Mechanical metamaterials: nonlinear beams and excess zero modes

Lubbers, L.A.

Author: Lubbers, L.A.
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On work presented in this thesis:

(8) L.A. Lubbers and M. van Hecke
Excess floppy modes in metamaterials with symmetries.
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(7) L.A. Lubbers, M. van Hecke and C. Coulais
A nonlinear beam model to describe the postbuckling of wide neo-Hookean beams.
J. Phys. Mech. Solids **106**, 191-206 (2017).

(6) C. Coulais, J.T.B. Overvelde, L.A. Lubbers, K. Bertoldi and M. van Hecke
Discontinuous buckling of wide beams and metabeams.
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On other work:

(5) S. Karpitschka, A. Pandey, L.A. Lubbers, J.H. Weijs, L. Botto, S. Das, B. Andreotti and J.H. Snoeijer
Dynamical theory of the inverted Cheerios effect.
Soft Matter **13**, 6000-6010 (2017).

(4) S. Karpitschka, A. Pandey, L.A. Lubbers, J.H. Weijs, L. Botto, S. Das, B. Andreotti and J.H. Snoeijer
Liquid drops attract or repel by the inverted Cheerios effect.
PNAS **113**, 7403 (2016).
Featured in The New York Times

(3) L.A. Lubbers, Q. Xu, S. Wilken, W.W. Zhang and H.M. Jaeger
Dense suspension splat: Monolayer spreading and hole formation after impact.
Phys. Rev. Lett. **113**, 044502 (2014).
Featured as a Synopsis in Physics

(2) L.A. Lubbers, J.H. Weijs, L. Botto, S. Das, B. Andreotti and J.H. Snoeijer
Drops on soft solids: free energy and double transition of contact angles.
J. Fluid Mech. **747**, R1 (2014).

(1) J. F. Hernández-Sánchez, L. A. Lubbers, A. Eddi and J. H. Snoeijer
Symmetric and asymmetric coalescence of drops on a substrate.
Phys. Rev. Lett. **109**, 184502 (2012).

Curriculum vitae

I was born on June 3rd, 1989 in Oldenzaal, the Netherlands. I grew up in the neighbouring village Denekamp and attended my secondary education at the *Twents Carmel College* with joint locations in Denekamp and Oldenzaal.

After obtaining my vwo degree in 2007, I directly started my BSc studies in Applied Physics at the *University of Twente* with a minor in Aircraft Engineering. In 2011, I completed my bachelors *cum laude* with a research project in the Optical Sciences group titled *Analyzing and simulating the coherent control model*. I then continued with the MSc programme in Applied Physics specializing in fluid physics, at the same university. During my masters I worked on three major projects, all of which have been published (see publications). The first project was an optional capita selecta project on drop coalescence performed at the Physics of Fluids (PoF) group in Twente, and the second project was an internship at the *James Franck Institute* in Chicago, on the impact of dense suspensions on solid surfaces. The third and final project was my master thesis titled *Liquid drops on soft solids*, again carried out within the PoF group. I received my MSc degree *cum laude* in November 2013.

Throughout my Bsc and MSc studies I have been teaching assistant for four different undergraduate courses, most of which I supervised three times. I was active simultaneously for *Twente Academy*, where I assisted high school students with their final school research project ('profielwerkstuk'), and organized activities at (primary) schools to introduce the world of science to school children of age 4-14, teachers and parents.

In December 2013, I left the Twente region to start my PhD in physics at *Leiden University* under the supervision of Prof. dr. Martin van Hecke. I presented my work at international conferences in Madrid (Spain), Oxford (United Kingdom), Denver, Purdue, San Antonio, Baltimore and New Orleans (United States). During my PhD, I was teaching assistant for the undergraduate course 'Statistical Physics' (2x) and the graduate course 'Mechanical Metamaterials', and (co)-supervised a number of undergraduate students.

In May 2018, I started working at TNO as Junior Scientist Innovator at the Explosions, Ballistics and Protection department located in Rijswijk.

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Corentin, without a doubt you made my start in Leiden much easier. As an office-mate you quickly familiarized me with beam theory and the simulation package *Abaqus*, helping me tremendously in getting my first project to run. We quickly became a team and our discussions on neo-Hookean elasticity in a later stadium of my PhD have been extremely useful for the work presented in this thesis and our related publications. I furthermore appreciate your great enthusiasm and solid faith that helped getting our beam theory published.

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