Extension of operators on pre-Riesz spaces

Zhang, F.

Author: Zhang, F.
Title: Extension of operators on pre-Riesz spaces
Issue Date: 2018-09-20

Chapter 3

Compact operators on pre-Riesz spaces

The theory of compact operators comes originally from integral equations theory. Recall that a compact operator is an operator which sends the closed unit ball of the domain space onto a relatively compact subset of the range space, i.e. this subset has a compact closure. An alternative definition is that an operator which maps every norm bounded sequence to a sequence with a norm convergent subsequence. Compact operators have more nicer properties than arbitrary operators. For example, the spectral property by B. de Pagter [20] holds, Schauder's Fixed Point Theorem of compact maps, and moreover compact operators form a two-sided ideal in the operator algebra, see the book by J. B. Conway [17]. For these reasons compact operators are more attractive for studying.

We list some remarkable results for compact operators on Banach lattices. For Banach lattices X, Y such that X', Y both have order continuous norms, if a positive operator $S\colon X \to Y$ is dominated by a compact operator $T\colon X \to Y$, i.e. $0 \leq S \leq T$, then S is compact, this was established by P. Dodds and D. Fremlin [21]. This positive domination property is also proved by A. W. Wickstead [57] in the situations that either X' or Y is atomic with an order continuous norm. This conclusion holds only in these three cases, due to A. W. Wickstead [58]. The domination property holds for Dunford-Pettis operators as well, as was established by N. Kalton and P. Saab [36]. Moreover, due to C. D. Aliprantis and O. Burkinshaw [5], if a positive operator on a Banach lattice is dominated by a compact operator, then its third power is a compact operator.

In this chapter, we mainly consider the similar positive compact domination property on pre-Riesz spaces. Namely, we explore under which suitable conditions for pre-Riesz spaces X and Y, we have that every positive operator dominated by a compact operator is compact. We also address the question whether a similar result concerning the third power of the operator is true for operators on pre-Riesz spaces.

To use the theory of pre-Riesz spaces and Riesz completions, one natural question is how we extend a norm on a pre-Riesz space to an order continuous norm on its Riesz completion. In [50, Theorem 3.43], the author states that a regular seminorm on a pre-Riesz space can be extended to a Riesz seminorm on the Riesz completion. So the difficulty comes down to the following. For an arbitrary element in the Riesz completion, find a directed net in the pre-Riesz space that order converges to it. In Section 3.1, we will settle this by providing the condition that the pre-Riesz space is pervasive Archimedean with the Riesz decomposition property.

Section 3.2 is concerned with a unique extension of an operator defined on a pre-Riesz space to its Dedekind completion, based on norm denseness and by means of order continuous norms.

In Section 3.3, we investigate how the positive domination property of compact operators introduced by P. Dodds and D. Fremlin in [21], and the theory of the third power compact operators by C. D. Aliprantis and O. Burkinshaw [5] can be generalized to operators on pre-Riesz spaces. We present some partial results.

3.1 Extension of order continuous norms

A seminorm p on an ordered vector space (X, K) is said to be **order continuous** if $x_\alpha \xrightarrow{o} x$ implies $p(x_\alpha - x) \to 0$.

Recall that Lemma 2.1.6 in Chapter 2 yields that for a pervasive pre-Riesz space with a monotone norm there exists a Riesz norm on the Riesz completion of the pre-Riesz space. In fact, such a norm can be obtained by a regular norm.

Definition 3.1.1. Let X be a directed partially ordered vector space with a seminorm p, then we say that p is **regular** if for every $x \in X$,

$$p(x) = \inf\{p(y);\, y \in X, -y \le x \le y\}. \tag{3.1}$$

On vector lattices, regular seminorms and Riesz seminorms coincide, this result is due to O. van Gaans [50, Theorem 3.40]. Moreover, due to [50, Theorem 3.43 and Corollary 3.45], one can extend the seminorm on a pre-Riesz space in the following way.

Theorem 3.1.2. Let (X, K) be a directed partially ordered vector space with a seminorm p. Let Y be a directed partially ordered vector space and $i\colon X \to Y$ a bipositive linear map, such that $i(X)$ is majorizing in Y. Define

$$p_r(y) := \inf\{p(x);\, x \in X \text{ such that } -i(x) \le y \le i(x)\},\ y \in Y. \tag{3.2}$$

The following statements hold.

(i) p_r is the greatest regular seminorm on Y with $p_r \le p$ on K.

(ii) $p_r = p$ on K if and only if p is monotone. Moreover, if p is monotone, then $p_r \ge \frac{1}{2}p$.

(iii) $p_r = p$ on X if and only if p is regular.

Remark 3.1.3. There are more situations in which regular norms are used. In the study of operator theory, if X and Y are two Banach lattices, then the space $\mathcal{L}_{\mathrm{b}}(X,Y)$ of all order bounded operators is merely a partially ordered vector space, and the operator norm is not a Riesz norm. As a subspace of $\mathcal{L}_{\mathrm{b}}(X,Y)$, the regular operator space $\mathcal{L}_{\mathrm{r}}(X,Y)$ is not complete with respect to the operator norm, but it is a Banach space for the regular norm $\|T\|_r := \inf\{\|S\|;\ S \in \mathcal{L}^+(X,Y), |Tx| \leq Sx, x \in X^+\}$, $T \in \mathcal{L}_{\mathrm{r}}(X,Y)$. If Y is additionally Dedekind complete, then $(\mathcal{L}_{\mathrm{r}}(X,Y), \|\cdot\|_r)$ is a Banach lattice, see [42, Proposition 1.3.6]. For more results on regular norms we refer the reader to the work of Y. A. Abramovich, Z. Chen, and A. W. Wickstead [2], W. Arendt [8], Z. Chen and A. W. Wickstead [15] and A. W. Wickstead [59].

Before we consider the extension of order continuous norms on pre-Riesz spaces, we point out the fact that for every pre-Riesz space X with the Riesz completion (X^ρ, i), the subspace $i(X)$ is order dense in X^ρ, but for a positive element in X^ρ, it is difficult to find an upward directed net of positive elements in $i(X)$ such that this net is order convergent to it, even if X is pervasive. In the following discussion, we try to construct an increasing net in $i(X)$ explicitly by using the Riesz decomposition property. The next result is due to B. Z. Vulikh [56, Lemma V.1.1].

Proposition 3.1.4. Let (X,K) be an ordered vector space. The following statements are equivalent:

(i) X has the Riesz decomposition property.

(ii) For arbitrary $x_1, x_2, x_3, x_4 \in X$ with $x_1, x_2 \leq x_3, x_4$, there exists $z \in X$ such that $x_1, x_2 \leq z \leq x_3, x_4$.

Notice that in Proposition 2.1.4, we have that if X is a pervasive Archimedean pre-Riesz space, then for an arbitrary $x \in (X^\rho)^+$, we can find some elements in $i(X)^+$ for which x is the supremum. If we assume that X has additionally the Riesz decomposition property, then we can construct a net which order converges to x.

Theorem 3.1.5. Let X be a pervasive Archimedean pre-Riesz space with the RDP, let (X^ρ, i) be the Riesz completion of X. For every $0 < y \in X^\rho$, if there exist $a_1, \ldots, a_n \in X^+$ such that $y = \bigwedge_{j=1}^n i(a_j)$, then there exists an increasing net in $i(X)^+$ which is order convergent to y. Moreover, for every $y \in X^\rho$ there exists a net in $i(X)$ which is order convergent to y.

Proof. Let $0 < y \in X^\rho$ and $y = \bigwedge_{j=1}^n i(a_j)$, $a_1, \ldots, a_n \in X^+$. As X is pervasive, the set

$$D = \{u \in i(X)^+; 0 < u \leq y\}$$

is not empty.

Let $u_1, u_2 \in D$ with $u_1, u_2 \leq a_1, \ldots, a_n$ in X. Since X has the RDP, there exists $u_3 \in i(X)$ such that

$$u_1, u_2 \leq u_3 \leq a_1, \ldots, a_n.$$

So $u_3 \leq \bigwedge_{k=1}^n i(a_k) = y_1$. Hence $u_3 \in D$. This implies that D is upward directed. Define $(z_\alpha)_{\alpha \in D}$ by $z_\alpha := \alpha$, $\alpha \in D$, clearly $(z_\alpha)_\alpha$ is an upward directed net in $i(X)$. As X is Archimedean, by Proposition 2.1.4 we have $y = \sup D$.

Moreover, let $y \in X^\rho$. If $y = 0$, it is obvious that we can define a net consisting of elements with value 0 such that it is order convergent to y. Let $y \neq 0$. Since $i(X)$ is an order dense subspace of X^ρ and generates X^ρ as a vector lattice, there exist a_j and b_k in X^+, $j = 1, \ldots, n$ and $k = 1, \ldots, m$, such that $y = \bigwedge_{j=1}^n i(a_j) - \bigwedge_{k=1}^m i(b_k)$. Let $y_1 = \bigwedge_{j=1}^n i(a_j)$, $y_2 = \bigwedge_{k=1}^m i(b_k)$, then $y_1, y_2 \in (X^\rho)^+$. Obviously, y_1 and y_2 can not be zero at the same time. Without loss of generality, let $y_1 \neq 0$. By the previous discussion, there exists $(z_\alpha)_\alpha$ in $i(X)^+$ with $z_\alpha \xrightarrow{o} y_1$. By the same reasoning, there exists a net $(w_\beta)_\beta$ with $w_\beta \xrightarrow{o} y_2$.

Define $s_{(\alpha,\beta)} = z_\alpha - w_\beta$. We claim that $s_{(\alpha,\beta)} \xrightarrow{o} y_1 - y_2$. In fact, there exist nets $(u_\alpha)_\alpha$, $(v_\beta)_\beta$ in $i(X)^+$ such that $-u_\alpha \leq z_\alpha - y_1 \leq u_\alpha$, $-v_\beta \leq w_\beta - y_2 \leq v_\beta$ and $u_\alpha \downarrow 0$, $v_\beta \downarrow 0$. So $-(u_\alpha + v_\beta) \leq (z_\alpha - w_\beta) - (y_1 - y_2) \leq u_\alpha + v_\beta$ and $(u_\alpha + v_\beta) \downarrow 0$. Hence $s_{(\alpha,\beta)} \xrightarrow{o} y_1 - y_2$. This completes the proof. □

Remark 3.1.6. Clearly, for $x = \bigvee_{k=1}^{n} i(a_k)$ in $(X^\rho)^+$, and $z \in i(X)$ with $z - x \in (X^\rho)^+$, by the proof of Theorem 3.1.5, one has a net $(u_\alpha)_{\alpha\in D}$ in $i(X)^+$ which is upward directed and convergent to $z - x$. Define $v_\alpha = z - u_\alpha$ for every $\alpha \in D$, then the net $(v_\alpha)_{\alpha\in D} \subseteq i(X)$ is downward directed and convergent to x.

In the next lemma, based on Zorn's lemma, it is shown that for a positive decreasing net in the Riesz completion (X^ρ, i) of a pre-Riesz space X, if its infimum exists, then there exists a downward directed net in $i(X)^+$ such that their infimum are equal, it is due to [35, Lemma 3.7.11].

Lemma 3.1.7. Let X be a pervasive Archimedean pre-Riesz space and (Y, i) a vector lattice cover of X. Let $(y_\alpha)_{\alpha\in I}$ be a net in Y such that $y_\alpha \downarrow 0$. There exists a net $(x_\beta)_{\beta\in J}$ in X with $x_\beta \downarrow 0$ and such that for every $\beta \in J$ there exists $\alpha_0 \in I$ such that for every $\alpha \geq \alpha_0$ we have $i(x_\beta) \geq y_\alpha$.

This lemma will be used to prove the following theorem, which extends order continuous norms on pre-Riesz spaces. Recall that a norm $\|\cdot\|$ on a partially ordered vector space X is called **semimonotone** if there exists $C \in \mathbb{R}^+$ such that for every $x, y \in X$ with $0 \leq x \leq y$ one has $\|x\| \leq C\|y\|$.

Theorem 3.1.8. Let X be a pervasive Archimedean pre-Riesz space, p a semimonotone seminorm on X, and (Y, i) a vector lattice cover of X. Let p_r on Y be defined by

$$p_r(y) := \inf\{p(x);\, x \in X \text{ such that } - i(x) \leq y \leq i(x)\},\ y \in Y.$$

If p on X is order continuous, then p_r is an order continuous seminorm on Y as well.

Proof. By Theorem 3.1.2 (i) it follows that p_r is a seminorm on Y. We show that p_r is order continuous.

Assume that $x_\alpha \xrightarrow{o} x$ in Y, i.e. there exists a net $(y_\beta)_{\beta\in B} \subset Y$ such that for every β there is α_0 such that for every $\alpha \geq \alpha_0$ we have $\pm(x_\alpha - x) \leq y_\beta \downarrow 0$. As by Theorem

3.1.2 (i) the seminorm p_r is regular we have $p_r(x_\alpha - x) \leq p_r(y_\beta)$. By Lemma 3.1.7 there exists a net $(z_\gamma)_\gamma \subset X$ such that for every γ there is β_0 such that for every $\beta \geq \beta_0$ we have $i(z_\gamma) \geq y_\beta$ and $z_\gamma \downarrow 0$. Since p is order continuous, $p(z_\gamma) \downarrow 0$. Since p is semimonotone, there is a constant C such that $p_r(i(v)) \leq Cp(v)$ for every $v \in X$. Then $p_r(x_\alpha - x) \leq p_r(y_\beta) \leq p_r(i(z_\gamma)) \leq Cp(z_\gamma) \to 0$. Hence p_r is order continuous on Y. □

Theorem 3.1.8 and Theorem 3.1.5 together yield the following corollary.

Corollary 3.1.9. Let X be a pervasive Archimedean pre-Riesz space with RDP with an order continuous semimonotone seminorm $\|\cdot\|_X$. Let (X^δ, j) be the Dedekind completion of X, and define $\|\cdot\|_{X^\delta}$ on X^δ by

$$\|y\|_{X^\delta} := \inf\{\|x\|_X;\ x \in X \text{ such that } -j(x) \leq y \leq j(x)\}.$$

Then $\|\cdot\|_{X^\delta}$ is an order continuous seminorm and $(X, \|\cdot\|_X)$ is norm dense in $(X^\delta, \|\cdot\|_{X^\delta})$. Furthermore, $\|\cdot\|_{X^\delta}$ extends $\|\cdot\|_X$ if $\|\cdot\|_X$ is regular.

Proof. The order continuity of $\|\cdot\|_{X^\delta}$ is same with the Theorem 3.1.8. It remains to show that $(X, \|\cdot\|_X)$ is norm dense in $(X^\delta, \|\cdot\|_{X^\delta})$. Let (X^ρ, i) be the Riesz completion of X. For every $y \in X^\rho$, according to Theorem 3.1.5, there is a net (x_α) in $i(X)$ with $x_\alpha \xrightarrow{o} y$. By Theorem 3.1.8, $\|\cdot\|_{X^\rho}$ is order continuous, and then $\|x_\alpha - y\|_{X^\rho} \to 0$, so that X is norm dense in X^ρ. Let $y \in (X^\delta)^+$, and $D := \{x \in X^\rho;\ 0 \leq x \leq y\}$. Since X^ρ is pervasive in X^δ and X^ρ is Archimedean, hence, by Proposition 2.1.4, we have D is upward directed and $\sup D = y$. Define the net $(z_\mu)_\mu$ by $z_\mu := \mu$, $\mu \in D$. Therefore $z_\mu \xrightarrow{o} y$ and $\|z_\mu - y\|_{X^\delta} \to 0$. Thus we conclude X^ρ is norm dense in X^δ. It follows that X is norm dense in X^δ.

Moreover, by Theorem 3.1.2 (iii) the seminorm $\|\cdot\|_{X^\delta}$ extends $\|\cdot\|_X$ if (and only if) $\|\cdot\|_X$ is regular. □

3.2 Extension of compact operators

In this section, we will show that a compact operator on a pre-Riesz space X with a suitable order continuous norm can be extended to a compact operator on the Dedekind completion X^δ of X. To make sure that the norm on X^δ indeed is an extension of the order continuous norm on X, the norm of X is required to be regular. Then we will use Corollary 3.1.9 to extend the operator, see the following theorem.

Theorem 3.2.1. Let $(X, \|\cdot\|_X)$ be a pervasive Archimedean pre-Riesz space with RDP equipped with an order continuous regular seminorm. Let (X^δ, i) be the Dedekind completion of X. Let Y be a Banach lattice with an order continuous norm. If T is a bounded operator in $L(X, Y)$, then there exists a unique bounded linear extension $\widehat{T} \in L(X^\delta, Y)$. If T is compact, then $\widehat{T}$ is compact as well.

Proof. Recall that by Theorem 3.1.2 (i) there exists the greatest regular seminorm $\|\cdot\|_{X^\delta}$ on X^δ which extends $\|\cdot\|_X$. Due to Corollary 3.1.9, X is dense with respect to the seminorm in X^δ. So there exists $\widehat{T} \in L(X^\delta, Y)$ which uniquely extends T by means of for $z \in X^\delta$, $\widehat{T}z = \lim Tz_n$, where $(z_n)_n \subseteq X$ norm converges to z.

Let $(x_n)_n$ be a norm bounded sequence in X^δ. It follows from the norm denseness of X in X^δ that there exists a norm bounded sequence $(y_n)_n$ in X such that $\|i(y_n) - x_n\|_{X^\delta} < \frac{1}{n}$ holds for all $n \in \mathbb{N}$. As T is compact, $(Ty_n)_n$ has a convergent subsequence $(Ty_{n_k})_{n_k}$. So there exists $y \in Y$ such that for $k \to \infty$ we have

$$\|Ty_{n_k} - y\|_Y \to 0.$$

For the subsequence $\left(\widehat{T}x_{n_k}\right)$ of $\left(\widehat{T}x_n\right)$, one has

$$\begin{aligned}\left\|\widehat{T}x_{n_k} - Ty_{n_k}\right\|_Y &= \left\|\widehat{T}x_{n_k} - \widehat{T}(i(y_{n_k}))\right\|_Y \\ &\le \left\|\widehat{T}\right\| \|x_{n_k} - i(y_{n_k})\|_{X^\delta} \\ &\le \left\|\widehat{T}\right\| \frac{1}{n_k} \to 0\end{aligned}$$

as $k \to \infty$. Then

$$\left\|\widehat{T}x_{n_k} - y\right\|_Y \le \left\|\widehat{T}x_{n_k} - Ty_{n_k}\right\|_Y + \|Ty_{n_k} - y\|_Y \to 0$$

as $k \to \infty$. So $\left(\widehat{T}x_n\right)$ has a convergent subsequence, and hence $\widehat{T}$ is compact. □

3.3 Compact domination results in pre-Riesz spaces

In this section, we will extend two results concerning the domination property of positive compact operators on Banach lattices to the setting of pre-Riesz spaces. We will consider appropriate norms and use Riesz completions or Dedekind completions of pre-Riesz spaces. In order to extend a compact operator on a pre-Riesz space to a compact operator on the Dedekind completion, we will use Theorem 3.2.1. Let us recall some preliminaries first.

Let (X, τ) be a topological vector space. The **topological dual** X' of (X, τ) is the vector space consisting of all τ-continuous linear functionals on X. For a partially ordered vector space X, the vector space of all order bounded linear functionals is called the **order dual** of X, and denoted by $X^\sim$. The next theorem is due to [6, Theorem 3.49].

Theorem 3.3.1. The topological dual X' of a locally convex-solid Riesz space (X, τ) is an ideal in its order dual $X^\sim$.

The classical domination property of compact operators between Banach lattices by [6, Theorem 5.20], which reads as follows.

Theorem 3.3.2. (Dodds-Fremlin) Let X be a Banach lattice, Y a Banach lattice with X' and Y having order continuous norms. If a positive operator $S\colon X \to Y$ is dominated by a compact operator, then S is a compact operator.

Remark 3.3.3. This result is proved originally by P. G. Dodds and D. H. Fremlin in [21]. An easier accessible proof is given in [6, Theorem 5.20]. In fact, note that in the proof of [6, Theorem 5.20], it does not use the norm completeness of X. As a consequence, we have the following corollary.

Corollary 3.3.4. Let X be a normed Riesz space, Y a Banach lattice with X' and Y having order continuous norms. If a positive operator $S\colon X \to Y$ is dominated by a compact operator, then S is a compact operator.

Now our aim comes down to extending Corollary 3.3.4 to pre-Riesz spaces. We firstly consider the domain space X to be a pre-Riesz space, and we suppose it is pervasive and Archimedean. Alternatively, if X is not norm complete, one could consider the order continuous regular seminorm on X, then use the same argument as in Theorem 3.2.1 and apply Corollary 3.3.4 to obtain compactness of S.

Theorem 3.3.5. Let X be a pervasive Archimedean pre-Riesz space with RDP equipped with an order continuous regular norm, (X^δ, i) the Dedekind completion of X. Let Y be a Banach lattice with an order continuous norm. Let T in $L(X, Y)$ be a positive compact operator and assume that $(X^\delta)'$ has order continuous norm. If $S \in L(X, Y)$ with $0 \leq S \leq T$, then S is compact.

Proof. Recall that by Theorem 3.1.2 (i) there exists the greatest regular seminorm on X^δ extending the norm on X. By Theorem 3.2.1 there exists a unique bounded linear extension $\widehat{T}$ of T on X^δ, which is compact. In fact, for $x \in X^\delta$, $\widehat{T}$ is given by $\widehat{T}x = \lim Tx_n$, where $(x_n)_n \subseteq X$ norm converges to x. We define $\widehat{S}$ in a similar way. So for $x \in (X^\delta)^+$ we have $(x_n)_n \subseteq X^+$ and $Sx_n \leq Tx_n$ for all $n \in \mathbb{N}$. It follows that

$$\widehat{S}x = \lim Sx_n \leq \lim Tx_n = \widehat{T}x.$$

The positivity of $\widehat{S}$ is clear. By Lemma 2.1.6 the seminorm on X^δ is actually a norm. Due to Corollary 3.3.4, we have $\widehat{S}$ is compact. Hence $S = \widehat{S}|_X$ is compact. □

For an operator between two pre-Riesz spaces X and Y, even though we could extend it to the Riesz completion, for the method in the proof of Theorem 3.3.2, we also need that the norms of $(X^\delta)'$ and Y^δ are order continuous. It is difficult to characterize the space structure of dual spaces of pre-Riesz spaces, for example, whether $(X')^\rho$ equals $(X^\rho)'$ or not. Fortunately, if we suppose that the codomain space Y is a directed Archimedean partially ordered vector space and complete with respect to the regular norm, then it has a Dedekind completion which is norm complete as well. Then we can embed Y into the Dedekind completion, and use Theorem 3.3.5, see the following two theorems.

Theorem 3.3.6. Let X be a directed Archimedean partially ordered vector space with a regular norm $\|\cdot\|$ such that X^+ is $\|\cdot\|$-closed. Let (X^δ, i) be the Dedekind completion of X, and define for $y \in X^\delta$,

$$\|y\|_r := \inf\{\|x\|;\, x \in X^+, -i(x) \le y \le i(x)\}.$$

If $(X, \|\cdot\|)$ is complete, then $(X^\delta, \|\cdot\|_r)$ is complete.

Proof. The proof is similar to [50, Theorem 2.12]. Assume that $(X, \|\cdot\|)$ is complete. Let $(y_n)_n$ be a sequence in X^δ such that $\sum_{n=1}^\infty 2^n \|y_n\|_r < \infty$. We show that there exists $y \in X^\delta$ such that $\left\|y - \sum_{n=1}^N y_n\right\|_r \to 0$ as $N \to \infty$. Take $x_n \in X^+$ with $-i(x_n) \le y_n \le i(x_n)$ and $\|y_n\|_r \ge \|x_n\| - \frac{1}{4^n}$. Then $\sum_{n=1}^\infty 2^n \|x_n\| < \infty$. Since X is norm complete, $u := \sum_{n=1}^\infty 2^n x_n$ exists in X. As for every $n \in \mathbb{N}$ we have $x_n \in X^+$ and X^+ is closed, we have $u \in X^+$ and $u \ge \sum_{n=1}^m 2^n x_n \ge 2^n x_n$ for every $m \in \mathbb{N}$. Now for $N > M$, we have $\sum_{n=1}^N y_n - \sum_{n=1}^M y_n = \sum_{n=M+1}^N y_n \le \sum_{n=M+1}^N i(x_n) \le \left(\sum_{n=M+1}^N 2^{-n}\right) i(u)$, and $\sum_{n=1}^N y_n - \sum_{n=1}^M y_n \ge -\left(\sum_{n=M+1}^N 2^{-n}\right) i(u)$. Since $\sum_{n=M+1}^N 2^{-n} \to 0$ when $M, N \to \infty$, we have that $\left(\sum_{n=1}^N y_n\right)_N$ is a relatively uniformly Cauchy sequence. Since X^δ is Dedekind complete and hence relatively uniformly complete, there exists $y \in X^\delta$ and there is a sequence $(\lambda_N)_N$ of reals with $\lambda_N \to 0$ such that $-\lambda_N i(u) \le y - \sum_{n=1}^N y_n \le \lambda_N i(u)$. Then $\left\|y - \sum_{n=1}^N y_n\right\|_r \le \lambda_N \|u\| \to 0$ as $N \to \infty$. Thus, $\left(X^\delta, \|\cdot\|_r\right)$ is complete. □

Then we will proof the positive domination property of compact operators on pre-Riesz spaces in the following theorem.

Theorem 3.3.7. Let X be a pervasive Archimedean pre-Riesz space with RDP equipped with an order continuous regular norm. Let Y be a directed Archimedean pre-Riesz space with an order continuous regular norm $\|\cdot\|_Y$ that Y is norm complete. Let X^δ be the Dedekind completion of X and $(X^\delta)'$ has order continuous norm. If a positive operator $S\colon X \to Y$ is dominated by T, i.e. $0 \leq S \leq T$, and T is compact, then S is compact as well.

Proof. Let Y^δ be the Dedekind completion of Y, and $i\colon Y \to Y^\delta$ be the natural embedding map. Since i is bipositive, we have $0 \leq i \circ S \leq i \circ T$ form X to Y^δ. As i is continuous and T is compact, $i \circ T$ is compact. By Theorem 3.1.2 (i) there exists the greatest regular seminorm on Y^δ (in fact, since Y has a norm, the seminorm on Y^δ is a norm) extending the norm on Y. Because Y has order continuous norm, by Theorem 3.1.8 the space Y^δ has order continuous norm as well, and by Theorem 3.3.6 the space Y^δ is norm complete. It follows from Theorem 3.3.5 that $i \circ S$ is compact. Let $(x_n)_n$ be a norm bounded sequence in X, then $(i \circ S)(x_n)_n$ has a norm convergent subsequence $(i \circ S)(x_{n_k})_{n_k}$ in Y^δ. Then $(i \circ S)(x_{n_k})_{n_k}$ is a Cauchy sequence, so $S(x_{n_k})_{n_k}$ is a Cauchy sequence in Y. Since $\|\cdot\|_Y$ is regular, by Theorem 3.1.2 (iii) we have $\|y\|_Y = \|i(y)\|_{Y^\delta}$ for every $y \in Y$. As Y is norm complete, $S(x_{n_k})$ is convergent in Y. Hence S is compact. □

Beside the domination property of compact operators as in Corollary 3.3.4, there is also a result in [5, Theorem 5.13] which clarifies the positive domination property of third power of compact operators on Banach lattices, see the following theorem.

Theorem 3.3.8. (Aliprantis-Burkinshaw) If a positive operator S on a Banach lattice is dominated by a compact operator, then S^3 is a compact operator.

Now we may ask whether or not this result can be extended to pre-Riesz spaces. The answer is affirmative if we additionally suppose, among other conditions, that the pre-Riesz space has an order unit and an order unit norm. To establish this result, let us recall some known preliminaries.

Recall that a subset A in a topological vector space (X,τ) is called **τ-totally bounded**, if for every τ-neighborhood V of zero there is a finite subset Φ of A such that $A \subseteq \bigcup_{x\in\Phi}(x+V) = \Phi + V$.

The following result can be found in [6, Theorem 3.3], and the next one is in [6, Theorem 5.10].

Theorem 3.3.9. Let $T\colon (X,\tau) \to (Y,\xi)$ be an operator between two topological vector spaces. If T is continuous on the τ-bounded subsets of X, then T carries τ-totally bounded sets to ξ-totally bounded sets.

Theorem 3.3.10. (Dodds-Fremlin) Let X and Y be two Riesz spaces with Y Dedekind complete. If τ is an order continuous locally convex solid topology on Y, then for each $x \in X^+$, the set

$$B = \{T \in L_b(X,Y);\ T[0,x] \text{ is } \tau\text{-totally bounded}\}$$

is a band in $L_b(X,Y)$.

Recall that for a Riesz space X and its order dual $X^\sim$, the **absolute weak topology** on X is defined by a collection of seminorms p_f via the formula

$$p_f(x) = |f|(|x|), \quad x \in X, f \in X^\sim,$$

and it is denoted by $|\sigma|(X,X^\sim)$. For a nonempty subset A of $X^\sim$, the **absolute weak topology generated by** A on X is the locally convex solid topology on X generated by the seminorms p_f defined via the formula

$$p_f(x) = |f|(|x|), \quad x \in X,\ f \in A.$$

Consider a dual system $\langle X, X'\rangle$. A locally convex topology τ on X is said to be **consistent** with the dual system if the topological dual of (X,τ) is precisely X'.

Definition 3.3.11. Let X be a Riesz space, and let X' be an ideal of $X^\sim$ separating the points of X. Then the pair $\langle X, X'\rangle$, under its natural duality

$\langle x, x' \rangle := x'(x)$, is said to be a **Riesz dual system**.

Let us recall a result by S. Kaplan [37, Theorem 3.50].

Theorem 3.3.12. (Kaplan) Let X be a Riesz space, and let A be a subset of $X^\sim$ separating the points of X. Then the topological dual of $(X, |\sigma|(X, A))$ is precisely the ideal generated by A in $X^\sim$.

As a result of the above theorem, we have the following corollary.

Corollary 3.3.13. Let X be a Riesz space and $\langle X, X' \rangle$ a Riesz dual system. Then the topological dual of $(X, |\sigma|(X, X'))$ is precisely X', and $|\sigma|(X, X')$ is consistent with $\langle X, X' \rangle$.

Let us recall some known results with respect to the Riesz dual system. The following one is due to [6, Theorem 3.57].

Theorem 3.3.14. For a Riesz dual system $\langle X, X' \rangle$, the following statements are equivalent.

(i) X is Dedekind complete and $\sigma(X, X')$ is order continuous.

(ii) X is an ideal of X''.

The following theorem is due to [6, Theorem 3.54].

Theorem 3.3.15. For a Riesz dual system $\langle X, X' \rangle$, the following statements are equivalent.

(i) Every consistent locally convex solid topology on X is order continuous.

(ii) $\sigma(X, X')$ is order continuous.

Also, recall that for a subset D of a topological vector space (X, τ), the **restriction topology**, in short r-topology, on D is such that $U \subseteq D$ is r-open if and only if there exists a $V \subseteq X$ which is τ-open and $U = V \cap X$. For a net $(x_\alpha)_{\alpha \in I}$ in D and $x \in D$, we then have $x_\alpha \xrightarrow{r} x$ in D if and only if $x_\alpha \xrightarrow{\tau} x$ in X.

Viewing the pre-Riesz space X as an order dense subspace of its Riesz completion (X^ρ, i). Our next goal is to establish the fact that if a positive operator from X to X is dominated by a compact operator, then it is continuous from the r-topology of $|\sigma|(X^\rho, (X^\rho)')$ to the norm topology. We will use $x_\alpha \xrightarrow{w} x$ to denote that x_α converges to x in X with respect to the weak topology, i.e. the $\sigma(X, X')$-topology. See the following lemma.

Lemma 3.3.16. Let X be an Archimedean pre-Riesz space endowed with a monotone norm $\|\cdot\|_X$, $S, T\colon X \to X$ bounded linear operators satisfying $0 \leq S \leq T$ with T compact. Let (X^ρ, i) be the Riesz completion of X with the seminorm $\|\cdot\|_{X^\rho}$ given by (3.2). Assume that there exists a positive compact operator $\widehat{T}\colon X^\rho \to X^\rho$ such that $\widehat{T} \circ i = i \circ T$. Then S is continuous for the r-topology of $|\sigma|(X^\rho, (X^\rho)')$ on norm bounded subsets of X to the norm topology.

Proof. Let $(x_\alpha)_\alpha$ be a norm bounded net in X with $x_\alpha \xrightarrow{r} 0$. It is enough to show that $\|S(x_\alpha)\|_X \to 0$ holds. To this end, from assumption of $x_\alpha \xrightarrow{r} 0$ in X it follows that $i(x_\alpha) \xrightarrow{|\sigma|(X^\rho,(X^\rho)')} 0$ in X^ρ, which means that for every $f \in (X^\rho)'$, we have $|f||i(x_\alpha)| \to 0$. It follows from $0 \leq |f(|i(x_\alpha)|)| \leq |f||i(x_\alpha)|$ that $|i(x_\alpha)| \xrightarrow{w} 0$ in X^ρ. According to Theorem 3.1.2 (ii), we have that $\|x\|_X \leq 2\|i(x)\|_{X^\rho}$ for every $x \in X$.

Since $\widehat{T}$ is compact and $(|i(x_\alpha)|)_\alpha$ is norm bounded, $\left(\widehat{T}(|i(x_\alpha)|)\right)_\alpha$ has a norm convergent subnet $\left(\widehat{T}y_\beta\right)_\beta$ such that $\widehat{T}y_\beta \xrightarrow{\|\cdot\|_{X^\rho}} z$ for some $z \in X^\rho$, and then $\widehat{T}y_\beta \xrightarrow{w} z$. As for every $f \in (X^\rho)'$ we have $\langle f, T(|i(x_\alpha)|)\rangle = \langle T^* f, |i(x_\alpha)|\rangle \to 0$. So $\widehat{T}(|i(x_\alpha)|) \xrightarrow{w} 0$ and then $z = 0$. Hence $\left(\widehat{T}(|i(x_\alpha)|)\right)_\alpha$ has a subnet which converges in norm to 0. Similarly, every subnet of $\left(\widehat{T}(|i(x_\alpha)|)\right)_\alpha$ has a subnet that norm converges to 0. Therefore, $\left\|\widehat{T}(|i(x_\alpha)|)\right\|_{X^\rho} \to 0$.

Since $0 \leq S \leq T$ and i is positive, we have $0 \leq \widehat{S} \circ i \leq \widehat{T} \circ i$. According to Theorem

3.1.2 (ii) we have that $\|x\|_X \leq 2\|i(x)\|_{X^\rho}$ for every $x \in X$. Thus, we have

$$\|S(x_\alpha)\|_X \leq 2\left\|(\widehat{S}\circ i)(x_\alpha)\right\|_{X^\rho} \leq 2\left\|\widehat{S}(|i(x_\alpha)|)\right\|_{X^\rho} \leq 2\left\|\widehat{T}(|i(x_\alpha)|)\right\|_{X^\rho} \to 0.$$

Hence S is continuous. □

We continue with a approximation property on the Riesz completion of a pre-Riesz space with respect to the regular norm.

Lemma 3.3.17. Let X, Y be two normed pre-Riesz spaces with a regular norm on Y, and $S, T\colon X \to Y$ such that $0 \leq S \leq T$. Let (Y^ρ, i) be the Riesz completion of Y. If T sends a subset A of X^+ to a norm totally bounded set, then for each $\epsilon > 0$ there exists some $u \in (Y^\rho)^+$ such that for all $x \in A$ we have

$$\left\|(i(Sx) - u)^+\right\|_{Y^\rho} \leq \epsilon.$$

Proof. By Theorem 3.1.2 one can define the seminorm on Y^ρ by

$$\|y\|_{Y^\rho} := \inf\{\|x\|_Y;\, x \in Y, -i(x) \leq y \leq i(x)\},\ y \in Y^\rho.$$

It is obvious that $\|i(y)\|_{Y^\rho} = \|y\|_Y$ for $y \in Y$, as $\|\cdot\|_Y$ is regular. Let $\epsilon > 0$. Since T sends a subset A of X^+ to a norm totally bounded set of Y, there exist $x_1, \ldots, x_n \in A$ such that for all $x \in A$ we have $\|Tx - Tx_j\|_Y < \epsilon$ for some j. Put $u = (i \circ T)\left(\sum_{j=1}^n x_j\right) \in (Y^\rho)^+$, then for $x \in A$ and j with $\|Tx - Tx_j\|_Y < \epsilon$ we have

$$\begin{aligned} 0 \leq (i(Sx) - u)^+ &= \left(i\big(Sx - T\sum_{j=1}^n x_j\big)\right)^+ \\ &\leq \left(i\big(Tx - T\sum_{j=1}^n x_j\big)\right)^+ \\ &\leq \big(i(Tx - Tx_j)\big)^+ \\ &\leq \left|\big(i(Tx - Tx_j)\big)^+\right|. \end{aligned}$$

Since regular norms are monotone we have

$$\begin{aligned} 0 \leq \left\|(i(Sx)-u)^+\right\|_{Y^\rho} &\leq \||i(Tx-Tx_j)|\|_{Y^\rho} \\ &= \|i(Tx-Tx_j)\|_{Y^\rho} \\ &= \|Tx-Tx_j\|_Y \leq \epsilon. \end{aligned}$$

Thus we have completed the proof. □

Next, we will show that for every Archimedean Riesz space X with an order unit u and an order unit norm $\|\cdot\|_u$, $\langle X'', X'\rangle$ is a Riesz dual system. To this end we need to show that X' is an ideal of $(X'')^\sim$ and use the theory of AL-spaces and AM-spaces.

Definition 3.3.18. A Banach lattice X is said to be

(1) an **AL-space** if $\|x+y\| = \|x\| + \|y\|$ for all $x, y \in X^+$ with $x \wedge y = 0$ and

(2) an **AM-space** if $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for all $x, y \in X^+$ with $x \wedge y = 0$.

The next lemma can be found in [42, Proposition 1.4.7]. We briefly recall the proof.

Lemma 3.3.19. Let X be an Archimedean Riesz space with an order unit u and an order unit norm $\|\cdot\|_u$. Then $(X', \|\cdot\|)$ is an AL-space.

Proof. For every $x \in X$ with $\|x\|_u \leq 1$, we have $-u \leq x \leq u$. So for $f \in X'$ with $f \geq 0$, we have $-f(u) \leq f(x) \leq f(u)$, and thus $|f(x)| \leq f(u)$. Hence $\|f\| \leq f(u)$. Since $f(u) \leq \|f\|\|u\| = \|f\|$, it follows that $\|f\| = f(u)$. Therefore

$$\|f+g\| = (f+g)(u) = f(u) + g(u) = \|f\| + \|g\|.$$

The norm dual of a normed Riesz space is a Banach lattice, so X' is an AL-space. □

Let us recall some know results. The following one is from [1, Corollary 3.7].

Corollary 3.3.20. Every AL-space has order continuous norm.

The next result is due to Nakano, see [6, Theorem 4.9].

Theorem 3.3.21. For a Banach lattice X the following statements are equivalent.

(i) X has order continuous norm.

(ii) X is an ideal of X''.

We cite the following result from [6, Corollary 4.5].

Corollary 3.3.22. (G. Birkhoff) The norm dual of a Banach lattice X coincides with its order dual, i.e., $X' = X^{\sim}$.

Lemma 3.3.23. Let X be an Archimedean Riesz space with an order unit norm. Then $\langle X'', X' \rangle$ is a Riesz dual system.

Proof. By Lemma 3.3.19, X' is an AL-space. By Corollary 3.3.20, every AL-space has an order continuous norm, so by Theorem 3.3.21 X' is an ideal of X'''. Since X'' is a Banach lattice, it follows from Corollary 3.3.22 that $X''' = (X'')^{\sim}$. Hence X' is an ideal in $(X'')^{\sim}$. Moreover, let $x \in X''$ with $x \neq 0$. Then there exists some $f \in X'$ satisfying $f(x) = x(f) \neq 0$.

Hence $\langle X'', X' \rangle$ is a Riesz dual system. □

Since a pre-Riesz space with an order unit and an order unit norm need not be norm complete, we can not use [6, Theorem 5.11] directly (in there it is required for the spaces to be Banach lattices). The next lemma is a modification of [6, Theorem 5.11], provided that the range space has an order unit and an order unit norm.

Lemma 3.3.24. Let X, Y be two normed Riesz spaces such that Y has an order unit and equipped with an order unit norm. Let $S, T\colon X \to Y$ be two positive operators with $0 \le S \le T$. If $T[0,x]$ is $|\sigma|(Y, Y')$-totally bounded for each $x \in X^+$, then $S[0,x]$ is likewise $|\sigma|(Y, Y')$-totally bounded for each $x \in X^+$.

Proof. By Lemma 3.3.23, the pair $\langle Y'', Y' \rangle$ is a Riesz dual system. Since Y'' is an ideal of Y'', it follows from Theorem 3.3.14 that $\sigma(Y'', Y')$ is an order continuous topology on Y''. By Theorem 3.3.15 we therefore have that every consistent locally convex solid topology on Y'' is order continuous. Observe that Corollary 3.3.13 it yields that $|\sigma|(Y'', Y')$ is consistent, and then $|\sigma|(Y'', Y')$ is an order continuous locally convex solid topology.

Let us view S, T as two operators from X to Y''. As $T[0,x]$ is $|\sigma|(Y, Y')$-totally bounded for each $x \in X^+$, it has $T[0,x]$ is $|\sigma|(Y'', Y')$-totally bounded for each $x \in X^+$. Since Y equipped with order unit norm, we have Y' is Dedekind complete. Hence, it follows from Theorem 3.3.10 that $S[0,x]$ is $|\sigma|(Y'', Y')$-totally bounded for each $x \in X^+$. So $S[0,x]$ is $|\sigma|(Y, Y')$-totally bounded for each $x \in X^+$. □

We are now in the position to extend the result of Theorem 3.3.8 to a setting with pre-Riesz spaces. To this end, first notice that for every order unit e in a pre-Riesz space X with the Riesz completion (X^ρ, i) the element $i(e)$ is an order unit in X^ρ.

Theorem 3.3.25. Let X be an Archimedean pre-Riesz space with an order unit e and the order unit norm $\|\cdot\|$ such that X is norm complete. Let $S, T\colon X \to X$ satisfy $0 \le S \le T$, and let T be compact. Assume that there exists a positive compact operator $\widehat{T}\colon X^\rho \to X^\rho$ such that $\widehat{T} \circ i = i \circ T$, where (X^ρ, i) is the Riesz completion of X equipped with the order unit norm $\|\cdot\|_{X^\rho}$. Then S^3 is compact.

Proof. Let $U = \{x \in X; \|x\| \le 1\}$ be the closed unit ball in X. Since the norm on X is an order unit norm we have

$$\|x\| = \inf\{\lambda \in \mathbb{R}^+;\ -\lambda e \le x \le \lambda e\},\ x \in X.$$

Let $x \in U$ be such that $\|x\| \leq 1$. Hence there exists $\lambda \leq 1$ with $-y \leq x \leq y$ and $y = \lambda e$. Due to $x = \frac{1}{2}(y+x) - \frac{1}{2}(y-x)$ it follows from $0 \leq y+x \leq 2y$ and $0 \leq y-x \leq 2y$ that $\|y+x\| \leq 2\|y\| \leq 2$ and $\|y-x\| \leq 2\|y\| \leq 2$, respectively. Thus $\frac{1}{2}(y+x), \frac{1}{2}(y-x) \in U \cap X^+$, and then $U \subseteq U^+ - U^+$ holds. Therefore, it is enough to show that $S^3(U^+)$ is a norm totally bounded set.

Let (X^ρ, i) be the Riesz completion of X, let $e \in X$ be an order unit, then $i(e)$ is an order unit in X^ρ, and the extension norm $\|\cdot\|_{X^\rho}$ is the order unit norm with respect to $i(e)$. By Lemma 3.3.17 there exists some $u \in (X^\rho)^+$ such that $\|(i(Sx)-u)^+\|_{X^\rho} \leq \epsilon$ for all $x \in U^+$. This implies that $0 \leq (i(Sx)-u)^+ \leq \epsilon i(e)$. Thus we have the following estimate,

$$\begin{aligned} i(Sx - \epsilon e) &= i(Sx) \wedge u + (i(Sx)-u)^+ - \epsilon i(e) \\ &\leq i(Sx) \wedge u \\ &\leq u. \end{aligned}$$

So $i(Sx) \in [0, u + \epsilon i(e)]$ for every $x \in U^+$. Take $v \in X$ with $i(v) \geq u$. Then for every $x \in U^+$ we have $i(Sx) \in [0, i(v+\epsilon e)]$. Hence

$$S(U^+) \subseteq [0, v+\epsilon e].$$

Therefore

$$S^2(U^+) \subseteq S[0, v+\epsilon e], \quad \text{and} \quad S^3(U^+) \subseteq S^2[0, v+\epsilon e]. \tag{$*$}$$

By Lemma 3.3.16 the operator S is continuous on norm bounded subsets of X with respect to the r-topology of the $|\sigma|(X^\rho, (X^\rho)')$ to the norm topology. By Theorem 3.3.9 the operator S maps totally bounded sets with respect to the r-topology of $|\sigma|(X^\rho, (X^\rho)')$ to norm totally bounded sets. Since $T[0, v+\epsilon e]$ is norm totally bounded, $T[0, v+\epsilon e]$ is totally bounded with respect to the $|\sigma|(X^\rho, (X^\rho)')$ topology. Hence, $T[0, v+\epsilon e]$ is totally bounded with respect to the r-topology of $|\sigma|(X^\rho, (X^\rho)')$. Then by Lemma 3.3.24 the set $S[0, v+\epsilon e]$ is likewise totally bounded with respect to the r-topology of $|\sigma|(X^\rho, (X^\rho)')$. Clearly, $S[0, v+\epsilon e] \subseteq$

$[0, S(v+\epsilon e)]$. Therefore, $S^2[0, v+\epsilon e] = S(S[0, v+\epsilon e])$ is a norm totally bounded set. By the second inclusion of $(*)$ we have $S^3(U^+)$ is a norm totally bounded set, as desired. □

Remark 3.3.26. In the view of Theorem 3.3.25, it is of interest to know under which condition such a positive compact operator $\widehat{T}$ exists. By Theorem 3.3.5 and Theorem 3.3.7, we know that this is true for X with order continuous norm. However, if X is a Banach lattice with an order unit e, then it can be renormed by

$$\|x\|_\infty = \inf\{\lambda > 0;\ |x| \leq \lambda e\},$$

and X becomes an AM-space. By Kakutani-Bohnenblust and M. Krein-S. Krein representation theorem [6, Theorem 4.29], we have X is lattice isometric to some $C(\Omega)$ for a (unique up to homeomorphism) Hausdorff compact topological space Ω, and the norm $\|\cdot\|_\infty$ on $C(\Omega)$ is not order continuous. So it is still an open question which choice of a norm on X leads to a similar result as Theorem 3.3.25.