Mechanical metamaterials: nonlinear beams and excess zero modes

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Introduction

> *Mechanical metamaterials are man-made materials which derive their unusual properties from their structure rather than their composition. Their spatial structure, or architecture, often consists of periodically arranged building blocks whose mutual interactions realize unusual properties, such as zero or negative elastic parameters [1].*

Anyone playing with a piece of rubber will have noticed that its sides expand when squeezing it. This is because rubber is essentially incompressible: Volume changes are energetically much more expensive than volume-preserving deformations. This example demonstrates material behaviour which is characterized by a positive Poisson's ratio (defined as the negative ratio of the transverse to axial strain [2]): Compressing (stretching) the material in one direction leads to the expansion (contraction) in directions transverse to the applied force [Fig. 1.1(a)]. Although a positive Poisson's ratio is a material property shared among the vast majority of conventional materials, the recent development of *mechanical metamaterials* realized the practical design of negative Poisson's ratio materials [3, 4]. These are counter-intuitive materials that either contract or expand in all directions when a force is applied [Fig. 1.1(b)], and are also known as *auxetic* materials or simply *auxetics*. Auxetics are widely studied because they feature enhanced properties in comparison to traditional materials, such as a higher indentation [5, 6] and fracture resistance [7], improved energy absorption [8] and the ability to perfectly wrap around objects such as spheres or domes [9–11]. The latter characteristic is for example exploited in industry to optimize the fit of footwear and prosthetics [12], but also for the design of curved aircraft wings and helicopter rotor blades [10, 13, 14].

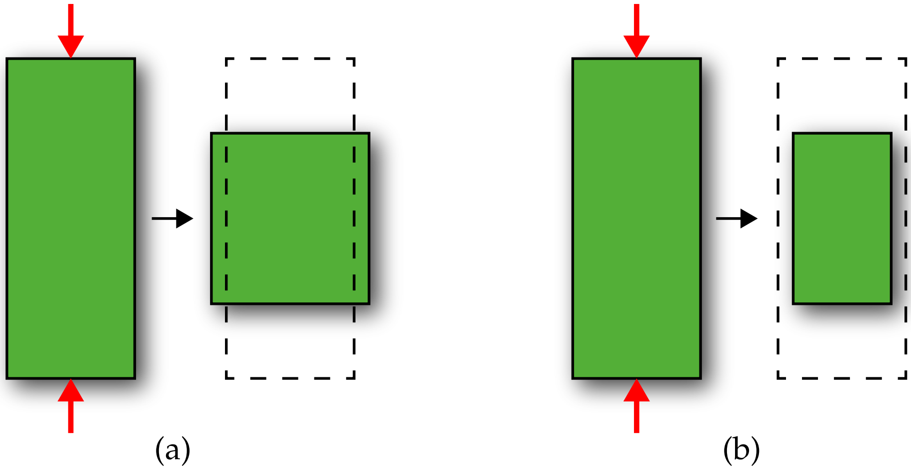

Figure 1.1: Initial and deformed shapes for the uniaxial compression of (a) conventional and (b) auxetic materials. Dashed lines plotted on top of the deformed shapes indicate initial material geometry.

The first development of an artificial auxetic metamaterial was reported in 1987 by Lakes [3]. His famous work outlines how auxetic foams can be produced from conventional foams by designing a structure consisting of three-dimensional, re-entrant unit cells (unit cells with inward deflected sides), and thus demonstrates how structure rather than composition determines the properties of the metamaterial. The regular, two-dimensional equivalent of the underlying re-entrant unit cell is shown in Fig. 1.2(a), which illustrates how a lattice of inverted honeycomb cells obtains its negative macroscopic Poisson's ratio. Followed by the work of Lakes, several other structures have been presented to architect auxetics [15–19]. Auxetic behaviour is particularly pronounced for the hinging motion of a collection of rigid squares [20] linked at their tips by flexible hinges [Fig. 1.2(b)]. This freely hinging, zero-energy motion, known as a *mechanism*, is characterized by the counter-rotations of squares (as indicated by circle-shaped arrows) that leads to a uniform contraction or expansion of the structure.

The key aspect that facilitates the auxetic behaviour for the model of hinged squares, are the sharp tips which connect adjacent squares [zoomed area in Fig. 1.2(b)]. These tips introduce strongly localized de-

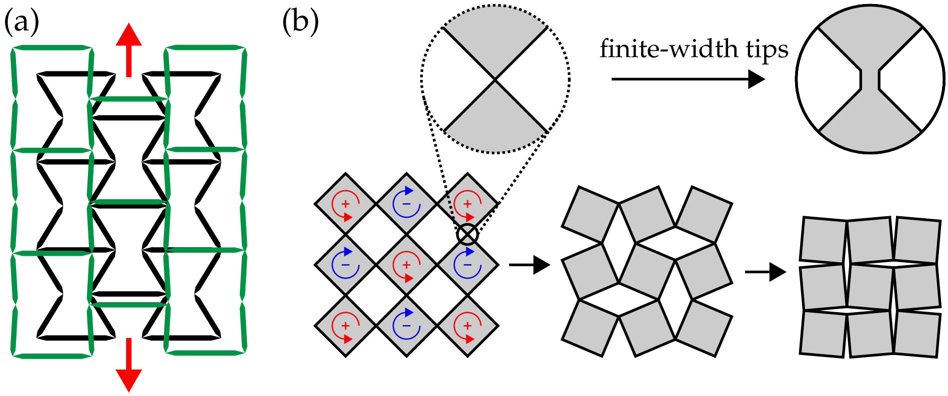

Figure 1.2: Auxetic metamaterials. (a) Structure with inverted honeycomb cells, which expands (shrinks) in the lateral direction when stretched (compressed) [21]. (b) A collection of squares linked at their tips exhibits a free hinging motion (indicated by the coloured arrows), which allows the structure to uniformly contract or expand [20]. Zoomed areas: The sharp tips localize bending and approximate ideal hinges (left). Finite-width tips can be regarded as beams (right).

formations, in contrast to the uniform deformation of a homogeneous sample of rubber. As a result deformations are non-affine [22, 23] and the macroscopic response becomes qualitatively different from its constituent material. Put simply, sharp tips approximate ideal hinges that allow for zero-energy rotational motion of the squares [1]. Moving away from the limit of sharp-tips, finite-width tips [Fig. 1.2(b)] lead to a finite-energy motion which closely resembles the underlying mechanism. Nonetheless, finite-width tips, which can effectively be seen as beams, add complexity to the mechanical properties of the metamaterial. Whereas deformations in sharp tips are entirely bending dominated, the beams in finite-width tips also undergo compressive deformations when excited by an external axial force. The mechanical complexity then arises from the energetic competition between compressive and bending deformations, and can lead to spontaneous symmetry breaking from straight to bent beams. This phenomenon is known as the buckling instability [24], which is associated with a strongly nonlinear relation between macroscopic stresses and strains. Metamaterials that encompass such a buckling instability are known as buckling-based metamaterials [1].

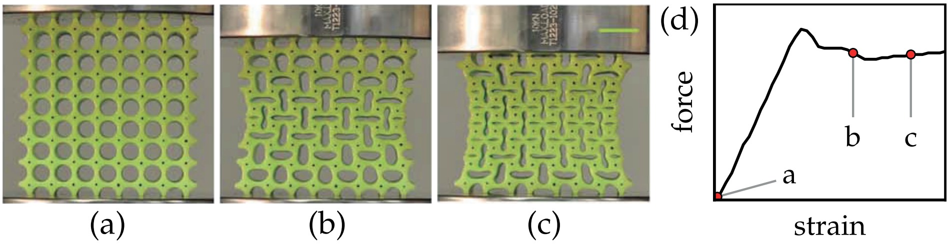

Figure 1.3: Buckling-based auxetic metamaterial. (a-c) Subsequent snapshots for the uniaxial compression of an elastic sheet patterned by circular holes. Under a sufficient load the beam elements undergo a collective buckling instability which induces a pattern transformation from holes to orthogonal ellipses. This pattern switch underlies the mechanism of rotating squares and results in an auxetic response (note how the sides move inwards). (d) Corresponding (experimental) force-strain relation with the (relative) forces and strains associated with panels (a-c) indicated by filled circles. Reproduced from reference [28] with permission from The Royal Society of Chemistry.

The simplest example of a buckling-based metamaterial is an elastic sheet patterned by a square array of circular holes [4, 25–28] as displayed in Fig. 1.3(a), which effectively resembles a structure of rigid squares connected by beams. When uniaxially compressed, the structure undergoes a pattern transformation, as shown in Fig. 1.3(a-c), which is triggered by a collective beam buckling instability. The scenario of collective beam buckling is reflected by the kink (at the maximum force) in the force-strain curve shown in panel (d), which is typical for a buckling instability [24]. Importantly, the characteristics of this pattern transformation are determined by the underlying mechanism. Hence, the properties of this metamaterial are determined by the interplay between the mechanical functionality of the beam elements, and the shape-changing properties of the underlying mechanism [29]. Both of these ingredients play a central role in this thesis, and in the sections hereafter we will discuss their properties in more detail.

1.1 Structural elastic instabilities

The loss of mechanical stability in elastic structures is widely studied and traditionally driven by the desire to design workarounds that can pre-

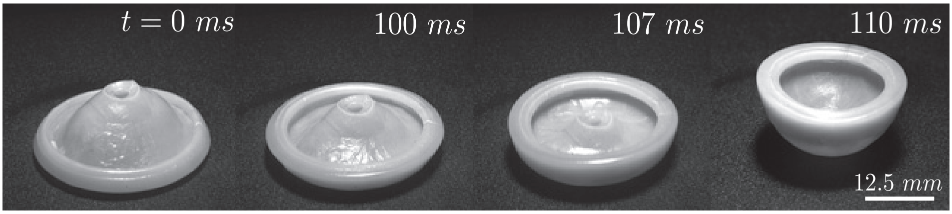

Figure 1.4: Snapping of a jumping popper toy. In order to jump, the shell like-structure first needs to be turned inside out to store elastic energy. Subsequently, the snap-through instability can be triggered by dropping it from a small height. The associated rapid change in curvature converts elastic energy into kinetic energy with an audible pop and causes the toy to jump. Image courtesy of D. Holmes, Boston University; the image has been published in reference [35].

vent structural failure [30]. Objects of focus have mainly been slender structural elements which are prone to buckling and snap-through instabilities, such as beams, plates, shells and frames [30]. More recently, structural instabilities of slender structures are recognized as an opportunity to generate new modes of functionalities in advanced materials [31]. In fact, biological systems make use of instabilities to obtain their functionality. Examples include the Venus fly trap [32] and the waterwheel plant [33], which exploit the rapid dynamics of a snap-through instability to catch their prey. Moreover, snapping lies at the basis of jumping bimetallic disks [34] and rubber popper toys [35] (Fig 1.4), which snap back towards their stable state after being turned inside-out.

Metamaterials that utilize structural instabilities for their functionality are known as instability-based metamaterials [1]. Buckling-based metamaterials [4, 25–28, 36–38] constitute a subclass of instability-based metamaterials that rely specifically on Euler buckling — known as the phenomenon where an elastic beam buckles under a sufficiently large compressive axial load [24], which is perhaps the simplest and most widespread instability. This type of metamaterials exhibit post-instabilities when subjected to compression, due to the collective buckling of beam ligaments which connect adjacent building blocks (e.g. as in Fig. 1.3). Although the pre-buckling regime and the onset of buckling is well understood, their post-buckling behaviour, which usually occurs far from equilibrium accompanied by large beam deformations, is not well developed yet [39]. In particular, the negative post-buckling stiffness, char-

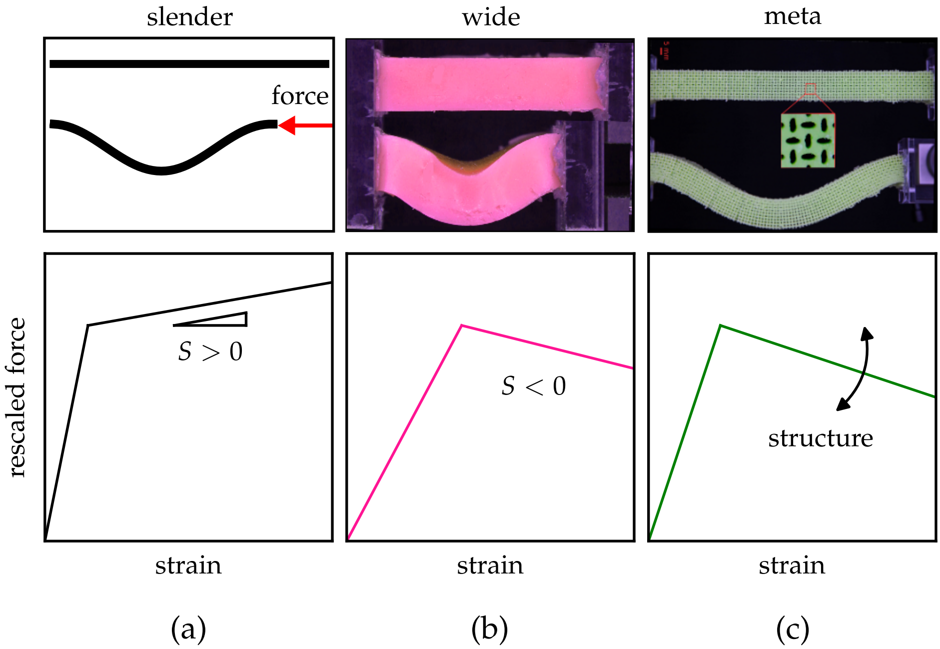

Figure 1.5: Force-displacement curves for the buckling of (a) slender beams, (b) wide beams and (c) metabeams. The post-buckling stiffness is quantified by the slope after buckling, S. For plain beams the post-buckling slope is a function of the beam width. The post-buckling slope of metabeams, however, can be tuned independently of beam width by changing their elliptical pore shapes. Other than utilizing instabilities, metamaterials can thus also be leveraged to change elastic instabilities. Image adapted from [40].

acterized by a decreasing force after buckling [Fig. 1.3(d)], is not well understood. Recently, we showed that the simplest possible setting in which the negative stiffness can be reproduced is on the level of a single beam ligament [40]. Fig. 1.5(a-b) shows how beam width crucially influences the post-buckling stiffness of beams: Slender beams display an increasing force after buckling (positive stiffness), but for wide beams the force after buckling decreases (negative stiffness). Metabeams [Fig. 1.5(c)], which are beams patterned with elliptical holes, can even be used to rationally design any post-buckling stiffness [40]. To our surprise negative post-buckling stiffness in the context of beams is not captured by existing models [41–43] and needless to say, a full understanding of the post-buckling behaviour of beams is necessary to take full advantage of the buckling instability in the design of buckling-based metamaterials.

In conclusion, sufficiently wide beams exhibit a negative post-buckling stiffness. This intriguing post-buckling behaviour is accompanied by large deformations, a combination which is theoretically not well described. Motivated by the role of (wide) beam ligaments in buckling-induced metamaterials, analysis of the post-buckling properties of beams plays a central role in this thesis. In chapter 2, we first identify the physical ingredient that induces negative post-buckling stiffness, followed by the development of a 1D model that accurately predicts the experimentally and numerically observed negative stiffness without adjustable parameters.

1.2 Role of geometry

In this section we zoom in on the role of geometry in the micro structure of mechanical metamaterials. We focus on the micro structure consisting of hinged square tiles [Fig. 1.2(b)], which composes the backbone of a range of 2D auxetic metamaterials [4, 20, 25–28] and which has inspired the design of programmable [44] and 3D mechanical metamaterials [36, 38]. Owing to its geometric design, the model of hinged squares provides a single zero-energy motion when connected by flexible linkers, also known as a mechanism or zero mode [1]. Consequently, metamaterials underlying the micro structure of hinged squares feature a single predefined mode of deformation. The natural question that arises is then whether new metamaterials can be constructed which are able to morph into multiple predefined shapes when excited by external forces.

To answer this question, we enlarge the design space and study mechanical metamaterials composed of aperiodic, rather than periodic micro structures. Recently, aperiodicity originating from unit cell orientations was applied to 3D mechanical metamaterials for the rational design of arbitrary pre-programmable shape changes [38] — featuring a single mode of deformation per chosen geometry. In this thesis, we construct 2D aperiodic micro structures by diluting (removing squares) the mechanism of hinged squares in order to pre-program multiple shape changes per geometry. Obviously, aperiodicity opens up pathways for the possible design of such structures, since the design space is greatly increased as a

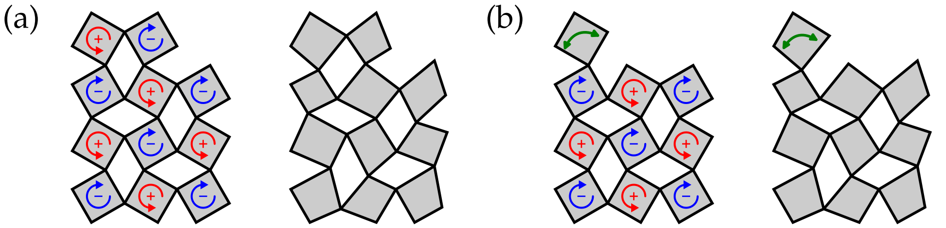

Figure 1.6: Symmetric versus generic systems. (a) The symmetric system (left) allows for a global hinging zero mode in which quads collectively counter rotate (indicated by the coloured arrows), but the generic system (right) is rigid. (b) Removing a quad from the top row allows the remaining quad on top to freely hinge (indicated by the green arrow), introducing an extra zero mode in both the symmetric and generic system.

result of square removal: For full filling, there exists only one way to tile space, but this number grows rapidly when squares are removed.

The resulting diluted systems comprise a two-fold problem. First, as mentioned, the diluted systems are interesting from the metamaterials perspective. Second, the non-generic nature of the square building blocks provides potential to unravel the rigidity of (randomly) diluted symmetric systems. The work presented in this thesis studies both of these aspects and will build a bridge between them. The rigidity of random spring networks has been widely studied [45–50], and is known as rigidity percolation. So far, the focus has been on generic disordered systems, in order to avoid degeneracies arising when symmetries are present. In this thesis however, we study the differences that arise between the rigidity of symmetric and generic systems.

Fig. 1.6 demonstrates two geometries for which the rigidity of systems composed by symmetric squares behave differently from systems composed by generic quads. First, panel (a) shows that sufficiently large generic systems are rigid, in contrast to symmetric systems that always exhibit a hinging mechanism, independently of system size. Using Maxwell counting [51], as will be discussed in chapter 3, it can readily be shown that the minimum system size to ensure rigidity of generic systems is 3×3. Next, panel (b) shows that removing a quad from the top row introduces an additional zero mode in both the symmetric and generic system. Thus, panel (a-b) show two examples in which symmetric systems exhibit one *excess zero mode* in comparison with its generic counterpart. It is conceivable, however, that the number of excess zero modes is not bounded to one, but can become larger for more complex geometries — examples will be discussed in chapter 3.

The above examples demonstrate some of the complexity occurring in the rigidity of symmetric systems and several open questions arise. Can the number of excess zero modes become arbitrary large? What are the necessary ingredients that facilitate the excess zero modes? How can we count the number of excess zero modes given a random dilution pattern? A comprehensive treatise on these questions, and more, are devoted to chapters 3 and 4 of this thesis.

1.3 In this thesis

In this thesis we investigate two aspects of mechanical metamaterials: The role of beam ligaments and the role of symmetries. As motivated above, both aspects play an important role in mechanical metamaterials, and give rise to several open questions.

We first elucidate the negative post-buckling stiffness of wide neo-Hookean beams in chapter 2. We show that the negative stiffness occurs in experiments, 3D simulations and simplified 2D simulations, demonstrating that negative post-buckling stiffness is a robust phenomenon that does not originate from boundary-induced singularities or 3D effects. Using the simplified 2D simulations we then identify the missing physical ingredient — which we will show to be a material nonlinearity — that underlies the negative post-buckling stiffness. Finally, we use this material nonlinearity to build a 1D nonlinear beam model that, without adjustable parameters, successfully captures the intriguing post-buckling behaviour of wide neo-Hookean beams.

In the remainder of this thesis we turn to collections of hinging quadrilaterals in order to probe the role of symmetries. In chapter 3 we investigate randomly diluted lattices of hinging squares, and compare these directly to generic systems featuring irregular quads, when using the same dilution pattern for both. We then demonstrate that symmetric systems exhibit excess zero modes as compared to the generic systems, with their multitude satisfying simple scaling relations with mean field exponents.

Finally, in chapter 4 we develop an approximate, yet accurate method to count the number of excess zero modes in diluted systems of hinging squares. We note that these modes are driven by densely connected patches of quads — clusters — and develop a procedure to separate any system into clusters, connectors and remaining quads. Using the topology of the clusters and their connectors we then iteratively estimate the number of (excess) zero modes, which allows us to obtain a tight lower bound on the exact, numerical results for their multitude.