Journal of Mathematical Behavior 45 (2017) 167–182

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Graphing formulas: Unraveling experts' recognition processes

Peter M.G.M. Kop[a,*], Fred J.J.M. Janssen[a], Paul H.M. Drijvers[b], Jan H. van Driel[c]

[a] Iclon, Leiden University, The Netherlands
[b] Freudenthal Institute, Utrecht University, The Netherlands
[c] Melbourne Graduate School of Education, The University of Melbourne, Australia

ARTICLE INFO

Article history:
Received 10 June 2016
Received in revised form 12 January 2017
Accepted 25 January 2017
Available online 9 February 2017

Keywords:
Graphing formulas
Experts' recognition
Function families
Prototypes and attributes

ABSTRACT

An instantly graphable formula (IGF) is a formula that a person can instantly visualize using a graph. These IGFs are personal and serve as building blocks for graphing formulas by hand. The questions addressed in this paper are what experts' repertoires of IGFs are and what experts attend to while recognizing these formulas. Three tasks were designed and administered to five experts. The data analysis, which was based on Barsalou and Schwarz and Hershkowitz, showed that experts' repertoires of IGFs could be described using function families that reflect the basic functions in secondary school curricula and revealed that experts' recognition could be described in terms of prototype, attribute, and part-whole reasoning. We give suggestions for teaching graphing formulas to students.

1. Introduction

Algebraic concepts, like functions, can be explored more deeply through linking different representations (Duval, 2006; Heid, Thomas, & Zbiek, 2012). Graphs and algebraic formulas are important representations of functions. Graphs seem to be more accessible than formulas (Leinhardt, Zaslavsky, & Stein, 1990; Moschkovich, Schoenfeld, & Arcavi, 1993). In addition, graphs give more direct information on covariation, that is, how the dependent variable changes as a result of changes of the independent variable (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). A graph shows features such as symmetry, intervals of increase or decrease, turning points, and infinity behavior. In this way, it visualizes the "story" that an algebraic formula tells. Therefore graphs are important in learning algebra, in particular in learning to read algebraic formulas (Eisenberg & Dreyfus, 1994; Kieran, 2006; Kilpatrick & Izsak, 2008; NCTM, 2000; Sfard & Linchevski, 1994).

Students have difficulties in seeing a function both as an input-output machine and as an object (Ayalon, Watson, & Lerman, 2015; Gray & Tall, 1994; Oehrtman, Carlson, & Thompson, 2008; Sfard, 1991). Graphs appeal to a gestalt-producing ability, and in this way can help to consolidate the functional relationship into a graphical entity (Kieran, 2006; Moschkovich et al., 1993). Graphs are also considered important in problem solving. Graphs are used for understanding the problem situation, recording information, exploring, and monitoring and evaluating results (Polya, 1945; Stylianou & Silver, 2004).

So, the ability to switch between representations, representation versatility, in particular conversions from algebraic formulas to graphs, is important in understanding algebra and in problem solving (Duval, 2006; NCTM, 2000; Stylianou, 2011; Thomas, Wilson, Corballis, Lim, & Yoon, 2010).

* Corresponding author.
E-mail address: koppmgm@iclon.leidenuniv.nl (P.M.G.M. Kop).

http://dx.doi.org/10.1016/j.jmathb.2017.01.002

In a previous study a framework was developed to describe strategies for graphing formulas without using technology (Kop, Janssen, Drijvers, Veenman, & Van Driel, 2015). In the framework, it is indicated how recognition guides heuristic search. When one has to graph a formula there are different possible levels of recognition: from complete recognition (one immediately knows the graph) to no recognition at all (one does not know anything about the graph). For every level of recognition the framework provides strong to weak heuristics.

For the two highest levels of recognition the graph is completely recognized or the formula is recognized as a member of a function family whose graph characteristics are known. For instance, at the highest level of recognition the graph of $y = x^2$ is instantly recognized as a parabola with minimum (0,0). At the second level of recognition, $y = 4{\cdot}0.75^x + 3$ is recognized as a member of the family of decreasing exponential functions, and so the horizontal asymptote is read from the formula. In this way the graph can be instantly visualized. Another example at this level: $y = -x^4 + 6x^2$ is recognized as a polynomial function of degree 4; because of the negative head coefficient its graph has an M-shape or an Λ-shape; a short investigation of, for instance, the zeroes will instantly give the graph.

At these two highest levels of recognition in the framework, formulas can be instantly linked to graphs. Therefore, these formulas are defined as *instantly graphable formulas* (IGF). A large set of IGFs is beneficial to proficiency in graphing formulas. The current study was focused on experts' recognition processes when dealing with IGFs. For this study we defined an expert as a person with at least a master's degree in mathematics and at least 10 years of experience teaching at the secondary or college level, with experience in graphing formulas by hand. Although these experts are expected to be able to instantly link many formulas to graphs, their repertoires of IGFs remain unknown. In addition, we investigated what experts attend to when recognizing IGFs. This information might give suggestions for a repertoire of IGFs for students and for a focus in teaching students IGFs.

2. Theory

2.1. Cognitive units as building blocks

IGFs can be seen as building blocks in thinking and reasoning with and about formulas and graphs. Barnard and Tall (1997) introduced the concept of "cognitive unit", an element of cognitive knowledge that can be the focus of attention altogether at one time. For experts, well-connected cognitive units can be compressed into a new single cognitive unit which can be used as just one step in a thinking process (Crowley & Tall, 1999). In this way experts' knowledge is well organized in hierarchical mental networks with complex cognitive units, which can be enlisted when necessary (Campitelli & Gobet, 2010; Chi, Feltovich, & Glaser, 1981; Chi, 2011).

As IGFs are cognitive units in graphing formulas, they can be combined (addition, multiplication, chaining, etc.) and can form new, more complex IGFs. For instance, when dealing with $y = -x^4 + 6x^2$, novices may recognize the IGFs $y = -x^4$ and $y = 6x^2$ and have to combine these two IGFs to draw a graph, whereas $y = -x^4 + 6x^2$ is an IGF for experts, who recognize a 4th degree polynomial function. For experts, a formula like $y = x^2 - 6x + 5$ can trigger other cognitive units, like "its graph is a parabola with a minimum value", and the equivalent formulas $y = (x-1)(x-5)$ and $y = (x-3)^2 - 4$, which can give information about the zeroes and the minimum value, etc. Experts are expected to have more, and more complex, IGFs than novices, which generally enable them to graph formulas with fewer demands on the working memory (Sweller, 1994).

The current study was focused on recognition: in particular, which formulas and/or function families were instantly recognized by experts and how the recognition processes can be described.

2.2. Recognition described using Barsalou's model with prototype, attribute, and part-whole reasoning

Barsalou (1992) showed how human knowledge is organized in categories or concepts. People construct these categories based on attributes. When a task requires a distinction to be drawn between exemplars of a category, people construct new attributes and in this way new categories (Barsalou, 1992). For instance, for the concept bird, attributes (variables) like size, color, and beak, with several values, can be used to distinguish different exemplars. Categories can have a large diversity of exemplars, but have a graded structure (Eysenck & Keane, 2000; Barsalou, 2008). Some exemplars in a category are more central to that category than others; these are called prototypes. For instance, a robin is considered a more typical example of a bird than, for instance, a chicken or a penguin. When dealing with exemplars of a category, people tend to associate prototypical features with these exemplars (Barsalou, 2008; Schwarz & Hershkowitz, 1999). The tendency to reason from prototypes can pose problems. Since concept formation is not necessarily done using pure definitions, Watson and Mason (2005) emphasized the need to go beyond prototypes and to search for the boundaries of a concept. In this way one becomes aware of the dimensions of possible variation and in each dimension of the range of permissible change (Bills et al., 2006; Sandefur, Mason, Stylianides, & Watson, 2013; Watson & Mason, 2005). The personal example space, the collection of examples and the interconnection between the examples a person has at his/her disposal (the accessible example space), play a major role in how a person makes sense of the tasks he/she is confronted with (Watson & Mason, 2005; Goldenberg & Mason, 2008). Vinner and Dreyfus (1989) used concept image to emphasize the personal character of people's mental networks. These concept images determine what a person "sees" when dealing with concepts or categories, and are used in rapid identification.

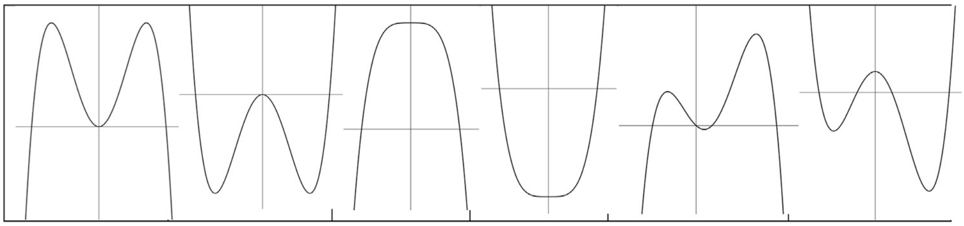


Fig. 1. Graphs of 4th degree polynomial functions.

Schwarz and Hershkowitz (1999) used prototypicality, attribute understanding, and part-whole reasoning as aspects to portray students' concept images of functions. We discuss these three aspects below.

Prototypicality refers to the prototypes (prototypical exemplars) a person knows and uses. Prototypes can be defined as the exemplar(s) with the set of highest frequency of attribute values in the category or with the highest correlation with other exemplars in the category (Barsalou, 1992). Prototypes are the examples that are acquired first and are usually the examples that have the longest list of attributes: the critical attributes of the category and the self-attributes (non-critical attributes) of the exemplar (Schwarz & Hershkowitz, 1999). Prototypes are used as a reference point for judging membership of the category: an exemplar is judged to be a member of a category if there is a good match between its attributes and those of the category prototype (Barsalou, 2008; Eysenck & Keane, 2000). When asked for a prototype of a category, it is expected that a person will not use a definition of prototype but will use a general idea about what prototypes are: namely, the most central exemplar(s) of a category from his/her personal perspective. As a consequence, when dealing with a category, the prototypes are the first examples that come to one's mind and are the natural examples that are used without any explanation. Examples in the domain of graphing formulas include prototypical formulas like $y = x^2$ and $y = x^3$, with their prototypical graphs. In this study we used the term prototype reasoning in this way.

Attribute understanding can be defined as the ability to recognize the attributes of a function across representations (Schwarz & Hershkowitz, 1999). For instance, from the formula $y = (x-1)(x-5)$, it is concluded that its graph is a parabola, it has zeroes at $x = 1$ and at $x = 5$ and a symmetry axis at $x = 3$. These attributes or properties of this function can be recognized in the graphical, tabular, and algebraic representations.

In his property-oriented view of functions, Slavit (1997) used properties (or attributes) like symmetry, monotonicity, horizontal and slant asymptotes, intercepts (zeroes), extrema, and points of inflection.

Depending on the task, people construct attributes to be able to distinguish exemplars: in this study, formulas and graphs (Barsalou, 1992). To distinguish different graphs of 4th degree polynomial functions in Fig. 1, one can use attributes like symmetry, infinity behavior, number of turning points, number of zeroes, and location of zeroes relative to the y-axis. When relating formulas and graphs, as in graphing formulas, one chooses or creates attributes to focus on features of formulas and graphs. We call this reasoning about attributes and their values attribute reasoning.

Part-whole reasoning refers to the ability to recognize that different formulas or different graphs relate to the same entity: in this case, to the same function. In the graphical representation, different scaling can result in different pictures of graphs belonging to the same function. In the algebraic representation, formula manipulation can result in different formulas of the same function: for instance, $y = x^2 - 4x$, $y = (x-2)^2 - 4$, and $y = x(x-4)$. From these different formulas different attributes of the graph can be read. Therefore, part-whole reasoning is important in the recognition of IGFs.

For attribute reasoning and part-whole reasoning one has to grasp the structure of a formula. In the literature this is called symbol sense (Arcavi, 1994). Symbol sense is a very general notion of "when and how" to use symbols and has several aspects, such as the ability to read through algebraic expressions, to see the expression as a whole rather than a concatenation of letters, and to recognize its global characteristics (Arcavi, 1994). Pierce and Stacey (2001) used algebraic insight to capture the symbol sense in transformational activities in the "solving" phase of problem solving (Pierce and Stacey, 2001). The algebraic insight is divided in two parts: algebraic expectation and the ability to link representations. Algebraic expectation has to do with recognition and identification of objects, forms, key features, dominant terms, and meanings of symbols (Kenney, 2008; Pierce & Stacey, 2001). Algebraic insight is shown when a person has expectations about graphs that are linked to features of the symbolic representation and when equivalent algebraic expressions are recognized (Ball, Stacey, & Pierce, 2003; Pierce and Stacey, 2001, 2004).

The three aspects prototype, attribute, and part-whole reasoning from Schwarz and Hershkowitz can be used to describe the recognition process in graphing formulas. A Barsalou model for recognizing IGFs is formulated in Fig. 2. In the case of graphing formulas, it is difficult to mention all possible values. For instance, the attribute "zeroes" can have values like 0,1,2,3, to indicate the number of zeroes, but also the location can be used as values of an attribute (for instance, a zero at $x = 5$). For the sake of readability, the values belonging to the attributes are omitted in Fig. 2.

The Barsalou model in Fig. 2 shows how function families are constructed by using value sets on a set of attributes and allows a detailed description of how formulas can be linked to graphs, and so of the recognition of IGFs. Starting with a formula (on the right side of Fig. 2), there are several possibilities: the formula can be manipulated (part-whole reasoning) into another formula, the formula can be recognized as a member of a function family, or the formula can be recognized as a

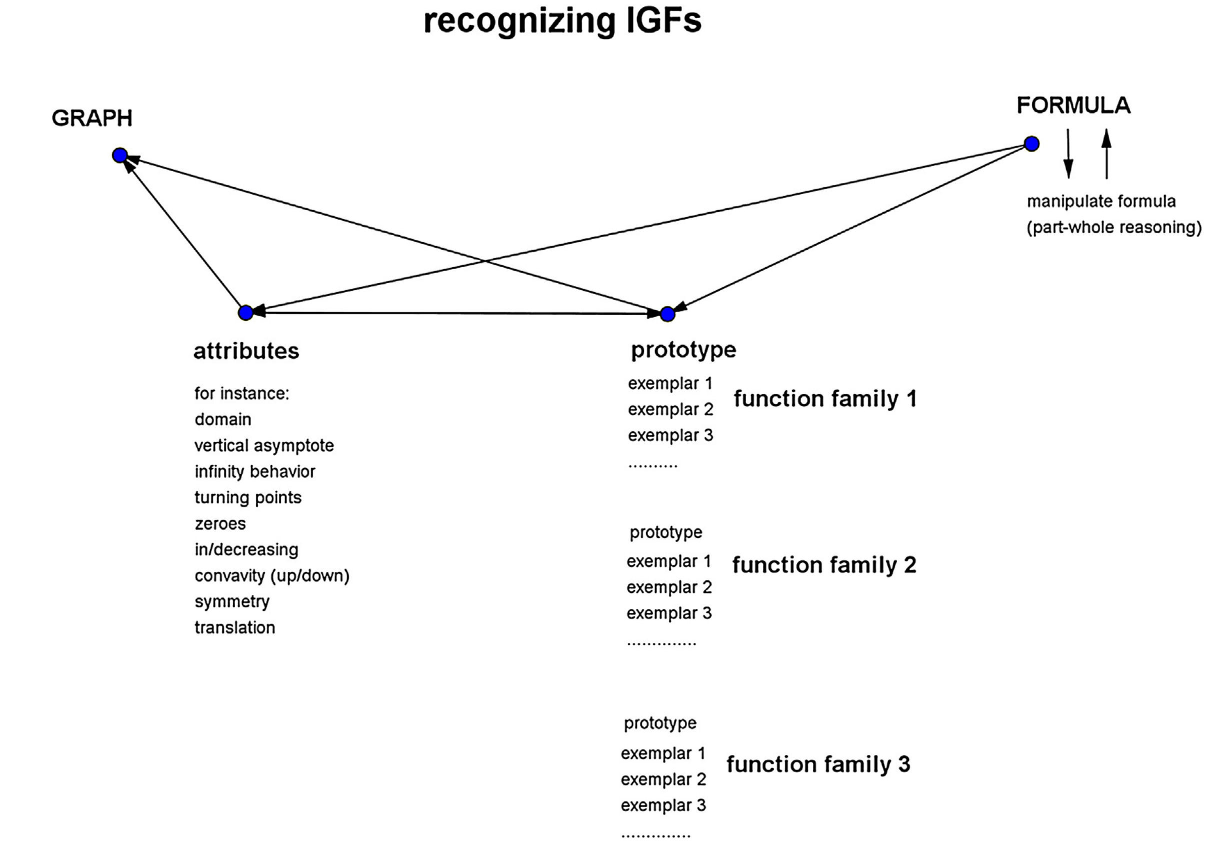


Fig. 2. IGFs in the form of a Barsalou model based on Schwarz and Hershkowitz (1999).

prototype of a function family. It is then possible that the graph is directly known, or that, using attribute reasoning, a graph can be visualized.

Some examples can illustrate this recognition process. In IGF $y = 4 \cdot 3^x + 2$ the prototype 3^x can be recognized (prototype reasoning), and via a translation (attribute reasoning) the graph can be visualized. In IGF $y = -2x(x-3)(x-6)$, the prototype x^3 can be recognized, $-x^3$ as a reversion (attribute reasoning), and via zeroes at $x=0$, $x=3$, $x=6$(attribute reasoning) the graph can be visualized. However, when $y = -2x(x-3)(x-6)$ is not recognized as a member of a function family or prototype of a function family, the formula is not an IGF (Kop et al., 2015). In this case the graph has to be constructed by, for instance, reasoning about attributes like infinity behavior and zeroes. If, when graphing $y = 4x^{-2}$, the formula can be rewritten to $y = 4/x^2$(part-whole reasoning) and recognized as a $1/x^2$(prototype reasoning), the formula is an IGF. But when from the formula $y = 4/x^2$ it is read that it has a vertical asymptote at $x = 0$, and that all outcomes are positive and when $x \to \pm\infty$ then $y = 0$ (infinity behavior), then we say that the graph is constructed through qualitative reasoning (Kop et al., 2015), and so the formula is not an IGF.

2.3. Global and local perspectives

Covariational reasoning is essential for graphing formulas. In covariational reasoning, one is able to imagine running through all input-output pairs simultaneously and so to reason about how a function is acting on an entire interval of input values (Carlson et al., 2002). In recognizing IGFs one has to have a picture of the function as an entity. In the literature this perspective of the function, seeing the function as a whole, is also addressed as the object or global perspective (Confrey & Smith, 1995; Even, 1998; Gray & Tall, 1994; Oehrtman et al., 2008; Sfard, 1991). There is also another perspective of the function, namely, to see a function as an input-output machine. This perspective has to do with the fundamental view on functions (what it means that a certain y-value belongs to a given x-value), and is addressed as the pointwise, process, or correspondence perspective. Switching between both kinds of perspective is necessary for reasoning about functions. Slavit (1997) spoke about the local and global nature of functional growth properties in addressing both kinds of perspective (Slavit, 1997). The global growth properties concern attributes like symmetry, monotonicity, horizontal and slant asymptotes, integrability, and invertibility, whereas the local properties are about extrema, intercepts, cusps, and points of inflection. In an in-between class, Slavit also mentioned continuity, sign, differentiability, domain, and range. Graphs can be described

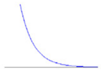
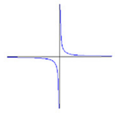
10^{-2x+5}	$1-5/(x+1)$	$2x^3(6-x)$	$6-2^x$
x^4-16x^2+28	$2\sqrt[3]{x}$	$3x^{-2}$	$4\sqrt{10-x}$
$(x^2-7)^2$	$(x+3)^4-9$	$(2x^{\frac{1}{3}})^5$	$(1-x)(2+x)+x^2$
$2x\sqrt{x}$	$(100x)^{\frac{1}{2}}$	$x-4/x$	$\sqrt{8-x^2}$
			

Fig. 3. A number of the cards used in task 1.

using these properties or attributes. Before the current research, it was unknown which attributes experts use in recognizing IGFs.

2.4. Research questions

In the current study we focused on experts' repertoires of formulas that can be instantly visualized using a graph (IGFs) and on their concept images of IGFs, with attributes, prototypes, and part-whole reasoning. We expected that experts would have large repertoires of IGFs that are structured in categories. However, we did not yet know what an expert repertoire of IGFs would be.

We expected experts to be able to manipulate algebraic formulas (part-whole reasoning), to use symbol sense and in particular algebraic insight, and to use sets of attributes with value sets to distinguish different graphs. However, we did not know which prototype, attribute, and part-whole reasoning they would use in linking formulas and graphs of IGFs.

This lead to the following research questions:

Can we describe experts' repertoires of instant graphable formulas (IGFs) using categories of function families?

What do experts attend to when linking formulas and graphs of IGFs, described in terms of prototype, attribute, and part-whole reasoning?

3. Method

The current study can be characterized as an exploratory study, in which we investigated "snapshots" of experts' concept images of function families with their algebraic formulas and graphs.

3.1. Tasks

Three different tasks were developed to elicit the experts' repertoires of IGFs and to explore the experts' prototype, attribute, and part-whole reasoning: a card-sorting task, a matching task, and a multiple choice task.

Card-sorting tasks are often used in eliciting structured knowledge (Chi et al., 1981; Jonassen, Beissner, & Yacci, 1993; de Jong & Ferguson-Hessler, 1986; Goldenberg & Mason, 2008; Sandefur et al., 2013).

In task 1, 60 formulas were given and the participants were asked to categorize them according to their graph. After this, they were asked to give a name and a prototypical formula for each of their categories. We structured this task by adding graphs to the cards showing the formulas. When such tasks are given without structuring beforehand, getting a complete picture or comparing the results can pose problems, because of the different criteria that can be used to sort the cards (Ruiz-Primo & Shavelson, 1996). Because we add four graphs to the 60 cards with formulas, the participants were explicitly compelled to focus on the graphs of the formulas. We did not indicate whether a participant should discriminate between parabolas with a maximum or minimum because the level of detail can be an indicator of expertise. Fig. 3 shows 20 cards from task 1. Most of these formulas, but not all, are related to one of the basic function families, which are studied in grades 10–12: $y=x^n$, $y=a^x$, $y=\log_2(x)$, $y=1/x$, $y=\sqrt{x}$, $y=\ln(x)$, $y=e^x$. Since, we used the basic functions from secondary school curricula, we expected that many formulas, but not all, would be IGFs for the experts. This categorization task gave information about dimensions of variation and the range of permissible change experts used in discriminating graphs. The names given for the different categories with the prototypes gave insight into the graph families and thus in the attribute and value sets experts used.

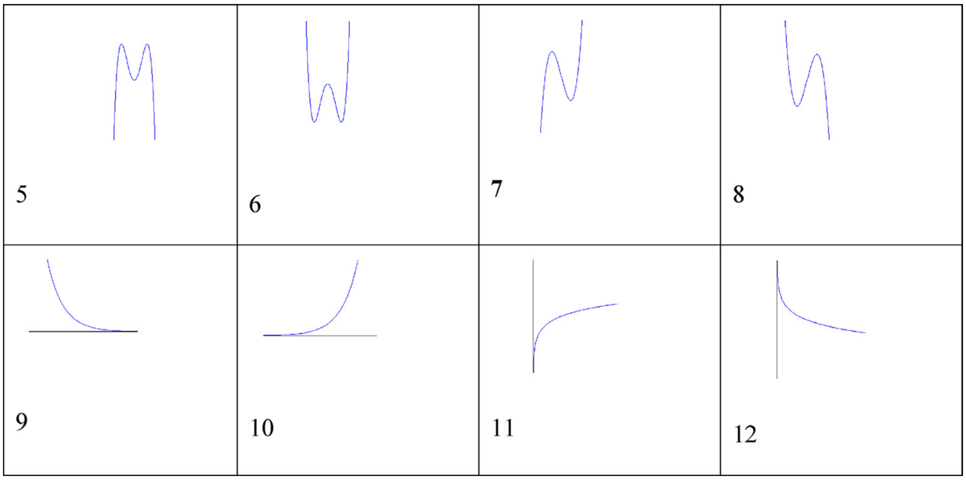


Fig. 4. Some alternatives of task 2.

In Task 2, the matching task, a list of 40 formulas was given and the participants were asked to select the correct alternative out of 21 alternatives: 20 graphs and one alternative stating "none of these". This last alternative was provided to discourage guessing. In this task the focus was on instant linking of formulas to the global shape of graphs. Therefore a strict time limit was used to encourage recognition and to discourage construction of a graph. We chose a matching task with many alternatives rather than a graphing task to indicate the level of detail that was needed: the experts had to recognize the global shape of the graph of the given formula.

The formulas used in this task resembled the formulas used in the first task. The following are some examples: $y = 2x(x-2)(x+4)$; $y = 6x^2 - 2x^4$; $y = e^{2x} + 1$; $y = x - 4/x$; $y = 4 - 2x + x/4$; $y = 4/2^x$; $y = \sqrt{x-6} + 2$; $y = \ln(e^2 \cdot x)$; $y = 2x^{-4}$; $y = 2(x-1)^4 - 4$;$y = \sqrt{8 - x^2}$; $y = \ln(4/x)$; $y = 9x/\sqrt[3]{x}$; $y = \ln(e^2 \cdot x)$. Eight of the alternative graphs are shown in Fig. 4.

Task 2 was also developed to elicit participants' repertoires of IGFs. Therefore some functions were added that do not belong to the function families of basic functions, for instance,$y = \sqrt{8 - x^2}$, $y = 30/(x^2 - 16)$, $y = x - 4/x$, because we wanted to investigate the boundaries of the experts' repertoires of IGFs. Because the formulas used were similar to those in task 1, this task was used to validate the results of task 1. When, for instance, in task 1 no distinction was made between increasing and decreasing parabola, but in task 2 this distinction was made, it was concluded that the participant could indeed make such a distinction.

Tasks 3A and 3B, thinking aloud multiple choice tasks, were developed to elicit the participants' prototype, attribute, and part-whole reasoning and in this way to get more detailed knowledge of the participants' concept images. The participants were asked to choose the correct alternative out of four alternatives. A similar task was used by Schwarz and Hershkowitz (1999) in their study of concept images of functions. Both tasks consisted of six items. In task 3 B a formula was given and the experts had to find the correct graph. In task 3A a graph was given and the experts had to provide a formula. In general, tasks like 3A are considered to be more challenging. But this is not clear when dealing with the function families of well-known basic functions. In this way we got more detailed information about the experts' concept images of IGFs. Three examples of this task are shown in Fig. 5.

The formulas were again chosen from the same set of functions as in tasks 1 and 2. Participants had to consider all alternatives because more than one alternative could be correct.

In tasks 1 and 2 the focus was on sketches of graphs; in this task, more detailed answers were needed. For instance, in tasks 1 and 2 it was not necessary to distinguish $y = -2x(x-2)(x-4)$ and $y = -2x(x+3)(x+6)$, but in task 3 this distinction had to be made (see task 3A-4 in Fig. 5).

3.2. Participants

Five mathematical experts were invited to participate in this study. We assigned the letters P, Q, R, S, and T to our five experts. The experts had different backgrounds: two mathematicians who had been teaching calculus and analysis to first-year students at university (Q, R), one author of a mathematics textbook series, who had been a teacher in secondary school (T), one math teacher who was involved in the National Math Exams and had been a secondary school teacher (S), and one math teacher educator in university (P). All had a Master's degree in mathematics and two had a PhD in mathematics (Q, R). All of them had been working as a teacher at university or in secondary education for more than 20 years and had been graphing many formulas without technology during their education and during their whole teaching career. Therefore, we considered them experts in graphing formulas.

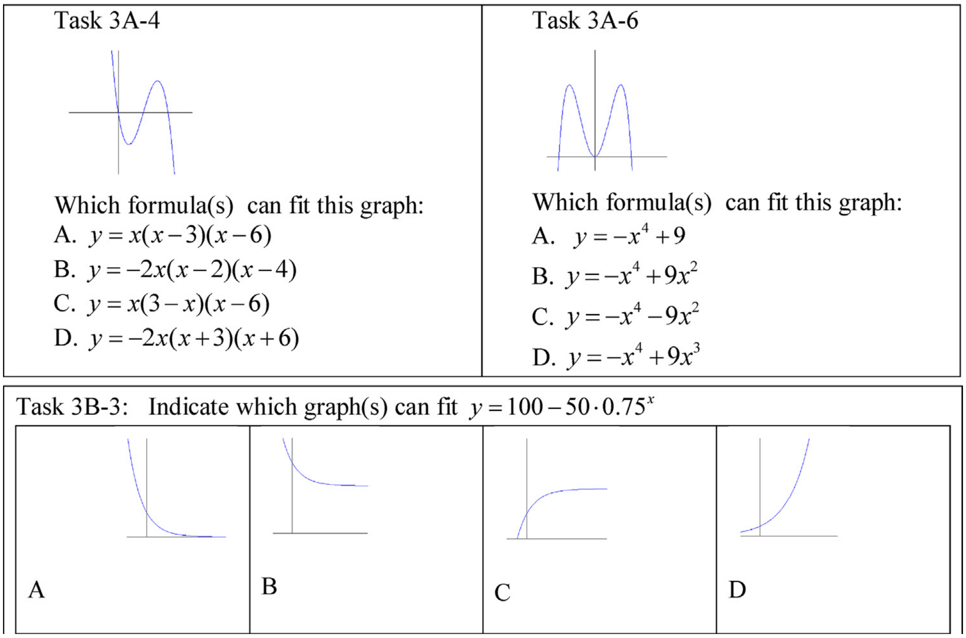


Fig. 5. Some examples of task 3: task 3A-4, 3A-6, 3B-3.

3.3. Data and data analysis

3.3.1. Data collection procedure

Written instructions were handed out for every task, together with an indication of the time needed to perform the task. For task 1, a time indication of maximum 40 min was given; for tasks 2 and 3, 20 min. For all tasks, the time needed was recorded, as the time required to perform a task can be an indication of expertise. During the tasks the first author only emphasized the need to keep on thinking aloud when the experts stopped talking. After each task, the first author asked the experts to look back and to describe the strategies, they had used in the task. The interviews were videotaped.

In task 1, the card-sorting task, 60 cards were laid on a table and the participants could physically group the formulas into different categories. Afterwards, the categories were glued on a large sheet. The participants then wrote the category names and the prototypical formulas for each category. In task 2 and task 3, the participants filled in the answers on a form.

During tasks 1 and 3 the participants were asked to think aloud; this was videotaped. Thinking aloud is considered to give reliable information about the problem-solving activities without disturbing the thinking process (Ericsson, 2006). For task 3 the thinking-aloud protocols were transcribed in order to analyze the prototype, attribute, and part-whole reasoning.

3.3.2. Data analysis

3.3.2.1. Task 1. The aim of task 1 was to gather information on which categories experts use in their repertoires of IGFs. It was expected that experts would use salient, global properties of graphs, like symmetry, in/decreasing, vertical asymptotes, infinity behavior, and number of turning points, to categorize their IGFs. Based on these salient properties, the first author made a theoretical, hypothetical experts' categorization before the start of this study. The categorizations of the five experts were compared with each other and with the first author's categorization. Based on these findings a common categorization was constructed. This was done in several steps. First, common elements in the categories and prototypes in the experts' categorizations and the first author's categorization were determined. From these findings a preliminary common expert categorization was formulated. In the second step, the level of detail was considered. A higher level of detail meant that subcategories were used. If one or more experts used a higher level of detail, then this level of detail was used in the (final) common expert categorization. In the last step, the distances between individual categorizations and the common expert categorization were calculated. We considered whether small adjustments in the common expert categorization would result in a lower minimum of the total of all distances. When no progression could be made, the final common expert categorization was found.

To determine the distance between an individual categorization and a common categorization, the following protocol was used:

- If the individual categorization had the "same" category but a formula was not mentioned or did not belong to that category, then the distance increased by +1

- If no subcategories were made in the individual categorization and the common expert categorization made a distinction between increasing and decreasing, then the distance increased by +2 (for instance, no subcategories between parabolas with maximum and parabolas with minimum gave an increase of the distance by +2 if the common expert categorization made this distinction)
- If two categories of the common expert categorization were merged in the individual categorization (other than the distinction between increasing and decreasing), then the distance increased by +4 (for instance, 3rd and 4th grade functions were put together in one category)
- If a completely new category, different from the common expert categorization, was formulated, then the distance increased by +6.

3.3.2.2. Task 2. In this task the numbers of mistakes per expert were counted. The mistakes were indicated in a table in order to see whether they were made in particular function families.

3.3.2.3. Task 3. To analyze the results of task 3, the transcripts were cut into fragments which contained crucial steps of explanations: idea units. Idea units are primitive elements in the justifications of participants (Schwarz & Hershkowitz, 1999). These idea units were encoded using the elements from Fig. 2: prototype, attribute, or part-whole reasoning.

Since prototypes are the natural examples of categories that can be used without any explanation, graphs and formulas that a participant used as the start of a reasoning process were considered prototypes for the expert. If no prototype reasoning was used or function family was mentioned, we said that the formula was not an IGF, and that the graph was constructed.

The fragments of the protocols were encoded as follows:

- pr (prototype reasoning): only a prototypical exemplar was mentioned; for instance, “it looks like a log”, “it is an x in the power 6”, “it is an expo”, “it is an oscillation”. If a function family was mentioned, like in “it is an exponential function” or “4th degree polynomial”, this was considered prototype reasoning
- att (attribute reasoning): an attribute was mentioned; for instance, “this one has a vertical asymptote at $x = 0$”, “it is always positive”, “it goes to minus infinity”.
- pw (part-whole reasoning): the formula was manipulated to an equivalent formula, for instance, $y = 4x^{-2}$ to $y = 4/x^2$
- con (construction): no function family or prototype was mentioned, the formula was not an IGF: the graph was constructed through, for instance, attribute reasoning or calculating points.

We give two examples of the encoding in Fig. 6.

4. Results

4.1. Results of task 1

The experts’ and authors’ categorizations are shown in Appendix A. The experts showed a great deal of agreement in their choices of categories, names of these categories, and prototypes of the categories. Only expert S used a different approach in his categorization of polynomial functions. He based his categorization on the number of turning points. The other experts all used the degree of polynomial functions. The 4th degree polynomial functions were divided into graphs with a W-form, a M-form, and a V-form (or as the experts mentioned, “increasing or decreasing”). No large differences were found on exponential functions and logarithmic functions, although some experts (P and R) made no distinction between “normal” and “reversed” graphs (for instance, $y = e^x$ versus $y = e^{-x}$ and $y = \ln(x)$ versus $y = -\ln(x)$). All experts agreed on linear broken functions and square-root functions. More differences were found in the categories of power functions, where only expert Q made distinctions based on domain and/or on concavity.

In the construction of the common expert categorization, the distances between individual categorizations and the common expert categorization were calculated. The final common expert categorization is shown in Table 1.

The following distances from the final categorization were found: 11, 3, 19, 20, and 15 (for P, Q, R, S, and T, respectively). The experts needed an average of 18 min: 23, 20, 11, 14, and 21 min (for P, Q, R, S, and T, respectively).

For this task the experts used a lot of part-whole reasoning in order to categorize, for instance, the following formulas correctly: $\ln(e^{2x})$, $(1-x)(2+x)+x^2$, $\ln(1/x)$, $x(x-1)/(x+1)(x-1)$, $\left(2x^{\frac{1}{3}}\right)^5$.

From the interviews and observations we know that the experts first made a global categorization. Later they looked in greater detail and used more attributes to discriminate between the formulas. The experts described their strategy as “from simple to more complex” (expert P), “I made a preliminary categorization based on the function families with which I was brought up: with polynomial, exponential, logarithmic, power, broken, and root functions and only after this I did focus on the graphs.” (expert Q), and “some I see at first sight, others only with second thoughts, like $x - 4/x$” (expert R). Most of the formulas in this task could be considered IGFs and the experts did not consider this task difficult: “not a daily task and nice to do, but not difficult” (expert T). Some experts mentioned the “things” they could instantly see from the formula, like definition domain, asymptotes, singularities, even/odd functions, infinity behavior. Some experts indicated that a more

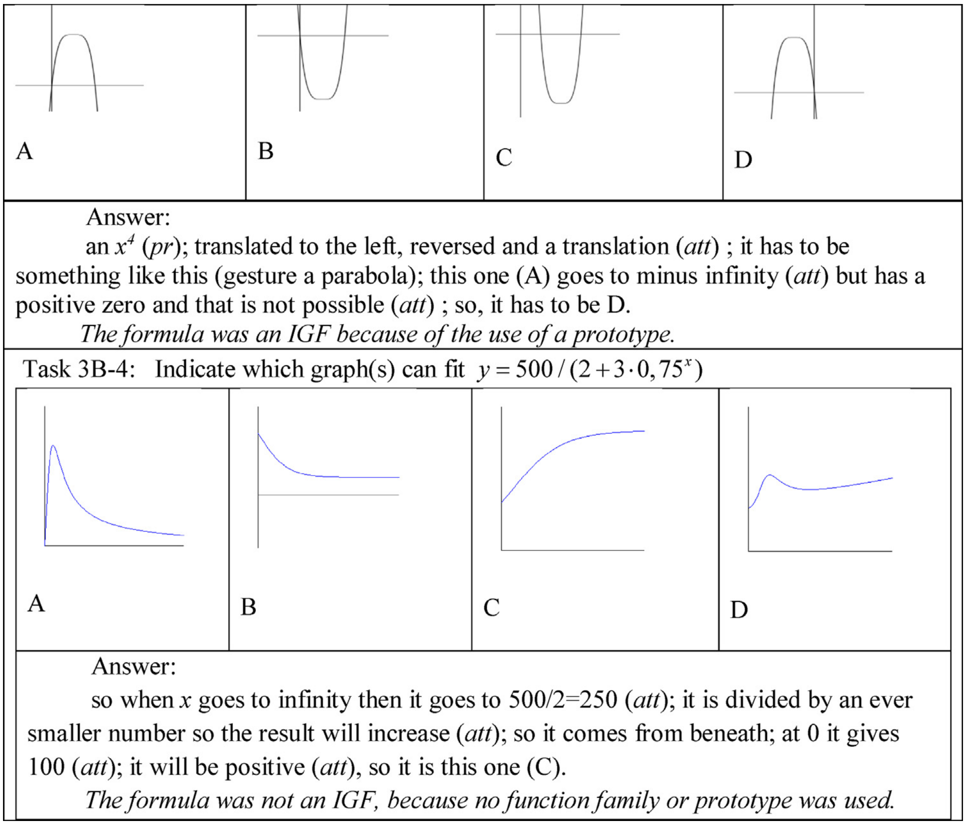


Fig. 6. Examples of the encoding of fragments of protocols of task 3.

Table 1
Common expert categorization.

Categories:
Linear
$x + 5(4 - x)$, $\ln(e^{2x})$,$(1 - x)(2 + x) + x^2$
Parabola
parabola with max: $x(9 - x)$, $-(x - 3)^2 + 2$,$(x + 5)(3 - x)$,$2x - 3(x + 2)(x - 2)$,$-(x - 1)^2 + 2(x - 1) + 6$;
parabola with min:$x^2 - 7(x - 5)$,$(6 - x)^2$,$x^2 + (-x + 1)^2$
3rd degree oscillation
$(x^2 - 7)(x - 5)$,$2(x - 3)^2(x + 3)$, $2x^3 + 4x^2 - 16x$,$x^3 - 9x$, $e^{3\ln(x)}$
4th degree
W-shape: $x^4 - 16x^2 + 28$, $(x^2 - 7)^2$;(6^e degree W-shape: $3(x^4 - 6)(x^2 - 8)$);
M-shape: $-3(x^2 - 4)(x^2 - 6)$, $2x^3(6 - x)$; V-shape: $(x + 3)^4 - 9$, $x^2(9 + x^2)$
Exponential
increasing: 4^{-3+x},$2.(\sqrt{2})^x$; decreasing: $18{\cdot}0.3^x$,2^{6-x},$8e^{-x}$,10^{-2x+5},$8/3^x$;
reversed exponential: $6 - 2^x$;$100 - e^x$
Logarithmic
inceasing: $\ln(e^2{\cdot}x)$,$1 + {}^2\log(x)$,$\ln(x) + \ln(2)$; decreasing: $-\ln(x)$,$\ln(1/x)$
distractor: $1/\ln(x)$
Hyperbola
hyperbola: $x(x - 1)/(x + 1)(x - 1)$,$(4x + 2)/x$,$1 - 5/(x + 1)$
power functions with negative odd power: $8x^{-3}$; with negative even power: $2/x^4$, $3x^{-2}$
slant asymptote: $x - 4/x$; two vertical asymptotes $(x^2 - 1)^{-1}$, $2/x - 3/(x - 1)$
'Roots'
increasing '$\sqrt{x}$-like': $3\sqrt{x + 6}$,$2\sqrt{x} - 6$;$(100x)^{\frac{1}{2}}$; decreasing '$\sqrt{x}$-like': $4\sqrt{10 - x}$,$(2 - x)^{\frac{1}{2}} + 2$
half a circle: $\sqrt{8 - x^2}$; V-shape: $\sqrt{8 + x^2}$
power functions
exponent '$\frac{1}{3}$-like' < 1: $2\sqrt[3]{x}$,$8\sqrt[3]{x^4}/2x$; exponent '$\frac{1}{3}$-like' > 1: $(2x^{\frac{1}{3}})^5$; exponent '$1\frac{1}{2}$-like': $2x\sqrt{x}$

Table 2
Results of task 2.

Participant	Number mistakes	Mistakes
P	3	$(4x+1)/(x+2)$; $7x\sqrt{x}$; $10/x^3$
Q	1	$(4x+1)/(x+2)$
R	1	$(4x+1)/(x+2)$
S	0	
T	5	$6x^2-2x^4$; $2x^{-4}$; $4-2x+x/4$; $(4x+1)/(x+2)$; $5x^7$

Table 3
Time needed for task 3A and 3B, the total number of mistakes, and the number of IGFs and constructions.

Participants	Time 3A	Time 3B	Number of mistakes	Number of IGFs	Number of constructions
P	4:54 min	7:44 min	0	10	2
Q	4:16 min	2:13 min	0	11	1
R	3:56 min	6:27 min	0	9	3
S	4:02 min	2:44 min	0	6	6
T	6:16 min	2:46 min	0	9	3

detailed categorization would be possible, but not without calculations: "In the next step I would have to make calculations; I would not trust myself to say more about this categorization off the top of my head" (expert Q).

4.2. Results of task 2

The results of task 2 (see Table 2) showed that three out of the five experts made no mistakes or only one mistake. Most mistakes were made with the formula $y=(4x+1)/(x+2)$. Four of our experts selected the alternative with the increasing hyperbola. From the other alternatives it could have been concluded that a distinction had to be made between an increasing and a decreasing hyperbola. Since a strict time limit of only 30 s for one formula was used and all experts finished this task easily within this time limit, it was concluded that all the formulas that did not belong to the alternative "none of these" could be considered IGFs for the experts.

From the observations and interviews we learned that all experts first examined the 20 graph alternatives and had a global view of the formulas to get an impression of which aspects would play a role in this task and what they had to focus on. All experts read almost all graphs by mentioning a function family that fitted the graph. When performing this task, they used part-whole reasoning if necessary, recognized a function family and used attribute reasoning to discriminate between different options of the same function family. For instance, $y=4^x-5y=3e^{-0{,}5x+4}-4$, $y=4/2^x$ were all recognized as members of the exponential function family, attribute reasoning, like infinity behavior and reversing a prototypical graph was used to choose the correct alternative.

4.3. Results of task 3

In task 3 the protocols were analyzed using prototype, attribute, and part-whole reasoning. From the encoded protocols, we found that experts often started with prototypes of function families, followed by attribute reasoning.

We give four examples (*pr* = prototype; *att* = attribute reasoning; *pw* = part-whole reasoning):

Example 1 *(: expert Q in task 3A-4 (3rd degree polynomial in Fig. 5)).* Something with a higher degree (pr), decreasing (att), let's see; this is something that increases (att), zeroes indeed at 0, 2, and 4 (att), that looks reliable; and this at 0, 3, and 6, and that will be possible (att); and this one increases, oh, no it decreases too (att); would be a possible alternative; and this one not, it has its zeroes on the wrong side (att).

Example 2 *(: expert Q in task 3A-6 (4th degree polynomial in Fig. 5)).* Let's see, 4th degree (pr), downwards (att); A. this one has no oscillations, and is only translated (att); B. is possible, where are the zeroes?, factorizing gives me $-x^2+9$ (pw), so zeroes at $x=3$ and $x=-3$(att); C. is not possible, because when I divided by x^2(pw) then no extra zeroes; d. when I divided by x^3(pw), it gave me only one more zero; so it has to be B

Example 3 *(: expert S in task 3B-3 (exponential function in Fig. 5)).* $100-50{\cdot}0.75^x$ is an exponential; function (pr) with $y=100$ as a horizontal asymptote (att); that leaves B. and C.; it is 100 minus . . .., so it comes from beneath the asymptote (att), so it has to be C.

Example 4 *(: expert T in task 3B-2 (Indicate which graph(s) can fit $y=-x(x-2)(x-4)$)).* This is a polynomial function of degree 3 (pr) and those graphs all look of degree 3 (pr); it is $-x^3$(att), so that means these alternatives are not possible (indicated A. and B.); these two are possible but it is only this one (C.) because D. has not the correct zeroes (att).

Experts made no mistakes in this task and worked fast: see Table 3. However, not all formulas could be considered IGFs for the experts, as some graphs had to be constructed by reasoning about attributes. In particular, the graph of the logistic function $y=500/(2+3{\cdot}0,75^x)$ (task 3B-4) had to be constructed by all our experts and the formula $y=6x^{-2}$(task 3B-5) was

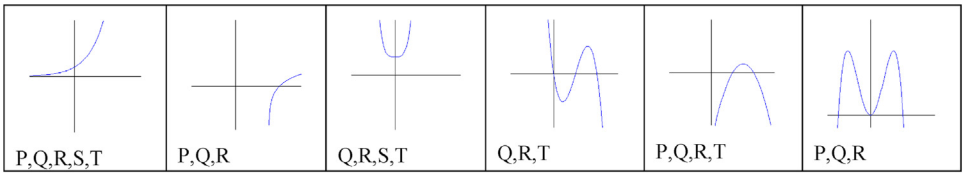


Fig. 7. Graphs recognized as a prototype of a function family in task 3A.

only recognized as an IGF by expert Q *("it is an $y = 1/x^2$")*. This description "it is an ..." suggested that Q saw $y = 6x^{-2}$ as a member of a function family, that was indicated by a prototype $y = 1/x^2$. The other experts did not show this prototype reasoning and instead used attribute reasoning about a vertical asymptote, and positive outcomes.

In task 3A the experts could work from graph to formula. This can only be done when a function family is recognized from the graph. In Example 1 and Example 2 above, it is shown that expert Q recognized the graph as a prototypical graph of a polynomial function of degree 3 respectively degree 4. In Fig. 7 it is indicated which experts started in task 3A their thinking aloud with mentioning a prototype of a function family. The other experts worked from the alternative formulas to the graph.

From the protocols we see that they experts used prototypical formulas and prototypical graphs of basic functions. They used prototypes of exponential, logarithmic, even, and polynomial of degree 2, 3 and 4 functions. Also, $y = \sqrt{a - x^2}$ (half a circle) was considered a function family. Only expert Q used $y = 1/x^2$ as a function family. Attributes that were used to discriminate between different alternatives were: increasing/decreasing of graph linked to positive/negative head coefficient, infinity behavior and horizontal asymptote, translations, vertical asymptote, number of zeroes and location of zeroes, reversing a graph, positive/negative outcomes, domain, and point of inflection.

Expert S seemed to use a lot of constructions, perhaps because a prototype or function family was not mentioned. From the protocols and results of the other tasks it was concluded that these function families were implicitly used by this expert.

From observations and interviews we learnt that the experts thought the functions used in task 3A were "easier" than those used in task 3B because they only required simple transformations. Another reason for the differences between task 3A and 3B was the amount of visual information in task 3B: "four formulas and one graph is easier to deal with than four graphs and one formula" (expert Q). Expert P mentioned that in general "it is more difficult to think from the graph than to think to the graph". Nevertheless, all experts indicated that both tasks required the same knowledge elements: namely, linking visual features of the graphs and features of the formulas.

5. Conclusions and discussion

5.1. Conclusions

The first aim of the current research was to describe experts' repertoires of IGFs. We hypothesized that experts would use categories to organize their knowledge of graphs and formulas. The experts' results in task 1 showed that the categories they constructed were very similar and also that the category descriptions were similar. These descriptions were closely related to the function families of basic functions that are taught in secondary school: linear functions, polynomial functions, exponential and logarithmic functions, broken functions, and power functions. Only expert S used descriptions containing numbers of turning points for the polynomial functions. Therefore a common categorization could be constructed. The distances between the individual categorizations and the final categorization varied from 3 to 20. Many of these differences could be explained by the absence of subcategories. For instance, some of the experts did not distinguish between increasing and decreasing exponential graphs or between parabola with a maximum or with a minimum. However, the experts' performances in task 2 confirmed that they could recognize these differences between subcategories as they made almost no mistakes in this task.

The time the experts needed to perform this categorization task varied from 11 to 23 min. When taking about 20 min to categorize 60 cards, the experts needed only 20 s per card to read, to recognize, to compare with others, and to group formulas with similar graphs. This meant that there was almost no time for the construction of new, unknown graphs. Some of the formulas, like $y = 1/\ln(x)$, $y = (x^2 - 1)^{-1}$, $y = x - 4/x$, $y = 2/x - 3/(x - 1)$ were categorized in a category with a single formula, often with a mention of some attributes, but without a graph. Therefore, it was concluded that the experts used the function families of the basic functions from secondary school to organize their categories of IGFs: linear functions; 2nd, 3rd and 4th degree polynomial, exponential, logarithmic and root functions with, in every function family, a distinction between increasing and decreasing; broken linear function; power functions x^n, with n odd/even, and $n = p/q$ with $p > q$, $p < q$. This should come as no surprise, since we used predominantly formulas of basic functions from the secondary school curricula. The experts were brought up with these categories, as they indicated in the interviews. They showed through their high proficiency that they had truly internalized this categorization of basic functions. The formulas seemed to be complex enough to capture the proficiency of the experts, as some formulas could not be instantly visualized or were not correctly categorized.

The second aim of the current study was to describe what experts attend to when linking formulas to graphs of IGFs. The recognition process when working from formulas to graphs can be well described using the Barsalou model of Fig. 2. It is shown in Table 3 that in recognizing IGFs, the experts often started with prototypes. This prototype reasoning was, when necessary, followed by attribute reasoning. For instance, $y = -2x(x-3)(x-6)$ is recognized as a prototypical "x^3", which is "reversed" and has zeroes at 0, 3, and 6; $y = \log_2(x+3)$ as a log translated to the left; $y = -(x+2)^4 + 16$ as an "x^4", reversed and translated; $y = \sqrt{6-x^2}$ as "half-a-circle". These examples were in line with the findings of Schwarz and Hershkowitz (1999), who found that proficient students used prototypes as levers for handling other examples and showed greater understanding of (critical) attributes.

The experts also recognized prototypical graphs for well-known function families, as they showed in task 2 and task 3A. For well-known function families there seemed to be little difference between working from formula to graph and working from graph to formula. When working with IGFs, the experts' concept images that were triggered by the given formula or given graph, seemed to contain equivalent formula(s), graph(s), attributes of graphs and of formulas, function family with prototypes, formulas of other functions in this function family.

In order to elicit experts' attribute reasoning, all attributes the experts used in task 3 were gathered: translation to the right/left and above/below, stretching horizontal or vertical, reversion (often indicated by reasoning about negative head coefficient), infinity behavior (with horizontal asymptotes), increasing/decreasing, number and location of turning points, location and number of zeroes, positive/negative, domain, point of inflection, and vertical asymptotes.

Particular attributes seemed to be linked to particular function families. As shown in task 3, these connections could work both ways: from function families to salient attributes of graphs and from graphs with salient attributes to function families. These salient attributes of a function family are characteristic of the members and prototypes of the function family. For instance, a vertical asymptote was directly linked to logarithmic functions or broken functions. And, when confronted with power functions with $n = p/q$, some instantly started with a focus on domain and concavity. For the different function families in our research, the experts used salient attributes: limited domain was linked to root functions, power functions and logarithmic functions; vertical asymptotes were linked to logarithmic functions and broken functions; horizontal asymptotes were linked to exponential functions and broken functions; symmetry was linked to even polynomial functions.

Experts used attributes appropriate to the tasks. For instance, when they had to link formulas to global graphs (tasks 1 and 2), they paid no attention to the factor 7 in$y = 7x\sqrt{x}$, or to the factor 3 and term 1 in $y = 4^{-x+3} + 1$. But when parameters influenced the global shape of the graph, these parameters were given ample attention. For instance, the minus signs of the head coefficient in $y = -x^4 + 9x^2$ and in $y = 2\sqrt{8-x}$ which reversed the prototypical graphs were directly noticed and mentioned. When more detailed graphs were requested, as in task 3, the experts again only used those attributes that were needed for the task. For instance, they did not mention anything about the factor 0.1 in $y = 0.1 \cdot x^2$ or about the term 12 in the formula $y = x^6 + 12$, because these were positive numbers. But when the task demanded it, the experts quickly noticed the attributes and values needed to graph the formulas. For instance, the experts instantly recognized the different locations of the zeroes in $y = x(3-x)(x-6)$ and $y = -2x(x+3)(x+6)$. These findings show that the experts worked efficiently and did not pay attention to "what is normal" (Chi et al., 1981; Chi, 2011).

The experts sometimes had to show their abilities in algebraic manipulation. As expected, they had no problems with this aspect of part-whole reasoning: this was shown in, for instance, $y = 6x^{-2}$ (in task 3B), and $y = 8x^{-3}$, $y = x(x-1)/((x+1)(x-1))$, and $y = (1-x)(2+x) + x^2$ (in task 1).

These results show that experts' processes of recognition of IGFs can be described using the model in Fig. 2: with prototypes, supported by attribute and part-whole reasoning.

5.2. Discussion and implications

5.2.1. A Barsalou model for recognition of IGFs

The current findings highlight the two highest levels of recognition of the framework for strategies in graphing formulas (Kop et al., 2015). We defined formulas at these levels as IGFs, instantly graphable formulas. We described the experts' repertoires of IGFs and described what experts attended to in recognizing IGFs. We showed that the experts used prototypes and attribute reasoning in recognizing IGFs and found how particular attribute and value sets were linked to particular function families. For instance, given a logarithmic formula such as $y = \log_3(2x+4) - 3$, a prototype $y = \log_3(x)$ or $y = \log(x)$ was instantly identified and attribute reasoning (translation, domain $x > -2$, and/or vertical asymptote at $x = -2$) resulted in a graph. We also found that for function families of basic functions, the experts could easily work from graph to a formula. Given a graph, they instantly recognized a function family that fitted the graph. For instance, a graph with attributes like domain $x > a$, a vertical asymptote at $x = a$ and concave down was instantly identified as a logarithmic function. This implies that the Barsalou model based on Schwarz and Hershkowitz in Fig. 2 can be expanded with linkages between attribute and value sets, prototypes and function families and with linkages from graph to attributes, prototypes, and function families. In Fig. 8, for some of the function families in the experts' categorizations (logarithmic, polynomial with degree 2, exponential, and broken functions), a prototype is described using attributes and values; for other examplars of the function family salient attributes are indicated.

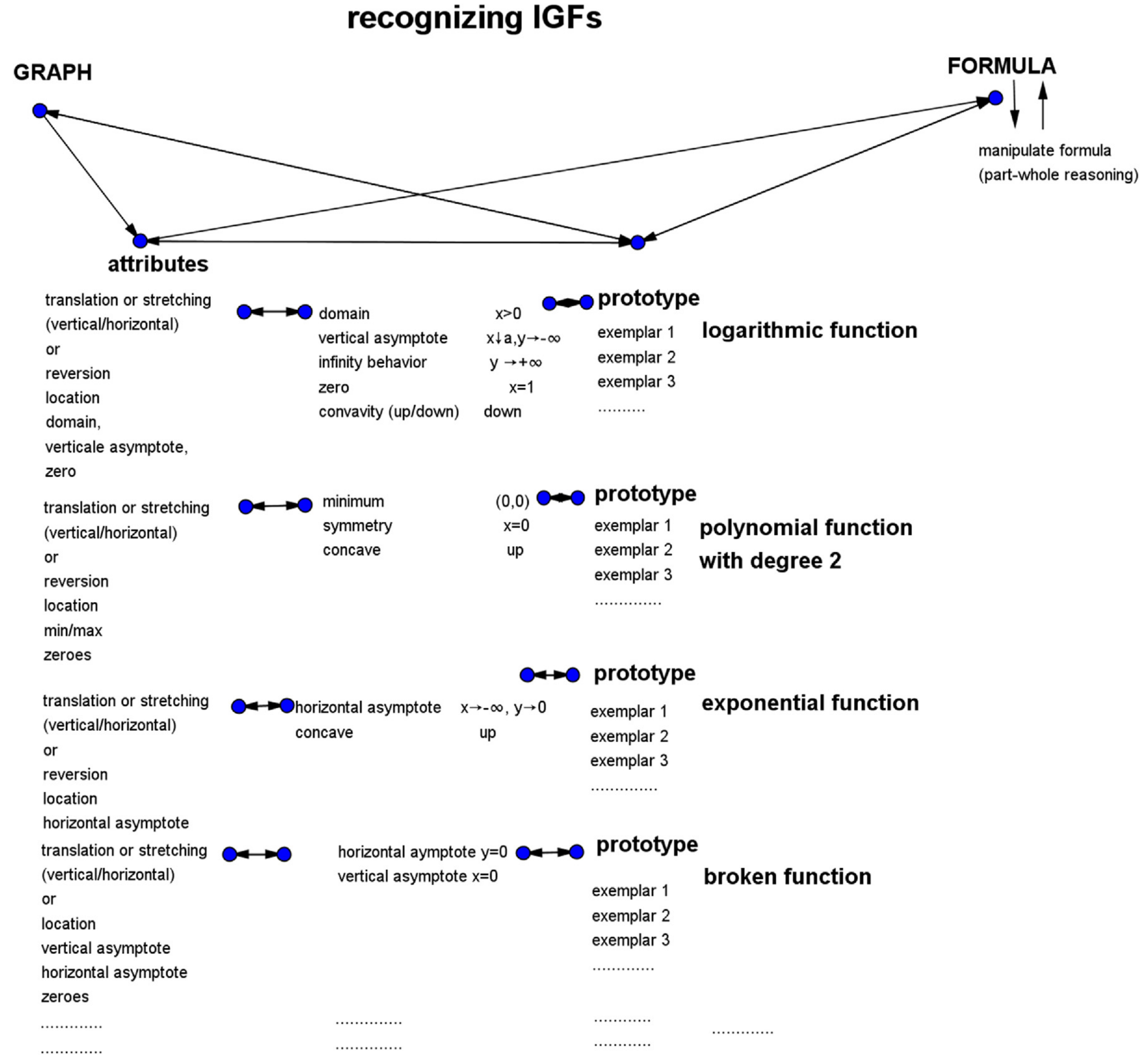


Fig. 8. A Barsalou model based on Schwarz and Hershkowitz with function families and their salient attributes.

5.2.2. Global properties in graphing formulas

The experts in our study focused on attributes and values that influenced the global shape of the graph. For instance, a parameter that reversed the prototypical graph of $y = x^4$ was given ample attention, like -2 in $y = -2x^4$, but a parameter that resulted in only a small change of the prototypical graph was not mentioned, such as 0.1 in $y = 0.1x^4$. In their attribute reasoning, the experts focused on attributes and values that gave a great deal of information about the whole graph. These attributes can be considered the global growth properties of Slavit's classification of function properties (Slavit, 1997). Starting with these global properties is considered to be more efficient in graphing formulas than using local properties (Even, 1998; Slavit, 1997).

In the current study we found that the experts used a set of attributes and values that differ from Slavit's global properties, partly because Slavit's focus was more on the function concept, whereas our focus was on the relation between formula and graph. Several global properties Slavit used, such as integrability and invertibility, were not mentioned at all by our experts. Based on the current results, we suggest that relevant global properties for recognizing IGFs may be symmetry, infinity behavior (including horizontal and slant asymptotes), vertical asymptotes, domain, increasing/decreasing on intervals, sign (reverse), and concavity. For local properties, we suggest zeroes, turning points, points of inflection, and individual points.

5.2.3. Suggestions for further research and teaching

In discussing their ideas about graphs and formulas, the experts used an ample repertoire of descriptions: "a valley", "it goes in the right direction‘, "it has to go downwards', "it runs flat", "tails go to minus infinity", "this one has no oscillations", “a reversed . . ..”, “it goes to infinity”, “it goes up”, “it comes from below”, “in infinity it is . . .”, “this one is only positive”, “log to the right”, “an oscillation downwards”, “a $-x^3$”.

These descriptions show that the experts often did not use the formal math attribute/property concepts, but used both pictures of the whole graph and action language such as “it (the graph) runs . . . . . .”. People talk ubiquitously about abstract concepts using concrete metaphors (Barsalou, 2008). Metonymies and metaphors are necessary for efficient communication and in the learning of mathematical concepts (Presmeg, 1998; Zandieh & Knapp, 2006). Further research is necessary to find out how these experts’ metonymies and metaphors can be helpful in the efficient teaching of graphing formulas.

A repertoire of IGFs is necessary for graphing formulas. Eisenberg and Dreyfus (1994) wrote about the need for a repertoire of basic functions and knowledge of the characteristics of the representations of these functions. Slavit (1997) speaks about “property noticing”, the ability to recognize and analyze functions by identifying the presence or absence of these properties and the need for a “library” of functional properties. Our findings show how experts used prototype and attribute reasoning for graphing formulas and so give an impression of an expert “library” of properties. Fig. 8, a Barsalou model based on Schwarz and Hershkowitz, shows how for IFGs these function families, prototypes, attributes, and part-whole reasoning are integrated in the experts’ concept images. Our findings may be helpful to further describe the recognition and identification of objects, forms, key features, and dominant terms used in Pierce and Stacey’s algebraic insight (Pierce and Stacey, 2001; Kenney, 2008). Not only in graphing formulas but also when using CAS or graphical calculators one needs this algebraic insight. For instance, Heid et al. (2012) showed how solving the equation $\ln(x) = 5\sin(x)$ required knowledge of the characteristics of function families of both formulas $y = \ln(x)$ and $y = 5\sin(x)$ and the ability to link the graph images to the formulas (Zbiek & Heid, 2011).

The results of this study can be relevant for teaching algebra and in particular functions. Students continue to experience difficulties with seeing the relationship between algebraic and graphical representations, although graphing technology can support students’ understanding in linking representations of functions (Kieran, 2006; Ruthven, Deaney, & Hennessy, 2009). In order to further improve education, we first need a domain-specific knowledge base (De Corte, 2010). Expertise research can provide such a knowledge base (De Corte, 2010; Campitelli & Gobet, 2010; Stylianou & Silver, 2004). The current findings show what knowledge experts used in recognizing IGFs: they used the basic functions to organize the function families, used prototypes to handle other exemplars of function families, and used prototypes and attributes to link graphs and formulas of function families. In secondary school curricula much attention is paid to basic functions, in particular to linear and quadratic functions. Our study suggests that only learning and practicing basic functions is not enough to become proficient in linking the formulas and graphs of functions. Students need to know how to handle parameters in formulas and need opportunities to integrate their knowledge of prototypes and attributes of function families into well-connected hierarchical mental networks. Besides such a knowledge-base for recognition, students need heuristic methods, like splitting formulas and qualitative reasoning, when recognition falls short (Kop et al., 2015).

For graphing formulas one has to be able to “read” algebraic formulas. Further research is necessary to investigate whether graphing formulas indeed improve symbol sense, in particular algebraic insight and how graphing formulas can be effectively and efficiently taught to students.

Appendix A.

Five experts’ categorizations and the researcher’s categorization with category names and prototypes.

P. 23:22 min	Q. 20:28 min	R. 11:25 min	S. 14:25 min	T. 21:00 min	Researcher’s categorization
Linear: $ax+$b	Straight lines: $ax+b$	Linear: $ax+b$	Linear $y=x$	Linear functions	Linear Increasing/decreasing
Degree 2: ax^2+bx+c	Parabola with max: $-x^2$ Parabola with min: x^2	Degree 2: ax^2+bx+c	1 turning point $y=x^2$	Degree 2	Parabola with max and with min
Polynomials: $\sum_{k=0}^{n} a_k x^k$ (defined on domain)	Degree 3 (odd): x^3	Degree 3: ax^3+bx^2+cx+d	2 turning points: $y=x^3-3x$	Degree 3	Degree 3 increasing
	Degree 4, decreasing: $-x^2(x^2-1)$ Degree 4, increasing: $x^2(x^2-1)$	Degree 4: a x^4 + b x^3..etc	3 turning points: $y=(x^2-1)^2$ 5 turning points $y=x^2(x^2-1)(x^2-2)$	Degree 4, with W-shape Degree 4 without W-shape	Degree 4 with W-shape M-shape V-shape Degree 6 with W-shape
$(ax+b)^k/(cx+d)^n$	Hyperbola: $1/x$ Quotientfunctions with more than 1 vertical asymptote: y = $1/(x^2-1)$	Broken functions	Vertical and horizontal asymptotes $y=1/x$ Verticale and slant asymptotes y = x − $4/x$ 2 vertical asymptotes: y = $1/(x^2-1)$	Linear broken: $y=(ax+b)/(cx+d)$	Hyperbola and Powerfunction with higher negative odd exponent 2 vertical asymptotes Slant asymptote

Powerfunction with negative exponent	Even hyperbola-like: $1/x^2$	Negative exponent: ax^{-n}		powerfunction	Powerfunction with higher negative and even exponent
Can be transformed to $b{\cdot}g^x+c$ HA $+\infty$ HA at $-\infty$)	Exponential increasing: e^x Exponential decreasing: e^{-x}	Exponential function: $a{\cdot}b^{x}+c$	Exponential: $y=2^x$	Elementary exponential function	Exponential increasing HA $x\to-\infty$ And decreasing (HA $x\to+\infty$) Reversed exponential: $6-2^x; 100-e^x$
Logarithmic function with transformation (increasing /decreasing and $1/\ln(x)$)	Logarithmic increasing: $\ln(x)$ Logarithmic decreasing: $-\ln(x)$	Logarithmic: $a\log(x)+c$ (included $1/\ln(x)$	Log: $y={}^2\log(x)$	Elementary logarithmic function	Logarithmic increasing and decreasing
Powerfunction, function with broken exponent, rootfunction with transformation: $c\sqrt{ax+b+d}$	Roots, domain to the left: $\sqrt{-x}$ Root, domain to the right: $\sqrt{x}$	Rootfunction: $\sqrt{ax^2+bx+c}$	$y=\sqrt{x}$	Elementary rootfunction, transformed $y=\sqrt{x}$	Roots increasing '$\sqrt{x}$-look-like' and decreasing '$\sqrt{x}$-look-like'
$\sqrt{ax^2+bx+c}$	Power function even $\sqrt[4]{x^2}$		Half a circle	Root	Half a cicle: $y=\sqrt{8-x^2}$ V-shape: $y=\sqrt{8+x^2}$
Broken powerfunction, not transformed from basic function	Oddpower function: $\sqrt[3]{x}$ Broken exponent, defined to the right: $x\sqrt{x}$	Broken exponent: $ax^{\frac{p}{q}}$		Powerfunction, positive exponent, no asymptote	Powerfunction with broken exponent, concave up/down
	Various: $1/\ln(x)$		Apart: $1/\ln(x)$	Rest $1/\ln(x)$; $(x^2-1)^{-1}$; $3(x^4-6)(x^2-8)$	distractor: $1/\ln(x)$

References

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics*, *14*(3), 24–35.

Ayalon, M., Watson, A., & Lerman, S. (2015). Functions represented as linear sequential data: relationships between presentation and student responses. *Educational Studies in Mathematics*, *90*(3), 321–339.

Ball, L., Pierce, R., & Stacey, K. (2003). Recognising equivalent algebraic expressions: An important component of algebraic expectation for working with CAS. *International Group for the Psychology of Mathematics Education*, *4*, 15–22.

Barnard, A. D., & Tall, D. O. (1997). Cognitive units, connections and mathematical proof. In E. Pehkonen (Ed.), *Proceedings of the twenty first international conference for the psychology of mathematics education* (Vol. 2) (pp. 41–48).

Barsalou, L. W. (1992). Frames, concepts, and conceptual fields. In A. Lehrer, & E. F. Kittay (Eds.), *Frames, fields, and contrasts: New essays in semantic and lexical organization.* (pp. 21–74). Hillsdale, NJ, England: Lawrence Erlbaum Associates, Inc [vi, 464 pp.].

Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology*, *59*, 617–645.

Bills, L., Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006). Exemplification in mathematics education. *Proceedings of the 30th conference of the international group for the psychology of mathematics education, Vol. 1*, 126–154.

Campitelli, G., & Gobet, F. (2010). Herbert Simon's decision-making approach: Investigation of cognitive processes in experts. *Review of General Psychology*, *14*(4), 354.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *22*(5), 352–378.

Chi, M. T., Feltovich, P. J., & Glaser, R. (1981). Categorization and representation of physics problems by experts and novices. *Cognitive Science*, *5*(2), 121–152.

Chi, M. T. (2011). Theoretical perspectives, methodological approaches, and trends in the study of expertise. In *Expertise in mathematics instruction*. pp. 17–39. US: Springer.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, 66–86.

Crowley, L., & Tall, D. O. (1999). The roles of cognitive units, connections and procedures in achieving goals in college algebra. In O. Zaslavsky (Ed.), *Proceedings of the 23rd conference of PME* (Vol. 2) (pp. 225–232).

De Corte, E. (2010). Historical developments in the understanding of learning. In H. Dumont, D. Instance, & F. Benavides (Eds.), *Educational research and innovation the nature of learning using research to inspire practice: Using research to inspire practice* (pp. 35–67). OECD Publishing.

de Jong, T., & Ferguson-Hessler, M. G. (1986). Cognitive structures of good and poor novice problem solvers in physics. *Journal of Educational Psychology*, *78*(4), 2.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*(1–2), 103–131.

Eisenberg, T., & Dreyfus, T. (1994). On understanding how students learn to visualize function transformations. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education* (Vol. 1) (pp. 45–68). Providence, RI: American Mathematical Society.

Ericsson, K. A. (2006). Protocol analysis and expert thought: Concurrent verbalizations of thinking during experts' performance on representative tasks. In K. A. Ericsson, N. Charness, P. J. Feltovich, & R. R. Hoffman (Eds.), *The Cambridge handbook of expertise and expert performance* (pp. 223–241). Cambridge: Cambridge University Press.

Even, R. (1998). Factors involved in linking representations of functions. *The Journal of Mathematical Behavior*, *17*(1), 105–121.

Eysenck, M. W., & Keane, M. T. (2000). *Cognitive psychology: a student's handbook*. Taylor & Francis.

Goldenberg, P., & Mason, J. (2008). Shedding light on and with example spaces. *Educational Studies in Mathematics*, *69*(2), 183–194.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity: and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, *25*, 116–140.

Heid, M. K., Thomas, M. O., & Zbiek, R. M. (2012). How might computer algebra systems change the role of algebra in the school curriculum? In *Third international handbook of mathematics education*. pp. 597–641. New York: Springer.
Jonassen, D. H., Beissner, K., & Yacci, M. (1993). Structural knowledge: Techniques for representing, conveying, and acquiring structural knowledge. *Psychology Press*.
R, K. (2008). *The influence of symbols on pre-calculus students' problem solving goals and activities (Doctoral dissertation)*. Available from Dissertations and Theses database. (UMI No. 3329205).
Kieran, C. (2006). Research on the learning and teaching of algebra. In A. Gutiérrez, & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 11–49). Rotterdam: Sense.
Kilpatrick, J., & Izsak, A. (2008). A history of algebra in the school curriculum. In C. E. Greens, & R. Rubenstein (Eds.), *Algebra and algebraic Thinking in school mathematics, seventieth yearbook* (pp. 3–18). NCTM.
Kop, P., Janssen, F., Drijvers, P., Veenman, M., & Van Driel, J. (2015). Identifying a framework for graphing formulas from expert strategies. *The Journal of Mathematical Behavior*, *39*, 121–134.
Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, *60*(1), 1–64.
Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 69–100). Hillsdale, NJ: Lawrence Erlbaum Associates.
NCTM. (2000). *Principles and standards for school mathematics*. (Accessed 14 october 2014). http://www.nctm.org/standards
Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson, & C. Rasmussen (Eds.), *Making the connection: research and teaching in undergraduate mathematics education* (pp. 27–42). Washington, D.C: Mathematical Association of America.
Pierce, R., & Stacey, K. (2001). Observations on students' responses to learning in a CAS environment. *Mathematics Education Research Journal*, *13*(1), 28–46.
Pierce, R., & Stacey, K. (2004). A framework for monitoring progress and planning teaching towards the effective use of computer algebra systems. *International Journal of Computers for Mathematical Learning*, *9*(1), 59–93.
Polya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.
Presmeg, N. C. (1998). Metaphoric and metonymic signification in mathematics. *The Journal of Mathematical Behavior*, *17*(1), 25–32.
Ruiz-Primo, M. A., & Shavelson, R. J. (1996). Problems and issues in the use of concept maps in science assessment. *Journal of Research in Science Teaching*, *33*(6), 569–600.
Ruthven, K., Deaney, R., & Hennessy, S. (2009). Using graphing software to teach about algebraic forms: A study of technology-supported practice in secondary-school mathematics. *Educational Studies in Mathematics*, *71*(3), 279–297.
Sandefur, J., Mason, J., Stylianides, G. J., & Watson, A. (2013). Generating and using examples in the proving process. *Educational Studies in Mathematics*, *83*(3), 323–340.
Schwarz, B. B., & Hershkowitz, R. (1999). Prototypes: Brakes or levers in learning the function concept? The role of computer tools. *Journal for Research in Mathematics Education*, 362–389.
Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics*, *26*, 191–228.
Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36.
Slavit, D. (1997). An alternate route to the reification of function. *Educational Studies in Mathematics*, *33*(3), 259–281.
Stylianou, D. A., & Silver, E. A. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning*, *6*(4), 353–387.
Stylianou, D. A. (2011). An examination of middle school students' representation practices in mathematical problem solving through the lens of expert work: Towards an organizing scheme. *Educational Studies in Mathematics*, *76*(3), 265–280.
Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design. *Learning and Instruction*, *4*(4), 295–312.
Thomas, M. O., Wilson, A. J., Corballis, M. C., Lim, V. K., & Yoon, C. (2010). Evidence from cognitive neuroscience for the role of graphical and algebraic representations in understanding function. *ZDM*, *42*(6), 607–619.
Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 356–366.
Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: learners generating examples*. Mahwah, NJ: Lawrence Erlbaum.
Zandieh, M. J., & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning: The concept of derivative as an example. *The Journal of Mathematical Behavior*, *25*(1), 1–17.
Zbiek, R. M., & Heid, M. K. (2011). Using technology to make sense of symbols and graphs and to reason about general cases. *Focus on Reasoning and Sense Making: Technology to Support Reasoning and Sense Making*, 19–31.