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Extension of operators on pre-Riesz spaces

Zhang, F.

Citation

Zhang, F. (2018, September 20). *Extension of operators on pre-Riesz spaces*. Retrieved from <https://hdl.handle.net/1887/65567>

Version: Not Applicable (or Unknown)

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Issue Date: 2018-09-20

Chapter 5

Dissipativity and positive off-diagonal property of operators

It is well known that the generator A of a positive contraction semigroup $T(t)_{t \geq 0}$ can be characterized by means of dissipativity of A with respect to some appropriate sublinear functionals on a Banach space X , see [11, Theorem 2.6], and [16, Proposition 7.13] for the case of X being an ordered Banach space with normal positive cone. In Banach lattices, the positivity of $T(t)_{t \geq 0}$ can be also characterized by its generator A which satisfies the “positive off-diagonal” property (it is called “positive minimum principle” in [10, Theorem 1.11]) This is also studied in an ordered Banach space with a cone with nonempty interior, [16, Theorem 7.27]. For more extensive treatments of dissipative operators, we refer the reader to K. J. Engel and R. Nagel [22]. The positive off-diagonal property of operators on ordered normed spaces is also studied by A. Kalauch [28, 29], S. Koshkin [38] and W. Arendt, P. R. Chernoff, and T. Kato [13] etc..

The goal of this chapter is to investigate the description of positivity and contractivity of semigroups through the dissipativity of generators with respect to corresponding sublinear functionals on ordered vector spaces. So the choice of sublinear functionals on ordered vector spaces is crucial. In Section 5.1, we give

two different ways of defining sublinear functionals which only involve the order structure and the norm of the space. One is given by a regular norm and the other one is obtained through a dual function. The latter one turns out to be more effective in studying the positivity and contractivity of semigroups on ordered Banach spaces in Section 5.2. Moreover, in Section 5.3 we will show a representation for positive functionals on Archimedean pre-Riesz spaces, which will be used to study the positive off-diagonal property of operators on the pre-Riesz space $C^1[0, 1]$.

5.1 Half-norms on ordered vector spaces

This section is mainly concerned with some basic definitions of sublinear functionals on ordered vector spaces, and some properties of operators. Two different types of sublinear functionals will be introduced specifically, and some intuitive properties of these functionals will be studied.

Definition 5.1.1. Let $(X, K, \|\cdot\|)$ be an ordered normed vector space. A sublinear function $p: X \rightarrow \mathbb{R}$ is called a **half-norm** if $p(f) + p(-f) > 0$ whenever $f \neq 0$ in X . If, in addition, there exists some constant $C > 0$ such that $p(f) + p(-f) \geq C\|f\|$ for all $f \in X$, then p is called a **strict half-norm**.

For an ordered normed vector space $(X, K, \|\cdot\|)$, let

$$\begin{aligned}\psi(x) &= \text{dist}(-x, K) = \inf\{\|x + y\|; y \in K\} \\ &= \inf\{\|y\|; x \leq y \in X\}, \quad x \in X,\end{aligned}\tag{5.1}$$

then ψ defines a half-norm on X . This ψ is called the **canonical half-norm**. Obviously, if X is a Banach lattice with a canonical half-norm p , then $p(x) = \|x^+\|$ for all $x \in X$.

Remark 5.1.2. Every ordered normed vector space X has a canonical half-norm ψ which is order preserving, i.e., if $x \leq y$ then $\psi(x - y) \leq 0$. Recall that a cone K in a normed space X is called normal if the norm on X is semimonotone, see Chapter

4. If K is closed then it is known that the following statements are equivalent: (1), the cone K is normal; (2), the canonical half-norm is strict; (3), the canonical half-norm is equivalent to the original norm. These properties are studied in [49]. By the open mapping theorem, X cannot be complete with respect to both norms unless K is normal.

For a sublinear function $p: X \rightarrow \mathbb{R}$, a bounded operator T on X is called **p -contractive** if $p(Tf) \leq p(f)$ for all $f \in X$. Similarly, a semigroup $T(t)_{t \geq 0}$ is called **p -contractive** if $T(t)$ is p -contractive for all $t \geq 0$.

We say that the **subdifferential** of p in $f \in X$, denoted by $dp(f)$, is defined by

$$dp(f) = \{\phi \in K'; \langle g, \phi \rangle \leq p(g) \text{ for all } g \in X, \langle f, \phi \rangle = p(f)\}. \quad (5.2)$$

Definition 5.1.3. An operator $A: X \supseteq \mathcal{D}(A) \rightarrow X$ is called **p -dissipative** if for all $f \in \mathcal{D}(A)$ there exists $\phi \in dp(f)$ such that $\langle Af, \phi \rangle \leq 0$; A is called **strictly p -dissipative** if for all $f \in \mathcal{D}(A)$ the inequality $\langle Af, \phi \rangle \leq 0$ holds for all $\phi \in dp(f)$.

If N is the norm function on a Banach lattice X , i.e. $N(f) = \|f\|$, define $N^+(f) = \|f^+\|$. An operator A on X is called **(strictly) dispersive** if A is (strictly) N^+ -dissipative, i.e., if for every $f \in \mathcal{D}(A)$, one has $\langle Af, \phi \rangle \leq 0$ for some (resp. all) $\phi \in dN^+(f)$, where $dN^+(f)$ is given by

$$dN^+(f) = \{\phi \in K'; \|\phi\| \leq 1, \langle f, \phi \rangle = \|f^+\|\}.$$

Obviously, if p is the canonical half norm, then the p -dissipative operators are dispersive.

Recall that the regular norm on an ordered vector space defined by (3.1), and studied in Chapter 3. By Theorem 3.1.2, there is a natural way to extend the regular norms on ordered vector spaces. In the following propositions, we will see that the sublinear functionals on ordered vector spaces induced by the regular norms have intriguing properties.

Proposition 5.1.4. Let (X, K) be a partially ordered vector space and $\|\cdot\|_r$ a regular seminorm on X . Let p be the sublinear functional on X defined by

$$p(x) = \inf\{\|y\|_r; y \in X, y \geq 0, y \geq x\}, \forall x \in X. \quad (5.3)$$

Then p is a strict half-norm on X . Moreover, let ψ on X be defined by (5.1), then $p(x) = \psi(x)$ for all $x \in K \cup (-K)$, in particular, $p(x) = \psi(x) = 0$ for $x \in (-K)$.

Proof. Let $x \neq 0$ be in X . If $y \geq 0$, $y \geq x$ and $z \geq 0$, $z \geq -x$, then $-(y+z) \leq x \leq (y+z)$. Since $\|\cdot\|_r$ is regular, we have $\|x\|_r \leq \|y+z\|_r$. So $\|y\|_r + \|z\|_r \geq \|y+z\|_r \geq \|x\|_r > 0$. Hence $p(x) + p(-x) \geq \|x\|_r > 0$. Thus p is a strict half-norm.

Moreover, if $0 \leq x \in X$, we can take $y = x \geq 0$ in (5.3), then $p(x) = \|x\|_r = \psi(x)$. It is clear that $p(x) = \psi(x) = 0$ for $x \in X$ with $x \leq 0$. $\square$

Proposition 5.1.5. Let X be a partially ordered vector space, $\|\cdot\|_r$ a regular norm on X , and p on X defined by (5.3). If $T \in L(X)^+$ is a contractive operator with respect to the regular norm $\|\cdot\|_r$, then T is p -contractive.

Proof. Let $T \in L(X)^+$. For $0 \leq y \in X$, suppose $\|Ty\|_r \leq \|y\|_r$. Then for $x \in X$,

$$\begin{aligned} p(x) &= \inf\{\|y\|_r; y \geq 0, y \geq x\} \\ &\geq \inf\{\|Ty\|_r; y \geq 0, y \geq x\} \\ &\geq \inf\{\|z\|_r; z \geq 0, z \geq Tx\} \\ &= p(Tx). \end{aligned}$$

So T is p -contractive. $\square$

We continue with a different approach of sublinear functionals on ordered vector spaces, which turns out to be more useful in dealing with the contractivity and positivity of semigroups.

Proposition 5.1.6. Let X be a partially ordered vector space. For every monotone sublinear functional $\phi \in \mathbb{R}^X$, i.e. $\phi(x) \leq \phi(y)$ whenever $0 \leq x \leq y$ in X , define the sublinear functional p_ϕ on X by

$$p_\phi(f) = \inf\{\langle g, \phi \rangle; g \in X, g \geq 0, g \geq f\}, \forall f \in X. \quad (5.4)$$

Then p_ϕ is sublinear. $p_\phi(f) = \langle f, \phi \rangle$ if $f \geq 0$, and $p_\phi(f) = 0$ if $f \leq 0$. Moreover, if X is a Banach lattice, then $p_\phi(f^+) = p_\phi(f)$ for $f \in X$.

Proof. The sublinearity of p_ϕ follows from ϕ is sublinear directly. It is clear that $p_\phi(f) = \langle f, \phi \rangle$ for $f \geq 0$, and $p_\phi(f) = 0$ for $f \leq 0$.

If X is a Banach lattice, let $f \in X$,

$$\begin{aligned} p_\phi(f^+) &= \inf\{\langle h, \phi \rangle; h \geq f^+, h \geq 0\} \\ &= \inf\{\langle h, \phi \rangle; h \geq \inf\{g; g \geq f, g \geq 0\}, h \geq 0\} \\ &\geq \inf\{\langle h, \phi \rangle; h \geq f, h \geq 0\} = p_\phi(f). \end{aligned}$$

Because $h \geq f$ and $h \geq 0$ implies $h \geq f^+$, and ϕ is monotone, we have $\{\langle h, \phi \rangle; h \geq f, h \geq 0\} \subseteq \{\langle h, \phi \rangle; h \geq f^+, h \geq 0\}$, and hence $\inf\{\langle h, \phi \rangle; h \geq f, h \geq 0\} \geq \inf\{\langle h, \phi \rangle; h \geq f^+, h \geq 0\}$. So we have $p(f) \geq p(f^+)$. $\square$

The differential function space $X = C^1[0, 1]$, or the Sobolev space $X = W^{n,p}$ with respect to its own norms are typical examples of ordered Banach spaces. On these spaces, the norms are not monotone, but we could define a sublinear functional p_ϕ by (5.4).

5.2 Contractivity and positivity of semigroups on ordered Banach spaces

In this section, we will give sufficient conditions on the generator A under which a semigroup $T(t)_{t \geq 0}$ is contractive with respect to p defined by (5.4), or $T(t)_{t \geq 0}$ is positive on an ordered Banach space.

Firstly, we will see that if A is supposed to be p -dissipative and $(I - \lambda A)$ is invertible for some $\lambda \geq 0$, then $T(t)_{t \geq 0}$ is p -contractive.

Theorem 5.2.1. Let X be an ordered Banach space, $\phi: X \rightarrow \mathbb{R}$ be a positive linear functional, and p_ϕ on X defined by (5.4). Let $A: X \supseteq \mathcal{D}(A) \rightarrow X$ be p_ϕ -dissipative. If $(I - \lambda A)$ is invertible for some $\lambda > 0$, then $(I - \lambda A)^{-1}$ is p_ϕ -contractive. In addition, if $T(t)_{t \geq 0}$ is a strongly continuous semigroup generated by A , then $T(t)$ is p_ϕ -contractive for every $t \geq 0$.

Proof. For a fixed $f \in \mathcal{D}(A)$, let $\psi \in dp_\phi(f)$ be such that $\langle Af, \psi \rangle \leq 0$. Then $\langle g, \psi \rangle \leq p(g)$ for all $g \in X$ and hence $|\langle g, \psi \rangle| \leq p(g)$. For some $\lambda_0 > 0$ one has that

$$\begin{aligned} p_\phi((\lambda_0 I - A)f) &\geq |(\lambda_0 I - A)f, \psi| \geq \operatorname{Re} \langle (\lambda_0 I - A)f, \psi \rangle \\ &= \operatorname{Re}(\langle \lambda_0 f, \psi \rangle - \langle Af, \psi \rangle) \geq \operatorname{Re} \langle \lambda_0 f, \psi \rangle \\ &= \operatorname{Re} \lambda_0 \langle f, \psi \rangle = \operatorname{Re} \lambda_0 p(f) \\ &= \lambda_0 p(f). \end{aligned}$$

So if $\lambda > 0$ is such that $(I - \lambda A)$ is invertible, then $(I - \lambda A)^{-1}$ is p_ϕ -contractive for some $\lambda \geq 0$.

In addition, for f in X and $t \geq 0$,

$$T(t)f = \lim_{n \rightarrow \infty} \left[\frac{n}{t} R\left(\frac{n}{t}, A\right) \right]^n f = \lim_{n \rightarrow \infty} \left(I - \frac{t}{n} A \right)^{-n} f.$$

So $T(t)$ is p_ϕ -contractive for every $t \geq 0$. □

Recall that a nonempty subset $\Phi \subseteq K'$ is called **total** if $\phi(x) \geq 0$ for each $\phi \in \Phi$ implies $x \geq 0$; the original definition is in [39, p. 102]. For example, if $X = C[0, 1]$, then $\Phi = \{\delta_t; t \in [0, 1]\}$ is total. To use this property, we claim that $dp(f) \cap \Phi \neq \emptyset$ for p in (5.4). In fact, for p defined in (5.4), let $t \in [0, 1]$ and $\phi = \delta_t$, then $p_t(f) = \inf\{\delta_t(g); g \geq 0, g \geq f\} = \inf\{g(t); g \geq 0, g \geq f\} = f^+(t)$. So if $f(t) \leq 0$ then $p_t(f) = 0$. We only need to consider $f \geq 0$. Let $\psi = \delta_t$, then $\psi(g) = \delta_t(g) = g(t) \leq g^+(t) = p_t(g)$, and $\psi(f) = \delta_t(f) = f(t) = f^+(t) = p_t(f)$. So $\psi \in dp(f)$. In this case, if A is the multiplication operator which multiplies a negative function, then A is p -dissipative.

The next theorem shows that the positivity of $T(t)_{t \geq 0}$ is obtained by supposing A is p_ϕ -dissipative for ϕ in a total set of X .

Theorem 5.2.2. Let X be an ordered Banach space. Let $A: X \supseteq \mathcal{D}(A) \rightarrow X$ be the generator of strongly continuous semigroup $T(t)_{t \geq 0}$. If A is p_ϕ -dissipative for all ϕ in a total set Φ and $(I - \lambda A)$ is invertible for some $\lambda \geq 0$, then $T(t)$ is positive for all $t \geq 0$.

Proof. Suppose that A is p_ϕ -dissipative for all $\phi \in \Phi$. Let $T(t)_{t \geq 0}$ be the semigroup generated by A . Let $\phi \in \Phi$, select $f \leq 0$ in X , by Theorem 5.2.1, one has that $p_\phi(T(t)f) \leq p_\phi(f)$, which means

$$\inf\{\langle g, \phi \rangle; g \geq T(t)f, g \geq 0\} \leq \inf\{\langle h, \phi \rangle; h \geq f, h \geq 0\}.$$

Take $h = 0$, then the right side of the above inequality is 0. It follows that

$$\inf\{\langle g, \phi \rangle; g \geq T(t)f, g \geq 0\} \leq 0.$$

Because of $g \geq T(t)f$, then $\langle g, \phi \rangle \geq \langle T(t)f, \phi \rangle$ for ϕ is positive. So

$$\langle T(t)f, \phi \rangle \leq \inf\{\langle g, \phi \rangle; g \geq T(t)f, g \geq 0\} \leq 0.$$

Since Φ is total, one has that $\langle T(t)f, \phi \rangle \leq 0$ for all $\phi \in \Phi$ implies $T(t)f \leq 0$. Thus $T(t)$ is positive. $\square$

Remark 5.2.3. Notice that [10, Theorem 1.2] says that, if A is densely defined on a Banach lattice, then $T(t)_{t \geq 0}$ is positive and contractive if and only if A is dispersive and $(\lambda - A)$ is surjective for some $\lambda > 0$. By Theorem 5.2.1, we could generalize one direction of this conclusion to ordered Banach spaces. We will illustrate this through an example of a second derivative operator with Dirichlet boundary condition. This example originally comes from [10, Example 1.5].

Example 5.2.4. Let $X = (C^n[0, 1], \|\cdot\|)$ be a Banach space, the densely defined operator A be the second derivative operator with Dirichlet boundary condition. Then the domain satisfies $\mathcal{D}(A) = \{f \in C^{n+2}[0, 1]; f(0) = f(1) = f''(0) = f''(1)\}$. Choose $p(f) = \|f^+\|_\infty$ for $f \in X$, then there exists $x \in (0, 1)$ such that $f(x) = \sup_{y \in [0, 1]} f(y) = \|f^+\|_\infty$. Since $\langle g, \delta_x \rangle = g(x) \leq \|g^+\|_\infty = p(g)$ for all $g \in X$, $\langle f, \delta_x \rangle = f(x) = \|f^+\|_\infty$, so $\delta_x \in dp(f)$. We have $\langle Af, \delta_x \rangle = f''(x) \leq 0$ since f has a maximum in X . So A is p -dissipative. Let $g \in X$, define $f_0(x) = \frac{1}{2}[e^x \int_x^1 e^{-y} g(y) dy - e^{-x} \int_x^1 e^y g(y) dy]$. Then there exist $m, n \in \mathbb{R}$ such that $f(x) = f_0(x) + me^x + ne^{-x}$ and $f(0) = f(1) = 0$, and then $f \in \mathcal{D}(A)$. Since $f - f'' = f_0 - f_0'' = g$, we have $(I - A)$ is surjective. For $f \in \mathcal{D}(A)$, suppose that $(I - A)f = 0$, then $f(x) = \alpha e^x + \beta e^{-x}$. Notice that $f(0) = f(1) = 0$ such that $\alpha = \beta = 0$, so $f(x) = 0$ for $x \in [0, 1]$. So $(I - A)$ is injective. Take $\phi = \|\cdot\|_\infty$ we have that p is given by (5.4). It follows from Theorem 5.2.1 that A is the generator of a contractive semigroup.

If $g \in X$ and $g \leq 0$, then $p(T(t)) \leq p(g) = \|g^+\|_\infty = 0$. So $\|(T(t)g)^+\|_\infty = 0$, so $T(t)g \leq 0$ for every $t \geq 0$. Hence $T(t)_{t \geq 0}$ is positive.

Remark 5.2.5. It is worth to mention that in an ordered Banach space, specifically $C^1[0, 1]$, the dispersivity of A will fail, in general, with respect to the original norm. However, by the above discussion, we still have flexibility to choose a functional p as in (5.4), which is only depends on a functional ϕ . This is also different from the arguments in [10, Example 1.5].

5.3 Positive off-diagonal property of operators on ordered vector spaces

In this section, we will introduce the positive off-diagonal property especially on pre-Riesz spaces, in particular $C^1(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ open. Explicitly, we investigate a representation theorem for positive functionals on Archimedean pre-Riesz spaces, which is also interesting independently.

Definition 5.3.1. The linear operator $A: X \supseteq \mathcal{D}(A) \rightarrow X$ is said to have the **positive off-diagonal property** if $\langle Au, \phi \rangle \geq 0$ whenever $0 \leq u \in \mathcal{D}(A)$ and $0 \leq \phi \in X^*$ with $\langle u, \phi \rangle = 0$.

The motivation of the positive off-diagonal property comes from matrix theory, where the off-diagonal elements of the matrix $A = (a_{ij})$ are positive, i.e., $a_{ij} \geq 0$ for all $i \neq j$. It is shown in [16, Lemma 7.23] that on an ordered Banach space with an order unit u such that $u \in \mathcal{D}(A)$, A has the positive off-diagonal property and $Au \leq 0$ if and only if A is Ψ_u -dissipative, where Ψ_u is the order unit function, i.e. $\Psi_u(x) = \inf\{\lambda \geq 0; x \leq \lambda u\}$, $x \in X$. However, the dissipativity and the positive off-diagonal property are independent, as the following example shows.

Example 5.3.2. In general, the properties that A has the positive off-diagonal property and A is p -dissipative do not imply each other. In fact, let $X = \mathbb{R}^2$, take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then A has the positive off-diagonal property. Take $f = (1, 0)$, then

$\phi = (1, 0) \in dp(f)$ but $\langle Af, \phi \rangle = 1$, so A is not dispersive. Let $A = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$, and $\mathcal{D}(A) = \{x = (x_1, x_2) \in X; x_1 \geq 0, x_2 = 0\}$. Take $\phi = (1, 0)$, then $\phi \in dN^+(f)$ for every $f \in \mathcal{D}(A)$. It is obvious that $\langle Af, \phi \rangle \leq 0$. So A is dispersive, but does not have positive off-diagonal property.

Next, we consider a representation theorem in pre-Riesz spaces.

Theorem 5.3.3. Let X be an Archimedean pre-Riesz space with order unit. Then there exists a compact Hausdorff space Ω and a bipositive linear map $i: X \rightarrow$

$C(\Omega)$ such that $i(X)$ is order dense in $C(\Omega)$. Moreover, for every positive linear functional $\phi: X \rightarrow \mathbb{R}^+$, there exists a regular Borel measure μ on Ω such that

$$\phi(x) = \int_{\Omega} i(x)(\omega) d\mu(\omega), \quad x \in X, \omega \in \Omega.$$

Proof. The result of first part of this theorem comes from [31, Lemma 6].

For the second part, let $C(\Omega)$ be the continuous function space where Ω is a compact Hausdorff space. Let $i: X \rightarrow C(\Omega)$ be a bipositive linear map such that $i(X)$ is an order dense subspace of $C(\Omega)$. So for a positive linear functional $\phi: X \rightarrow \mathbb{R}$, one has that $\phi \circ i^{-1}: i(X) \rightarrow \mathbb{R}^+$ is a positive linear functional on $i(X)$. Since $\mathbb{R}$ is Dedekind complete, and $i(X)$ is a majorizing subspace of $C(\Omega)$, by Theorem 1.3.6 (Kantorovich), there exists an extension of $\phi \circ i^{-1}$ to a positive functional $\psi: C(\Omega) \rightarrow \mathbb{R}$. By the Riesz representation theorem, for ψ on $C(\Omega)$, there exists a unique regular Borel measure μ on Ω such that

$$\psi(f) = \int_{\Omega} f(\omega) d\mu(\omega), \quad \forall f \in C(\Omega), \omega \in \Omega.$$

So for every $x \in X$, one has $\phi \circ i^{-1}(i(x)) = \psi(i(x))$. If we take $f = i(x)$, then

$$\phi \circ i^{-1}(i(x)) = \psi(i(x)) = \int_{\Omega} i(x)(\omega) d\mu(\omega).$$

Thus we get the conclusion. □

We give an example to illustrate that the positive off-diagonal property of A can be generalized to a special kind of partially ordered vector space, in particular pre-Riesz space $C^1[0, 1]$.

Example 5.3.4. Let $C[0, 1]$ be the real continuous functions. Let $X = C^1[0, 1]$ which is an Archimedean pre-Riesz space, then X is an order dense subspace of $C[0, 1]$. Let $A \in L(X)$ be a densely defined operator, we claim that A has positive off-diagonal property if and only if $(Au)(\omega) \geq 0$ whenever $0 \leq u \in \mathcal{D}(A)$ and $\omega \in [0, 1]$ with $u(\omega) = 0$. In fact, first suppose that A has the positive off-diagonal

property and $0 \leq u \in \mathcal{D}(A)$, $\omega \in \Omega$ with $u(\omega) = 0$. Take $0 \leq \phi_\omega \in X^*$ to be the point evaluation at ω , then it follows from $u(\omega) = \langle u, \phi_\omega \rangle = 0$ that $\langle Au, \phi_\omega \rangle \geq 0$, i.e. $(Au)(\omega) \geq 0$. Conversely, assume $0 \leq u \in \mathcal{D}(A)$, and $0 \leq \phi \in X^*$ with $\langle u, \phi \rangle = 0$. Then by the above Theorem 5.3.3, there exists a regular Borel measure μ that represents ϕ , i.e. $\langle u, \phi \rangle = \int_0^1 i(u)(\omega) d\mu(\omega)$. By assumption, we have $i(u)(\omega) = 0$ and $u(\omega) = 0$ for all ω in the support of μ , then $(Au)(\omega) \geq 0$ and $i(Au)(\omega) \geq 0$. Hence $\langle Au, \phi \rangle = \int_0^1 i(Au)(\lambda) d\mu(\lambda) \geq 0$. This shows that A has the positive off-diagonal property.