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## Extension of operators on pre-Riesz spaces

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### Citation

Zhang, F. (2018, September 20). *Extension of operators on pre-Riesz spaces*. Retrieved from <https://hdl.handle.net/1887/65567>

Version: Not Applicable (or Unknown)

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**Title:** Extension of operators on pre-Riesz spaces

**Issue Date:** 2018-09-20

# Chapter 1

## Preliminaries

In this chapter, we will introduce some of the basic background information and results that will be used throughout this text. For the terminology of Riesz spaces we mainly refer to W. A. J. Luxemburg & A. C. Zaanen [40], E. de Jonge & A. C. M. van Rooij [18], A. C. Zaanen [60], for Banach lattices and positive operators to [6, 42], and for topological vector spaces to [47]. The embedding theory of partially ordered vector spaces has been studied by G. Buskes & A. C. M. van Rooij [14], M. van Haandel [54] and, more recently, by O. van Gaans [50] and A. Kalauch [29]. For the terminology of partially ordered vector spaces and pre-Riesz spaces, we refer to these papers and monographs. For the readers' convenience, we will also introduce some concepts where they are needed in later chapters.

This chapter consists of three sections.

The first section includes two subsections, the one with the terminology of partially ordered vector spaces and the other one with concepts of vector lattices. The partial order of a vector space is induced by a positive cone, and such an order appears throughout most of definitions and properties of spaces in this thesis. The structure of vector lattices is given in the second subsection, e.g. ideals, bands etc..

Section 1.2 is concerned with the terminology of pre-Riesz spaces and Riesz com-

pletions. A pre-Riesz space is one kind of partially ordered vector space, which can be embedded order densely into a vector lattice cover, and the smallest such a cover is called the Riesz completion. This theory was established by [54], and considered as a cornerstone of this thesis. The importance of the embedding map from a pre-Riesz space to a vector lattice will show up in the main body of this text.

Section 1.3 contains definitions of different properties of linear operators between ordered vector spaces. In addition, it presents the Hahn-Banach extension theorem and the Kantorovich extension theorem.

## 1.1 Ordered vector spaces and vector lattices

In this section, we will introduce some basic terminology of ordered vector spaces and vector lattices.

### 1.1.1 Ordered vector spaces

**Definition 1.1.1.** Let  $X$  be a real vector space.

- (i) A reflexive, transitive and antisymmetric relation “ $\leq$ ” on  $X$  is called a **vector space order** if
  - (a)  $x, y, z \in X$  and  $x \leq y$  imply  $x + z \leq y + z$ ,
  - (b)  $x \in X$ ,  $0 \leq x$  and  $\lambda \in \mathbb{R}^+$  imply  $0 \leq \lambda x$ .

Then  $(X, \leq)$  is called a **partially ordered vector space**, in short, POVS. For  $x, y \in X$ , the relation  $x \leq y$  can be also written as  $y \geq x$ . An element  $x \in X$  is called **positive** if  $0 \leq x$ . The relation  $x > 0$  means  $x \geq 0$  and  $x \neq 0$ , and  $x$  is then called **strictly positive**.

- (ii) A non-empty subset  $K \neq \{0\}$  of  $X$  is called a **wedge** if  $x, y \in K$  and  $\lambda, \mu \in \mathbb{R}^+$  imply  $\lambda x + \mu y \in K$ . A wedge is called a **cone** if  $K \cap (-K) = 0$ . A wedge  $K$  in  $X$  is called **generating** if  $X = K - K$ .

The following proposition is straightforward.

**Proposition 1.1.2.** Let  $X$  be a real vector space.

- (i) Let  $K$  be a cone in  $X$  and  $\leq$  on  $X$  defined by means of

$$x \leq y \Leftrightarrow y - x \in K. \quad (1.1)$$

Then  $\leq$  is a vector space order.

- (ii) Let  $\leq$  be a vector space order on  $X$ . Then the set

$$X^+ := \{x \in X; 0 \leq x\} \quad (1.2)$$

of all positive elements in  $X$  is a cone.

- (iii) Let  $K$  be a cone in  $X$ ,  $\leq$  be the order defined by (1.1) and  $X^+$  be the corresponding cone in (1.2). Then  $K = X^+$ .

There are many ways to choose a cone  $K$  in a vector space. This is shown by the following example.

*Example 1.1.3.* (1) Let  $X = \mathbb{R}^2$ .

- (a) The set

$$K_1 := \{(x_1, x_2); x_1 \geq 0, x_2 \geq 0\}$$

is a cone. Here  $K_1$  is the so-called **standard cone**. The partial order induced by  $K_1$  is called the **standard order**.

- (b) The set

$$K_2 := \{(x_1, x_2); x_2 > 0\} \cup \{(x_1, 0); x_1 \geq 0\}$$

is a cone in  $X$ .

(2) Let  $X = \mathbb{R}^3$ . Then the set

$$L_{\mathbb{R}^2} := \{(x_1, x_2, x_3); x_1^2 + x_2^2 \leq x_3^2, x_3 \geq 0\}$$

is a cone in  $X$ .

In a real vector space  $X$ , let  $K \subseteq X$  be a given cone. To stress that the vector space order  $\leq$  on  $X$  is induced by the cone  $K$ , we will use  $(X, K)$  to denote the ordered vector space. Occasionally, we write merely  $X$  instead of  $(X, K)$  if no ambiguousness can arise.

**Definition 1.1.4.** Let  $(X, K)$  be a partially ordered vector space.

- (i)  $X$  is called **directed** if for every  $x, y \in X$  there exists  $z \in X$  such that  $z \geq x, y$ .
- (ii) For  $x, y \in X$ ,  $x \leq y$ , we denote

$$[x, y] := \{z \in X; x \leq z \leq y\},$$

and we call  $[x, y]$  an **order interval**.

- (iii) A non-empty subset  $M \subseteq X$  is called **bounded above**, if there exists  $x \in X$  such that for all  $m \in M$  we have  $m \leq x$ , respectively, **bounded below** if  $m \geq x$ .  $M$  is called **order bounded** if there exist  $x, y \in X$  such that  $M \subseteq [x, y]$ . The set of all **upper bounds** is denoted by

$$M^u = \{x \in X; x \geq m \text{ for all } m \in M\},$$

respectively,  $M^l$  for **lower bounds**.

- (iv) A non-empty subset  $M$  of  $X$  is called **majorizing** if for every  $x \in X$  there exists  $m \in M$  such that  $m \geq x$ .
- (v) An element  $u > 0$  is called an **order unit** if for every  $x \in X$  there is a  $\lambda \in \mathbb{R}^+$  such that  $x \in [-\lambda u, \lambda u]$ .  $X$  is called an **order unit space** if it has an order unit.

- (vi)  $X$  is called **Archimedean** if for every  $x, y \in X$  with  $nx \leq y$  for every  $n \in \mathbb{N}$  we have  $x \leq 0$ .
- (vii)  $X$  is called **Dedekind complete** whenever every non-empty bounded above (bounded below) subset of  $X$  has a supremum (infimum).
- (viii)  $X$  is called  **$\sigma$ -Dedekind complete** if every non-empty finite or countable subset of  $X$  that is bounded above (bounded below) subset has a supremum (infimum).

**Note 1.1.5.** It should be noticed that, in the above Definition 1.1.4 (iii), the order of  $M \subseteq X$  is inherited from  $X$ . In this text, if not specified otherwise subspaces of  $X$  are always equipped with the inherited order.

In a partially ordered vector space  $(X, K)$ , the cone  $K$  is generating if and only if  $X$  is directed. The existence of an order unit  $u > 0$  in  $X$  implies that  $K$  is generating. Indeed, for every  $x \in X$  there is  $\lambda \in \mathbb{R}^+ \setminus \{0\}$  such that  $\lambda u - x \in K$ , moreover  $\lambda u \in K$  and

$$x = \lambda u - (\lambda u - x).$$

Clearly, if  $X$  is Dedekind complete, then  $X$  is  $\sigma$ -Dedekind complete. Moreover, if  $X$  is  $\sigma$ -Dedekind complete, then  $X$  is Archimedean.

**Definition 1.1.6.** Let  $(X, K)$  be a partially ordered vector space, and  $I$  a directed index set.

- (i) A net  $(x_\alpha)_{\alpha \in I}$  in  $X$  is called **increasing**, if  $\alpha \geq \beta$  implies  $x_\alpha \geq x_\beta$ , respectively, **decreasing** if the net  $(-x_\alpha)_{\alpha \in I}$  is increasing. We will use  $x_\alpha \downarrow x$  to denote that  $(x_\alpha)$  is decreasing and  $\inf\{x_\alpha; \alpha \in I\} = x$ . Similarly, we use  $x_\alpha \uparrow x$ .
- (ii) A net  $(x_\alpha)_{\alpha \in I} \subset X$  is said to **order converges**, in short, **o-converges** to  $x \in X$  (denoted by  $x_\alpha \xrightarrow{o} x$ ), if there is a net  $(y_\beta)_{\beta \in J} \subset X$  such that for

every  $\beta$  there is  $\alpha_0$  such that for every  $\alpha \geq \alpha_0$  we have  $\pm(x_\alpha - x) \leq y_\beta \downarrow 0$ .

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- (iii) A non-empty subset  $M \subseteq X$  is called **order closed**, in short, **o-closed**, if for each net  $(x_\alpha)_{\alpha \in I}$  in  $M$  which o-converges to  $x \in X$  one has that  $x \in M$ .
- (iv) A sequence  $(x_n)$  in  $X$  is said to be **relatively uniformly convergent** to  $x \in X$  (denoted by  $x_n \xrightarrow{\text{ru}} x$ ), if there exist some  $v > 0$  and  $\lambda_n \downarrow 0$  in  $\mathbb{R}$  such that  $\pm(x_n - x) \leq \lambda_n v$  for all  $n \in \mathbb{N}$ . A subset  $M \subseteq X$  is called **relatively uniformly closed** if it is closed under relative uniform convergence of sequences. By the **relative uniform closure** of a set  $M \subseteq X$  we mean the smallest relatively uniformly closed set in  $X$  which contains  $M$ . For  $u > 0$ , the sequence  $(x_n)$  is called an **u-uniformly Cauchy sequence** if for any  $\epsilon > 0$  there exists an element  $n(\epsilon) \in \mathbb{N}$  such that  $\pm(x_m - x_n) \leq \epsilon u$  for all  $m, n \geq n(\epsilon)$ . The Archimedean ordered vector space  $X$  is called **uniformly complete** if, for every  $u > 0$  in  $X$ , every u-uniformly Cauchy sequence has a relatively uniform limit.

The main properties of order convergence are listed in the following lemma.

**Lemma 1.1.7.** Let  $(X, K)$  be a partially ordered vector space.

- (i) If  $x_\alpha \downarrow x$ , then  $\lambda x_\alpha \downarrow \lambda x$  for all  $\lambda \in \mathbb{R}^+$ .
- (ii) If  $x_\alpha \downarrow x$  or  $x_\alpha \uparrow x$ , then  $x_\alpha \xrightarrow{\circ} x$ .
- (iii) If  $(x_\alpha)_{\alpha \in I}$  and  $(y_\beta)_{\beta \in J}$  are decreasing, then for  $z_{(\alpha, \beta)} := x_\alpha + y_\beta$ ,  $\alpha \in I, \beta \in J$ , we have that  $(z_{(\alpha, \beta)})_{(\alpha, \beta) \in I \otimes J}$  is decreasing, where the index set  $I \otimes J$  is entry-wise directed, i.e.  $(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$  if and only if  $\alpha_1 \leq \alpha_2$  and  $\beta_1 \leq \beta_2$ . Moreover, if  $x_\alpha \downarrow 0$  and  $y_\beta \downarrow 0$ , then  $x_\alpha + y_\beta = z_{(\alpha, \beta)} \downarrow 0$ .
- (iv) If  $x_\alpha \xrightarrow{\circ} x$  and  $x_\alpha \xrightarrow{\circ} y$ , then  $x = y$ .

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<sup>1</sup>There is another definition for order convergence in partially ordered vector space  $X$ . A net  $(x_\alpha)_{\alpha \in I}$  in  $X$  is said to order converges to  $x \in X$  if there is a net  $(y_\alpha)_{\alpha \in I} \subset X$  such that  $y_\alpha \downarrow 0$  and for all  $\alpha \in I$  one has  $\pm(x_\alpha - x) \leq y_\alpha$ . The Definition 1.1.6 (ii) is stronger, [24, Proposition 3.6]. In this thesis, we will use Definition 1.1.6 (ii).

- (v) For  $x_\alpha \xrightarrow{\circ} x$  and  $y_\beta \xrightarrow{\circ} y$  then  $\lambda x_\alpha + \mu y_\beta \xrightarrow{\circ} \lambda x + \mu y$  (where the according index set  $\{(\alpha, \beta)\}$  is entry-wise ordered) for every  $\lambda, \mu \in \mathbb{R}$ .

*Proof.* (i), (ii) and (iii) are clear.

- (iv) It follows from  $\pm(x - x_\alpha) \leq u_\alpha \downarrow 0$  and  $\pm(y - x_\alpha) \leq v_\alpha \downarrow 0$  that

$$\pm(x - y) = \pm(x - x_\alpha + x_\alpha - y) \leq u_\alpha + v_\alpha \downarrow 0.$$

Hence,  $x = y$ .

- (v) Firstly, we show if  $x_\alpha \xrightarrow{\circ} x$ , then  $\lambda x_\alpha \xrightarrow{\circ} \lambda x$  for arbitrary  $\lambda \in \mathbb{R}$ . Because there exists a net  $(u_\alpha)_\alpha \downarrow 0$  such that  $\pm(x_\alpha - x) \leq u_\alpha$ , we have

$$\pm\lambda(x_\alpha - x) \leq \pm\lambda u_\alpha.$$

Then  $\pm\lambda(x_\alpha - x) \leq |\lambda|u_\alpha \downarrow 0$ , so  $\lambda x_\alpha \xrightarrow{\circ} \lambda x$ .

Secondly, we show if  $x_\alpha \xrightarrow{\circ} x$  and  $y_\beta \xrightarrow{\circ} y$ , then  $x_\alpha + y_\beta \xrightarrow{\circ} x + y$ . It follows from  $\pm(x - x_\alpha) \leq u_\alpha \downarrow 0$  and  $\pm(y - y_\beta) \leq v_\beta \downarrow 0$  that

$$\pm(x_\alpha + y_\beta - x - y) = \pm[(x_\alpha - x) + (y_\beta - y)] \leq u_\alpha + v_\beta.$$

Let  $w_{(\alpha, \beta)} := u_\alpha + v_\beta$ . By (iii) we have  $w_{(\alpha, \beta)} \downarrow 0$ .

□

## 1.1.2 Vector lattices

**Definition 1.1.8.** Let  $(X, K)$  be a partially ordered vector space.  $X$  is called a **vector lattice** or **Riesz space** if every subset consisting of two elements has a supremum and an infimum. The supremum and infimum are denoted by

$$x \vee y := \sup\{x, y\} \quad \text{and} \quad x \wedge y := \inf\{x, y\}, \quad \forall x, y \in X.$$

The properties of vector lattices and the lattice operations on it can be found in almost any textbook of Riesz space and Banach lattices, see, e.g. [6, 60].

For any vector  $x$  in a vector lattice  $X$ , define

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0, \quad \text{and} \quad |x| := x \vee (-x).$$

The element  $x^+$  is called the **positive part**,  $x^-$  is called the **negative part**, and  $|x|$  is called the **absolute value** of  $x$ .

Next, we will give definitions of disjointness, ideal and band in a vector lattice. These definitions will be generalized to pre-Riesz space in the next section.

**Definition 1.1.9.** Let  $X$  be a vector lattice.

- (i) Two elements  $x, y \in X$  are called **disjoint** if  $|x| \wedge |y| = 0$ , denoted by  $x \perp y$ .

Let  $M$  be a subset of  $X$ .

- (ii) The set

$$M^d = \{x \in X; x \perp y \text{ for all } y \in M\}$$

is called the **disjoint complement** of  $M$ .

- (iii)  $M$  is called **solid** if  $x \in X$ ,  $y \in M$  and  $|x| \leq |y|$  imply  $x \in M$ .

- (iv)  $M$  is called an **ideal** in  $X$  if  $M$  is a solid linear subspace of  $X$ .

- (v) The ideal  $M$  is called a **band** in  $X$  if for any subset  $D$  of  $M$  which has a supremum in  $X$ , this supremum is in  $M$ , in other words, it follows from  $D \subseteq M$  and  $f = \sup D$  that  $f \in M$ .

The following theorem is due to [6, Theorem 1.7].

**Theorem 1.1.10.** Let  $X$  be a vector lattice, and  $x, y \in X$ . Then  $x \perp y$  if and only if  $|x + y| = |x - y|$ .

**Note 1.1.11.** It should be noticed that in an Archimedean vector lattice  $X$ ,  $M^d$  is always a band and  $M$  is a band if and only if  $M = M^{\text{dd}}$ , where  $M^{\text{dd}} = (M^d)^d$ .

## 1.2 Pre-Riesz spaces and Riesz completions

Pre-Riesz spaces are those partially ordered vector spaces which have suitable vector lattice completions. Pre-Riesz spaces have been introduced by M. van Haandel in [54] firstly. Lately, they have been studied recently by O. van Gaans and A. Kalauch [32, 34, 53], most of our notations come from their papers.

**Definition 1.2.1.** A partially ordered vector space  $(X, K)$  is called **pre-Riesz** if for every  $x, y, z \in X$  the inclusion  $\{x + y, x + z\}^u \subseteq \{y, z\}^u$  implies  $x \in K$ .

The following proposition comes from [54, Theorem 1.7(ii)].

**Proposition 1.2.2.** Every pre-Riesz space is directed and every directed Archimedean partially ordered vector space is pre-Riesz.

Clearly, each vector lattice is pre-Riesz. However, there are many examples of pre-Riesz spaces which are not vector lattices, see [31, 33] and the following examples.

*Example 1.2.3.* (1) Let  $X = \mathbb{R}^2$  with the cone defined by  $K = \{(x_1, x_2); x_2 > 0\} \cup \{(x_1, 0); x_1 \geq 0\}$ , then  $(X, K)$  is a pre-Riesz space but not Archimedean.

(2) Let  $X = C^1[0, 1]$  be the differentiable functions space on  $[0, 1]$  with the natural cone  $K = \{f \in X; f(x) \geq 0 \text{ for all } x \in [0, 1]\}$ , then  $(X, K)$  is directed and Archimedean, hence a pre-Riesz space.

(3) Let  $X = \text{Pol}^2(\mathbb{R})$  be the ordered vector space of all real polynomial functions on  $\mathbb{R}$ , ordered by the natural cone  $K = \{f \in X; f(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$ . Then  $(X, K)$  is a pre-Riesz space.

(4) Let  $X = \{\alpha \mathbf{1} + v; \alpha \in \mathbb{R}, v \in C[0, 1], v(0) = 0, \int_0^1 v(t) dt = 0\}$  ordered by the natural cone, then  $X$  is a pre-Riesz space, where  $\mathbf{1}$  denotes the constant-one function.

**Definition 1.2.4.** A linear subspace  $D$  of a partially ordered vector space  $X$  is called **order dense** in  $X$  if for every  $x \in X$  we have

$$x = \inf\{y \in D; y \geq x\}.$$

*Remark 1.2.5.* The order denseness in Definition 1.2.4 is slightly different from the classical meaning of order denseness in vector lattices. To distinguish them, we use the term ‘property (p)’ for the classical definition in vector lattices. A vector sublattice  $G$  of a vector lattice  $E$  is said to have **property (p)** whenever for each  $0 < x \in E$ , there exists some  $y \in G$  with  $0 < y \leq x$ , see [60]. However, Definition 1.2.4 originally comes from [14]. In vector lattices, the order denseness in the sense of Definition 1.2.4 implies the property (p), but the property (p) does not implies the order denseness, see Example [30, Example 3.3.12]. If, moreover, the vector lattice is Archimedean, then these two concepts are equivalent, [30, Example 3.3.13].

In this text, the meaning of order denseness in partially ordered vector spaces always refers to Definition 1.2.4 if not explicitly stated otherwise.

It is obvious that  $D$  is order dense in  $X$  if and only if for every  $x \in X$  one has

$$x = \sup\{y \in D; y \leq x\}.$$

Moreover, any order dense subspace is majorizing.

**Definition 1.2.6.** Let  $X$  and  $Y$  be two partially ordered vector spaces. A linear map  $i: X \rightarrow Y$  is called **bipositive** if for every  $x \in X$  one has

$$i(x) \geq 0 \text{ if and only if } x \geq 0.$$

For sets  $L \subseteq X$ ,  $M \subseteq Y$  and a mapping  $i: X \rightarrow Y$ , we denote

$$i(L) := \{i(x); x \in L\} \text{ and } i^{-1}(M) := \{x \in X; i(x) \in M\}.$$

We say that a subspace  $X$  of a vector lattice  $Y$  **generates**  $Y$  as a vector lattice if for every  $y \in Y$  there exist  $a_1, \dots, a_m, b_1, \dots, b_n \in X$  such that

$$y = \bigvee_{i=1}^m a_i - \bigvee_{i=1}^n b_i.$$

It turns out that a pre-Riesz space can be always embedded as an order dense subspace in a vector lattice. This is shown by the following theorem [54, Corollaries 4.9-11 and Theorems 3.5, 3.7, 4.13], which plays a fundamental role in our research.

**Theorem 1.2.7.** Let  $X$  be a partially ordered vector space. The following statements are equivalent.

- (i)  $X$  is a pre-Riesz space.
- (ii) There exist a vector lattice  $Y$  and a bipositive linear map  $i: X \rightarrow Y$  such that  $i(X)$  is order dense in  $Y$ .
- (iii) There exist a vector lattice  $Y$  and a bipositive linear map  $i: X \rightarrow Y$  such that  $i(X)$  is order dense in  $Y$  and  $i(X)$  generates  $Y$  as a vector lattice.

A pair  $(Y, i)$  as in (ii) is called a **vector lattice cover** of  $X$ . All spaces  $Y$  as in (iii) are isomorphically determined as vector lattices, i.e. if  $j$  is an isomorphism from  $X$  onto an order dense subspace of a vector lattice  $Z$ , then there is an isomorphism  $k$  from  $Z$  onto  $Y$  such that  $k \circ j = i$ . In the sense of isomorphism, we will say  $(Y, i)$  be **the Riesz completion** of  $X$ , denoted by  $X^\rho$ .

The construction of the Riesz completion of  $X$  can be found in [54]. Throughout this text, we will use  $(X^\rho, i)$  to denote the Riesz completion of a pre-Riesz space  $(X, K)$ .

Obviously, the order denseness of  $i(X)$  in  $X^\rho$  implies that  $i(X)$  is majorizing in  $X^\rho$ .

*Example 1.2.8.* (1) The Riesz completion of  $X = C^1[0, 1]$  with the cone  $K = \{f \in X; f(x) \geq 0 \text{ for all } x \in [0, 1]\}$  is the space of piecewise differentiable functions on  $[0, 1]$ .

(2) The Riesz completion of  $X = \text{Pol}^2(\mathbb{R})$  with the cone  $K = \{f \in X; f(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$  is the space of piecewise polynomial functions on  $[0, 1]$ .

- (3) [33, Example 4.8] Let  $X = \mathbb{R}^3$  and let the positive cone  $K$  be the positive linear span of  $x_1 = (1, 0, 1)$ ,  $x_2 = (0, 1, 1)$ ,  $x_3 = (-1, 0, 1)$ ,  $x_4 = (0, -1, 1)$ . The map  $i: \mathbb{R}^3 \rightarrow \mathbb{R}^4$  is given by

$$i: x \mapsto (f_1(x), f_2(x), f_3(x), f_4(x)),$$

with

$$f_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, f_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, f_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Then  $i$  is a bipositive linear map and  $i(X)$  is order dense in  $\mathbb{R}^4$ . Hence,  $i$  embeds the partially ordered vector space  $(\mathbb{R}^3, K)$  into the vector lattice  $(\mathbb{R}^4, \mathbb{R}_+^4)$ .

**Note 1.2.9.** It is worth mentioning that the classical Dedekind completion is in general more involved than Riesz completions. For an Archimedean directed partially ordered vector space  $X$ , there is a Dedekind complete vector lattice  $Y$  and an order isomorphism  $j$  from  $X$  onto a subspace  $j(X)$  of  $Y$  such that  $j(X)$  is order dense in  $Y$ , i.e. for every  $y \in Y$ , one has

$$y = \sup\{j(x); x \in X, j(x) \leq y\} = \inf\{j(x); x \in X, j(x) \geq y\}. \quad (1.3)$$

$(Y, j)$  is called a **Dedekind completion** of  $X$ , see [35, Proposition 2.1.4]. We will use  $(X^\delta, j)$  to denote the Dedekind completion of  $X$ . If  $X$  is a vector lattice, then  $j(X)$  is a sublattice of  $X^\delta$ .

In general, the Dedekind completion is larger than the Riesz completion, see [32, Example 3.5].

The following theorem shows that Dedekind completions of a directed Archimedean partially ordered space are isomorphically determined, [54, Theorem 4.14].

**Theorem 1.2.10.** Let  $X$  and  $Y$  be two Archimedean directed partially ordered

spaces, and  $i: X \rightarrow Y$  a bipositive linear map such that  $i[X]$  order dense in  $Y$ . Then their Dedekind completions  $X^\delta$  and  $Y^\delta$  are order isomorphic.

If  $X$  is not Archimedean, then  $X^\delta$  can still be constructed as a partially ordered set, but it fails to be a vector space [32, p. 577]. If  $X$  is Archimedean, the Riesz completion  $X^\rho$  is the vector sublattice of  $X^\delta$  generated by  $j(X)$  and therefore  $X^\rho$  is Archimedean [32, Remark 3.4]. So we have the following fact which is due to [30, Proposition 1.4.7].

**Proposition 1.2.11.** If a pre-Riesz space  $(X, K)$  is Archimedean, then  $X^\rho$  is Archimedean as well.

Next, we will introduce the concepts of ideal and band in partially ordered vector spaces. Similar to the case of vector lattices, an ideal of a partially ordered vector space is induced by means of a solid subset.

**Definition 1.2.12.** Let  $(X, K)$  be a partially ordered vector space and  $M$  is a subset of  $X$ .

- (i)  $M$  is called **solid** if for every  $x \in X$  and  $y \in M$ , one has  $\{x, -x\}^u \supseteq \{y, -y\}^u$  implies  $x \in M$ .
- (ii)  $M$  is called **ideal** if  $M$  is a solid linear subspace of  $X$ .

Disjointness in a partially ordered vector space  $(X, K)$  is introduced in [32]. The definition of band will be given by means of disjoint complements of subsets.

**Definition 1.2.13.** Let  $X$  be a partially ordered vector space. Two elements  $x, y \in X$  are called **disjoint**, in symbols  $x \perp y$ , if

$$\{x + y, -x - y\}^u = \{x - y, -x + y\}^u.$$

If  $X$  is a vector lattice, then this notion of disjointness coincides with the usual one in vector lattices in Definition 1.1.9. Next, we will show an example of disjoint elements in partially ordered vector spaces, which is cited from [32, Example 4.6].

*Example 1.2.14.* Let  $X = \mathbb{R}^3$  ordered by the cone  $K$  which is the positive linear span of  $x_1 = (1, 0, 1)$ ,  $x_2 = (0, 1, 1)$ ,  $x_3 = (-1, 0, 1)$ ,  $x_4 = (0, -1, 1)$ . Then

$$\{x_1 + x_3, -x_1 - x_3\}^u = \{x_1 - x_3, -x_1 + x_3\}^u.$$

Hence,  $x_1 \perp x_3$ . Similarly,  $x_2 \perp x_4$ .

The following proposition is due to [32, Proposition 2.1].

**Proposition 1.2.15.** Let  $X$  and  $Y$  be two partially ordered vector spaces and  $x, y \in X$ .

- (1) If  $X$  is a subspace of  $Y$ , then  $x \perp y$  in  $Y$  implies  $x \perp y$  in  $X$ .
- (2) If  $X$  is an order dense subspace of  $Y$ , then  $x \perp y$  in  $Y$  if and only if  $x \perp y$  in  $X$ .

Let  $X$  be a pre-Riesz space and  $(Y, i)$  a vector lattice cover of  $X$ . Then from the above proposition it follows that for every  $x, y \in X$  we have  $x \perp y$  if and only if  $i(x) \perp i(y)$  in  $Y$ .

Thus we could define disjoint complements in partially ordered vector spaces similar to the vector lattice case.

**Definition 1.2.16.** Let  $X$  be a partially ordered vector space. The **disjoint complement** of a subset  $M \subseteq X$  is the set

$$M^d = \{y \in X; y \perp x \text{ for all } x \in M\}.$$

We give the definition of a band in partially ordered vector spaces.

**Definition 1.2.17.** A subspace  $B$  of a partially ordered vector space  $X$  is called a **band** in  $X$  if  $B = B^{dd}$ .

For a subset  $S \subseteq X$ , Proposition 1.2.15 implies that

$$S^{\text{d}} = i^{-1} \left( i(S)^{\text{d}} \right). \quad (1.4)$$

Thus, disjoint complements in pre-Riesz spaces have properties as in the vector lattice setting, namely  $S^{\text{d}}$  is solid and o-closed, see [33, Theorem 5.10]. In particular, a disjoint complement is a linear subspace in  $X$ , more than that, it is a band.

Note that the notion of band coincides with the usual one provided  $X$  is an Archimedean vector lattice.

We will need the following technical observation. If  $D$  is a majorizing subspace of a vector lattice  $Y$  and  $u \in Y$  is such that  $u \perp d$  for every  $d \in D$ , then  $u = 0$ . Indeed, there is  $w \in D$  such that  $|u| \leq w$ . Hence  $|u| = |u| \wedge w = 0$ , consequently  $u = 0$ .

Since  $i(X)$  is majorizing in  $X^\rho$ , we have the following immediate result.

**Lemma 1.2.18.** If  $X$  is a pre-Riesz space and  $u \in X^\rho$  is such that  $u \perp x$  for every  $x \in i(X)$ , then  $u = 0$ .

## 1.3 Positive operators

In this section, we will give definitions of some different classes of operators between ordered vector spaces and list some basic properties of operators.

**Definition 1.3.1.** An operator  $T: X \rightarrow Y$  between two ordered vector spaces  $X$  and  $Y$  is called **positive** if  $T(x) \geq 0$  for all  $x \geq 0$ , denoted by  $T \geq 0$  or  $0 \leq T$ . The vector space of all linear operators between vector spaces  $X$  and  $Y$  is denoted by  $\mathcal{L}(X, Y)$ . Usually  $\mathcal{L}(X)$  stands for  $\mathcal{L}(X, X)$

**Definition 1.3.2.** An operator  $T: X \rightarrow Y$  between two ordered vector spaces  $X$  and  $Y$  is called **order bounded** if it maps order bounded subsets of  $X$  to order

bounded subsets of  $Y$ . The vector space of order bounded operators between two vector spaces  $X$  and  $Y$  is denoted by  $\mathcal{L}_b(X, Y)$ . Moreover  $\mathcal{L}_b(X)$  stands for  $\mathcal{L}_b(X, X)$ .

**Definition 1.3.3.** An operator  $T: X \rightarrow Y$  between two ordered vector spaces  $X$  and  $Y$  is called **regular** if there exists  $T_1 \geq 0$  and  $T_2 \geq 0$  such that  $T = T_1 - T_2$ . The vector space of regular operators between two vector spaces  $X$  and  $Y$  is denoted by  $\mathcal{L}_r(X, Y)$ . Moreover  $\mathcal{L}_r(X)$  means  $\mathcal{L}_r(X, X)$ .

With the above notations, we have the following inclusions,

$$\mathcal{L}_r(X, Y) \subseteq \mathcal{L}_b(X, Y) \subseteq \mathcal{L}(X, Y).$$

The inclusion  $\mathcal{L}_r(X, Y) \subseteq \mathcal{L}_b(X, Y)$  can be proper, see [6, Example 1.16].

**Definition 1.3.4.** An operator  $T: X \rightarrow Y$  between two ordered vector spaces is called **disjointness preserving** if for every  $x, y \in X$  from  $x \perp y$  in  $X$  it follows that  $Tx \perp Ty$  in  $Y$ .

Let us recall two important extension theorems, which are frequently used in this text. The first one we cite from [6, Theorem 1.25].

**Theorem 1.3.5. (Hahn-Banach)** Let  $X$  be a vector space,  $Y$  a Dedekind complete Riesz space, and let  $p: X \rightarrow Y$  be a sublinear function. If  $U$  is a subspace of  $X$  and  $S: U \rightarrow Y$  is an operator with  $S(x) \leq p(x)$  for all  $x \in U$ , then there exists some operator  $T: X \rightarrow Y$  such that

- (i)  $T = S$  on  $U$ , i.e.  $T$  is a linear extension of  $S$  to all of  $X$ .
- (ii)  $T(x) \leq p(x)$  for all  $x \in X$ .

The second one we cite from [6, Theorem 1.32].

**Theorem 1.3.6. (Kantorovich)** Let  $X$  be an ordered vector space,  $Y$  a Dedekind complete Riesz space. Every positive linear operator  $T: D \rightarrow Y$  defined on a majorizing subspace  $D \subseteq X$  extends to all of  $X$  as a positive linear operator.