Extension of operators on pre-Riesz spaces
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Title: Extension of operators on pre-Riesz spaces
Issue Date: 2018-09-20

Chapter 4

Disjointness preserving semigroups

This chapter has been published as:

A. Kalauch, O. van Gaans and F. Zhang. Disjointness preserving C_0-semigroups and local operators on ordered Banach spaces. *Indag. Math. (N.S.)*, 29(2):535-547 2018.

In the theory of C_0-semigroups, many results involve the order structure of the underlying Banach space. For instance, a rich theory of disjointness preserving semigroups and semigroups with local generators has been developed in [10, 12]. The Banach spaces in these works are Banach lattices. In this chapter, we investigate how results on disjointness preserving C_0-semigroups on Banach lattices can be generalized to the more general setting of ordered Banach spaces.

Our main result of this chapter is that the generator of a disjointness preserving C_0-semigroup on a suitable ordered Banach space is local, as in the Banach lattice case. It turns out that the choice of norms is the main issue of the analysis. We consider semimonotone norms for which the cone of positive elements is closed. On Banach spaces, those norms are equivalent to regular norms, which are a natural generalization of lattice norms.

This chapter is organized as follows. Properties of the norms are discussed in Section 4.1. Section 4.2 contains a discussion of local operators in different settings and some of their basic properties. In Section 4.3, we present our main result of this chapter. Moreover, we establish two results on local operators generating disjointness preserving C_0-semigroups. The first one considers a bounded generator and uses Taylor series, the second one uses resolvent operators and Yosida approximations.

4.1 Normed partially ordered vector spaces

Let (X, K) be a partially ordered vector space with a seminorm p, p is called **semimonotone** if there exists a constant $C \in \mathbb{R}^+$ such that for every $x, y \in X$ with $0 \leq x \leq y$ one has $p(x) \leq Cp(y)$. The cone K is said to be **normal** if p is semimonotone. Recall that a seminorm p on X is monotone if $p(x) \leq p(y)$ whenever $x, y \in X$ with $0 \leq x \leq y$.

We start the results of this section by a proposition, due to [56, Theorems IV.2.1 and IV.2.4], which clarifies the relation of semimonotone norms and monotone norms on partially ordered vector spaces.

Proposition 4.1.1. If $\|\cdot\|$ is a semimonotone norm on ordered vector space X, then there exists an equivalent monotone norm on X.

By using this proposition, we could extend the semimonotone norm on a directed partially ordered vector space to a regular seminorm. In fact, such an extension is exactly same with the formula which was given by (3.2) in the Theorem 3.1.2. The only difference is we consider semimonotone norms at here. The details can be seen in the next lemma.

Lemma 4.1.2. Let (X, K) be a directed partially ordered vector space and let $\|\cdot\|$ be a semimonotone norm on X. Let Y be a directed partially ordered vector space and $i\colon X \to Y$ a bipositive linear map, such that $i(X)$ is majorizing in Y. For $y \in Y$ let

$$\rho(y) := \inf\left\{\|x\|;\ x \in X,\ -i(x) \leq y \leq i(x)\right\}. \tag{4.1}$$

If $Y = X$, K is closed and $(X, \|\cdot\|)$ is complete, then $\rho \circ i$ is a regular norm on X that is equivalent to $\|\cdot\|$.

Proof. By Proposition 4.1.1, $\|\cdot\|$ is equivalent to a monotone norm and then, if $(X, \|\cdot\|)$ is complete and K is closed, [51, Corollary 6.4(ii)] says that $\|\cdot\|$ is equivalent to the regular norm $\rho \circ i$. □

The space of Lemma 4.1.2 is actually an ordered Banach space. By an **ordered Banach space** $(X, K, \|\cdot\|)$ we mean a Banach space $(X, \|\cdot\|)$ with a closed generating cone K. Since K is closed, the space (X, K) is Archimedean, and since K is generating, (X, K) is directed. Consequently, (X, K) is a pre-Riesz space and can be embedded into its Riesz completion X^ρ, see Theorem 1.2.7.

The next lemma shows how disjointness can be detected with the aid of the extension given by (4.1).

Lemma 4.1.3. Let $(X, K, \|\cdot\|)$ be a normed partially ordered vector space such that K is closed and generating. Let Y be a vector lattice and $i\colon X \to Y$ a bipositive linear map, such that $i(X)$ is majorizing in Y. For $y \in Y$ let $\rho(y)$ be defined by (4.1). If $u, v \in X$ are such that $\rho(|i(u)| \wedge |i(v)|) = 0$, then $u \perp v$.

Proof. Let $u, v \in X$ be such that $\rho(|i(u)| \wedge |i(v)|) = 0$. As

$$|i(u)| \wedge |i(v)| = \tfrac{1}{2}\left|\,|i(u) + i(v)| - |i(u) - i(v)|\,\right|$$

one obtains that $\rho\left(|i(u) + i(v)| - |i(u) - i(v)|\right) = 0$. Hence for every $n \in \mathbb{N}$ there exists an $x_n \in X$ such that

$$-i(x_n) \leq |i(u) + i(v)| - |i(u) - i(v)| \leq i(x_n) \tag{4.2}$$

and $\lim_{n\to\infty}\|x_n\| = 0$. The first inequality in (4.2) yields

$$\pm(i(u) - i(v)) - i(x_n) \leq |i(u) + i(v)|,$$

hence for $x \geq \pm(u + v)$ it follows that $x \geq \pm(u - v) - x_n$ for every $n \in \mathbb{N}$. Since K is closed and $x_n \to 0$, one obtains $x \geq \pm(u - v)$. We conclude

$$\{u + v, -u - v\}^{\mathrm{u}} \subseteq \{u - v, -u + v\}^{\mathrm{u}}. \tag{4.3}$$

From the second inequality in (4.2) it follows that

$$\pm(i(u) + i(v)) - i(x_n) \leq |i(u) - i(v)|,$$

and an analogous argumentation yields equality in (4.3), which implies that u and v are disjoint. □

Recall that a subspace $B \subseteq X$ is a band in an ordered vector space X if $B = B^{\mathrm{dd}}$. It turns out that bands are closed for regular norms.

Lemma 4.1.4. If

(i) (X, K) is a pre-Riesz space with a regular norm $\|\cdot\|$ such that K is closed, or

(ii) $(X, K, \|\cdot\|)$ is an ordered Banach space with a semimonotone norm,

then every band in X is closed.

Proof. According to Lemma 4.1.2, the conditions in (ii) yield that the norm $\|\cdot\|$ is equivalent to a regular norm, so it suffices to give a proof under condition (i).

Let B be a band in X, let (x_n) be a sequence in B and let $x \in X$ be such that $\|x_n - x\| \to 0$. Let (X^ρ, i) be the Riesz completion of X, as in Theorem 1.2.7. Let ρ be the Riesz seminorm on $Y = X^\rho$ defined as in (4.1).

Let $y \in B^{\mathrm{d}}$. Then for every $n \in \mathbb{N}$ one has $x_n \perp y$, so that by Proposition 1.2.15 one has $i(x_n) \perp i(y)$, which implies $|i(x_n)| \wedge |i(y)| = 0$. Since $\rho(i(x_n)) - i(x)) = \|x_n - x\| \to 0$, the continuity of the lattice operations with respect to ρ yields that $\rho(|i(x)| \wedge |i(y)|) = 0$. Hence, by Lemma 4.1.3, we obtain $x \perp y$. It follows that $x \in B^{\mathrm{dd}} = B$. Hence B is closed. □

4.2 Local operators

In this section we introduce a notion of local operators on pre-Riesz spaces. Also band preserving operators are a well-established class of operators in the theory of vector lattices, see e.g. [6, 42]. In [10, (5.4)], local operators are defined on Banach lattices. Below we use disjointness to define local operators on partially ordered vector space in the spirit of [10].

Throughout this chapter, we will use $\mathcal{D}(T)$ to denote the domain space of the operator T, and $L(X)$ to denote the bounded linear operators on X.

Definition 4.2.1. Let X be a partially ordered vector space and let $T\colon X \supseteq \mathcal{D}(T) \to X$ be a linear operator.

(i) T is called **band preserving** if for every band B in X one has $T(B \cap \mathcal{D}(T)) \subseteq B$.

(ii) T is called **local** if for every $x \in \mathcal{D}(T)$, $y \in X$ with $x \perp y$ it follows that $Tx \perp y$.

Local operators turn out to be a special class of disjointness preserving operators. The class of local operators plays a role in the theory of differential equations, where the notion 'local' is used ambiguously (see Remark 4.2.5 below).

With the above definition, we observe that local operators and band preserving operators coincide in a pre-Riesz space.

Proposition 4.2.2. Suppose that X is a pre-Riesz space and let $T\colon X \supseteq \mathcal{D}(T) \to X$ be a linear operator. T is local if and only if T is band preserving.

Proof. Let T be local. For every $x \in \mathcal{D}(T)$ one obtains $Tx \in \{x\}^{\mathrm{dd}}$. Indeed, for every $y \in \{x\}^{\mathrm{d}}$ it follows that $Tx \perp y$, therefore $Tx \in \{x\}^{\mathrm{dd}}$. Let B be a band in X and $x \in B \cap \mathcal{D}(T)$. Then $\{x\}^{\mathrm{dd}} \subseteq B^{\mathrm{dd}} = B$, hence $Tx \in B$. We conclude that T is band preserving.

Now let T be band preserving, $x \in \mathcal{D}(T)$ and $y \in X$ such that $x \perp y$. Then $\{y\}^{\mathrm{d}}$ is a band in X, and $x \in \{y\}^{\mathrm{d}} \cap \mathcal{D}(T)$, hence $Tx \in \{y\}^{\mathrm{d}}$, which yields $Tx \perp y$. Consequently T is local. □

Typical examples of local operators are differential operators and multiplication operators. We discuss some of these settings in the subsequent remarks.

Remark 4.2.3. If X is a Banach lattice, every bounded local operator on X is contained in the *center*

$$\mathrm{Z}(X) := \{T \in L(X);\ \exists \alpha > 0 :\ -\alpha I \leq T \leq \alpha I\}.$$

More precisely, the following assertions are equivalent for a bounded linear operator $T\colon X \to X$ (see e.g. [44, Section 9]):

(i) T is local,

(ii) $-\|T\|I \leq T \leq \|T\|I$,

(iii) for every ideal J in X one has $T[J] \subseteq J$.

In [44, Section 9] the following two examples are given to illustrate that local operators are closely related to multiplication operators.

(a) Let (Ω, μ) be a σ-finite measure space and $X = L^p(\Omega, \mu)$ $(1 \leq p \leq \infty)$, then $\mathrm{Z}(X)$ is isomorphic to $L^\infty(\Omega, \mu)$ via the identification of $y \in L^\infty(\Omega, \mu)$ with the operator from X to X given by $x \mapsto yx$.

(b) Let Ω be a locally compact Hausdorff space and X the space $C_0(\Omega)$ of all continuous functions x on Ω vanishing at infinity (i.e. for every $\varepsilon > 0$ there is a compact set $S \subseteq \Omega$ such that for every $t \in \Omega \setminus S$ one has $|x(t)| < \varepsilon$), endowed with the supremum norm. $\mathrm{Z}(X)$ is isomorphic to the space $C^{\mathrm{b}}(\Omega)$ of bounded continuous functions via the identification of $y \in C^{\mathrm{b}}(\Omega)$ with the operator from X to X given by $x \mapsto yx$.

Remark 4.2.4. We continue the discussion of case (a) above for unbounded T. Let $X = L^p(\Omega, \mu)$ $(1 \leq p \leq \infty)$ and $T\colon L^p(\Omega, \mu) \supseteq \mathcal{D}(T) \to L^p(\Omega, \mu)$. A notion of 'locality' in this setting is defined in [23, I.4.13(8)]. To avoid confusion, we denote this property by (L):

(L) For every measurable set $S \subseteq \Omega$ and for every $x \in \mathcal{D}(T)$ with $x = 0$ almost everywhere on S it follows that $Tx = 0$ almost everywhere on S.

T is local if and only if (L) is satisfied. Indeed, suppose that (L) is satisfied and let $x \in \mathcal{D}(T)$, $y \in X$ and $x \perp y$. Then $S := \{t \in \Omega;\, y(t) \neq 0\}$ is measurable and $x = 0$ almost everywhere on S, hence $Tx = 0$ almost everywhere on S, which implies $Tx \perp y$. Therefore T is local. Now suppose that T is local and let $S \subseteq \Omega$ be measurable and $x \in \mathcal{D}(T)$ be such that $x = 0$ almost everywhere on S. Then $x \perp \chi_S$, hence $Tx \perp \chi_S$, which implies $Tx = 0$ on S. Consequently (L) is satisfied. In the present setting, a local operator need not be a multiplication operator, consider e.g. the operator $T\colon L^p[0,1] \supseteq C^1[0,1] \to L^p[0,1]$, $x \mapsto x'$.

Remark 4.2.5. We continue the discussion of (b) in Remark 4.2.3, where we now consider the special case where Ω is a compact Hausdorff space and $X = C(\Omega)$. For every bounded local operator $T\colon C(\Omega) \to C(\Omega)$ there is $y \in C(\Omega)$ such that $T\colon x \mapsto yx$. (This can be deduced from the fact that $C(\Omega)$ is an Archimedean f-algebra with unit and that T is band preserving and order bounded, see [6, Theorem 8.27].) We discuss several notions of locality for unbounded operators T.

(I) In [12, Theorem 3.7] a linear operator $T\colon C(\Omega) \supseteq \mathcal{D}(T) \to C(\Omega)$ is called *local* if the following property is satisfied:

$$\forall\, x \in \mathcal{D}(T),\ x \geq 0,\ \omega \in \Omega \ \text{ with } \ x(\omega) = 0 \ \Rightarrow \ (Tx)(\omega) = 0. \tag{4.4}$$

If (4.4) is satisfied for an operator T, then T is local in the sense of Definition 4.2.1. The converse implication is not true, in general. Indeed, consider $\mathcal{D}(T) := \{x \in C^2[0,1]; x'(0) = x'(1) = 0\}$ and

$$T\colon C[0,1] \supseteq \mathcal{D}(T) \to C[0,1],\ x \mapsto x''$$

(cf. the one-dimensional diffusion semigroup in [43, 2.7]). On one hand, T does not satisfy (4.4), take e.g. $x(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2$, then $x(t)|_{t=0} = 0$ but $(Tx)(t)|_{t=0} = -1$. On the other hand, T is local. Indeed, let $x \in \mathcal{D}(T)$, $y \in \mathrm{C}[0,1]$ with $x \perp y$, and $N := \{t \in [0,1]; x(t) = 0\}$. For $t \in \operatorname{int} N$ one has $(Tx)(t) = 0$, whereas for $t \in \partial N$ and every $n \in \mathbb{N}$ there is $t_n \in \left[t - \frac{1}{n}, t + \frac{1}{n}\right] \cap [0,1]$ such that $x(t_n) \neq 0$, i.e. $y(t_n) = 0$, which implies $y(t) = 0$. Hence $Tx \perp y$, consequently T is local.

(II) An operator $T\colon C(\Omega) \supseteq \mathcal{D}(T) \to C(\Omega)$ satisfies (4.4) if and only if for every open set $O \subseteq \Omega$ one has $T(I_O \cap \mathcal{D}(T)) \subseteq I_O$, where

$$I_O = \{x \in C(\Omega); \forall \omega \in \Omega \setminus O\colon x(\omega) = 0\},$$

i.e. T preserves for every open set $O \subseteq \Omega$ the largest ideal having O as its carrier.

(III) We relate (4.4) to a notion of locality given in [12, p.147, second Remark]. Let (X, K) be a partially ordered vector space with a norm $\|\cdot\|$. For a linear operator $T\colon X \supseteq \mathcal{D}(T) \to X$ the following property is considered:

$$\forall\, x \in \mathcal{D}(T) \cap K,\ f \in K' \ \text{ with } \ f(x) = 0 \ \Rightarrow \ f(Tx) = 0. \tag{4.5}$$

Note that (4.5) is satisfied if and only if T and $-T$ are *positive-off diagonal* (for a definition of this notion see [16, Definition 7.18] (we will study the positive-off diagonal property in Chapter 5, so this definition can been seen in the next chapter); cf. also the *positive minimum principle* in [10, Definition 1.6]).

Before we link (4.4) and (4.5), we reformulate (4.5) for the case that X is an Archimedean partially ordered vector space with an order unit u. The norm induced by u, which will be denoted by $\|\cdot\|_u$, is defined by

$$\|x\|_u = \inf\{\alpha \in [0,\infty);\ -\alpha u \leq x \leq \alpha u\},\ x \in X.$$

Let

$$\Sigma := \{f\colon X \to \mathbb{R};\ f \text{ positive linear, and } f(u) = 1\},$$

and denote by Λ the set of extreme points of Σ. For $x \in \mathcal{D}(T) \cap K$ the set $N = \{f \in \Sigma;\ f(x) = 0\}$ is the weak* closure of the convex hull of $N \cap \Lambda$ (see also [29, (3.6)]). Therefore (4.5) holds if and only if

$$\forall\, x \in \mathcal{D}(T) \cap K,\ f \in \Lambda \text{ with } f(x) = 0 \Rightarrow f(Tx) = 0. \tag{4.6}$$

For $X = \mathrm{C}(\Omega)$ as above and u the constant-one function, Λ is homeomorphic to Ω, hence the conditions (4.4) and (4.6) are equivalent. This implies the equivalence of (4.4) and (4.5).

Now let us consider some basic results on local operators on pre-Riesz spaces. The following lemma will be needed in the proof of Theorem 4.3.4.

Lemma 4.2.6. Let X be a pre-Riesz space.

(i) If $S\colon X \supseteq \mathcal{D}(S) \to X$ and $T\colon X \supseteq \mathcal{D}(T) \to X$ are local operators and $\alpha, \beta \in \mathbb{R}$, then $\alpha S + \beta T\colon X \supseteq \mathcal{D}(S) \cap \mathcal{D}(T) \to X$ is a local operator.

(ii) If $S\colon X \supseteq \mathcal{D}(S) \to X$ and $T\colon X \supseteq \mathcal{D}(T) \to \mathcal{D}(S) \subseteq X$ are local operators, then $ST\colon X \supseteq \mathcal{D}(T) \to X$ is a local operator.

Proof. (i) Let $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$ and $y \in X$ be such that $x \perp y$. Then $Sx \perp y$ and $Tx \perp y$, so that $Sx, Tx \in \{y\}^{\mathrm{d}}$. Since $\{y\}^{\mathrm{d}}$ is a linear subspace, it follows that $\alpha Sx + \beta Tx \perp y$. Hence $\alpha S + \beta T$ is local.

(ii) Let $x \in \mathcal{D}(T), y \in X$ be such that $x \perp y$. Then $Tx \perp y$ as T is local, so that $STx \perp y$ as S is local and $Tx \in \mathcal{D}(S)$. Hence ST is local. □

Next we consider a result on the inverse of a local operator, which we need in the proof of Corollary 4.3.7 below. It turns out that the inverse of a local operator T is local if both T and T^{-1} are positive.

Proposition 4.2.7. Let (X, K) be a pre-Riesz space, let $T\colon X \supseteq \mathcal{D}(T) \to X$ be a bijective linear operator such that the inclusion map $i\colon \mathcal{D}(T) \to X$ is a Riesz* homomorphism. If both T and T^{-1} are positive and T is local, then T^{-1} is also local.

Proof. As a first step, for $x \in X, y \in \mathcal{D}(T)$ with $x \perp y$ we show that $T^{-1}x \perp y$ in $\mathcal{D}(T)$, which comes down to $\{T^{-1}x + y, -T^{-1}x - y\}^{\mathrm{u}} \cap \mathcal{D}(T) = \{T^{-1}x - y, -T^{-1}x + y\}^{\mathrm{u}} \cap \mathcal{D}(T)$. Let $z \in \{T^{-1}x + y, -T^{-1}x - y\}^{\mathrm{u}} \cap \mathcal{D}(T)$. Then, as T is positive, $Tz \geq x + Ty, -x - Ty$. Since T is local and therefore $Ty \perp x$, we have $Tz \geq x - Ty, -x + Ty$, so $z \geq T^{-1}x - y, -T^{-1}x + y$, as T^{-1} is positive. Therefore, $\{T^{-1}x + y, -T^{-1}x - y\}^{\mathrm{u}} \cap \mathcal{D}(T) \subseteq \{T^{-1}x - y, -T^{-1}x + y\}^{\mathrm{u}} \cap \mathcal{D}(T)$. A similar proof yields the converse inclusion, so that $\{T^{-1}x + y, -T^{-1}x - y\}^{\mathrm{u}} \cap \mathcal{D}(T) = \{T^{-1}x - y, -T^{-1}x + y\}^{\mathrm{u}} \cap \mathcal{D}(T)$. This shows that $T^{-1}x \perp y$ in $\mathcal{D}(T)$.

Since the inclusion map $i\colon \mathcal{D}(T) \to X$ is a Riesz* homomorphism, Lemma 2.2.4 yields that $i(T^{-1}x) \perp i(y)$, hence $T^{-1}x \perp y$ in X. Thus, T^{-1} is local. □

We conclude this section by the following simple observation.

Lemma 4.2.8. Let (X, K) be a pre-Riesz space and $T\colon X \to X$ a bipositive linear bijection. Then T is disjointness preserving.

Proof. Let $x, y \in X$ be such that $x \perp y$. Then $\{x+y, -x-y\}^{\mathrm{u}} = \{x-y, -x+y\}^{\mathrm{u}}$. Hence for every $u \in X$ we have

$$u \geq x + y, -x - y \iff u \geq x - y, -x + y.$$

As T is bipositive, the latter is equivalent to

$$Tu \geq Tx + Ty, -Tx - Ty \iff Tu \geq Tx - Ty, -Tx + Ty.$$

Since T is surjective, this comes down to $\{Tx+Ty, -Tx-Ty\}^{\mathrm{u}} = \{Tx-Ty, -Tx+Ty\}^{\mathrm{u}}$. Hence $Tx \perp Ty$. □

4.3 Disjointness preserving C_0-semigroups

Disjointness preserving C_0-semigroups on Banach lattices are discussed e.g. in [10, Section 5], where, in particular, it is shown that the generator of a disjointness preserving C_0-semigroup is local. We prove the analogous result for disjointness preserving C_0-semigroups on ordered Banach spaces with a semimonotone norm.

Let us see some necessary definitions in the theory of C_0-semigroups.

Definition 4.3.1. Let $(X, K, \|\cdot\|)$ be an ordered Banach space.

(i) A subset $\{T(t) : t \in \mathbb{R}^+\}$ of $L(X)$ is called a **one-parameter semigroup** on X if $T(0) = I$, $T(s+t) = T(s)T(t)$ for all $s, t \in \mathbb{R}^+$, and usually written as $T(t)_{t\geq 0}$. Here I stands for the identity operator.

(ii) The **generator** of $T(t)_{t\geq 0}$ is given by $A\colon X \supseteq \mathcal{D}(A) \to X, x \mapsto \lim_{t\downarrow 0} \frac{T(t)x-x}{t}$, where $\mathcal{D}(A)$ is

$$\mathcal{D}(A) := \{x \in X; \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists in } X\}.$$

(iii) $T(t)_{t\geq 0}$ is called **strongly continuous**, or also C_0**-semigroup**, if the map $t \mapsto T(t)$ is continuous for the strong topology on $L(X)$, i.e. $\lim_{t\to t_0} \|T(t)f - T(t_0)f\| = 0$ for every $f \in X$ and $t_0 \geq 0$.

We say that a C_0-semigroup $T : [0, \infty) \to L(X)$ is called **disjointness preserving** if for every $t \in [0, \infty)$, the operator $T(t)$ is disjointness preserving. Then we have the following result.

Theorem 4.3.2. Let $(X, K, \|\cdot\|)$ be an ordered Banach space with a semimonotone norm and let $T\colon [0,\infty) \to L(X)$ be a disjointness preserving C_0-semigroup with generator A. Then A is local.

Proof. Let (X^ρ, i) be the Riesz completion of X. Because of Lemma 4.1.2, the semimonotone norm $\|\cdot\|$ is equivalent to the regular norm $\rho \circ i$, where the regular norm ρ on $Y = X^\rho$ is given by (4.1). As Y is a vector lattice, ρ is a Riesz norm and (4.1) comes down to $\rho(y) = \inf\{\|x\|;\, x \in X,\, |y| \leq i(x)\}$, $y \in Y$. Let $x \in \mathcal{D}(A)$ and $y \in X$ be such that $x \perp y$. Then $i(x) \perp i(y)$ in Y. Now the line of reasoning is as in the proof of [10, Proposition 5.4]. For every $t > 0$ one has

$$\begin{aligned}
\left|\tfrac{1}{t} i(T(t)x - x)\right| \wedge |i(y)| &\leq \tfrac{1}{t}\,|i(T(t)x)| \wedge |i(y)| + \tfrac{1}{t}\,|i(x)| \wedge |i(y)| \\
&= \tfrac{1}{t}\,|i(T(t)x)| \wedge |i(T(t)y - y) - i(T(t)y)| \\
&\leq \tfrac{1}{t}\,|i(T(t)x)| \wedge |i(T(t)y - y)| \\
&\leq |i(T(t)y - y)|.
\end{aligned}$$

Here we used that Y is distributive and T is disjointness preserving. We conclude

$$\rho\left(\left|\tfrac{1}{t} i(T(t)x - x)\right| \wedge |i(y)|\right) \leq \rho(|i(T(t)y - y)|).$$

For $t \downarrow 0$ one has $T(t)y - y \to 0$ in X, hence $\rho(|i(T(t)y - y)|) \to 0$, which implies

$$\rho\left(\left|\tfrac{1}{t} i(T(t)x - x)\right| \wedge |i(y)|\right) \to 0.$$

Further,

$$\begin{aligned}
&\left|\rho\left(|i(Ax)| \wedge |i(y)|\right) - \rho\left(\left|\tfrac{1}{t} i(T(t)x - x)\right| \wedge |i(y)|\right)\right| \\
\leq &\rho\left(\left||i(Ax)| \wedge |i(y)| - \left|\tfrac{1}{t} i(T(t)x - x)\right| \wedge |i(y)|\right|\right) \\
\leq &\rho\left(\left||i(Ax)| - \left|i\left(\tfrac{1}{t}(T(t)x - x)\right)\right|\right|\right) \\
\leq &\rho\left(\left|i\left(Ax - \tfrac{1}{t}(T(t)x - x)\right)\right|\right) \to 0,
\end{aligned}$$

for $t \downarrow 0$, since $\|Ax - \frac{1}{t}(T(t)x - x)\| \to 0$. Therefore $\rho\left(|i(Ax)| \wedge |i(y)|\right) = 0$. Now Lemma 4.1.3 implies that $Ax \perp y$, hence A is local. □

Notice that the converse of the statement of 4.3.2 is not true in general, not even in Banach lattices. We illustrate this by an example from [43, 23].

Example 4.3.3. Let A be the second derivative operator given in Remark 4.2.5(I). We have already shown that A is local. The one-dimensional diffusion semigroup generated by A is given by

$$T(t)f(x) = \int_0^1 K_t(x,y)f(x)dy,$$

with kernel

$$K_t(x,y) = 1 + 2\sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t)\cos(\pi n x) \cdot \cos(\pi n y),$$

see [43, 2.7] or [23, 2.12]. There it is also shown that $K_t(\cdot,\cdot)$ is a positive, continuous function on $[0,1]^2$. Obviously, for $t \in (0,\infty)$, $T(t)$ is not disjointness preserving on $C[0,1]$.

It is nevertheless interesting to investigate a converse of Theorem 4.3.2. We consider two cases in which a C_0-semigroup with a local generator will be disjointness preserving. The first one considers as extra condition that the generator is a bounded operator and uses the Taylor series for the semigroup. The second case assumes that also the resolvent operators are local and uses the Yosida approximations. It turns out that in both cases the semigroup even consists of local operators.

We begin with the case of a bounded generator.

Theorem 4.3.4. Let $(X, K, \|\cdot\|)$ be an ordered Banach space with a semimonotone norm. If $A \in \mathcal{L}(X)$ is local, then e^{tA} is local for every $t \in \mathbb{R}$.

Proof. Let $x, y \in X$ be such that $x \perp y$. Let $t \in \mathbb{R}$. For every $N \in \mathbb{N}$, Lemma

4.2.6 yields that $\sum_{k=0}^{N} \frac{t^n}{n!} A^n x \perp y$. According to Lemma 4.1.4, $\{y\}^{\mathrm{d}}$ is closed, so that $e^{tA}x = \sum_{k=0}^{\infty} \frac{t^n}{n!} A^n x \perp y$. Hence e^{tA} is local. □

We proceed by investigating unbounded local generators. Typical examples of local operators are differential operators and multiplication operators. Recall that Example 4.3.3 shows an example of a differential operator that generates a C_0-semigroup that is not disjointness preserving. The next example presents a differential operator that generates a C_0-semigroup which is disjointess preserving. It also presents a multiplication operator as generator.

Example 4.3.5. (i) Translation Semigroup. We consider the Banach space $X := C_{\mathrm{ub}}(\mathbb{R})$ of all uniformly continuous, bounded functions on $\mathbb{R}$. For $t \in [0, \infty)$, the left translation operator $T_l(t)\colon X \to X$ is given by

$$T_l(t)x(s) := x(s+t),\ s \in \mathbb{R},\ x \in X.$$

Then $T_l\colon [0,\infty) \to \mathcal{L}(X)$ is a C_0-semigroup on X with generator A given by differentiation,

$$Ax := x'$$

with domain $\mathcal{D}(A) = \{x \in X;\ x \text{ differentiable and } x' \in X\}$. Then A is local (and unbounded) and T is disjointness preserving, but not local.

(ii) Multiplication Semigroup. Let $X := C_0(\Omega)$, where Ω is a locally compact Hausdorff space, as defined in Remark 4.2.3(b). Let $q\colon \Omega \to \mathbb{R}$ be continuous and bounded above. Define for $t \in [0, \infty)$ the operator $T_q(t)\colon X \to X$ by

$$T_q(t)x = e^{tq(t)}x,\ x \in X.$$

Then $T_q\colon [0,\infty) \to L(X)$ is a C_0-semigroup with generator A given by $Ax = qx$, $x \in \mathcal{D}(A)$ and $\mathcal{D}(A) = \{x \in X\colon\ qx \in X\}$. Then A is local and $T_q(t)$ is local for every $t \in [0, \infty)$.

An interesting difference between the generators A in Example (i) and (ii) above is that in (ii) also the inverse of A is local. It turns out that a C_0-semigroup is local whenever both A and A^{-1} are local, which is a special case of the following theorem.

Theorem 4.3.6. Let $(X, K, \|\cdot\|)$ be an ordered Banach space with semimonotone norm and let $T\colon [0,\infty) \to L(X)$ be a C_0-semigroup with generator A. If $A\colon X \supseteq \mathcal{D}(A) \to X$ is local and there exists a $\lambda_0 \in \rho(A) \cap \mathbb{R}$ such that for every $\lambda \in \rho(A)$ with $\lambda \geq \lambda_0$ we have that $(\lambda I - A)^{-1}\colon X \to \mathcal{D}(A) \subseteq X$ is local, then $T(t)$ is local for every $t \in [0,\infty)$.

Proof. By Lemma 4.2.6, the Yosida approximation $A_\lambda = A(\lambda I - A)^{-1}$ is local for every $\lambda \in \rho(A)$ with $\lambda \geq \lambda_0$. Let $t \in [0,\infty)$. Since A_λ is bounded, due to Theorem 4.3.4 we obtain that e^{tA_λ} is local. For $x \in X$, we have $T(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda}x$. We infer that $T(t)$ is local. Indeed, if $x, y \in X$ are such that $x \perp y$, then $e^{tA_\lambda}x \perp y$. Since the band $\{y\}^{\mathrm{d}}$ is closed by Lemma 4.1.4, it follows that $T(t)x \perp y$. Thus, $T(t)$ is local. □

Corollary 4.3.7. Let $(X, K, \|\cdot\|)$ be an ordered Banach space with semimonotone norm and let $T\colon [0,\infty) \to L(X)$ be a C_0-semigroup with generator A such that the inclusion map $i\colon \mathcal{D}(A) \to X$ is a Riesz* homomorphism. If there exists a $\lambda_0 \in \rho(A) \cap \mathbb{R}$ such that $\lambda_0 I - A\colon \mathcal{D}(A) \to X$ is positive and local and for every $\lambda \in \rho(A)$ with $\lambda \geq \lambda_0$ we have that $(\lambda I - A)^{-1}$ is positive, then $T(t)$ is local for every $t \in [0,\infty)$.

Proof. Let $\lambda \in \rho(A)$ with $\lambda \geq \lambda_0$. Then $\lambda I - A\colon \mathcal{D}(A) \to X$ is positive and bijective and, by assumption, $(\lambda I - A)^{-1}$ is positive. By Lemma 4.2.6, $\lambda I - A = \lambda_0 I - A + (\lambda - \lambda_0)I$ is local. Proposition 4.2.7 then yields that $(\lambda I - A)^{-1}$ is local. Hence we can apply Theorem 4.3.6 and obtain that $T(t)$ is local for every $t \in [0,\infty)$. □

Corollary 4.3.8. Let $(X, K, \|\cdot\|)$ be an ordered Banach space with semimonotone norm and let $T\colon [0,\infty) \to L(X)$ be a C_0-semigroup with generator A such that the inclusion map $i\colon \mathcal{D}(A) \to X$ is a Riesz* homomorphism. If A is positive and local and there exists a $\lambda_0 \in \rho(A) \cap \mathbb{R}$ such that $A \leq \lambda_0 I$, then $T(t)$ is local for every $t \in [0, \infty)$.

Proof. Assume that A is positive. Then $T(t)$ is positive for every $t \in [0, \infty)$, so there exists $\lambda_1 \in \mathbb{R}$ such that $(\lambda I - A)^{-1}$ is positive for every $\lambda \in \rho(A)$ with $\lambda \geq \lambda_1$, due to [16, Chapter 7]. As $\lambda_0 I - A\colon \mathcal{D}(A) \to X$ is positive and local, Corollary 4.3.7 yields that $T(t)$ is local for every $t \in [0, \infty)$. □

Merely as illustration, we present an example of an ordered Banach space satisfying the conditions of Corollary 4.3.7, on which there exists a non-trivial multiplication operator A which generates a C_0-semigroup.

Example 4.3.9. Consider the locally compact Hausdorff space $[0, 1)$ and

$$X = \left\{ x \in C_0[0, 1);\ x|_{[0, \frac{1}{2}]} \in \mathrm{Pol}^2[0, \tfrac{1}{2}] \right\},$$

where $C_0[0, 1)$ is defined in Remark 4.2.3(b) and $\mathrm{Pol}^2[a, b]$ is the space of polynomial functions of at most degree 2 on $[a, b]$. As in the proof of [32, Example 3.5], it can be verified that X is order dense in $C_0[0, 1)$. Thus, X is a pre-Riesz space and the embedding map $i\colon X \to C_0[0, 1)$ is a Riesz* homomorphism.

Moreover, X is a closed subspace of $(C_0[0, 1), \|\cdot\|_\infty)$. Indeed, let (x_n) be a sequence in X and $x \in C_0[0, 1)$ be such that $\|x_n - x\|_\infty \to 0$. Then $(x_n|_{[0, \frac{1}{2}]})$ is a sequence in $\mathrm{Pol}^2[0, \frac{1}{2}]$ and

$$\left\| x_n|_{[0, \frac{1}{2}]} - x|_{[0, \frac{1}{2}]} \right\| \leq \|x_n - x\|_\infty \to 0.$$

Since $(\mathrm{Pol}^2[0, \frac{1}{2}], \|\cdot\|_\infty)$ is finite dimensional, it is closed in $(C[0, \frac{1}{2}], \|\cdot\|_\infty)$. Thus, it follows that $x|_{[0, \frac{1}{2}]} \in \mathrm{Pol}^2[0, \frac{1}{2}]$ and hence $x \in X$.

Since the cone in X is closed and generating, X is an ordered Banach space.

Let $q \in C[0, 1)$ be bounded above and constant on $[0, \frac{1}{2}]$. As in Example 4.3.5 (ii), let $Ax = qx$ for $x \in \mathcal{D}(A) = \{x \in X;\ s \mapsto q(s)x(s) \in X\}$. The elements of X that vanish on an interval $[1 - \varepsilon, 1)$ for some $\varepsilon > 0$ are norm dense in X and they are in $\mathcal{D}(A)$, hence $\mathcal{D}(A)$ is norm dense in X. Also, A is closed. Indeed, let $x_n \in \mathcal{D}(A)$ and $x, y \in X$ be such that $\|x_n - x\|_\infty \to 0$ and $\|Ax_n - y\|_\infty \to 0$. Then for every $t \in [0,1)$ we have that $x_n(t) \to x(t)$, hence $q(t)x_n(t) \to q(t)x(t)$, and $(qx_n)(t) \to y(t)$, and therefore $q(t)x(t) = y(t)$. Hence $x \in \mathcal{D}(A)$ and $Ax = y$.

Next we show that $A\colon X \supseteq \mathcal{D}(A) \to X$ is local. Let $x \in \mathcal{D}(A)$ and $y \in X$ be such that $x \perp y$. Since X is order dense in $C_0[0,1)$, it follows that $x \perp y$ in $C_0[0,1)$. Therefore, for every $t \in [0,1)$ we have $x(t) = 0$ or $y(t) = 0$, hence $q(t)x(t) = 0$ or $y(t) = 0$, which yields that $Ax \perp y$ in $C_0[0,1)$ and hence in X.

Let $\lambda_0 \in [0,\infty)$ with $\lambda_0 > \sup_s q(s)$. Then we have that $\lambda_0 I - A$ is positive and local and for every $\lambda \geq \lambda_0$ we have that $(\lambda I - A)^{-1}$ is positive. Now A satisfies all conditions of Corollary 4.3.7, provided A generates a C_0-semigroup T. This is in fact the case, namely T is given by $(T(t)x)(s) = e^{q(s)t}x(s)$, $s \in [0,1]$, $t \in [0,\infty)$. Clearly, $T(t)$ is local for every $t \in [0,\infty)$.

The conditions in our analysis that the norm is semimonotone and the space is norm complete appear fairly weak, but exclude in fact many interesting examples such as C^k-spaces and Sobolev spaces. A general theory of disjointness preserving C_0-semigroups on such spaces will be an interesting topic of further research.