Mechanical metamaterials: nonlinear beams and excess zero modes

Lubbers, L.A.

Author: Lubbers, L.A.
Title: Mechanical metamaterials: nonlinear beams and excess zero modes
Issue Date: 2018-09-13

Summary

Mechanical metamaterials are man-made materials which derive their unusual properties from their structure rather than their composition. Their structure, or architecture, often consists of periodically arranged building blocks whose mutual interactions realize unusual properties. In this thesis, we study the role of two aspects of mechanical metamaterials: (i) the beam ligaments and (ii) the microstructures of hinging squares. Both provide functionality to a wide variety of mechanical metamaterials [4, 20, 25–29, 36–38, 40, 44]. However, as we motivate in the introductory chapter of this thesis, several open problems arise on both aspects. First, although the mechanical behaviour of slender beam ligaments is well understood, the finite-width ligaments that often occur in mechanical metamaterials lead to new physics; wide beams exhibit a negative post-buckling stiffness, characterized by a decreasing force after buckling, which is not well understood. Second, fully filled microstructures of hinging squares constitute an auxetic mechanism [1, 20], but possible new zero modes derived from (diluted) microstructures with missing squares remain largely unexplored. How do the number of zero modes increase in diluted systems of hinging squares, can we count these, and what is the spatial structure of such modes? In this thesis, we address these open problems, thereby providing the necessary understanding to fully leverage the characteristics of wide beam ligaments and diluted collections of hinging squares for the design of novel mechanical functionalities.

In **chapter 2** we focus on beams and develop a 1D nonlinear model to describe the negative post-buckling stiffness, or subcritical buckling, of wide neo-Hookean [52] beams. We start by demonstrating that subcritical buckling is a robust phenomenon that does not originate from boundary-induced singularities nor from 3D effects. To this end, we compare experiments and fully realistic 3D numerical simulations against 2D simulations with idealized boundary conditions. In all cases, we find that the post-buckling stiffness of wide beams varies systematically with the beams aspect ratio t, and becomes negative for $t \gtrsim 0.12$. This allows us to focus on the simpler 2D setting to pinpoint the physical mechanism at stake in subcritical buckling. Specifically, we show that the missing crucial ingredient to account for subcritical buckling is the material nonlinearity in the axial stress-strain relation, which is due to the large deformations involved in wide beam buckling. We then construct an 1D energy density

functional by combining the Mindlin-Reissner beam description [41] with a nonlinearity in the axial stress-strain relation, and derive a closed set of beam equations by minimizing the beam's elastic energy. We have solved these equations analytically to determine the post-buckling stiffness in this model, and find excellent agreement between theory, experiments and simulations, without adjustable parameters. Altogether, the work presented in this chapter extends the understanding of the post-buckling of structures featuring wide elastic beams and opens up avenues for the design of post-instabilities in metamaterials.

In **chapter 3** we study the anomalous excess zero modes that arise in randomly diluted collections of rigid quadrilaterals, linked at their tips. The most basic example of an excess zero mode occurs for systems of full filling (no quads removed). Whereas large systems of generic quads are rigid, in contrast, large symmetric systems featuring regular, identical squares posses one global hinging zero mode [1, 20], irrespective of size. Here we focus on the number of excess zero modes, defined as the difference between the number of zero modes in symmetric and generic systems with identical dilution patterns. We perform a large number of independent simulations and show that the average number of excess zero modes as function of the number of removed quads exhibits a peak that exceeds one; this indicates there exist dilution patterns featuring more than one excess zero mode. By quantifying this (average) maximum as function of system size and the fraction of removed quads, we demonstrate that the number of excess zero modes is an intrinsic quantity, which exhibits finite size scaling with simple mean field exponents. Furthermore, we periodically tile a 6×6 unit cell to design dilution geometries with a density of zero modes that is six times higher than the peak value for random cutting, independent of system size. Lastly, we study the occurrence of excess zero modes for random bond removal and find strong similarities with the scaling behaviour for quad removal. In summary, this chapter demonstrates the existence of an arbitrary number of excess zero modes in randomly diluted systems of hinging squares.

In **chapter 4**, the final chapter of this thesis, we develop an approximate method to count the number of (excess) zero modes in systems of hinging squares. Starting from the observation that the occurrence of excess modes is driven by densely connected patches of quads (which have one zero mode in the symmetric case, but no zero modes in the generic case), we develop a procedure to partition a given system into clusters, connectors and remaining quads. We demonstrate that the remaining quads contribute very similarly to the number of zero modes in the generic and symmetric case, and are thus irrelevant for the number of excess zero modes. This allows us to focus on the simplified, pruned systems, which solely consist of clusters and connectors. To estimate the number of (excess) zero modes in the pruned systems we treat the clusters as 'black boxes' with four degrees of freedom (translation, rotation and hinging), and demonstrate how their motions are constrained by the presence of three type of connectors. One subtle feature of strongly connected clusters is that these exhibit inter-cluster self-stresses due to redundant connectors, and we eliminate most of these self-stresses by the iterative merging of clusters. This procedure yields an iterative discrete algorithm that estimates the number of zero modes. We finally compare these predictions against exact Hessian-based results, and find that our estimate is a tight lower bound on the number of (excess) zero modes of the pruned systems. Hence, we are able to predictively understand the number of (excess) zero modes in diluted systems of hinging squares.