Extension of operators on pre-Riesz spaces

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Chapter 2

Disjointness preserving operators on ordered vector space

Disjointness preserving operators on vector lattices have been studied by Y. A. Abramovich and A. K. Kitover [3, 4], W. Arendt [9], B. de Pagter [19], C. B. Huijsmans and A. W. Wickstead [27]. In the problem section of [26], Y. A. Abramovich raises a question: for an invertible disjointness preserving operator $T\colon X \to Y$ with X, Y being vector lattices, when does T^{-1} preserve disjointness? An affirmative answer is given by C. B. Huijsmans and B. de Pagter in [25] by showing that X being a uniformly complete vector lattice and Y a normed vector lattice is a sufficient condition, see Theorem 2.1.1.

In this chapter, we mainly deal with the above question in the case of pre-Riesz spaces. To generalize the result by C. B. Huijsmans and B. de Pagter [25], our idea is to use that every pre-Riesz space can be embedded order densely into the Riesz completion, and then we use the theory of Riesz spaces. It turns out that the main difficulty is to deal with compatibility of order convergence and norm convergence in both the pre-Riesz space and the Riesz completion. We will impose suitable conditions on the pre-Riesz space, for instance pervasive, fordable, etc..

This chapter includes two main sections.

In Section 2.1, we will discuss the generalization of the result by C. B. Huijsmans and B. de Pagter [25]. It is separated into two parts. Subsection 2.1.1 concerns the range space Y being a pre-Riesz space. To use the theory of the Riesz completion Y^ρ, it is necessary to extend the norm of Y to a Riesz norm on Y^ρ. This comes true if Y is a pervasive pre-Riesz space with a monotone norm. Then, to achieve the goal, we use the fact that two elements are disjoint in Y if and only if they are disjoint in Y^ρ. As an independent interesting result, moreover, we generalize a theorem by B. de Pagter from [19]. The second subsection deals with the situation of the domain space X being a pervasive pre-Riesz space. This turns out to be more difficult than the first part, and we have to add more conditions, e.g. a denseness condition in the sense that for a positive element there exists a positive sequence which is convergent from below.

Section 2.2 is concerned with exploring more sufficient conditions of extending disjointness preserving operators on pre-Riesz spaces to Riesz completions. One is that Riesz* homomorphisms on pre-Riesz spaces preserve disjointness. The other one is that we can show that the inverse of T is a disjointness preserving operator, provided that X is a fordable pre-Riesz space and T satisfies the condition (β). Moreover, by using the Hahn-Banach theorem, we establish that an order bounded disjointness preserving operator T can be extended to an order bounded disjointness preserving operator on the Riesz completion if the pre-Riesz space has the Riesz decomposition property.

2.1 A generalization of disjointness preserving operators on Riesz spaces

In this section, the disjointness preserving operators and their inverses between pre-Riesz spaces will be considered. An important result on inverses of disjointness preserving operators is given in [25, Theorem 2.1, Corollary 2.2], which reads as follows.

Theorem 2.1.1. Let X be a uniformly complete vector lattice, Y a normed vector lattice, and $T\colon X \to Y$ an injective and disjointness preserving operator, then $Tx_1 \perp Tx_2$ implies $x_1 \perp x_2$ in X. Moreover, if T is bijective, then T^{-1} is disjointness preserving as well.

We will use two steps to generalize this result to pre-Riesz spaces. Firstly, we consider the range space X being a pre-Riesz space, and secondly the domain space Y being a pre-Riesz space.

2.1.1 Pervasive pre-Riesz spaces as ranges

Let us recall the definition of pervasiveness in pre-Riesz spaces. The definition was firstly given by O. van Gaans and A. Kalauch [53, Definition 2.3] in studying the restriction of bands in pre-Riesz spaces.

Definition 2.1.2. A pre-Riesz space X with Riesz completion (X^ρ, i) is called **pervasive** if for every $y \in X^\rho$ with $y \geq 0$, $y \neq 0$ there exists $x \in X$, $x \neq 0$, such that $0 < i(x) \leq y$.

It should be noticed that the term of pervasive is a property of a pre-Riesz space in its Riesz completion, and the definition of property (p), which appeared in Remark 1.2.5, is a similar property in vector lattices. We just use different terms to distinguish them in different situations. Here are some examples of pre-Riesz spaces which do or do not have the pervasive property.

Example 2.1.3. (1) The pre-Riesz space $X = C^1[0,1]$ with the cone $K = \{f \in X;\ f(x) \geq 0 \text{ for all } x \in [0,1]\}$ is pervasive.

(2) The pre-Riesz space $X = \mathrm{Pol}^2(\mathbb{R})$ with the cone $K = \{f \in X;\ f(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$ is not pervasive.

(3) [30, Example 3.3.22] The pre-Riesz space $X = \{\alpha\mathbf{1} + v;\ v \in C[0,1],\ v(0) = 0,\ \alpha \in \mathbb{R},\ \int_0^1 v(t)dt = 0\}$ ordered by a natural cone is not pervasive.

It is known that for a Riesz subspace of Y of an Archimedean Riesz space X, Y has the property (p) (i.e. order dense in Riesz space) in X if and only if for each $x \in X^+$, it has $x = \sup\{y \in Y; 0 \leq y \leq x\}$, see [6, Theorem 1.34]. In the proof of this conclusion, it does not use the lattice operations. So one has the similar conclusion in the Archimedean pre-Riesz space case. In fact, this was observed by J. van Waaij [55, Theorem 4.15, Corollary 4.16], and by H. Malinowski [41, Lemma 89], see the following proposition.

Proposition 2.1.4. For an Archimedean pre-Riesz space X, let (X^ρ, i) be the Riesz completion. The following are equivalent.

(i) X is pervasive.

(ii) For all $0 < y \in X^\rho$, it holds $y = \sup\{x \in i(X); 0 < x \leq y\}$.

It yields that in Riesz spaces, pervasiveness and order denseness are the same.

Now we turn to our main purpose of this subsection, which is replacing the range space Y in Theorem 2.1.1 by a pre-Riesz space. The idea is to consider the Riesz completion (Y^ρ, i) of Y and apply Theorem 2.1.1 to $i \circ T$. For that purpose we need a Riesz norm on Y^ρ.

Recall that for an ordered vector space (X, K) with a norm $\|\cdot\|$, we say that $(X, K, \|\cdot\|)$ is an **ordered normed space**. In some cases of this thesis, we write $\|\cdot\|_X$ to emphasize the norm of X.

Definition 2.1.5. Let (X, K) be a partially ordered vector space with a seminorm p.

(i) p is called **monotone** if for every $x, y \in X$ with $0 \leq x \leq y$ one has $p(x) \leq p(y)$.

(ii) If X is a Riesz space, p is called **Riesz** if it is monotone and $p(|x|) = p(x)$ for every $x \in X$.

The next lemma presents conditions on a pre-Riesz space that provide a Riesz norm on the Riesz completion.

Lemma 2.1.6. Let X be a pervasive pre-Riesz space, let (Y, i) be a vector lattice cover of X, and let $\|\cdot\|_X$ be a monotone norm on X. Define for $y \in Y$

$$\|y\|_Y = \inf\left\{\|x\|_X\,;\, x \in X,\ |y| \le i(x)\right\}. \tag{2.1}$$

Then $\|\cdot\|_Y$ is a Riesz norm on Y.

Proof. Let us first prove that $\|\cdot\|_Y$ is a seminorm. Let $y_1, y_2 \in Y$ and $u, v \in X$ be such that $|y_1| \le i(u)$, $|y_2| \le i(v)$. Then $|y_1 + y_2| \le |y_1| + |y_2| \le i(u+v)$. So we have

$$\begin{aligned}
\|y_1 + y_2\|_Y &= \inf\left\{\|u+v\|_X\,;\, u+v \in X, |y_1+y_2| \le i(u+v)\right\}\\
&\le \inf\left\{\|u+v\|_X\,;\, u, v \in X, |y_1| \le i(u), |y_2| \le i(v)\right\}\\
&\le \inf\left\{\|u\|_X + \|v\|_X\,;\, u, v \in X, |y_1| \le i(u), |y_2| \le i(v)\right\}\\
&\le \inf\left\{\|u\|_X\,;\, u \in X, |y_1| \le i(u)\right\} + \inf\left\{\|v\|_X\,;\, v \in X, |y_2| \le i(v)\right\}\\
&\le \|y_1\|_Y + \|y_2\|_Y\,.
\end{aligned}$$

The absolute homogeneity and non-negativity are clear. So $\|\cdot\|_Y$ is a seminorm.

Note that since $i(X)$ is majorizing in Y, the set of which the infimum is taken in (2.1) is nonempty. The equality $\|u\|_Y = \||u|\|_Y$ is clearly true for any $u \in Y$. To show $\|\cdot\|_Y$ is monotone, we suppose $x, y \in Y$ with $|x| \le |y|$. For $v \in X$ we have that $|y| \le i(v)$ implies $|x| \le i(v)$. So we have

$$\|x\|_Y = \inf\left\{\|v\|_X\,;\, v \in X,\ |x| \le i(v)\right\} \le \inf\left\{\|v\|_X\,;\, v \in X,\ |y| \le i(v)\right\} = \|y\|_Y\,.$$

Therefore, $\|\cdot\|_Y$ is monotone, hence, Riesz. It remains to show that $\|\cdot\|_Y$ is a norm.

Observe that for $v \in X$, $v \ge 0$, we have for every $y \in X$ with $i(y) \ge |i(v)|$ that $y \ge v \ge 0$. As $\|\cdot\|_X$ is monotone, hence $\|y\|_X \ge \|v\|_X$. So $\|i(v)\|_Y \ge \|v\|_X$. Let $z \in Y$ be such that $z \ne 0$. Since X is pervasive, there exists $y \in X$ with

$0 < i(y) \leq |z|$. Then $\|z\|_Y = \||z|\|_Y \geq \|i(y)\|_Y \geq \|y\|_X > 0$. Hence $\|\cdot\|_Y$ is a Riesz norm. □

We can now extend Theorem 2.1.1 to a setting with the range space being a pre-Riesz space.

Theorem 2.1.7. Let X be a uniformly complete vector lattice, Y a pervasive pre-Riesz space with a monotone norm, and $T\colon X \to Y$ an injective and disjointness preserving operator. Then for every $x_1, x_2 \in X$ we have that $Tx_1 \perp Tx_2$ implies $x_1 \perp x_2$.

Proof. Let (Y^ρ, i) be the Riesz completion of Y. Since $T\colon X \to Y$ is injective, we have that $i \circ T\colon X \to Y^\rho$ is injective as well. As T is disjointness preserving, by means of Proposition 1.2.15 we have that $i \circ T$ is disjointness preserving. With the aid of Lemma 2.1.6, the monotone norm of Y yields a Riesz norm on Y^ρ.

Let $x_1, x_2 \in X$ be such that $Tx_1 \perp Tx_2$. Then $(i \circ T)x_1 \perp (i \circ T)x_2$. We apply Theorem 2.1.1 and obtain that $x_1 \perp x_2$. □

Corollary 2.1.8. Let X be a uniformly complete vector lattice, Y a pervasive pre-Riesz space with a monotone norm, and $T\colon X \to Y$ a bijective and disjointness preserving operator. Then $T^{-1}\colon Y \to X$ is disjointness preserving as well.

Proof. Let $y_1, y_2 \in Y$ be such that $y_1 \perp y_2$. Take $x_1, x_2 \in X$ with $Tx_1 = y_1$ and $Tx_2 = y_2$. We have $Tx_1 \perp Tx_2$, hence according to Theorem 2.1.7 we obtain $x_1 \perp x_2$. Therefore, T^{-1} is disjointness preserving. □

A key role in the proof of Theorem 2.1.1 is played by the next result which is due to B. de Pagter, see [19, Theorem 8].

Theorem 2.1.9 (B. de Pagter)**.** Let X be a uniformly complete Archimedean Riesz space and let Y be an Archimedean Riesz space such that for every disjoint sequence $(w_n)_n$ in Y with $w_n > 0$ $(n \in \mathbb{N})$ there exist positive real numbers λ_n

($n \in \mathbb{N}$) such that the set $\{\lambda_n w_n;\, n \in \mathbb{N}\}$ is not order bounded in Y. Then for every disjointness preserving operator $T\colon X \to Y$, there exists an order dense ideal in X on which T is order bounded.

One could try to generalize the range space in Theorem 2.1.9 to a more general pre-Riesz space Y and thus try to generalize Theorem 2.1.1. It turns out that the conditions on Y needed for this approach are not more general than those of the approach above. Nevertheless, our extension of Theorem 2.1.9 might be of independent interest.

We need the following simple observation, which follows from the fact that $i(Y)$ is majorizing in its Riesz completion Y^ρ.

Lemma 2.1.10. Let Y be a pre-Riesz space and let (Y^ρ, i) be its Riesz completion. For every subset $A \subset Y$, one has that A is order bounded in Y if and only if $i(A)$ is order bounded in Y^ρ.

We arrive at the following extension of Theorem 2.1.9.

Theorem 2.1.11. Let X be a uniformly complete Archimedean Riesz space and let Y be a pervasive Archimedean pre-Riesz space such that for every disjoint sequence $(w_n)_n$ in Y with $w_n > 0$ ($n \in \mathbb{N}$) there exist positive real numbers λ_n ($n \in \mathbb{N}$) such that the set $\{\lambda_n w_n;\, n \in \mathbb{N}\}$ is not order bounded in Y. Then for every disjointness preserving operator $T\colon X \to Y$, there exists an order dense ideal in X on which T is order bounded.

Proof. Let $T\colon X \to Y$ be a disjointness preserving operator. Let (Y^ρ, i) denote the Riesz completion of Y. By Theorem 1.2.15, we have for every $x_1, x_2 \in X$ that $i(Tx_1) \perp i(Tx_2)$ in Y^ρ if and only if $Tx_1 \perp Tx_2$ in Y, so $i \circ T\colon X \to Y^\rho$ is disjointness preserving as well.

Let $(w_n)_n$ be a disjoint sequence in Y^ρ with $w_n > 0$ ($n \in \mathbb{N}$). Since Y is pervasive, for every $n \in \mathbb{N}$ there exists $y_n \in Y$ with $0 < i(y_n) \leq w_n$. Then $(y_n)_n$ is a disjoint sequence in Y, so there exist positive real numbers λ_n ($n \in \mathbb{N}$) such that

$\{\lambda_n y_n;\, n \in \mathbb{N}\}$ is not order bounded in Y. With the aid of Lemma 2.1.10, it follows that $\{\lambda_n i(y_n);\, n \in \mathbb{N}\}$ is not order bounded and hence $\{\lambda_n w_n;\, n \in \mathbb{N}\}$ is not order bounded.

Theorem 2.1.9 now yields that there exists an order dense ideal D in X on which $i \circ T$ is order bounded. Then it follows from Lemma 2.1.10 that T is order bounded on D. □

The condition in Theorem 2.1.9 and Theorem 2.1.11 involving the disjoint sequence $(w_n)_n$ is satisfied if the space Y can be equipped with a monotone norm. The next lemma provides the details of the simple verification of this fact.

Lemma 2.1.12. If Y is a pre-Riesz space with a monotone norm $\|\cdot\|_Y$, then for every sequence $(w_n)_n$ in Y with $w_n > 0$ $(n \in \mathbb{N})$, there exist positive real numbers λ_n $(n \in \mathbb{N})$ such that the set $\{\lambda_n w_n;\, n \in \mathbb{N}\}$ is not order bounded in Y.

Proof. For the sequence $(w_n)_n$ in Y, let $\lambda_n := \frac{n}{\|w_n\|_Y}$. Then $\|\lambda_n w_n\|_Y = \frac{n}{\|w_n\|_Y} \|w_n\|_Y$ $= n$, so $\{\lambda_n w_n;\, n \in \mathbb{N}\}$ is not norm bounded. Since the norm of Y is monotone, every order bounded set in Y is norm bounded. Hence $\{\lambda_n w_n;\, n \in \mathbb{N}\}$ is not order bounded in Y. □

Observe that if X is a Banach lattice then X is a uniformly complete Archimedean Riesz space due to P. Meyer-Nieberg [42, Proposition 1.1.8(iv)]. By combining Theorem 2.1.11 with Lemma 2.1.12, we immediately obtain the following corollary.

Corollary 2.1.13. If X is a Banach lattice and Y a pervasive pre-Riesz space with a monotone norm, then for every disjointness preserving operator $T\colon X \to Y$ there exists an order dense ideal in X on which T is order bounded.

2.1.2 Pre-Riesz spaces as domains

In this subsection, we present some results of disjointness preserving operators on pre-Riesz spaces. In the following two theorems, we consider extending disjointness preserving operators in two different ways, i.e. order continuous operators and norm continuous operators.

Recall that an operator $T\colon X \to Y$ between two ordered vector spaces is said to be **order continuous** if $Tx_\alpha \xrightarrow{o} 0$ whenever $x_\alpha \xrightarrow{o} 0$.

Theorem 2.1.14. Let X be a pre-Riesz space with the Riesz completion (X^ρ, i). Let Y be a vector lattice. Assume that for every $x \in X^\rho$ with $x \geq 0$, there exists a sequence $(x_n)_{n=1}^\infty$ in X with $i(x_n) \geq 0$ for every n such that $i(x_n) \uparrow x$. If $\widehat{T}\colon X^\rho \to Y$ is an order continuous operator such that $(\widehat{T} \circ i)\colon X \to Y$ is a positive linear disjointness preserving operator, then $\widehat{T}$ is also disjointness preserving on X^ρ.

Proof. We only need to show the conclusion holds for positive elements in X^ρ. Let $x, y \in (X^\rho)^+$ be such that $x \perp y$. Then there exist sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in X^+ such that $i(x_n) \uparrow x$ and $i(y_n) \uparrow y$. It follows immediately that $x_n \perp y_n$ in X^+. Since $\widehat{T}$ is order continuous, and the lattice operations are order continuous [35, Proposition 1.1.34], we get

$$\begin{aligned}\widehat{T}x \wedge \widehat{T}y &= \left(\widehat{T} \lim i(x_n)\right) \wedge \left(\widehat{T} \lim i(y_n)\right)\\ &= \left(\lim \left(\widehat{T} \circ i\right) x_n\right) \wedge \left(\lim \left(\widehat{T} \circ i\right) y_n\right)\\ &= \lim \left(Tx_n \wedge Ty_n\right) = 0.\end{aligned}$$

We conclude $\widehat{T}x \perp \widehat{T}y$. □

Theorem 2.1.15. Let $(X, \|\cdot\|_X)$ be a normed pre-Riesz space. Let $(Y, \|\cdot\|_Y)$ be a normed vector lattice. Let (X^ρ, i) be the Riesz completion of X with norm $\|\cdot\|_{X^\rho}$. Assume that for every $x \in X^\rho \setminus \{0\}$, $x \geq 0$, there exists an increasing sequence $(x_n)_{n=1}^\infty$ in X with $0 \leq i(x_n) \leq x$ for every n, and $\|i(x_n) - x\|_{X^\rho} \to 0$. If $\widehat{T}\colon X^\rho \to Y$ is norm continuous, $\left(\widehat{T} \circ i\right) : X \to Y$ is a positive linear disjointness preserving operator, then $\widehat{T}$ is also disjointness preserving on X^ρ.

Proof. Let x, y in $(X^\rho)^+$ with $x \perp y$. By assumption, there exist two increasing sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ in X with $0 \leq i(x_n) \leq x$, $0 \leq i(y_n) \leq y$ for every n, and $\|i(x_n) - x\|_{X^\rho} \to 0$, $\|i(y_n) - y\|_{X^\rho} \to 0$. Hence, $x_n \perp y_n$ in X^+ for all $n \in \mathbb{N}$. So it has $\left(\widehat{T} \circ i\right)(x_n) \perp \left(\widehat{T} \circ i\right)(y_n)$. Since $\widehat{T}$ is norm continuous, it follows that $\left\|\left(\widehat{T} \circ i\right)(x_n) - \widehat{T}(x)\right\|_Y \to 0$ and $\left\|\left(\widehat{T} \circ i\right)(y_n) - \widehat{T}(y)\right\|_Y \to 0$. By the fact that the lattice operations are norm continuous [35, Proposition 3.6.19], it has

$$\left\|\left|\left(\widehat{T} \circ i\right)(x_n)\right| \wedge \left|\left(\widehat{T} \circ i\right)(y_n)\right| - \left|\widehat{T}(x)\right| \wedge \left|\widehat{T}(y)\right|\right\|_Y \to 0.$$

Therefore $\left|\widehat{T}(x)\right| \wedge \left|\widehat{T}(y)\right| = 0$, and hence $\widehat{T}$ is disjointness preserving. □

It turns out the extension of operator as in the above two theorems is continuous.

Theorem 2.1.16. Let X be a pervasive pre-Riesz space with Riesz completion (X^ρ, i_X), let $\|\cdot\|_X$ be a monotone norm on X, and let $\|\cdot\|_{X^\rho}$ be defined as in (2.1). Let Y be a partially ordered vector space with a monotone norm $\|\cdot\|_Y$, and let $T\colon X \to Y$ be a positive and continuous linear map. Then every positive linear map $\widehat{T}\colon X^\rho \to Y$ that extends T in the sense that $\widehat{T} \circ i_X = T$ is continuous with respect to $\|\cdot\|_{X^\rho}$ and $\left\|\widehat{T}\right\| \leq \|T\|$.

Proof. As $\|\cdot\|_X$ is monotone, it follows from Lemma 2.1.6 that $\|\cdot\|_{X^\rho}$ is a Riesz norm. Let $u \in X^\rho$ and $u \geq 0$. Take $x \in X$ with $u \leq i(x)$, as T and $\widehat{T}$ are positive, it has $0 \leq \widehat{T}(u) \leq \left(\widehat{T} \circ i\right)(x) = T(x)$. Since $\|\cdot\|_Y$ is a monotone norm, we have $\left\|\widehat{T}u\right\|_Y \leq \|Tx\|_Y \leq \|T\| \, \|x\|_X$, and then $\left\|\widehat{T}u\right\|_Y \leq \|T\| \, \|u\|_{X^\rho}$. Thus T is continuous and $\left\|\widehat{T}\right\| \leq \|T\|$. □

Since a pre-Riesz space is a majorizing subspace of the Riesz completion, we could use the Kantorovich's extension theorem to extend a positive linear operator, which is from an Archimedean pre-Riesz space to a Dedekind complete Riesz space, to a positive linear operator, which is from the Riesz completion to a Dedekind complete Riesz space.

To start with, we recall that Definition 1.1.6 (iv) for uniformly complete ordered vector space X. We will use X^u to denote the uniformly completion of X.

Theorem 2.1.17. Let X be a pervasive pre-Riesz space with Riesz completion (X^ρ, i_X), let $\|\cdot\|_X$ be a monotone norm on X, and let $\|\cdot\|_{X^\rho}$ be the norm on X^ρ defined as in (2.1). Assume that for every $x \in X^\rho \setminus \{0\}$, $x \geq 0$, there exists an increasing sequence $(x_n)_n$ in X with $0 \leq i_X(x_n) \leq x$ for every n, and $\|i_X(x_n) - x\|_{X^\rho} \to 0$. Let Y be a pervasive pre-Riesz space with Dedekind completion (Y^δ, i_Y), let $\|\cdot\|_Y$ be a monotone norm on Y, and let $\|\cdot\|_{Y^\delta}$ be the norm on Y^δ defined as in (2.1). If $T\colon X \to Y$ is a positive linear map that is continuous, disjointness preserving and injective, then there exists a positive linear extension $\widehat{T}\colon X^\rho \to Y^\delta$ of $i_Y \circ T \circ i_X^{-1}\colon i_X(X) \to Y^\delta$ that is continuous, disjointness preserving and injective as well. Moreover, if X has an order unit and $X^{\rho u}$ denotes the uniform completion of X^ρ, then there exists a positive linear extension $T_u\colon X^{\rho u} \to Y^\delta$ of $i_Y \circ T \circ i_X^{-1}\colon i_X(X) \to Y^\delta$ that is disjointness preserving and injective.

Proof. Due to Theorem 1.3.6 (Kantorovich), there exists a positive operator $\widehat{T}\colon X^\rho \to Y^\delta$ extending $i_Y \circ T \circ i_X^{-1}\colon i_X(X) \to Y^\delta$. By Lemma 2.1.16, $\widehat{T}$ is continuous with respect to the Riesz norm $\|\cdot\|_{X^\rho}$ on X^ρ and $\|\cdot\|_{Y^\delta}$ on Y^δ. By Theorem 2.1.15, $\widehat{T}$ is disjointness preserving.

Next we show that $\widehat{T}$ is injective. Let $v \in X^\rho$ with $\widehat{T}v = 0$, then $\widehat{T}v^+ - \widehat{T}v^- = 0$. As $\widehat{T}$ is disjointness preserving, $\widehat{T}v^+ = 0$ and $\widehat{T}v^- = 0$. Suppose that $v \neq 0$, then either $v^+ \neq 0$ or $v^- \neq 0$. Assume without loss of generality that $v^+ \neq 0$. As X is pervasive, there is an element $x \in X \setminus \{0\}$ with $0 \leq i_X(x) \leq v^+$. So we have $0 \leq (i_Y \circ T)(x) = (\widehat{T} \circ i_X)(x) \leq \widehat{T}v^+ = 0$ and hence $Tx = 0$. This contradicts that T is injective. Therefore $v = 0$ and hence $\widehat{T}$ is injective. So $\widehat{T}\colon X^\rho \to Y^\delta$ is a positive disjointness preserving injective operator.

As X^ρ is a majorizing subspace of $X^{\rho u}$, by Theorem 1.3.6, $\widehat{T}$ can be extended to a positive operator $T_u\colon X^{\rho u} \to Y^\delta$.

We show as an intermediate step that every element in $X^{\rho u}$ can be approximated from below in the relative uniform topology by a sequence from $i(X)$. For $z \in$

$(X^{\rho u})^+$, $z \neq 0$, we have an element $u \in (X^\rho)^+$, a sequence $(z_n)_n \in (X^\rho)^+$ and a sequence $(\lambda_n)_n$ in $\mathbb{R}$ with $\lambda_n \downarrow 0$ and $|z_n - z| \leq \lambda_n u$ for every $n \in \mathbb{N}$. Then $z_n - \lambda_n u \leq z$ for every n. Take $w_n = (z_n - \lambda_n u)^+$ in X^ρ. We have $w_n \in X^\rho$, $0 \leq w_n \leq z$, and

$$|w_n - z| = |(z_n - \lambda u)^+ - z^+| \leq |z_n - \lambda_n u - z| \leq 2\lambda_n u,$$

so $w_n \to z$ in the relative uniform topology.

Next we show that T_u is disjointness preserving. Let $v, w \in X^{\rho u}$ with $v \perp w$. By the previous discussion, there exist v_n, w_n in X^ρ such that $0 \leq v_n \leq |x|$, $0 \leq w_n \leq |y|$ and $v_n \to |v|$ and $w_n \to |w|$ in the relative uniform topology. Then $0 \leq v_n \wedge w_n \leq |v| \wedge |w| = 0$, so $v_n \perp w_n$ and therefore $\widehat{T} v_n \perp \widehat{T} w_n$. Also, it follows that there exist sequences $(\alpha_n)_n$ and $(\beta_n)_n$ in $\mathbb{R}$ with $\alpha_n \downarrow 0$ and $\beta_n \downarrow 0$ and $u_1, u_2 \in X^\rho$ such that $|v_n - |v|| \leq \alpha_n u_1$ and $|w_n - |w|| \leq \beta_n u_2$ for every n. Take $u = u_1 \vee u_2$. We obtain that $|v| \leq ||v| - v_n| + |v_n| \leq \alpha_n u + v_n$ and $|w| \leq ||w| - w_n| + |w_n| \leq \beta_n u + w_n$. It follows that

$$\begin{aligned}
T_u(|v|) \wedge T_u(|w|) &\leq T_u(\alpha_n u + v_n) \wedge T_u(\beta_n u + w_n) \\
&\leq (\alpha_n T_u(u) + T_u(v_n)) \wedge (\beta_n T_u(u) + T_u(w_n)) \\
&\leq ((\alpha_n \vee \beta_n) T_u(u) + T_u(v_n)) \wedge ((\alpha_n \vee \beta_n) T_u(u) + T_u(w_n)) \\
&= (\alpha_n \vee \beta_n) T_u(u) + (T_u(v_n) \wedge T_u(w_n)) \\
&= (\alpha_n \vee \beta_n) \widehat{T}(u) + (\widehat{T}(v_n) \wedge \widehat{T}(w_n)). \\
&= (\alpha_n \vee \beta_n) \widehat{T}(u).
\end{aligned}$$

Since Y^δ is Archimedean and $\alpha_n \vee \beta_n \downarrow 0$, we infer that $T_u(|v|) \wedge T_u(|w|) = 0$. Hence $T_u \colon X^{\rho u} \to Y^\delta$ is disjointness preserving.

By using the same argument as for $\widehat{T}$, we get that T_u is injective as well. □

Proposition 2.1.18. In the setting of Theorem 2.1.17, $T_u \colon X^{\rho u} \to T_u(X^{\rho u})$ has a disjointness preserving inverse $T_u^{-1} \colon T_u(X^{\rho u}) \to X^{\rho u}$.

Proof. Note that T_u is a positive disjointness preserving operator, hence a Riesz homomorphism. Therefore, $T_u(X^{\rho u})$ is a Riesz subspace of Y^δ. We can extend the norm of Y to a monotone norm on Y^δ by (2.1). Since T_u is injective, Corollary 2.1.8 yields that $T_u^{-1}\colon T_u(X^{\rho u}) \to X^{\rho u}$ is disjointness preserving. □

In the following example, we will give an application for Theorem 2.1.17.

Example 2.1.19. Let $m \in \mathbb{N}$ and let $C^m[0,1]$ be the subspace of $C[0,1]$ consisting of all m times continuously differentiable functions on $[0,1]$. For every $f \in C[0,1]^+$ there exists a sequence $(f_n)_n$ in $C^m[0,1]^+$ with $0 \le f_n \le f$ and $\|f_n - f\|_\infty \to 0$ and also $f_n \uparrow f$. We will give a proof of this statement in six steps below. The main idea is to approximate f for a given $\varepsilon > 0$ up to 6ε from below by a $g \in C^m[0,1]^+$, by choosing g to be 0 where $f \le 4\varepsilon$, choosing g between $f - 4\varepsilon$ and $f - 3\varepsilon$ where $f > 6\varepsilon$ and glue these pieces of g smoothly together. Some technical precautions are needed to make sure that our construction involves only finitely many subintervals and that the smooth connecting parts of g are between 0 and f.

(a) Firstly, note that for every $f \in C[0,1]$ and every $\varepsilon > 0$ there exists $g \in C^m[0,1]$ such that $\|f - g\|_\infty < 4\varepsilon$ and $g \le f - 3\varepsilon$. Indeed, according to Weierstrass's approximation theorem, there exists $g \in C^m[0,1]$ such that $\|(f - (7/2)\varepsilon) - g\|_\infty < \varepsilon/2$ and then $g < (f - (7/2)\varepsilon) + \varepsilon/2 = f - 3\varepsilon$ and $\|f - g\|_\infty < 4\varepsilon$.

(b) Secondly, observe the following elementary gluing result. If $\varepsilon > 0$ and $p, q, r, s \in [0,1]$ are such that $p < q < r < s$ and $g\colon [p,q] \cup [r,s] \to \mathbb{R}$ is a C^m function, then there exists $h \in C^m[p,s]$ such that $h = g$ on $[p,q] \cup [r,s]$ and $\min\{g(q), g(r)\} - \varepsilon \le h(t) \le \max\{g(q), g(r)\} + \varepsilon$ for every $t \in (q,r)$.

(c) Thirdly, observe that the following variations on (b) are also true. If $\varepsilon > 0$ and $p, q, r, s \in [0,1]$ are such that $p < q < r < s$ and $g\colon [p,q] \cup [r,s] \to \mathbb{R}$ is C^m, $g(q) > \varepsilon$, and $g = 0$ on $[r,s]$, then there exists $h \in C^m[p,s]$ such that $h = g$ on $[p,q] \cup [r,s]$ and $0 \le h(t) < g(q) + \varepsilon$ for every $t \in (q,r)$.

If $\varepsilon > 0$ and $p, q, r, s \in [0,1]$ are such that $p < q < r < s$ and $g\colon [p,q] \cup [r,s] \to \mathbb{R}$

is C^m, $g(r) > \varepsilon$, and $g = 0$ on $[p, q]$ then there exists $h \in C^m[p, s]$ such that $h = g$ on $[p, q] \cup [r, s]$ and $0 \le h(t) < g(r) + \varepsilon$ for every $t \in (q, r)$.

(d) Next, let $f \in C[0, 1]^+$ be such that $f(0) > 0$ and $f(1) > 0$ and let $\varepsilon > 0$. We will construct a $g \in C^m[0, 1]$ such that $0 \le g \le f$ and $\|g - f\|_\infty \le 6\varepsilon$. Without loss of generality we may assume that ε is so small that $f(0) > 6\varepsilon$ and $f(1) > 6\varepsilon$.

Define $\tau_0 = 0$ and for $k \in \mathbb{N}$ define, inductively,

$$\sigma_k := \inf\{t \in [\tau_{k-1}, 1];\ f(t) < 5\varepsilon \text{ or } t = 1\}, \text{ and}$$
$$\tau_k := \inf\{t \in [\sigma_k, 1];\ f(t) > 6\varepsilon\}.$$

We have $\sigma_1 > 0$ and for $k \in \mathbb{N}$ with $\sigma_k < 1$ we have $\sigma_k < \tau_k$, since $f(0), f(1) > 6\varepsilon$ and f is continuous. If $\sigma_k = 1$, then $\tau_k = 1$. Similarly, for every $k \in \mathbb{N}$ we have $\tau_k < \sigma_{k+1}$ or $\tau_k = \sigma_{k+1} = 1$. For $k \in \mathbb{N}$ with $\sigma_k < 1$ we have $f(\sigma_k) = 5\varepsilon$ and if $\tau_k < 1$ then $f(\tau_k) = 6\varepsilon$. There exists $N \in \mathbb{N}$ such that $\sigma_k = \tau_k = 1$ for every $k \ge N$, since otherwise there would be a convergent subsequence $(\sigma_{k_j})_j$ with $\sigma_{k_j} < 1$ and hence $f(\sigma_{k_j}) = 5\varepsilon$ for every j, and then $(\tau_{k_j})_j$ would converge to the same limit while $f(\tau_{k_j}) = 6\varepsilon$, which contradicts the continuity of f.

Now we are ready to construct the desired function g. Let $k \in \mathbb{N}$ be such that $\tau_{k-1} < 1$. On $[\tau_{k-1}, \sigma_k]$, we the aid of (a), we take g to be C^m and such that

$$g \le f - 3\varepsilon \text{ on } [\tau_{k-1}, \sigma_k]$$

and $\sup_{t\in[\tau_{k-1},\sigma_k]} |f(t) - g(t)| < 4\varepsilon$. Since $f \ge 5\varepsilon$ on $[\tau_{k-1}, \sigma_k]$, we have

$$g > \varepsilon \text{ on } [\tau_{k-1}, \sigma_k].$$

If $\sigma_1 = 1$, then we have thus defined g on all of $[0, 1]$. Otherwise, let $k \in \mathbb{N}$ be such that $\sigma_k < 1$. We will define g on (σ_k, τ_k). Recall that $\tau_k < 1$, so that g has already been defined on $[\tau_{k-1}, \sigma_k] \cup [\tau_k, \sigma_{k+1}]$. Observe that $f \le 6\varepsilon$ on $[\sigma_k, \tau_k]$.

If $f \ge 4\varepsilon$ on $[\sigma_k, \tau_k]$, then with the aid of step (b), we take g on (σ_k, τ_k) such that

g is C^m on $[\tau_{k-1}, \sigma_{k+1}]$ and $\min\{g(\sigma_k), g(\tau_k)\} - \varepsilon \le g(t) \le \max\{g(\sigma_k), g(\tau_k)\} + \varepsilon$ for every $t \in (\sigma_k, \tau_k)$. As $g(\sigma_k), g(\tau_k) > \varepsilon$, it follows that $g > 0$ on (σ_k, τ_k). Since $g \le f - 3\varepsilon \le 3\varepsilon$ at σ_k and at τ_k, we also have that $g(t) < 4\varepsilon \le f(t)$ for every $t \in (\sigma_k, \tau_k)$.

If we do not have that $f \ge 4\varepsilon$ on $[\sigma_k, \tau_k]$, then we define

$$\pi_k := \inf\{t \in [\sigma_k, \tau_k];\ f(t) < 4\epsilon\} \text{ and}$$
$$\rho_k := \sup\{t \in [\pi_k, \tau_k];\ f(t) < 4\varepsilon\}.$$

Note that $\pi_k < \rho_k$ and $f \ge 4\varepsilon$ on $[\sigma_k, \pi_k] \cup [\rho_k, \tau_k]$. On $[\pi_k, \rho_k]$ we take $g = 0$. Recall that $g(\sigma_k) > \varepsilon$. With the aid of step (c), we take g on (σ_k, π_k) such that g is a C^m function on $[\tau_{k-1}, \rho_k]$ and $0 \le g(t) < g(\sigma_k) + \varepsilon$ for every $t \in (\sigma_k, \pi_k)$. Then for every $t \in (\sigma_k, \pi_k)$ we have $g(t) < f(\sigma_k) - 3\varepsilon + \varepsilon = 3\varepsilon < f(t)$.

Similarly, on (ρ_k, τ_k) we take g such that g is C^m on $[\pi_k, \sigma_{k+1}]$ and $0 \le g(t) < g(\tau_k) + \varepsilon$ for every $t \in (\rho_k, \tau_k)$. Then for every $t \in (\rho_k, \tau_k)$ we have $g(t) < f(\tau_k) - 3\varepsilon + \varepsilon = 4\varepsilon \le f(t)$. Thus, we have constructed g on (σ_k, τ_k) such that g is C^m, $g \ge 0$ and $g \le f$ on $[\sigma_k, \tau_k]$. Since $f \le 6\varepsilon$ on $[\sigma_k, \tau_k]$, it follows that $\sup_{t \in [\sigma_k, \tau_k]} |f(t) - g(t)| \le 6\varepsilon$.

In conclusion, $g \in C^m[0,1]$, $0 \le g \le f$, and $\|f - g\|_\infty \le 6\varepsilon$.

(e) We show that for every $f \in C[0,1]^+$ and $\varepsilon > 0$ there exists a $g \in C^m[0,1]$ such that $0 \le g \le f$ and $\|g - f\|_\infty \le 3\varepsilon$. Due to (d), it only remains to deal with the case where $f(0) = 0$ or $f(1) = 0$. If $f \le 3\varepsilon$ on $[0,1]$, then we can take $g = 0$, so we may assume that there exists $t \in [0,1]$ with $f(t) > 3\varepsilon$. We first consider the case where $f(0) = 0$ and $f(1) > 0$. Without loss of generality we assume that $f(1) > 2\varepsilon$. Define

$$\tau := \inf\{t \in [0,1];\ f(t) > 3\varepsilon\} \text{ and}$$
$$\sigma := \sup\{t \in [0,\tau];\ f(t) < 2\varepsilon\}.$$

Observe that $0 < \sigma < \tau$, $f \ge 2\varepsilon$ on $[\sigma, \tau]$, and $f(\tau) = 3\varepsilon$. Then, according to (d), there exists a C^m function g on $[\tau, 1]$ such that $0 \leq g \leq (f - 2\varepsilon)^+$ and $\|(f - 2\varepsilon)^+ - g\|_\infty < \varepsilon$. We take $g = 0$ on $[0, \sigma]$ and with the aid of (c) we choose g on (σ, τ) such that g is a C^m function on $[0, 1]$ and such that for every $t \in (\sigma, \tau)$ we have $0 \leq g(t) \leq g(\tau) + \varepsilon$. Then, for every $t \in (\sigma, \tau)$ we have

$$g(t) \leq g(\tau) + \varepsilon \leq (f(\tau) - 2\varepsilon)^+ + \varepsilon \leq 2\varepsilon \leq f(t).$$

Thus we have constructed a $g \in C^m[0, 1]$ such that $0 \leq g \leq f$ with $\|f - g\|_\infty \leq 3\varepsilon$.

The cases where $f(0) = 0$ and $f(1) = 0$, or $f(0) > 0$ and $f(1) = 0$ can be dealt with in a similar fashion.

(f) Let $f \in C[0, 1]^+$. We construct a sequence $(f_n)_n$ as announced above. By means of (e), we choose $g_1 \in C^m[0, 1]$ with $0 \leq g_1 \leq f$ and $\|f - g_1\|_\infty < 2^{-1}$. Inductively, for $n \in \mathbb{N}$ we choose $g_{n+1} \in C^m[0, 1]$ with $0 \leq g_{n+1} \leq f - \sum_{j=1}^n g_j$ and $\|(f - \sum_{j=1}^n g_j) - g_{n+1}\|_\infty < 2^{-(n+1)}$. Let

$$f_n := \sum_{j=1}^n g_j, \; n \in \mathbb{N}.$$

Then, as every g_j is positive, $(f_n)_n$ is an increasing sequence in $C^m[0, 1]^+$. Further, since $g_{n+1} \leq f - f_n$, we have $f_n \leq f - g_{n+1} \leq f$. Moreover, $\|f - f_{n+1}\|_\infty < 2^{-(n+1)} \to 0$ as $n \to \infty$. Finally, since $f_n \uparrow$ and $\|f - f_n\|_\infty \to 0$, it follows that $f = \sup_n f_n$ in $C[0, 1]$.

If $T\colon C[0, 1] \to C[0, 1]$ is an order continuous operator and $T|_{C^\infty[0,1]}$ is disjointness preserving, then T is disjointness preserving. Indeed, if $f, g \in C[0, 1]^+$ are disjoint, take $f_n, g_n \in C^\infty[0, 1]^+$ as above, then $Tf_n \perp Tg_n$ for all $n \in \mathbb{N}$. Since T is order continuous, we have $Tf_n \to Tf$ and $Tg_n \to Tg$ as $n \to \infty$, so $Tf \perp Tg$. Hence T is disjointness preserving.

In addition, if $T\colon C^\infty[0, 1] \to C^\infty[0, 1]$ is positive disjointness preserving and continuous with respect to the norm $\|\cdot\|_\infty$ and injective, then Theorem 2.1.17 yields that there exists $\widehat{T}\colon C[0, 1] \to C[0, 1]$ which is disjointness preserving and such that $\widehat{T}|_{C^\infty[0,1]} = T$.

Combining Theorem 2.1.1 and Theorem 2.1.17 we obtain the following.

Theorem 2.1.20. In the setting of Theorem 2.1.17, if $T\colon X \to Y$ is bijective, then T^{-1} is disjointness preserving.

As for a Riesz subspace Z of Y^δ, order denseness and pervasiveness are equivalent, we have that Z is pervasive in Y^δ. The disjointness preserving of T in Z and Y^δ are equivalent as well, and their norms are identical. So we can use Corollary 2.1.8 to obtain a corollary.

Corollary 2.1.21. Assume the setting of Theorem 2.1.17, and let Z be a Riesz subspace of Y^δ and $T_u\colon X^{\rho u} \to Z$. Then $(T_u)^{-1}$ is disjointness preserving.

In Theorem 2.1.15, it is required that for every $x \in (X^\rho)^+$, there exists a sequence $(x_n)_{n=1}^\infty$ in X which converges to x from below. In general, however, this condition is not always satisfied. For example, let $X = \mathrm{Pol}^2[0,1]$, then X^ρ is the subspace of $C[0,1]$ consisting of piecewise polynomial functions. It is easy to find a positive $x \in X^\rho \setminus \{0\}$ which vanishes on a subinterval of $[0,1]$ and then there is no sequence in X^+ that converges to x from below.

We note that not every disjointness preserving operator on a pre-Riesz space can be extended to its Riesz completion without some strong conditions and keep the property of being disjointness preserving. For example, if $X = \mathrm{Pol}^2[0,1]$ the operator T defined by $(Tx)(t) = \int_0^t x(s)ds, t \in [0,1]$, is not disjointness preserving on X^ρ but it is disjointness preserving on X, since two elements $x, y \in X$ are disjoint in X only if $x = 0$ or $y = 0$.

2.2 More conditions for extending disjointness preserving operators

This section is a joint work with A. Kalauch (TU Dresden).

In this section, we will explore disjointness preserving operators on pre-Riesz spaces from three different angles.

The first part is concerned with the question whether Riesz* homomorphisms between two pre-Riesz spaces are disjointness preserving operators on pre-Riesz spaces. Then in the second part, a condition which is called 'condition (β)', will be considered and it will be shown to imply that the inverse of T is disjointness preserving. Moreover, we study when an operator from a pervasive Archimedean pre-Riesz space with the Riesz decomposition property to a Dedekind complete vector lattice can be extended to an order bounded and disjointness preserving operator on the Riesz completion.

2.2.1 Riesz* homomorphism

Let us recall the definition of Riesz homomorphisms on Riesz spaces first.

Definition 2.2.1. Let X and Y be two Riesz spaces, an operator $T\colon X \to Y$ is said to be **Riesz homomorphism** if it preserves the lattice operations, i.e. whenever $T(x \wedge y) = T(x) \wedge T(y)$ holds for all $x, y \in X$.

The definition of Riesz* homomorphism on pre-Riesz spaces is given by M. van Haandel [54, Definition 5.1].

Definition 2.2.2. Let X and Y be two pre-Riesz spaces, an operator $T\colon X \to Y$ is said to be **Riesz* homomorphism** if for every $x_1, \ldots, x_n \in X$ it has $T(\{x_1, \ldots, x_n\}^{\mathrm{ul}}) \subseteq \{Tx_n, \ldots, Tx_n\}^{\mathrm{ul}}$ in Y.

The next theorem, due to M. van Haandel [54, Theorem 5.6], characterizes that the Riesz* homomorphisms on pre-Riesz spaces are exactly the restriction of Riesz homomorphisms on Riesz completions.

Theorem 2.2.3. Let $T\colon X \to Y$ between two pre-Riesz spaces be a linear operator, and X^ρ and Y^ρ the Riesz completion of X and Y, respectively. Then T

is a Riesz* homomorphism if and only if it extends to a Riesz homomorphism $T_\rho\colon X^\rho \to Y^\rho$.

Concerning disjointness preserving operators in vector lattices, Riesz homomorphisms are typical examples. Therefore, in view of Theorem 2.2.3, one expects that in pre-Riesz spaces Riesz* homomorphisms preserve disjointness.

Lemma 2.2.4. If X and Y are pre-Riesz spaces and $T\colon X \to Y$ is a bipositive Riesz* homomorphism, then for every $x, y \in X$ we have $x \perp y$ if and only if $Tx \perp Ty$.

Proof. If $Tx \perp Ty$ in Y, then it follows directly from the definition of disjointness that $Tx \perp Ty$ in $T[X]$ (see Proposition 1.2.15), so that bipositivity of T yields that $x \perp y$.

Assume $x \perp y$. Let (X^ρ, i_X) and (Y^ρ, i_Y) be the Riesz completions of X and Y, respectively. Due to Theorem 2.2.3 there is a Riesz homomorphism $T_\rho\colon X^\rho \to Y^\rho$ that extends T in the sense that $T_\rho \circ i_X = i_Y \circ T$. In particular, T_ρ is positive and disjointness preserving. Recall that $x \perp y$ in X if and only if $i_X(x) \perp i_X(y)$ in X^ρ, and similarly for elements in Y. For $x, y \in X$ with $x \perp y$ we thus have $i_X(x) \perp i_X(y)$ and therefore $T_\rho(i_X(x)) \perp T_\rho(i_X(y))$, so that $i_Y(T(x)) \perp i_Y(T(y))$, which yields that $T(x) \perp T(y)$. □

2.2.2 Condition (β)

In the setting of vector lattices, in [4, Definition 2.2] a condition (β) for T is introduced by means of disjoint complements, and it is shown that (β) implies that T^{-1} is disjointness preserving. We deal with the analogous condition (β) in pre-Riesz spaces, see the subsequent definition. Recall that for elements x, y of a pre-Riesz space X one has $\{x\}^{\mathrm{dd}} \subseteq \{y\}^{\mathrm{dd}}$ if and only if $x \in \{y\}^{\mathrm{dd}}$.

Definition 2.2.5. Let X and Y be pre-Riesz spaces and let $T\colon X \to Y$ be a linear operator. T is said to satisfy **condition** (β) if for every $x, y \in X$ with $\{x\}^{\mathrm{dd}} \subseteq \{y\}^{\mathrm{dd}}$ it follows that $\{Tx\}^{\mathrm{dd}} \subseteq \{Ty\}^{\mathrm{dd}}$.

The idea of this condition was introduced by B. Randrianantoanina [46]. In function spaces, (β) means the following: if the support of a function is contained in the support of another function, then the same is true for the supports of their images. We will show that (β) for T implies T^{-1} being disjointness preserving, provided X and Y are pre-Riesz spaces and X is, in addition, fordable.

Definition 2.2.6. Let X be a pre-Riesz space and (X^ρ, i) its Riesz completion. X is called **fordable** if for every $y \in X^\rho$, $y \geq 0$, there is $M \subseteq X$ such that $\{y\}^{\mathrm{d}} = i(M)^{\mathrm{d}}$.

The property 'fordable' emerges firstly in [53, Lemma 2.4]. The content of this lemma is stated as follows.

Lemma 2.2.7. Every pervasive pre-Riesz space is fordable.

However, the converse of above lemma is not true in general. In [30, Example 3.3.22], it shows that $X = \{\alpha\mathbf{1} + v; \alpha \in \mathbb{R}, v \in C[0,1], v(0) = 0, \int_0^1 v(t)dt = 0\}$ is a fordable pre-Riesz space, but not pervasive. In addition, the pre-Riesz space $X = \mathrm{Pol}^2(\mathbb{R})$ is not fordable, hence not pervasive, see [30, Example 3.3.23].

With the aid of the fordable condition, we show some results about disjoint elements in pre-Riesz space.

Lemma 2.2.8. Let X be a pre-Riesz space and let $x, y \in X$. Then the following statements hold:

(i) If $x \perp y$ then $\{x\}^{\mathrm{dd}} \cap \{y\}^{\mathrm{dd}} = \{0\}$.

(ii) If X is, in addition, fordable, then from $\{x\}^{\mathrm{dd}} \cap \{y\}^{\mathrm{dd}} = \{0\}$ it follows that $x \perp y$.

Proof. (i) Assume that $x \perp y$, i.e. $y \in \{x\}^{\mathrm{d}}$. Hence, $\{y\}^{\mathrm{dd}} \subseteq \{x\}^{\mathrm{ddd}} = \{x\}^{\mathrm{d}}$, which implies

$$\{x\}^{\mathrm{dd}} \cap \{y\}^{\mathrm{dd}} \subseteq \{x\}^{\mathrm{dd}} \cap \{x\}^{\mathrm{d}} = \{0\}.$$

(ii) Assume that $\{x\}^{\mathrm{dd}} \cap \{y\}^{\mathrm{dd}} = \{0\}$. Let (X^ρ, i) be the Riesz completion of X and define $u := |i(x)| \wedge |i(y)|$. Since X is fordable, there is $S \subseteq X$ such that $\{u\}^{\mathrm{d}} = i(S)^{\mathrm{d}}$ in X^ρ. From Proposition 1.2.15 it follows that

$$i^{-1}\left[\{u\}^{\mathrm{d}}\right] = i^{-1}\left[i(S)^{\mathrm{d}}\right] = S^{\mathrm{d}}. \tag{2.2}$$

We show that $S^{\mathrm{dd}} \subseteq \{x\}^{\mathrm{dd}}$. Indeed, let $z \in \{x\}^{\mathrm{d}}$, then $i(z) \perp i(x)$, hence $i(z) \perp u$. Due to (2.2), $z \in i^{-1}\left[\{u\}^{\mathrm{d}}\right] = S^{\mathrm{d}}$. It follows that $\{x\}^{\mathrm{d}} \subseteq S^{\mathrm{d}}$, and therefore $S^{\mathrm{dd}} \subseteq \{x\}^{\mathrm{dd}}$.

Analogously, one obtains $S^{\mathrm{dd}} \subseteq \{y\}^{\mathrm{dd}}$. The assumption yields $S \subseteq S^{\mathrm{dd}} \subseteq \{0\}$. Now (2.2) implies that $i(X) \subseteq \{u\}^{\mathrm{d}}$, hence from Lemma 1.2.18 it follows that $u = 0$. Consequently, $i(x) \perp i(y)$, which implies $x \perp y$. □

If X is not fordable, then Lemma 2.2.8 (ii) is not true, in general. Indeed, consider in [33, Example 4.8] the elements $x = (1,0,1)^{\mathrm{T}}$ and $y = (0,1,1)^{\mathrm{T}}$ in $\mathbb{R}^3$, then $\{x\}^{\mathrm{dd}} \cap \{y\}^{\mathrm{dd}} = \{0\}$, but $x \not\perp y$.

We continue now with the main result of this subsection.

Theorem 2.2.9. Let X and Y be pre-Riesz spaces and let $T\colon X \to Y$ be a linear injective operator.

(i) If X is, in addition, fordable and T satisfies condition (β), then $T^{-1}\colon T[X] \to X$ is disjointness preserving.

(ii) Let T be surjective and disjointness preserving. If T^{-1} is disjointness preserving then T satisfies (β).

Proof. (i) Let $y_1, y_2 \in T[X]$ be such that $y_1 \perp y_2$ in Y and let $x_1, x_2 \in X$ be such that $Tx_1 = y_1$ and $Tx_2 = y_2$. Due to Lemma 2.2.8 (i) one obtains

$$\{y_1\}^{\mathrm{dd}} \cap \{y_2\}^{\mathrm{dd}} = \{0\}.$$

Let $u \in \{x_1\}^{\mathrm{dd}} \cap \{x_2\}^{\mathrm{dd}}$. From $u \in \{x_1\}^{\mathrm{dd}}$ it follows that $\{u\}^{\mathrm{dd}} \subseteq \{x_1\}^{\mathrm{dd}}$, hence property (β) yields that $\{Tu\}^{\mathrm{dd}} \subseteq \{Tx_1\}^{\mathrm{dd}}$. Analogously, $\{Tu\}^{\mathrm{dd}} \subseteq \{Tx_2\}^{\mathrm{dd}}$, therefore

$$\{Tu\}^{\mathrm{dd}} \subseteq \{Tx_1\}^{\mathrm{dd}} \cap \{Tx_2\}^{\mathrm{dd}} = \{y_1\}^{\mathrm{dd}} \cap \{y_2\}^{\mathrm{dd}} = \{0\},$$

which yields $Tu = 0$. As T is injective it follows that $u = 0$. Thus we obtain that $\{x_1\}^{\mathrm{dd}} \cap \{x_2\}^{\mathrm{dd}} = \{0\}$. Since X is fordable, Lemma 2.2.8 (ii) yields that $x_1 \perp x_2$. Consequently, T^{-1} is disjointness preserving.

(ii) The line of reasoning here is similar to the proof of [4, Proposition 3.3]. Let $x_1, x_2 \in X$ be such that $\{x_1\}^{\mathrm{dd}} \subseteq \{x_2\}^{\mathrm{dd}}$, and assume that $\{Tx_1\}^{\mathrm{dd}} \nsubseteq \{Tx_2\}^{\mathrm{dd}}$. This means $Tx_1 \notin \{Tx_2\}^{\mathrm{dd}}$, i.e. there is a $y \in \{Tx_2\}^{\mathrm{d}}$, $y \neq 0$, such that $Tx_1 \not\perp y$. In particular, one has $y \perp Tx_2$. Since T is bijective, there is an $x \in X$, $x \neq 0$, such that $Tx = y$. Since T^{-1} is disjointness preserving, one obtains $x \perp x_2$. On the other hand, since T is disjointness preserving, one gets $x \not\perp x_1$. This contradicts the assumption, since $x \in \{x_2\}^{\mathrm{d}} \subseteq \{x_1\}^{\mathrm{d}}$. □

2.2.3 Order bounded and disjointness preserving extension

The purpose of this subsection is to extend operators on pre-Riesz spaces to Riesz completions with a different point of view than in previous sections. It turns out that this extension could preserve the order boundedness and disjointness, providing that X be a pervasive Archimedean pre-Riesz space with the Riesz decomposition property and Y a Dedekind complete vector lattice.

Definition 2.2.10. The ordered vector space (X, K) has the **Riesz decomposition property**, in short **RDP**, if for every $x_1, x_2, z \in K$ with $z \leq x_1 + x_2$, there exist $z_1, z_2 \in K$ such that $z = z_1 + z_2$ with $z_1 \leq x_1$ and $z_2 \leq x_2$.

The extension is characterized by the following theorem.

Theorem 2.2.11. Let X be a pervasive Archimedean pre-Riesz space with the RDP, Y a Dedekind complete vector lattice. If $T \colon X \to Y$ is order bounded and disjointness preserving, then there exists an extension to all of X^ρ which is order bounded and disjointness preserving.

Proof. Let $i\colon X \to X^\rho$ be the bipositive linear map as in Theorem 1.2.7. For a fixed $y \in X^\rho$ with $y > 0$, since X is pervasive there exists some $x \in X$ with $0 < i(x) \leq y$. As X is Archimedean, it follows from Proposition 2.1.4 that $y = \sup\{i(x) \in i(X); 0 < i(x) \leq y\}$. Because of $i(X)$ is majorizing in X^ρ, there exists $z \in X$ such that $y \leq i(z)$. So $i(x) \leq y \leq i(z)$ and $x \leq z$. The order boundedness of T implies that $\{Tx; x \in X, 0 < i(x) \leq y\}$ is bounded. Thus $\sup\{Tx; x \in X, 0 < i(x) \leq y\}$ exists in Y. So one can define a mapping $\widehat{T}\colon X^\rho \to Y$ via the formula

$$\widehat{T}y = \sup\{Tx;\ x \in X, 0 \leq i(x) \leq |y|\},\ y \in X^\rho. \tag{2.3}$$

Clearly, this $\widehat{T}$ is order bounded in the sense that it maps order bounded sets to order bounded sets. For $0 \leq i(x) \leq |y_1 + y_2| \leq |y_1| + |y_2|$, because X has RDP, there exist $x_1, x_2 \in X$ with $x_1 + x_2 = x$, $0 \leq i(x_1) \leq |y_1|$ and $0 \leq i(x_2) \leq |y_2|$. Thus $\widehat{T}(y_1+y_2) = \sup\{Tx; x \in X, 0 \leq i(x) \leq |y_1+y_2|\} \leq \sup\{T(x_1+x_2); x_1, x_2 \in X, 0 \leq i(x_1) \leq |y_1|, 0 \leq i(x_2) \leq |y_2|\} \leq \widehat{T}y_1+\widehat{T}y_2$. So $\widehat{T}$ is sublinear, and it is clear that $T(x) \leq \widehat{T}(x)$ holds for all $x \in X$. By Hahn-Banach extension theorem, the operator T has a linear extension S to X^ρ satisfying $S(u) \leq \widehat{T}(u)$ for all $u \in X^\rho$.

An easy argument shows that S is also order bounded. We only need to prove S is disjointness preserving. Let $y_1, y_2 \in X^\rho$ with $y_1 \perp y_2$. So $|y_1| \perp |y_2|$. By the above discussion, there exists $0 \leq x_j \in X$ such that $0 \leq i(x_j) \leq |y_j|$, $j = 1, 2$. Hence, $i(x_1) \perp i(x_2)$ and $x_1 \perp x_2$, the disjointness preserving of T implies that $Tx_1 \perp Tx_2$. Thus $\widehat{T}y_1 \perp \widehat{T}y_2$. Since $S(y_1) \leq \widehat{T}(y_1)$ and $-S(y_1) = S(-y_1) \leq \widehat{T}(-y_1) = \widehat{T}(y_1)$, we have $|S(y_1)| \leq \widehat{T}(y_1)$. Similarly, $|S(y_2)| \leq \widehat{T}(y_2)$. Hence, $|S(y_1)| \wedge |S(y_2)| \leq \widehat{T}(y_1) \wedge \widehat{T}(y_2) = 0$. Thus $S(y_1) \perp S(y_2)$. Thus we complete the proof. □

It is a remarkable fact that in the case of vector lattices, every order bounded and disjointness preserving operator is regular [25]. However, this is not true for operators on pre-Riesz spaces in general. In [52] an example of an order bounded, non-regular linear functional on a directed partially ordered vector space was given. We will use this example to construct an order bounded disjointness preserving operator which is not regular.

Example 2.2.12. For $A \subseteq [0,\infty)$ let χ_A denote the corresponding indicator function. Define for $n,k \in \mathbb{N}$,

$$\begin{aligned} e_n : [0,\infty) \to \mathbb{R}, &\qquad t \mapsto \chi_{[n-1,n)}(t), \\ u_{n,k} : [0,\infty) \to \mathbb{R}, &\qquad t \mapsto nt\chi_{[0,\frac{1}{n}]}(t) + \tfrac{1}{k}\chi_{\left\{n+\frac{1}{k}\right\}}(t), \end{aligned}$$

and consider the subspace $X := \operatorname{span}\{e_n, u_{n,k};\ n,k \in \mathbb{N}\}$ of $\mathbb{R}^{[0,\infty)}$ with pointwise order. For every $x \in X$ there exists $t_0 > 0$ such that x is affine and, hence, differentiable on $(0,t_0)$. Define

$$T : X \to X, \quad x \mapsto \left(\lim_{t \downarrow 0} x'(t)\right) e_1.$$

For the sake of completeness, we list all relevant properties of X and T, where (i), (iv) and (v) are already dealt with in [52].

(i) X is directed. Indeed, every element in X is bounded and has bounded support. For $x,y \in X$ there is $n \in \mathbb{N}$ such that $x,y \le n\sum_{i=1}^{n} e_i$, hence X is directed.

(ii) X is a pre-Riesz space. Indeed, $\mathbb{R}^{[0,\infty)}$ is Archimedean, therefore its subspace X is Archimedean as well. [54, Theorem 1.7(ii)] yields that X is a pre-Riesz space.

(iii) T is disjointness preserving. Indeed, let $x^{(1)}, x^{(2)} \in X$ with $x^{(1)} \perp x^{(2)}$. There is $M \in \mathbb{N}$ such that for $i \in \{1,2\}$

$$x^{(i)} = \sum_{n=1}^{M} \alpha_n^{(i)} e_n + \sum_{n,k=1}^{M} \lambda_{n,k}^{(i)} u_{n,k},$$

i.e. $x^{(1)}$ and $x^{(2)}$ are affine on $\left[0,\frac{1}{M}\right]$. Let without loss of generality $x^{(1)} = 0$ on $\left[0,\frac{1}{M}\right]$, then $Tx^{(1)} = 0$ and hence $Tx^{(1)} \perp Tx^{(2)}$.

(iv) T is order bounded. Indeed, for $v, w \in X$ with $v \leq w$ there are $N \in \mathbb{N}$ and $C \in (0, \infty)$ such that $\pm v, \pm w \leq C \sum_{i=1}^{N} e_i$. An element $x \in X$ with $v \leq x \leq w$ is affine on $\left[0, \frac{1}{N}\right]$, hence $-2NCe_1 \leq Tx \leq 2NCe_1$.

(v) T is not regular. Indeed, assume that there is a positive linear operator $S\colon X \to X$ such that $S \geq T$. For $n, k \in \mathbb{N}$ one has that $0 \leq u_{n,k} \leq e_1 + \frac{1}{k} e_{n+1}$, hence

$$T(u_{n,k}) = ne_1 \leq S(u_{n,k}) \leq S(e_1) + \tfrac{1}{k} S(e_{n+1}),$$

and therefore $k(ne_1 - S(e_1)) \leq S(e_{n+1})$ for every $k \in \mathbb{N}$. Since X is Archimedean, it follows that $ne_1 - S(e_1) \leq 0$. From $ne_1 \leq S(e_1)$ for every $n \in \mathbb{N}$ one obtains $e_1 \leq 0$, a contradiction.