \|\tilde{v}\|_{H^1} \|\tilde{v}\|_{L^2}. \quad (4.3.47)$$

In addition, for any pair $(\vartheta^A, \vartheta^B) \in \mathbb{R}^2$ and any pair

$$(\tilde{v}^A, \tilde{v}^B) \in H^1(\mathbb{R}; (\mathbb{R}^d)^5) \times H^1(\mathbb{R}; (\mathbb{R}^d)^5) \quad (4.3.48)$$

with $\|\tilde{v}^A\|_{H^1} \leq 1$ and $\|\tilde{v}^B\|_{H^1} \leq 1$, we have the Lipschitz estimate

$$\begin{aligned} \|\mathcal{N}_{f;\vartheta^A}(\tilde{v}^A) - \mathcal{N}_{f;\vartheta^B}(\tilde{v}^B)\|_{L^2} &\leq K \left[\|\tilde{v}^A\|_{H^1} + \|\tilde{v}^B\|_{H^1} + |\vartheta^A - \vartheta^B| \right] \\ &\quad \left[\|\tilde{v}^A - \tilde{v}^B\|_{L^2} + |\vartheta^A - \vartheta^B| \right]. \end{aligned} \quad (4.3.49)$$

Proof. Upon writing

$$\mathcal{M}_{f;\tilde{\Phi}}(\tilde{v}) = f(\tau\tilde{\Phi} + \tilde{v}) - Df(\tau\tilde{\Phi})\tilde{v} - f(\tau\tilde{\Phi}), \quad (4.3.50)$$

we readily see that

$$\mathcal{N}_{f;\vartheta}(\tilde{v}) = \mathcal{M}_{f;T_\vartheta\Phi_*}(\tilde{v}). \quad (4.3.51)$$

Since f is at least C^2 -smooth, there exists $C_1 > 0$ so that the pointwise bound

$$|\mathcal{M}_{f;\tilde{\Phi}}(\tilde{v})(\xi)| \leq C_1 |\tilde{v}(\xi)|^2 \quad (4.3.52)$$

holds whenever

$$\|\tilde{\Phi}\|_{H^1} \leq \|\Phi_*\|_{H^1} + 1, \quad \|\tilde{v}\|_{H^1} \leq 1. \quad (4.3.53)$$

This yields

$$\|\mathcal{M}_{f;\tilde{\Phi}}(\tilde{v})\|_{L^2} \leq C_2 \|\tilde{v}\|_{H^1} \|\tilde{v}\|_{L^2} \quad (4.3.54)$$

for some $C_2 > 0$, from which (4.3.47) follows.

Upon writing

$$\Delta_{\mathcal{M}} = \mathcal{M}_{f;\tilde{\Phi}^A}(\tilde{v}^A) - \mathcal{M}_{f;\tilde{\Phi}^B}(\tilde{v}^B), \quad (4.3.55)$$

a short computation shows that

$$\begin{aligned} \Delta_{\mathcal{M}} &= \mathcal{M}_{f;\tilde{\Phi}^B + \tilde{v}^B}(\tilde{\Phi}^A + \tilde{v}^A - \tilde{\Phi}^B - \tilde{v}^B) - \mathcal{M}_{f;\tilde{\Phi}^B}(\tilde{\Phi}^A - \tilde{\Phi}^B) \\ &\quad + (Df(\tau\tilde{\Phi}^B + \tilde{v}^B) - Df(\tau\tilde{\Phi}^B))[\tilde{\Phi}^A + \tilde{v}^A - \tilde{\Phi}^B - \tilde{v}^B] \\ &\quad + (Df(\tau\tilde{\Phi}^B) - Df(\tau\tilde{\Phi}^A))\tilde{v}^A. \end{aligned} \quad (4.3.56)$$

Under the assumption that (4.3.53) holds for both $(\tilde{\Phi}^A, v^A)$ and $(\tilde{\Phi}^B, v^B)$, we hence find

$$\|\Delta_{\mathcal{M}}\|_{L^2} \leq C_3 \left[\|\tilde{\Phi}^A - \tilde{\Phi}^B\|_{H^1} + \|\tilde{v}^A\|_{H^1} + \|\tilde{v}^B\|_{H^1} \right] \left[\|\tilde{v}^A - \tilde{v}^B\|_{L^2} + \|\tilde{\Phi}^A - \tilde{\Phi}^B\|_{L^2} \right] \quad (4.3.57)$$

for some $C_3 > 0$. The bound (4.3.49) now follows from the fact that

$$\|T_{\vartheta^B}\Phi_* - T_{\vartheta^A}\Phi_*\|_{H^1} \leq C_4 |\vartheta^B - \vartheta^A| \quad (4.3.58)$$

for some $C_4 > 0$. □

Lemma 4.3.3. Assume that (Hf), ($H\Phi$) and (HS1) hold and pick a sufficiently large constant $K > 0$. Then for any pair $\vartheta \in \mathbb{R}$ and $\theta_d \in \mathbb{R}$ we have the bound

$$\|\mathcal{N}_{\Phi_*; \vartheta}(\theta_d)\|_{H^1} \leq K |\theta_d|^2. \quad (4.3.59)$$

In addition, for any pair $(\vartheta^A, \vartheta^B) \in \mathbb{R}^2$ and any pair $(\theta_d^A, \theta_d^B) \in \mathbb{R}^2$ we have the bound

$$\|\mathcal{N}_{\Phi_*; \vartheta^A}(\theta_d^A) - \mathcal{N}_{\Phi_*; \vartheta^B}(\theta_d^B)\|_{H^1} \leq K [|\vartheta^A - \vartheta^B| + |\theta_d^A| + |\theta_d^B|] [|\vartheta^A - \vartheta^B| + |\theta_d^A - \theta_d^B|]. \quad (4.3.60)$$

Proof. Assumptions (Hf), ($H\Phi$) and (HS1) imply that Φ_*''' is a continuous function that decays exponentially, from which (4.3.59) follows. Writing

$$\Delta_{\mathcal{N}} = \mathcal{N}_{\Phi_*; \vartheta^A}(\theta_d^A) - \mathcal{N}_{\Phi_*; \vartheta^B}(\theta_d^B), \quad (4.3.61)$$

one can compute

$$\begin{aligned} \Delta_{\mathcal{N}} &= \mathcal{N}_{\Phi_*; \vartheta^B + \theta_d^B}(\vartheta^A + \theta_d^A - \vartheta^B - \theta_d^B) - \mathcal{N}_{\Phi_*; \vartheta^B}(\vartheta^A - \vartheta^B) \\ &\quad + (T_{\vartheta^B + \theta_d^B} \Phi'_* - T_{\vartheta^B} \Phi'_*)[\vartheta^A + \theta_d^A - \vartheta^B - \theta_d^B] \\ &\quad + (T_{\vartheta^B} \Phi'_* - T_{\vartheta^A} \Phi'_*)\theta_d^A. \end{aligned} \quad (4.3.62)$$

The inequality (4.3.60) follows directly from this representation. $\square$

Corollary 4.3.4. Assume that (Hf), ($H\Phi$) and (HS1) hold and pick a sufficiently large constant $K > 0$. Then for any triplet of sequences

$$(\theta, v, w) : \mathbb{Z} \rightarrow \mathbb{R} \times H^1 \times H^1 \quad (4.3.63)$$

that has

$$\|v_l\|_{H^1} + \|w_l\|_{H^1} \leq 1 \quad (4.3.64)$$

for all $l \in \mathbb{Z}$, we have the bounds

$$\begin{aligned} \|\mathcal{R}(\text{ev}_l(\theta, v, w))\|_{H^1 \times L^2} &\leq K [|\theta_{l+1} - \theta_l| + \|\text{ev}_l(v, w)\|_{\mathbf{H}^1 \times \mathbf{H}^1}] \\ &\quad \times [|\theta_{l+1} - \theta_l| + \|\text{ev}_l(v, w)\|_{\mathbf{L}^2 \times \mathbf{L}^2}] \\ &\quad + |c - c_*| \|v_l\|_{H^1}, \\ |\mathcal{S}(\text{ev}_l(\theta, v, w))| &\leq K |\theta_{l+1} - \theta_l| [\|v_{l+1}\|_{L^2} + |\theta_{l+1} - \theta_l|] \end{aligned} \quad (4.3.65)$$

for all $l \in \mathbb{Z}$. In addition, for any pair of triplets

$$(\theta^A, v^A, w^A) : \mathbb{Z} \rightarrow \mathbb{R} \times H^1 \times H^1, \quad (\theta^B, v^B, w^B) : \mathbb{Z} \rightarrow \mathbb{R} \times H^1 \times H^1 \quad (4.3.66)$$

that has

$$\|v_l^A\|_{H^1} + \|w_l^A\|_{H^1} \leq 1, \quad \|v_l^B\|_{H^1} + \|w_l^B\|_{H^1} \leq 1 \quad (4.3.67)$$

for all $l \in \mathbb{Z}$, the quantities

$$\begin{aligned}\Delta_{\mathcal{R}} &= \mathcal{R}(\text{ev}_l(\theta^A, v^A, w^A)) - \mathcal{R}(\text{ev}_l(\theta^B, v^B, w^B)), \\ \Delta_{\mathcal{S}} &= \mathcal{S}(\text{ev}_l(\theta^A, v^A, w^A)) - \mathcal{S}(\text{ev}_l(\theta^B, v^B, w^B))\end{aligned}\tag{4.3.68}$$

satisfy the bounds

$$\begin{aligned}\|\Delta_{\mathcal{R}}\|_{H^1 \times L^2} &\leq K \left[|\theta_{l+1}^A - \theta_l^A| + |\theta_{l+1}^B - \theta_l^B| + \|\text{ev}_l(v^A, w^A)\|_{\mathbf{H}^1 \times \mathbf{H}^1} \right. \\ &\quad \left. + \|\text{ev}_l(v^B, w^B)\|_{\mathbf{H}^1 \times \mathbf{H}^1} \right] \times \left[\|\text{ev}_l(v^A - v^B)\|_{\mathbf{L}^2} \right. \\ &\quad \left. + \|\text{ev}_l(w^A - w^B)\|_{\mathbf{L}^2} + |\text{ev}_l(\theta^A - \theta^B)| \right] \\ &\quad + |c - c_*| \|v_l^A - v_l^B\|_{H^1}, \\ |\Delta_{\mathcal{S}}| &\leq \left[|\theta_{l+1}^A - \theta_l^A| + |\theta_{l+1}^B - \theta_l^B| + \|\text{ev}_l v^A\|_{\mathbf{L}^2} + \|\text{ev}_l v^B\|_{\mathbf{L}^2} \right] \\ &\quad \times \left[\|\text{ev}_l(v^A - v^B)\|_{\mathbf{L}^2} + |\text{ev}_l(\theta^A - \theta^B)| \right].\end{aligned}\tag{4.3.69}$$

Proof. This follows from Lemma's 4.3.2-4.3.3 upon inspecting the definitions of $\mathcal{S}$ and $\mathcal{R}$. $\square$

We are now in a position to state our main center manifold result. The fact that this manifold is two dimensional is related to the observation that the linear problem

$$\mathcal{D}(0)[v, w] = (I - P_0^{(1)})\mathcal{T}(0)[v, w]\tag{4.3.70}$$

has the constant solutions $(\Phi'_*, 0)$ and $([\partial_z \phi_z]_{z=0}, \Phi'_*)$, as we will see in §4.5.

Proposition 4.3.5 (see §4.7). Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) all hold, recall the integer r defined in (Hf) and pick sufficiently small constants $\delta_c > 0$ and $\delta > 0$. Then there exists a function

$$h : \mathbb{R}^2 \times (c_* - \delta_c, c_* + \delta_c) \mapsto H^1 \times H^1\tag{4.3.71}$$

together with functions

$$f_\theta, f_\kappa : \mathbb{R} \times (c_* - \delta_c, c_* + \delta_c) \rightarrow \mathbb{R}\tag{4.3.72}$$

such that the following properties are satisfied.

- (i) The function h is C^{r-1} smooth. In addition, we have the behaviour

$$h(\vartheta, \kappa, c) = O(|\kappa|^2 + |c - c_*|)\tag{4.3.73}$$

for $\kappa \rightarrow 0$ and $c \rightarrow c_*$, uniformly for $\vartheta \in \mathbb{R}$.

- (ii) The functions f_θ and f_κ are C^{r-1} -smooth. In addition, we have the behaviour

$$f_\kappa(\kappa, c) = 2[\partial_z^2 \lambda_z]_{z=0}^{-1} \left[c - c_* - \frac{1}{2}[\partial_z^2 d_\varphi]_{\varphi=0} \kappa^2 \right] + O((c - c_*)^2 + (c - c_*)\kappa + \kappa^3)\tag{4.3.74}$$

as $\kappa \rightarrow 0$ and $c \rightarrow c_*$.

- (iii) For each small $\phi \geq 0$ there is a unique $\kappa_\phi \in [0, \delta]$ for which $f_\kappa(\kappa_\phi, d_\phi) = 0$. Similarly, whenever $-\phi \geq 0$ is small there is a unique $\kappa_\phi \in [-\delta, 0]$ for which $f_\kappa(\kappa_\phi, d_\phi) = 0$. In both cases we have $f_\theta(\kappa_\phi, d_\phi) = \tan \phi$.
- (iv) Pick a $c \in (c_* - \delta, c_* + \delta)$ and consider a triplet of sequences

$$(\theta, v, w) : \mathbb{Z} \rightarrow \mathbb{R} \times H^1 \times H^1 \quad (4.3.75)$$

that satisfies (4.3.42) and admits the bound

$$\|v_l\|_{H^1} + \|w_l\|_{H^1} \leq \delta \quad (4.3.76)$$

for all $l \in \mathbb{Z}$. Then upon writing

$$\kappa_l = Q_{\theta_l} w_l = \langle T_{\theta_l} \psi_*, w_l \rangle_{L^2}, \quad (4.3.77)$$

the identity

$$(v, w)_l = \kappa_l T_{\theta_l}([\partial_z \phi_z]_{z=0}, \Phi'_*) + h(\theta_l, \kappa_l, c) \quad (4.3.78)$$

together with the difference equation

$$\begin{aligned} \theta_{l+1} - \theta_l &= f_\theta(\kappa_l, c), \\ \kappa_{l+1} - \kappa_l &= f_\kappa(\kappa_l, c) \end{aligned} \quad (4.3.79)$$

are both satisfied for all $l \in \mathbb{Z}$.

- (v) Pick a speed $c \in (c_* - \delta, c_* + \delta)$ and pair of sequences

$$(\theta, \kappa) : \mathbb{Z} \rightarrow \mathbb{R} \times \mathbb{R} \quad (4.3.80)$$

that satisfies (4.3.79) and admits the bound

$$\|\kappa_l\| \leq \delta \quad (4.3.81)$$

for all $l \in \mathbb{Z}$. Then the triplet (θ, v, w) obtained by applying the identity (4.3.78) satisfies (4.3.42).

Proof of Theorem 4.2.3. For explicitness, we assume that $[\partial_\varphi^2 d_\varphi]_{\varphi=0} > 0$ and $[\partial_z^2 \lambda_z]_{z=0} > 0$. Whenever $c - c_* > 0$ is sufficiently small, the identity $[\partial_\varphi d_\varphi]_{\varphi=0} = 0$ allows us to use a Taylor expansion to show that there exist $\varphi_- < 0 < \varphi_+$ for which $d_{\varphi_-} = d_{\varphi_+} = c$. Noting that

$$\varphi_\pm = O(\sqrt{c - c_*}), \quad (4.3.82)$$

we use (iii) of Proposition 4.3.5 to define two quantities $\kappa_\pm = \kappa_{\varphi_\pm}$ for which we have the identity

$$f_\kappa(\kappa_{\varphi_\pm}, c) = 0. \quad (4.3.83)$$

In addition, possibly after further restricting the size of $c - c_*$, we can ensure that the inequalities

$$f_\kappa(\kappa, c) > 0, \quad |D_1 f_\kappa(\kappa, c)| < \frac{1}{2} \quad (4.3.84)$$

hold for all $\kappa \in (\kappa_-, \kappa_+)$. As a consequence, for any such κ we have the bounds

$$\begin{aligned}
 \kappa - \kappa_- &\leq f_\kappa(\kappa, c) - f_\kappa(\kappa_-, c) \\
 &= f_\kappa(\kappa, c) \\
 &= f_\kappa(\kappa, c) - f_\kappa(\kappa_+, c) \\
 &\leq \kappa_+ - \kappa.
 \end{aligned} \tag{4.3.85}$$

In particular, for any $\kappa \in (\kappa_-, \kappa_+)$ one can apply the contraction mapping principle to the fixed point problem

$$\tilde{\kappa} = \kappa - f_\kappa(\tilde{\kappa}, c) \tag{4.3.86}$$

and obtain a unique solution in $\tilde{\kappa} \in (\kappa_-, \kappa_+)$.

For any choice of $\tilde{\kappa}_0 \in (\kappa_-, \kappa_+)$, the problem

$$\kappa_{l+1} - \kappa_l = f_\kappa(\kappa, c), \quad \kappa_0 = \tilde{\kappa}_0 \tag{4.3.87}$$

can therefore be iterated backwards and forwards with respect to l to yield a solution $\kappa : \mathbb{Z} \rightarrow (\kappa_-, \kappa_+)$. This solution is strictly increasing and satisfies the limits

$$\lim_{l \rightarrow \pm\infty} \kappa_l = \kappa_\pm. \tag{4.3.88}$$

By applying the representation (4.3.78) one can now construct the desired solution (θ, v) . $\square$

4.4 Preliminaries

In this section we obtain a number of preliminary results related to the constant-coefficient linear system

$$\mathcal{D}(0)[V] - \mathcal{T}(0)[V] = H. \tag{4.4.1}$$

In particular, we study the Fourier symbol $\Delta(z)$ associated to this system and obtain a representation formula for solutions that are allowed to grow at a small exponential rate.

As a preparation, we introduce the notation

$$\mathcal{L}_* = \mathcal{L}_0 = \mathcal{L}_*^{(0)}, \quad Q_* = Q_0 = \langle \psi_*, \cdot \rangle \tag{4.4.2}$$

for the linearization of the travelling wave MFDE (4.2.8) around Φ_* and the corresponding projection onto the kernel element Φ'_* . In addition, for any $z \in \mathbb{C}$ we define the vector

$$s_z^\diamond = \left(-\sum_{j=0}^{\sigma_B-1} e^{-zj}, \sum_{j=1}^{\sigma_A} e^{zj}, \sum_{j=1}^{\sigma_B} e^{zj}, -\sum_{j=0}^{\sigma_A-1} e^{-zj}, 0 \right) \in \mathbb{C}^5, \tag{4.4.3}$$

recalling the convention that sums where the lower index is strictly larger than the upper index are set to zero. By construction, this allows us to write

$$s^\diamond[e^{z\cdot}w] = e^{z\cdot}s_z^\diamond w \quad (4.4.4)$$

for any $w \in H^1$.

Upon introducing the linear operators $\Delta(z) : H^1 \times H^1 \rightarrow H^1 \times L^2$ that act as

$$\Delta(z) = \begin{pmatrix} e^z - 1 & -e^z \\ \mathcal{L}_* & Df(\tau\Phi_*)\bar{\tau}s_z^\diamond \end{pmatrix}, \quad (4.4.5)$$

we hence have the identity

$$\mathcal{D}(0)[e^{z\cdot}V] - \mathcal{T}(0)[e^{z\cdot}V] = e^{z\cdot}\Delta(z)V \quad (4.4.6)$$

for any $V \in H^1 \times H^1$. Our first main result shows that these operators are invertible along vertical lines that are close to the imaginary axis.

Proposition 4.4.1. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are all satisfied and pick a sufficiently small $\eta_{\max} > 0$. Then there exists a constant $K > 0$ so that for every η with $0 < |\eta| < \eta_{\max}$ and every $\omega \in [0, 2\pi]$, the operator

$$\Delta(\eta + i\omega) : H^1 \times H^1 \rightarrow H^1 \times L^2 \quad (4.4.7)$$

is invertible and satisfies the bound

$$\|\Delta(\eta + i\omega)^{-1}\|_{\mathcal{L}(H^1 \times L^2; H^1 \times H^1)} \leq K\eta^{-2}. \quad (4.4.8)$$

In order to gain insight regarding solutions to the homogeneous linear system

$$\mathcal{D}(0)[V] - \mathcal{T}(0)[V] = 0, \quad (4.4.9)$$

we briefly discuss the maximal Jordan chain associated to $\Delta(z)$ at $z = 0$. In particular, we set out to construct an analytic function $z \mapsto \mathcal{J}(z) \in H^1 \times H^1$ with $\mathcal{J}(0) \neq 0$ so that

$$\Delta(z)\mathcal{J}(z) = O(z^m) \quad (4.4.10)$$

for the largest possible value of m .

As a preparation, we note that

$$s_z^\diamond = \frac{e^z}{1-e^z} \left(e^{-\sigma_B z} - 1, e^{\sigma_A z} - 1, e^{\sigma_B z} - 1, e^{-\sigma_A z} - 1, 0 \right) \quad (4.4.11)$$

whenever $z \neq 0$. Recalling the definition (4.2.16), we hence see that

$$\mathcal{L}_z = \mathcal{L}_* + e^{-z}(e^z - 1)Df(\tau\Phi_*)\bar{\tau}s_z^\diamond. \quad (4.4.12)$$

Upon introducing the notation

$$A_k = [\partial_z^k \mathcal{L}_z]_{z=0} \quad (4.4.13)$$

for any integer $k \geq 1$, we may differentiate (4.4.12) to find

$$\begin{aligned} Df(\tau\Phi_*)\bar{\tau}s_0^\diamond &= A_1, \\ Df(\tau\Phi_*)\bar{\tau}[\partial_z s_z^\diamond]_{z=0} &= \frac{1}{2}(A_2 + A_1). \end{aligned} \quad (4.4.14)$$

In particular, we may write

$$\Delta(0) = \begin{pmatrix} 0 & -1 \\ \mathcal{L}_* & A_1 \end{pmatrix}, \quad \Delta'(0) = \begin{pmatrix} 1 & -1 \\ 0 & \frac{1}{2}(A_1 + A_2) \end{pmatrix}. \quad (4.4.15)$$

Using (HS3) we immediately see

$$\text{Ker}(\Delta(0)) = \text{span}\{(\Phi'_*, 0)^T\}, \quad (4.4.16)$$

which allows us to pick $\mathcal{J}(0) = (\Phi'_*, 0)$. This chain can be extended by exploiting the following preliminary identities.

Lemma 4.4.2. Assume that (Hf), $(H\Phi)$ and (HS1)-(HS3) are satisfied. Then we have the identities

$$\begin{aligned} \mathcal{L}_*[\partial_z \phi_z]_{z=0} &= -A_1 \Phi'_* + [\partial_z \lambda_z]_{z=0} \Phi'_*, \\ \mathcal{L}_*[\partial_z^2 \phi_z]_{z=0} &= -A_2 \Phi'_* - 2A_1[\partial_z \phi_z]_{z=0} + [\partial_z^2 \lambda_z]_{z=0} \Phi'_* + 2[\partial_z \lambda_z]_{z=0}[\partial_z \phi_z]_{z=0}, \\ \mathcal{L}_*[\partial_z^3 \phi_z]_{z=0} &= -(A_3 \Phi'_* + 3A_2[\partial_z \phi_z]_{z=0} + 3A_1[\partial_z^2 \phi_z]_{z=0}) \\ &\quad + [\partial_z^3 \lambda_z]_{z=0} \Phi'_* + 3[\partial_z^2 \lambda_z]_{z=0}[\partial_z \phi_z]_{z=0} + 3[\partial_z \lambda_z]_{z=0}[\partial_z^2 \phi_z]_{z=0}, \end{aligned} \quad (4.4.17)$$

together with

$$\begin{aligned} [\partial_z \lambda_z]_{z=0} &= \langle \psi_*, A_1 \Phi'_* \rangle, \\ [\partial_z^2 \lambda_z]_{z=0} &= \langle \psi_*, A_2 \Phi'_* + 2A_1[\partial_z \phi_z]_{z=0} \rangle, \\ [\partial_z^3 \lambda_z]_{z=0} &= \langle \psi_*, A_3 \Phi'_* + 3A_2[\partial_z \phi_z]_{z=0} + 3A_1[\partial_z^2 \phi_z]_{z=0} \rangle. \end{aligned} \quad (4.4.18)$$

Proof. Differentiating the definition $\mathcal{L}_z \phi_z = \lambda_z \phi_z$, we obtain the identities

$$\begin{aligned} [\partial_z \mathcal{L}_z] \phi_z + \mathcal{L}_z [\partial_z \phi_z] &= [\partial_z \lambda_z] \phi_z + \lambda_z [\partial_z \phi_z], \\ [\partial_z^2 \mathcal{L}_z] \phi_z + 2[\partial_z \mathcal{L}_z] \partial_z \phi_z + \mathcal{L}_z [\partial_z^2 \phi_z] &= [\partial_z^2 \lambda_z] \phi_z + 2[\partial_z \lambda_z] \partial_z \phi_z \\ &\quad + \lambda_z [\partial_z^2 \phi_z], \\ [\partial_z^3 \mathcal{L}_z] \phi_z + 3[\partial_z^2 \mathcal{L}_z] \partial_z \phi_z + 3[\partial_z \mathcal{L}_z] \partial_z^2 \phi_z + \mathcal{L}_z [\partial_z^3 \phi_z] &= [\partial_z^3 \lambda_z] \phi_z + 3[\partial_z^2 \lambda_z] \partial_z \phi_z \\ &\quad + 3[\partial_z \lambda_z] \partial_z^2 \phi_z + \lambda_z [\partial_z^3 \phi_z]. \end{aligned} \quad (4.4.19)$$

Evaluating these expressions at $z = 0$ we find (4.4.17). The derivatives (4.4.18) can then be obtained by recalling the normalization $\langle \psi_*, \phi_z \rangle = 1$ and using the fact that $\langle \psi_*, \mathcal{L}_* y \rangle = 0$ for all $y \in H^1$. $\square$

Indeed, we now write

$$\mathcal{J}(z) = (\Phi'_*, 0) + z([\partial_z \phi_z]_{z=0}, \Phi'_*) + z^2(v(z), w(z)) \quad (4.4.20)$$

for a pair of analytic functions $z \mapsto (v(z), w(z)) \in H^1 \times H^1$. Using the identity

$$\Delta''(0) = \begin{pmatrix} 1 & -1 \\ 0 & Df(\tau\Phi_*)\bar{\tau}[\partial_z^2 s_z]_{z=0} \end{pmatrix}, \quad (4.4.21)$$

we may exploit (4.4.17) to compute

$$\begin{aligned} \Delta(z)\mathcal{J}(z) &= \Delta(0)(\Phi'_*, 0)^T + z\left(\Delta'(0)(\Phi'_*, 0)^T + \Delta(0)([\partial_z \phi_z]_{z=0}, \Phi'_*)^T\right) \\ &\quad + z^2\left(\frac{1}{2}\Delta''(0)(\Phi'_*, 0)^T + \Delta'(0)([\partial_z \phi_z]_{z=0}, \Phi'_*)^T + \Delta(0)V(0)\right) + O(z^3) \\ &= z^2\left([\partial_z \phi_z]_{z=0} - \frac{1}{2}\Phi'_*, \frac{1}{2}(A_1 + A_2)\Phi'_*\right)^T + \Delta(0)(v(0), w(0))^T + O(z^3). \end{aligned} \quad (4.4.22)$$

In particular, we have achieved $m = 2$ in (4.4.10). This corresponds with the presence of two solutions

$$[V_{\text{hom}}^A]_l = (\Phi'_*, 0), \quad [V_{\text{hom}}^B]_l = (l\Phi'_*, 0) + ([\partial_z \phi_z]_{z=0}, \Phi'_*) \quad (4.4.23)$$

to the linear homogeneous problem (4.4.9). However, it is not possible to achieve $m = 3$. Indeed, setting the $O(z^2)$ term in (4.4.22) to zero, we obtain

$$w(0) = \frac{1}{2}\Phi'_* - [\partial_z \phi_z]_{z=0} \quad (4.4.24)$$

and hence

$$\mathcal{L}_* v(0) = \frac{1}{2}A_2\Phi'_* + A_1[\partial_z \phi_z]_{z=0}. \quad (4.4.25)$$

Taking the inner product with ψ_* , we may use (HM) to obtain the contradiction

$$0 = \langle \psi_*, \frac{1}{2}A_2\Phi'_* + A_1[\partial_z \phi_z]_{z=0} \rangle = \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \neq 0. \quad (4.4.26)$$

Our second main result confirms that there are no other linearly independent solutions to (4.4.9) that are bounded by $e^{\eta_{\max}|l|}$. In addition, it provides a representation formula for solutions to the inhomogeneous system (4.4.1) that share such an exponential bound.

In order to formulate this conveniently, we introduce the family of sequence spaces

$$BX_{\mu,\nu}(\mathcal{H}) := \left\{ V : \mathbb{Z} \mapsto \mathcal{H} : \sup_{l < 0} e^{-\mu l} \|V_l\|_{\mathcal{H}} + \sup_{l \geq 0} e^{-\nu l} \|V_l\|_{\mathcal{H}} < \infty \right\} \quad (4.4.27)$$

for any $\mu, \nu \in \mathbb{R}$ and any Hilbert space $\mathcal{H}$. In addition, for any $H \in BX_{\mu,\nu}$ we introduce the forward discrete Laplace transform

$$\mathcal{Z}_+[H](z) = \sum_{n=0}^{\infty} e^{-zn} H_n, \quad \operatorname{Re} z > \nu \quad (4.4.28)$$

together with the backward discrete Laplace transform

$$\mathcal{Z}_-[H](z) = \sum_{n=1}^{\infty} e^{zn} H_{-n}, \quad \operatorname{Re} z < \mu. \quad (4.4.29)$$

Proposition 4.4.3. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are all satisfied. Fix four constants $\gamma_- < \eta_- < \eta_+ < \gamma_+$ for which $0 < |\eta_{\pm}| < \eta_{\max}$ and $0 < |\gamma_{\pm}| < \eta_{\max}$ all hold, together with

$$\operatorname{sign}(\gamma_-) = \operatorname{sign}(\eta_-), \quad \operatorname{sign}(\gamma_+) = \operatorname{sign}(\eta_+). \quad (4.4.30)$$

Consider any $V \in BX_{\eta_-, \eta_+}(H^1 \times H^1)$ and write

$$H = \mathcal{D}(0)[V] - \mathcal{T}(0)[V]. \quad (4.4.31)$$

Then we have the representation formula

$$\begin{aligned} V_l = & \frac{1}{2\pi i} \int_{\gamma_+ - \pi i}^{\gamma_+ + \pi i} e^{zl} \Delta(z)^{-1} \mathcal{Z}_+[H](z) dz + \frac{1}{2\pi i} \int_{\gamma_- - \pi i}^{\gamma_- + \pi i} e^{zl} \Delta(z)^{-1} \mathcal{Z}_-[H](z) dz \\ & + [V_{\text{hom}}^A]_l \tilde{P}_A[\operatorname{ev}_0 V] + [V_{\text{hom}}^B]_l [\tilde{P}_B \operatorname{ev}_0 V] \end{aligned} \quad (4.4.32)$$

for some pair of bounded linear maps

$$(\tilde{P}_A, \tilde{P}_B) : \mathbf{H}^1 \times \mathbf{H}^1 \rightarrow \mathbb{R} \times \mathbb{R} \quad (4.4.33)$$

that satisfy

$$\tilde{P}_A = \tilde{P}_B = 0 \quad (4.4.34)$$

when $\operatorname{sign}(\eta_-) = \operatorname{sign}(\eta_+)$ and

$$\tilde{P}_A[\operatorname{ev}_0 V_{\text{hom}}^A] = \tilde{P}_B[\operatorname{ev}_0 V_{\text{hom}}^B] = 1, \quad \tilde{P}_A[\operatorname{ev}_0 V_{\text{hom}}^B] = \tilde{P}_B[\operatorname{ev}_0 V_{\text{hom}}^A] = 0 \quad (4.4.35)$$

otherwise.

Whenever $\mathcal{L}_z : H^1 \rightarrow L^2$ is invertible, a short calculation shows that the same is true for $\Delta(z)$ with

$$\Delta(z)^{-1} = \begin{pmatrix} \mathcal{L}_z^{-1} Df(\tau\Phi_*) \bar{\tau} s_z^\diamond e^{-z} & \mathcal{L}_z^{-1} \\ -e^{-z} + e^{-z}(e^z - 1)\mathcal{L}_z^{-1} Df(\tau\Phi_*) \bar{\tau} s_z^\diamond e^{-z} & e^{-z}(e^z - 1)\mathcal{L}_z^{-1} \end{pmatrix}. \quad (4.4.36)$$

It is hence crucial to understand the behaviour of $\mathcal{L}_z^{-1}$ for small $|z| > 0$, which we set out to do by exploiting the Fredholm properties of $\mathcal{L}_*$.

As a preparation, we introduce the notation

$$\mathcal{L}_*^{\text{qinv}} f = v \quad (4.4.37)$$

for the unique $v \in H^1$ that has $\langle \psi_*, v \rangle = 0$ and satisfies the problem

$$\mathcal{L}_* v = f - \langle \psi_*, f \rangle \Phi'_*. \quad (4.4.38)$$

This allows us to rephrase the identities (4.4.17) in a more explicit form.

Corollary 4.4.4. Assume that (Hf), $(H\Phi)$ and (HS1)-(HS3) are satisfied. Then we have the identities

$$\begin{aligned} [\partial_z \phi_z]_{z=0} &= -\mathcal{L}^{\text{qinv}} A_1 \Phi'_*, \\ [\partial_z^2 \phi_z]_{z=0} &= -\mathcal{L}^{\text{qinv}} \left[A_2 \Phi'_* + 2A_1 [\partial_z \phi_z]_{z=0} - 2[\partial_z \lambda_z]_{z=0} [\partial_z \phi_z]_{z=0} \right]. \end{aligned} \quad (4.4.39)$$

Proof. These expressions follow from (4.4.17), noting that $\mathcal{L}^{\text{qinv}} \Phi'_* = 0$. $\square$

At this point, it is natural to briefly turn our attention to the angular dependence of the waves $(c_\varphi, \Phi_\varphi)$, which can be analyzed using techniques that are similar to those used above. Unfortunately, the expressions for the second derivatives are somewhat more involved. In order to accomodate this, we introduce the notation

$$B_1 p = Df(\tau\Phi_*)(-\sigma_A, -\sigma_B, \sigma_A, \sigma_B, 0)[\bar{\tau} p] \quad (4.4.40)$$

for any $p \in H^1$.

Lemma 4.4.5. Assume that (Hf), $(H\Phi)$ and (HS1)-(HS3) are satisfied. Then the statements in Lemma 4.2.2 hold true. In addition, we have the identities

$$\begin{aligned} \mathcal{L}_* [\partial_\varphi \Phi_\varphi]_{\varphi=0} &= -A_1 \Phi'_* + [\partial_\varphi c_\varphi]_{\varphi=0} \Phi'_*, \\ \mathcal{L}_* [\partial_\varphi^2 \Phi_\varphi]_{\varphi=0} &= [\partial_\varphi^2 c_\varphi]_{\varphi=0} \Phi'_* + 2[\partial_\varphi c_\varphi]_{\varphi=0} [\partial_\varphi \Phi']_{\varphi=0} \\ &\quad - D^2 f(\tau\Phi_*) \left[s_0^\diamond \tau \Phi'_* + \tau [\partial_\varphi \Phi_\varphi]_{\varphi=0}, s_0^\diamond \tau \Phi'_* + \tau [\partial_\varphi \Phi_\varphi]_{\varphi=0} \right] \\ &\quad - A_2 \Phi''_* - B_1 \Phi'_* - 2A_1 [\partial_\varphi \Phi'_\varphi]_{\varphi=0}, \end{aligned} \quad (4.4.41)$$

together with

$$\begin{aligned}
[\partial_\varphi c_\varphi]_{\varphi=0} &= \langle \psi_*, A_1 \Phi_*' \rangle, \\
[\partial_\varphi^2 c_\varphi]_{\varphi=0} &= \langle \psi_*, D^2 f(\tau \Phi_*) [s_0^\diamond \tau \Phi_*' + \tau [\partial_\varphi \Phi_\varphi]_{\varphi=0}, s_0^\diamond \tau \Phi_* + \tau [\partial_\varphi \Phi_\varphi]_{\varphi=0}] \rangle \\
&\quad + \langle \psi_*, A_2 \Phi_*'' \rangle + \langle \psi_*, B_1 \Phi_*' \rangle + 2 \langle \psi_*, A_1 [\partial_\varphi \Phi_\varphi']_{\varphi=0} \rangle \\
&\quad - 2 [\partial_\varphi c_\varphi]_{\varphi=0} \langle \psi_*, [\partial_\varphi \Phi_\varphi']_{\varphi=0} \rangle.
\end{aligned} \tag{4.4.42}$$

Proof. Items (i)-(iii) of Lemma 4.2.2 can be established as in [65, Prop 3.7]. Differentiating the travelling wave MFDE (4.2.32), we find

$$[\partial_\varphi c_\varphi] \Phi_\varphi' = -c_\varphi [\partial_\varphi \Phi_\varphi'] + Df(\tau_\varphi \Phi_\varphi) [\tau_\varphi [\partial_\varphi \Phi_\varphi] + [\partial_\varphi \tau_\varphi] \Phi_\varphi], \tag{4.4.43}$$

together with

$$\begin{aligned}
[\partial_\varphi^2 c_\varphi] \Phi_\varphi' &= -c_\varphi [\partial_\varphi^2 \Phi_\varphi'] - 2 [\partial_\varphi c_\varphi] [\partial_\varphi \Phi_\varphi'] \\
&\quad + Df(\tau_\varphi \Phi_\varphi) [\tau_\varphi [\partial_\varphi^2 \Phi_\varphi] + 2 [\partial_\varphi \tau_\varphi] [\partial_\varphi \Phi_\varphi] + [\partial_\varphi^2 \tau_\varphi] \Phi_\varphi] \\
&\quad + D^2 f(\tau_\varphi \Phi_\varphi) [[\partial_\varphi \tau_\varphi] \Phi_\varphi + \tau_\varphi [\partial_\varphi \Phi_\varphi], [\partial_\varphi \tau_\varphi] \Phi_\varphi + \tau_\varphi [\partial_\varphi \Phi_\varphi]].
\end{aligned} \tag{4.4.44}$$

For any $p \in H^1$, we can compute

$$\begin{aligned}
[\partial_\varphi \tau_\varphi p]_{\varphi=0} &= \sigma_* \bar{\tau} (-\sin \phi_*, \cos \phi_*, \sin \phi_*, -\cos \phi_*) p' \\
&= \bar{\tau} (-\sigma_B, \sigma_A, \sigma_B, -\sigma_A) p' \\
&= \bar{\tau} s_0^\diamond p',
\end{aligned} \tag{4.4.45}$$

together with

$$\begin{aligned}
[\partial_\varphi^2 \tau_\varphi p]_{\varphi=0} &= \sigma_*^2 \bar{\tau} (\sin(\phi_*)^2, \cos(\phi_*)^2, \sin(\phi_*)^2, \cos(\phi_*)^2) p'' \\
&\quad + \sigma_* \bar{\tau} (-\cos(\phi_*), -\sin(\phi_*), \cos(\phi_*), \sin(\phi_*)) p' \\
&= \bar{\tau} (\sigma_B^2, \sigma_A^2, \sigma_B^2, \sigma_A^2) p'' + \bar{\tau} (-\sigma_A, -\sigma_B, \sigma_A, \sigma_B, 0) p' \\
&= \bar{\tau} (2[\partial_z s_z]_{z=0} - s_0^\diamond) p'' + \bar{\tau} (-\sigma_A, -\sigma_B, \sigma_A, \sigma_B, 0) p'.
\end{aligned} \tag{4.4.46}$$

Based on (4.4.14), we may hence write

$$\begin{aligned}
Df(\tau \Phi_*) [\partial_\varphi \tau_\varphi p]_{\varphi=0} &= A_1 p', \\
Df(\tau \Phi_*) [\partial_\varphi^2 \tau_\varphi p]_{\varphi=0} &= A_2 p'' + B_1 p'.
\end{aligned} \tag{4.4.47}$$

Using these identities to evaluate the expressions (4.4.43)-(4.4.44) at $\varphi = 0$ readily leads to the identities (4.4.41). The derivatives (4.4.42) can then be obtained by using the fact that $\langle \psi_*, \mathcal{L}_* y \rangle = 0$ for all $y \in H^1$. Item (iv) of Lemma 4.2.2 follows directly by comparing the first identities in (4.4.17) and (4.4.18) with those in (4.4.41) and (4.4.42). $\square$

We now construct a preliminary inverse for $\mathcal{L}_z$ that behaves as z^{-2} as $z \mapsto 0$. As a preparation, we implicitly define the remainder expressions $R_{\mathcal{L};i}$ by writing

$$\begin{aligned}\mathcal{L}_z &= \mathcal{L}_* + zR_{\mathcal{L};1}(z) \\ &= \mathcal{L}_* + zA_1 + z^2R_{\mathcal{L};2}(z) \\ &= \mathcal{L}_* + zA_1 + \tfrac{1}{2}z^2A_2 + z^3R_{\mathcal{L};3}(z).\end{aligned}\tag{4.4.48}$$

Lemma 4.4.6. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are satisfied. Pick a sufficiently large $K > 0$ together with a sufficiently small $\delta_z > 0$. Then there exists an analytic map

$$\{z \in \mathbb{C} : |z| < \delta_z\} \ni z \mapsto B_2(z) \in \mathcal{L}(L^2; H^1) \tag{4.4.49}$$

so that $v = B_2(z)h$ is the unique $v \in H^1$ that satisfies

$$\mathcal{L}_z v = z^2 h \tag{4.4.50}$$

whenever $h \in L^2$ and $0 < |z| < \delta_z$.

Proof. We set out to seek a solution a solution to (4.4.50) of the form

$$v = \kappa[\Phi'_* + z[\partial_z \phi_z]_{z=0}] + z^2 w \tag{4.4.51}$$

for some $\kappa \in \mathbb{R}$ and $w \in H^1$ that satisfies $\langle \psi_*, w \rangle = 0$. Writing

$$S(z) = R_{\mathcal{L};3}(z)\Phi'_* + R_{\mathcal{L};2}(z)[\partial_z \phi_z]_{z=0}, \tag{4.4.52}$$

we may use (4.4.17) to compute

$$z^{-2}\mathcal{L}_z v = \kappa\left(\tfrac{1}{2}A_2\Phi'_* + A_1[\partial_z \phi_z]_{z=0} + zS(z)\right) + \mathcal{L}_* w + zR_{\mathcal{L};1}(z)w. \tag{4.4.53}$$

The identity (4.4.50) is hence equivalent to the system

$$\begin{aligned}Q_* h &= zQ_* R_{\mathcal{L};1}(z)w + \kappa\left(\tfrac{1}{2}[\partial_z^2 \lambda_z]_{z=0} + zQ_* S(z)\right), \\ w &= \mathcal{L}^{\text{qinv}}[h - zR_{\mathcal{L};1}(z)w] + \tfrac{1}{2}\kappa[\partial_z^2 \phi_z]_{z=0} - z\kappa\mathcal{L}^{\text{qinv}}S(z).\end{aligned}\tag{4.4.54}$$

Whenever $|z|$ is sufficiently small, we may use the quantity

$$\nu(z) = \left[\tfrac{1}{2}[\partial_z^2 \lambda_z]_{z=0} + zQ_* S(z)\right]^{-1} \tag{4.4.55}$$

to rewrite the first line of (4.4.54) in the form

$$\kappa = \nu(z)Q_* h - z\nu(z)Q_* R_{\mathcal{L};1}(z)w. \tag{4.4.56}$$

Substituting this into the second line of (4.4.54), we find

$$w + zM(z)w = \mathcal{L}^{\text{qinv}}h + \tfrac{1}{2}\nu(z)[Q_* h][\partial_z^2 \phi_z]_{z=0} - z\nu(z)[Q_* h]\mathcal{L}^{\text{qinv}}S(z), \tag{4.4.57}$$

in which

$$M(z)w = \mathcal{L}^{\text{qinv}} R_{\mathcal{L};1}(z)w + z\nu(z)[Q_* R_{\mathcal{L};1}(z)w] \mathcal{L}^{\text{qinv}} S(z) - \frac{1}{2}\nu(z)[Q_* R_{\mathcal{L};1}(z)w][\partial_z^2 \phi_z]_{z=0}. \quad (4.4.58)$$

The desired properties now follow from the fact that $I + zM(z)$ is invertible whenever $|z|$ is sufficiently small. $\square$

In the following result we explicitly identify the singular $O(z^{-2})$ and $O(z^{-1})$ terms in the expansion of $\mathcal{L}_z^{-1}$. In order to express these in a convenient fashion, we introduce the operator $\Gamma_* : L^2 \rightarrow \mathbb{R}$ that acts as

$$\Gamma_* h = -Q_* A_1 \mathcal{L}^{\text{qinv}} h - \frac{1}{3} \frac{[\partial_z^3 \lambda_z]_{z=0}}{[\partial_z^2 \lambda_z]_{z=0}} Q_* h. \quad (4.4.59)$$

Lemma 4.4.7. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are satisfied and pick a sufficiently small constant $\delta_z > 0$. Then there exists an analytic function

$$\{z \in \mathbb{C} : |z| < \delta_z\} \ni z \mapsto \mathcal{J}_z \in \mathcal{L}(L^2; H^1) \quad (4.4.60)$$

so that we have

$$\mathcal{L}_z^{-1} = 2[\partial_z^2 \lambda_z]_{z=0}^{-1} \left[z^{-2}(\Phi'_* + z[\partial_z \phi_z]_{z=0})Q_* + z^{-1}\Phi'_* \Gamma_* \right] + \mathcal{J}_z \quad (4.4.61)$$

whenever $0 < |z| < \delta_z$.

Proof. Fix $z \neq 0$ together with $h \in L^2$ and consider the functions

$$\begin{aligned} v_A &= 2[\partial_z^2 \lambda_z]_{z=0}^{-1} \left[z^{-2}[\Phi'_* + z[\partial_z \phi_z]_{z=0}]Q_* h + z^{-1}\Phi'_* \Gamma_* h \right], \\ v_B &= \mathcal{L}^{\text{qinv}} h - z\mathcal{L}^{\text{qinv}} A_1 \mathcal{L}^{\text{qinv}} h, \\ v_C &= 2[\partial_z^2 \lambda_z]_{z=0}^{-1} \left[\frac{1}{2}[\partial_z^2 \phi_z]_{z=0}Q_* f + [\partial_z \phi_z]_{z=0}\Gamma_* f \right] \\ &\quad + 2z[\partial_z^2 \lambda_z]_{z=0}^{-1} \left[\frac{1}{6}[\partial_z^3 \phi_z]_{z=0}Q_* h + \frac{1}{2}[\partial_z^2 \phi_z]_{z=0}\Gamma_* h \right] \\ &\quad - z\mathcal{L}^{\text{qinv}} [\partial_z \phi_z]_{z=0}Q_* h. \end{aligned} \quad (4.4.62)$$

Performing the expansion

$$\mathcal{L}_z v_{\#} = \mathcal{E}_{\#;0} + z\mathcal{E}_{\#;1} + z^2\mathcal{R}_{\#;2}(z) \quad (4.4.63)$$

and demanding that $\mathcal{R}_{\#;2}(z)$ is analytic in zero for $\# \in \{A, B, C\}$, we can use Lemma 4.4.2 to compute

$$\begin{aligned} \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0}\mathcal{E}_{A;0} &= \left(\frac{1}{2}A_2\Phi'_* + A_1[\partial_z \phi_z]_{z=0} \right) Q_* h + A_1\Phi'_* \Gamma_* h, \\ \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0}\mathcal{E}_{A;1} &= \left(\frac{1}{6}A_3\Phi'_* + \frac{1}{2}A_2[\partial_z \phi_z]_{z=0} \right) Q_* h + \frac{1}{2}A_2\Phi'_* \Gamma_* h, \end{aligned} \quad (4.4.64)$$

together with

$$\begin{aligned}\mathcal{E}_{B;0} &= h - \Phi'_* Q_* h, \\ \mathcal{E}_{B;1} &= \Phi'_* Q_* A_1 \mathcal{L}^{\text{qinv}} h\end{aligned}\tag{4.4.65}$$

and finally

$$\begin{aligned}\frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \mathcal{E}_{C;0} &= -\left(\frac{1}{2}A_2 \Phi'_* + A_1[\partial_z \phi_z]_{z=0}\right) Q_* f + \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \Phi'_* Q_* f - A_1 \Phi'_* \Gamma_* h, \\ \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \mathcal{E}_{C;1} &= \frac{1}{2}A_1[\partial_z^2 \phi_z]_{z=0} Q_* h + A_1[\partial_z \phi_z]_{z=0} \Gamma_* h \\ &\quad - \left(\frac{1}{6}A_3 \Phi'_* + \frac{1}{2}A_2[\partial_z \phi_z]_{z=0} + \frac{1}{2}A_1[\partial_z^2 \phi_z]_{z=0}\right) Q_* h + \frac{1}{6}[\partial_z^3 \lambda_z]_{z=0} \Phi'_* Q_* h \\ &\quad - \left(\frac{1}{2}A_2 \Phi'_* + A_1[\partial_z \phi_z]_{z=0}\right) \Gamma_* h + \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \Phi'_* \Gamma_* h.\end{aligned}\tag{4.4.66}$$

Summing these expressions we see that

$$\begin{aligned}\mathcal{E}_{A;0} + \mathcal{E}_{B;0} + \mathcal{E}_{C;0} &= h, \\ \mathcal{E}_{A;1} + \mathcal{E}_{B;1} + \mathcal{E}_{C;1} &= \Phi'_* Q_* A_1 \mathcal{L}^{\text{qinv}} h + \Phi'_* \Gamma_* h + \frac{1}{3}[\partial_z^2 \lambda_z]_{z=0}^{-1} [\partial_z^3 \lambda_z]_{z=0} \Phi'_* Q_* h \\ &= 0,\end{aligned}\tag{4.4.67}$$

where we used (4.4.59) to simplify the second expression.

In particular, Lemma 4.4.6 allows us to write

$$\mathcal{L}_z^{-1} h = v_A + v_B + v_C - B_2(z) [\mathcal{R}_{A;2}(z) + \mathcal{R}_{B;2}(z) + \mathcal{R}_{C;2}(z)],\tag{4.4.68}$$

from which the desired expansion follows. $\square$

Proof of Proposition 4.4.1. For any $\delta_z > 0$ and sufficiently small $\eta_{\max} > 0$, (HS1) implies that $\mathcal{L}_z$ is invertible for $|\operatorname{Re} z| \leq \eta_{\max}$ and $\delta_z \leq |\operatorname{Im} z| \leq \pi$. The result now follows from the expansion obtained for $\mathcal{L}_z^{-1}$ in Lemma 4.4.7. $\square$

We now proceed towards establishing the representation formula (4.4.32). To this end, we introduce the left-shift operator S that acts on sequences as

$$[SV]_l = V_{l+1}.\tag{4.4.69}$$

Lemma 4.4.8. Consider any sequence $y \in BX_{\mu,\nu}(\mathcal{H})$ and pick an integer $k \geq 1$. Then we have the identities

$$\begin{aligned}\mathcal{Z}_+[S^k y](z) &= e^{zk} \mathcal{Z}_+[y](z) - \sum_{j=0}^{k-1} e^{z(k-j)} v_j, \\ \mathcal{Z}_+[S^{-k} y](z) &= e^{-zk} \mathcal{Z}_+[y](z) + \sum_{j=1}^k e^{-z(k-j)} v_{-j}\end{aligned}\tag{4.4.70}$$

whenever $\operatorname{Re} z > \nu$, together with

$$\begin{aligned} \mathcal{Z}_-[S^k y](z) &= e^{zk} \mathcal{Z}_-[y](z) + \sum_{j=0}^{k-1} e^{z(k-j)} v_j, \\ \mathcal{Z}_-[S^{-k} y](z) &= e^{-zk} \mathcal{Z}_-[y](z) - \sum_{j=1}^k e^{-z(k-j)} v_{-j} \end{aligned} \quad (4.4.71)$$

whenever $\operatorname{Re} z < \mu$.

Proof. Upon computing

$$\begin{aligned} \sum_{j \geq 0} e^{-zj} y_{j+k} &= \sum_{j' \geq k} e^{zk} e^{-zj'} y_{j'} \\ &= e^{zk} \sum_{j' \geq 0} e^{-zj'} y_{j'} - e^{zk} \sum_{j'=0}^{k-1} e^{-zj'} y_{j'} \end{aligned} \quad (4.4.72)$$

together with

$$\begin{aligned} \sum_{j \geq 0} e^{-zj} y_{j-k} &= \sum_{j' \geq -k} e^{-zk} e^{-zj'} y_{j'} \\ &= e^{-zk} \sum_{j' \geq 0} e^{-zj'} y_{j'} + e^{-zk} \sum_{j=1}^k e^{zj} y_{-j}, \end{aligned} \quad (4.4.73)$$

the identities (4.4.70) readily follow. In addition, we compute

$$\begin{aligned} \sum_{j \geq 1} e^{zj} y_{-j+k} &= \sum_{j' \geq 1-k} e^{zk} e^{zj'} y_{-j'} \\ &= e^{zk} \sum_{j' \geq 1} e^{zj'} y_{-j'} + e^{zk} \sum_{j=0}^{k-1} e^{-zj} y_j \end{aligned} \quad (4.4.74)$$

together with

$$\begin{aligned} \sum_{j \geq 1} e^{zj} y_{-j-k} &= \sum_{j' \geq k+1} e^{-zk} e^{zj'} y_{-j'} \\ &= e^{-zk} \sum_{j' \geq 1} e^{zj'} y_{-j'} - e^{-zk} \sum_{j'=1}^k e^{zj'} y_{-j'}, \end{aligned} \quad (4.4.75)$$

from which (4.4.71) follows. $\square$

We now introduce for any $z \in \mathbb{C}$ and any $w : \{-\sigma_* + 1, \dots, \sigma_*\} \rightarrow H^1$ the notation

$$\begin{aligned} q_z[w] = & \left(\sum_{j=1}^{\sigma_B-1} w_{-j} \sum_{k=j}^{\sigma_B-1} e^{-z(k-j)}, \sum_{j=0}^{\sigma_A-1} w_j \sum_{k=j+1}^{\sigma_A} e^{z(k-j)}, \right. \\ & \left. \sum_{j=0}^{\sigma_B-1} w_j \sum_{k=j+1}^{\sigma_B} e^{z(k-j)}, \sum_{j=1}^{\sigma_A-1} w_{-j} \sum_{k=j}^{\sigma_A-1} e^{-z(k-j)}, 0 \right). \end{aligned} \quad (4.4.76)$$

Corollary 4.4.9. Consider any sequence $w \in BX_{\mu,\nu}(H^1)$ and pick an integer $k \geq 1$. Then we have the identity

$$\mathcal{Z}_+[s^\diamond[w]](z) = s_z^\diamond \mathcal{Z}_+[w](z) - q_z \operatorname{ev}_0 w \quad (4.4.77)$$

whenever $\operatorname{Re} z > \nu$, together with

$$\mathcal{Z}_-[s^\diamond[w]](z) = s_z^\diamond \mathcal{Z}_-[w](z) + q_z \operatorname{ev}_0 w \quad (4.4.78)$$

whenever $\operatorname{Re} z < \mu$.

Proof. For the first compoment, we compute

$$\begin{aligned}
 \sum_{j=0}^{\sigma_B-1} \mathcal{Z}_+[S^{-j}w](z) &= \sum_{j=0}^{\sigma_B-1} e^{-jz} \mathcal{Z}_+[w](z) + \sum_{j=1}^{\sigma_B-1} \sum_{j'=1}^j e^{-z(j-j')} w_{-j'} \\
 &= \sum_{j=0}^{\sigma_B-1} e^{-jz} \mathcal{Z}_+[w](z) + \sum_{j'=1}^{\sigma_B-1} w_{-j'} \sum_{j=j'}^{\sigma_B-1} e^{-z(j-j')}.
 \end{aligned} \tag{4.4.79}$$

For the second component, we note that

$$\begin{aligned}
 \sum_{j=1}^{\sigma_A} \mathcal{Z}_+[S^jw](z) &= \sum_{j=1}^{\sigma_A} e^{jz} \mathcal{Z}_+[w](z) - \sum_{j=1}^{\sigma_A} \sum_{j'=0}^{j-1} e^{z(j-j')} w_{j'} \\
 &= \sum_{j=1}^{\sigma_A} e^{jz} \mathcal{Z}_+[w](z) - \sum_{j'=0}^{\sigma_A-1} w_{j'} \sum_{j=j'+1}^{\sigma_A} e^{z(j-j')}.
 \end{aligned} \tag{4.4.80}$$

The desired expression follows directly from these computations, noting that the third and fourth components can be obtained by flipping σ_A and σ_B in the expressions for the second respectively first component. $\square$

Corollary 4.4.10. Consider any $w \in H^1$. Then we have the identities

$$\begin{aligned}
 q_0[w\mathbf{1}] &= [\partial_z s_z^\diamond]_{z=0} w, \\
 [\partial_z q_z]_{z=0}[w\mathbf{1}] &= \frac{1}{2} \left([\partial_z s_z^\diamond]_{z=0} + [\partial_z^2 s_z^\diamond]_{z=0} \right) w.
 \end{aligned} \tag{4.4.81}$$

Proof. The expressions follow from the direct computation

$$\begin{aligned}
 q_0[w\mathbf{1}] &= \left(\sum_{j=1}^{\sigma_B-1} (\sigma_B - j), \sum_{j=0}^{\sigma_A-1} (\sigma_A - j), \sum_{j=0}^{\sigma_B-1} (\sigma_B - j), \sum_{j=1}^{\sigma_A-1} (\sigma_A - j), 0 \right) w \\
 &= \left(\sum_{j=1}^{\sigma_B-1} j, \sum_{j=1}^{\sigma_A} j, \sum_{j=1}^{\sigma_B} j, \sum_{j=1}^{\sigma_A-1} j, 0 \right) w \\
 &= [\partial_z s_z^\diamond]_{z=0} w,
 \end{aligned} \tag{4.4.82}$$

together with

$$\begin{aligned}
[\partial_z q_z]_{z=0}[w\mathbf{1}] &= \left(- \sum_{j=1}^{\sigma_B-1} \sum_{k=j}^{\sigma_B-1} (k-j), \sum_{j=0}^{\sigma_A-1} \sum_{k=j+1}^{\sigma_A} (k-j), \right. \\
&\quad \left. \sum_{j=0}^{\sigma_B-1} \sum_{k=j+1}^{\sigma_B} (k-j), - \sum_{j=1}^{\sigma_A-1} \sum_{k=j}^{\sigma_A-1} (k-j), 0 \right) w \\
&= \frac{1}{2} \left(- \sum_{j=1}^{\sigma_B-1} (\sigma_B - 1 - j)(\sigma_B - j), \sum_{j=0}^{\sigma_A-1} \frac{1}{2} (\sigma_A - j)(\sigma_A + 1 - j), \right. \\
&\quad \left. \sum_{j=0}^{\sigma_B-1} \frac{1}{2} (\sigma_B - j)(\sigma_B + 1 - j), - \sum_{j=1}^{\sigma_A-1} (\sigma_A - 1 - j)(\sigma_A - j), 0 \right) w \\
&= \frac{1}{2} q_0[w\mathbf{1}] + \frac{1}{2} \left(- \sum_{j=1}^{\sigma_B-1} (\sigma_B - j)^2, \sum_{j=0}^{\sigma_A-1} (\sigma_A - j)^2, \right. \\
&\quad \left. \sum_{j=0}^{\sigma_B-1} (\sigma_B - j)^2, - \sum_{j=1}^{\sigma_A-1} (\sigma_A - j)^2, 0 \right) w \\
&= \frac{1}{2} \left([\partial_z s_z^\diamond]_{z=0} + [\partial_z^2 s_z^\diamond]_{z=0} \right) w.
\end{aligned} \tag{4.4.83}$$

□

For any pair of sequences

$$(v, w) : \{-\sigma_* + 1, \dots, \sigma_*\} \rightarrow H^1 \times H^1 \tag{4.4.84}$$

we introduce the notation

$$\mathcal{Q}(z)[v, w] = \left((v_0 - w_0)e^z, Df(\tau\Phi_*)\bar{\tau}q_z w \right)^T. \tag{4.4.85}$$

This allows us to take the two discrete Laplace transforms of the main linear system (4.4.1).

Corollary 4.4.11. Assume that (Hf) and $(H\Phi)$ are satisfied. Pick a pair $\mu, \nu \in \mathbb{R}$, consider any $V \in BX_{\mu, \nu}(H^1 \times H^1)$ and write

$$H = \mathcal{D}(0)[V] - \mathcal{T}(0)[V]. \tag{4.4.86}$$

Then we have the identity

$$\Delta(z)\mathcal{Z}_+[V](z) = \mathcal{Q}(z)\text{ev}_0 V + \mathcal{Z}_+[H](z) \tag{4.4.87}$$

whenever $\text{Re } s > \nu$, together with

$$\Delta(z)\mathcal{Z}_-[V](z) = -\mathcal{Q}(z)\text{ev}_0 V + \mathcal{Z}_-[H](z) \tag{4.4.88}$$

whenever $\text{Re } z < \mu$.

Proof. Writing $V = (v, w)$ and $H = (g, h)$, we may use Lemma 4.4.8 and Corollary 4.4.9 to compute

$$\begin{aligned} (e^z - 1)\mathcal{Z}_+[v](z) - v_0 e^z &= e^z(\mathcal{Z}_+[w](z) - w_0) + \mathcal{Z}_+[g](z), \\ Df(\tau\Phi_*)\bar{\tau}[s_z^\diamond \mathcal{Z}_+[w](z) - q_z w] &= -\mathcal{L}_* \mathcal{Z}_+[v](z) + \mathcal{Z}_+[h](z), \end{aligned} \quad (4.4.89)$$

which is equivalent to (4.4.87). The remaining identity (4.4.88) follows in a similar fashion. $\square$

For convenience, we introduce the linear operators

$$\begin{aligned} M_0[v, w] &= Df(\tau\Phi_*)\bar{\tau}\left[s_0^\diamond(v_0 - w_0) + q_0^\diamond w\right], \\ M_1[v, w] &= Df(\tau\Phi_*)\bar{\tau}\left[\partial_z s_z^\diamond|_{z=0}(v_0 - w_0) + [\partial_z q_z^\diamond]_{z=0} w\right], \end{aligned} \quad (4.4.90)$$

together with the projections

$$\begin{aligned} \pi_A &= 2[\partial_z^2 \lambda_z]_{z=0}^{-1} [Q_* M_1 + \Gamma_* M_0], \\ \pi_B &= 2[\partial_z^2 \lambda_z]_{z=0}^{-1} Q_* M_0. \end{aligned} \quad (4.4.91)$$

Lemma 4.4.12. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are satisfied. Then we have the identity

$$\text{Res}_{z=0} e^{zl} \Delta(z)^{-1} \mathcal{Q}(z) = [V_{\text{hom}}^A]_l \pi_A + [V_{\text{hom}}^B]_l \pi_B. \quad (4.4.92)$$

Proof. Pick any $z \in \mathbb{Z}$ for which $\mathcal{L}_z$ is invertible. Upon introducing the representation

$$e^{zl} \Delta(z)^{-1} \mathcal{Q}(z)[v, w] = \mathcal{B}_l(z)(v_0 - w_0, w)^T \quad (4.4.93)$$

with

$$\mathcal{B}_l(z) = \begin{pmatrix} [\mathcal{B}_l(z)]_{1,1} & [\mathcal{B}_l(z)]_{1,2} \\ [\mathcal{B}_l(z)]_{2,1} & [\mathcal{B}_l(z)]_{2,2} \end{pmatrix}, \quad (4.4.94)$$

one can use (4.4.36) and (4.4.85) to verify the expressions

$$\begin{aligned} [\mathcal{B}_l(z)]_{1,1} &= e^{zl} \mathcal{L}_z^{-1} Df(\tau\Phi_*)\bar{\tau} s_z^\diamond, \\ [\mathcal{B}_l(z)]_{1,2} &= e^{zl} \mathcal{L}_z^{-1} Df(\tau\Phi_*)\bar{\tau} q_z^\diamond, \\ [\mathcal{B}_l(z)]_{2,1} &= -e^{zl} + e^{zl}(1 - e^{-z}) \mathcal{L}_z^{-1} Df(\tau\Phi_*)\bar{\tau} s_z^\diamond, \\ [\mathcal{B}_l(z)]_{2,2} &= e^{zl}(1 - e^{-z}) \mathcal{L}_z^{-1} Df(\tau\Phi_*)\bar{\tau} q_z^\diamond. \end{aligned} \quad (4.4.95)$$

Using the Laurent expansion (4.4.61), we readily compute

$$\begin{aligned} \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \text{Res}_{z=0} [\mathcal{B}_l(z)]_{1,1} &= l\Phi'_* Q_* Df(\tau\Phi_*)\bar{\tau} s_0^\diamond + \Phi'_* Q_* Df(\tau\Phi_*)\bar{\tau} [\partial_z s_z^\diamond]_{z=0} \\ &\quad + [\partial_z \phi_z]_{z=0} Q_* Df(\tau\Phi_*)\bar{\tau} s_0^\diamond + \Phi'_* \Gamma_* Df(\tau\Phi_*)\bar{\tau} s_0^\diamond, \\ \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \text{Res}_{z=0} [\mathcal{B}_l(z)]_{2,1} &= \Phi'_* Q_* Df(\tau\Phi_*)\bar{\tau} s_0^\diamond, \end{aligned} \quad (4.4.96)$$

together with

$$\begin{aligned}
\frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \text{Res}_{z=0}[\mathcal{B}_l(z)]_{1,2} &= l\Phi'_* Q_* Df(\tau\Phi_*) \bar{\tau} q_0^\diamond + \Phi'_* Q_* Df(\tau\Phi_*) \bar{\tau} [\partial_z q_z^\diamond]_{z=0} \\
&\quad + [\partial_z \phi_z]_{z=0} Q_* Df(\tau\Phi_*) \bar{\tau} q_0^\diamond + \Phi'_* \Gamma_* Df(\tau\Phi_*) \bar{\tau} q_0^\diamond, \\
\frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \text{Res}_{z=0}[\mathcal{B}_l(z)]_{2,2} &= \Phi'_* Q_* Df(\tau\Phi_*) \bar{\tau} q_0^\diamond.
\end{aligned} \tag{4.4.97}$$

The desired expressions now follow readily. $\square$

Lemma 4.4.13. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are satisfied. Then we have the identities

$$\pi_A[\text{ev}_0 V_{\text{hom}}^A] = \pi_B[\text{ev}_0 V_{\text{hom}}^B] = 1, \quad \pi_A[\text{ev}_0 V_{\text{hom}}^B] = \pi_B[\text{ev}_0 V_{\text{hom}}^A] = 0. \tag{4.4.98}$$

Proof. As preparations, we use Lemma 4.4.2 to compute

$$\begin{aligned}
Q_* A_1 \Phi'_* &= 0, \\
Q_* A_1 [\partial_z \phi_z]_{z=0} &= \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} - \frac{1}{2} Q_* A_2 \Phi'_*, \\
Q_* A_1 [\partial_z^2 \phi_z]_{z=0} &= \frac{1}{3}[\partial_z^3 \lambda_z]_{z=0} - \frac{1}{3} Q_* A_3 \Phi'_* - Q_* A_2 [\partial_z \phi_z]_{z=0},
\end{aligned} \tag{4.4.99}$$

together with

$$\begin{aligned}
\Gamma_* A_1 \Phi'_* &= Q_* A_1 [\partial_z \phi_z]_{z=0} \\
&= \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} - \frac{1}{2} Q_* A_2 \Phi'_*, \\
\Gamma_* A_2 \Phi'_* &= -\frac{1}{3}[\partial_z^2 \lambda_z]_{z=0}^{-1} [\partial_z^3 \lambda_z]_{z=0} Q_* A_2 \Phi'_* - Q_* A_1 \mathcal{L}^{\text{qinv}} A_2 \Phi'_*, \\
\Gamma_* A_1 \phi_0^{(1)} &= -\frac{1}{3}[\partial_z^2 \lambda_z]_{z=0}^{-1} [\partial_z^3 \lambda_z]_{z=0} Q_* A_1 [\partial_z \phi_z]_{z=0} + \frac{1}{2} Q_* A_1 [\partial_z^2 \phi_z]_{z=0} \\
&\quad + \frac{1}{2} Q_* A_1 \mathcal{L}^{\text{qinv}} A_2 \Phi'_* \\
&= +\frac{1}{6}[\partial_z^2 \lambda_z]_{z=0}^{-1} [\partial_z^3 \lambda_z]_{z=0} Q_* A_2 \Phi'_* - \frac{1}{6} Q_* A_3 \Phi'_* - \frac{1}{2} Q_* A_2 [\partial_z \phi_z]_{z=0} \\
&\quad + \frac{1}{2} Q_* A_1 \mathcal{L}^{\text{qinv}} A_2 \Phi'_*.
\end{aligned} \tag{4.4.100}$$

Using Corollary 4.4.10 together with (4.4.14), we also compute

$$\begin{aligned}
M_0 \text{ev}_0 V_{\text{hom}}^A &= A_1 \Phi'_*, \\
M_1 \text{ev}_0 V_{\text{hom}}^A &= \frac{1}{2}(A_1 + A_2) \Phi'_*,
\end{aligned} \tag{4.4.101}$$

together with

$$\begin{aligned}
M_0 \text{ev}_0 V_{\text{hom}}^B &= A_1([\partial_z \phi_z]_{z=0} - \Phi'_*) + \frac{1}{2}(A_1 + A_2)\Phi'_* \\
&= A_1[\partial_z \phi_z]_{z=0} + \frac{1}{2}(A_2 - A_1)\Phi'_*, \\
M_1 \text{ev}_0 V_{\text{hom}}^B &= \frac{1}{2}(A_1 + A_2)([\partial_z \phi_z]_{z=0} - \Phi'_*) + \frac{1}{6}(A_3 + 3A_2 + 2A_1)\Phi'_* \\
&= \frac{1}{2}(A_1 + A_2)[\partial_z \phi_z]_{z=0} + \frac{1}{6}(A_3 - A_1)\Phi'_*.
\end{aligned} \tag{4.4.102}$$

In particular, we find

$$\begin{aligned}
\frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \pi_A [V_{\text{hom}}^A] &= \frac{1}{2}Q_* A_2 \Phi'_* + \Gamma_* A_1 \Phi'_* = \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0}, \\
\frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \pi_B [V_{\text{hom}}^A] &= 0,
\end{aligned} \tag{4.4.103}$$

together with

$$\begin{aligned}
\frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \pi_A [V_{\text{hom}}^B] &= \frac{1}{2}Q_*(A_1 + A_2)[\partial_z \phi_z]_{z=0} + \frac{1}{6}Q_* A_3 \Phi'_* + \Gamma_* A_1 [\partial_z \phi_z]_{z=0} \\
&\quad + \frac{1}{2}\Gamma_*(A_2 - A_1)\Phi'_* \\
&= \frac{1}{4}[\partial_z^2 \lambda_z]_{z=0} - \frac{1}{4}Q_* A_2 \Phi'_* + \frac{1}{2}Q_* A_2 [\partial_z \phi_z]_{z=0} + \frac{1}{6}Q_* A_3 \Phi'_* \\
&\quad + \frac{1}{6} \frac{[\partial_z^3 \lambda_z]_{z=0}}{[\partial_z^2 \lambda_z]_{z=0}} Q_* A_2 \Phi'_* - \frac{1}{6}Q_* A_3 \Phi'_* - \frac{1}{2}Q_* A_2 [\partial_z \phi_z]_{z=0} \\
&\quad + \frac{1}{2}Q_* A_1 \mathcal{L}^{\text{qinv}} A_2 \Phi'_* - \frac{1}{6} \frac{[\partial_z^3 \lambda_z]_{z=0}}{[\partial_z^2 \lambda_z]_{z=0}} Q_* A_2 \Phi'_* \\
&\quad - \frac{1}{2}Q_* A_1 \mathcal{L}^{\text{qinv}} A_2 \Phi'_* - \frac{1}{4}[\partial_z^2 \lambda_z]_{z=0} + \frac{1}{4}Q_* A_2 \Phi'_* \\
&= 0, \\
\frac{1}{2}[\partial_z^2 \lambda_z]_{z=0} \pi_B [V_{\text{hom}}^B] &= Q_* A_1 [\partial_z \phi_z]_{z=0} + \frac{1}{2}Q_* A_2 \Phi'_* \\
&= \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0}.
\end{aligned} \tag{4.4.104}$$

□

Proof of Proposition 4.4.3. Applying the inverse Laplace transform to the identity (4.4.87), the representation formula (4.4.32) follows directly from the Cauchy integral formula and Lemmas 4.4.12-4.4.13. □

4.5 Inhomogeneous linear system for constant θ

In this section we are interested in the constant-coefficient linear system

$$\mathcal{D}(\vartheta \mathbf{1})[V] = [1 - P_\vartheta^{(1)}]\mathcal{T}(\vartheta \mathbf{1})[V] + H, \tag{4.5.1}$$

in which $\vartheta \in \mathbb{R}$. This equation can be seen as the linear part of the (v, w) system in our main problem (4.3.42) for the special case of a constant sequence $\theta_l = \vartheta$.

Our main result constructs a solution operator for this system acting on sequences that are allowed to grow at a small exponential rate. In order to accomodate this, we introduce the notation

$$\begin{aligned} BS_\eta(\mathcal{H}) &= BX_{-\eta, \eta}(\mathcal{H}) \\ &= \{V : \mathbb{Z} \rightarrow \mathcal{H} \text{ for which } \|V\|_\eta := \sup e^{-\eta|j|} \|V_j\|_{\mathcal{H}} < \infty\} \end{aligned} \quad (4.5.2)$$

for any Hilbert space $\mathcal{H}$. In addition, we recall the definitions (4.3.40) for the projections $Q_\vartheta^{(1)}$ and $Q_\vartheta^{(2)}$.

Proposition 4.5.1. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are satisfied and recall the constant r appearing in (Hf). Pick a sufficiently small $\eta_{\max} > 0$ together with a sufficiently large $K > 0$. Then for every $0 < \eta < \eta_{\max}$ there exists a C^{r-1} -smooth map

$$\mathcal{K}_\eta^{\text{cc}} : \mathbb{R} \mapsto \mathcal{L}(BS_\eta(H^1 \times L^2); BS_\eta(H^1 \times H^1)) \quad (4.5.3)$$

that satisfies the following properties.

- (i) Pick any $0 < \eta < \eta_{\max}$ and $\vartheta \in \mathbb{R}$. For any $H \in BS_\eta(H^1 \times L^2)$, the function $V = \mathcal{K}_\eta^{\text{cc}}(\vartheta)H$ satisfies (4.5.1) and admits the orthogonality conditions

$$Q_\vartheta^{(1)} V_0 = Q_\vartheta^{(2)} V_0 = 0. \quad (4.5.4)$$

- (ii) Pick any $0 < \eta < \eta_{\max}$ and assume that $V \in BS_\eta(H^1 \times H^1)$ satisfies (4.5.1) with $H = 0$ for some $\vartheta \in \mathbb{R}$. Then there exists a pair $(a_1, a_2) \in \mathbb{R}^2$ for which we have

$$V_j = a_1 T_\vartheta(\Phi'_{**}, 0) + a_2 T_\vartheta([\partial_z \phi_z]_{z=0}, \Phi'_*). \quad (4.5.5)$$

- (iii) For any $0 < \eta < \eta_{\max}$ and $\vartheta \in \mathbb{R}$ we have the bound

$$\|\mathcal{K}_\eta^{\text{cc}}(\vartheta)\|_{\mathcal{L}(BS_\eta(H^1 \times L^2); BS_\eta(H^1 \times H^1))} < K\eta^{-3}. \quad (4.5.6)$$

- (iv) Pick any $0 < \eta < \eta_{\max}$. Then for any pair $(\vartheta_1, \vartheta_2) \in \mathbb{R}^2$, we have the identity

$$\mathcal{K}_\eta^{\text{cc}}(\vartheta_1) = T_{\vartheta_1 - \vartheta_2} \mathcal{K}_\eta^{\text{cc}}(\vartheta_2) T_{\vartheta_2 - \vartheta_1}. \quad (4.5.7)$$

- (v) Consider a pair $(\eta_1, \eta_2) \in (0, \eta_{\max})^2$ together with a function

$$H \in BS_{\eta_1}(H^1 \times L^2) \cap BS_{\eta_2}(H^1 \times L^2). \quad (4.5.8)$$

Then for any $\vartheta \in \mathbb{R}$ we have

$$\mathcal{K}_{\eta_1}^{\text{cc}}(\vartheta)H = \mathcal{K}_{\eta_2}^{\text{cc}}(\vartheta)H. \quad (4.5.9)$$

Our strategy is to exploit the representation formula derived in §4.4 for the unprojected problem

$$\mathcal{D}(0)[V] - \mathcal{T}(0)[V] = H. \quad (4.5.10)$$

In particular, we first use the Fourier symbols $\Delta(z)$ defined in (4.4.5) to construct an inverse in the sequence spaces

$$\ell_\eta^2(\mathcal{H}) = \{V : \mathbb{Z} \rightarrow \mathcal{H} \text{ for which } \|V\|_{\ell_\eta^2}^2 := \sum_{l \in \mathbb{Z}} e^{-2\eta l} \|V_l\|_{\mathcal{H}}^2 < \infty\}, \quad (4.5.11)$$

where again $\mathcal{H}$ is a Hilbert space. This can subsequently be used to obtain an inverse in the spaces

$$\ell_\eta^\infty(\mathcal{H}) = \{V : \mathbb{Z} \rightarrow \mathcal{H} \text{ for which } \|V\|_{\ell_\eta^\infty} := \sup_{l \in \mathbb{Z}} e^{-\eta l} \|V_l\|_{\mathcal{H}} < \infty\} \quad (4.5.12)$$

by exploiting the fact that interactions between lattice sites decay exponentially with respect to the separation distance.

Lemma 4.5.2. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are satisfied and pick a sufficiently small $\eta_{\max} > 0$ together with a sufficiently large $K > 0$. Then for every η with $0 < |\eta| < \eta_{\max}$, there exists a bounded operator

$$\Lambda_\eta^{\text{inv}} : \ell_\eta^2(H^1 \times L^2) \rightarrow \ell_\eta^2(H^1 \times H^1) \quad (4.5.13)$$

that satisfies the following properties.

(i) For any $H \in \ell_\eta^2(H^1 \times L^2)$, the sequence $V = \Lambda_\eta^{\text{inv}} H$ satisfies (4.5.10).

(ii) We have the bound

$$\|\Lambda_\eta^{\text{inv}}\|_{\mathcal{L}(\ell_\eta^2(H^1 \times L^2), \ell_\eta^2(H^1 \times H^1))} \leq K\eta^{-2}. \quad (4.5.14)$$

(iii) We have the explicit expression

$$[\Lambda_\eta^{\text{inv}} H]_l = \frac{1}{2\pi i} \int_{\eta-i\pi}^{\eta+i\pi} e^{zl} \Delta(z)^{-1} [\mathcal{Z}_+[H](z) + \mathcal{Z}_-[H](z)] dz. \quad (4.5.15)$$

Proof. This follows directly from Proposition 4.4.1 and standard properties of the Fourier transform; see for example [76, §3]. $\square$

Lemma 4.5.3. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are all satisfied. Pick a sufficiently small $\eta_{\max} > 0$ together with a sufficiently large $K > 0$. Then for every pair $0 < \eta_1 < \eta_2 < \eta_{\max}$ and any

$$H \in \ell_{\eta_1}^\infty(H^1 \times L^2) \cap \ell_{\eta_2}^2(H^1 \times L^2) \quad (4.5.16)$$

we have the inclusion $\Lambda_{\eta_2}^{\text{inv}} H \in \ell_{\eta_1}^\infty(H^1 \times H^1)$, together with the bound

$$\|\Lambda_{\eta_2}^{\text{inv}} H\|_{\ell_{\eta_1}^\infty(H^1 \times H^1)} \leq K\eta_1^{-3} \|H\|_{\ell_{\eta_1}^\infty(H^1 \times L^2)}. \quad (4.5.17)$$

In addition, for any

$$H \in \ell_{-\eta_1}^\infty(H^1 \times L^2) \cap \ell_{-\eta_2}^2(H^1 \times L^2) \quad (4.5.18)$$

we have the inclusion $\Lambda_{-\eta_2}^{\text{inv}} H \in \ell_{-\eta_1}^\infty(H^1 \times H^1)$, together with the bound

$$\|\Lambda_{-\eta_2}^{\text{inv}} H\|_{\ell_{-\eta_1}^\infty(H^1 \times H^1)} \leq K\eta_1^{-3} \|H\|_{\ell_{-\eta_1}^\infty(H^1 \times L^2)}. \quad (4.5.19)$$

Proof. Following the approach in [71, Lem. 5.8], we introduce the sequences

$$H^{(k)} : \mathbb{Z} \rightarrow H^1 \times L^2, \quad V^{(k)} : \mathbb{Z} \rightarrow H^1 \times H^1 \quad (4.5.20)$$

for $k \in \mathbb{Z}$ by writing

$$H_l^{(k)} = \delta_{kl} H_l, \quad V^{(k)} = \Lambda_{\eta_2}^{\text{inv}} H^{(k)}. \quad (4.5.21)$$

In view of the convergence

$$\sum_{k \in \mathbb{Z}} H^{(k)} = H \in \ell_{\eta_2}^2(H^1 \times L^2), \quad (4.5.22)$$

the boundedness of $\Lambda_{\eta_2}^{\text{inv}}$ implies that also

$$\sum_{k \in \mathbb{Z}} V^{(k)} = V \in \ell_{\eta_2}^2(H^1 \times H^1) \quad (4.5.23)$$

and hence

$$\sum_{k \in \mathbb{Z}} V_l^{(k)} = V_l \in H^1 \times H^1 \quad (4.5.24)$$

for all $l \in \mathbb{Z}$.

We now pick two constants $\eta_\pm$ in such a way that

$$0 < \eta_- < \eta_1 < \eta_2 < \eta_+ < \eta_{\max}. \quad (4.5.25)$$

By construction, we have

$$H^{(k)} \in \ell_{\eta_-}^2(H^1 \times L^2) \cap \ell_{\eta_+}^2(H^1 \times L^2) \cap \ell_{\eta_2}^2(H^1 \times L^2). \quad (4.5.26)$$

Recalling the left-shift operator S defined in (4.4.69), we note that

$$S^k \Lambda_{\eta_2}^{\text{inv}} = \Lambda_{\eta_2}^{\text{inv}} S^k \quad (4.5.27)$$

In particular, we have

$$S^k V^{(k)} = S^k \Lambda_{\eta_2}^{\text{inv}} H^{(k)} = \Lambda_{\eta_2}^{\text{inv}} S^k H^{(k)} = \Lambda_{\eta_\pm}^{\text{inv}} S^k H^{(k)}. \quad (4.5.28)$$

Here the last identity follows from Proposition 4.4.3, since the sequence

$$Y = \Lambda_{\eta_2}^{\text{inv}} S^k H^{(k)} - \Lambda_{\eta_{\pm}}^{\text{inv}} S^k H^{(k)} \quad (4.5.29)$$

satisfies the inclusions

$$Y \in \ell_{\eta_2}^2(H^1 \times H^1) + \ell_{\eta_{\pm}}^2(H^1 \times H^1) \subset BX_{\eta_-, \eta_+}(H^1 \times H^1), \quad (4.5.30)$$

together with the homogeneous problem

$$\mathcal{D}(0)Y = \mathcal{T}(0)Y. \quad (4.5.31)$$

We are now able to use item (ii) of Lemma 4.5.2 to compute

$$\begin{aligned} \left\| e^{-\eta_1 l} V_l^{(k)} \right\|_{H^1 \times H^1} &= \left\| e^{-\eta_1 l} [S_k V^{(k)}]_{l-k} \right\|_{H^1 \times H^1} \\ &= \left\| e^{-\eta_1 l} [\Lambda_{\eta_{\pm}}^{\text{inv}} S_k H^{(k)}]_{l-k} \right\|_{H^1 \times H^1} \\ &\leq C_1 \eta_{\pm}^{-2} e^{-\eta_1 l} e^{-\eta_{\pm}(k-l)} \left\| S_k H^{(k)} \right\|_{\ell_{\eta_{\pm}}^2(H^1 \times L^2)} \\ &= C_1 \eta_{\pm}^{-2} e^{-\eta_1 l} e^{-\eta_{\pm}(k-l)} \|H_k\|_{H^1 \times L^2} \\ &= C_1 \eta_{\pm}^{-2} e^{-(\eta_{\pm} - \eta_1)(k-l)} e^{-\eta_1 k} \|H_k\|_{H^1 \times L^2} \\ &\leq C_1 \eta_{\pm}^{-2} e^{-(\eta_{\pm} - \eta_1)(k-l)} \|H\|_{\ell_{\eta_1}^{\infty}(H^1 \times L^2)} \end{aligned} \quad (4.5.32)$$

for some $C_1 > 0$. Summing over k , we hence find

$$\begin{aligned} \left\| e^{-\eta_1 l} V_l \right\|_{H^1 \times H^1} &\leq C_1 \|H\|_{\ell_{\eta_1}^{\infty}(H^1 \times L^2)} \\ &\quad \times \left[\eta_+^{-2} \sum_{k \geq l} e^{-(\eta_+ - \eta_1)(k-l)} + \sum_{k < l} \eta_-^{-2} e^{-(\eta_1 - \eta_-)(l-k)} \right] \\ &\leq C_1 \left[\eta_+^{-2} (\eta_+ - \eta_1)^{-1} + \eta_-^{-2} (\eta_1 - \eta_-)^{-1} \right] \|H\|_{\ell_{\eta_1}^{\infty}(H^1 \times L^2)}. \end{aligned} \quad (4.5.33)$$

The result follows directly from this bound, possibly after decreasing the size of $\eta_{\max} > 0$. $\square$

For any $H \in BS_{\eta}(H^1 \times L^2)$ we now introduce the splitting

$$H = H_{\geq 0} + H_{< 0} \quad (4.5.34)$$

by writing

$$[H_{\geq 0}]_l = \mathbf{1}_{l \geq 0} H_l, \quad [H_{< 0}]_l = \mathbf{1}_{l < 0} H_l. \quad (4.5.35)$$

We subsequently write

$$\mathcal{K}_{\eta; l}^{\text{up}} H = \Lambda_{\eta+\epsilon}^{\text{inv}} H_{\geq 0} + \Lambda_{-\eta-\epsilon}^{\text{inv}} H_{< 0} \quad (4.5.36)$$

for some small $\epsilon > 0$, which by construction implies that $V = \mathcal{K}_{\eta;I}^{\text{up}} H$ satisfies the unprojected problem (4.5.10) with $\vartheta = 0$. In addition, Lemma 4.5.3 implies that $V \in BS_{\eta}(H^1 \times H^1)$. In order to allow for any $\vartheta \in \mathbb{R}$, we introduce the operator

$$\mathcal{K}_{\eta;II}^{\text{up}}(\vartheta)H = T_{\vartheta}\mathcal{K}_{\eta;I}^{\text{up}}T_{-\vartheta}H. \quad (4.5.37)$$

In view of the orthogonality conditions (4.5.4), we finally write

$$\mathcal{K}_{\eta}^{\text{up}}(\vartheta)H = \mathcal{K}_{\eta;II}^{\text{up}}(\vartheta)H - T_{\vartheta}V_{\text{hom}}^A Q_{\vartheta}^{(1)} \text{pev}_0 \mathcal{K}_{\eta;II}^{\text{up}}(\vartheta)H - T_{\vartheta}V_{\text{hom}}^B Q_{\vartheta}^{(2)} \text{pev}_0 \mathcal{K}_{\eta;II}^{\text{up}}(\vartheta)H. \quad (4.5.38)$$

Lemma 4.5.4. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are all satisfied. Pick a sufficiently small $\eta_{\text{max}} > 0$ together with a sufficiently large $K > 0$. Then for any $0 < \eta < \eta_{\text{max}}$, any $\vartheta \in \mathbb{R}$ and any $H \in BS_{\eta}(H^1 \times L^2)$, the function $V = \mathcal{K}_{\eta}^{\text{up}}(\vartheta)H$ satisfies the unprojected problem (4.5.10) and admits the orthogonality conditions

$$Q_{\vartheta}^{(1)}V_0 = Q_{\vartheta}^{(2)}V_0 = 0. \quad (4.5.39)$$

In addition, properties (iii) - (v) from Proposition 4.5.1 are satisfied after replacing $\mathcal{K}_{\eta}^{\text{cc}}$ by $\mathcal{K}_{\eta}^{\text{up}}$.

Proof. In view of the discussion above, the statements follow directly from the fact that the set of solutions to the homogeneous problem (4.4.9) in $BS_{\eta}(H^1 \times H^1)$ is two-dimensional as a consequence of Proposition 4.4.3. A detailed discussion can be found in the proof of [71, Prop. 5.1]. $\square$

We now set out to lift the results above from the unprojected system (4.5.10) to the full system (4.5.1). A key role is reserved for the summation operator $\mathcal{J}$ that acts on a sequence W as

$$\begin{aligned} \mathcal{J}[W]_l &= \sum_{j=1}^l W_{j-1} - \sum_{j=1}^{-l} W_{-j} \\ &= \sum_{j=0}^{l-1} W_j - \sum_{j=1}^{-l} W_{-j}, \end{aligned} \quad (4.5.40)$$

with the usual remark that sums are set to zero when the lower bound is strictly larger than the upper bound.

Lemma 4.5.5. Pick a Hilbert space $\mathcal{H}$ together with a constant $\eta > 0$. Then for any $W \in BS_{\eta}(\mathcal{H})$, we have $\mathcal{J}[W] \in BS_{\eta}(\mathcal{H})$, with

$$(S - I)\mathcal{J}[W] = W. \quad (4.5.41)$$

Proof. These statements follow directly by inspecting the definition (4.5.40). $\square$

We now write

$$\mathcal{N}_{\eta}^{\text{cc}} = \{V \in BS_{\eta}(H^1 \times H^1) : \mathcal{D}(0)[V] = (I - P_0^{(1)})\mathcal{T}(0)[V]\} \quad (4.5.42)$$

for the set of solutions to the homogenous version of (4.5.1). By relating this set to its counterpart for (4.4.9) we show that $\mathcal{N}_{\eta}^{\text{cc}}$ is also two-dimensional for small $\eta > 0$.

Lemma 4.5.6. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are all satisfied. Then for all sufficiently small $\eta > 0$, we have the identification

$$\mathcal{N}_\eta^{\text{cc}} = \text{span}\{(\Phi'_*, 0), ([\partial_z \phi_z]_{z=0}, \Phi'_*)\}. \quad (4.5.43)$$

Proof. A direct computation shows that

$$\mathcal{D}(0)[\Phi'_*, 0] = \mathcal{T}(0)[\Phi'_*, 0] = 0, \quad (4.5.44)$$

together with

$$\begin{aligned} \mathcal{T}(0)[[\partial_z \phi_z]_{z=0}, \Phi'_*] &= (\Phi'_*, A_1 \Phi'_*), \\ \mathcal{D}(0)[[\partial_z \phi_z]_{z=0}, \Phi'_*] &= (0, Df(\tau \Phi_*) \bar{\tau} s^\diamond[\Phi'_*] \mathbf{1}) \\ &= (0, Df(\tau \Phi_*) \bar{\tau} s_0^\diamond \Phi'_*) \\ &= (0, A_1 \Phi'_*). \end{aligned} \quad (4.5.45)$$

In particular, we find

$$(I - P_0^{(1)})\mathcal{T}(0)[[\partial_z \phi_z]_{z=0}, \Phi'_*] = (0, A_1 \Phi'_*), \quad (4.5.46)$$

which verifies that the right-hand-side of (4.5.43) is indeed contained in $\mathcal{N}_\eta^{\text{cc}}$.

Conversely, let us consider a sequence $W \in \mathcal{N}_\eta^{\text{cc}}$, which implies that

$$\begin{aligned} \mathcal{D}(0)[P_0^{(1)}\mathcal{J}[\mathcal{T}(0)[W]]] &= (S - I)P_0^{(1)}\mathcal{J}[\mathcal{T}(0)[W]] \\ &= P_0^{(1)}(S - I)\mathcal{J}[\mathcal{T}(0)[W]] \\ &= P_0^{(1)}\mathcal{T}(0)[W]. \end{aligned} \quad (4.5.47)$$

Upon writing

$$V = W + P_0^{(1)}\mathcal{J}[\mathcal{T}(0)[W]], \quad (4.5.48)$$

we note that the identity $\mathcal{T}(0)P_0^{(1)} = 0$ implies that $\mathcal{T}(0)[V] = \mathcal{T}(0)[W]$. We may hence compute

$$\begin{aligned} \mathcal{D}(0)[V] &= \mathcal{D}(0)[W] + \mathcal{D}(0)P_0^{(1)}\mathcal{J}[\mathcal{T}(0)[W]] \\ &= (I - P_0^{(1)})\mathcal{T}(0)[W] + P_0^{(1)}[\mathcal{T}(0)W] \\ &= (I - P_0^{(1)})\mathcal{T}(0)[V] + P_0^{(1)}[\mathcal{T}(0)V] \\ &= \mathcal{T}(0)[V]. \end{aligned} \quad (4.5.49)$$

Notice that the map $W \mapsto W + P_0^{(1)}\mathcal{J}[\mathcal{T}(0)[W]]$ maps $BS_\eta(H^1 \times H^1)$ into itself. It is also injective, which can be easily seen by looking at the second component. This implies that $\mathcal{N}_\eta^{\text{cc}}$ is at most two-dimensional. $\square$

Proof of Proposition 4.5.1. Pick $H \in BS_\eta(H^1 \times L^2)$ and write

$$V = [I - P_0^{(1)}]\mathcal{K}^{\text{up}}(0)[I - P_0^{(1)}]H + P_0^{(1)}\mathcal{J}[H]. \quad (4.5.50)$$

Using $\mathcal{T}(0)P_0 = 0$ we may compute

$$\mathcal{T}(0)V = \mathcal{T}(0)\mathcal{K}_\eta^{\text{up}}(0)[I - P_0^{(1)}]H. \quad (4.5.51)$$

Using the commutation relation $P_0^{(1)}\mathcal{D}(0) = \mathcal{D}(0)P_0^{(1)}$ together with (4.5.41), we see that

$$\mathcal{D}(0)P_0^{(1)}\mathcal{J}[H] = P_0^{(1)}\mathcal{D}(0)\mathcal{J}[H] = P_0^{(1)}(S - I)\mathcal{J}[H] = P_0^{(1)}H. \quad (4.5.52)$$

In particular, we may compute

$$\begin{aligned} \mathcal{D}(0)[V] &= [I - P_0^{(1)}]\mathcal{T}(0)\mathcal{K}_\eta^{\text{up}}(0)[I - P_0^{(1)}]H + [I - P_0^{(1)}]H + P_0^{(1)}H \\ &= [I - P_0^{(1)}]\mathcal{T}(0)V + H. \end{aligned} \quad (4.5.53)$$

Upon writing

$$\mathcal{K}_\eta^{\text{cc}}(\vartheta) = [I - P_\vartheta^{(1)}]\mathcal{K}_\eta^{\text{up}}(\vartheta)[I - P_\vartheta^{(1)}]H + P_\vartheta^{(1)}\mathcal{J}[H], \quad (4.5.54)$$

the desired properties now follow readily from Lemmas 4.5.4 and 4.5.6. $\square$

4.6 Slowly varying coefficients

In this section we study the properties of the bounded linear operator

$$\Lambda(\theta) : BS_\eta(H^1 \times L^2) \mapsto BS_\eta(L^2 \times L^2) \quad (4.6.1)$$

that for any sequence $\theta : \mathbb{Z} \mapsto \mathbb{R}$ acts as

$$[\Lambda(\theta)V]_l = \text{pev}_l \mathcal{D}(\theta)[V] - (1 - P_{\theta_l}^{(1)})\text{pev}_l \mathcal{T}(\theta)[V]. \quad (4.6.2)$$

We are specially interested in cases where the sequence θ varies slowly with respect to $l \in \mathbb{Z}$, which means

$$\|(S - I)\theta\|_\infty < \delta_\theta \quad (4.6.3)$$

for some small $\delta_\theta > 0$.

Our first main result states that the kernel of $\Lambda(\theta)$ is again two-dimensional, provided that (4.6.3) holds. For technical reasons, we also extend the two basis functions for the kernel to situations where (4.6.3) fails to hold.

Proposition 4.6.1. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are all satisfied. Pick a sufficiently small constant $\delta_\theta > 0$ together with a sufficiently small $\eta > 0$. Then for every $\theta : \mathbb{Z} \rightarrow \mathbb{R}$ there exist two functions

$$V_{\text{hom}}^A(\theta) \in \bigcap_{0 < \eta < \eta_{\max}} BS_\eta(H^1 \times H^1), \quad V_{\text{hom}}^B(\theta) \in \bigcap_{0 < \eta < \eta_{\max}} BS_\eta(H^1 \times H^1) \quad (4.6.4)$$

that satisfy the following properties.

- (i) For any $\theta : \mathbb{Z} \mapsto \mathbb{R}$ with $\|(S - I)\theta\|_\infty < \delta_\theta$, we have the identities

$$\Lambda(\theta)V_{\text{hom}}^A(\theta) = \Lambda(\theta)V_{\text{hom}}^B(\theta) = 0. \quad (4.6.5)$$

- (ii) The normalization conditions

$$Q_{\theta_0}^{(1)}[V_{\text{hom}}^A(\theta)]_0 = Q_{\theta_0}^{(2)}[V_{\text{hom}}^B(\theta)]_0 = 1, \quad Q_{\theta_0}^{(1)}[V_{\text{hom}}^B(\theta)]_0 = Q_{\theta_0}^{(2)}[V_{\text{hom}}^A(\theta)]_0 = 0 \quad (4.6.6)$$

hold for all $\theta : \mathbb{Z} \rightarrow \mathbb{R}$.

- (iii) Pick any $0 < \eta < \eta_{\max}$ and suppose that $\Lambda(\theta)V = 0$ for some $V \in BS_\eta(H^1 \times H^1)$ and $\theta : \mathbb{Z} \rightarrow \mathbb{R}$ for which $\|(S - I)\theta\|_\infty < \delta_\theta$. Then we have the identity

$$V = V_{\text{hom}}^A(\theta)Q_{\theta_0}^{(1)}V_0 + V_{\text{hom}}^B(\theta)Q_{\theta_0}^{(2)}V_0. \quad (4.6.7)$$

Our second main result constructs operators $\mathcal{K}_\eta(\theta)$ that can be seen as an inverse for $\Lambda(\theta)$ whenever (4.6.3) holds. Naturally, the kernel elements above obtained in the result above need to be projected out, which is performed in (4.6.9). Special care needs to be taken when considering the smoothness with respect to θ . Indeed, the smoothness criteria below are based on the arguments involving nested Banach spaces argument that are traditionally used to establish the smoothness of center manifolds; see for example [36, §IX.7]. We remark that the notation $\mathcal{L}^{(p)}$ stands for bounded p -linear maps.

Proposition 4.6.2. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are all satisfied. Recall the integer r appearing in (Hf) and pick two sufficiently small constants $0 < \eta_{\min} < 2r\eta_{\min} < \eta_{\max}$. Then for every $\eta_{\min} < \eta < \eta_{\max}$, there exists a map

$$\mathcal{K}_\eta : \{\theta : \mathbb{Z} \rightarrow \mathbb{R}\} \mapsto \mathcal{L}\left(BS_\eta(H^1 \times L^2); BS_\eta(H^1 \times H^1)\right) \quad (4.6.8)$$

that satisfies the following properties.

- (i) There exists a constant $\delta_\theta > 0$ so that for any $\eta_{\min} < \eta < \eta_{\max}$, any $H \in BS_\eta(H^1 \times L^2)$ and any $\theta : \mathbb{Z} \rightarrow \mathbb{R}$ for which $\|(S - I)\theta\|_\infty < \delta_\theta$, the function $V = \mathcal{K}_\eta(\theta)H$ satisfies $\Lambda(\theta)V = H$.

- (ii) Pick any $\eta_{\min} < \eta < \eta_{\max}$ and $\theta : \mathbb{Z} \rightarrow \mathbb{R}$. Then for any $H \in BS_\eta(H^1 \times L^2)$, the function $V = \mathcal{K}_\eta(\theta)H$ satisfies the orthogonality conditions

$$Q_{\theta_0}^{(1)} V_0 = Q_{\theta_0}^{(2)} V_0 = 0. \quad (4.6.9)$$

- (iii) For any $\eta_{\min} < \eta < \eta_{\max}$ and $\theta : \mathbb{Z} \rightarrow \mathbb{R}$ we have the bound

$$\|\mathcal{K}_\eta(\theta)\|_{\mathcal{L}(BS_\eta(H^1 \times L^2); BS_\eta(H^1 \times H^1))} < K. \quad (4.6.10)$$

- (iv) Pick a triplet

$$(\eta_1, \eta_2, \eta_3) \in (\eta_{\min}, \eta_{\max})^3 \quad (4.6.11)$$

for which $\eta_1 + \eta_2 \leq \eta_3$. Then for any pair $(\theta^A, \theta^B) \in BS_{\eta_1}(\mathbb{R})^2$ and any $H \in BS_{\eta_2}(H^1 \times L^2)$ we have the estimate

$$\|\mathcal{K}_{\eta_2}(\theta^A)H - \mathcal{K}_{\eta_2}(\theta^B)H\|_{BS_{\eta_3}(H^1 \times H^1)} \leq K \|\theta^A - \theta^B\|_{BS_{\eta_1}(\mathbb{R})} \|H\|_{BS_{\eta_2}(H^1 \times L^2)}. \quad (4.6.12)$$

- (v) Consider a pair $(\eta_1, \eta_2) \in (0, \eta_{\max})^2$ together with a function

$$H \in BS_{\eta_1}(H^1 \times L^2) \cap BS_{\eta_2}(H^1 \times L^2). \quad (4.6.13)$$

Then for any $\theta : \mathbb{Z} \rightarrow \mathbb{R}$ we have

$$\mathcal{K}_{\eta_1}(\theta)H = \mathcal{K}_{\eta_2}(\theta)H. \quad (4.6.14)$$

- (vi) Pick an integer $1 \leq m \leq r - 1$ together with a triplet

$$(\eta_1, \eta_2, \eta_3) \in (\eta_{\min}, \eta_{\max})^3 \quad (4.6.15)$$

for which $m(\eta_1 + \eta_2) \leq \eta_3$. Then the map

$$BS_{\eta_1}(\mathbb{R}) \ni \theta \mapsto \mathcal{K}(\theta) \in \mathcal{L}(BS_{\eta_2}(H^1 \times L^2); BS_{\eta_3}(H^1 \times H^1)) \quad (4.6.16)$$

is C^m -smooth. In addition, for any integer $1 \leq p \leq m$, the derivative $D^p \mathcal{K}$ can be seen as a map

$$D^p \mathcal{K} : BS_{\eta_1}(\mathbb{R}) \mapsto \mathcal{L}^{(p)}\left(BS_{\eta_1}(\mathbb{R})^p; \mathcal{L}(BS_{\eta_2}(H^1 \times L^2); BS_\eta(H^1 \times H^1))\right) \quad (4.6.17)$$

for every $\eta \geq p\eta_1 + \eta_2$. This map is continuous in the first variable if $\eta > p\eta_1 + \eta_2$.

Our strategy is to exploit the inverses $\mathcal{K}_\eta^{\text{cc}}(\vartheta)$ for the constant-coefficient problem (4.5.1) to introduce an approximate inverse

$$\mathcal{K}_\eta^{\text{apx}}(\theta) : BS_\eta(H^1 \times L^2) \rightarrow BS_\eta(H^1 \times H^1) \quad (4.6.18)$$

for $\Lambda(\theta)$ by writing

$$[\mathcal{K}_\eta^{\text{apx}}(\theta)H]_j = \text{pev}_j \mathcal{K}_\eta^{\text{cc}}(\theta_j)H. \quad (4.6.19)$$

In order to turn this into an actual inverse, we need to establish bounds for the remainder term

$$S_\eta^{\text{rm}}(\theta)H = \Lambda(\theta)\mathcal{K}_\eta^{\text{apx}}(\theta)H - H. \quad (4.6.20)$$

To this end, we introduce the coordinate projection $\pi_2[v, w] = w$ together with the sequence

$$[\Delta_s^\diamond(\theta)H]_l = [s^\diamond \pi_2 \mathcal{K}_\eta^{\text{apx}}(\theta)H]_l - [s^\diamond \pi_2 \mathcal{K}_\eta^{\text{cc}}(\theta_l)H]_l. \quad (4.6.21)$$

A short computation shows that

$$\begin{aligned} [\Delta_s^\diamond(\theta)H]_l = & \left(- \sum_{j=0}^{\sigma_B-1} [\mathcal{K}_\eta^{\text{cc}}(\theta_{l-j})H - \mathcal{K}_\eta^{\text{cc}}(\theta_l)H]_{l-j}, \sum_{j=1}^{\sigma_A} [\mathcal{K}_\eta^{\text{cc}}(\theta_{l+j})H - \mathcal{K}_\eta^{\text{cc}}(\theta_l)H]_{l+j}, \right. \\ & \sum_{j=1}^{\sigma_B} [\mathcal{K}_\eta^{\text{cc}}(\theta_{l+j})H - \mathcal{K}_\eta^{\text{cc}}(\theta_l)H]_{l+j}, \\ & \left. - \sum_{j=0}^{\sigma_A-1} [\mathcal{K}_\eta^{\text{cc}}(\theta_{l-j})H - \mathcal{K}_\eta^{\text{cc}}(\theta_l)H]_{l-j}, 0 \right). \end{aligned} \quad (4.6.22)$$

In a similar spirit, we introduce the sequences

$$\begin{aligned} [\Delta_{\mathcal{T}}(\theta)H]_l &= [\mathcal{T}(\theta)\mathcal{K}_\eta^{\text{apx}}(\theta)H]_l - [\mathcal{T}(\mathbf{1}\theta_l)\mathcal{K}_\eta^{\text{cc}}(\theta_l)H]_l, \\ [\Delta_{\mathcal{D}}(\theta)H]_l &= [\mathcal{D}(\theta)\mathcal{K}_\eta^{\text{apx}}(\theta)H]_l - [\mathcal{D}(\mathbf{1}\theta_l)\mathcal{K}_\eta^{\text{cc}}(\theta_l)H]_l \end{aligned} \quad (4.6.23)$$

and compute

$$\begin{aligned} [\Delta_{\mathcal{D}}(\theta)H]_l &= \left(\pi_1[\mathcal{K}_\eta^{\text{cc}}(\theta_{l+1})H - \mathcal{K}_\eta^{\text{cc}}(\theta_l)H]_{l+1}, Df(T_{\theta_l}\tau\Phi_*)\bar{\tau}[\Delta_s^\diamond(\theta)H]_l \right), \\ [\Delta_{\mathcal{T}}(\theta)H]_l &= \left(\pi_2[\mathcal{K}_\eta^{\text{cc}}(\theta_{l+1})H - \mathcal{K}_\eta^{\text{cc}}(\theta_l)H]_{l+1}, 0 \right). \end{aligned} \quad (4.6.24)$$

These computations allows us to obtain the identity

$$\begin{aligned} [S_\eta^{\text{rm}}(\theta)]_l &= \text{pev}_l \Lambda(\theta)\mathcal{K}_\eta^{\text{apx}}(\theta)H - \text{pev}_l \Lambda(\mathbf{1}\theta_l)\mathcal{K}_\eta^{\text{cc}}(\theta_l)H \\ &= [\Delta_{\mathcal{D}}(\theta)H]_l - [I - P_{\theta_l}^{(1)}][\Delta_{\mathcal{T}}(\theta)H]_l. \end{aligned} \quad (4.6.25)$$

In order to formulate appropriate bounds for this expression, we introduce the notation

$$\begin{aligned} \text{cev}_l \theta &= \text{ev}_l \theta - \mathbf{1}\theta_l \\ &= (\theta_{l-\sigma_*+1} - \theta_l, \dots, \theta_{l+\sigma_*} - \theta_l). \end{aligned} \quad (4.6.26)$$

Lemma 4.6.3. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are all satisfied. Pick a sufficiently small constant $\eta_{\max} > 0$ together with a sufficiently large $K > 0$. Then for any $0 < \eta < \eta_{\max}$ and any $H \in BS_\eta(H^1 \times L^2)$, the following estimates hold.

(i) For any sequence $\theta : \mathbb{Z} \rightarrow \mathbb{R}$ we have the bound

$$\|\text{pev}_l S_\eta^{\text{rm}}(\theta)\|_{H^1 \times L^2} \leq K\eta^{-3} e^{\eta|l|} |\text{cev}_l \theta| \|H\|_{BS_\eta(H^1 \times L^2)}. \quad (4.6.27)$$

(ii) For any pair of sequences $(\theta^A, \theta^B) : \mathbb{Z} \rightarrow \mathbb{R}^2$ we have the bound

$$\|\text{pev}_l [S_\eta^{\text{rm}}(\theta^A) - S_\eta^{\text{rm}}(\theta^B)]\|_{H^1 \times L^2} \leq K\eta^{-3} e^{\eta|l|} |\text{ev}_l(\theta^A - \theta^B)| \|H\|_{BS_\eta(H^1 \times L^2)}. \quad (4.6.28)$$

Proof. As a consequence of the bound (4.5.6) and the smoothness of the map $\vartheta \mapsto \mathcal{K}_\eta^{\text{cc}}(\vartheta)$, there exists $C_1 > 0$ for which

$$\|\text{pev}_l \mathcal{K}_\eta^{\text{cc}}(\vartheta^A)H - \text{pev}_l \mathcal{K}_\eta^{\text{cc}}(\vartheta^B)H\|_{H^1 \times H^1} \leq C_1 \eta^{-3} |\vartheta^A - \vartheta^B| e^{\eta|l|} \|H\|_{BS_\eta(H^1 \times L^2)}. \quad (4.6.29)$$

In particular, we see that for all $|j| \leq \sigma_*$ we have

$$\|\text{pev}_{l+j} [\mathcal{K}_\eta^{\text{cc}}(\theta_{l+j})H - \mathcal{K}_\eta^{\text{cc}}(\theta_l)H]\|_{H^1 \times H^1} \leq C_2 \eta^{-3} e^{\eta|l|} |\text{cev}_l \theta| \|H\|_{BS_\eta(H^1 \times L^2)}. \quad (4.6.30)$$

The bound (4.6.27) hence follows immediately from the representations (4.6.22), (4.6.24) and (4.6.25).

In addition, (4.6.29) also implies that

$$\|\text{ev}_l [\mathcal{K}_\eta^{\text{apx}}(\theta^A)H - \mathcal{K}_\eta^{\text{apx}}(\theta^B)H]\|_{H^1 \times H^1} \leq C_3 \eta^{-3} |\text{ev}_l(\theta^A - \theta^B)| e^{\eta|l|} \|H\|_{BS_\eta(H^1 \times L^2)}. \quad (4.6.31)$$

The second bound (4.6.28) readily follows from this. $\square$

In order to restrict the size of the remainder term, we need to introduce an appropriate cut-off function. To this end, we pick an arbitrary C^∞ -smooth function $\chi : [0, \infty) \rightarrow \mathbb{R}$ that has $\chi(\zeta) = 1$ for $0 \leq \zeta \leq 1$ and $\chi(\zeta) = 0$ for $\zeta \geq 2$. For any $\delta > 0$, we subsequently write χ_δ for the function

$$\chi_\delta(\zeta) = \chi(\zeta/\delta). \quad (4.6.32)$$

With this definition in hand, we pick a constant $\tilde{\delta}_\vartheta > 0$ and introduce the cut-off remainder term

$$[S_\eta^{\text{rm};c}(\theta)H]_l = \chi_{\tilde{\delta}_\vartheta}(|\text{cev}_l \theta|) [S_\eta^{\text{rm}}(\theta)H]_l. \quad (4.6.33)$$

The pointwise estimates in Lemma 4.6.3 immediately yield the following bounds on this new remainder term.

Corollary 4.6.4. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are all satisfied. Pick a sufficiently small constant $\eta_{\max} > 0$ together with a sufficiently large $K > 0$. Then for any $0 < \eta < \eta_{\max}$ and any $H \in BS_\eta(H^1 \times L^2)$ the following estimates are valid.

(i) For any sequence $\theta : \mathbb{Z} \rightarrow \mathbb{R}$ we have the bound

$$\|S_\eta^{\text{rm};c}(\theta)\|_{BS_\eta(H^1 \times L^2)_H} \leq K\tilde{\delta}_\theta \|H\|_{BS_\eta(H^1 \times L^2)}. \quad (4.6.34)$$

(ii) For any $\eta_1 > 0$ and any pair of sequences $\theta^A, \theta^B \in BS_{\eta_1}(\mathbb{R})$, we have the bound

$$\begin{aligned} \|S_\eta^{\text{rm};c}(\theta^A) - S_\eta^{\text{rm};c}(\theta^B)\|_{BS_{\eta_1+\eta}(H^1 \times L^2)} &\leq K\eta^{-3} \|\text{ev}_l(\theta^A - \theta^B)\|_{BS_{\eta_1}(\mathbb{R})} \\ &\quad \times \|H\|_{BS_\eta(H^1 \times L^2)}. \end{aligned} \quad (4.6.35)$$

After picking $\tilde{\delta}_\theta > 0$ to be sufficiently small, the bound (4.6.35) allows us to define the full inverse

$$\mathcal{K}_\eta(\theta) = \mathcal{K}_\eta^{\text{apx}}(\theta)[I + S_\eta^{\text{rm};c}(\theta)]^{-1}. \quad (4.6.36)$$

We remark that the normalization conditions (4.6.9) hold as a direct consequence of (4.5.4) and the construction of $\mathcal{K}_\eta^{\text{apx}}$.

Proof of Proposition 4.6.2. Items (i), (ii), (iii) and (v) follow directly from the discussion above. To obtain (iv) and (vi) one can use the bounds (4.6.31) and (4.6.35); see e.g. the proof of [71, Prop. 6.1]. $\square$

Turning to identify the kernel of $\Lambda(\theta)$, we introduce the operator

$$\mathcal{E}(\theta) : BS_\eta(H^1 \times H^1) \rightarrow BS_\eta(H^1 \times L^2) \quad (4.6.37)$$

that acts as

$$\mathcal{E}(\theta)[V] = \Lambda(\theta)[V] - \Lambda(\theta_0 \mathbf{1})[V] \quad (4.6.38)$$

for any sequence $\theta : \mathbb{Z} \rightarrow \mathbb{R}$. We now write

$$\begin{aligned} V_{\text{hom}}^A(\theta) &= T_{\theta_0}(\Phi'_*, 0) - \mathcal{K}_\eta(\theta)\mathcal{E}(\theta)T_{\theta_0}(\Phi'_*, 0)\mathbf{1}, \\ V_{\text{hom}}^B(\theta) &= T_{\theta_0}([\partial_z \phi_z]_{z=0}, \Phi'_*) - \mathcal{K}_\eta(\theta)\mathcal{E}(\theta)T_{\theta_0}([\partial_z \phi_z]_{z=0}, \Phi'_*)\mathbf{1}. \end{aligned} \quad (4.6.39)$$

Proof of Proposition 4.6.1. Item (i) follows from the fact that

$$\mathcal{E}(\theta)T_{\theta_0}(\Phi'_*, 0)\mathbf{1} = \Lambda(\theta)T_{\theta_0}(\Phi'_*, 0)\mathbf{1}, \quad (4.6.40)$$

together with a similar identity for $V_{\text{hom}}^B(\theta)$. Item (ii) follows directly from the normalization (4.6.9). Finally, (iii) can be established by following the proof of [71, Lem. 6.4]. $\square$

4.7 The center manifold

Our goal here is to construct and analyze a global center manifold for the system (4.3.42) that captures all the solutions where the pair (v, w) remains small. In particular, we set out to establish Proposition 4.3.5. While the main spirit of the ideas in [71, §7] can be used to establish the existence of the manifold, we need to take special care to identify the reduced equation that is satisfied on the center space. The key issue is that we wish to recover a first order difference equation from a differential-difference system of order $2\sigma_*$.

Let us choose two small constants $\delta_c > 0$ and $\delta_v > 0$ and recall the constant $\delta_\theta > 0$ defined in Proposition 4.6.2. The main idea is that we look for solutions to (4.3.42) that can be written in the form

$$\text{ev}_l V = \Psi_l^\alpha T_{\Psi_l^\theta}(\Phi'_*, 0) \mathbf{1} + \Psi_l^\beta T_{\Psi_l^\theta}([\partial_z \phi_z]_{z=0}, \Phi'_*) \mathbf{1} + h(\Psi_l, c), \quad (4.7.1)$$

for a triplet of scalar functions

$$\Psi = (\Psi^\theta, \Psi^\alpha, \Psi^\beta) : \mathbb{Z} \rightarrow \mathbb{R}^3 \quad (4.7.2)$$

and a map

$$h : \mathbb{R} \times (-\delta_v, \delta_v)^2 \times (c_* - \delta_c, c_* + \delta_c) \rightarrow \mathbf{H}^1 \times \mathbf{H}^1 \quad (4.7.3)$$

that satisfies

$$Q_\vartheta^{(1)} \text{pev}_0 h(\vartheta, \alpha, \beta) = Q_\vartheta^{(2)} \text{pev}_0 h(\vartheta, \alpha, \beta) = 0. \quad (4.7.4)$$

In order to setup an appropriate fixed point argument for the pair (Ψ, h) , we first need to add cut-offs to the nonlinearities defined in (4.3.33) and (4.3.39). In particular, for any triplet $\psi = (\psi_\theta, \psi_\alpha, \psi_\beta) \in \mathbb{R}^3$, we recall the cut-off function (4.6.32) and introduce the notation

$$\mathcal{I}(\psi, h) \in \mathbf{H}^1 \times \mathbf{H}^1 \quad (4.7.5)$$

for the functions

$$\mathcal{I}(\psi; h) = \chi_{\delta_v}(|\psi_\alpha| + |\psi_\beta|) \left[\psi_\alpha T_{\psi_\theta}(\Phi'_*, 0) \mathbf{1} + \psi_\beta T_{\psi_\theta}([\partial_z \phi_z]_{z=0}, \Phi'_*) \mathbf{1} + h(\psi, c) \right]. \quad (4.7.6)$$

This allows us to introduce the new nonlinearities $\mathcal{R}^c$ and $\mathcal{S}^c$ that act as

$$\begin{aligned} [\mathcal{R}^c(\Psi; h)]_l &= \chi_{\delta_\theta}(|\Psi_{l+1}^\theta - \Psi_l^\theta|) \mathcal{R}(\text{ev}_l \Psi^\theta, \mathcal{I}(\Psi_l; h)), \\ [\mathcal{S}^c(\Psi; h)]_l &= \chi_{\delta_\theta}(|\Psi_{l+1}^\theta - \Psi_l^\theta|) \mathcal{S}(\text{ev}_l \Psi^\theta, \mathcal{I}(\Psi_l; h)). \end{aligned} \quad (4.7.7)$$

Let us now assume that we have a solution to (4.3.42) of the form (4.7.1) for some $\Psi \in BS_\eta(\mathbb{R}^3)$ that satisfies

$$\|(S - I)\Psi^\theta\|_\infty < \delta_\theta, \quad \|\Psi^\alpha\|_\infty + \|\Psi^\beta\|_\infty < \delta_v. \quad (4.7.8)$$

Then upon writing $\psi = \Psi(0)$, we must have

$$V = \psi_\alpha V_{\text{hom}}^A(\Psi^\theta) + \psi_\beta V_{\text{hom}}^B(\Psi^\theta) + \mathcal{K}_\eta(\Psi^\theta) \left[(c - c_*)(0, T_{\Psi^\theta} \Phi'_*) + \mathcal{R}^c(\Psi; h) \right]. \quad (4.7.9)$$

Upon introducing the sequence

$$(\alpha, \beta)_l = (Q_{\Psi_l^\theta}^{(1)}, Q_{\Psi_l^\theta}^{(2)})V_l, \quad (4.7.10)$$

item (iii) of Proposition 4.6.1 implies that

$$\begin{aligned} V_{l_0+l} &= \alpha_{l_0} \text{pev}_l V_{\text{hom}}^A(S_{l_0} \Psi^\theta) + \beta_{l_0} \text{pev}_l V_{\text{hom}}^B(S_{l_0} \Psi^\theta) \\ &\quad + \text{pev}_l \mathcal{K}(S_{l_0} \Psi^\theta) \left[(c - c_*)(0, T_{S_{l_0} \Psi^\theta} \Phi'_*) + \mathcal{R}^c(S_{l_0} \Psi; h) \right] \end{aligned} \quad (4.7.11)$$

for any pair $(l_0, l) \in \mathbb{Z}^2$.

We now introduce the notation

$$\begin{aligned} \mathcal{J}_{V;\alpha}(\Psi^\theta) &= -\mathcal{K}_\eta(\Psi^\theta) \mathcal{E}(\Psi^\theta) T_{\Psi_0^\theta}(\Phi'_*, 0) \mathbf{1}, \\ \mathcal{J}_{V;\beta}(\Psi^\theta) &= -\mathcal{K}_\eta(\Psi^\theta) \mathcal{E}(\Psi^\theta) T_{\Psi_0^\theta}([\partial_z \phi_z]_{z=0}, \Phi'_*) \mathbf{1}, \end{aligned} \quad (4.7.12)$$

together with

$$\begin{aligned} \mathcal{A}_V(\Psi^\theta) &= \mathcal{K}_\eta(\Psi^\theta)(0, T_{\Psi^\theta} \Phi'_*), \\ \mathcal{M}_V(\Psi; h) &= \mathcal{K}_\eta(\Psi^\theta) \mathcal{R}^c(\Psi; h). \end{aligned} \quad (4.7.13)$$

The representation (4.6.39) allows us to rewrite (4.7.11) as

$$\begin{aligned} V_{l_0+l} &= \alpha_{l_0} T_{\Psi_{l_0}^\theta}(\Phi'_*, 0) + \beta_{l_0} T_{\Psi_{l_0}^\theta}([\partial_z \phi_z]_{z=0}, \Phi'_*) + \alpha_{l_0} \text{pev}_l \mathcal{J}_{V;\alpha}(S_{l_0} \Psi^\theta) \\ &\quad + \beta_{l_0} \text{pev}_l \mathcal{J}_{V;\beta}(S_{l_0} \Psi^\theta) + (c - c_*) \text{pev}_l \mathcal{A}_V(S_{l_0} \Psi^\theta) + \text{pev}_l \mathcal{M}_V(S_{l_0} \Psi; h). \end{aligned} \quad (4.7.14)$$

For any $y \in \mathbb{R}^n$ we now introduce the auxilliary cut-off function

$$\bar{\chi}_\delta(y) = \chi_\delta(|y|)y, \quad (4.7.15)$$

together with the notation

$$\begin{aligned} \mathcal{J}_{\alpha\beta}(\Psi^\theta)[\alpha, \beta] &= \bar{\chi}_{\delta_v}(\alpha) \bar{\chi}_{\delta_\theta}([Q_{\Psi_1^\theta} - Q_{\Psi_0^\theta}] T_{\Psi_0^\theta}(\Phi'_*, 0)) \\ &\quad + \bar{\chi}_{\delta_v}(\beta) \bar{\chi}_{\delta_\theta}([Q_{\Psi_1^\theta} - Q_{\Psi_0^\theta}] T_{\Psi_0^\theta}([\partial_z \phi_z]_{z=0}, \Phi'_*)) \\ &\quad + \bar{\chi}_{\delta_v}(\alpha) \bar{\chi}_{\delta_\theta}(Q_{\Psi_1^\theta} \text{pev}_1 \mathcal{J}_{V;\alpha}(\Psi^\theta)) \\ &\quad + \bar{\chi}_{\delta_v}(\beta) \bar{\chi}_{\delta_\theta}(Q_{\Psi_1^\theta} \text{pev}_1 \mathcal{J}_{V;\beta}(\Psi^\theta)) \end{aligned} \quad (4.7.16)$$

and also

$$\begin{aligned} \mathcal{A}_{\alpha\beta}(\Psi^\theta) &= Q_{\Psi_1^\theta} \text{pev}_1 \mathcal{A}_V(\Psi^\theta), \\ \mathcal{M}_{\alpha\beta}(\Psi; h) &= Q_{\Psi_1^\theta} \text{pev}_1 \mathcal{M}_V(\Psi; h). \end{aligned} \quad (4.7.17)$$

As a reminder, there exists a constant $K > 0$ so that

$$\left\| \mathcal{E}(\Psi^\theta) T_{\Psi_0^\theta}(\Phi'_*, 0) \right\|_{BS_\eta(H^1 \times L^2)} \leq K \left[\|(S - I)\Psi^\theta\|_\infty L_0 + e^{-\eta L_0} \right]. \quad (4.7.18)$$

holds for any $L_0 \geq 1$. In particular, if we pick a sufficiently small $\tilde{\epsilon} > 0$ and strengthen the assumption (4.7.8) by demanding

$$\|(S - I)\Psi^\theta\|_\infty + \|\alpha\|_\infty + \|\beta\|_\infty < \tilde{\epsilon}, \quad (4.7.19)$$

then in fact

$$(\alpha, \beta)_{l_0+1} = (\alpha, \beta)_{l_0} + \mathcal{J}_{\alpha\beta}(S_{l_0}\Psi^\theta)[\alpha_{l_0}, \beta_{l_0}] + (c - c_*)\mathcal{A}_{\alpha\beta}(S_{l_0}\Psi^\theta) + \mathcal{M}_{\alpha\beta}(S_{l_0}\Psi; h) \quad (4.7.20)$$

holds for every $l_0 \in \mathbb{Z}$.

Inspired by the identity (4.3.34), we introduce the expressions

$$\begin{aligned} \mathcal{J}_\theta(\Psi^\theta)[\alpha, \beta] &= +\bar{\chi}_{\delta_v}(\alpha)\bar{\chi}_{\delta_\theta}(Q_{\Psi_0^\theta}^{(2)}\text{pev}_1\mathcal{J}_{V;\alpha}(\Psi^\theta)) \\ &\quad +\bar{\chi}_{\delta_v}(\beta)\bar{\chi}_{\delta_\theta}(Q_{\Psi_0^\theta}^{(2)}\text{pev}_1\mathcal{J}_{V;\beta}(\Psi^\theta)), \\ \mathcal{A}_\theta(\Psi^\theta) &= Q_{\Psi_0^\theta}^{(2)}\text{pev}_1\mathcal{A}_V(\Psi^\theta), \\ \mathcal{M}_\theta(\Psi; h) &= Q_{\Psi_0^\theta}^{(2)}\text{pev}_1\mathcal{M}_V(\Psi; h) + \text{pev}_0\mathcal{S}^c(\Psi; h). \end{aligned} \quad (4.7.21)$$

This allows us to augment the difference equation (4.7.20) for the pair (α, β) by introducing a new sequence $\theta : \mathbb{Z} \rightarrow \mathbb{R}$ that must satisfy

$$\theta_{l_0+1} - \theta_{l_0} = \beta_{l_0} + \mathcal{J}_\theta(S_{l_0}\Psi^\theta)[\alpha_{l_0}, \beta_{l_0}] + (c - c_*)\mathcal{A}_\theta(S_{l_0}\Psi^\theta) + \mathcal{M}_\theta(S_{l_0}\Psi; h). \quad (4.7.22)$$

In view of the original Ansatz (4.7.1), it is natural to look for solutions to (4.7.20) and (4.7.22) that have $(\theta, \alpha, \beta) = \Psi$. This leads to the system

$$\begin{aligned} \text{pev}_{l_0}(S - I)(\Psi^\alpha, \Psi^\beta) &= \mathcal{J}_{\alpha\beta}(S_{l_0}\Psi^\theta)\text{pev}_{l_0}[\Psi^\alpha, \Psi^\beta] \\ &\quad + (c - c_*)\mathcal{A}_{\alpha\beta}(S_{l_0}\Psi^\theta) + \mathcal{M}_{\alpha\beta}(S_{l_0}\Psi; h), \\ \text{pev}_{l_0}(S - I)\Psi^\theta &= \beta_{l_0} + \mathcal{J}_\theta(S_{l_0}\Psi^\theta)\text{pev}_{l_0}[\Psi^\alpha, \Psi^\beta] \\ &\quad + (c - c_*)\mathcal{A}_\theta(S_{l_0}\Psi^\theta) + \mathcal{M}_\theta(S_{l_0}\Psi; h). \end{aligned} \quad (4.7.23)$$

In addition, we can combine (4.7.1) and (4.7.14) to arrive at

$$\begin{aligned} h(\psi, c) &= \psi_\alpha \text{ev}_0 \mathcal{J}_{V;\alpha}(\Psi) + \psi_\beta \text{ev}_0 \mathcal{J}_{V;\beta}(\Psi) \\ &\quad + (c - c_*) \text{ev}_0 \mathcal{A}_W(\Psi) + \text{ev}_0 \mathcal{M}_V(\Psi; h), \end{aligned} \quad (4.7.24)$$

which can be seen as a consistency condition for the function h . Our main result here states that the global center manifold for (4.3.42) can be constructed by solving the fixed point problem generated by (4.7.23)-(4.7.24).

Proposition 4.7.1. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are all satisfied and pick two sufficiently small constants $0 < \eta_{\min} < \eta_{\max}$. In addition, pick a sufficiently large constant $K > 0$ together with a sufficiently small constant $\delta_v > 0$ and write

$$\delta_c = \delta_v^2, \quad \delta_\theta = \delta_v^{4/7}. \quad (4.7.25)$$

Then there exist C^{r-1} -smooth maps

$$\begin{aligned}\Psi_* &: \mathbb{R}^3 \times (c_* - \delta_c, c_* + \delta_c) && \rightarrow BS_{\eta_{\min}}(\mathbb{R}^3), \\ h_* &: \mathbb{R} \times (-\delta_v, \delta_v)^2 \times (c_* - \delta_c, c_* + \delta_c) && \rightarrow \mathbf{H}^1 \times \mathbf{H}^1\end{aligned}\quad (4.7.26)$$

so that the following properties hold true.

- (i) Pick any $\psi \in \mathbb{R}^3$ and $c \in (c_* - \delta_c, c_* + \delta_c)$. Then the identity (4.7.24) holds upon writing

$$\Psi = \Psi_*(\psi, c), \quad h = h_*. \quad (4.7.27)$$

- (ii) Pick any $\psi \in \mathbb{R}^3$ and $c \in (c_* - \delta_c, c_* + \delta_c)$. Then for any $\eta \in (\eta_{\min}, \eta_{\max})$, the function $\Psi = \Psi_*(\psi, c)$ is the unique solution in $BC_\eta(\mathbb{R}; \mathbb{R}^3)$ to the problem (4.7.23) with $\text{pev}_0 \Psi = \psi$ and $h = h^*$.

- (iii) Pick any $\psi \in \mathbb{R}^3$ and $c \in (c_* - \delta_c, c_* + \delta_c)$ together with a pair $(l_0, l) \in \mathbb{Z}^2$. Then we have the identity

$$\text{pev}_{l+l_0} \Psi_*(\psi, c) = \text{pev}_l \Psi_*(\text{pev}_{l_0} \Psi_*(\psi, c), c). \quad (4.7.28)$$

- (iv) We have $Q^{(1)} \text{pev}_0 h_* = Q^{(2)} \text{pev}_0 h_* = 0$ and we have

$$\|\text{pev}_0 h_*(\alpha, \beta, \theta)\|_{H^1 \times H^1} \leq K[|c - c_*| + \alpha^2 + \beta^2]. \quad (4.7.29)$$

- (v) Pick any $\psi \in \mathbb{R}^3$ and $c \in (c_* - \delta_c, c_* + \delta_c)$ and suppose that

$$\|\Psi_*^\alpha(\psi, c)\|_\infty + \|\Psi_*^\beta(\psi, c)\|_\infty < \delta_v. \quad (4.7.30)$$

Then upon writing $\Psi = \Psi_*(\psi, c)$ together with

$$V_l = T_{\theta_l}(\Phi'_*, 0)\Psi_l^\alpha + T_{\theta_l}([\partial_z \phi_z]_{z=0}, \Phi'_*)\Psi_l^\beta + \text{pev}_0 h_*(\Psi_l, c), \quad (4.7.31)$$

the pair (Ψ^θ, V) is a solution to the full system (4.3.42).

- (vi) Consider two sequences

$$\theta : \mathbb{Z} \rightarrow \mathbb{R}, \quad V : \mathbb{Z} \rightarrow H^1 \times H^1 \quad (4.7.32)$$

that satisfy (4.3.42) and admit the bound

$$\|\text{pev}_l V\|_{H^1 \times H^1} \leq \delta_v \quad (4.7.33)$$

for all $l \in \mathbb{Z}$. Then upon writing

$$\alpha_l = Q_{\theta_l}^{(1)} \text{pev}_l V, \quad \beta_l = Q_{\theta_l}^{(2)} \text{pev}_l V, \quad (4.7.34)$$

we have

$$V_l = \alpha_l T_{\theta_l}(\Phi'_*, 0) + \beta_l T_{\theta_l}([\partial_z \phi_z]_{z=0}, \Phi'_*, 0) + h_*(\theta_l, \alpha_l, \beta_l, c) \quad (4.7.35)$$

for all $l \in \mathbb{Z}$.

Proof. In view of the bounds for $\mathcal{R}$ and $\mathcal{S}$ obtained in Corollary 4.3.4 and the properties of $\mathcal{K}_\eta$ described in Proposition 4.6.2, the arguments used to establish [71, Lem. 7.2-7.5] can also be applied to construct solutions of the fixed point problem (4.7.23)-(4.7.24) and to show that these solutions admit the stated properties. $\square$

Upon demanding that

$$Q_{\theta_0}^{(1)} V_0 = 0 \quad (4.7.36)$$

any solution to (4.3.42) by construction has $Q_{\theta_l}^{(1)} V_l = 0$ for all $l \in \mathbb{Z}$. In particular, imposing the initial condition $\alpha_0 = 0$ and using (4.7.28) to eliminate the nonlocal terms appearing in $\mathcal{J}_{\alpha\beta}$, $\mathcal{A}_{\alpha\beta}$ and $\mathcal{M}_{\alpha\beta}$ and their counterparts for θ , the problem (4.7.23) with $h = h^*$ can be reformulated in the form

$$\begin{aligned} \beta_{l+1} - \beta_l &= f_\beta(\beta, c), \\ \theta_{l+1} - \theta_l &= f_\theta(\beta, c). \end{aligned} \quad (4.7.37)$$

The fact that f_β and f_θ do not depend on θ is a direct consequence of the shift-invariance of the system. We have the expansions

$$\begin{aligned} f_\beta(\beta, c) &= \nu_1(c - c_*) + \nu_2\beta_l^2 + O((c - c_*)^2 + (c - c_*)\beta + \beta^3), \\ f_\theta(\beta, c) &= \beta_l + \nu_3(c - c_*) + O(\beta_l^2 + (c - c_*)^2). \end{aligned} \quad (4.7.38)$$

We first set out to compute ν_1 and ν_3 , which by inspection of the definitions (4.7.17) and (4.7.21) for $\mathcal{A}_{\alpha\beta}$ and $\mathcal{A}_\theta$ can be seen to satisfy

$$\nu_1 = \nu_3 = Q_0^{(2)} \text{pev}_1 \mathcal{A}_V(0). \quad (4.7.39)$$

Using the identity (4.5.50), we may compute

$$\begin{aligned} \mathcal{A}_V(0) &= \mathcal{K}_\eta(0)(0, \Phi'_*) \\ &= \mathcal{K}_\eta^{\text{cc}}(0)(0, \Phi'_*) \\ &= [I - P_0^{(1)}] \mathcal{K}_\eta^{\text{up}}(0)(0, \Phi'_*). \end{aligned} \quad (4.7.40)$$

This latter expression can be evaluated by a direct computation involving an Ansatz that is polynomial in l . This is performed in the next result, which directly implies that

$$\nu_1 = \nu_3 = 2[\partial_z^2 \lambda_z]_{z=0}^{-1}. \quad (4.7.41)$$

Lemma 4.7.2. Assume that (Hf), $(H\Phi)$, (HS1)-(HS3) and (HM) are all satisfied. Then there exists $v_* \in H^1$ for which we have

$$\begin{aligned} \text{pev}_l \mathcal{K}_\eta^{\text{up}}(0)(0, \Phi'_*) \mathbf{1} &= 2[\partial_z^2 \lambda_z]_{z=0}^{-1} \left[\frac{1}{2} l^2 (\Phi'_*, 0) + l([\partial_z \phi_z]_{z=0}, \Phi'_*) \right. \\ &\quad \left. + \left(v_*, [\partial_z \phi_z]_{z=0} - \frac{1}{2} \Phi'_* \right) + \frac{1}{2} V_{\text{hom}}^B \right]. \end{aligned} \quad (4.7.42)$$

In particular, we have

$$Q_0^{(2)} \text{pev}_1 \mathcal{K}_\eta^{\text{up}}(0)(0, \Phi'_*) \mathbf{1} = 2[\partial_z^2 \lambda_z]_{z=0}^{-1}. \quad (4.7.43)$$

Proof. For convenience, we introduce the polynomial sequences

$$p_l^{(0)} = 1, \quad p_l^{(1)} = l \quad p_l^{(2)} = l^2. \quad (4.7.44)$$

For any $w \in H^1$, we can compute

$$\text{pev}_l s^\diamond[p^{(0)}w] = s_0^\diamond w, \quad (4.7.45)$$

together with

$$\begin{aligned} \text{pev}_l s^\diamond[p^{(1)}w] &= l(-\sigma_B, \sigma_A, \sigma_B, -\sigma_A)w \\ &\quad + (\sum_{j=0}^{\sigma_B-1} l, \sum_{j=1}^{\sigma_A} l, \sum_{j=1}^{\sigma_B} l, \sum_{j=0}^{\sigma_A-1} l)w \\ &= ls_0^\diamond w + [\partial_z s_z^\diamond]_{z=0} w. \end{aligned} \quad (4.7.46)$$

In particular, for any pair $(v, w) \in H^1 \times H^1$ we find

$$\text{pev}_l \mathcal{D}(0)[p^{(0)}(v, w)] = (0, A_1 w), \quad (4.7.47)$$

together with

$$\text{pev}_l \mathcal{D}(0)[p^{(1)}(v, w)] = l(0, A_1 w) + (v, \frac{1}{2}(A_1 + A_2)w). \quad (4.7.48)$$

Finally, we have

$$\text{pev}_l \mathcal{D}(0)[p^{(2)}(v, 0)] = (2l + 1)(v, 0). \quad (4.7.49)$$

Upon writing

$$W_l = \frac{1}{2}l^2(\Phi'_*, 0) + l([\partial_z \phi_z]_{z=0}, \Phi'_*) + (v_*, w_*), \quad (4.7.50)$$

we may hence compute

$$\text{pev}_l[\mathcal{D}(0)W] = \frac{1}{2}(2l + 1)(\Phi'_*, 0) + l(0, A_1 \Phi'_*) + ([\partial_z \phi_z]_{z=0}, \frac{1}{2}(A_1 + A_2)\Phi'_*) + (0, A_1 w_*), \quad (4.7.51)$$

together with

$$\begin{aligned} \text{pev}_l[\mathcal{T}(0)W] &= (w_* + (l + 1)\Phi'_*, -l\mathcal{L}_*[\partial_z \phi_z]_{z=0} + \mathcal{L}_* v_*) \\ &= (w_* + (l + 1)\Phi'_*, +lA_1 \Phi'_* - \mathcal{L}_* v_*). \end{aligned} \quad (4.7.52)$$

In particular, we find

$$\text{pev}_l[\mathcal{D}(0)W - \mathcal{T}(0)W] = ([\partial_z \phi_z]_{z=0} - \frac{1}{2}\Phi'_* - w_*, \frac{1}{2}(A_1 + A_2)\Phi'_* + A_1 w_* + \mathcal{L} v_*). \quad (4.7.53)$$

Upon writing

$$w_* = [\partial_z \phi_z]_{z=0} - \frac{1}{2} \Phi'_* \quad (4.7.54)$$

and picking a constant $\mu \in \mathbb{R}$, we hence see that

$$\text{pev}_l[\mathcal{D}(0)W - \mathcal{T}(0)W] = \mu(0, \Phi'_*) \quad (4.7.55)$$

holds if and only if

$$\mathcal{L}v_* + \frac{1}{2}A_2\Phi'_* + A_1[\partial_z \phi_z]_{z=0} = \mu\Phi'_*. \quad (4.7.56)$$

This latter equation has a solution $v_* \in H^1$ if and only if

$$\mu = \langle \psi_*, \frac{1}{2}A_2\Phi'_* + A_1[\partial_z \phi_z]_{z=0} \rangle = \frac{1}{2}[\partial_z^2 \lambda_z]_{z=0}. \quad (4.7.57)$$

We hence see that

$$\mathcal{K}_\eta^{\text{up}}(0)(0, \Phi'_*)\mathbf{1} = \frac{2}{[\partial_z^2 \lambda_z]_{z=0}}W + \nu_A V_{\text{hom}}^A + \nu_B V_{\text{hom}}^B \quad (4.7.58)$$

for some pair $(\nu_A, \nu_B) \in \mathbb{R}^2$ that ensures that

$$Q_0^{(1)}\text{pev}_0\mathcal{K}_\eta^{\text{up}}(0)(0, \Phi'_*)\mathbf{1} = Q_0^{(2)}\text{pev}_0\mathcal{K}_\eta^{\text{up}}(0)(0, \Phi'_*)\mathbf{1} = 0. \quad (4.7.59)$$

Upon imposing the normalization $Q_0v_* = 0$, the desired result follows by noting that

$$Q_0^{(1)}\text{pev}_0W = Q_0v_* = 0, \quad Q_0^{(2)}\text{pev}_0W = Q_0w_* = -\frac{1}{2}. \quad (4.7.60)$$

□

We now set out to compute the coefficient ν_2 by examining the zeroes of the functions f_β and f_θ . In particular, recalling the direction-dependent waves $(c_\varphi, \Phi_\varphi)$ defined in Lemma 4.2.2, we introduce the sequences

$$\Xi^{(\varphi)} : \mathbb{Z} \rightarrow H^1 \quad (4.7.61)$$

that are given by

$$\Xi_l^{(\varphi)}(\xi) = \Phi_\varphi\left(\cos \varphi[\xi + l \tan \varphi]\right) \quad (4.7.62)$$

for any small $|\varphi|$. This allows us to rewrite the planar wave solutions (4.2.31) in the form

$$\begin{aligned} u_{nl}(t) &= \Phi_\varphi(n \cos \varphi + l \sin \varphi l + c_\varphi t) \\ &= \Phi_\varphi(\cos \varphi[n + \tan \varphi l + d_\varphi t]) \\ &= \Xi_l^{(\varphi)}(n + d_\varphi t). \end{aligned} \quad (4.7.63)$$

We now pick a constant $\theta_0^{(\varphi)}$ in such a way that

$$\int_{-\infty}^{\infty} \psi_*(\xi) [\Phi_\varphi(\xi \cos \varphi) - \Phi_*(\xi + \theta_0^{(\varphi)})] d\xi = 0, \quad (4.7.64)$$

which is possible for small $|\varphi|$ on account of the normalization $\langle \psi_*, \Phi_*' \rangle = 1$. In addition, we introduce three sequences

$$(\theta^{(\varphi)}, v^{(\varphi)}, w^{(\varphi)}) : \mathbb{Z} \rightarrow \mathbb{R} \times H^1 \times H^1 \quad (4.7.65)$$

by writing

$$\theta_l^{(\varphi)} = \theta_0^{(\varphi)} + l \tan \varphi, \quad (4.7.66)$$

together with

$$\begin{aligned} v_l^{(\varphi)} &= T_{\theta_l^{(\varphi)}} [\Phi_\varphi(\cos \varphi [\cdot - \theta_0^{(\varphi)}]) - \Phi_*], \\ w_l^{(\varphi)} &= T_{\theta_l^{(\varphi)}} [\Phi_\varphi(\cos \varphi [\cdot - \theta_0^{(\varphi)}]) - \Phi_\varphi(\cos \varphi [\cdot - \theta_0^{(\varphi)} - \tan \varphi])]. \end{aligned} \quad (4.7.67)$$

By construction, we have

$$Q_{\theta_l^{(\varphi)}} v_l^{(\varphi)} = 0, \quad (4.7.68)$$

together with

$$\begin{aligned} \Xi_l^{(\phi)} &= T_{\theta_l^{(\varphi)}} \Phi_* + v_l^{(\varphi)}, \\ \Xi_l^{(\varphi)} - \Xi_{l-1}^{(\varphi)} &= w_l^{(\varphi)}. \end{aligned} \quad (4.7.69)$$

Applying Lemma 4.3.1, we hence see that the triplet $(\theta^{(\varphi)}, v^{(\varphi)}, w^{(\varphi)})$ is a solution to the problem (4.3.42) with $c = d_\varphi$.

Upon writing

$$\bar{\beta}_\varphi = Q_0 [\Phi_\varphi(\cos \varphi [\cdot - \theta_0^{(\varphi)}]) - \Phi_\varphi(\cos \varphi [\cdot - \theta_0^{(\varphi)} - \tan \varphi])], \quad (4.7.70)$$

we see that

$$Q_{\theta_l^{(\varphi)}} w_l = \bar{\beta}_\varphi \quad (4.7.71)$$

for all $l \in \mathbb{Z}$. Whenever $|\varphi|$ is sufficiently small, item (vi) of Proposition 4.7.1 now implies that

$$f_\beta(\bar{\beta}_\varphi, d_\varphi) = 0, \quad (4.7.72)$$

together with

$$f_\theta(\bar{\beta}_\varphi, d_\varphi) = \tan \varphi. \quad (4.7.73)$$

In view of the expansion

$$\bar{\beta}_\varphi = \varphi + O(\varphi^2), \quad (4.7.74)$$

we find

$$\frac{1}{2} \nu_1 [\partial_\varphi^2 d_\varphi]_{\varphi=0} + \nu_2 = 0 \quad (4.7.75)$$

and hence

$$\nu_2 = -[\partial_z^2 \lambda_z]_{z=0}^{-1} [\partial_\varphi^2 d_\varphi]_{\varphi=0}. \quad (4.7.76)$$

Proof of Proposition 4.3.5. We define the function h via

$$h(\theta, \kappa, c) = h_*(\theta, 0, \kappa, c). \quad (4.7.77)$$

The results now follow directly from Proposition 4.7.1 together with the expressions (4.7.41) and (4.7.76) for ν_1 and ν_2 . $\square$

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Samenvatting

Laten we ons voorstellen dat we in een stadion naar een wedstrijd van ons favoriete team kijken. Tijdens het kijken van de wedstrijd vinden we het fijn om te zitten of te staan, maar alles daartussenin is erg oncomfortabel. Plots begint iemand een *Wave*. In een kwestie van een paar seconden staat iedereen één voor één op, zodat de golf langs het hele stadion beweegt.

Elke keer dat we ergens aan denken, worden de neuronen in onze hersenen geactiveerd en beginnen ze elektrische signalen af te vuren via hun axonen. Het axon is een zenuwvezel die bedekt is met een myeline-coating, maar die op regelmatige afstanden (ongeveer elke 1mm) openingen toelaat. Deze openingen zijn ongeveer $1\mu\text{m}$ lang en worden *Nodes of Ranvier* (knoppunten van Ranvier) genoemd. Als we besluiten deze elektrische signalen alleen in de Nodes of Ranvier te observeren, zien we een spanningspiek die met een constante snelheid van een knooppunt naar de volgende beweegt.

In dit proefschrift bespreken we lopende golf oplossingen voor de Nagumo-vergelijking wanneer deze wordt beschouwd op een eendimensionale discrete ruimte (zoals de rij van Nodes of Ranvier op een zenuwvezel of een enkele rij toeschouwers in het stadion) of tweedimensionale discrete ruimtes (zoals een oneindig groot zitgedeelte in een stadion). Zulke problemen worden Lattice Differential Equations (LDE's) genoemd.

Als we de continue tegenhangers beschouwen, zoals het elektrische signaal dat door het volledige axon beweegt, of normale golven op het zeeoppervlak, kunnen we een redelijk goed ontwikkelde theorie gebruiken. Dit type problemen behoren tot de klasse van Partial Differential Equations (PDE's). In het geval van discrete ruimtes, is onze kennis echter vrij beperkt in vergelijking met het continue geval.

Het verschil in ruimtelijke structuur vereist verschillende wiskundige benaderingen, die verschillende conclusies kunnen geven. We zullen de resultaten voor LDE's in twee typen verdelen.

Aan de ene kant kunnen we proberen om ideeën en technieken te gebruiken die zijn ontwikkeld voor PDE's, waarbij we vaak technische problemen tegenkomen. Stel dat de Wave in het stadion wordt gestart door een groep mensen die niet goed is in het starten van Waves, waardoor een soort hoek midden vooraan wordt gemaakt. In Hoofdstuk 4 zullen we enkele resultaten presenteren over hoe dit soort gebroken golven zich door het hele stadion kunnen verspreiden. Het onderwerp is een goed

uitgangspunt als we geïnteresseerd zijn in het bestuderen van de dynamische processen achter de groei van kristallen.

Aan de andere kant kunnen we denken aan fenomenen die alleen in discrete ruimtes voorkomen, maar die geen directe tegenhanger hebben in de wereld van PDE's. In Hoofdstuk 2 bestuderen we het gedrag van Waves in een rij toeschouwers in een stadion waarin lange en korte mensen worden afgewisseld. Om precies te zijn, zullen we het bestaan, de stabiliteit en de snelheid van de zogenaamde *bichromatische* (2-kleurige) golven bespreken, die de homogene vaste toestanden van 0 en 1 met 2— periodieke heterogene toestanden verbinden. Zoals we zullen zien, komen dit soort golven alleen op roosters voor, omdat het helemaal niet duidelijk is hoe een continue limiet van dergelijke ruimtelijke structuren zou kunnen worden gedefinieerd.

In Hoofdstuk 3 zullen we de theorie van Hoofdstuk 2 uitbreiden naar n — periodieke vaste stabiele toestanden voor $n \geq 3$. We zullen zien dat naarmate we de periodiciteit vergroten, steeds meer verschijnselen optreden. Voor $n \geq 3$ vinden we bijvoorbeeld dat heterogene vaste toestanden verdwijnen en vervolgens weer verschijnen als de diffusiecoëfficiënt toeneemt. Ook vinden we 4-kleurige golven die op elkaar kunnen botsen en dan worden omgezet in bewegende monochromatische golven, degenen die simpelweg 0 met 1 verbinden.

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Curriculum Vitae

Leonardo Morelli was born in Rome, Italy, on the 17th October 1989. He completed his high school studies at *Liceo Scientifico Augusto Righi* in 2008. He subsequently enrolled in the BSc in Mathematics at *La Sapienza University* in Rome. After he graduated in 2011, he continued his studies with a MSc in Applied Mathematics in the same university. In 2014 he graduated under the supervision of Prof. Dr. Marco Isopi, by defending his thesis "*Rumor spreading and SIR model on random evolving graphs*". In September 2014 he moved to Leiden, the Netherlands, to start his Ph.D. in the dynamical systems group under the supervision of Dr. Hermen Jan Hupkes at *Leiden University*. In 2016 he helped organizing the *NDNS+ PhD Days* conference in Lunteren. Between 2014 and 2018 he served as teaching assistant in several English and Dutch spoken courses. He also presented his research at several conferences in the Netherlands and the United States of America.

