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Extension of Operators on Pre-Riesz Spaces

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To my beloved parents.

Preface

The theory of Riesz spaces, as a branch of functional analysis, has been developed extensively in the last decades. It was first considered by F. Riesz, L. Kantorovic, and H. Freudenthal in the middle nineteen thirties, and subsequently studied by scholars from the Soviet Union, Japan and the United States. Then the theory has grown rapidly in Dutch and German schools since the middle of the nineteen seventies, and it has also attracted scholars from the United Kingdom and Spain. Later on, more and more researchers have joined in the developments, for instance from China, South Africa, Slovenia and so on.

Most vector spaces admit a partial order in a pointwise manner or in another way. In a nutshell, Riesz spaces (vector lattices) are the partially ordered vector spaces in which any pair of elements has a least upper bound with respect to the order (lattice operation). Even though this is a very concise fact, the study of Riesz spaces in conjunction with Banach's theory of normed vector spaces has progressed slowly and steadily from the mid-1930s to 1960s. The last 30 years of the previous century was a fruitful age. The theory of Riesz space has systematically been established in *Riesz Spaces I* by W. A. J. Luxemburg and A. C. Zaanen [40]. H. H. Schaefer [48] has bridged the gap between the theory of positive operators in Banach lattices and the mainstream of operator theory in Banach spaces in *Banach Lattices and Positive Operators*. The book *Positive Operators* by C. D. Aliprantis and O. Burkinshaw [6], and the book *Banach Lattices* by P. Meyer-Nieberg [42] were remarkable next steps. There are too many monographs to list them one by one at here, we refer the reader to the previously mentioned four books for more history of Riesz spaces. In the area of dynamic systems, the theory of Riesz spaces and positive operators has been applied as well. We only mention two books which are related to this thesis. One is *One-parameter Semigroups* [16] by Dutch scholars, and the other one is *One-parameter Semigroups of Positive Operators* [45] by German scholars.

Compared with the high level of accomplishment in Riesz space theory, there are only few results for general partially ordered vector spaces, due to the lack of lattice structure. Nonetheless, some scholars, e.g. C. D. Aliprantis and E. Langford [7] in 1984, M. van Haandel [54] in 1993, have developed a theory of the vector lattice covers of partially ordered vector spaces in a categorical approach. Based on the theory in the latter approach, O. van Gaans and A. Kalauch have extended some basic results from Riesz spaces, viz. the concepts of ideals, bands and disjointness etc. to a new setting of pre-Riesz spaces. This thesis proceeds their works. It extends the study of operators on pre-Riesz spaces.

This thesis includes five main chapters. The first chapter introduces some basic terminologies in ordered vector spaces and vector lattices. The second chapter investigates disjointness preserving operators on partially ordered vector spaces, in particular pre-Riesz spaces. The third chapter is concerned with then extension of the theory of compact operators, in particular the positive domination property in pre-Riesz spaces. The fourth chapter studies disjointness preserving C_0-semigroups in partially ordered vector spaces with respect to a suitable norm. In the last chapter, the dissipativity and positive off-diagonal property of operators on ordered vector spaces are considered.

Contents

Chapter 1

Preliminaries

In this chapter, we will introduce some of the basic background information and results that will be used throughout this text. For the terminology of Riesz spaces we mainly refer to W. A. J. Luxemburg & A. C. Zaanen [40], E. de Jonge & A. C. M. van Rooij [18], A. C. Zaanen [60], for Banach lattices and positive operators to [6, 42], and for topological vector spaces to [47]. The embedding theory of partially ordered vector spaces has been studied by G. Buskes & A. C. M. van Rooij [14], M. van Haandel [54] and, more recently, by O. van Gaans [50] and A. Kalauch [29]. For the terminology of partially ordered vector spaces and pre-Riesz spaces, we refer to these papers and monographs. For the readers' convenience, we will also introduce some concepts where they are needed in later chapters.

This chapter consists of three sections.

The first section includes two subsections, the one with the terminology of partially ordered vector spaces and the other one with concepts of vector lattices. The partial order of a vector space is induced by a positive cone, and such an order appears throughout most of definitions and properties of spaces in this thesis. The structure of vector lattices is given in the second subsection, e.g. ideals, bands etc..

Section 1.2 is concerned with the terminology of pre-Riesz spaces and Riesz completions. A pre-Riesz space is one kind of partially ordered vector space, which can be embedded order densely into a vector lattice cover, and the smallest such a cover is called the Riesz completion. This theory was established by [54], and considered as a cornerstone of this thesis. The importance of the embedding map from a pre-Riesz space to a vector lattice will show up in the main body of this text.

Section 1.3 contains definitions of different properties of linear operators between ordered vector spaces. In addition, it presents the Hahn-Banach extension theorem and the Kantorovich extension theorem.

1.1 Ordered vector spaces and vector lattices

In this section, we will introduce some basic terminology of ordered vector spaces and vector lattices.

1.1.1 Ordered vector spaces

Definition 1.1.1. Let X be a real vector space.

(i) A reflexive, transitive and antisymmetric relation "$\leq$" on X is called a **vector space order** if

  (a) $x, y, z \in X$ and $x \leq y$ imply $x + z \leq y + z$,

  (b) $x \in X$, $0 \leq x$ and $\lambda \in \mathbb{R}^+$ imply $0 \leq \lambda x$.

Then $(X, \leq)$ is called a **partially ordered vector space**, in short, POVS. For $x, y \in X$, the relation $x \leq y$ can be also written as $y \geq x$. An element $x \in X$ is called **positive** if $0 \leq x$. The relation $x > 0$ means $x \geq 0$ and $x \neq 0$, and x is then called **strictly positive**.

(ii) A non-empty subset $K \neq \{0\}$ of X is called a **wedge** if $x, y \in K$ and $\lambda, \mu \in \mathbb{R}^+$ imply $\lambda x + \mu y \in K$. A wedge is called a **cone** if $K \cap (-K) = 0$. A wedge K in X is called **generating** if $X = K - K$.

The following proposition is straightforward.

Proposition 1.1.2. Let X be a real vector space.

(i) Let K be a cone in X and $\leq$ on X defined by means of

$$x \leq y \;:\Leftrightarrow\; y - x \in K. \tag{1.1}$$

Then $\leq$ is a vector space order.

(ii) Let $\leq$ be a vector space order on X. Then the set

$$X^+ := \{x \in X;\, 0 \leq x\} \tag{1.2}$$

of all positive elements in X is a cone.

(iii) Let K be a cone in X, $\leq$ be the order defined by (1.1) and X^+ be the corresponding cone in (1.2). Then $K = X^+$.

There are many ways to choose a cone K in a vector space. This is shown by the following example.

Example 1.1.3. (1) Let $X = \mathbb{R}^2$.

(a) The set

$$K_1 := \{(x_1, x_2);\, x_1 \geq 0, x_2 \geq 0\}$$

is a cone. Here K_1 is the so-called **standard cone**. The partial order induced by K_1 is called the **standard order**.

(b) The set

$$K_2 := \{(x_1, x_2);\, x_2 > 0\} \cup \{(x_1, 0);\, x_1 \geq 0\}$$

is a cone in X.

(2) Let $X = \mathbb{R}^3$. Then the set

$$\mathrm{L}_{\mathbb{R}^2} := \{(x_1, x_2, x_3);\ x_1^2 + x_2^2 \leq x_3^2,\ x_3 \geq 0\}$$

is a cone in X.

In a real vector space X, let $K \subseteq X$ be a given cone. To stress that the vector space order $\leq$ on X is induced by the cone K, we will use (X, K) to denote the ordered vector space. Occasionally, we write merely X instead of (X, K) if no ambiguousness can arise.

Definition 1.1.4. Let (X, K) be a partially ordered vector space.

(i) X is called **directed** if for every $x, y \in X$ there exists $z \in X$ such that $z \geq x, y$.

(ii) For $x, y \in X$, $x \leq y$, we denote

$$[x, y] := \{z \in X;\ x \leq z \leq y\},$$

and we call $[x, y]$ an **order interval**.

(iii) A non-empty subset $M \subseteq X$ is called **bounded above**, if there exists $x \in X$ such that for all $m \in M$ we have $m \leq x$, respectively, **bounded below** if $m \geq x$. M is called **order bounded** if there exist $x, y \in X$ such that $M \subseteq [x, y]$. The set of all **upper bounds** is denoted by

$$M^{\mathrm{u}} = \{x \in X;\ x \geq m \text{ for all } m \in M\},$$

respectively, M^{l} for **lower bounds**.

(iv) A non-empty subset M of X is called **majorizing** if for every $x \in X$ there exists $m \in M$ such that $m \geq x$.

(v) An element $u > 0$ is called an **order unit** if for every $x \in X$ there is a $\lambda \in \mathbb{R}^+$ such that $x \in [-\lambda u, \lambda u]$. X is called an **order unit space** if it has an order unit.

(vi) X is called **Archimedean** if for every $x, y \in X$ with $nx \leq y$ for every $n \in \mathbb{N}$ we have $x \leq 0$.

(vii) X is called **Dedekind complete** whenever every non-empty bounded above (bounded below) subset of X has a supremum (infimum).

(viii) X is called **σ-Dedekind complete** if every non-empty finite or countable subset of X that is bounded above (bounded below) subset has a supremum (infimum).

Note 1.1.5. It should be noticed that, in the above Definition 1.1.4 (iii), the order of $M \subseteq X$ is inherited from X. In this text, if not specified otherwise subspaces of X are always equipped with the inherited order.

In a partially ordered vector space (X, K), the cone K is generating if and only if X is directed. The existence of an order unit $u > 0$ in X implies that K is generating. Indeed, for every $x \in X$ there is $\lambda \in \mathbb{R}^+ \setminus \{0\}$ such that $\lambda u - x \in K$, moreover $\lambda u \in K$ and

$$x = \lambda u - (\lambda u - x).$$

Clearly, if X is Dedekind complete, then X is σ-Dedekind complete. Moreover, if X is σ-Dedekind complete, then X is Archimedean.

Definition 1.1.6. Let (X, K) be a partially ordered vector space, and I a directed index set.

(i) A net $(x_\alpha)_{\alpha \in I}$ in X is called **increasing**, if $\alpha \geq \beta$ implies $x_\alpha \geq x_\beta$, respectively, **decreasing** if the net $(-x_\alpha)_{\alpha \in I}$ is increasing. We will use $x_\alpha \downarrow x$ to denote that (x_α) is decreasing and $\inf\{x_\alpha; \alpha \in I\} = x$. Similarly, we use $x_\alpha \uparrow x$.

(ii) A net $(x_\alpha)_{\alpha \in I} \subset X$ is said to **order converges**, in short, **o-converges** to $x \in X$ (denoted by $x_\alpha \xrightarrow{o} x$), if there is a net $(y_\beta)_{\beta \in J} \subset X$ such that for every β there is α_0 such that for every $\alpha \geq \alpha_0$ we have $\pm(x_\alpha - x) \leq y_\beta \downarrow 0$. [1]

(iii) A non-empty subset $M \subseteq X$ is called **order closed**, in short, **o-closed**, if for each net $(x_\alpha)_{\alpha \in I}$ in M which o-converges to $x \in X$ one has that $x \in M$.

(iv) A sequence (x_n) in X is said to be **relatively uniformly convergent** to $x \in X$ (denoted by $x_n \xrightarrow{\text{ru}} x$), if there exist some $v > 0$ and $\lambda_n \downarrow 0$ in $\mathbb{R}$ such that $\pm(x_n - x) \leq \lambda_n v$ for all $n \in \mathbb{N}$. A subset $M \subseteq X$ is called **relatively uniformly closed** if it is closed under relative uniform convergence of sequences. By the **relative uniform closure** of a set $M \subseteq X$ we mean the smallest relatively uniformly closed set in X which contains M. For $u > 0$, the sequence (x_n) is called an **u-uniformly Cauchy sequence** if for any $\epsilon > 0$ there exists an element $n(\epsilon) \in \mathbb{N}$ such that $\pm(x_m - x_n) \leq \epsilon u$ for all $m, n \geq n(\epsilon)$. The Archimedean ordered vector space X is called **uniformly complete** if, for every $u > 0$ in X, every u-uniformly Cauchy sequence has a relatively uniform limit.

The main properties of order convergence are listed in the following lemma.

Lemma 1.1.7. Let (X, K) be a partially ordered vector space.

(i) If $x_\alpha \downarrow x$, then $\lambda x_\alpha \downarrow \lambda x$ for all $\lambda \in \mathbb{R}^+$.

(ii) If $x_\alpha \downarrow x$ or $x_\alpha \uparrow x$, then $x_\alpha \xrightarrow{\text{o}} x$.

(iii) If $(x_\alpha)_{\alpha \in I}$ and $(y_\beta)_{\beta \in J}$ are decreasing, then for $z_{(\alpha,\beta)} := x_\alpha + y_\beta$, $\alpha \in I, \beta \in J$, we have that $\left(z_{(\alpha,\beta)}\right)_{(\alpha,\beta) \in I \otimes J}$ is decreasing, where the index set $I \otimes J$ is entry-wise directed, i.e. $(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$ if and only if $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. Moreover, if $x_\alpha \downarrow 0$ and $y_\beta \downarrow 0$, then $x_\alpha + y_\beta = z_{(\alpha,\beta)} \downarrow 0$.

(iv) If $x_\alpha \xrightarrow{\text{o}} x$ and $x_\alpha \xrightarrow{\text{o}} y$, then $x = y$.

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[1] There is another definition for order convergence in partially ordered vector space X. A net $(x_\alpha)_{\alpha \in I}$ in X is said to order converges to $x \in X$ if there is a net $(y_\alpha)_{\alpha \in I} \subset X$ such that $y_\alpha \downarrow 0$ and for all $\alpha \in I$ one has $\pm(x_\alpha - x) \leq y_\alpha$. The Definition 1.1.6 (ii) is stronger, [24, Proposition 3.6]. In this thesis, we will use Definition 1.1.6 (ii).

(v) For $x_\alpha \xrightarrow{o} x$ and $y_\beta \xrightarrow{o} y$ then $\lambda x_\alpha + \mu y_\beta \xrightarrow{o} \lambda x + \mu y$ (where the according index set $\{(\alpha, \beta)\}$ is entry-wise ordered) for every $\lambda, \mu \in \mathbb{R}$.

Proof. (i), (ii) and (iii) are clear.

(iv) It follows from $\pm(x - x_\alpha) \leq u_\alpha \downarrow 0$ and $\pm(y - x_\alpha) \leq v_\alpha \downarrow 0$ that

$$\pm(x - y) = \pm(x - x_\alpha + x_\alpha - y) \leq u_\alpha + v_\alpha \downarrow 0.$$

Hence, $x = y$.

(v) Firstly, we show if $x_\alpha \xrightarrow{o} x$, then $\lambda x_\alpha \xrightarrow{o} \lambda x$ for arbitrary $\lambda \in \mathbb{R}$. Because there exists a net $(u_\alpha)_\alpha \downarrow 0$ such that $\pm(x_\alpha - x) \leq u_\alpha$, we have

$$\pm\lambda(x_\alpha - x) \leq \pm\lambda u_\alpha.$$

Then $\pm\lambda(x_\alpha - x) \leq |\lambda| u_\alpha \downarrow 0$, so $\lambda x_\alpha \xrightarrow{o} \lambda x$.

Secondly, we show if $x_\alpha \xrightarrow{o} x$ and $y_\beta \xrightarrow{o} y$, then $x_\alpha + y_\beta \xrightarrow{o} x + y$. It follows from $\pm(x - x_\alpha) \leq u_\alpha \downarrow 0$ and $\pm(y - y_\beta) \leq v_\beta \downarrow 0$ that

$$\pm(x_\alpha + y_\beta - x - y) = \pm[(x_\alpha - x) + (y_\beta - y)] \leq u_\alpha + v_\beta.$$

Let $w_{(\alpha,\beta)} := u_\alpha + v_\beta$. By (iii) we have $w_{(\alpha,\beta)} \downarrow 0$.

□

1.1.2 Vector lattices

Definition 1.1.8. Let (X, K) be a partially ordered vector space. X is called a **vector lattice** or **Riesz space** if every subset consisting of two elements has a supremum and an infimum. The supremum and infimum are denoted by

$$x \vee y := \sup\{x, y\} \quad \text{and} \quad x \wedge y := \inf\{x, y\}, \ \forall x, y \in X.$$

The properties of vector lattices and the lattice operations on it can be found in almost any textbook of Riesz space and Banach lattices, see, e.g. [6, 60].

For any vector x in a vector lattice X, define

$$x^+ := x \vee 0, \; x^- := (-x) \vee 0, \text{ and } |x| := x \vee (-x).$$

The element x^+ is called the **positive part**, x^- is called the **negative part**, and $|x|$ is called the **absolute value** of x.

Next, we will give definitions of disjointness, ideal and band in a vector lattice. These definitions will be generalized to pre-Riesz space in the next section.

Definition 1.1.9. Let X be a vector lattice.

(i) Two elements $x, y \in X$ are called **disjoint** if $|x| \wedge |y| = 0$, denoted by $x \perp y$.

Let M be a subset of X.

(ii) The set

$$M^{\mathrm{d}} = \{x \in X;\, x \perp y \text{ for all } y \in M\}$$

is called the **disjoint complement** of M.

(iii) M is called **solid** if $x \in X$, $y \in M$ and $|x| \leq |y|$ imply $x \in M$.

(iv) M is called an **ideal** in X if M is a solid linear subspace of X.

(v) The ideal M is called a **band** in X if for any subset of M which has a supremum in X, this supremum is in M, in other words, it follows from $D \subseteq M$ and $f = \sup D$ that $f \in M$.

The following theorem is due to [6, Theorem 1.7].

Theorem 1.1.10. Let X be a vector lattice, and $x, y \in X$. Then $x \perp y$ if and only if $|x + y| = |x - y|$.

Note 1.1.11. It should be noticed that in an Archimedean vector lattice X, M^{d} is always a band and M is a band if and only if $M = M^{\mathrm{dd}}$, where $M^{\mathrm{dd}} = \left(M^{\mathrm{d}}\right)^{\mathrm{d}}$.

1.2 Pre-Riesz spaces and Riesz completions

Pre-Riesz spaces are those partially ordered vector spaces which have suitable vector lattice completions. Pre-Riesz space have been introduced by M. van Haandel in [54] firstly. Lately, they have been studied recently by O. van Gaans and A. Kalauch [32, 34, 53], most of our notations come from their papers.

Definition 1.2.1. A partially ordered vector space (X, K) is called **pre-Riesz** if for every $x, y, z \in X$ the inclusion $\{x + y, x + z\}^{\mathrm{u}} \subseteq \{y, z\}^{\mathrm{u}}$ implies $x \in K$.

The following proposition comes from [54, Theorem 1.7(ii)].

Proposition 1.2.2. Every pre-Riesz space is directed and every directed Archimedean partially ordered vector space is pre-Riesz.

Clearly, each vector lattice is pre-Riesz. However, there are many examples of pre-Riesz spaces which are not vector lattices, see [31, 33] and the following examples.

Example 1.2.3.

(1) Let $X = \mathbb{R}^2$ with the cone defined by $K = \{(x_1, x_2);\ x_2 > 0\} \cup \{(x_1, 0);\ x_1 \geq 0\}$, then (X, K) is a pre-Riesz space but not Archimedean.

(2) Let $X = C^1[0, 1]$ be the differentiable functions space on $[0, 1]$ with the natural cone $K = \{f \in X;\ f(x) \geq 0 \text{ for all } x \in [0, 1]\}$, then (X, K) is directed and Archimedean, hence a pre-Riesz space.

(3) Let $X = \mathrm{Pol}^2(\mathbb{R})$ be the ordered vector space of all real polynomial functions on $\mathbb{R}$, ordered by the natural cone $K = \{f \in X;\ f(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$. Then (X, K) is a pre-Riesz space.

(4) Let $X = \{\alpha\mathbf{1} + v;\ \alpha \in \mathbb{R}, v \in C[0, 1], v(0) = 0, \int_0^1 v(t)dt = 0\}$ ordered by the natural cone, then X is a pre-Riesz space, where $\mathbf{1}$ denotes the constant-one function.

Definition 1.2.4. A linear subspace D of a partially ordered vector space X is called **order dense** in X if for every $x \in X$ we have

$$x = \inf\{y \in D;\ y \geq x\}.$$

Remark 1.2.5. The order denseness in Definition 1.2.4 is slightly different from the classical meaning of order denseness in vector lattices. To distinguish them, we use the term 'property (p)' for the classical definition in vector lattices. A vector sublattice G of a vector lattice E is said to have **property (p)** whenever for each $0 < x \in E$, there exists some $y \in G$ with $0 < y \leq x$, see [60]. However, Definition 1.2.4 originally comes from [14]. In vector lattices, the order denseness in the sense of Definition 1.2.4 implies the property (p), but the property (p) does not implies the order denseness, see Example [30, Example 3.3.12]. If, moreover, the vector lattice is Archimedean, then these two concepts are equivalent, [30, Example 3.3.13].

In this text, the meaning of order denseness in partially ordered vector spaces always refers to Definition 1.2.4 if not explicitly stated otherwise.

It is obvious that D is order dense in X if and only if for every $x \in X$ one has

$$x = \sup\{y \in D;\, y \leq x\}.$$

Moreover, any order dense subspace is majorizing.

Definition 1.2.6. Let X and Y be two partially ordered vector spaces. A linear map $i\colon X \to Y$ is called **bipositive** if for every $x \in X$ one has

$$i(x) \geq 0 \text{ if and only if } x \geq 0.$$

For sets $L \subseteq X$, $M \subseteq Y$ and a mapping $i\colon X \to Y$, we denote

$$i(L) := \{i(x);\, x \in L\} \text{ and } i^{-1}(M) := \{x \in X;\, i(x) \in M\}.$$

We say that a subspace X of a vector lattice Y **generates** Y as a vector lattice if for every $y \in Y$ there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in X$ such that

$$y = \bigvee_{i=1}^{m} a_i - \bigvee_{i=1}^{n} b_i.$$

It turns out that a pre-Riesz space can be always embedded as an order dense subspace in a vector lattice. This is shown by the following theorem [54, Corollaries 4.9-11 and Theorems 3.5, 3.7, 4.13], which plays a fundamental role in our research.

Theorem 1.2.7. Let X be a partially ordered vector space. The following statements are equivalent.

(i) X is a pre-Riesz space.

(ii) There exist a vector lattice Y and a bipositive linear map $i\colon X \to Y$ such that $i(X)$ is order dense in Y.

(iii) There exist a vector lattice Y and a bipositive linear map $i\colon X \to Y$ such that $i(X)$ is order dense in Y and $i(X)$ generates Y as a vector lattice.

A pair (Y, i) as in (ii) is called a **vector lattice cover** of X. All spaces Y as in (iii) are isomorphically determined as vector lattices, i.e. if j is an isomorphism from X onto an order dense subspace of a vector lattice Z, then there is an isomorphism k from Z onto Y such that $k \circ j = i$. In the sense of isomorphism, we will say (Y, i) be **the Riesz completion** of X, denoted by X^ρ.

The construction of the Riesz completion of X can be found in [54]. Throughout this text, we will use (X^ρ, i) to denote the Riesz completion of a pre-Riesz space (X, K).

Obviously, the order denseness of $i(X)$ in X^ρ implies that $i(X)$ is majorizing in X^ρ.

Example 1.2.8. (1) The Riesz completion of $X = C^1[0,1]$ with the cone $K = \{f \in X;\ f(x) \geq 0 \text{ for all } x \in [0,1]\}$ is the space of piecewise differentiable functions on $[0,1]$.

(2) The Riesz completion of $X = \mathrm{Pol}^2(\mathbb{R})$ with the cone $K = \{f \in X;\ f(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$ is the space of piecewise polynomial functions on $[0,1]$.

(3) [33, Example 4.8] Let $X = \mathbb{R}^3$ and let the positive cone K be the positive linear span of $x_1 = (1, 0, 1)$, $x_2 = (0, 1, 1)$, $x_3 = (-1, 0, 1)$, $x_4 = (0, -1, 1)$. The map $i \colon \mathbb{R}^3 \to \mathbb{R}^4$ is given by

$$i \colon x \mapsto (f_1(x), f_2(x), f_3(x), f_4(x)),$$

with

$$f_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \ f_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \ f_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ f_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Then i is a bipositive linear map and $i(X)$ is order dense in $\mathbb{R}^4$. Hence, i embeds the partially ordered vector space $(\mathbb{R}^3, K)$ into the vector lattice $(\mathbb{R}^4, \mathbb{R}^4_+)$.

Note 1.2.9. It is worth mentioning that the classical Dedekind completion is in general more involved than Riesz completions. For an Archimedean directed partially ordered vector space X, there is a Dedekind complete vector lattice Y and an order isomorphism j from X onto a subspace $j(X)$ of Y such that $j(X)$ is order dense in Y, i.e. for every $y \in Y$, one has

$$y = \sup\{j(x);\, x \in X,\, j(x) \leq y\} = \inf\{j(x);\, x \in X,\, j(x) \geq y\}. \tag{1.3}$$

(Y, j) is called a **Dedekind completion** of X, see [35, Proposition 2.1.4]. We will use (X^δ, j) to denote the Dedekind completion of X. If X is a vector lattice, then $j(X)$ is a sublattice of X^δ.

In general, the Dedekind completion is larger than the Riesz completion, see [32, Example 3.5].

The following theorem shows that Dedekind completions of a directed Archimedean partially ordered space are isomorphically determined, [54, Theorem 4.14].

Theorem 1.2.10. Let X and Y be two Archimedean directed partially ordered spaces, and $i\colon X \to Y$ a bipositive linear map such that $i[X]$ order dense in Y. Then their Dedekind completions X^δ and Y^δ are order isomorphic.

If X is not Archimedean, then X^δ can still be constructed as a partially ordered set, but it fails to be a vector space [32, p. 577]. If X is Archimedean, the Riesz completion X^ρ is the vector sublattice of X^δ generated by $j(X)$ and therefore X^ρ is Archimedean [32, Remark 3.4]. So we have the following fact which is due to [30, Proposition 1.4.7].

Proposition 1.2.11. If a pre-Riesz space (X, K) is Archimedean, then X^ρ is Archimedean as well.

Next, we will introduce the concepts of ideal and band in partially ordered vector spaces. Similar to the case of vector lattices, an ideal of a partially ordered vector space is induced by means of a solid subset.

Definition 1.2.12. Let (X, K) be a partially ordered vector space and M is a subset of X.

(i) M is called **solid** if for every $x \in X$ and $y \in M$, one has $\{x, -x\}^{\mathrm{u}} \supseteq \{y, -y\}^{\mathrm{u}}$ implies $x \in M$.

(ii) M is called **ideal** if M is a solid linear subspace of X.

Disjointness in a partially ordered vector space (X, K) is introduced in [32]. The definition of band will be given by means of disjoint complements of subsets.

Definition 1.2.13. Let X be a partially ordered vector space. Two elements $x, y \in X$ are called **disjoint**, in symbols $x \perp y$, if

$$\{x + y, -x - y\}^{\mathrm{u}} = \{x - y, -x + y\}^{\mathrm{u}}.$$

If X is a vector lattice, then this notion of disjointness coincides with the usual one in vector lattices in Definition 1.1.9. Next, we will show an example of disjoint elements in partially ordered vector spaces, which is cited from [32, Example 4.6].

Example 1.2.14. Let $X = \mathbb{R}^3$ ordered by the cone K which is the positive linear span of $x_1 = (1, 0, 1)$, $x_2 = (0, 1, 1)$, $x_3 = (-1, 0, 1)$, $x_4 = (0, -1, 1)$. Then

$$\{x_1 + x_3, -x_1 - x_3\}^{\mathrm{u}} = \{x_1 - x_3, -x_1 + x_3\}^{\mathrm{u}}.$$

Hence, $x_1 \perp x_3$. Similarly, $x_2 \perp x_4$.

The following proposition is due to [32, Proposition 2.1].

Proposition 1.2.15. Let X and Y be two partially ordered vector spaces and $x, y \in X$.

(1) If X is a subspace of Y, then $x \perp y$ in Y implies $x \perp y$ in X.

(2) If X is an order dense subspace of Y, then $x \perp y$ in Y if and only if $x \perp y$ in X.

Let X be a pre-Riesz space and (Y, i) a vector lattice cover of X. Then from the above proposition it follows that for every $x, y \in X$ we have $x \perp y$ if and only if $i(x) \perp i(y)$ in Y.

Thus we could define disjoint complements in partially ordered vector spaces similar to the vector lattice case.

Definition 1.2.16. Let X be a partially ordered vector space. The **disjoint complement** of a subset $M \subseteq X$ is the set

$$M^{\mathrm{d}} = \{y \in X;\ y \perp x \text{ for all } x \in M\}.$$

We give the definition of a band in partially ordered vector spaces.

Definition 1.2.17. A subspace B of a partially ordered vector space X is called a **band** in X if $B = B^{\mathrm{dd}}$.

For a subset $S \subseteq X$, Proposition 1.2.15 implies that

$$S^{\mathrm{d}} = i^{-1}\left(i(S)^{\mathrm{d}}\right). \tag{1.4}$$

Thus, disjoint complements in pre-Riesz spaces have properties as in the vector lattice setting, namely S^{d} is solid and o-closed, see [33, Theorem 5.10]. In particular, a disjoint complement is a linear subspace in X, more than that, it is a band.

Note that the notion of band coincides with the usual one provided X is an Archimedean vector lattice.

We will need the following technical observation. If D is a majorizing subspace of a vector lattice Y and $u \in Y$ is such that $u \perp d$ for every $d \in D$, then $u = 0$. Indeed, there is $w \in D$ such that $|u| \leq w$. Hence $|u| = |u| \wedge w = 0$, consequently $u = 0$.

Since $i(X)$ is majorizing in X^ρ, we have the following immediate result.

Lemma 1.2.18. If X is a pre-Riesz space and $u \in X^\rho$ is such that $u \perp x$ for every $x \in i(X)$, then $u = 0$.

1.3 Positive operators

In this section, we will give definitions of some different classes of operators between ordered vector spaces and list some basic properties of operators.

Definition 1.3.1. An operator $T\colon X \to Y$ between two ordered vector spaces X and Y is called **positive** if $T(x) \geq 0$ for all $x \geq 0$, denoted by $T \geq 0$ or $0 \leq T$. The vector space of all linear operators between vector spaces X and Y is denoted by $\mathcal{L}(X, Y)$. Usually $\mathcal{L}(X)$ stands for $\mathcal{L}(X, X)$

Definition 1.3.2. An operator $T\colon X \to Y$ between two ordered vector spaces X and Y is called **order bounded** if it maps order bounded subsets of X to order bounded subsets of Y. The vector space of order bounded operators between two vector spaces X and Y is denoted by $\mathcal{L}_{\mathrm{b}}(X,Y)$. Moreover $\mathcal{L}_{\mathrm{b}}(X)$ stands for $\mathcal{L}_{\mathrm{b}}(X,X)$.

Definition 1.3.3. An operator $T\colon X \to Y$ between two ordered vector spaces X and Y is called **regular** if there exists $T_1 \geq 0$ and $T_2 \geq 0$ such that $T = T_1 - T_2$. The vector space of regular operators between two vector spaces X and Y is denoted by $\mathcal{L}_{\mathrm{r}}(X,Y)$. Moreover $\mathcal{L}_{\mathrm{r}}(X)$ means $\mathcal{L}_{\mathrm{r}}(X,X)$.

With the above notations, we have the following inclusions,

$$\mathcal{L}_{\mathrm{r}}(X,Y) \subseteq \mathcal{L}_{\mathrm{b}}(X,Y) \subseteq \mathcal{L}(X,Y).$$

The inclusion $\mathcal{L}_{\mathrm{r}}(X,Y) \subseteq \mathcal{L}_{\mathrm{b}}(X,Y)$ can be proper, see [6, Example 1.16].

Definition 1.3.4. An operator $T\colon X \to Y$ between two ordered vector spaces is called **disjointness preserving** if for every $x, y \in X$ from $x \perp y$ in X it follows that $Tx \perp Ty$ in Y.

Let us recall two important extension theorems, which are frequently used in this text. The first one we cite from [6, Theorem 1.25].

Theorem 1.3.5. (Hahn-Banach) Let X be a vector space, Y a Dedekind complete Riesz space, and let $p\colon X \to Y$ be a sublinear function. If U is a subspace of X and $S\colon U \to Y$ is an operator with $S(x) \leq p(x)$ for all $x \in U$, then there exists some operator $T\colon X \to Y$ such that

(i) $T = S$ on U, i.e. T is a linear extension of S to all of X.

(ii) $T(x) \leq p(x)$ for all $x \in X$.

The second one we cite from [6, Theorem 1.32].

Theorem 1.3.6. (Kantorovich) Let X be an ordered vector space, Y a Dedekind complete Riesz space. Every positive linear operator $T\colon D \to Y$ defined on a majorizing subspace $D \subseteq X$ extends to all of X as a positive linear operator.

Chapter 2

Disjointness preserving operators on ordered vector space

Disjointness preserving operators on vector lattices have been studied by Y. A. Abramovich and A. K. Kitover [3, 4], W. Arendt [9], B. de Pagter [19], C. B. Huijsmans and A. W. Wickstead [27]. In the problem section of [26], Y. A. Abramovich raises a question: for an invertible disjointness preserving operator $T\colon X \to Y$ with X, Y being vector lattices, when does T^{-1} preserve disjointness? An affirmative answer is given by C. B. Huijsmans and B. de Pagter in [25] by showing that X being a uniformly complete vector lattice and Y a normed vector lattice is a sufficient condition, see Theorem 2.1.1.

In this chapter, we mainly deal with the above question in the case of pre-Riesz spaces. To generalize the result by C. B. Huijsmans and B. de Pagter [25], our idea is to use that every pre-Riesz space can be embedded order densely into the Riesz completion, and then we use the theory of Riesz spaces. It turns out that the main difficulty is to deal with compatibility of order convergence and norm convergence in both the pre-Riesz space and the Riesz completion. We will impose suitable conditions on the pre-Riesz space, for instance pervasive, fordable, etc..

This chapter includes two main sections.

In Section 2.1, we will discuss the generalization of the result by C. B. Huijsmans and B. de Pagter [25]. It is separated into two parts. Subsection 2.1.1 concerns the range space Y being a pre-Riesz space. To use the theory of the Riesz completion Y^ρ, it is necessary to extend the norm of Y to a Riesz norm on Y^ρ. This comes true if Y is a pervasive pre-Riesz space with a monotone norm. Then, to achieve the goal, we use the fact that two elements are disjoint in Y if and only if they are disjoint in Y^ρ. As an independent interesting result, moreover, we generalize a theorem by B. de Pagter from [19]. The second subsection deals with the situation of the domain space X being a pervasive pre-Riesz space. This turns out to be more difficult than the first part, and we have to add more conditions, e.g. a denseness condition in the sense that for a positive element there exists a positive sequence which is convergent from below.

Section 2.2 is concerned with exploring more sufficient conditions of extending disjointness preserving operators on pre-Riesz spaces to Riesz completions. One is that Riesz* homomorphisms on pre-Riesz spaces preserve disjointness. The other one is that we can show that the inverse of T is a disjointness preserving operator, provided that X is a fordable pre-Riesz space and T satisfies the condition (β). Moreover, by using the Hahn-Banach theorem, we establish that an order bounded disjointness preserving operator T can be extended to an order bounded disjointness preserving operator on the Riesz completion if the pre-Riesz space has the Riesz decomposition property.

2.1 A generalization of disjointness preserving operators on Riesz spaces

In this section, the disjointness preserving operators and their inverses between pre-Riesz spaces will be considered. An important result on inverses of disjointness preserving operators is given in [25, Theorem 2.1, Corollary 2.2], which reads as follows.

Theorem 2.1.1. Let X be a uniformly complete vector lattice, Y a normed vector lattice, and $T\colon X \to Y$ an injective and disjointness preserving operator, then $Tx_1 \perp Tx_2$ implies $x_1 \perp x_2$ in X. Moreover, if T is bijective, then T^{-1} is disjointness preserving as well.

We will use two steps to generalize this result to pre-Riesz spaces. Firstly, we consider the range space X being a pre-Riesz space, and secondly the domain space Y being a pre-Riesz space.

2.1.1 Pervasive pre-Riesz spaces as ranges

Let us recall the definition of pervasiveness in pre-Riesz spaces. The definition was firstly given by O. van Gaans and A. Kalauch [53, Definition 2.3] in studying the restriction of bands in pre-Riesz spaces.

Definition 2.1.2. A pre-Riesz space X with Riesz completion (X^ρ, i) is called **pervasive** if for every $y \in X^\rho$ with $y \geq 0$, $y \neq 0$ there exists $x \in X$, $x \neq 0$, such that $0 < i(x) \leq y$.

It should be noticed that the term of pervasive is a property of a pre-Riesz space in its Riesz completion, and the definition of property (p), which appeared in Remark 1.2.5, is a similar property in vector lattices. We just use different terms to distinguish them in different situations. Here are some examples of pre-Riesz spaces which do or do not have the pervasive property.

Example 2.1.3. (1) The pre-Riesz space $X = C^1[0,1]$ with the cone $K = \{f \in X;\, f(x) \geq 0 \text{ for all } x \in [0,1]\}$ is pervasive.

(2) The pre-Riesz space $X = \mathrm{Pol}^2(\mathbb{R})$ with the cone $K = \{f \in X;\, f(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$ is not pervasive.

(3) [30, Example 3.3.22] The pre-Riesz space $X = \{\alpha\mathbf{1} + v;\, v \in C[0,1],\, v(0) = 0,\, \alpha \in \mathbb{R},\, \int_0^1 v(t)dt = 0\}$ ordered by a natural cone is not pervasive.

It is known that for a Riesz subspace of Y of an Archimedean Riesz space X, Y has the property (p) (i.e. order dense in Riesz space) in X if and only if for each $x \in X^+$, it has $x = \sup\{y \in Y; 0 \leq y \leq x\}$, see [6, Theorem 1.34]. In the proof of this conclusion, it does not use the lattice operations. So one has the similar conclusion in the Archimedean pre-Riesz space case. In fact, this was observed by J. van Waaij [55, Theorem 4.15, Corollary 4.16], and by H. Malinowski [41, Lemma 89], see the following proposition.

Proposition 2.1.4. For an Archimedean pre-Riesz space X, let (X^ρ, i) be the Riesz completion. The following are equivalent.

(i) X is pervasive.

(ii) For all $0 < y \in X^\rho$, it holds $y = \sup\{x \in i(X); 0 < x \leq y\}$.

It yields that in Riesz spaces, pervasiveness and order denseness are the same.

Now we turn to our main purpose of this subsection, which is replacing the range space Y in Theorem 2.1.1 by a pre-Riesz space. The idea is to consider the Riesz completion (Y^ρ, i) of Y and apply Theorem 2.1.1 to $i \circ T$. For that purpose we need a Riesz norm on Y^ρ.

Recall that for an ordered vector space (X, K) with a norm $\|\cdot\|$, we say that $(X, K, \|\cdot\|)$ is an **ordered normed space**. In some cases of this thesis, we write $\|\cdot\|_X$ to emphasize the norm of X.

Definition 2.1.5. Let (X, K) be a partially ordered vector space with a seminorm p.

(i) p is called **monotone** if for every $x, y \in X$ with $0 \leq x \leq y$ one has $p(x) \leq p(y)$.

(ii) If X is a Riesz space, p is called **Riesz** if it is monotone and $p(|x|) = p(x)$ for every $x \in X$.

The next lemma presents conditions on a pre-Riesz space that provide a Riesz norm on the Riesz completion.

Lemma 2.1.6. Let X be a pervasive pre-Riesz space, let (Y,i) be a vector lattice cover of X, and let $\|\cdot\|_X$ be a monotone norm on X. Define for $y \in Y$

$$\|y\|_Y = \inf\left\{\|x\|_X\,;\, x \in X,\ |y| \leq i(x)\right\}. \tag{2.1}$$

Then $\|\cdot\|_Y$ is a Riesz norm on Y.

Proof. Let us first prove that $\|\cdot\|_Y$ is a seminorm. Let $y_1, y_2 \in Y$ and $u, v \in X$ be such that $|y_1| \leq i(u)$, $|y_2| \leq i(v)$. Then $|y_1+y_2| \leq |y_1| + |y_2| \leq i(u+v)$. So we have

$$\begin{aligned}
\|y_1+y_2\|_Y &= \inf\left\{\|u+v\|_X\,;\, u+v \in X, |y_1+y_2| \leq i(u+v)\right\}\\
&\leq \inf\left\{\|u+v\|_X\,;\, u,v \in X, |y_1| \leq i(u), |y_2| \leq i(v)\right\}\\
&\leq \inf\left\{\|u\|_X + \|v\|_X\,;\, u,v \in X, |y_1| \leq i(u), |y_2| \leq i(v)\right\}\\
&\leq \inf\left\{\|u\|_X\,;\, u \in X, |y_1| \leq i(u)\right\} + \inf\left\{\|v\|_X\,;\, v \in X, |y_2| \leq i(v)\right\}\\
&\leq \|y_1\|_Y + \|y_2\|_Y\,.
\end{aligned}$$

The absolute homogeneity and non-negativity are clear. So $\|\cdot\|_Y$ is a seminorm.

Note that since $i(X)$ is majorizing in Y, the set of which the infimum is taken in (2.1) is nonempty. The equality $\|u\|_Y = \||u|\|_Y$ is clearly true for any $u \in Y$. To show $\|\cdot\|_Y$ is monotone, we suppose $x, y \in Y$ with $|x| \leq |y|$. For $v \in X$ we have that $|y| \leq i(v)$ implies $|x| \leq i(v)$. So we have

$$\|x\|_Y = \inf\left\{\|v\|_X\,;\, v \in X,\ |x| \leq i(v)\right\} \leq \inf\left\{\|v\|_X\,;\, v \in X,\ |y| \leq i(v)\right\} = \|y\|_Y\,.$$

Therefore, $\|\cdot\|_Y$ is monotone, hence, Riesz. It remains to show that $\|\cdot\|_Y$ is a norm.

Observe that for $v \in X$, $v \geq 0$, we have for every $y \in X$ with $i(y) \geq |i(v)|$ that $y \geq v \geq 0$. As $\|\cdot\|_X$ is monotone, hence $\|y\|_X \geq \|v\|_X$. So $\|i(v)\|_Y \geq \|v\|_X$. Let $z \in Y$ be such that $z \neq 0$. Since X is pervasive, there exists $y \in X$ with

$0 < i(y) \leq |z|$. Then $\|z\|_Y = \||z|\|_Y \geq \|i(y)\|_Y \geq \|y\|_X > 0$. Hence $\|\cdot\|_Y$ is a Riesz norm. □

We can now extend Theorem 2.1.1 to a setting with the range space being a pre-Riesz space.

Theorem 2.1.7. Let X be a uniformly complete vector lattice, Y a pervasive pre-Riesz space with a monotone norm, and $T\colon X \to Y$ an injective and disjointness preserving operator. Then for every $x_1, x_2 \in X$ we have that $Tx_1 \perp Tx_2$ implies $x_1 \perp x_2$.

Proof. Let (Y^ρ, i) be the Riesz completion of Y. Since $T\colon X \to Y$ is injective, we have that $i \circ T\colon X \to Y^\rho$ is injective as well. As T is disjointness preserving, by means of Proposition 1.2.15 we have that $i \circ T$ is disjointness preserving. With the aid of Lemma 2.1.6, the monotone norm of Y yields a Riesz norm on Y^ρ.

Let $x_1, x_2 \in X$ be such that $Tx_1 \perp Tx_2$. Then $(i \circ T)x_1 \perp (i \circ T)x_2$. We apply Theorem 2.1.1 and obtain that $x_1 \perp x_2$. □

Corollary 2.1.8. Let X be a uniformly complete vector lattice, Y a pervasive pre-Riesz space with a monotone norm, and $T\colon X \to Y$ a bijective and disjointness preserving operator. Then $T^{-1}\colon Y \to X$ is disjointness preserving as well.

Proof. Let $y_1, y_2 \in Y$ be such that $y_1 \perp y_2$. Take $x_1, x_2 \in X$ with $Tx_1 = y_1$ and $Tx_2 = y_2$. We have $Tx_1 \perp Tx_2$, hence according to Theorem 2.1.7 we obtain $x_1 \perp x_2$. Therefore, T^{-1} is disjointness preserving. □

A key role in the proof of Theorem 2.1.1 is played by the next result which is due to B. de Pagter, see [19, Theorem 8].

Theorem 2.1.9 (B. de Pagter)**.** Let X be a uniformly complete Archimedean Riesz space and let Y be an Archimedean Riesz space such that for every disjoint sequence $(w_n)_n$ in Y with $w_n > 0$ ($n \in \mathbb{N}$) there exist positive real numbers λ_n

($n \in \mathbb{N}$) such that the set $\{\lambda_n w_n;\, n \in \mathbb{N}\}$ is not order bounded in Y. Then for every disjointness preserving operator $T\colon X \to Y$, there exists an order dense ideal in X on which T is order bounded.

One could try to generalize the range space in Theorem 2.1.9 to a more general pre-Riesz space Y and thus try to generalize Theorem 2.1.1. It turns out that the conditions on Y needed for this approach are not more general than those of the approach above. Nevertheless, our extension of Theorem 2.1.9 might be of independent interest.

We need the following simple observation, which follows from the fact that $i(Y)$ is majorizing in its Riesz completion Y^ρ.

Lemma 2.1.10. Let Y be a pre-Riesz space and let (Y^ρ, i) be its Riesz completion. For every subset $A \subset Y$, one has that A is order bounded in Y if and only if $i(A)$ is order bounded in Y^ρ.

We arrive at the following extension of Theorem 2.1.9.

Theorem 2.1.11. Let X be a uniformly complete Archimedean Riesz space and let Y be a pervasive Archimedean pre-Riesz space such that for every disjoint sequence $(w_n)_n$ in Y with $w_n > 0$ ($n \in \mathbb{N}$) there exist positive real numbers λ_n ($n \in \mathbb{N}$) such that the set $\{\lambda_n w_n;\, n \in \mathbb{N}\}$ is not order bounded in Y. Then for every disjointness preserving operator $T\colon X \to Y$, there exists an order dense ideal in X on which T is order bounded.

Proof. Let $T\colon X \to Y$ be a disjointness preserving operator. Let (Y^ρ, i) denote the Riesz completion of Y. By Theorem 1.2.15, we have for every $x_1, x_2 \in X$ that $i(Tx_1) \perp i(Tx_2)$ in Y^ρ if and only if $Tx_1 \perp Tx_2$ in Y, so $i \circ T\colon X \to Y^\rho$ is disjointness preserving as well.

Let $(w_n)_n$ be a disjoint sequence in Y^ρ with $w_n > 0$ ($n \in \mathbb{N}$). Since Y is pervasive, for every $n \in \mathbb{N}$ there exists $y_n \in Y$ with $0 < i(y_n) \leq w_n$. Then $(y_n)_n$ is a disjoint sequence in Y, so there exist positive real numbers λ_n ($n \in \mathbb{N}$) such that

$\{\lambda_n y_n;\, n \in \mathbb{N}\}$ is not order bounded in Y. With the aid of Lemma 2.1.10, it follows that $\{\lambda_n i(y_n);\, n \in \mathbb{N}\}$ is not order bounded and hence $\{\lambda_n w_n;\, n \in \mathbb{N}\}$ is not order bounded.

Theorem 2.1.9 now yields that there exists an order dense ideal D in X on which $i \circ T$ is order bounded. Then it follows from Lemma 2.1.10 that T is order bounded on D. □

The condition in Theorem 2.1.9 and Theorem 2.1.11 involving the disjoint sequence $(w_n)_n$ is satisfied if the space Y can be equipped with a monotone norm. The next lemma provides the details of the simple verification of this fact.

Lemma 2.1.12. If Y is a pre-Riesz space with a monotone norm $\|\cdot\|_Y$, then for every sequence $(w_n)_n$ in Y with $w_n > 0$ ($n \in \mathbb{N}$), there exist positive real numbers λ_n ($n \in \mathbb{N}$) such that the set $\{\lambda_n w_n;\, n \in \mathbb{N}\}$ is not order bounded in Y.

Proof. For the sequence $(w_n)_n$ in Y, let $\lambda_n := \frac{n}{\|w_n\|_Y}$. Then $\|\lambda_n w_n\|_Y = \frac{n}{\|w_n\|_Y} \|w_n\|_Y = n$, so $\{\lambda_n w_n;\, n \in \mathbb{N}\}$ is not norm bounded. Since the norm of Y is monotone, every order bounded set in Y is norm bounded. Hence $\{\lambda_n w_n;\, n \in \mathbb{N}\}$ is not order bounded in Y. □

Observe that if X is a Banach lattice then X is a uniformly complete Archimedean Riesz space due to P. Meyer-Nieberg [42, Proposition 1.1.8(iv)]. By combining Theorem 2.1.11 with Lemma 2.1.12, we immediately obtain the following corollary.

Corollary 2.1.13. If X is a Banach lattice and Y a pervasive pre-Riesz space with a monotone norm, then for every disjointness preserving operator $T\colon X \to Y$ there exists an order dense ideal in X on which T is order bounded.

2.1.2 Pre-Riesz spaces as domains

In this subsection, we present some results of disjointness preserving operators on pre-Riesz spaces. In the following two theorems, we consider extending disjointness preserving operators in two different ways, i.e. order continuous operators and norm continuous operators.

Recall that an operator $T\colon X \to Y$ between two ordered vector spaces is said to be **order continuous** if $Tx_\alpha \xrightarrow{\text{o}} 0$ whenever $x_\alpha \xrightarrow{\text{o}} 0$.

Theorem 2.1.14. Let X be a pre-Riesz space with the Riesz completion (X^ρ, i). Let Y be a vector lattice. Assume that for every $x \in X^\rho$ with $x \geq 0$, there exists a sequence $(x_n)_{n=1}^\infty$ in X with $i(x_n) \geq 0$ for every n such that $i(x_n) \uparrow x$. If $\widehat{T}\colon X^\rho \to Y$ is an order continuous operator such that $(\widehat{T} \circ i)\colon X \to Y$ is a positive linear disjointness preserving operator, then $\widehat{T}$ is also disjointness preserving on X^ρ.

Proof. We only need to show the conclusion holds for positive elements in X^ρ. Let $x, y \in (X^\rho)^+$ be such that $x \perp y$. Then there exist sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in X^+ such that $i(x_n) \uparrow x$ and $i(y_n) \uparrow y$. It follows immediately that $x_n \perp y_n$ in X^+. Since $\widehat{T}$ is order continuous, and the lattice operations are order continuous [35, Proposition 1.1.34], we get

$$\begin{aligned}
\widehat{T}x \wedge \widehat{T}y &= \left(\widehat{T} \lim i(x_n)\right) \wedge \left(\widehat{T} \lim i(y_n)\right) \\
&= \left(\lim \left(\widehat{T} \circ i\right) x_n\right) \wedge \left(\lim \left(\widehat{T} \circ i\right) y_n\right) \\
&= \lim \left(Tx_n \wedge Ty_n\right) = 0.
\end{aligned}$$

We conclude $\widehat{T}x \perp \widehat{T}y$. ☐

Theorem 2.1.15. Let $(X, \|\cdot\|_X)$ be a normed pre-Riesz space. Let $(Y, \|\cdot\|_Y)$ be a normed vector lattice. Let (X^ρ, i) be the Riesz completion of X with norm $\|\cdot\|_{X^\rho}$. Assume that for every $x \in X^\rho \setminus \{0\}$, $x \geq 0$, there exists an increasing sequence $(x_n)_{n=1}^\infty$ in X with $0 \leq i(x_n) \leq x$ for every n, and $\|i(x_n) - x\|_{X^\rho} \to 0$. If $\widehat{T}\colon X^\rho \to Y$ is norm continuous, $\left(\widehat{T} \circ i\right)\colon X \to Y$ is a positive linear disjointness preserving operator, then $\widehat{T}$ is also disjointness preserving on X^ρ.

Proof. Let x, y in $(X^\rho)^+$ with $x \perp y$. By assumption, there exist two increasing sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ in X with $0 \leq i(x_n) \leq x$, $0 \leq i(y_n) \leq y$ for every n, and $\|i(x_n) - x\|_{X^\rho} \to 0$, $\|i(y_n) - y\|_{X^\rho} \to 0$. Hence, $x_n \perp y_n$ in X^+ for all $n \in \mathbb{N}$. So it has $\left(\widehat{T} \circ i\right)(x_n) \perp \left(\widehat{T} \circ i\right)(y_n)$. Since $\widehat{T}$ is norm continuous, it follows that $\left\|\left(\widehat{T} \circ i\right)(x_n) - \widehat{T}(x)\right\|_Y \to 0$ and $\left\|\left(\widehat{T} \circ i\right)(y_n) - \widehat{T}(y)\right\|_Y \to 0$. By the fact that the lattice operations are norm continuous [35, Proposition 3.6.19], it has

$$\left\|\left|\left(\widehat{T} \circ i\right)(x_n)\right| \wedge \left|\left(\widehat{T} \circ i\right)(y_n)\right| - \left|\widehat{T}(x)\right| \wedge \left|\widehat{T}(y)\right|\right\|_Y \to 0.$$

Therefore $\left|\widehat{T}(x)\right| \wedge \left|\widehat{T}(y)\right| = 0$, and hence $\widehat{T}$ is disjointness preserving. □

It turns out the extension of operator as in the above two theorems is continuous.

Theorem 2.1.16. Let X be a pervasive pre-Riesz space with Riesz completion (X^ρ, i_X), let $\|\cdot\|_X$ be a monotone norm on X, and let $\|\cdot\|_{X^\rho}$ be defined as in (2.1). Let Y be a partially ordered vector space with a monotone norm $\|\cdot\|_Y$, and let $T\colon X \to Y$ be a positive and continuous linear map. Then every positive linear map $\widehat{T}\colon X^\rho \to Y$ that extends T in the sense that $\widehat{T} \circ i_X = T$ is continuous with respect to $\|\cdot\|_{X^\rho}$ and $\left\|\widehat{T}\right\| \leq \|T\|$.

Proof. As $\|\cdot\|_X$ is monotone, it follows from Lemma 2.1.6 that $\|\cdot\|_{X^\rho}$ is a Riesz norm. Let $u \in X^\rho$ and $u \geq 0$. Take $x \in X$ with $u \leq i(x)$, as T and $\widehat{T}$ are positive, it has $0 \leq \widehat{T}(u) \leq \left(\widehat{T} \circ i\right)(x) = T(x)$. Since $\|\cdot\|_Y$ is a monotone norm, we have $\left\|\widehat{T}u\right\|_Y \leq \|Tx\|_Y \leq \|T\| \, \|x\|_X$, and then $\left\|\widehat{T}u\right\|_Y \leq \|T\| \, \|u\|_{X^\rho}$. Thus T is continuous and $\left\|\widehat{T}\right\| \leq \|T\|$. □

Since a pre-Riesz space is a majorizing subspace of the Riesz completion, we could use the Kantorovich's extension theorem to extend a positive linear operator, which is from an Archimedean pre-Riesz space to a Dedekind complete Riesz space, to a positive linear operator, which is from the Riesz completion to a Dedekind complete Riesz space.

To start with, we recall that Definition 1.1.6 (iv) for uniformly complete ordered vector space X. We will use X^u to denote the uniformly completion of X.

Theorem 2.1.17. Let X be a pervasive pre-Riesz space with Riesz completion (X^ρ, i_X), let $\|\cdot\|_X$ be a monotone norm on X, and let $\|\cdot\|_{X^\rho}$ be the norm on X^ρ defined as in (2.1). Assume that for every $x \in X^\rho \setminus \{0\}$, $x \geq 0$, there exists an increasing sequence $(x_n)_n$ in X with $0 \leq i_X(x_n) \leq x$ for every n, and $\|i_X(x_n) - x\|_{X^\rho} \to 0$. Let Y be a pervasive pre-Riesz space with Dedekind completion (Y^δ, i_Y), let $\|\cdot\|_Y$ be a monotone norm on Y, and let $\|\cdot\|_{Y^\delta}$ be the norm on Y^δ defined as in (2.1). If $T\colon X \to Y$ is a positive linear map that is continuous, disjointness preserving and injective, then there exists a positive linear extension $\widehat{T}\colon X^\rho \to Y^\delta$ of $i_Y \circ T \circ i_X^{-1}\colon i_X(X) \to Y^\delta$ that is continuous, disjointness preserving and injective as well. Moreover, if X has an order unit and $X^{\rho u}$ denotes the uniform completion of X^ρ, then there exists a positive linear extension $T_u\colon X^{\rho u} \to Y^\delta$ of $i_Y \circ T \circ i_X^{-1}\colon i_X(X) \to Y^\delta$ that is disjointness preserving and injective.

Proof. Due to Theorem 1.3.6 (Kantorovich), there exists a positive operator $\widehat{T}\colon X^\rho \to Y^\delta$ extending $i_Y \circ T \circ i_X^{-1}\colon i_X(X) \to Y^\delta$. By Lemma 2.1.16, $\widehat{T}$ is continuous with respect to the Riesz norm $\|\cdot\|_{X^\rho}$ on X^ρ and $\|\cdot\|_{Y^\delta}$ on Y^δ. By Theorem 2.1.15, $\widehat{T}$ is disjointness preserving.

Next we show that $\widehat{T}$ is injective. Let $v \in X^\rho$ with $\widehat{T}v = 0$, then $\widehat{T}v^+ - \widehat{T}v^- = 0$. As $\widehat{T}$ is disjointness preserving, $\widehat{T}v^+ = 0$ and $\widehat{T}v^- = 0$. Suppose that $v \neq 0$, then either $v^+ \neq 0$ or $v^- \neq 0$. Assume without loss of generality that $v^+ \neq 0$. As X is pervasive, there is an element $x \in X \setminus \{0\}$ with $0 \leq i_X(x) \leq v^+$. So we have $0 \leq (i_Y \circ T)(x) = (\widehat{T} \circ i_X)(x) \leq \widehat{T}v^+ = 0$ and hence $Tx = 0$. This contradicts that T is injective. Therefore $v = 0$ and hence $\widehat{T}$ is injective. So $\widehat{T}\colon X^\rho \to Y^\delta$ is a positive disjointness preserving injective operator.

As X^ρ is a majorizing subspace of $X^{\rho u}$, by Theorem 1.3.6, $\widehat{T}$ can be extended to a positive operator $T_u\colon X^{\rho u} \to Y^\delta$.

We show as an intermediate step that every element in $X^{\rho u}$ can be approximated from below in the relative uniform topology by a sequence from $i(X)$. For $z \in$

$(X^{\rho u})^+$, $z \neq 0$, we have an element $u \in (X^\rho)^+$, a sequence $(z_n)_n \in (X^\rho)^+$ and a sequence $(\lambda_n)_n$ in $\mathbb{R}$ with $\lambda_n \downarrow 0$ and $|z_n - z| \leq \lambda_n u$ for every $n \in \mathbb{N}$. Then $z_n - \lambda_n u \leq z$ for every n. Take $w_n = (z_n - \lambda_n u)^+$ in X^ρ. We have $w_n \in X^\rho$, $0 \leq w_n \leq z$, and

$$|w_n - z| = |(z_n - \lambda u)^+ - z^+| \leq |z_n - \lambda_n u - z| \leq 2\lambda_n u,$$

so $w_n \to z$ in the relative uniform topology.

Next we show that T_u is disjointness preserving. Let $v, w \in X^{\rho u}$ with $v \perp w$. By the previous discussion, there exist v_n, w_n in X^ρ such that $0 \leq v_n \leq |x|$, $0 \leq w_n \leq |y|$ and $v_n \to |v|$ and $w_n \to |w|$ in the relative uniform topology. Then $0 \leq v_n \wedge w_n \leq |v| \wedge |w| = 0$, so $v_n \perp w_n$ and therefore $\widehat{T}v_n \perp \widehat{T}w_n$. Also, it follows that there exist sequences $(\alpha_n)_n$ and $(\beta_n)_n$ in $\mathbb{R}$ with $\alpha_n \downarrow 0$ and $\beta_n \downarrow 0$ and $u_1, u_2 \in X^\rho$ such that $|v_n - |v|| \leq \alpha_n u_1$ and $|w_n - |w|| \leq \beta_n u_2$ for every n. Take $u = u_1 \vee u_2$. We obtain that $|v| \leq ||v| - v_n| + |v_n| \leq \alpha_n u + v_n$ and $|w| \leq ||w| - w_n| + |w_n| \leq \beta_n u + w_n$. It follows that

$$\begin{aligned}
T_u(|v|) \wedge T_u(|w|) &\leq T_u(\alpha_n u + v_n) \wedge T_u(\beta_n u + w_n)\\
&\leq (\alpha_n T_u(u) + T_u(v_n)) \wedge (\beta_n T_u(u) + T_u(w_n))\\
&\leq ((\alpha_n \vee \beta_n) T_u(u) + T_u(v_n)) \wedge ((\alpha_n \vee \beta_n) T_u(u) + T_u(w_n))\\
&= (\alpha_n \vee \beta_n) T_u(u) + (T_u(v_n) \wedge T_u(w_n))\\
&= (\alpha_n \vee \beta_n) \widehat{T}(u) + (\widehat{T}(v_n) \wedge \widehat{T}(w_n)).\\
&= (\alpha_n \vee \beta_n) \widehat{T}(u).
\end{aligned}$$

Since Y^δ is Archimedean and $\alpha_n \vee \beta_n \downarrow 0$, we infer that $T_u(|v|) \wedge T_u(|w|) = 0$. Hence $T_u \colon X^{\rho u} \to Y^\delta$ is disjointness preserving.

By using the same argument as for $\widehat{T}$, we get that T_u is injective as well. □

Proposition 2.1.18. In the setting of Theorem 2.1.17, $T_u \colon X^{\rho u} \to T_u(X^{\rho u})$ has a disjointness preserving inverse $T_u^{-1} \colon T_u(X^{\rho u}) \to X^{\rho u}$.

Proof. Note that T_u is a positive disjointness preserving operator, hence a Riesz homomorphism. Therefore, $T_u(X^{\rho u})$ is a Riesz subspace of Y^δ. We can extend the norm of Y to a monotone norm on Y^δ by (2.1). Since T_u is injective, Corollary 2.1.8 yields that $T_u^{-1}\colon T_u(X^{\rho u}) \to X^{\rho u}$ is disjointness preserving. □

In the following example, we will give an application for Theorem 2.1.17.

Example 2.1.19. Let $m \in \mathbb{N}$ and let $C^m[0,1]$ be the subspace of $C[0,1]$ consisting of all m times continuously differentiable functions on $[0,1]$. For every $f \in C[0,1]^+$ there exists a sequence $(f_n)_n$ in $C^m[0,1]^+$ with $0 \le f_n \le f$ and $\|f_n - f\|_\infty \to 0$ and also $f_n \uparrow f$. We will give a proof of this statement in six steps below. The main idea is to approximate f for a given $\varepsilon > 0$ up to 6ε from below by a $g \in C^m[0,1]^+$, by choosing g to be 0 where $f \le 4\varepsilon$, choosing g between $f - 4\varepsilon$ and $f - 3\varepsilon$ where $f > 6\varepsilon$ and glue these pieces of g smoothly together. Some technical precautions are needed to make sure that our construction involves only finitely many subintervals and that the smooth connecting parts of g are between 0 and f.

(a) Firstly, note that for every $f \in C[0,1]$ and every $\varepsilon > 0$ there exists $g \in C^m[0,1]$ such that $\|f - g\|_\infty < 4\varepsilon$ and $g \le f - 3\varepsilon$. Indeed, according to Weierstrass's approximation theorem, there exists $g \in C^m[0,1]$ such that $\|(f - (7/2)\varepsilon) - g\|_\infty < \varepsilon/2$ and then $g < (f - (7/2)\varepsilon) + \varepsilon/2 = f - 3\varepsilon$ and $\|f - g\|_\infty < 4\varepsilon$.

(b) Secondly, observe the following elementary gluing result. If $\varepsilon > 0$ and $p, q, r, s \in [0,1]$ are such that $p < q < r < s$ and $g\colon [p,q] \cup [r,s] \to \mathbb{R}$ is a C^m function, then there exists $h \in C^m[p,s]$ such that $h = g$ on $[p,q] \cup [r,s]$ and $\min\{g(q), g(r)\} - \varepsilon \le h(t) \le \max\{g(q), g(r)\} + \varepsilon$ for every $t \in (q,r)$.

(c) Thirdly, observe that the following variations on (b) are also true. If $\varepsilon > 0$ and $p, q, r, s \in [0,1]$ are such that $p < q < r < s$ and $g\colon [p,q] \cup [r,s] \to \mathbb{R}$ is C^m, $g(q) > \varepsilon$, and $g = 0$ on $[r,s]$, then there exists $h \in C^m[p,s]$ such that $h = g$ on $[p,q] \cup [r,s]$ and $0 \le h(t) < g(q) + \varepsilon$ for every $t \in (q,r)$.

If $\varepsilon > 0$ and $p, q, r, s \in [0,1]$ are such that $p < q < r < s$ and $g\colon [p,q] \cup [r,s] \to \mathbb{R}$

is C^m, $g(r) > \varepsilon$, and $g = 0$ on $[p,q]$ then there exists $h \in C^m[p,s]$ such that $h = g$ on $[p,q] \cup [r,s]$ and $0 \leq h(t) < g(r) + \varepsilon$ for every $t \in (q,r)$.

(d) Next, let $f \in C[0,1]^+$ be such that $f(0) > 0$ and $f(1) > 0$ and let $\varepsilon > 0$. We will construct a $g \in C^m[0,1]$ such that $0 \leq g \leq f$ and $\|g - f\|_\infty \leq 6\varepsilon$. Without loss of generality we may assume that ε is so small that $f(0) > 6\varepsilon$ and $f(1) > 6\varepsilon$.

Define $\tau_0 = 0$ and for $k \in \mathbb{N}$ define, inductively,

$$\begin{aligned} \sigma_k &:= \inf\{t \in [\tau_{k-1}, 1];\ f(t) < 5\varepsilon \text{ or } t = 1\}, \text{ and} \\ \tau_k &:= \inf\{t \in [\sigma_k, 1];\ f(t) > 6\varepsilon\}. \end{aligned}$$

We have $\sigma_1 > 0$ and for $k \in \mathbb{N}$ with $\sigma_k < 1$ we have $\sigma_k < \tau_k$, since $f(0), f(1) > 6\varepsilon$ and f is continuous. If $\sigma_k = 1$, then $\tau_k = 1$. Similarly, for every $k \in \mathbb{N}$ we have $\tau_k < \sigma_{k+1}$ or $\tau_k = \sigma_{k+1} = 1$. For $k \in \mathbb{N}$ with $\sigma_k < 1$ we have $f(\sigma_k) = 5\varepsilon$ and if $\tau_k < 1$ then $f(\tau_k) = 6\varepsilon$. There exists $N \in \mathbb{N}$ such that $\sigma_k = \tau_k = 1$ for every $k \geq N$, since otherwise there would be a convergent subsequence $(\sigma_{k_j})_j$ with $\sigma_{k_j} < 1$ and hence $f(\sigma_{k_j}) = 5\varepsilon$ for every j, and then $(\tau_{k_j})_j$ would converge to the same limit while $f(\tau_{k_j}) = 6\varepsilon$, which contradicts the continuity of f.

Now we are ready to construct the desired function g. Let $k \in \mathbb{N}$ be such that $\tau_{k-1} < 1$. On $[\tau_{k-1}, \sigma_k]$, we the aid of (a), we take g to be C^m and such that

$$g \leq f - 3\varepsilon \text{ on } [\tau_{k-1}, \sigma_k]$$

and $\sup_{t \in [\tau_{k-1}, \sigma_k]} |f(t) - g(t)| < 4\varepsilon$. Since $f \geq 5\varepsilon$ on $[\tau_{k-1}, \sigma_k]$, we have

$$g > \varepsilon \text{ on } [\tau_{k-1}, \sigma_k].$$

If $\sigma_1 = 1$, then we have thus defined g on all of $[0,1]$. Otherwise, let $k \in \mathbb{N}$ be such that $\sigma_k < 1$. We will define g on (σ_k, τ_k). Recall that $\tau_k < 1$, so that g has already been defined on $[\tau_{k-1}, \sigma_k] \cup [\tau_k, \sigma_{k+1}]$. Observe that $f \leq 6\varepsilon$ on $[\sigma_k, \tau_k]$.

If $f \geq 4\varepsilon$ on $[\sigma_k, \tau_k]$, then with the aid of step (b), we take g on (σ_k, τ_k) such that

g is C^m on $[\tau_{k-1}, \sigma_{k+1}]$ and $\min\{g(\sigma_k), g(\tau_k)\} - \varepsilon \leq g(t) \leq \max\{g(\sigma_k), g(\tau_k)\} + \varepsilon$ for every $t \in (\sigma_k, \tau_k)$. As $g(\sigma_k), g(\tau_k) > \varepsilon$, it follows that $g > 0$ on (σ_k, τ_k). Since $g \leq f - 3\varepsilon \leq 3\varepsilon$ at σ_k and at τ_k, we also have that $g(t) < 4\varepsilon \leq f(t)$ for every $t \in (\sigma_k, \tau_k)$.

If we do not have that $f \geq 4\varepsilon$ on $[\sigma_k, \tau_k]$, then we define

$$\pi_k := \inf\{t \in [\sigma_k, \tau_k];\ f(t) < 4\epsilon\} \text{ and}$$
$$\rho_k := \sup\{t \in [\pi_k, \tau_k];\ f(t) < 4\varepsilon\}.$$

Note that $\pi_k < \rho_k$ and $f \geq 4\varepsilon$ on $[\sigma_k, \pi_k] \cup [\rho_k, \tau_k]$. On $[\pi_k, \rho_k]$ we take $g = 0$. Recall that $g(\sigma_k) > \varepsilon$. With the aid of step (c), we take g on (σ_k, π_k) such that g is a C^m function on $[\tau_{k-1}, \rho_k]$ and $0 \leq g(t) < g(\sigma_k) + \varepsilon$ for every $t \in (\sigma_k, \pi_k)$. Then for every $t \in (\sigma_k, \pi_k)$ we have $g(t) < f(\sigma_k) - 3\varepsilon + \varepsilon = 3\varepsilon < f(t)$.

Similarly, on (ρ_k, τ_k) we take g such that g is C^m on $[\pi_k, \sigma_{k+1}]$ and $0 \leq g(t) < g(\tau_k) + \varepsilon$ for every $t \in (\rho_k, \tau_k)$. Then for every $t \in (\rho_k, \tau_k)$ we have $g(t) < f(\tau_k) - 3\varepsilon + \varepsilon = 4\varepsilon \leq f(t)$. Thus, we have constructed g on (σ_k, τ_k) such that g is C^m, $g \geq 0$ and $g \leq f$ on $[\sigma_k, \tau_k]$. Since $f \leq 6\varepsilon$ on $[\sigma_k, \tau_k]$, it follows that $\sup_{t \in [\sigma_k, \tau_k]} |f(t) - g(t)| \leq 6\varepsilon$.

In conclusion, $g \in C^m[0,1]$, $0 \leq g \leq f$, and $\|f - g\|_\infty \leq 6\varepsilon$.

(e) We show that for every $f \in C[0,1]^+$ and $\varepsilon > 0$ there exists a $g \in C^m[0,1]$ such that $0 \leq g \leq f$ and $\|g - f\|_\infty \leq 3\varepsilon$. Due to (d), it only remains to deal with the case where $f(0) = 0$ or $f(1) = 0$. If $f \leq 3\varepsilon$ on $[0,1]$, then we can take $g = 0$, so we may assume that there exists $t \in [0,1]$ with $f(t) > 3\varepsilon$. We first consider the case where $f(0) = 0$ and $f(1) > 0$. Without loss of generality we assume that $f(1) > 2\varepsilon$. Define

$$\tau := \inf\{t \in [0,1];\ f(t) > 3\varepsilon\} \text{ and}$$
$$\sigma := \sup\{t \in [0,\tau];\ f(t) < 2\varepsilon\}.$$

Observe that $0 < \sigma < \tau$, $f \geq 2\varepsilon$ on $[\sigma, \tau]$, and $f(\tau) = 3\varepsilon$. Then, according to (d), there exists a C^m function g on $[\tau, 1]$ such that $0 \leq g \leq (f - 2\varepsilon)^+$ and $\|(f - 2\varepsilon)^+ - g\|_\infty < \varepsilon$. We take $g = 0$ on $[0, \sigma]$ and with the aid of (c) we choose g on (σ, τ) such that g is a C^m function on $[0, 1]$ and such that for every $t \in (\sigma, \tau)$ we have $0 \leq g(t) \leq g(\tau) + \varepsilon$. Then, for every $t \in (\sigma, \tau)$ we have

$$g(t) \leq g(\tau) + \varepsilon \leq (f(\tau) - 2\varepsilon)^+ + \varepsilon \leq 2\varepsilon \leq f(t).$$

Thus we have constructed a $g \in C^m[0, 1]$ such that $0 \leq g \leq f$ with $\|f - g\|_\infty \leq 3\varepsilon$.

The cases where $f(0) = 0$ and $f(1) = 0$, or $f(0) > 0$ and $f(1) = 0$ can be dealt with in a similar fashion.

(f) Let $f \in C[0, 1]^+$. We construct a sequence $(f_n)_n$ as announced above. By means of (e), we choose $g_1 \in C^m[0, 1]$ with $0 \leq g_1 \leq f$ and $\|f - g_1\|_\infty < 2^{-1}$. Inductively, for $n \in \mathbb{N}$ we choose $g_{n+1} \in C^m[0, 1]$ with $0 \leq g_{n+1} \leq f - \sum_{j=1}^n g_j$ and $\|(f - \sum_{j=1}^n g_j) - g_{n+1}\|_\infty < 2^{-(n+1)}$. Let

$$f_n := \sum_{j=1}^{n} g_j, \ n \in \mathbb{N}.$$

Then, as every g_j is positive, $(f_n)_n$ is an increasing sequence in $C^m[0, 1]^+$. Further, since $g_{n+1} \leq f - f_n$, we have $f_n \leq f - g_{n+1} \leq f$. Moreover, $\|f - f_{n+1}\|_\infty < 2^{-(n+1)} \to 0$ as $n \to \infty$. Finally, since $f_n \uparrow$ and $\|f - f_n\|_\infty \to 0$, it follows that $f = \sup_n f_n$ in $C[0, 1]$.

If $T\colon C[0, 1] \to C[0, 1]$ is an order continuous operator and $T|_{C^\infty[0,1]}$ is disjointness preserving, then T is disjointness preserving. Indeed, if $f, g \in C[0, 1]^+$ are disjoint, take $f_n, g_n \in C^\infty[0, 1]^+$ as above, then $Tf_n \perp Tg_n$ for all $n \in \mathbb{N}$. Since T is order continuous, we have $Tf_n \to Tf$ and $Tg_n \to Tg$ as $n \to \infty$, so $Tf \perp Tg$. Hence T is disjointness preserving.

In addition, if $T\colon C^\infty[0, 1] \to C^\infty[0, 1]$ is positive disjointness preserving and continuous with respect to the norm $\|\cdot\|_\infty$ and injective, then Theorem 2.1.17 yields that there exists $\widehat{T}\colon C[0, 1] \to C[0, 1]$ which is disjointness preserving and such that $\widehat{T}|_{C^\infty[0,1]} = T$.

Combining Theorem 2.1.1 and Theorem 2.1.17 we obtain the following.

Theorem 2.1.20. In the setting of Theorem 2.1.17, if $T\colon X \to Y$ is bijective, then T^{-1} is disjointness preserving.

As for a Riesz subspace Z of Y^δ, order denseness and pervasiveness are equivalent, we have that Z is pervasive in Y^δ. The disjointness preserving of T in Z and Y^δ are equivalent as well, and their norms are identical. So we can use Corollary 2.1.8 to obtain a corollary.

Corollary 2.1.21. Assume the setting of Theorem 2.1.17, and let Z be a Riesz subspace of Y^δ and $T_u\colon X^{\rho u} \to Z$. Then $(T_u)^{-1}$ is disjointness preserving.

In Theorem 2.1.15, it is required that for every $x \in (X^\rho)^+$, there exists a sequence $(x_n)_{n=1}^\infty$ in X which converges to x from below. In general, however, this condition is not always satisfied. For example, let $X = \mathrm{Pol}^2[0,1]$, then X^ρ is the subspace of $C[0,1]$ consisting of piecewise polynomial functions. It is easy to find a positive $x \in X^\rho \setminus \{0\}$ which vanishes on a subinterval of $[0,1]$ and then there is no sequence in X^+ that converges to x from below.

We note that not every disjointness preserving operator on a pre-Riesz space can be extended to its Riesz completion without some strong conditions and keep the property of being disjointness preserving. For example, if $X = \mathrm{Pol}^2[0,1]$ the operator T defined by $(Tx)(t) = \int_0^t x(s)ds, t \in [0,1]$, is not disjointness preserving on X^ρ but it is disjointness preserving on X, since two elements $x, y \in X$ are disjoint in X only if $x = 0$ or $y = 0$.

2.2 More conditions for extending disjointness preserving operators

This section is a joint work with A. Kalauch (TU Dresden).

In this section, we will explore disjointness preserving operators on pre-Riesz spaces from three different angles.

The first part is concerned with the question whether Riesz* homomorphisms between two pre-Riesz spaces are disjointness preserving operators on pre-Riesz spaces. Then in the second part, a condition which is called 'condition (β)', will be considered and it will be shown to imply that the inverse of T is disjointness preserving. Moreover, we study when an operator from a pervasive Archimedean pre-Riesz space with the Riesz decomposition property to a Dedekind complete vector lattice can be extended to an order bounded and disjointness preserving operator on the Riesz completion.

2.2.1 Riesz* homomorphism

Let us recall the definition of Riesz homomorphisms on Riesz spaces first.

Definition 2.2.1. Let X and Y be two Riesz spaces, an operator $T\colon X \to Y$ is said to be **Riesz homomorphism** if it preserves the lattice operations, i.e. whenever $T(x \wedge y) = T(x) \wedge T(y)$ holds for all $x, y \in X$.

The definition of Riesz* homomorphism on pre-Riesz spaces is given by M. van Haandel [54, Definition 5.1].

Definition 2.2.2. Let X and Y be two pre-Riesz spaces, an operator $T\colon X \to Y$ is said to be **Riesz* homomorphism** if for every $x_1, \ldots, x_n \in X$ it has $T(\{x_1, \ldots, x_n\}^{\mathrm{ul}}) \subseteq \{Tx_n, \ldots, Tx_n\}^{\mathrm{ul}}$ in Y.

The next theorem, due to M. van Haandel [54, Theorem 5.6], characterizes that the Riesz* homomorphisms on pre-Riesz spaces are exactly the restriction of Riesz homomorphisms on Riesz completions.

Theorem 2.2.3. Let $T\colon X \to Y$ between two pre-Riesz spaces be a linear operator, and X^ρ and Y^ρ the Riesz completion of X and Y, respectively. Then T

is a Riesz* homomorphism if and only if it extends to a Riesz homomorphism $T_\rho\colon X^\rho \to Y^\rho$.

Concerning disjointness preserving operators in vector lattices, Riesz homomorphisms are typical examples. Therefore, in view of Theorem 2.2.3, one expects that in pre-Riesz spaces Riesz* homomorphisms preserve disjointness.

Lemma 2.2.4. If X and Y are pre-Riesz spaces and $T\colon X \to Y$ is a bipositive Riesz* homomorphism, then for every $x, y \in X$ we have $x \perp y$ if and only if $Tx \perp Ty$.

Proof. If $Tx \perp Ty$ in Y, then it follows directly from the definition of disjointness that $Tx \perp Ty$ in $T[X]$ (see Proposition 1.2.15), so that bipositivity of T yields that $x \perp y$.

Assume $x \perp y$. Let (X^ρ, i_X) and (Y^ρ, i_Y) be the Riesz completions of X and Y, respectively. Due to Theorem 2.2.3 there is a Riesz homomorphism $T_\rho\colon X^\rho \to Y^\rho$ that extends T in the sense that $T_\rho \circ i_X = i_Y \circ T$. In particular, T_ρ is positive and disjointness preserving. Recall that $x \perp y$ in X if and only if $i_X(x) \perp i_X(y)$ in X^ρ, and similarly for elements in Y. For $x, y \in X$ with $x \perp y$ we thus have $i_X(x) \perp i_X(y)$ and therefore $T_\rho(i_X(x)) \perp T_\rho(i_X(y))$, so that $i_Y(T(x)) \perp i_Y(T(y))$, which yields that $T(x) \perp T(y)$. ☐

2.2.2 Condition (β)

In the setting of vector lattices, in [4, Definition 2.2] a condition (β) for T is introduced by means of disjoint complements, and it is shown that (β) implies that T^{-1} is disjointness preserving. We deal with the analogous condition (β) in pre-Riesz spaces, see the subsequent definition. Recall that for elements x, y of a pre-Riesz space X one has $\{x\}^{\mathrm{dd}} \subseteq \{y\}^{\mathrm{dd}}$ if and only if $x \in \{y\}^{\mathrm{dd}}$.

Definition 2.2.5. Let X and Y be pre-Riesz spaces and let $T\colon X \to Y$ be a linear operator. T is said to satisfy **condition** (β) if for every $x, y \in X$ with $\{x\}^{\mathrm{dd}} \subseteq \{y\}^{\mathrm{dd}}$ it follows that $\{Tx\}^{\mathrm{dd}} \subseteq \{Ty\}^{\mathrm{dd}}$.

The idea of this condition was introduced by B. Randrianantoanina [46]. In function spaces, (β) means the following: if the support of a function is contained in the support of another function, then the same is true for the supports of their images. We will show that (β) for T implies T^{-1} being disjointness preserving, provided X and Y are pre-Riesz spaces and X is, in addition, fordable.

Definition 2.2.6. Let X be a pre-Riesz space and (X^ρ, i) its Riesz completion. X is called **fordable** if for every $y \in X^\rho$, $y \geq 0$, there is $M \subseteq X$ such that $\{y\}^{\mathrm{d}} = i(M)^{\mathrm{d}}$.

The property 'fordable' emerges firstly in [53, Lemma 2.4]. The content of this lemma is stated as follows.

Lemma 2.2.7. Every pervasive pre-Riesz space is fordable.

However, the converse of above lemma is not true in general. In [30, Example 3.3.22], it shows that $X = \{\alpha \mathbf{1} + v;\, \alpha \in \mathbb{R}, v \in C[0,1], v(0) = 0, \int_0^1 v(t)dt = 0\}$ is a fordable pre-Riesz space, but not pervasive. In addition, the pre-Riesz space $X = \mathrm{Pol}^2(\mathbb{R})$ is not fordable, hence not pervasive, see [30, Example 3.3.23].

With the aid of the fordable condition, we show some results about disjoint elements in pre-Riesz space.

Lemma 2.2.8. Let X be a pre-Riesz space and let $x, y \in X$. Then the following statements hold:

(i) If $x \perp y$ then $\{x\}^{\mathrm{dd}} \cap \{y\}^{\mathrm{dd}} = \{0\}$.

(ii) If X is, in addition, fordable, then from $\{x\}^{\mathrm{dd}} \cap \{y\}^{\mathrm{dd}} = \{0\}$ it follows that $x \perp y$.

Proof. (i) Assume that $x \perp y$, i.e. $y \in \{x\}^{\mathrm{d}}$. Hence, $\{y\}^{\mathrm{dd}} \subseteq \{x\}^{\mathrm{ddd}} = \{x\}^{\mathrm{d}}$, which implies

$$\{x\}^{\mathrm{dd}} \cap \{y\}^{\mathrm{dd}} \subseteq \{x\}^{\mathrm{dd}} \cap \{x\}^{\mathrm{d}} = \{0\}.$$

(ii) Assume that $\{x\}^{\mathrm{dd}} \cap \{y\}^{\mathrm{dd}} = \{0\}$. Let (X^ρ, i) be the Riesz completion of X and define $u := |i(x)| \wedge |i(y)|$. Since X is fordable, there is $S \subseteq X$ such that $\{u\}^{\mathrm{d}} = i(S)^{\mathrm{d}}$ in X^ρ. From Proposition 1.2.15 it follows that

$$i^{-1}\left[\{u\}^{\mathrm{d}}\right] = i^{-1}\left[i(S)^{\mathrm{d}}\right] = S^{\mathrm{d}}. \tag{2.2}$$

We show that $S^{\mathrm{dd}} \subseteq \{x\}^{\mathrm{dd}}$. Indeed, let $z \in \{x\}^{\mathrm{d}}$, then $i(z) \perp i(x)$, hence $i(z) \perp u$. Due to (2.2), $z \in i^{-1}\left[\{u\}^{\mathrm{d}}\right] = S^{\mathrm{d}}$. It follows that $\{x\}^{\mathrm{d}} \subseteq S^{\mathrm{d}}$, and therefore $S^{\mathrm{dd}} \subseteq \{x\}^{\mathrm{dd}}$.

Analogously, one obtains $S^{\mathrm{dd}} \subseteq \{y\}^{\mathrm{dd}}$. The assumption yields $S \subseteq S^{\mathrm{dd}} \subseteq \{0\}$. Now (2.2) implies that $i(X) \subseteq \{u\}^{\mathrm{d}}$, hence from Lemma 1.2.18 it follows that $u = 0$. Consequently, $i(x) \perp i(y)$, which implies $x \perp y$. □

If X is not fordable, then Lemma 2.2.8 (ii) is not true, in general. Indeed, consider in [33, Example 4.8] the elements $x = (1,0,1)^{\mathrm{T}}$ and $y = (0,1,1)^{\mathrm{T}}$ in $\mathbb{R}^3$, then $\{x\}^{\mathrm{dd}} \cap \{y\}^{\mathrm{dd}} = \{0\}$, but $x \not\perp y$.

We continue now with the main result of this subsection.

Theorem 2.2.9. Let X and Y be pre-Riesz spaces and let $T\colon X \to Y$ be a linear injective operator.

(i) If X is, in addition, fordable and T satisfies condition (β), then $T^{-1}\colon T[X] \to X$ is disjointness preserving.

(ii) Let T be surjective and disjointness preserving. If T^{-1} is disjointness preserving then T satisfies (β).

Proof. (i) Let $y_1, y_2 \in T[X]$ be such that $y_1 \perp y_2$ in Y and let $x_1, x_2 \in X$ be such that $Tx_1 = y_1$ and $Tx_2 = y_2$. Due to Lemma 2.2.8 (i) one obtains

$$\{y_1\}^{\mathrm{dd}} \cap \{y_2\}^{\mathrm{dd}} = \{0\}.$$

Let $u \in \{x_1\}^{\mathrm{dd}} \cap \{x_2\}^{\mathrm{dd}}$. From $u \in \{x_1\}^{\mathrm{dd}}$ it follows that $\{u\}^{\mathrm{dd}} \subseteq \{x_1\}^{\mathrm{dd}}$, hence property ($\beta$) yields that $\{Tu\}^{\mathrm{dd}} \subseteq \{Tx_1\}^{\mathrm{dd}}$. Analogously, $\{Tu\}^{\mathrm{dd}} \subseteq \{Tx_2\}^{\mathrm{dd}}$, therefore

$$\{Tu\}^{\mathrm{dd}} \subseteq \{Tx_1\}^{\mathrm{dd}} \cap \{Tx_2\}^{\mathrm{dd}} = \{y_1\}^{\mathrm{dd}} \cap \{y_2\}^{\mathrm{dd}} = \{0\},$$

which yields $Tu = 0$. As T is injective it follows that $u = 0$. Thus we obtain that $\{x_1\}^{\mathrm{dd}} \cap \{x_2\}^{\mathrm{dd}} = \{0\}$. Since X is fordable, Lemma 2.2.8 (ii) yields that $x_1 \perp x_2$. Consequently, T^{-1} is disjointness preserving.

(ii) The line of reasoning here is similar to the proof of [4, Proposition 3.3]. Let $x_1, x_2 \in X$ be such that $\{x_1\}^{\mathrm{dd}} \subseteq \{x_2\}^{\mathrm{dd}}$, and assume that $\{Tx_1\}^{\mathrm{dd}} \nsubseteq \{Tx_2\}^{\mathrm{dd}}$. This means $Tx_1 \notin \{Tx_2\}^{\mathrm{dd}}$, i.e. there is a $y \in \{Tx_2\}^{\mathrm{d}}$, $y \neq 0$, such that $Tx_1 \not\perp y$. In particular, one has $y \perp Tx_2$. Since T is bijective, there is an $x \in X$, $x \neq 0$, such that $Tx = y$. Since T^{-1} is disjointness preserving, one obtains $x \perp x_2$. On the other hand, since T is disjointness preserving, one gets $x \not\perp x_1$. This contradicts the assumption, since $x \in \{x_2\}^{\mathrm{d}} \subseteq \{x_1\}^{\mathrm{d}}$. □

2.2.3 Order bounded and disjointness preserving extension

The purpose of this subsection is to extend operators on pre-Riesz spaces to Riesz completions with a different point of view than in previous sections. It turns out that this extension could preserve the order boundedness and disjointness, providing that X be a pervasive Archimedean pre-Riesz space with the Riesz decomposition property and Y a Dedekind complete vector lattice.

Definition 2.2.10. The ordered vector space (X, K) has the **Riesz decomposition property**, in short **RDP**, if for every $x_1, x_2, z \in K$ with $z \leq x_1 + x_2$, there exist $z_1, z_2 \in K$ such that $z = z_1 + z_2$ with $z_1 \leq x_1$ and $z_2 \leq x_2$.

The extension is characterized by the following theorem.

Theorem 2.2.11. Let X be a pervasive Archimedean pre-Riesz space with the RDP, Y a Dedekind complete vector lattice. If $T\colon X \to Y$ is order bounded and disjointness preserving, then there exists an extension to all of X^ρ which is order bounded and disjointness preserving.

Proof. Let $i\colon X \to X^\rho$ be the bipositive linear map as in Theorem 1.2.7. For a fixed $y \in X^\rho$ with $y > 0$, since X is pervasive there exists some $x \in X$ with $0 < i(x) \leq y$. As X is Archimedean, it follows from Proposition 2.1.4 that $y = \sup\{i(x) \in i(X); 0 < i(x) \leq y\}$. Because of $i(X)$ is majorizing in X^ρ, there exists $z \in X$ such that $y \leq i(z)$. So $i(x) \leq y \leq i(z)$ and $x \leq z$. The order boundedness of T implies that $\{Tx; x \in X, 0 < i(x) \leq y\}$ is bounded. Thus $\sup\{Tx; x \in X, 0 < i(x) \leq y\}$ exists in Y. So one can define a mapping $\widehat{T}\colon X^\rho \to Y$ via the formula

$$\widehat{T}y = \sup\{Tx;\ x \in X, 0 \leq i(x) \leq |y|\},\ y \in X^\rho. \tag{2.3}$$

Clearly, this $\widehat{T}$ is order bounded in the sense that it maps order bounded sets to order bounded sets. For $0 \leq i(x) \leq |y_1 + y_2| \leq |y_1| + |y_2|$, because X has RDP, there exist $x_1, x_2 \in X$ with $x_1 + x_2 = x$, $0 \leq i(x_1) \leq |y_1|$ and $0 \leq i(x_2) \leq |y_2|$. Thus $\widehat{T}(y_1+y_2) = \sup\{Tx; x \in X, 0 \leq i(x) \leq |y_1+y_2|\} \leq \sup\{T(x_1+x_2); x_1, x_2 \in X, 0 \leq i(x_1) \leq |y_1|, 0 \leq i(x_2) \leq |y_2|\} \leq \widehat{T}y_1+\widehat{T}y_2$. So $\widehat{T}$ is sublinear, and it is clear that $T(x) \leq \widehat{T}(x)$ holds for all $x \in X$. By Hahn-Banach extension theorem, the operator T has a linear extension S to X^ρ satisfying $S(u) \leq \widehat{T}(u)$ for all $u \in X^\rho$.

An easy argument shows that S is also order bounded. We only need to prove S is disjointness preserving. Let $y_1, y_2 \in X^\rho$ with $y_1 \perp y_2$. So $|y_1| \perp |y_2|$. By the above discussion, there exists $0 \leq x_j \in X$ such that $0 \leq i(x_j) \leq |y_j|$, $j = 1, 2$. Hence, $i(x_1) \perp i(x_2)$ and $x_1 \perp x_2$, the disjointness preserving of T implies that $Tx_1 \perp Tx_2$. Thus $\widehat{T}y_1 \perp \widehat{T}y_2$. Since $S(y_1) \leq \widehat{T}(y_1)$ and $-S(y_1) = S(-y_1) \leq \widehat{T}(-y_1) = \widehat{T}(y_1)$, we have $|S(y_1)| \leq \widehat{T}(y_1)$. Similarly, $|S(y_2)| \leq \widehat{T}(y_2)$. Hence, $|S(y_1)| \wedge |S(y_2)| \leq \widehat{T}(y_1) \wedge \widehat{T}(y_2) = 0$. Thus $S(y_1) \perp S(y_2)$. Thus we complete the proof. $\square$

It is a remarkable fact that in the case of vector lattices, every order bounded and disjointness preserving operator is regular [25]. However, this is not true for operators on pre-Riesz spaces in general. In [52] an example of an order bounded, non-regular linear functional on a directed partially ordered vector space was given. We will use this example to construct an order bounded disjointness preserving operator which is not regular.

Example 2.2.12. For $A \subseteq [0,\infty)$ let χ_A denote the corresponding indicator function. Define for $n, k \in \mathbb{N}$,

$$\begin{aligned} e_n : [0,\infty) &\to \mathbb{R}, \qquad t \mapsto \chi_{[n-1,n)}(t), \\ u_{n,k} \colon [0,\infty) &\to \mathbb{R}, \qquad t \mapsto nt\chi_{\left[0,\frac{1}{n}\right]}(t) + \tfrac{1}{k}\chi_{\left\{n+\frac{1}{k}\right\}}(t), \end{aligned}$$

and consider the subspace $X := \operatorname{span}\{e_n, u_{n,k};\, n,k \in \mathbb{N}\}$ of $\mathbb{R}^{[0,\infty)}$ with pointwise order. For every $x \in X$ there exists $t_0 > 0$ such that x is affine and, hence, differentiable on $(0, t_0)$. Define

$$T \colon X \to X, \quad x \mapsto \left(\lim_{t\downarrow 0} x'(t)\right) e_1.$$

For the sake of completeness, we list all relevant properties of X and T, where (i), (iv) and (v) are already dealt with in [52].

(i) X is directed. Indeed, every element in X is bounded and has bounded support. For $x, y \in X$ there is $n \in \mathbb{N}$ such that $x, y \leq n\sum_{i=1}^{n} e_i$, hence X is directed.

(ii) X is a pre-Riesz space. Indeed, $\mathbb{R}^{[0,\infty)}$ is Archimedean, therefore its subspace X is Archimedean as well. [54, Theorem 1.7(ii)] yields that X is a pre-Riesz space.

(iii) T is disjointness preserving. Indeed, let $x^{(1)}, x^{(2)} \in X$ with $x^{(1)} \perp x^{(2)}$. There is $M \in \mathbb{N}$ such that for $i \in \{1,2\}$

$$x^{(i)} = \sum_{n=1}^{M} \alpha_n^{(i)} e_n + \sum_{n,k=1}^{M} \lambda_{n,k}^{(i)} u_{n,k},$$

i.e. $x^{(1)}$ and $x^{(2)}$ are affine on $\left[0, \frac{1}{M}\right]$. Let without loss of generality $x^{(1)} = 0$ on $\left[0, \frac{1}{M}\right]$, then $Tx^{(1)} = 0$ and hence $Tx^{(1)} \perp Tx^{(2)}$.

(iv) T is order bounded. Indeed, for $v, w \in X$ with $v \leq w$ there are $N \in \mathbb{N}$ and $C \in (0, \infty)$ such that $\pm v, \pm w \leq C \sum_{i=1}^{N} e_i$. An element $x \in X$ with $v \leq x \leq w$ is affine on $\left[0, \frac{1}{N}\right]$, hence $-2NCe_1 \leq Tx \leq 2NCe_1$.

(v) T is not regular. Indeed, assume that there is a positive linear operator $S\colon X \to X$ such that $S \geq T$. For $n, k \in \mathbb{N}$ one has that $0 \leq u_{n,k} \leq e_1 + \frac{1}{k} e_{n+1}$, hence

$$T(u_{n,k}) = ne_1 \leq S(u_{n,k}) \leq S(e_1) + \tfrac{1}{k} S(e_{n+1}),$$

and therefore $k(ne_1 - S(e_1)) \leq S(e_{n+1})$ for every $k \in \mathbb{N}$. Since X is Archimedean, it follows that $ne_1 - S(e_1) \leq 0$. From $ne_1 \leq S(e_1)$ for every $n \in \mathbb{N}$ one obtains $e_1 \leq 0$, a contradiction.

Chapter 3

Compact operators on pre-Riesz spaces

The theory of compact operators comes originally from integral equations theory. Recall that a compact operator is an operator which sends the closed unit ball of the domain space onto a relatively compact subset of the range space, i.e. this subset has a compact closure. An alternative definition is that an operator which maps every norm bounded sequence to a sequence with a norm convergent subsequence. Compact operators have more nicer properties than arbitrary operators. For example, the spectral property by B. de Pagter [20] holds, Schauder's Fixed Point Theorem of compact maps, and moreover compact operators form a two-sided ideal in the operator algebra, see the book by J. B. Conway [17]. For these reasons compact operators are more attractive for studying.

We list some remarkable results for compact operators on Banach lattices. For Banach lattices X, Y such that X', Y both have order continuous norms, if a positive operator $S\colon X \to Y$ is dominated by a compact operator $T\colon X \to Y$, i.e. $0 \leq S \leq T$, then S is compact, this was established by P. Dodds and D. Fremlin [21]. This positive domination property is also proved by A. W. Wickstead [57] in the situations that either X' or Y is atomic with an order continuous norm. This conclusion holds only in these three cases, due to A. W. Wickstead [58]. The domination property holds for Dunford-Pettis operators as well, as was established by N. Kalton and P. Saab [36]. Moreover, due to C. D. Aliprantis and O. Burkinshaw [5], if a positive operator on a Banach lattice is dominated by a compact operator, then its third power is a compact operator.

In this chapter, we mainly consider the similar positive compact domination property on pre-Riesz spaces. Namely, we explore under which suitable conditions for pre-Riesz spaces X and Y, we have that every positive operator dominated by a compact operator is compact. We also address the question whether a similar result concerning the third power of the operator is true for operators on pre-Riesz spaces.

To use the theory of pre-Riesz spaces and Riesz completions, one natural question is how we extend a norm on a pre-Riesz space to an order continuous norm on its Riesz completion. In [50, Theorem 3.43], the author states that a regular seminorm on a pre-Riesz space can be extended to a Riesz seminorm on the Riesz completion. So the difficulty comes down to the following. For an arbitrary element in the Riesz completion, find a directed net in the pre-Riesz space that order converges to it. In Section 3.1, we will settle this by providing the condition that the pre-Riesz space is pervasive Archimedean with the Riesz decomposition property.

Section 3.2 is concerned with a unique extension of an operator defined on a pre-Riesz space to its Dedekind completion, based on norm denseness and by means of order continuous norms.

In Section 3.3, we investigate how the positive domination property of compact operators introduced by P. Dodds and D. Fremlin in [21], and the theory of the third power compact operators by C. D. Aliprantis and O. Burkinshaw [5] can be generalized to operators on pre-Riesz spaces. We present some partial results.

3.1 Extension of order continuous norms

A seminorm p on an ordered vector space (X, K) is said to be **order continuous** if $x_\alpha \xrightarrow{o} x$ implies $p(x_\alpha - x) \to 0$.

Recall that Lemma 2.1.6 in Chapter 2 yields that for a pervasive pre-Riesz space with a monotone norm there exists a Riesz norm on the Riesz completion of the pre-Riesz space. In fact, such a norm can be obtained by a regular norm.

Definition 3.1.1. Let X be a directed partially ordered vector space with a seminorm p, then we say that p is **regular** if for every $x \in X$,

$$p(x) = \inf\{p(y);\, y \in X, -y \leq x \leq y\}. \tag{3.1}$$

On vector lattices, regular seminorms and Riesz seminorms coincide, this result is due to O. van Gaans [50, Theorem 3.40]. Moreover, due to [50, Theorem 3.43 and Corollary 3.45], one can extend the seminorm on a pre-Riesz space in the following way.

Theorem 3.1.2. Let (X, K) be a directed partially ordered vector space with a seminorm p. Let Y be a directed partially ordered vector space and $i\colon X \to Y$ a bipositive linear map, such that $i(X)$ is majorizing in Y. Define

$$p_r(y) := \inf\{p(x);\, x \in X \text{ such that } -i(x) \leq y \leq i(x)\},\ y \in Y. \tag{3.2}$$

The following statements hold.

(i) p_r is the greatest regular seminorm on Y with $p_r \leq p$ on K.

(ii) $p_r = p$ on K if and only if p is monotone. Moreover, if p is monotone, then $p_r \geq \frac{1}{2}p$.

(iii) $p_r = p$ on X if and only if p is regular.

Remark 3.1.3. There are more situations in which regular norms are used. In the study of operator theory, if X and Y are two Banach lattices, then the space $\mathcal{L}_{\mathrm{b}}(X,Y)$ of all order bounded operators is merely a partially ordered vector space, and the operator norm is not a Riesz norm. As a subspace of $\mathcal{L}_{\mathrm{b}}(X,Y)$, the regular operator space $\mathcal{L}_{\mathrm{r}}(X,Y)$ is not complete with respect to the operator norm, but it is a Banach space for the regular norm $\|T\|_r := \inf\{\|S\|;\, S \in \mathcal{L}^+(X,Y), |Tx| \leq Sx, x \in X^+\}$, $T \in \mathcal{L}_{\mathrm{r}}(X,Y)$. If Y is additionally Dedekind complete, then $(\mathcal{L}_{\mathrm{r}}(X,Y), \|\cdot\|_r)$ is a Banach lattice, see [42, Proposition 1.3.6]. For more results on regular norms we refer the reader to the work of Y. A. Abramovich, Z. Chen, and A. W. Wickstead [2], W. Arendt [8], Z. Chen and A. W. Wickstead [15] and A. W. Wickstead [59].

Before we consider the extension of order continuous norms on pre-Riesz spaces, we point out the fact that for every pre-Riesz space X with the Riesz completion (X^ρ, i), the subspace $i(X)$ is order dense in X^ρ, but for a positive element in X^ρ, it is difficult to find an upward directed net of positive elements in $i(X)$ such that this net is order convergent to it, even if X is pervasive. In the following discussion, we try to construct an increasing net in $i(X)$ explicitly by using the Riesz decomposition property. The next result is due to B. Z. Vulikh [56, Lemma V.1.1].

Proposition 3.1.4. Let (X,K) be an ordered vector space. The following statements are equivalent:

(i) X has the Riesz decomposition property.

(ii) For arbitrary $x_1, x_2, x_3, x_4 \in X$ with $x_1, x_2 \leq x_3, x_4$, there exists $z \in X$ such that $x_1, x_2 \leq z \leq x_3, x_4$.

Notice that in Proposition 2.1.4, we have that if X is a pervasive Archimedean pre-Riesz space, then for an arbitrary $x \in (X^\rho)^+$, we can find some elements in $i(X)^+$ for which x is the supremum. If we assume that X has additionally the Riesz decomposition property, then we can construct a net which order converges to x.

Theorem 3.1.5. Let X be a pervasive Archimedean pre-Riesz space with the RDP, let (X^ρ, i) be the Riesz completion of X. For every $0 < y \in X^\rho$, if there exist $a_1, \ldots, a_n \in X^+$ such that $y = \bigwedge_{j=1}^n i(a_j)$, then there exists an increasing net in $i(X)^+$ which is order convergent to y. Moreover, for every $y \in X^\rho$ there exists a net in $i(X)$ which is order convergent to y.

Proof. Let $0 < y \in X^\rho$ and $y = \bigwedge_{j=1}^n i(a_j)$, $a_1, \ldots, a_n \in X^+$. As X is pervasive, the set

$$D = \{u \in i(X)^+;\, 0 < u \leq y\}$$

is not empty.

Let $u_1, u_2 \in D$ with $u_1, u_2 \leq a_1, \ldots, a_n$ in X. Since X has the RDP, there exists $u_3 \in i(X)$ such that

$$u_1, u_2 \leq u_3 \leq a_1, \ldots, a_n.$$

So $u_3 \leq \bigwedge_{k=1}^n i(a_k) = y_1$. Hence $u_3 \in D$. This implies that D is upward directed. Define $(z_\alpha)_{\alpha \in D}$ by $z_\alpha := \alpha$, $\alpha \in D$, clearly $(z_\alpha)_\alpha$ is an upward directed net in $i(X)$. As X is Archimedean, by Proposition 2.1.4 we have $y = \sup D$.

Moreover, let $y \in X^\rho$. If $y = 0$, it is obvious that we can define a net consisting of elements with value 0 such that it is order convergent to y. Let $y \neq 0$. Since $i(X)$ is an order dense subspace of X^ρ and generates X^ρ as a vector lattice, there exist a_j and b_k in X^+, $j = 1, \ldots, n$ and $k = 1, \ldots, m$, such that $y = \bigwedge_{j=1}^n i(a_j) - \bigwedge_{k=1}^m i(b_k)$. Let $y_1 = \bigwedge_{j=1}^n i(a_j)$, $y_2 = \bigwedge_{k=1}^m i(b_k)$, then $y_1, y_2 \in (X^\rho)^+$. Obviously, y_1 and y_2 can not be zero at the same time. Without loss of generality, let $y_1 \neq 0$. By the previous discussion, there exists $(z_\alpha)_\alpha$ in $i(X)^+$ with $z_\alpha \xrightarrow{o} y_1$. By the same reasoning, there exists a net $(w_\beta)_\beta$ with $w_\beta \xrightarrow{o} y_2$.

Define $s_{(\alpha,\beta)} = z_\alpha - w_\beta$. We claim that $s_{(\alpha,\beta)} \xrightarrow{o} y_1 - y_2$. In fact, there exist nets $(u_\alpha)_\alpha$, $(v_\beta)_\beta$ in $i(X)^+$ such that $-u_\alpha \leq z_\alpha - y_1 \leq u_\alpha$, $-v_\beta \leq w_\beta - y_2 \leq v_\beta$ and $u_\alpha \downarrow 0$, $v_\beta \downarrow 0$. So $-(u_\alpha + v_\beta) \leq (z_\alpha - w_\beta) - (y_1 - y_2) \leq u_\alpha + v_\beta$ and $(u_\alpha + v_\beta) \downarrow 0$. Hence $s_{(\alpha,\beta)} \xrightarrow{o} y_1 - y_2$. This completes the proof. $\square$

Remark 3.1.6. Clearly, for $x = \bigvee_{k=1}^{n} i(a_k)$ in $(X^\rho)^+$, and $z \in i(X)$ with $z - x \in (X^\rho)^+$, by the proof of Theorem 3.1.5, one has a net $(u_\alpha)_{\alpha \in D}$ in $i(X)^+$ which is upward directed and convergent to $z - x$. Define $v_\alpha = z - u_\alpha$ for every $\alpha \in D$, then the net $(v_\alpha)_{\alpha \in D} \subseteq i(X)$ is downward directed and convergent to x.

In the next lemma, based on Zorn's lemma, it is shown that for a positive decreasing net in the Riesz completion (X^ρ, i) of a pre-Riesz space X, if its infimum exists, then there exists a downward directed net in $i(X)^+$ such that their infimum are equal, it is due to [35, Lemma 3.7.11].

Lemma 3.1.7. Let X be a pervasive Archimedean pre-Riesz space and (Y, i) a vector lattice cover of X. Let $(y_\alpha)_{\alpha \in I}$ be a net in Y such that $y_\alpha \downarrow 0$. There exists a net $(x_\beta)_{\beta \in J}$ in X with $x_\beta \downarrow 0$ and such that for every $\beta \in J$ there exists $\alpha_0 \in I$ such that for every $\alpha \geq \alpha_0$ we have $i(x_\beta) \geq y_\alpha$.

This lemma will be used to prove the following theorem, which extends order continuous norms on pre-Riesz spaces. Recall that a norm $\|\cdot\|$ on a partially ordered vector space X is called **semimonotone** if there exists $C \in \mathbb{R}^+$ such that for every $x, y \in X$ with $0 \leq x \leq y$ one has $\|x\| \leq C\|y\|$.

Theorem 3.1.8. Let X be a pervasive Archimedean pre-Riesz space, p a semimonotone seminorm on X, and (Y, i) a vector lattice cover of X. Let p_r on Y be defined by

$$p_r(y) := \inf\{p(x);\, x \in X \text{ such that } -i(x) \leq y \leq i(x)\},\ y \in Y.$$

If p on X is order continuous, then p_r is an order continuous seminorm on Y as well.

Proof. By Theorem 3.1.2 (i) it follows that p_r is a seminorm on Y. We show that p_r is order continuous.

Assume that $x_\alpha \xrightarrow{o} x$ in Y, i.e. there exists a net $(y_\beta)_{\beta \in B} \subset Y$ such that for every β there is α_0 such that for every $\alpha \geq \alpha_0$ we have $\pm(x_\alpha - x) \leq y_\beta \downarrow 0$. As by Theorem

3.1.2 (i) the seminorm p_r is regular we have $p_r(x_\alpha - x) \leq p_r(y_\beta)$. By Lemma 3.1.7 there exists a net $(z_\gamma)_\gamma \subset X$ such that for every γ there is β_0 such that for every $\beta \geq \beta_0$ we have $i(z_\gamma) \geq y_\beta$ and $z_\gamma \downarrow 0$. Since p is order continuous, $p(z_\gamma) \downarrow 0$. Since p is semimonotone, there is a constant C such that $p_r(i(v)) \leq Cp(v)$ for every $v \in X$. Then $p_r(x_\alpha - x) \leq p_r(y_\beta) \leq p_r(i(z_\gamma)) \leq Cp(z_\gamma) \to 0$. Hence p_r is order continuous on Y. □

Theorem 3.1.8 and Theorem 3.1.5 together yield the following corollary.

Corollary 3.1.9. Let X be a pervasive Archimedean pre-Riesz space with RDP with an order continuous semimonotone seminorm $\|\cdot\|_X$. Let (X^δ, j) be the Dedekind completion of X, and define $\|\cdot\|_{X^\delta}$ on X^δ by

$$\|y\|_{X^\delta} := \inf\{\|x\|_X;\ x \in X \text{ such that } -j(x) \leq y \leq j(x)\}.$$

Then $\|\cdot\|_{X^\delta}$ is an order continuous seminorm and $(X, \|\cdot\|_X)$ is norm dense in $(X^\delta, \|\cdot\|_{X^\delta})$. Furthermore, $\|\cdot\|_{X^\delta}$ extends $\|\cdot\|_X$ if $\|\cdot\|_X$ is regular.

Proof. The order continuity of $\|\cdot\|_{X^\delta}$ is same with the Theorem 3.1.8. It remains to show that $(X, \|\cdot\|_X)$ is norm dense in $(X^\delta, \|\cdot\|_{X^\delta})$. Let (X^ρ, i) be the Riesz completion of X. For every $y \in X^\rho$, according to Theorem 3.1.5, there is a net (x_α) in $i(X)$ with $x_\alpha \xrightarrow{o} y$. By Theorem 3.1.8, $\|\cdot\|_{X^\rho}$ is order continuous, and then $\|x_\alpha - y\|_{X^\rho} \to 0$, so that X is norm dense in X^ρ. Let $y \in (X^\delta)^+$, and $D := \{x \in X^\rho;\ 0 \leq x \leq y\}$. Since X^ρ is pervasive in X^δ and X^ρ is Archimedean, hence, by Proposition 2.1.4, we have D is upward directed and $\sup D = y$. Define the net $(z_\mu)_\mu$ by $z_\mu := \mu$, $\mu \in D$. Therefore $z_\mu \xrightarrow{o} y$ and $\|z_\mu - y\|_{X^\delta} \to 0$. Thus we conclude X^ρ is norm dense in X^δ. It follows that X is norm dense in X^δ.

Moreover, by Theorem 3.1.2 (iii) the seminorm $\|\cdot\|_{X^\delta}$ extends $\|\cdot\|_X$ if (and only if) $\|\cdot\|_X$ is regular. □

3.2 Extension of compact operators

In this section, we will show that a compact operator on a pre-Riesz space X with a suitable order continuous norm can be extended to a compact operator on the Dedekind completion X^δ of X. To make sure that the norm on X^δ indeed is an extension of the order continuous norm on X, the norm of X is required to be regular. Then we will use Corollary 3.1.9 to extend the operator, see the following theorem.

Theorem 3.2.1. Let $(X, \|\cdot\|_X)$ be a pervasive Archimedean pre-Riesz space with RDP equipped with an order continuous regular seminorm. Let (X^δ, i) be the Dedekind completion of X. Let Y be a Banach lattice with an order continuous norm. If T is a bounded operator in $L(X, Y)$, then there exists a unique bounded linear extension $\widehat{T} \in L(X^\delta, Y)$. If T is compact, then $\widehat{T}$ is compact as well.

Proof. Recall that by Theorem 3.1.2 (i) there exists the greatest regular seminorm $\|\cdot\|_{X^\delta}$ on X^δ which extends $\|\cdot\|_X$. Due to Corollary 3.1.9, X is dense with respect to the seminorm in X^δ. So there exists $\widehat{T} \in L(X^\delta, Y)$ which uniquely extends T by means of for $z \in X^\delta$, $\widehat{T}z = \lim Tz_n$, where $(z_n)_n \subseteq X$ norm converges to z.

Let $(x_n)_n$ be a norm bounded sequence in X^δ. It follows from the norm denseness of X in X^δ that there exists a norm bounded sequence $(y_n)_n$ in X such that $\|i(y_n) - x_n\|_{X^\delta} < \frac{1}{n}$ holds for all $n \in \mathbb{N}$. As T is compact, $(Ty_n)_n$ has a convergent subsequence $(Ty_{n_k})_{n_k}$. So there exists $y \in Y$ such that for $k \to \infty$ we have

$$\|Ty_{n_k} - y\|_Y \to 0.$$

For the subsequence $\left(\widehat{T}x_{n_k}\right)$ of $\left(\widehat{T}x_n\right)$, one has

$$\begin{aligned}\left\|\widehat{T}x_{n_k} - Ty_{n_k}\right\|_Y &= \left\|\widehat{T}x_{n_k} - \widehat{T}(i(y_{n_k}))\right\|_Y \\ &\leq \left\|\widehat{T}\right\| \|x_{n_k} - i(y_{n_k})\|_{X^\delta} \\ &\leq \left\|\widehat{T}\right\| \frac{1}{n_k} \to 0\end{aligned}$$

as $k \to \infty$. Then

$$\left\| \widehat{T}x_{n_k} - y \right\|_Y \leq \left\| \widehat{T}x_{n_k} - Ty_{n_k} \right\|_Y + \|Ty_{n_k} - y\|_Y \to 0$$

as $k \to \infty$. So $\left(\widehat{T}x_n\right)$ has a convergent subsequence, and hence $\widehat{T}$ is compact. □

3.3 Compact domination results in pre-Riesz spaces

In this section, we will extend two results concerning the domination property of positive compact operators on Banach lattices to the setting of pre-Riesz spaces. We will consider appropriate norms and use Riesz completions or Dedekind completions of pre-Riesz spaces. In order to extend a compact operator on a pre-Riesz space to a compact operator on the Dedekind completion, we will use Theorem 3.2.1. Let us recall some preliminaries first.

Let (X, τ) be a topological vector space. The **topological dual** X' of (X, τ) is the vector space consisting of all τ-continuous linear functionals on X. For a partially ordered vector space X, the vector space of all order bounded linear functionals is called the **order dual** of X, and denoted by $X^{\sim}$. The next theorem is due to [6, Theorem 3.49].

Theorem 3.3.1. The topological dual X' of a locally convex-solid Riesz space (X, τ) is an ideal in its order dual $X^{\sim}$.

The classical domination property of compact operators between Banach lattices by [6, Theorem 5.20], which reads as follows.

Theorem 3.3.2. (Dodds-Fremlin) Let X be a Banach lattice, Y a Banach lattice with X' and Y having order continuous norms. If a positive operator $S\colon X \to Y$ is dominated by a compact operator, then S is a compact operator.

Remark 3.3.3. This result is proved originally by P. G. Dodds and D. H. Fremlin in [21]. An easier accessible proof is given in [6, Theorem 5.20]. In fact, note that in the proof of [6, Theorem 5.20], it does not use the norm completeness of X. As a consequence, we have the following corollary.

Corollary 3.3.4. Let X be a normed Riesz space, Y a Banach lattice with X' and Y having order continuous norms. If a positive operator $S\colon X \to Y$ is dominated by a compact operator, then S is a compact operator.

Now our aim comes down to extending Corollary 3.3.4 to pre-Riesz spaces. We firstly consider the domain space X to be a pre-Riesz space, and we suppose it is pervasive and Archimedean. Alternatively, if X is not norm complete, one could consider the order continuous regular seminorm on X, then use the same argument as in Theorem 3.2.1 and apply Corollary 3.3.4 to obtain compactness of S.

Theorem 3.3.5. Let X be a pervasive Archimedean pre-Riesz space with RDP equipped with an order continuous regular norm, (X^δ, i) the Dedekind completion of X. Let Y be a Banach lattice with an order continuous norm. Let T in $L(X,Y)$ be a positive compact operator and assume that $(X^\delta)'$ has order continuous norm. If $S \in L(X,Y)$ with $0 \leq S \leq T$, then S is compact.

Proof. Recall that by Theorem 3.1.2 (i) there exists the greatest regular seminorm on X^δ extending the norm on X. By Theorem 3.2.1 there exists a unique bounded linear extension $\widehat{T}$ of T on X^δ, which is compact. In fact, for $x \in X^\delta$, $\widehat{T}$ is given by $\widehat{T}x = \lim Tx_n$, where $(x_n)_n \subseteq X$ norm converges to x. We define $\widehat{S}$ in a similar way. So for $x \in (X^\delta)^+$ we have $(x_n)_n \subseteq X^+$ and $Sx_n \leq Tx_n$ for all $n \in \mathbb{N}$. It follows that

$$\widehat{S}x = \lim Sx_n \leq \lim Tx_n = \widehat{T}x.$$

The positivity of $\widehat{S}$ is clear. By Lemma 2.1.6 the seminorm on X^δ is actually a norm. Due to Corollary 3.3.4, we have $\widehat{S}$ is compact. Hence $S = \widehat{S}|_X$ is compact. □

For an operator between two pre-Riesz spaces X and Y, even though we could extend it to the Riesz completion, for the method in the proof of Theorem 3.3.2, we also need that the norms of $(X^\delta)'$ and Y^δ are order continuous. It is difficult to characterize the space structure of dual spaces of pre-Riesz spaces, for example, whether $(X')^\rho$ equals $(X^\rho)'$ or not. Fortunately, if we suppose that the codomain space Y is a directed Archimedean partially ordered vector space and complete with respect to the regular norm, then it has a Dedekind completion which is norm complete as well. Then we can embed Y into the Dedekind completion, and use Theorem 3.3.5, see the following two theorems.

Theorem 3.3.6. Let X be a directed Archimedean partially ordered vector space with a regular norm $\|\cdot\|$ such that X^+ is $\|\cdot\|$-closed. Let (X^δ, i) be the Dedekind completion of X, and define for $y \in X^\delta$,

$$\|y\|_r := \inf\{\|x\|;\, x \in X^+, -i(x) \leq y \leq i(x)\}.$$

If $(X, \|\cdot\|)$ is complete, then $(X^\delta, \|\cdot\|_r)$ is complete.

Proof. The proof is similar to [50, Theorem 2.12]. Assume that $(X, \|\cdot\|)$ is complete. Let $(y_n)_n$ be a sequence in X^δ such that $\sum_{n=1}^\infty 2^n \|y_n\|_r < \infty$. We show that there exists $y \in X^\delta$ such that $\left\|y - \sum_{n=1}^N y_n\right\|_r \to 0$ as $N \to \infty$. Take $x_n \in X^+$ with $-i(x_n) \leq y_n \leq i(x_n)$ and $\|y_n\|_r \geq \|x_n\| - \frac{1}{4^n}$. Then $\sum_{n=1}^\infty 2^n \|x_n\| < \infty$. Since X is norm complete, $u := \sum_{n=1}^\infty 2^n x_n$ exists in X. As for every $n \in \mathbb{N}$ we have $x_n \in X^+$ and X^+ is closed, we have $u \in X^+$ and $u \geq \sum_{n=1}^m 2^n x_n \geq 2^n x_n$ for every $m \in \mathbb{N}$. Now for $N > M$, we have $\sum_{n=1}^N y_n - \sum_{n=1}^M y_n = \sum_{n=M+1}^N y_n \leq \sum_{n=M+1}^N i(x_n) \leq \left(\sum_{n=M+1}^N 2^{-n}\right) i(u)$, and $\sum_{n=1}^N y_n - \sum_{n=1}^M y_n \geq -\left(\sum_{n=M+1}^N 2^{-n}\right) i(u)$. Since $\sum_{n=M+1}^N 2^{-n} \to 0$ when $M, N \to \infty$, we have that $\left(\sum_{n=1}^N y_n\right)_N$ is a relatively uniformly Cauchy sequence. Since X^δ is Dedekind complete and hence relatively uniformly complete, there exists $y \in X^\delta$ and there is a sequence $(\lambda_N)_N$ of reals with $\lambda_N \to 0$ such that $-\lambda_N i(u) \leq y - \sum_{n=1}^N y_n \leq \lambda_N i(u)$. Then $\left\|y - \sum_{n=1}^N y_n\right\|_r \leq \lambda_N \|u\| \to 0$ as $N \to \infty$. Thus, $\left(X^\delta, \|\cdot\|_r\right)$ is complete. □

Then we will proof the positive domination property of compact operators on pre-Riesz spaces in the following theorem.

Theorem 3.3.7. Let X be a pervasive Archimedean pre-Riesz space with RDP equipped with an order continuous regular norm. Let Y be a directed Archimedean pre-Riesz space with an order continuous regular norm $\|\cdot\|_Y$ that Y is norm complete. Let X^δ be the Dedekind completion of X and $(X^\delta)'$ has order continuous norm. If a positive operator $S\colon X \to Y$ is dominated by T, i.e. $0 \leq S \leq T$, and T is compact, then S is compact as well.

Proof. Let Y^δ be the Dedekind completion of Y, and $i\colon Y \to Y^\delta$ be the natural embedding map. Since i is bipositive, we have $0 \leq i \circ S \leq i \circ T$ form X to Y^δ. As i is continuous and T is compact, $i \circ T$ is compact. By Theorem 3.1.2 (i) there exists the greatest regular seminorm on Y^δ (in fact, since Y has a norm, the seminorm on Y^δ is a norm) extending the norm on Y. Because Y has order continuous norm, by Theorem 3.1.8 the space Y^δ has order continuous norm as well, and by Theorem 3.3.6 the space Y^δ is norm complete. It follows from Theorem 3.3.5 that $i \circ S$ is compact. Let $(x_n)_n$ be a norm bounded sequence in X, then $(i \circ S)(x_n)_n$ has a norm convergent subsequence $(i \circ S)(x_{n_k})_{n_k}$ in Y^δ. Then $(i \circ S)(x_{n_k})_{n_k}$ is a Cauchy sequence, so $S(x_{n_k})_{n_k}$ is a Cauchy sequence in Y. Since $\|\cdot\|_Y$ is regular, by Theorem 3.1.2 (iii) we have $\|y\|_Y = \|i(y)\|_{Y^\delta}$ for every $y \in Y$. As Y is norm complete, $S(x_{n_k})$ is convergent in Y. Hence S is compact. □

Beside the domination property of compact operators as in Corollary 3.3.4, there is also a result in [5, Theorem 5.13] which clarifies the positive domination property of third power of compact operators on Banach lattices, see the following theorem.

Theorem 3.3.8. (Aliprantis-Burkinshaw) If a positive operator S on a Banach lattice is dominated by a compact operator, then S^3 is a compact operator.

Now we may ask whether or not this result can be extended to pre-Riesz spaces. The answer is affirmative if we additionally suppose, among other conditions, that the pre-Riesz space has an order unit and an order unit norm. To establish this result, let us recall some known preliminaries.

Recall that a subset A in a topological vector space (X,τ) is called **τ-totally bounded**, if for every τ-neighborhood V of zero there is a finite subset Φ of A such that $A \subseteq \bigcup_{x\in\Phi}(x+V) = \Phi + V$.

The following result can be found in [6, Theorem 3.3], and the next one is in [6, Theorem 5.10].

Theorem 3.3.9. Let $T\colon (X,\tau) \to (Y,\xi)$ be an operator between two topological vector spaces. If T is continuous on the τ-bounded subsets of X, then T carries τ-totally bounded sets to ξ-totally bounded sets.

Theorem 3.3.10. (Dodds-Fremlin) Let X and Y be two Riesz spaces with Y Dedekind complete. If τ is an order continuous locally convex solid topology on Y, then for each $x \in X^+$, the set

$$B = \{T \in L_b(X,Y);\ T[0,x] \text{ is } \tau\text{-totally bounded}\}$$

is a band in $L_b(X,Y)$.

Recall that for a Riesz space X and its order dual $X^\sim$, the **absolute weak topology** on X is defined by a collection of seminorms p_f via the formula

$$p_f(x) = |f|(|x|), \quad x \in X, f \in X^\sim,$$

and it is denoted by $|\sigma|(X,X^\sim)$. For a nonempty subset A of $X^\sim$, the **absolute weak topology generated by** A on X is the locally convex solid topology on X generated by the seminorms p_f defined via the formula

$$p_f(x) = |f|(|x|), \quad x \in X,\ f \in A.$$

Consider a dual system $\langle X, X'\rangle$. A locally convex topology τ on X is said to be **consistent** with the dual system if the topological dual of (X,τ) is precisely X'.

Definition 3.3.11. Let X be a Riesz space, and let X' be an ideal of $X^\sim$ separating the points of X. Then the pair $\langle X, X'\rangle$, under its natural duality

$\langle x, x' \rangle := x'(x)$, is said to be a **Riesz dual system**.

Let us recall a result by S. Kaplan [37, Theorem 3.50].

Theorem 3.3.12. (Kaplan) Let X be a Riesz space, and let A be a subset of $X^{\sim}$ separating the points of X. Then the topological dual of $(X, |\sigma|(X, A))$ is precisely the ideal generated by A in $X^{\sim}$.

As a result of the above theorem, we have the following corollary.

Corollary 3.3.13. Let X be a Riesz space and $\langle X, X' \rangle$ a Riesz dual system. Then the topological dual of $(X, |\sigma|(X, X'))$ is precisely X', and $|\sigma|(X, X')$ is consistent with $\langle X, X' \rangle$.

Let us recall some known results with respect to the Riesz dual system. The following one is due to [6, Theorem 3.57].

Theorem 3.3.14. For a Riesz dual system $\langle X, X' \rangle$, the following statements are equivalent.

(i) X is Dedekind complete and $\sigma(X, X')$ is order continuous.

(ii) X is an ideal of X''.

The following theorem is due to [6, Theorem 3.54].

Theorem 3.3.15. For a Riesz dual system $\langle X, X' \rangle$, the following statements are equivalent.

(i) Every consistent locally convex solid topology on X is order continuous.

(ii) $\sigma(X, X')$ is order continuous.

Also, recall that for a subset D of a topological vector space (X, τ), the **restriction topology**, in short r-topology, on D is such that $U \subseteq D$ is r-open if and only if there exists a $V \subseteq X$ which is τ-open and $U = V \cap X$. For a net $(x_\alpha)_{\alpha\in I}$ in D and $x \in D$, we then have $x_\alpha \xrightarrow{r} x$ in D if and only if $x_\alpha \xrightarrow{\tau} x$ in X.

Viewing the pre-Riesz space X as an order dense subspace of its Riesz completion (X^ρ, i). Our next goal is to establish the fact that if a positive operator from X to X is dominated by a compact operator, then it is continuous from the r-topology of $|\sigma|(X^\rho, (X^\rho)')$ to the norm topology. We will use $x_\alpha \xrightarrow{w} x$ to denote that x_α converges to x in X with respect to the weak topology, i.e. the $\sigma(X, X')$-topology. See the following lemma.

Lemma 3.3.16. Let X be an Archimedean pre-Riesz space endowed with a monotone norm $\|\cdot\|_X$, $S, T\colon X \to X$ bounded linear operators satisfying $0 \le S \le T$ with T compact. Let (X^ρ, i) be the Riesz completion of X with the seminorm $\|\cdot\|_{X^\rho}$ given by (3.2). Assume that there exists a positive compact operator $\widehat{T}\colon X^\rho \to X^\rho$ such that $\widehat{T} \circ i = i \circ T$. Then S is continuous for the r-topology of $|\sigma|(X^\rho, (X^\rho)')$ on norm bounded subsets of X to the norm topology.

Proof. Let $(x_\alpha)_\alpha$ be a norm bounded net in X with $x_\alpha \xrightarrow{r} 0$. It is enough to show that $\|S(x_\alpha)\|_X \to 0$ holds. To this end, from assumption of $x_\alpha \xrightarrow{r} 0$ in X it follows that $i(x_\alpha) \xrightarrow{|\sigma|(X^\rho,(X^\rho)')} 0$ in X^ρ, which means that for every $f \in (X^\rho)'$, we have $|f||i(x_\alpha)| \to 0$. It follows from $0 \le |f(|i(x_\alpha)|)| \le |f||i(x_\alpha)|$ that $|i(x_\alpha)| \xrightarrow{w} 0$ in X^ρ. According to Theorem 3.1.2 (ii), we have that $\|x\|_X \le 2\|i(x)\|_{X^\rho}$ for every $x \in X$.

Since $\widehat{T}$ is compact and $(|i(x_\alpha)|)_\alpha$ is norm bounded, $\left(\widehat{T}(|i(x_\alpha)|)\right)_\alpha$ has a norm convergent subnet $\left(\widehat{T}y_\beta\right)_\beta$ such that $\widehat{T}y_\beta \xrightarrow{\|\cdot\|_{X^\rho}} z$ for some $z \in X^\rho$, and then $\widehat{T}y_\beta \xrightarrow{w} z$. As for every $f \in (X^\rho)'$ we have $\langle f, T(|i(x_\alpha)|)\rangle = \langle T^* f, |i(x_\alpha)|\rangle \to 0$. So $\widehat{T}(|i(x_\alpha)|) \xrightarrow{w} 0$ and then $z = 0$. Hence $\left(\widehat{T}(|i(x_\alpha)|)\right)_\alpha$ has a subnet which converges in norm to 0. Similarly, every subnet of $\left(\widehat{T}(|i(x_\alpha)|)\right)_\alpha$ has a subnet that norm converges to 0. Therefore, $\left\|\widehat{T}(|i(x_\alpha)|)\right\|_{X^\rho} \to 0$.

Since $0 \le S \le T$ and i is positive, we have $0 \le \widehat{S} \circ i \le \widehat{T} \circ i$. According to Theorem

3.1.2 (ii) we have that $\|x\|_X \leq 2\|i(x)\|_{X^\rho}$ for every $x \in X$. Thus, we have

$$\|S(x_\alpha)\|_X \leq 2 \left\|(\widehat{S} \circ i)(x_\alpha)\right\|_{X^\rho} \leq 2 \left\|\widehat{S}(|i(x_\alpha)|)\right\|_{X^\rho} \leq 2 \left\|\widehat{T}(|i(x_\alpha)|)\right\|_{X^\rho} \to 0.$$

Hence S is continuous. □

We continue with a approximation property on the Riesz completion of a pre-Riesz space with respect to the regular norm.

Lemma 3.3.17. Let X, Y be two normed pre-Riesz spaces with a regular norm on Y, and $S, T\colon X \to Y$ such that $0 \leq S \leq T$. Let (Y^ρ, i) be the Riesz completion of Y. If T sends a subset A of X^+ to a norm totally bounded set, then for each $\epsilon > 0$ there exists some $u \in (Y^\rho)^+$ such that for all $x \in A$ we have

$$\left\|(i(Sx) - u)^+\right\|_{Y^\rho} \leq \epsilon.$$

Proof. By Theorem 3.1.2 one can define the seminorm on Y^ρ by

$$\|y\|_{Y^\rho} := \inf\{\|x\|_Y;\, x \in Y, -i(x) \leq y \leq i(x)\},\ y \in Y^\rho.$$

It is obvious that $\|i(y)\|_{Y^\rho} = \|y\|_Y$ for $y \in Y$, as $\|\cdot\|_Y$ is regular. Let $\epsilon > 0$. Since T sends a subset A of X^+ to a norm totally bounded set of Y, there exist $x_1, \ldots, x_n \in A$ such that for all $x \in A$ we have $\|Tx - Tx_j\|_Y < \epsilon$ for some j. Put $u = (i \circ T)\left(\sum_{j=1}^n x_j\right) \in (Y^\rho)^+$, then for $x \in A$ and j with $\|Tx - Tx_j\|_Y < \epsilon$ we have

$$\begin{aligned} 0 \leq (i(Sx) - u)^+ &= \left(i\big(Sx - T\sum_{j=1}^n x_j\big)\right)^+ \\ &\leq \left(i\big(Tx - T\sum_{j=1}^n x_j\big)\right)^+ \\ &\leq \big(i(Tx - Tx_j)\big)^+ \\ &\leq \left|\big(i(Tx - Tx_j)\big)^+\right|. \end{aligned}$$

Since regular norms are monotone we have

$$\begin{aligned} 0 \leq \left\|(i(Sx)-u)^+\right\|_{Y^\rho} &\leq \||i(Tx-Tx_j)|\|_{Y^\rho} \\ &= \|i(Tx-Tx_j)\|_{Y^\rho} \\ &= \|Tx-Tx_j\|_Y \leq \epsilon. \end{aligned}$$

Thus we have completed the proof. ☐

Next, we will show that for every Archimedean Riesz space X with an order unit u and an order unit norm $\|\cdot\|_u$, $\langle X'', X'\rangle$ is a Riesz dual system. To this end we need to show that X' is an ideal of $(X'')^\sim$ and use the theory of AL-spaces and AM-spaces.

Definition 3.3.18. A Banach lattice X is said to be

(1) an **AL-space** if $\|x+y\| = \|x\| + \|y\|$ for all $x, y \in X^+$ with $x \wedge y = 0$ and

(2) an **AM-space** if $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for all $x, y \in X^+$ with $x \wedge y = 0$.

The next lemma can be found in [42, Proposition 1.4.7]. We briefly recall the proof.

Lemma 3.3.19. Let X be an Archimedean Riesz space with an order unit u and an order unit norm $\|\cdot\|_u$. Then $(X', \|\cdot\|)$ is an AL-space.

Proof. For every $x \in X$ with $\|x\|_u \leq 1$, we have $-u \leq x \leq u$. So for $f \in X'$ with $f \geq 0$, we have $-f(u) \leq f(x) \leq f(u)$, and thus $|f(x)| \leq f(u)$. Hence $\|f\| \leq f(u)$. Since $f(u) \leq \|f\|\|u\| = \|f\|$, it follows that $\|f\| = f(u)$. Therefore

$$\|f+g\| = (f+g)(u) = f(u) + g(u) = \|f\| + \|g\|.$$

The norm dual of a normed Riesz space is a Banach lattice, so X' is an AL-space. ☐

Let us recall some know results. The following one is from [1, Corollary 3.7].

Corollary 3.3.20. Every AL-space has order continuous norm.

The next result is due to Nakano, see [6, Theorem 4.9].

Theorem 3.3.21. For a Banach lattice X the following statements are equivalent.

(i) X has order continuous norm.

(ii) X is an ideal of X''.

We cite the following result from [6, Corollary 4.5].

Corollary 3.3.22. (G. Birkhoff) The norm dual of a Banach lattice X coincides with its order dual, i.e., $X' = X^{\sim}$.

Lemma 3.3.23. Let X be an Archimedean Riesz space with an order unit norm. Then $\langle X'', X' \rangle$ is a Riesz dual system.

Proof. By Lemma 3.3.19, X' is an AL-space. By Corollary 3.3.20, every AL-space has an order continuous norm, so by Theorem 3.3.21 X' is an ideal of X'''. Since X'' is a Banach lattice, it follows from Corollary 3.3.22 that $X''' = (X'')^{\sim}$. Hence X' is an ideal in $(X'')^{\sim}$. Moreover, let $x \in X''$ with $x \neq 0$. Then there exists some $f \in X'$ satisfying $f(x) = x(f) \neq 0$.

Hence $\langle X'', X' \rangle$ is a Riesz dual system. □

Since a pre-Riesz space with an order unit and an order unit norm need not be norm complete, we can not use [6, Theorem 5.11] directly (in there it is required for the spaces to be Banach lattices). The next lemma is a modification of [6, Theorem 5.11], provided that the range space has an order unit and an order unit norm.

Lemma 3.3.24. Let X, Y be two normed Riesz spaces such that Y has an order unit and equipped with an order unit norm. Let $S, T\colon X \to Y$ be two positive operators with $0 \leq S \leq T$. If $T[0,x]$ is $|\sigma|(Y, Y')$-totally bounded for each $x \in X^+$, then $S[0,x]$ is likewise $|\sigma|(Y, Y')$-totally bounded for each $x \in X^+$.

Proof. By Lemma 3.3.23, the pair $\langle Y'', Y' \rangle$ is a Riesz dual system. Since Y'' is an ideal of Y'', it follows from Theorem 3.3.14 that $\sigma(Y'', Y')$ is an order continuous topology on Y''. By Theorem 3.3.15 we therefore have that every consistent locally convex solid topology on Y'' is order continuous. Observe that Corollary 3.3.13 it yields that $|\sigma|(Y'', Y')$ is consistent, and then $|\sigma|(Y'', Y')$ is an order continuous locally convex solid topology.

Let us view S, T as two operators from X to Y''. As $T[0,x]$ is $|\sigma|(Y, Y')$-totally bounded for each $x \in X^+$, it has $T[0,x]$ is $|\sigma|(Y'', Y')$-totally bounded for each $x \in X^+$. Since Y equipped with order unit norm, we have Y' is Dedekind complete. Hence, it follows from Theorem 3.3.10 that $S[0,x]$ is $|\sigma|(Y'', Y')$-totally bounded for each $x \in X^+$. So $S[0,x]$ is $|\sigma|(Y, Y')$-totally bounded for each $x \in X^+$. $\square$

We are now in the position to extend the result of Theorem 3.3.8 to a setting with pre-Riesz spaces. To this end, first notice that for every order unit e in a pre-Riesz space X with the Riesz completion (X^ρ, i) the element $i(e)$ is an order unit in X^ρ.

Theorem 3.3.25. Let X be an Archimedean pre-Riesz space with an order unit e and the order unit norm $\|\cdot\|$ such that X is norm complete. Let $S, T\colon X \to X$ satisfy $0 \leq S \leq T$, and let T be compact. Assume that there exists a positive compact operator $\widehat{T}\colon X^\rho \to X^\rho$ such that $\widehat{T} \circ i = i \circ T$, where (X^ρ, i) is the Riesz completion of X equipped with the order unit norm $\|\cdot\|_{X^\rho}$. Then S^3 is compact.

Proof. Let $U = \{x \in X; \|x\| \leq 1\}$ be the closed unit ball in X. Since the norm on X is an order unit norm we have

$$\|x\| = \inf\{\lambda \in \mathbb{R}^+;\ -\lambda e \leq x \leq \lambda e\},\ x \in X.$$

Let $x \in U$ be such that $\|x\| \leq 1$. Hence there exists $\lambda \leq 1$ with $-y \leq x \leq y$ and $y = \lambda e$. Due to $x = \frac{1}{2}(y+x) - \frac{1}{2}(y-x)$ it follows from $0 \leq y+x \leq 2y$ and $0 \leq y-x \leq 2y$ that $\|y+x\| \leq 2\|y\| \leq 2$ and $\|y-x\| \leq 2\|y\| \leq 2$, respectively. Thus $\frac{1}{2}(y+x), \frac{1}{2}(y-x) \in U \cap X^+$, and then $U \subseteq U^+ - U^+$ holds. Therefore, it is enough to show that $S^3(U^+)$ is a norm totally bounded set.

Let (X^ρ, i) be the Riesz completion of X, let $e \in X$ be an order unit, then $i(e)$ is an order unit in X^ρ, and the extension norm $\|\cdot\|_{X^\rho}$ is the order unit norm with respect to $i(e)$. By Lemma 3.3.17 there exists some $u \in (X^\rho)^+$ such that $\|(i(Sx)-u)^+\|_{X^\rho} \leq \epsilon$ for all $x \in U^+$. This implies that $0 \leq (i(Sx)-u)^+ \leq \epsilon i(e)$. Thus we have the following estimate,

$$\begin{aligned} i(Sx - \epsilon e) &= i(Sx) \wedge u + (i(Sx)-u)^+ - \epsilon i(e) \\ &\leq i(Sx) \wedge u \\ &\leq u. \end{aligned}$$

So $i(Sx) \in [0, u + \epsilon i(e)]$ for every $x \in U^+$. Take $v \in X$ with $i(v) \geq u$. Then for every $x \in U^+$ we have $i(Sx) \in [0, i(v+\epsilon e)]$. Hence

$$S(U^+) \subseteq [0, v + \epsilon e].$$

Therefore

$$S^2(U^+) \subseteq S[0, v+\epsilon e], \quad \text{and} \quad S^3(U^+) \subseteq S^2[0, v+\epsilon e]. \tag{$*$}$$

By Lemma 3.3.16 the operator S is continuous on norm bounded subsets of X with respect to the r-topology of the $|\sigma|(X^\rho, (X^\rho)')$ to the norm topology. By Theorem 3.3.9 the operator S maps totally bounded sets with respect to the r-topology of $|\sigma|(X^\rho, (X^\rho)')$ to norm totally bounded sets. Since $T[0, v+\epsilon e]$ is norm totally bounded, $T[0, v+\epsilon e]$ is totally bounded with respect to the $|\sigma|(X^\rho, (X^\rho)')$ topology. Hence, $T[0, v+\epsilon e]$ is totally bounded with respect to the r-topology of $|\sigma|(X^\rho, (X^\rho)')$. Then by Lemma 3.3.24 the set $S[0, v+\epsilon e]$ is likewise totally bounded with respect to the r-topology of $|\sigma|(X^\rho, (X^\rho)')$. Clearly, $S[0, v+\epsilon e] \subseteq$

$[0, S(v+\epsilon e)]$. Therefore, $S^2[0, v+\epsilon e] = S(S[0, v+\epsilon e])$ is a norm totally bounded set. By the second inclusion of $(*)$ we have $S^3(U^+)$ is a norm totally bounded set, as desired. □

Remark 3.3.26. In the view of Theorem 3.3.25, it is of interest to know under which condition such a positive compact operator $\widehat{T}$ exists. By Theorem 3.3.5 and Theorem 3.3.7, we know that this is true for X with order continuous norm. However, if X is a Banach lattice with an order unit e, then it can be renormed by

$$\|x\|_\infty = \inf\{\lambda > 0;\ |x| \leq \lambda e\},$$

and X becomes an AM-space. By Kakutani-Bohnenblust and M. Krein-S. Krein representation theorem [6, Theorem 4.29], we have X is lattice isometric to some $C(\Omega)$ for a (unique up to homeomorphism) Hausdorff compact topological space Ω, and the norm $\|\cdot\|_\infty$ on $C(\Omega)$ is not order continuous. So it is still an open question which choice of a norm on X leads to a similar result as Theorem 3.3.25.

Chapter 4

Disjointness preserving semigroups

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In the theory of C_0-semigroups, many results involve the order structure of the underlying Banach space. For instance, a rich theory of disjointness preserving semigroups and semigroups with local generators has been developed in [10, 12]. The Banach spaces in these works are Banach lattices. In this chapter, we investigate how results on disjointness preserving C_0-semigroups on Banach lattices can be generalized to the more general setting of ordered Banach spaces.

Our main result of this chapter is that the generator of a disjointness preserving C_0-semigroup on a suitable ordered Banach space is local, as in the Banach lattice case. It turns out that the choice of norms is the main issue of the analysis. We consider semimonotone norms for which the cone of positive elements is closed. On Banach spaces, those norms are equivalent to regular norms, which are a natural generalization of lattice norms.

This chapter is organized as follows. Properties of the norms are discussed in Section 4.1. Section 4.2 contains a discussion of local operators in different settings and some of their basic properties. In Section 4.3, we present our main result of this chapter. Moreover, we establish two results on local operators generating disjointness preserving C$_0$-semigroups. The first one considers a bounded generator and uses Taylor series, the second one uses resolvent operators and Yosida approximations.

4.1 Normed partially ordered vector spaces

Let (X, K) be a partially ordered vector space with a seminorm p, p is called **semimonotone** if there exists a constant $C \in \mathbb{R}^+$ such that for every $x, y \in X$ with $0 \leq x \leq y$ one has $p(x) \leq Cp(y)$. The cone K is said to be **normal** if p is semimonotone. Recall that a seminorm p on X is monotone if $p(x) \leq p(y)$ whenever $x, y \in X$ with $0 \leq x \leq y$.

We start the results of this section by a proposition, due to [56, Theorems IV.2.1 and IV.2.4], which clarifies the relation of semimonotone norms and monotone norms on partially ordered vector spaces.

Proposition 4.1.1. If $\|\cdot\|$ is a semimonotone norm on ordered vector space X, then there exists an equivalent monotone norm on X.

By using this proposition, we could extend the semimonotone norm on a directed partially ordered vector space to a regular seminorm. In fact, such an extension is exactly same with the formula which was given by (3.2) in the Theorem 3.1.2. The only difference is we consider semimonotone norms at here. The details can be seen in the next lemma.

Lemma 4.1.2. Let (X, K) be a directed partially ordered vector space and let $\|\cdot\|$ be a semimonotone norm on X. Let Y be a directed partially ordered vector space and $i\colon X \to Y$ a bipositive linear map, such that $i(X)$ is majorizing in Y. For $y \in Y$ let

$$\rho(y) := \inf\left\{\|x\|;\ x \in X,\ -i(x) \le y \le i(x)\right\}. \tag{4.1}$$

If $Y = X$, K is closed and $(X, \|\cdot\|)$ is complete, then $\rho \circ i$ is a regular norm on X that is equivalent to $\|\cdot\|$.

Proof. By Proposition 4.1.1, $\|\cdot\|$ is equivalent to a monotone norm and then, if $(X, \|\cdot\|)$ is complete and K is closed, [51, Corollary 6.4(ii)] says that $\|\cdot\|$ is equivalent to the regular norm $\rho \circ i$. ☐

The space of Lemma 4.1.2 is actually an ordered Banach space. By an **ordered Banach space** $(X, K, \|\cdot\|)$ we mean a Banach space $(X, \|\cdot\|)$ with a closed generating cone K. Since K is closed, the space (X, K) is Archimedean, and since K is generating, (X, K) is directed. Consequently, (X, K) is a pre-Riesz space and can be embedded into its Riesz completion X^ρ, see Theorem 1.2.7.

The next lemma shows how disjointness can be detected with the aid of the extension given by (4.1).

Lemma 4.1.3. Let $(X, K, \|\cdot\|)$ be a normed partially ordered vector space such that K is closed and generating. Let Y be a vector lattice and $i\colon X \to Y$ a bipositive linear map, such that $i(X)$ is majorizing in Y. For $y \in Y$ let $\rho(y)$ be defined by (4.1). If $u, v \in X$ are such that $\rho(|i(u)| \wedge |i(v)|) = 0$, then $u \perp v$.

Proof. Let $u, v \in X$ be such that $\rho(|i(u)| \wedge |i(v)|) = 0$. As

$$|i(u)| \wedge |i(v)| = \tfrac{1}{2}\,\big|\,|i(u) + i(v)| - |i(u) - i(v)|\,\big|$$

one obtains that $\rho\left(|i(u) + i(v)| - |i(u) - i(v)|\right) = 0$. Hence for every $n \in \mathbb{N}$ there exists an $x_n \in X$ such that

$$-i(x_n) \le |i(u) + i(v)| - |i(u) - i(v)| \le i(x_n) \tag{4.2}$$

and $\lim_{n\to\infty}\|x_n\|=0$. The first inequality in (4.2) yields

$$\pm(i(u)-i(v))-i(x_n)\leq|i(u)+i(v)|,$$

hence for $x\geq\pm(u+v)$ it follows that $x\geq\pm(u-v)-x_n$ for every $n\in\mathbb{N}$. Since K is closed and $x_n\to 0$, one obtains $x\geq\pm(u-v)$. We conclude

$$\{u+v,-u-v\}^{\mathrm{u}}\subseteq\{u-v,-u+v\}^{\mathrm{u}}. \tag{4.3}$$

From the second inequality in (4.2) it follows that

$$\pm(i(u)+i(v))-i(x_n)\leq|i(u)-i(v)|,$$

and an analogous argumentation yields equality in (4.3), which implies that u and v are disjoint. □

Recall that a subspace $B\subseteq X$ is a band in an ordered vector space X if $B=B^{\mathrm{dd}}$. It turns out that bands are closed for regular norms.

Lemma 4.1.4. If

(i) (X,K) is a pre-Riesz space with a regular norm $\|\cdot\|$ such that K is closed, or

(ii) $(X,K,\|\cdot\|)$ is an ordered Banach space with a semimonotone norm,

then every band in X is closed.

Proof. According to Lemma 4.1.2, the conditions in (ii) yield that the norm $\|\cdot\|$ is equivalent to a regular norm, so it suffices to give a proof under condition (i).

Let B be a band in X, let (x_n) be a sequence in B and let $x\in X$ be such that $\|x_n-x\|\to 0$. Let (X^ρ,i) be the Riesz completion of X, as in Theorem 1.2.7. Let ρ be the Riesz seminorm on $Y=X^\rho$ defined as in (4.1).

Let $y \in B^{\mathrm{d}}$. Then for every $n \in \mathbb{N}$ one has $x_n \perp y$, so that by Proposition 1.2.15 one has $i(x_n) \perp i(y)$, which implies $|i(x_n)| \wedge |i(y)| = 0$. Since $\rho(i(x_n)) - i(x)) = \|x_n - x\| \to 0$, the continuity of the lattice operations with respect to ρ yields that $\rho(|i(x)| \wedge |i(y)|) = 0$. Hence, by Lemma 4.1.3, we obtain $x \perp y$. It follows that $x \in B^{\mathrm{dd}} = B$. Hence B is closed. □

4.2 Local operators

In this section we introduce a notion of local operators on pre-Riesz spaces. Also band preserving operators are a well-established class of operators in the theory of vector lattices, see e.g. [6, 42]. In [10, (5.4)], local operators are defined on Banach lattices. Below we use disjointness to define local operators on partially ordered vector space in the spirit of [10].

Throughout this chapter, we will use $\mathcal{D}(T)$ to denote the domain space of the operator T, and $L(X)$ to denote the bounded linear operators on X.

Definition 4.2.1. Let X be a partially ordered vector space and let $T\colon X \supseteq \mathcal{D}(T) \to X$ be a linear operator.

(i) T is called **band preserving** if for every band B in X one has $T(B \cap \mathcal{D}(T)) \subseteq B$.

(ii) T is called **local** if for every $x \in \mathcal{D}(T)$, $y \in X$ with $x \perp y$ it follows that $Tx \perp y$.

Local operators turn out to be a special class of disjointness preserving operators. The class of local operators plays a role in the theory of differential equations, where the notion 'local' is used ambiguously (see Remark 4.2.5 below).

With the above definition, we observe that local operators and band preserving operators coincide in a pre-Riesz space.

Proposition 4.2.2. Suppose that X is a pre-Riesz space and let $T\colon X \supseteq \mathcal{D}(T) \to X$ be a linear operator. T is local if and only if T is band preserving.

Proof. Let T be local. For every $x \in \mathcal{D}(T)$ one obtains $Tx \in \{x\}^{\mathrm{dd}}$. Indeed, for every $y \in \{x\}^{\mathrm{d}}$ it follows that $Tx \perp y$, therefore $Tx \in \{x\}^{\mathrm{dd}}$. Let B be a band in X and $x \in B \cap \mathcal{D}(T)$. Then $\{x\}^{\mathrm{dd}} \subseteq B^{\mathrm{dd}} = B$, hence $Tx \in B$. We conclude that T is band preserving.

Now let T be band preserving, $x \in \mathcal{D}(T)$ and $y \in X$ such that $x \perp y$. Then $\{y\}^{\mathrm{d}}$ is a band in X, and $x \in \{y\}^{\mathrm{d}} \cap \mathcal{D}(T)$, hence $Tx \in \{y\}^{\mathrm{d}}$, which yields $Tx \perp y$. Consequently T is local. ☐

Typical examples of local operators are differential operators and multiplication operators. We discuss some of these settings in the subsequent remarks.

Remark 4.2.3. If X is a Banach lattice, every bounded local operator on X is contained in the *center*

$$\mathrm{Z}(X) := \{T \in L(X);\ \exists \alpha > 0 :\ -\alpha I \leq T \leq \alpha I\}.$$

More precisely, the following assertions are equivalent for a bounded linear operator $T\colon X \to X$ (see e.g. [44, Section 9]):

(i) T is local,

(ii) $-\|T\| I \leq T \leq \|T\| I$,

(iii) for every ideal J in X one has $T[J] \subseteq J$.

In [44, Section 9] the following two examples are given to illustrate that local operators are closely related to multiplication operators.

(a) Let (Ω, μ) be a σ-finite measure space and $X = L^p(\Omega, \mu)$ $(1 \leq p \leq \infty)$, then $\mathrm{Z}(X)$ is isomorphic to $L^\infty(\Omega, \mu)$ via the identification of $y \in L^\infty(\Omega, \mu)$ with the operator from X to X given by $x \mapsto yx$.

(b) Let Ω be a locally compact Hausdorff space and X the space $C_0(\Omega)$ of all continuous functions x on Ω vanishing at infinity (i.e. for every $\varepsilon > 0$ there is a compact set $S \subseteq \Omega$ such that for every $t \in \Omega \setminus S$ one has $|x(t)| < \varepsilon$), endowed with the supremum norm. $\mathrm{Z}(X)$ is isomorphic to the space $C^{\mathrm{b}}(\Omega)$ of bounded continuous functions via the identification of $y \in C^{\mathrm{b}}(\Omega)$ with the operator from X to X given by $x \mapsto yx$.

Remark 4.2.4. We continue the discussion of case (a) above for unbounded T. Let $X = L^p(\Omega, \mu)$ $(1 \leq p \leq \infty)$ and $T\colon L^p(\Omega, \mu) \supseteq \mathcal{D}(T) \to L^p(\Omega, \mu)$. A notion of 'locality' in this setting is defined in [23, I.4.13(8)]. To avoid confusion, we denote this property by (L):

(L) For every measurable set $S \subseteq \Omega$ and for every $x \in \mathcal{D}(T)$ with $x = 0$ almost everywhere on S it follows that $Tx = 0$ almost everywhere on S.

T is local if and only if (L) is satisfied. Indeed, suppose that (L) is satisfied and let $x \in \mathcal{D}(T)$, $y \in X$ and $x \perp y$. Then $S := \{t \in \Omega;\, y(t) \neq 0\}$ is measurable and $x = 0$ almost everywhere on S, hence $Tx = 0$ almost everywhere on S, which implies $Tx \perp y$. Therefore T is local. Now suppose that T is local and let $S \subseteq \Omega$ be measurable and $x \in \mathcal{D}(T)$ be such that $x = 0$ almost everywhere on S. Then $x \perp \chi_S$, hence $Tx \perp \chi_S$, which implies $Tx = 0$ on S. Consequently (L) is satisfied. In the present setting, a local operator need not be a multiplication operator, consider e.g. the operator $T\colon L^p[0,1] \supseteq C^1[0,1] \to L^p[0,1]$, $x \mapsto x'$.

Remark 4.2.5. We continue the discussion of (b) in Remark 4.2.3, where we now consider the special case where Ω is a compact Hausdorff space and $X = C(\Omega)$. For every bounded local operator $T\colon C(\Omega) \to C(\Omega)$ there is $y \in C(\Omega)$ such that $T\colon x \mapsto yx$. (This can be deduced from the fact that $C(\Omega)$ is an Archimedean f-algebra with unit and that T is band preserving and order bounded, see [6, Theorem 8.27].) We discuss several notions of locality for unbounded operators T.

(I) In [12, Theorem 3.7] a linear operator $T\colon C(\Omega) \supseteq \mathcal{D}(T) \to C(\Omega)$ is called *local* if the following property is satisfied:

$$\forall x \in \mathcal{D}(T),\ x \geq 0,\ \omega \in \Omega \ \text{ with } \ x(\omega) = 0 \ \Rightarrow \ (Tx)(\omega) = 0. \tag{4.4}$$

If (4.4) is satisfied for an operator T, then T is local in the sense of Definition 4.2.1. The converse implication is not true, in general. Indeed, consider $\mathcal{D}(T) := \{x \in C^2[0,1]; x'(0) = x'(1) = 0\}$ and

$$T\colon C[0,1] \supseteq \mathcal{D}(T) \to C[0,1],\ x \mapsto x''$$

(cf. the one-dimensional diffusion semigroup in [43, 2.7]). On one hand, T does not satisfy (4.4), take e.g. $x(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2$, then $x(t)|_{t=0} = 0$ but $(Tx)(t)|_{t=0} = -1$. On the other hand, T is local. Indeed, let $x \in \mathcal{D}(T)$, $y \in \mathrm{C}[0,1]$ with $x \perp y$, and $N := \{t \in [0,1]; x(t) = 0\}$. For $t \in \operatorname{int} N$ one has $(Tx)(t) = 0$, whereas for $t \in \partial N$ and every $n \in \mathbb{N}$ there is $t_n \in \left[t - \frac{1}{n}, t + \frac{1}{n}\right] \cap [0,1]$ such that $x(t_n) \neq 0$, i.e. $y(t_n) = 0$, which implies $y(t) = 0$. Hence $Tx \perp y$, consequently T is local.

(II) An operator $T\colon C(\Omega) \supseteq \mathcal{D}(T) \to C(\Omega)$ satisfies (4.4) if and only if for every open set $O \subseteq \Omega$ one has $T(I_O \cap \mathcal{D}(T)) \subseteq I_O$, where

$$I_O = \{x \in C(\Omega); \forall \omega \in \Omega \setminus O\colon x(\omega) = 0\},$$

i.e. T preserves for every open set $O \subseteq \Omega$ the largest ideal having O as its carrier.

(III) We relate (4.4) to a notion of locality given in [12, p.147, second Remark]. Let (X, K) be a partially ordered vector space with a norm $\|\cdot\|$. For a linear operator $T\colon X \supseteq \mathcal{D}(T) \to X$ the following property is considered:

$$\forall x \in \mathcal{D}(T) \cap K,\ f \in K' \ \text{ with } \ f(x) = 0 \ \Rightarrow \ f(Tx) = 0. \tag{4.5}$$

Note that (4.5) is satisfied if and only if T and $-T$ are *positive-off diagonal* (for a definition of this notion see [16, Definition 7.18] (we will study the positive-off diagonal property in Chapter 5, so this definition can been seen in the next chapter); cf. also the *positive minimum principle* in [10, Definition 1.6]).

Before we link (4.4) and (4.5), we reformulate (4.5) for the case that X is an Archimedean partially ordered vector space with an order unit u. The norm induced by u, which will be denoted by $\|\cdot\|_u$, is defined by

$$\|x\|_u = \inf\{\alpha \in [0,\infty);\ -\alpha u \leq x \leq \alpha u\},\ x \in X.$$

Let

$$\Sigma := \{f\colon X \to \mathbb{R};\ f \text{ positive linear, and } f(u) = 1\},$$

and denote by Λ the set of extreme points of Σ. For $x \in \mathcal{D}(T) \cap K$ the set $N = \{f \in \Sigma;\ f(x) = 0\}$ is the weak* closure of the convex hull of $N \cap \Lambda$ (see also [29, (3.6)]). Therefore (4.5) holds if and only if

$$\forall\, x \in \mathcal{D}(T) \cap K,\ f \in \Lambda \text{ with } f(x) = 0 \Rightarrow f(Tx) = 0. \tag{4.6}$$

For $X = \mathrm{C}(\Omega)$ as above and u the constant-one function, Λ is homeomorphic to Ω, hence the conditions (4.4) and (4.6) are equivalent. This implies the equivalence of (4.4) and (4.5).

Now let us consider some basic results on local operators on pre-Riesz spaces. The following lemma will be needed in the proof of Theorem 4.3.4.

Lemma 4.2.6. Let X be a pre-Riesz space.

(i) If $S\colon X \supseteq \mathcal{D}(S) \to X$ and $T\colon X \supseteq \mathcal{D}(T) \to X$ are local operators and $\alpha, \beta \in \mathbb{R}$, then $\alpha S + \beta T\colon X \supseteq \mathcal{D}(S) \cap \mathcal{D}(T) \to X$ is a local operator.

(ii) If $S\colon X \supseteq \mathcal{D}(S) \to X$ and $T\colon X \supseteq \mathcal{D}(T) \to \mathcal{D}(S) \subseteq X$ are local operators, then $ST\colon X \supseteq \mathcal{D}(T) \to X$ is a local operator.

Proof. (i) Let $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$ and $y \in X$ be such that $x \perp y$. Then $Sx \perp y$ and $Tx \perp y$, so that $Sx, Tx \in \{y\}^{\mathrm{d}}$. Since $\{y\}^{\mathrm{d}}$ is a linear subspace, it follows that $\alpha Sx + \beta Tx \perp y$. Hence $\alpha S + \beta T$ is local.

(ii) Let $x \in \mathcal{D}(T), y \in X$ be such that $x \perp y$. Then $Tx \perp y$ as T is local, so that $STx \perp y$ as S is local and $Tx \in \mathcal{D}(S)$. Hence ST is local. $\square$

Next we consider a result on the inverse of a local operator, which we need in the proof of Corollary 4.3.7 below. It turns out that the inverse of a local operator T is local if both T and T^{-1} are positive.

Proposition 4.2.7. Let (X, K) be a pre-Riesz space, let $T\colon X \supseteq \mathcal{D}(T) \to X$ be a bijective linear operator such that the inclusion map $i\colon \mathcal{D}(T) \to X$ is a Riesz* homomorphism. If both T and T^{-1} are positive and T is local, then T^{-1} is also local.

Proof. As a first step, for $x \in X, y \in \mathcal{D}(T)$ with $x \perp y$ we show that $T^{-1}x \perp y$ in $\mathcal{D}(T)$, which comes down to $\{T^{-1}x + y, -T^{-1}x - y\}^{\mathrm{u}} \cap \mathcal{D}(T) = \{T^{-1}x - y, -T^{-1}x + y\}^{\mathrm{u}} \cap \mathcal{D}(T)$. Let $z \in \{T^{-1}x + y, -T^{-1}x - y\}^{\mathrm{u}} \cap \mathcal{D}(T)$. Then, as T is positive, $Tz \geq x + Ty, -x - Ty$. Since T is local and therefore $Ty \perp x$, we have $Tz \geq x - Ty, -x + Ty$, so $z \geq T^{-1}x - y, -T^{-1}x + y$, as T^{-1} is positive. Therefore, $\{T^{-1}x + y, -T^{-1}x - y\}^{\mathrm{u}} \cap \mathcal{D}(T) \subseteq \{T^{-1}x - y, -T^{-1}x + y\}^{\mathrm{u}} \cap \mathcal{D}(T)$. A similar proof yields the converse inclusion, so that $\{T^{-1}x + y, -T^{-1}x - y\}^{\mathrm{u}} \cap \mathcal{D}(T) = \{T^{-1}x - y, -T^{-1}x + y\}^{\mathrm{u}} \cap \mathcal{D}(T)$. This shows that $T^{-1}x \perp y$ in $\mathcal{D}(T)$.

Since the inclusion map $i\colon \mathcal{D}(T) \to X$ is a Riesz* homomorphism, Lemma 2.2.4 yields that $i(T^{-1}x) \perp i(y)$, hence $T^{-1}x \perp y$ in X. Thus, T^{-1} is local. $\square$

We conclude this section by the following simple observation.

Lemma 4.2.8. Let (X, K) be a pre-Riesz space and $T\colon X \to X$ a bipositive linear bijection. Then T is disjointness preserving.

Proof. Let $x, y \in X$ be such that $x \perp y$. Then $\{x+y, -x-y\}^{\mathrm{u}} = \{x-y, -x+y\}^{\mathrm{u}}$. Hence for every $u \in X$ we have

$$u \geq x + y, -x - y \iff u \geq x - y, -x + y.$$

As T is bipositive, the latter is equivalent to

$$Tu \geq Tx + Ty, -Tx - Ty \iff Tu \geq Tx - Ty, -Tx + Ty.$$

Since T is surjective, this comes down to $\{Tx+Ty, -Tx-Ty\}^{\mathrm{u}} = \{Tx-Ty, -Tx+Ty\}^{\mathrm{u}}$. Hence $Tx \perp Ty$. $\square$

4.3 Disjointness preserving C$_0$-semigroups

Disjointness preserving C$_0$-semigroups on Banach lattices are discussed e.g. in [10, Section 5], where, in particular, it is shown that the generator of a disjointness preserving C$_0$-semigroup is local. We prove the analogous result for disjointness preserving C$_0$-semigroups on ordered Banach spaces with a semimonotone norm.

Let us see some necessary definitions in the theory of C$_0$-semigroups.

Definition 4.3.1. Let $(X, K, \|\cdot\|)$ be an ordered Banach space.

(i) A subset $\{T(t) : t \in \mathbb{R}^+\}$ of $L(X)$ is called a **one-parameter semigroup** on X if $T(0) = I$, $T(s+t) = T(s)T(t)$ for all $s, t \in \mathbb{R}^+$, and usually written as $T(t)_{t\geq 0}$. Here I stands for the identity operator.

(ii) The **generator** of $T(t)_{t\geq 0}$ is given by $A\colon X \supseteq \mathcal{D}(A) \to X, x \mapsto \lim_{t\downarrow 0} \frac{T(t)x-x}{t}$, where $\mathcal{D}(A)$ is

$$\mathcal{D}(A) := \{x \in X;\ \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists in } X\}.$$

(iii) $T(t)_{t\geq 0}$ is called **strongly continuous**, or also C$_0$**-semigroup**, if the map $t \mapsto T(t)$ is continuous for the strong topology on $L(X)$, i.e. $\lim_{t\to t_0} \|T(t)f - T(t_0)f\| = 0$ for every $f \in X$ and $t_0 \geq 0$.

We say that a C$_0$-semigroup $T : [0, \infty) \to L(X)$ is called **disjointness preserving** if for every $t \in [0, \infty)$, the operator $T(t)$ is disjointness preserving. Then we have the following result.

Theorem 4.3.2. Let $(X, K, \|\cdot\|)$ be an ordered Banach space with a semimonotone norm and let $T\colon [0,\infty) \to L(X)$ be a disjointness preserving C_0-semigroup with generator A. Then A is local.

Proof. Let (X^ρ, i) be the Riesz completion of X. Because of Lemma 4.1.2, the semimonotone norm $\|\cdot\|$ is equivalent to the regular norm $\rho \circ i$, where the regular norm ρ on $Y = X^\rho$ is given by (4.1). As Y is a vector lattice, ρ is a Riesz norm and (4.1) comes down to $\rho(y) = \inf\left\{\|x\|;\, x \in X,\, |y| \leq i(x)\right\},\ y \in Y$. Let $x \in \mathcal{D}(A)$ and $y \in X$ be such that $x \perp y$. Then $i(x) \perp i(y)$ in Y. Now the line of reasoning is as in the proof of [10, Proposition 5.4]. For every $t > 0$ one has

$$\begin{aligned}
\left|\tfrac{1}{t} i(T(t)x - x)\right| \wedge |i(y)| &\leq \tfrac{1}{t}\,|i(T(t)x)| \wedge |i(y)| + \tfrac{1}{t}\,|i(x)| \wedge |i(y)| \\
&= \tfrac{1}{t}\,|i(T(t)x)| \wedge |i(T(t)y - y) - i(T(t)y)| \\
&\leq \tfrac{1}{t}\,|i(T(t)x)| \wedge |i(T(t)y - y)| \\
&\leq |i(T(t)y - y)|.
\end{aligned}$$

Here we used that Y is distributive and T is disjointness preserving. We conclude

$$\rho\left(\left|\tfrac{1}{t} i(T(t)x - x)\right| \wedge |i(y)|\right) \leq \rho(|i(T(t)y - y)|).$$

For $t \downarrow 0$ one has $T(t)y - y \to 0$ in X, hence $\rho(|i(T(t)y - y)|) \to 0$, which implies

$$\rho\left(\left|\tfrac{1}{t} i(T(t)x - x)\right| \wedge |i(y)|\right) \to 0.$$

Further,

$$\begin{aligned}
&\left|\rho\left(|i(Ax)| \wedge |i(y)|\right) - \rho\left(\left|\tfrac{1}{t} i(T(t)x - x)\right| \wedge |i(y)|\right)\right| \\
\leq &\rho\left(\left||i(Ax)| \wedge |i(y)| - \left|\tfrac{1}{t} i(T(t)x - x)\right| \wedge |i(y)|\right|\right) \\
\leq &\rho\left(\left||i(Ax)| - \left|i\left(\tfrac{1}{t}(T(t)x - x)\right)\right|\right|\right) \\
\leq &\rho\left(\left|i\left(Ax - \tfrac{1}{t}(T(t)x - x)\right)\right|\right) \to 0,
\end{aligned}$$

for $t \downarrow 0$, since $\|Ax - \frac{1}{t}(T(t)x - x)\| \to 0$. Therefore $\rho\left(|i(Ax)| \wedge |i(y)|\right) = 0$. Now Lemma 4.1.3 implies that $Ax \perp y$, hence A is local. □

Notice that the converse of the statement of 4.3.2 is not true in general, not even in Banach lattices. We illustrate this by an example from [43, 23].

Example 4.3.3. Let A be the second derivative operator given in Remark 4.2.5(I). We have already shown that A is local. The one-dimensional diffusion semigroup generated by A is given by

$$T(t)f(x) = \int_0^1 K_t(x,y)f(x)dy,$$

with kernel

$$K_t(x,y) = 1 + 2\sum_{n=1}^{\infty} \exp(-\pi^2 n^2 t)\cos(\pi n x)\cdot\cos(\pi n y),$$

see [43, 2.7] or [23, 2.12]. There it is also shown that $K_t(\cdot,\cdot)$ is a positive, continuous function on $[0,1]^2$. Obviously, for $t \in (0,\infty)$, $T(t)$ is not disjointness preserving on $C[0,1]$.

It is nevertheless interesting to investigate a converse of Theorem 4.3.2. We consider two cases in which a C$_0$-semigroup with a local generator will be disjointness preserving. The first one considers as extra condition that the generator is a bounded operator and uses the Taylor series for the semigroup. The second case assumes that also the resolvent operators are local and uses the Yosida approximations. It turns out that in both cases the semigroup even consists of local operators.

We begin with the case of a bounded generator.

Theorem 4.3.4. Let $(X, K, \|\cdot\|)$ be an ordered Banach space with a semimonotone norm. If $A \in \mathcal{L}(X)$ is local, then e^{tA} is local for every $t \in \mathbb{R}$.

Proof. Let $x, y \in X$ be such that $x \perp y$. Let $t \in \mathbb{R}$. For every $N \in \mathbb{N}$, Lemma

4.2.6 yields that $\sum_{k=0}^{N} \frac{t^n}{n!} A^n x \perp y$. According to Lemma 4.1.4, $\{y\}^{\mathrm{d}}$ is closed, so that $e^{tA}x = \sum_{k=0}^{\infty} \frac{t^n}{n!} A^n x \perp y$. Hence e^{tA} is local. □

We proceed by investigating unbounded local generators. Typical examples of local operators are differential operators and multiplication operators. Recall that Example 4.3.3 shows an example of a differential operator that generates a C$_0$-semigroup that is not disjointness preserving. The next example presents a differential operator that generates a C$_0$-semigroup which is disjointess preserving. It also presents a multiplication operator as generator.

Example 4.3.5. (i) Translation Semigroup. We consider the Banach space $X := C_{\mathrm{ub}}(\mathbb{R})$ of all uniformly continuous, bounded functions on $\mathbb{R}$. For $t \in [0, \infty)$, the left translation operator $T_l(t) \colon X \to X$ is given by

$$T_l(t)x(s) := x(s+t), \; s \in \mathbb{R}, \; x \in X.$$

Then $T_l \colon [0, \infty) \to \mathcal{L}(X)$ is a C$_0$-semigroup on X with generator A given by differentiation,

$$Ax := x'$$

with domain $\mathcal{D}(A) = \{x \in X;\, x \text{ differentiable and } x' \in X\}$. Then A is local (and unbounded) and T is disjointness preserving, but not local.

(ii) Multiplication Semigroup. Let $X := C_0(\Omega)$, where Ω is a locally compact Hausdorff space, as defined in Remark 4.2.3(b). Let $q \colon \Omega \to \mathbb{R}$ be continuous and bounded above. Define for $t \in [0, \infty)$ the operator $T_q(t) \colon X \to X$ by

$$T_q(t)x = e^{tq(t)}x, \; x \in X.$$

Then $T_q \colon [0, \infty) \to L(X)$ is a C_0-semigroup with generator A given by $Ax = qx$, $x \in \mathcal{D}(A)$ and $\mathcal{D}(A) = \{x \in X \colon \; qx \in X\}$. Then A is local and $T_q(t)$ is local for every $t \in [0, \infty)$.

An interesting difference between the generators A in Example (i) and (ii) above is that in (ii) also the inverse of A is local. It turns out that a C$_0$-semigroup is local whenever both A and A^{-1} are local, which is a special case of the following theorem.

Theorem 4.3.6. Let $(X, K, \|\cdot\|)$ be an ordered Banach space with semimonotone norm and let $T\colon [0,\infty) \to L(X)$ be a C$_0$-semigroup with generator A. If $A\colon X \supseteq \mathcal{D}(A) \to X$ is local and there exists a $\lambda_0 \in \rho(A) \cap \mathbb{R}$ such that for every $\lambda \in \rho(A)$ with $\lambda \geq \lambda_0$ we have that $(\lambda I - A)^{-1}\colon X \to \mathcal{D}(A) \subseteq X$ is local, then $T(t)$ is local for every $t \in [0,\infty)$.

Proof. By Lemma 4.2.6, the Yosida approximation $A_\lambda = A(\lambda I - A)^{-1}$ is local for every $\lambda \in \rho(A)$ with $\lambda \geq \lambda_0$. Let $t \in [0,\infty)$. Since A_λ is bounded, due to Theorem 4.3.4 we obtain that e^{tA_λ} is local. For $x \in X$, we have $T(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda}x$. We infer that $T(t)$ is local. Indeed, if $x, y \in X$ are such that $x \perp y$, then $e^{tA_\lambda}x \perp y$. Since the band $\{y\}^{\mathrm{d}}$ is closed by Lemma 4.1.4, it follows that $T(t)x \perp y$. Thus, $T(t)$ is local. □

Corollary 4.3.7. Let $(X, K, \|\cdot\|)$ be an ordered Banach space with semimonotone norm and let $T\colon [0,\infty) \to L(X)$ be a C$_0$-semigroup with generator A such that the inclusion map $i\colon \mathcal{D}(A) \to X$ is a Riesz* homomorphism. If there exists a $\lambda_0 \in \rho(A) \cap \mathbb{R}$ such that $\lambda_0 I - A\colon \mathcal{D}(A) \to X$ is positive and local and for every $\lambda \in \rho(A)$ with $\lambda \geq \lambda_0$ we have that $(\lambda I - A)^{-1}$ is positive, then $T(t)$ is local for every $t \in [0,\infty)$.

Proof. Let $\lambda \in \rho(A)$ with $\lambda \geq \lambda_0$. Then $\lambda I - A\colon \mathcal{D}(A) \to X$ is positive and bijective and, by assumption, $(\lambda I - A)^{-1}$ is positive. By Lemma 4.2.6, $\lambda I - A = \lambda_0 I - A + (\lambda - \lambda_0)I$ is local. Proposition 4.2.7 then yields that $(\lambda I - A)^{-1}$ is local. Hence we can apply Theorem 4.3.6 and obtain that $T(t)$ is local for every $t \in [0,\infty)$. □

Corollary 4.3.8. Let $(X, K, \|\cdot\|)$ be an ordered Banach space with semimonotone norm and let $T\colon [0,\infty) \to L(X)$ be a C$_0$-semigroup with generator A such that the inclusion map $i\colon \mathcal{D}(A) \to X$ is a Riesz* homomorphism. If A is positive and local and there exists a $\lambda_0 \in \rho(A) \cap \mathbb{R}$ such that $A \leq \lambda_0 I$, then $T(t)$ is local for every $t \in [0, \infty)$.

Proof. Assume that A is positive. Then $T(t)$ is positive for every $t \in [0, \infty)$, so there exists $\lambda_1 \in \mathbb{R}$ such that $(\lambda I - A)^{-1}$ is positive for every $\lambda \in \rho(A)$ with $\lambda \geq \lambda_1$, due to [16, Chapter 7]. As $\lambda_0 I - A\colon \mathcal{D}(A) \to X$ is positive and local, Corollary 4.3.7 yields that $T(t)$ is local for every $t \in [0, \infty)$. ☐

Merely as illustration, we present an example of an ordered Banach space satisfying the conditions of Corollary 4.3.7, on which there exists a non-trivial multiplication operator A which generates a C_0-semigroup.

Example 4.3.9. Consider the locally compact Hausdorff space $[0, 1)$ and

$$X = \left\{ x \in C_0[0, 1);\ x|_{[0,\frac{1}{2}]} \in \mathrm{Pol}^2[0, \tfrac{1}{2}] \right\},$$

where $C_0[0, 1)$ is defined in Remark 4.2.3(b) and $\mathrm{Pol}^2[a, b]$ is the space of polynomial functions of at most degree 2 on $[a, b]$. As in the proof of [32, Example 3.5], it can be verified that X is order dense in $C_0[0, 1)$. Thus, X is a pre-Riesz space and the embedding map $i\colon X \to C_0[0, 1)$ is a Riesz* homomorphism.

Moreover, X is a closed subspace of $(C_0[0, 1), \|\cdot\|_\infty)$. Indeed, let (x_n) be a sequence in X and $x \in C_0[0, 1)$ be such that $\|x_n - x\|_\infty \to 0$. Then $(x_n|_{[0,\frac{1}{2}]})$ is a sequence in $\mathrm{Pol}^2[0, \frac{1}{2}]$ and

$$\left\| x_n|_{[0,\frac{1}{2}]} - x|_{[0,\frac{1}{2}]} \right\| \leq \|x_n - x\|_\infty \to 0.$$

Since $(\mathrm{Pol}^2[0, \frac{1}{2}], \|\cdot\|_\infty)$ is finite dimensional, it is closed in $(C[0, \frac{1}{2}], \|\cdot\|_\infty)$. Thus, it follows that $x|_{[0,\,\frac{1}{2}]} \in \mathrm{Pol}^2[0, \frac{1}{2}]$ and hence $x \in X$.

Since the cone in X is closed and generating, X is an ordered Banach space.

Let $q \in C[0, 1)$ be bounded above and constant on $[0, \frac{1}{2}]$. As in Example 4.3.5 (ii), let $Ax = qx$ for $x \in \mathcal{D}(A) = \{x \in X;\ s \mapsto q(s)x(s) \in X\}$. The elements of X that vanish on an interval $[1 - \varepsilon, 1)$ for some $\varepsilon > 0$ are norm dense in X and they are in $\mathcal{D}(A)$, hence $\mathcal{D}(A)$ is norm dense in X. Also, A is closed. Indeed, let $x_n \in \mathcal{D}(A)$ and $x, y \in X$ be such that $\|x_n - x\|_\infty \to 0$ and $\|Ax_n - y\|_\infty \to 0$. Then for every $t \in [0, 1)$ we have that $x_n(t) \to x(t)$, hence $q(t)x_n(t) \to q(t)x(t)$, and $(qx_n)(t) \to y(t)$, and therefore $q(t)x(t) = y(t)$. Hence $x \in \mathcal{D}(A)$ and $Ax = y$.

Next we show that $A\colon X \supseteq \mathcal{D}(A) \to X$ is local. Let $x \in \mathcal{D}(A)$ and $y \in X$ be such that $x \perp y$. Since X is order dense in $C_0[0, 1)$, it follows that $x \perp y$ in $C_0[0, 1)$. Therefore, for every $t \in [0, 1)$ we have $x(t) = 0$ or $y(t) = 0$, hence $q(t)x(t) = 0$ or $y(t) = 0$, which yields that $Ax \perp y$ in $C_0[0, 1)$ and hence in X.

Let $\lambda_0 \in [0, \infty)$ with $\lambda_0 > \sup_s q(s)$. Then we have that $\lambda_0 I - A$ is positive and local and for every $\lambda \geq \lambda_0$ we have that $(\lambda I - A)^{-1}$ is positive. Now A satisfies all conditions of Corollary 4.3.7, provided A generates a C$_0$-semigroup T. This is in fact the case, namely T is given by $(T(t)x)(s) = e^{q(s)t}x(s)$, $s \in [0, 1]$, $t \in [0, \infty)$. Clearly, $T(t)$ is local for every $t \in [0, \infty)$.

The conditions in our analysis that the norm is semimonotone and the space is norm complete appear fairly weak, but exclude in fact many interesting examples such as C^k-spaces and Sobolev spaces. A general theory of disjointness preserving C$_0$-semigroups on such spaces will be an interesting topic of further research.

Chapter 5

Dissipativity and positive off-diagonal property of operators

It is well know that the generator A of a positive contraction semigroup $T(t)_{t\geq 0}$ can be characterized by means of dissipativity of A with respect to some appropriate sublinear functionals on a Banach space X, see [11, Theorem 2.6], and [16, Proposition 7.13] for the case of X being an ordered Banach space with normal positive cone. In Banach lattices, the positivity of $T(t)_{t\geq 0}$ can be also characterized by its generator A which satisfies the "positive off-diagonal" property (it is called "positive minimum principle" in [10, Theorem 1.11]) This is also studied in an ordered Banach space with a cone with nonempty interior, [16, Theorem 7.27]. For more extensive treatments of dissipative operators, we refer the reader to K. J. Engel and R. Nagel [22]. The positive off-diagonal property of operators on ordered normed spaces is also studied by A. Kalauch [28, 29], S. Koshkin [38] and W. Arendt, P. R. Chernoff, and T. Kato [13] etc..

The goal of this chapter is to investigate the description of positivity and contractivity of semigroups through the dissipativity of generators with respect to corresponding sublinear functionals on ordered vector spaces. So the choice of sublinear functionals on ordered vector spaces is crucial. In Section 5.1, we give two different ways of defining sublinear functionals which only involve the order structure and the norm of the space. One is given by a regular norm and the other one is obtained through a dual function. The latter one turns out to be more effective in studying the positivity and contractivity of semigroups on ordered Banach spaces in Section 5.2. Moreover, in Section 5.3 we will show a representation for positive functionals on Archimedean pre-Riesz spaces, which will be used to study the positive off-diagonal property of operators on the pre-Riesz space $C^1[0,1]$.

5.1 Half-norms on ordered vector spaces

This section is mainly concerned with some basic definitions of sublinear functionals on ordered vector spaces, and some properties of operators. Two different types of sublinear functionals will be introduced specifically, and some intuitive properties of these functionals will be studied.

Definition 5.1.1. Let $(X, K, \|\cdot\|)$ be an ordered normed vector space. A sublinear function $p\colon X \to \mathbb{R}$ is called a **half-norm** if $p(f)+p(-f) > 0$ whenever $f \neq 0$ in X. If, in addition, there exists some constant $C > 0$ such that $p(f) + p(-f) \geq C\|f\|$ for all $f \in X$, then p is called a **strict half-norm**.

For an ordered normed vector space $(X, K, \|\cdot\|)$, let

$$\begin{aligned}\psi(x) = \operatorname{dist}(-x, K) &= \inf\{\|x+y\|;\ y \in K\}\\ &= \inf\{\|y\|;\ x \leq y \in X\},\ x \in X, \qquad (5.1)\end{aligned}$$

then ψ defines a half-norm on X. This ψ is called the **canonical half-norm**. Obviously, if X is a Banach lattice with a canonical half-norm p, then $p(x) = \|x^+\|$ for all $x \in X$.

Remark 5.1.2. Every ordered normed vector space X has a canonical half-norm ψ which is order preserving, i.e., if $x \leq y$ then $\psi(x-y) \leq 0$. Recall that a cone K in a normed space X is called normal if the norm on X is semimonotone, see Chapter

4. If K is closed then it is known that the following statements are equivalent: (1), the cone K is normal; (2), the canonical half-norm is strict; (3), the canonical half-norm is equivalent to the original norm. These properties are studied in [49]. By the open mapping theorem, X cannot be complete with respect to both norms unless K is normal.

For a sublinear function $p\colon X \to \mathbb{R}$, a bounded operator T on X is called p-**contractive** if $p(Tf) \leq p(f)$ for all $f \in X$. Similarly, a semigroup $T(t)_{t\geq 0}$ is called p-**contractive** if $T(t)$ is p-contractive for all $t \geq 0$.

We say that the **subdifferential** of p in $f \in X$, denoted by $dp(f)$, is defined by

$$dp(f) = \{\phi \in K';\ \langle g, \phi\rangle \leq p(g) \text{ for all } g \in X, \langle f, \phi\rangle = p(f)\}. \tag{5.2}$$

Definition 5.1.3. An operator $A\colon X \supseteq \mathcal{D}(A) \to X$ is called p-**dissipative** if for all $f \in \mathcal{D}(A)$ there exists $\phi \in dp(f)$ such that $\langle Af, \phi\rangle \leq 0$; A is called **strictly** p-**dissipative** if for all $f \in \mathcal{D}(A)$ the inequality $\langle Af, \phi\rangle \leq 0$ holds for all $\phi \in dp(f)$.

If N is the norm function on a Banach lattice X, i.e. $N(f) = \|f\|$, define $N^+(f) = \|f^+\|$. An operator A on X is called **(strictly) dispersive** if A is (strictly) N^+-dissipative, i.e., if for every $f \in \mathcal{D}(A)$, one has $\langle Af, \phi\rangle \leq 0$ for some (resp. all) $\phi \in dN^+(f)$, where $dN^+(f)$ is given by

$$dN^+(f) = \{\phi \in K';\ \|\phi\| \leq 1, \langle f, \phi\rangle = \|f^+\|\}.$$

Obviously, if p is the canonical half norm, then the p-dissipative operators are dispersive.

Recall that the regular norm on an ordered vector space defined by (3.1), and studied in Chapter 3. By Theorem 3.1.2, there is a natural way to extend the regular norms on ordered vector spaces. In the following propositions, we will see that the sublinear functionals on ordered vector spaces induced by the regular norms have intriguing properties.

Proposition 5.1.4. Let (X, K) be a partially ordered vector space and $\|\cdot\|_r$ a regular seminorm on X. Let p be the sublinear functional on X defined by

$$p(x) = \inf\{\|y\|_r;\ y \in X, y \geq 0, y \geq x\}, \forall x \in X. \tag{5.3}$$

Then p is a strict half-norm on X. Moreover, let ψ on X be defined by (5.1), then $p(x) = \psi(x)$ for all $x \in K \cup (-K)$, in particular, $p(x) = \psi(x) = 0$ for $x \in (-K)$.

Proof. Let $x \neq 0$ be in X. If $y \geq 0$, $y \geq x$ and $z \geq 0$, $z \geq -x$, then $-(y+z) \leq x \leq (y+z)$. Since $\|\cdot\|_r$ is regular, we have $\|x\|_r \leq \|y+z\|_r$. So $\|y\|_r + \|z\|_r \geq \|y+z\|_r \geq \|x\|_r > 0$. Hence $p(x)+p(-x) \geq \|x\|_r > 0$. Thus p is a strict half-norm.

Moreover, if $0 \leq x \in X$, we can take $y = x \geq 0$ in (5.3), then $p(x) = \|x\|_r = \psi(x)$. It is clear that $p(x) = \psi(x) = 0$ for $x \in X$ with $x \leq 0$. □

Proposition 5.1.5. Let X be a partially ordered vector space, $\|\cdot\|_r$ a regular norm on X, and p on X defined by (5.3). If $T \in L(X)^+$ is a contractive operator with respect to the regular norm $\|\cdot\|_r$, then T is p-contractive.

Proof. Let $T \in L(X)^+$. For $0 \leq y \in X$, suppose $\|Ty\|_r \leq \|y\|_r$. Then for $x \in X$,

$$\begin{aligned} p(x) &= \inf\{\|y\|_r;\ y \geq 0, y \geq x\} \\ &\geq \inf\{\|Ty\|_r;\ y \geq 0, y \geq x\} \\ &\geq \inf\{\|z\|_r;\ z \geq 0, z \geq Tx\} \\ &= p(Tx). \end{aligned}$$

So T is p-contractive. □

We continue with a different approach of sublinear functionals on ordered vector spaces, which turns out to be more useful in dealing with the contractivity and positivity of semigroups.

Proposition 5.1.6. Let X be a partially ordered vector space. For every monotone sublinear functional $\phi \in \mathbb{R}^X$, i.e. $\phi(x) \leq \phi(y)$ whenever $0 \leq x \leq y$ in X, define the sublinear functional p_ϕ on X by

$$p_\phi(f) = \inf\{\langle g, \phi\rangle;\, g \in X, g \geq 0, g \geq f\}, \, \forall\, f \in X. \tag{5.4}$$

Then p_ϕ is sublinear. $p_\phi(f) = \langle f, \phi\rangle$ if $f \geq 0$, and $p_\phi(f) = 0$ if $f \leq 0$. Moreover, if X is a Banach lattice, then $p_\phi(f^+) = p_\phi(f)$ for $f \in X$.

Proof. The sublinearity of p_ϕ follows from ϕ is sublinear directly. It is clear that $p_\phi(f) = \langle f, \phi\rangle$ for $f \geq 0$, and $p_\phi(f) = 0$ for $f \leq 0$.

If X is a Banach lattice, let $f \in X$,

$$\begin{aligned} p_\phi(f^+) &= \inf\{\langle h, \phi\rangle;\, h \geq f^+, h \geq 0\} \\ &= \inf\{\langle h, \phi\rangle;\, h \geq \inf\{g; g \geq f, g \geq 0\}, h \geq 0\} \\ &\geq \inf\{\langle h, \phi\rangle;\, h \geq f, h \geq 0\} = p_\phi(f). \end{aligned}$$

Because $h \geq f$ and $h \geq 0$ implies $h \geq f^+$, and ϕ is monotone, we have $\{\langle h, \phi\rangle;\, h \geq f, h \geq 0\} \subseteq \{\langle h, \phi\rangle;\, h \geq f^+, h \geq 0\}$, and hence $\inf\{\langle h, \phi\rangle;\, h \geq f, h \geq 0\} \geq \inf\{\langle h, \phi\rangle;\, h \geq f^+, h \geq 0\}$. So we have $p(f) \geq p(f^+)$. ☐

The differential function space $X = C^1[0, 1]$, or the Sobolev space $X = W^{n,p}$ with respect to its own norms are typical examples of ordered Banach spaces. On these spaces, the norms are not monotone, but we could define a sublinear functional p_ϕ by (5.4).

5.2 Contractivity and positivity of semigroups on ordered Banach spaces

In this section, we will give sufficient conditions on the generator A under which a semigroup $T(t)_{t\geq 0}$ is contractive with respect to p defined by (5.4), or $T(t)_{t\geq 0}$ is positive on an ordered Banach space.

Firstly, we will see that if A is supposed to be p-dissipative and $(I-\lambda A)$ is invertible for some $\lambda \geq 0$, then $T(t)_{t\geq 0}$ is p-contractive.

Theorem 5.2.1. Let X be an ordered Banach space, $\phi\colon X \to \mathbb{R}$ be a positive linear functional, and p_ϕ on X defined by (5.4). Let $A\colon X \supseteq \mathcal{D}(A) \to X$ be p_ϕ-dissipative. If $(I - \lambda A)$ is invertible for some $\lambda > 0$, then $(I - \lambda A)^{-1}$ is p_ϕ-contractive. In addition, if $T(t)_{t\geq 0}$ is a strongly continuous semigroup generated by A, then $T(t)$ is p_ϕ-contractive for every $t \geq 0$.

Proof. For a fixed $f \in \mathcal{D}(A)$, let $\psi \in dp_\phi(f)$ be such that $\langle Af, \psi\rangle \leq 0$. Then $\langle g, \psi\rangle \leq p(g)$ for all $g \in X$ and hence $|\langle g, \psi\rangle| \leq p(g)$. For some $\lambda_0 > 0$ one has that

$$\begin{aligned} p_\phi((\lambda_0 I - A)f) &\geq |\langle(\lambda_0 I - A)f, \psi\rangle| \geq \mathrm{Re}\langle(\lambda_0 I - A)f, \psi\rangle \\ &= \mathrm{Re}(\langle\lambda_0 f, \psi\rangle - \langle Af, \psi\rangle) \geq \mathrm{Re}\langle\lambda_0 f, \psi\rangle \\ &= \mathrm{Re}\lambda_0\langle f, \psi\rangle = \mathrm{Re}\lambda_0 p(f) \\ &= \lambda_0 p(f). \end{aligned}$$

So if $\lambda > 0$ is such that $(I - \lambda A)$ is invertible, then $(I - \lambda A)^{-1}$ is p_ϕ-contractive for some $\lambda \geq 0$.

In addition, for f in X and $t \geq 0$,

$$T(t)f = \lim_{n\to\infty}[\frac{n}{t}R(\frac{n}{t}, A)]^n f = \lim_{n\to\infty}(I - \frac{t}{n}A)^{-n} f.$$

So $T(t)$ is p_ϕ-contractive for every $t \geq 0$. □

Recall that a nonempty subset $\Phi \subseteq K'$ is called **total** if $\phi(x) \geq 0$ for each $\phi \in \Phi$ implies $x \geq 0$; the original definition is in [39, p. 102]. For example, if $X = C[0,1]$, then $\Phi = \{\delta_t;\, t \in [0,1]\}$ is total. To use this property, we claim that $dp(f) \cap \Phi \neq \emptyset$ for p in (5.4). In fact, for p defined in (5.4), let $t \in [0,1]$ and $\phi = \delta_t$, then $p_t(f) = \inf\{\delta_t(g);\, g \geq 0, g \geq f\} = \inf\{g(t);\, g \geq 0, g \geq f\} = f^+(t)$. So if $f(t) \leq 0$ then $p_t(f) = 0$. We only need to consider $f \geq 0$. Let $\psi = \delta_t$, then $\psi(g) = \delta_t(g) = g(t) \leq g^+(t) = p_t(g)$, and $\psi(f) = \delta_t(f) = f(t) = f^+(t) = p_t(f)$. So $\psi \in dp(f)$. In this case, if A is the multiplication operator which multiplies a negative function, then A is p-dissipative.

The next theorem shows that the positivity of $T(t)_{t\geq 0}$ is obtained by supposing A is p_ϕ-dissipative for ϕ in a total set of X.

Theorem 5.2.2. Let X be an ordered Banach space. Let $A\colon X \supseteq \mathcal{D}(A) \to X$ be the generator of strongly continuous semigroup $T(t)_{t\geq 0}$. If A is p_ϕ-dissipative for all ϕ in a total set Φ and $(I - \lambda A)$ is invertible for some $\lambda \geq 0$, then $T(t)$ is positive for all $t \geq 0$.

Proof. Suppose that A is p_ϕ-dissipative for all $\phi \in \Phi$. Let $T(t)_{t\geq 0}$ be the semigroup generated by A. Let $\phi \in \Phi$, select $f \leq 0$ in X, by Theorem 5.2.1, one has that $p_\phi(T(t)f) \leq p_\phi(f)$, which means

$$\inf\{\langle g, \phi\rangle;\, g \geq T(t)f, g \geq 0\} \leq \inf\{\langle h, \phi\rangle;\, h \geq f, h \geq 0\}.$$

Take $h = 0$, then the right side of the above inequality is 0. It follows that

$$\inf\{\langle g, \phi\rangle;\, g \geq T(t)f, g \geq 0\} \leq 0.$$

Because of $g \geq T(t)f$, then $\langle g, \phi\rangle \geq \langle T(t)f, \phi\rangle$ for ϕ is positive. So

$$\langle T(t)f, \phi\rangle \leq \inf\{\langle g, \phi\rangle;\, g \geq T(t)f, g \geq 0\} \leq 0.$$

Since Φ is total, one has that $\langle T(t)f, \phi\rangle \leq 0$ for all $\phi \in \Phi$ implies $T(t)f \leq 0$. Thus $T(t)$ is positive. $\square$

Remark 5.2.3. Notice that [10, Theorem 1.2] says that, if A is densely defined on a Banach lattice, then $T(t)_{t\geq 0}$ is positive and contractive if and only if A is dispersive and $(\lambda - A)$ is surjective for some $\lambda > 0$. By Theorem 5.2.1, we could generalize one direction of this conclusion to ordered Banach spaces. We will illustrate this through an example of a second derivative operator with Dirichlet boundary condition. This example originally comes from [10, Example 1.5].

Example 5.2.4. Let $X = (C^n[0,1], \|\cdot\|)$ be a Banach space, the densely defined operator A be the second derivative operator with Dirichlet boundary condition. Then the domain satisfies $\mathcal{D}(A) = \{f \in C^{n+2}[0,1];\, f(0) = f(1) = f''(0) = f''(1)\}$. Choose $p(f) = \|f^+\|_\infty$ for $f \in X$, then there exists $x \in (0,1)$ such that $f(x) = \sup_{y\in[0,1]} f(y) = \|f^+\|_\infty$. Since $\langle g, \delta_x\rangle = g(x) \leq \|g^+\|_\infty = p(g)$ for all $g \in X$, $\langle f, \delta_x\rangle = f(x) = \|f^+\|_\infty$, so $\delta_x \in dp(f)$. We have $\langle Af, \delta_x\rangle = f''(x) \leq 0$ since f has a maximum in X. So A is p-dissipative. Let $g \in X$, define $f_0(x) = \frac{1}{2}[e^x \int_x^1 e^{-y} g(y)dy - e^{-x} \int_x^1 e^y g(y)dy]$. Then there exist $m, n \in \mathbb{R}$ such that $f(x) = f_0(x) + me^x + ne^{-x}$ and $f(0) = f(1) = 0$, and then $f \in \mathcal{D}(A)$. Since $f - f'' = f_0 - f_0'' = g$, we have $(I - A)$ is surjective. For $f \in \mathcal{D}(A)$, suppose that $(I - A)f = 0$, then $f(x) = \alpha e^x + \beta e^{-x}$. Notice that $f(0) = f(1) = 0$ such that $\alpha = \beta = 0$, so $f(x) = 0$ for $x \in [0,1]$. So $(I - A)$ is injective. Take $\phi = \|\cdot\|_\infty$ we have that p is given by (5.4). It follows from Theorem 5.2.1 that A is the generator of a contractive semigroup.

If $g \in X$ and $g \leq 0$, then $p(T(t)) \leq p(g) = \|g^+\|_\infty = 0$. So $\|(T(t)g)^+\|_\infty = 0$, so $T(t)g \leq 0$ for every $t \geq 0$. Hence $T(t)_{t\geq 0}$ is positive.

Remark 5.2.5. It is worth to mention that in an ordered Banach space, specifically $C^1[0,1]$, the dispersivity of A will fail, in general, with respect to the original norm. However, by the above discussion, we still have flexibility to choose a functional p as in (5.4), which is only depends on a functional ϕ. This is also different from the arguments in [10, Example 1.5].

5.3 Positive off-diagonal property of operators on ordered vector spaces

In this section, we will introduce the positive off-diagonal property especially on pre-Riesz spaces, in particular $C^1(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ open. Explicitly, we investigate a representation theorem for positive functionals on Archimedean pre-Riesz spaces, which is also interesting independently.

Definition 5.3.1. The linear operator $A\colon X \supseteq \mathcal{D}(A) \to X$ is said to have the **positive off-diagonal property** if $\langle Au, \phi\rangle \geq 0$ whenever $0 \leq u \in \mathcal{D}(A)$ and $0 \leq \phi \in X^*$ with $\langle u, \phi\rangle = 0$.

The motivation of the positive off-diagonal property comes from matrix theory, where the off-diagonal elements of the matrix $A = (a_{ij})$ are positive, i.e., $a_{ij} \geq 0$ for all $i \neq j$. It is shown in [16, Lemma 7.23] that on an ordered Banach space with an order unit u such that $u \in \mathcal{D}(A)$, A has the positive off-diagonal property and $Au \leq 0$ if and only if A is Ψ_u-dissipative, where Ψ_u is the order unit function, i.e. $\Psi_u(x) = \inf\{\lambda \geq 0; x \leq \lambda u\}$, $x \in X$. However, the dissipativity and the positive off-diagonal property are independent, as the following example shows.

Example 5.3.2. In general, the properties that A has the positive off-diagonal property and A is p-dissipative do not imply each other. In fact, let $X = \mathbb{R}^2$, take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then A has the positive off-diagonal property. Take $f = (1, 0)$, then $\phi = (1, 0) \in dp(f)$ but $\langle Af, \phi\rangle = 1$, so A is not dispersive. Let $A = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$, and $\mathcal{D}(A) = \{x = (x_1, x_2) \in X; x_1 \geq 0, x_2 = 0\}$. Take $\phi = (1, 0)$, then $\phi \in dN^+(f)$ for every $f \in \mathcal{D}(A)$. It is obvious that $\langle Af, \phi\rangle \leq 0$. So A is dispersive, but does not have positive off-diagonal property.

Next, we consider a representation theorem in pre-Riesz spaces.

Theorem 5.3.3. Let X be an Archimedean pre-Riesz space with order unit. Then there exists a compact Hausdorff space Ω and a bipositive linear map $i\colon X \to$

$C(\Omega)$ such that $i(X)$ is order dense in $C(\Omega)$. Moreover, for every positive linear functional $\phi\colon X \to \mathbb{R}^+$, there exists a regular Borel measure μ on Ω such that

$$\phi(x) = \int_\Omega i(x)(\omega) d\mu(\omega), \quad x \in X,\ \omega \in \Omega.$$

Proof. The result of first part of this theorem comes from [31, Lemma 6].

For the second part, let $C(\Omega)$ be the continuous function space where Ω is a compact Housdorff space. Let $i\colon X \to C(\Omega)$ be a bipositive linear map such that $i(X)$ is an order dense subspace of $C(\Omega)$. So for a positive linear functional $\phi\colon X \to \mathbb{R}$, one has that $\phi \circ i^{-1}\colon i(X) \to \mathbb{R}^+$ is a positive linear functional on $i(X)$. Since $\mathbb{R}$ is Dedekind complete, and $i(X)$ is a majorizing subspace of $C(\Omega)$, by Theorem 1.3.6 (Kantorovich), there exists an extension of $\phi \circ i^{-1}$ to a positive functional $\psi\colon C(\Omega) \to \mathbb{R}$. By the Riesz representation theorem, for ψ on $C(\Omega)$, there exists a unique regular Borel measure μ on Ω such that

$$\psi(f) = \int_\Omega f(\omega) d\mu(\omega), \quad \forall f \in C(\Omega),\ \omega \in \Omega.$$

So for every $x \in X$, one has $\phi \circ i^{-1}(i(x)) = \psi(i(x))$. If we take $f = i(x)$, then

$$\phi \circ i^{-1}(i(x)) = \psi(i(x)) = \int_\Omega i(x)(\omega) d\mu(\omega).$$

Thus we get the conclusion. $\square$

We give an example to illustrate that the positive off-diagonal property of A can be generalized to a special kind of partially ordered vector space, in particular pre-Riesz space $C^1[0,1]$.

Example 5.3.4. Let $C[0,1]$ be the real continuous functions. Let $X = C^1[0,1]$ which is an Archimedean pre-Riesz space, then X is an order dense subspace of $C[0,1]$. Let $A \in L(X)$ be a densely defined operator, we claim that A has positive off-diagonal property if and only if $(Au)(\omega) \geq 0$ whenever $0 \leq u \in \mathcal{D}(A)$ and $\omega \in [0,1]$ with $u(\omega) = 0$. In fact, first suppose that A has the positive off-diagonal property and $0 \leq u \in \mathcal{D}(A)$, $\omega \in \Omega$ with $u(\omega) = 0$. Take $0 \leq \phi_\omega \in X^*$ to be the point evaluation at ω, then it follows from $u(\omega) = \langle u, \phi_\omega \rangle = 0$ that $\langle Au, \phi_\omega \rangle \geq 0$, i.e. $(Au)(\omega) \geq 0$. Conversely, assume $0 \leq u \in \mathcal{D}(A)$, and $0 \leq \phi \in X^*$ with $\langle u, \phi \rangle = 0$. Then by the above Theorem 5.3.3, there exists a regular Borel measure μ that represents ϕ, i.e. $\langle u, \phi \rangle = \int_0^1 i(u)(\omega) d\mu(\omega)$. By assumption, we have $i(u)(\omega) = 0$ and $u(\omega) = 0$ for all ω in the support of μ, then $(Au)(\omega) \geq 0$ and $i(Au)(\omega) \geq 0$. Hence $\langle Au, \phi \rangle = \int_0^1 i(Au)(\lambda) d\mu(\lambda) \geq 0$. This shows that A has the positive off-diagonal property.

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Index

Summary

This thesis mainly extends the theory of positive operators on Riesz spaces to a setting of pre-Riesz spaces. It includes five chapters. The first chapter is on preliminaries, and the rest of the four chapters present main results. These results can be roughly divided into two parts: the second and third chapter extend work on disjointness preserving operators and compact operators to pre-Riesz spaces, and the last two chapters concern the theory of C_0-semigroups in ordered Banach spaces.

Our work is based on the theory which was established by M. van Haandel in 1993, which yields that every directed Archimedean partially ordered vector space (pre-Riesz space) owns a vector lattice cover, that is, it can be embedded order densely into a Riesz space. We establish that for a pervasive pre-Riesz space X with monotone norm, there exists a Riesz norm on the Riesz completion X^ρ. Moreover, we show that if for every nonzero positive element x in X^ρ, there exists an increasing sequence which is less equal than x and norm converges to x, then every positive linear operator $T : X \to Y$ which is disjointness preserving and injective can be extended to $\widehat{T} : X^\rho \to Y^\delta$ where $\widehat{T}$ is disjointness preserving and injective, and where X, Y are pervasive pre-Riesz spaces with monotone norms, X^ρ is the Riesz completion of X and Y^δ is the Dedekind completion of Y. In addition, we explore that for an operator $T : X \to Y$ with X a pervasive Archimedean pre-Riesz space with the Riesz decomposition property and Y a Dedekind complete vector lattice, there exists an extension of T to all of X^ρ which preserves order boundedness and disjointness. In a pervasive Archimedean pre-Riesz space X with the Riesz decomposition property, for a positive element in X^ρ, we construct a net in X which is order convergent to it. Using this result, we extend order continuous norms on pre-Riesz spaces. Furthermore, we show that the compact domination property of positive operators from X to Y, where X and Y are Banach lattices, holds as well in pervasive Archimedean pre-Riesz spaces with the Riesz decomposition property and regular norms, provided $(X^\delta)'$ and Y having order continuous norms.

In the second part of the results, we establish that a disjointness preserving C$_0$-semigroup $T(t)_{t\geq 0}$ on an ordered Banach space with a semimonotone norm always admits a local generator A. We show that the reverse direction of this implication holds if A and A^{-1} are local. We also consider the contractivity and positivity of semigroups $T(t)_{t\geq 0}$ on ordered Banach spaces, which are related to the dissipativity of A with respect to a suitable sublinear functional.

We provide many examples to support our results, where one typical example of a pre-Riesz space is $C^1[0,1]$.

Samenvatting

Dit proefschrift breidt de theorie van positieve operatoren op Rieszruimten uit naar pre-Rieszruimten. De tekst bestaat grofweg uit twee delen. De eerste twee hoofdstukken breiden bestaand werk over disjunctheidsbewarende operatoren en compacte operatoren uit naar pre-Rieszruimten. De laatste twee hoofdstukken gaan over de theorie van C_0-halfgroepen op geordende Banachruimten.

Het werk bouwt voort op de theorie van M. van Haandel uit 1993. Deze theorie laat zien dat iedere gerichte Archimedische partieel geordende vectorruimte (pre-Rieszruimte) een overdekkend vectorrooster heeft. Dat wil zeggen dat zo'n ruimte orde-dicht ingebed kan worden in een Rieszruimte. We tonen aan dat er voor een pervasieve pre-Rieszruimte X met monotone norm een Riesznorm op de Rieszcompletering X^ρ bestaat. Bovendien laten we zien dat er voor ieder postief element x ongelijk aan nul een stijgende rij bestaat onder x die in norm naar x convergeert. Dan kan iedere postieve injectieve disjunctheidsbewarende operator $T\colon X \to Y$ uitgebreid worden naar een operator $\hat{T}\colon X^\rho \to Y^\delta$ die disjunctheidsbewarend en injectief is, aangenomen dat X en Y pervasieve pre-Rieszruimten zijn met monotone normen, X^ρ de Riesz completering is van X en Y^δ de Dedekind-completering van Y. Verder gaan we na dat er voor een operator $T\colon X \to Y$, met X een pervasieve Archimedische pre-Rieszruimte met de Rieszdecompositie-eigenschap en Y een Dedekind volledig vectorrooster, een uitbreiding van T naar de hele ruimte X^ρ bestaat die orde-begrensdheid en disjunctheid bewaart. In een pervasieve Archimedische pre-Rieszruimte met de Rieszdecompositie-eigenschap construeren we voor ieder positief element x in X^ρ een net in X dat in orde naar x convergeert. Met behulp van dit resultaat breiden we orde-continue normen op pre-Rieszruimten uit. Verder laten we zien dat de domineringseigenschap voor compacte positieve operatoren van X naar Y, waar X en Y Banachroosters zijn, ook geldt in pervasieve Archimedische pre-Rieszruimten met de Rieszdecompositie-eigenschap en reguliere normen, aangenomen dat $(X^\delta)'$ en Y orde-continue normen hebben.

In het tweede deel bewijzen we dat een disjunctheidsbewarende C_0-halfgroep

$T(t)_{t\geq 0}$ op een geordende Banachruimte met een semimonotone norm een generator heeft die lokaal is. We laten zien dat het omgekeerde geldt als A en A^{-1} beide lokaal zijn. We beschouwen ook de contractiviteit en positiviteit van halfgroepen op geordende Banachruimten en brengen die in verband met de dissipativiteit van hun generatoren ten opzichte van geschikt gekozen sublineaire functionalen.

We geven veel voorbeelden om de resultaten te illustreren, waar de ruimte $C^1[0,1]$ een typisch voorbeeld van een pre-Rieszruimte is.

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Curriculum Vitae

Feng Zhang was born on 18th March 1988 in Lvliang, Shanxi, China.

In 2007, he started his bachelor study in mathematics at the Jinzhong University in Shanxi. In 2010, he received the bachelor degree and started his master study in School of Mathematics at the Southwest Jiaotong University. In 2014, he received his master degree with his master thesis “Global conservative solutions of a modified two-component Camassa-Holm equation” under the supervision of professor Han Yang.

In 2014, he received a four years’ scholarship from China Scholarship Council to support him doing Ph.D. research in Mathematical Institute, Leiden University, under the supervision of Dr. Onno van Gaans. His Ph.D. research focused on the functional analysis, extending the operator theory in Banach lattice to more general settings, explicitly, pre-Riesz spaces.