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ABSTRACT

In this paper, we study a system of two singular second order differential equations, which arises from the theory of harmonic analysis on complex symmetric spaces. First of all, the distributional solutions on an neighborhood of zero in $\mathbb{R}^2$ are determined. Next, some new function spaces are introduced and the system is solved in the duals of these new spaces.

INTRODUCTION

Suppose Ω is a self-conjugate, open, simply connected neighbourhood of zero in $\mathbb{C}$. We will indentify $\mathbb{C}$ and $\mathbb{R}^2$ at our convenience. Let, as usual, $\mathscr{D}(\Omega)$ denote the space of C^∞-functions on Ω with compact support, endowed with the Schwartz-topology. Write $\mathscr{D}'(\Omega)$ for its dual, the distributions on Ω. For any function $f:\Omega\to\mathbb{C}$ we define $\tilde{f}:\Omega\to\mathbb{C}$ by

(0) $\qquad \tilde{f}(z)=f(\bar{z}).$

Furthermore, let a and b be holomorphic functions on Ω, subject to the following conditions:

(1) $\qquad a(z)=0\Leftrightarrow z=0;\ \dfrac{\partial a}{\partial z}(0)=1$

(2) $\qquad b(0)=\mu+1 \quad (\mu\in\mathbb{R}).$

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Let L and $\tilde{L}$ be differential operators, defined as:

$$L = a\frac{\partial^2}{\partial z^2} + b\frac{\partial}{\partial z}; \ \tilde{L} = \tilde{a}\frac{\partial^2}{\partial \bar{z}^2} + \tilde{b}\frac{\partial}{\partial \bar{z}}$$

where

$$\frac{\partial}{\partial z} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \bar{z}} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We will solve the system

$$(3) \qquad \begin{cases} (L-\alpha)T = 0 \\ (\tilde{L}-\beta)T = 0 \end{cases} \quad (T \in \mathcal{D}'(\Omega))$$

for all $\alpha, \beta \in \mathbb{C}$.

Having done this, we will introduce some new function spaces which occur in harmonic analysis on complex symmetric spaces. The results on the distributional solutions of (3) will enable us to solve (3) in the duals of these new spaces.

1. DISTRIBUTIONAL SOLUTIONS

1.1. Strategy and result

Let $\Omega_1 = \Omega - \{0\}$ and $\Omega_2 = \Omega - \{x | x \leq 0\}$.

Since $L-\alpha$ is elliptic on Ω_1 by (1) and since its coefficients are real-analytic functions, any solution of $(L-\alpha)T = 0$ on Ω_1 is in fact a real-analytic function. We are thus led to examine real-analytic functions that satisfy (3) on Ω_1. In order to find these, we first determine the functions that satisfy (3) on Ω_2. This solution space S, say, is 4-dimensional. Let $s_1, \ldots, s_4$ be a basis. The s_i are, in general, multiple-valued which implies that, if a general solution $\sum_{i=1}^4 c_i s_i$ of (3) on Ω_2 is to be continued to a solution on Ω_1, not all combinations of the c_i are allowed. We will find necessary and sufficient (linear) conditions for the c_i for this to be possible. Having thus found the solutions on Ω_1 we extend these to distributions on Ω. If T is one such, then $(L-\alpha)T$ and $(\tilde{L}-\beta)T$ have support contained in $\{0\}$, so, finally, we try to add a distribution with support in $\{0\}$ that makes up for the difference (or is a solution itself in case T is a solution of (3)).

The solution space of (3) is two- or four-dimensional.

1.2. Solutions on Ω_1

Consider the equation

$$(4) \qquad (L-\gamma)\psi = 0 \quad (\gamma \in \mathbb{C}).$$

If $U \subset \Omega$ is open, let $H(U)$ be the holomorphic functions on U and let $H_{s,\gamma}(U)$ be the holomorphic solutions of (4) on U. From the material in [Whittaker and Watson, 10.1–10.3] we infer the following result without too much trouble:

LEMMA 1.

(i) *If* $\mu \notin \mathbb{Z}$, *then there exist* Φ_γ *and* $W_{1,\gamma}$ *in* $H(\Omega)$ *such that* Φ_γ *and*

$$W_\gamma \equiv z^{-\mu} W_{1,\gamma}$$

constitute a basis for $H_{s,\gamma}(\Omega_2)$.

(ii) *If* $\mu = 0, 1, 2, \ldots,$ *then there exist* Φ_γ *and* $W_{1,\gamma}$ *in* $H(\Omega)$ *and* $\lambda_\gamma \in \mathbb{C}$ *such that* Φ_γ *and*

$$W_\gamma \equiv z^{-\mu} W_{1,\gamma} + \lambda_\gamma \Phi_\gamma \log z$$

constitute a basis for $H_{s,\gamma}(\Omega_2)$.

(iii) *If* $\mu = -1, -2, -3, \ldots,$ *then there exist* Φ_γ *and* $W_{1,\gamma}$ *in* $H(\Omega)$ *and* $\lambda_\gamma \in \mathbb{C}$ *such that* $z^{-\mu}\Phi_\gamma$ *and*

$$W_\gamma \equiv W_{1,\gamma} + \lambda_\gamma z^{-\mu} \Phi_\gamma \log z$$

constitute a basis for $H_{s,\gamma}(\Omega_2)$.

(iv) *For all values of* μ, *we may assume:*

(5) $$\Phi_\gamma(0) = W_{1,\gamma}(0) = 1.$$

For $\mu = 0$, *we may assume in addition:*

(6) $$\lambda_\gamma = 1.$$

(v) *For any basis* $\{\psi_{\gamma,1}, \psi_{\gamma,2}\}$ *of* $H_{s,\gamma}(\Omega_2)$ *and for any connected open subset* U *of* Ω_2, *the restrictions of* $\psi_{\gamma,1}$ *and* $\psi_{\gamma,2}$ *to* U *constitute a basis for* $H_{s,\gamma}(U)$.

REMARKS

(i) In the Lemma, as well as in the rest of this paper, log denotes the principal branch of the logarithm. For $x<0$ we define $\log x = \log|x| + \pi i$. All exponential functions occurring will be defined with respect to this choice.

(ii) For $\mu = 1, 2, \ldots,$ W_γ is not uniquely determined, even if (5) is satisfied. However, it will be assumed in the sequel that for all μ and γ a choice has been made in such a way that (5) and (6) are satisfied.

If $U \subset \Omega_1$ is open, let $A_{s,\alpha,\beta}(U)$ be the solution space of (3) on U. All functions in $A_{s,\alpha,\beta}(U)$ are real-analytic.

Let $\{\psi_{\alpha,1}, \psi_{\alpha,2}\}$ and $\{\psi_{\beta,1}, \psi_{\beta,2}\}$ be bases for $H_{s,\alpha}(\Omega_2)$ and $H_{s,\beta}(\Omega_2)$ respectively. We will proceed to show that $\{\psi_{\alpha,1}\tilde{\psi}_{\beta,1}, \psi_{\alpha,1}\tilde{\psi}_{\beta,2}, \psi_{\alpha,2}\tilde{\psi}_{\beta,1}, \psi_{\alpha,2}\tilde{\psi}_{\beta,2}\}$ is a basis for $A_{s,\alpha,\beta}(\Omega_2)$.

We will need the following fact:

Let $U \subset \mathbb{C}$ be open and $f: U \to \mathbb{C}$ smooth. Let $\tilde{U}$ be the conjugate of U and define $\tilde{f}: \tilde{U} \to \mathbb{C}$ as in (0). Then:

$$\frac{\partial \tilde{f}}{\partial z} = \left(\frac{\partial f}{\partial \bar{z}}\right)^{\sim} \text{ and } \frac{\partial \tilde{f}}{\partial \bar{z}} = \left(\frac{\partial f}{\partial z}\right)^{\sim} \text{ on } \tilde{U}.$$

LEMMA 2. *Let $\psi \in A_{s,\alpha,\beta}(\Omega_2)$. Fix $c \in \Omega_2$. Then there exists an open neighbourhood U of c and $c_1, \ldots, c_4 \in \mathbb{C}$ such that*

$$\psi = c_1 \psi_{\alpha,1} \tilde{\psi}_{\beta,1} + c_2 \psi_{\alpha,1} \tilde{\psi}_{\beta,2} + c_3 \psi_{\alpha,2} \tilde{\psi}_{\beta,1} + c_4 \psi_{\alpha,2} \tilde{\psi}_{\beta,2} \text{ on } U.$$

PROOF. Since ψ is real-analytic we can choose an open disk U around c such that

$$\psi(z) = \sum_{l=0}^{\infty} f_l(z)(\bar{z} - \bar{c})^l \text{ on } U,$$

where the series is absolutely and uniformly convergent on U and $f_l \in H(U)$ for all l. Since $(L-\alpha)\psi = 0$ and $(L-\alpha)f_l \in H(U)$ for all l, we have $(L-\alpha)f_l = 0$ for all l. By Lemma 1 (v) there exist $p_l, q_l \in \mathbb{C}$ such that $f_l = p_l \psi_{\alpha,1} + q_l \psi_{\alpha,2}$ on U.

Define $g_1, g_2 \in H(\tilde{U})$ by

$$g_1(z) = \sum_{l=0}^{\infty} p_l (z - \bar{c})^l; \; g_2(z) = \sum_{l=0}^{\infty} g_l (z - \bar{c})^l \quad (z \in \tilde{U}).$$

Then

$$\psi = \psi_{\alpha,1} \tilde{g}_1 + \psi_{\alpha,2} \tilde{g}_2 \text{ on } U.$$

Using the definition of $\tilde{L}$ and the remark preceding the Lemma, we see that

$$\psi_{\alpha,1}((L-\beta)g_1)^{\sim} + \psi_{\alpha,2}((L-\beta)g_2)^{\sim} = 0 \text{ on } U.$$

We claim that $((L-\beta)g_1)^{\sim} = ((L-\beta)g_2)^{\sim} = 0$ on U. If $z_0 \in U$ and, say, $(L-\beta)g_1(\bar{z}_0) \neq 0$, we can write

$$\frac{\psi_{\alpha,1}}{\psi_{\alpha,2}} = -\frac{((L-\beta)g_2)^{\sim}}{((L-\beta)g_1)^{\sim}}$$

on some neighbourhood of z_0.

The left side is a power series in $z - z_0$ and the right side is a power series in $\bar{z} - \bar{z}_0$, so both must be a constant, contradicting Lemma 1 (v).

Therefore, by Lemma 1 (v), there are $c_1, \ldots, c_4 \in \mathbb{C}$ such that

$$g_1 = c_1 \psi_{\beta,1} + c_2 \psi_{\beta,2}; \; g_2 = c_3 \psi_{\beta,1} + c_4 \psi_{\beta,2} \text{ on } \tilde{U},$$

which proves the Lemma.

LEMMA 3. *$\{\psi_{\alpha,1} \tilde{\psi}_{\beta,1}, \psi_{\alpha,1} \tilde{\psi}_{\beta,2}, \psi_{\alpha,2} \tilde{\psi}_{\beta,1}, \psi_{\alpha,2} \tilde{\psi}_{\beta,2}\}$ is a basis for $A_{s,\alpha,\beta}(\Omega_2)$.*

PROOF. Let $\psi \in A_{s,\alpha,\beta}(\Omega_2)$. Choose $c \in \Omega_2$ and write

$$\psi = c_1 \psi_{\alpha,1} \tilde{\psi}_{\beta,1} + c_2 \psi_{\alpha,1} \tilde{\psi}_{\beta,2} + c_3 \psi_{\alpha,2} \tilde{\psi}_{\beta,1} + c_4 \psi_{\alpha,2} \tilde{\psi}_{\beta,2}$$

on some open neighbourhood of c. Now the right side above is obviously in $A_{s,\alpha,\beta}(\Omega_2)$, so by unicity of analytic continuation, it equals ψ on Ω_2. The linear independence is established by a power series argument as in Lemma 2. ●

LEMMA 4. *The following functions constitute a basis for* $A_{s,\alpha,\beta}(\Omega_1)$:
If $\mu\notin\mathbb{Z}$: $\Phi_\alpha\tilde{\Phi}_\beta$

$$|z|^{-2\mu}W_{1,\alpha}\tilde{W}_{1,\beta}.$$

If $\mu=0,1,2,\dots$ *and* $(\lambda_\alpha,\lambda_\beta)\neq(0,0)$: $\Phi_\alpha\tilde{\Phi}_\beta$

$$\lambda_\alpha\Phi_\alpha\bar{z}^{-\mu}\tilde{W}_{1,\beta}+2\lambda_\alpha\lambda_\beta\Phi_\alpha\tilde{\Phi}_\beta\log|z|+\lambda_\beta z^{-\mu}W_{1,\alpha}\tilde{\Phi}_\beta.$$

If $\mu=1,2,3,\dots$ *and* $(\lambda_\alpha,\lambda_\beta)=(0,0)$: $\Phi_\alpha\tilde{\Phi}_\beta$

$$\bar{z}^{-\mu}\Phi_\alpha\tilde{W}_{1,\beta}$$

$$z^{-\mu}W_{1,\alpha}\tilde{\Phi}_\beta$$

$$|z|^{-2\mu}W_{1,\alpha}\tilde{W}_{1,\beta}.$$

If $\mu=-1,-2,-3,\dots$ *and* $(\lambda_\alpha,\lambda_\beta)\neq(0,0)$: $|z|^{-2\mu}\Phi_\alpha\tilde{\Phi}_\beta$

$$\lambda_\alpha z^{-\mu}\Phi_\alpha\tilde{W}_{1,\beta}+2\lambda_\alpha\lambda_\beta|z|^{-2\mu}\Phi_\alpha\tilde{\Phi}_\beta\log|z|+\lambda_\beta W_{1,\alpha}\bar{z}^{-\mu}\tilde{\Phi}_\beta.$$

If $\mu=-1,-2,-3,\dots$ *and* $(\lambda_\alpha,\lambda_\beta)=(0,0)$: $|z|^{-2\mu}\Phi_\alpha\tilde{\Phi}_\beta$

$$z^{-\mu}\Phi_\alpha\tilde{W}_{1,\beta}$$

$$\bar{z}^{-\mu}W_{1,\alpha}\tilde{\Phi}_\beta$$

$$W_{1,\alpha}\tilde{W}_{1,\beta}.$$

PROOF. Let $\mu\notin\mathbb{Z}$ and $\psi\in A_{s,\alpha,\beta}(\Omega_1)$. By Lemma 1 and 3 we can write:

$$\psi=c_1\Phi_\alpha\tilde{\Phi}_\beta+c_2\Phi_\alpha\bar{z}^{-\mu}\tilde{W}_{1,\beta}+c_3z^{-\mu}W_{1,\alpha}\tilde{\Phi}_\beta+c_4|z|^{-2\mu}W_{1,\alpha}\tilde{W}_{1,\beta}\text{ on }\Omega_2.$$

The right hand side must have a smooth continuation to Ω_1. For all $x<0$ we let $z\to x$ through the upper and lower halfplane. Equating the obtained expressions yields:

$$c_2\Phi_\alpha(x)W_{1,\beta}(x)=c_3\Phi_\beta(x)W_{1,\alpha}(x)\text{ for all }x<0.$$

Now (5) implies: $c_2=c_3$.

Applying the same procedure to $\partial\psi/\partial z$ gives:

$$c_2\frac{\partial\Phi_\alpha}{\partial z}(x)W_{1,\beta}(x)+c_3\left\{\mu x^{-1}W_{1,\alpha}(x)-\frac{\partial W_{1,\alpha}}{\partial z}(x)\right\}\Phi_\beta(x)=0\text{ for all }x<0.$$

Since the first term is bounded as $x\to0$, we have $c_3=0$. This gives

$$\psi=c_1\Phi_\alpha\tilde{\Phi}_\beta+c_4|z|^{-2\mu}W_{1,\alpha}\tilde{W}_{1,\beta}$$

and this expression obviously has a smooth extension to Ω_1.

The remaining cases are treated similarly (for $\mu=-1,-2,-3,\dots$ $\partial\psi/\partial\bar{z}$ has to be considered as well). ●

1.3. Reflection of distributions; the partie finie in the two-dimensional case

In order to cut down the size of the calculations we want to exploit the fact that L and $\tilde{L}$ are the same "up to a reflection".

Let $T \in \mathscr{D}'(\Omega)$. Define $\tilde{T} \in \mathscr{D}'(\Omega)$ by:

$$\langle \tilde{T}, \psi \rangle = \langle T, \tilde{\psi} \rangle \quad (\psi \in \mathscr{D}(\Omega)).$$

Write

$$\frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} \delta \text{ as } \delta^{(k,l)}.$$

The following Lemma is easily established.

LEMMA 5. *Let $T \in \mathscr{D}'(\Omega)$. Then:*

(i) $(\chi T)^{\sim} = \tilde{\chi}\tilde{T}$ *for all* $\chi \in C^\infty(\Omega)$.

(ii) $\dfrac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} \tilde{T} = \left(\dfrac{\partial^{k+l}}{\partial z^l \partial \bar{z}^k} T \right)^{\sim}$ *for all* $k, l \geq 0$.

(iii) $(\delta^{(k,l)})^{\sim} = \delta^{(l,k)}$ *for all* $k, l \geq 0$.

(iv) *If T is represented by a function f, then $\tilde{T}$ is represented by $\tilde{f}$.*

(v) $(L-\gamma)T = ((\tilde{L}-\gamma)\tilde{T})^{\sim}$ *for all* $\gamma \in \mathbb{C}$.

(vi) *If $\chi \in H(\Omega)$, then* $(L-\gamma)\{\tilde{\chi}T\} = \tilde{\chi}\{(L-\gamma)T\}$ *for all* $\gamma \in \mathbb{C}$.

Partie finie

Consider the distribution $z^{-\mu}\bar{z}^{-\nu}$ $(\mu, \nu \in \mathbb{R})$ on Ω_1. We extend this distribution to Ω.

For $\varrho > 0$, let $\Omega_\varrho = \{z \in \Omega \,|\, |z| \geq \varrho\}$.

LEMMA 6. *Let $\psi \in \mathscr{D}(\Omega)$. If $\mu+\nu \notin \mathbb{Z}$, then there are unique constants c_k $(k=0,2,3,\ldots,[\mu+\nu])$ such that*

$$\int_{\Omega_\varrho} z^{-\mu}\bar{z}^{-\nu}\psi d(x,y) = \sum_{k=2}^{[\mu+\nu]} c_k \varrho^{-\mu-\nu+k} + c_0 + o(1) \quad (\varrho \downarrow 0). \tag{8}$$

If $\mu+\nu \in \mathbb{Z}$, then there are unique constants c_k $(k=0,1,2,\ldots,\mu+\nu-1)$ such that

$$\int_{\Omega_\varrho} z^{-\mu}\bar{z}^{-\nu}\psi d(x,y) = \sum_{k=2}^{\mu+\nu-1} c_k \varrho^{-\mu-\nu+k} + c_1 \log \varrho + c_0 + o(1) \quad (\varrho \downarrow 0). \tag{9}$$

In both cases we can define a distribution on Ω that extends $z^{-\mu}\bar{z}^{-\nu}$ by

$$\langle Pf z^{-\mu}\bar{z}^{-\nu}, \psi \rangle = c_0.$$

PROOF. Fix $\varrho_0 > 0$. Then, for $\varrho < \varrho_0$:

$$\int_{\Omega_\varrho} z^{-\mu}\bar{z}^{-\nu}\psi d(x,y) = \int_{\Omega_{\varrho_0}} z^{-\mu}\bar{z}^{-\nu}\psi d(x,y) + \int_{\varrho \leq |z| \leq \varrho_0} z^{-\mu}\bar{z}^{-\nu}\psi d(x,y).$$

Now expand ψ in a Taylor series around 0 such that the remainder term tends sufficiently rapidly to 0 if $|z| \to 0$ and introduce polar coordinates. ●

We will write $Pf(f) = c_0$ for any function f of ϱ having an asymptotic expansion as in (8) or (9).

It is easily checked that

$$(10) \qquad (Pf z^{-\mu}\bar{z}^{-\nu})^{\sim} = Pf z^{-\nu}\bar{z}^{-\mu}.$$

The following lemmas will be handy later on.

Let C_ϱ be the negatively oriented circle with 0 as its centre and radius ϱ.

LEMMA 7. *Let $k, l \in \mathbb{Z}$. Then:*

$$\text{(i)} \qquad Pf \int_{C_\varrho} \tfrac{1}{2} z^k \bar{z}^l (i dx + dy) = \begin{cases} -\pi \text{ if } (k,l) = (0,-1) \\ 0 \text{ otherwise} \end{cases}$$

$$\text{(ii)} \qquad Pf \int_{C_\varrho} \tfrac{1}{2} |z|^{\pm 2\mu} z^k \bar{z}^l (i dx + dy) = 0 \text{ if } \mu \notin \mathbb{Z}.$$

PROOF. Use polar coordinates. ●

LEMMA 8. *Suppose χ satisfies $(L-\gamma)\chi = 0$ on Ω_1 ($\gamma \in \mathbb{C}$). Let ψ be smooth on Ω_1 with compact support in Ω. With C_ϱ and Ω_ϱ as above we have:*

$$\int_{\Omega_\varrho} \chi(L^t - \gamma)\psi d(x,y) = \int_{C_\varrho} \tfrac{1}{2}[\psi, \chi](i dx + dy)$$

where

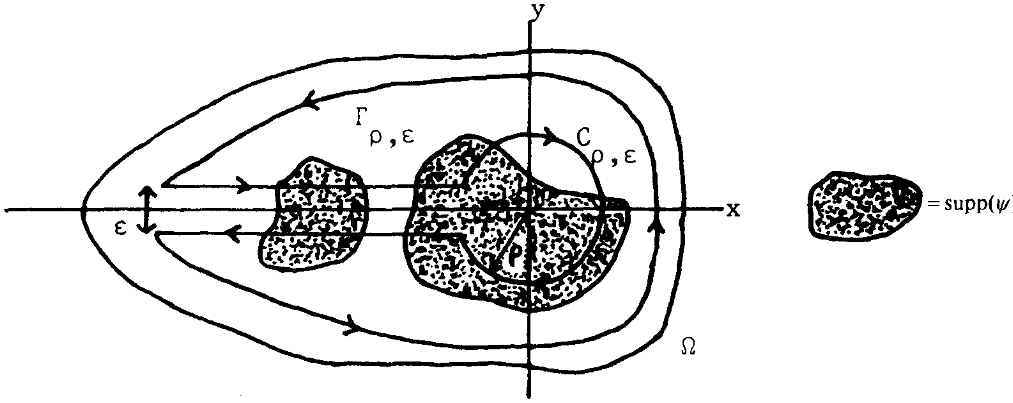
$$L^t \psi = \frac{\partial^2}{\partial z^2}(a\psi) - \frac{\partial}{\partial z}(b\psi)$$

$$(11) \qquad [\psi, \chi] = a\left(\frac{\partial \psi}{\partial z}\chi - \psi\frac{\partial \chi}{\partial z}\right) + \left(\frac{\partial a}{\partial z} - b\right)\chi\psi.$$

PROOF. As in [M.T. Kosters, p. 35] we have:

$$\chi L^t \psi - (L\chi)\psi = \frac{\partial}{\partial z}[\psi, \chi].$$ ●

Consider the following contours $\Gamma_{\varrho,\varepsilon}$ in Ω_1. ($\Gamma_{\varrho,0}$ surrounds supp(ψ)).



Let $\Omega_{\varrho,\varepsilon}$ be the interior of $\Gamma_{\varrho,\varepsilon}$. Using Green's Theorem in the fourth step we have:

$$\int_{\Omega_\varrho} \chi(L^t - \gamma)\psi d(x, y) = \int_{\Omega_\varrho} \chi(L^t \psi) - (L\chi)\psi d(x, y) =$$

$$= \int_{\Omega_\varrho} \frac{\partial}{\partial z} [\psi, \chi] d(x, y) = \lim_{\varepsilon \downarrow 0} \left\{ \int_{\Omega_{\varrho,\varepsilon}} \tfrac{1}{2} \frac{\partial}{\partial x} [\psi, \chi] d(x, y) - \right.$$

$$\left. - \int_{\Omega_{\varrho,\varepsilon}} \tfrac{1}{2} i \frac{\partial}{\partial y} [\psi, \chi] d(x, y) \right\} = \lim_{\varepsilon \downarrow 0} \left\{ \int_{\Gamma_{\varrho,\varepsilon}} \tfrac{1}{2}[\psi, \chi] dy + \int_{\Gamma_{\varrho,\varepsilon}} \tfrac{1}{2} i [\psi, \chi] dx \right\} =$$

$$= \int_{C_\varrho} \tfrac{1}{2}[\psi, \chi](idx + dy)$$

by the choice of the $\Gamma_{\varrho,\varepsilon}$.

1.4. Solutions of (3) on Ω

Define the following distributions on Ω (extensions of the bases in Lemma 4): For all $\mu \neq -1, -2, -3, \ldots$:

$$T_{1,\alpha,\beta} = \Phi_\alpha \tilde{\Phi}_\beta.$$

For $\mu = 1, 2, 3, \ldots$ and $(\lambda_\alpha, \lambda_\beta) = (0, 0)$ or $\mu \notin \mathbb{Z}$:

$$T_{2,\alpha,\beta} = W_{1,\alpha} \tilde{W}_{1,\beta} Pf |z|^{-2\mu}.$$

For $\mu = 0, 1, 2, \ldots$ and $(\lambda_\alpha, \lambda_\beta) \neq (0, 0)$:

$$T_{3,\alpha,\beta} = \lambda_\alpha \Phi_\alpha \tilde{W}_{1,\beta} Pf \bar{z}^{-\mu} + 2\lambda_\alpha \lambda_\beta \Phi_\alpha \tilde{\Phi}_\beta \log |z| + \lambda_\beta W_{1,\alpha} \tilde{\Phi}_\beta Pf z^{-\mu}.$$

For $\mu = 1, 2, 3, \ldots$ and $(\lambda_\alpha, \lambda_\beta) = (0, 0)$:

$$T_{4,\alpha,\beta} = \Phi_\alpha \tilde{W}_{1,\beta} Pf \bar{z}^{-\mu} \text{ and } T_{5,\alpha,\beta} = W_{1,\alpha} \tilde{\Phi}_\beta Pf z^{-\mu}.$$

For $\mu = -1, -2, -3, \ldots$ and all $(\lambda_\alpha, \lambda_\beta)$:

$$T_{6,\alpha,\beta} = |z|^{-2\mu} \Phi_\alpha \tilde{\Phi}_\beta.$$

For $\mu = -1, -2, -3, \ldots$ and $(\lambda_\alpha, \lambda_\beta) \neq (0, 0)$:

$$T_{7,\alpha,\beta} = \lambda_\alpha z^{-\mu} \Phi_\alpha \tilde{W}_{1,\beta} + 2\lambda_\alpha \lambda_\beta |z|^{-2\mu} \Phi_\alpha \tilde{\Phi}_\beta \log |z| + \lambda_\beta W_{1,\alpha} \bar{z}^{-\mu} \tilde{\Phi}_\beta.$$

For $\mu = -1, -2, -3, \ldots$ and $(\lambda_\alpha, \lambda_\beta) = (0, 0)$:

$$T_{8,\alpha,\beta} = z^{-\mu} \Phi_\alpha \tilde{W}_{1,\beta}.$$

$$T_{9,\alpha,\beta} = \bar{z}^{-\mu} W_{1,\alpha} \tilde{\Phi}_\beta.$$

$$T_{10,\alpha,\beta} = W_{1,\alpha} \tilde{W}_{1,\beta}.$$

Table 1.

Conditions	Distributions involved
$\mu \notin \mathbb{Z}$	$T_{1,\alpha,\beta}, T_{2,\alpha,\beta}$
$\mu = 0,1,2,\ldots;\ (\lambda_\alpha, \lambda_\beta) \neq (0,0)$	$T_{1,\alpha,\beta}, T_{3,\alpha,\beta}$
$\mu = 1,2,3,\ldots;\ (\lambda_\alpha, \lambda_\beta) = (0,0)$	$T_{1,\alpha,\beta}, T_{2,\alpha,\beta}, T_{4,\alpha,\beta}, T_{5,\alpha,\beta}$
$\mu = -1,-2,-3,\ldots;\ (\lambda_\alpha, \lambda_\beta) \neq (0,0)$	$T_{6,\alpha,\beta}, T_{7,\alpha,\beta}$
$\mu = -1,-2,-3,\ldots;\ (\lambda_\alpha, \lambda_\beta) = (0,0)$	$T_{6,\alpha,\beta}, T_{8,\alpha,\beta}, T_{9,\alpha,\beta}, T_{10,\alpha,\beta}$

We now calculate $(L-\alpha)T$ and $(\tilde{L}-\beta)T$ for all distributions involved.

LEMMA 9. *When they are defined,* $T_{1,\alpha,\beta}, T_{6,\alpha,\beta}, T_{8,\alpha,\beta}, T_{9,\alpha,\beta}$ *and* $T_{10,\alpha,\beta}$ *are solutions of* (3).

PROOF. Obvious, since the functions defining these distributions are smooth on Ω and satisfy (3) on Ω_1. ●

LEMMA 10.

(i) *If* $\mu \notin \mathbb{Z}$, *then* $(L-\alpha)T_{2,\alpha,\beta} = (\tilde{L}-\beta)T_{2,\alpha,\beta} = 0$ *for all* α, β.

(ii) *If* $\mu = 1,2,3,\ldots,$ *then:*

$$(L-\alpha)T_{2,\alpha,\beta} = \frac{\pi}{(\mu-1)!(\mu+1)!}\,\delta^{(\mu,\mu-1)} + \sum_{k,l=0}^{\mu-1} c_{kl}\delta^{(k,l)}$$

and

$$(\tilde{L}-\beta)T_{2,\alpha,\beta} = \frac{\pi}{(\mu-1)!(\mu+1)!}\,\delta^{(\mu-1,\mu)} + \sum_{k,l=0}^{\mu-1} c'_{k,l}\delta^{(k,l)}$$

for all (α,β) *such that* $(\lambda_\alpha, \lambda_\beta) = (0,0)$ (*the* c_{kl} *and* c'_{kl} *are complex constants depending on* (α,β)).

PROOF. (i) We first prove $(L-\alpha)T_{2,\alpha,\beta} = 0$ for all α,β. By Lemma 5 (vi), it is sufficient to show that

$$(L-\alpha)W_{1,\alpha}Pf|z|^{-2\mu} = 0 \text{ for all } \alpha, \beta.$$

Let $\psi \in \mathscr{D}(\Omega)$. Using Lemma 8 we have:

$$\langle (L-\alpha)W_{1,\alpha}Pf|z|^{-2\mu}, \psi\rangle = Pf \int_{C_\varrho} \tfrac{1}{2}[\psi, |z|^{-2\mu}W_{1,\alpha}](idx+dy).$$

When we expand $[\cdot,\cdot]$ in a Taylor series around 0 we get terms of the type $z^k\bar{z}^l|z|^{-2\mu}$ or

$$z^k\bar{z}^l\frac{\partial}{\partial z}|z|^{-2\mu} = -\mu z^k\bar{z}^{l+1}|z|^{-2(\mu+1)} \quad (k,l \geq 0),$$

neither of which contributes to the partie finie by Lemma 7. This proves the first statement in (i).

Now we have for all α, β:

$$(\tilde{L}-\beta)T_{2,\alpha,\beta}=\{(L-\beta)\tilde{T}_{2,\alpha,\beta}\}^{\sim} \qquad \text{by Lemma 5 (v)}$$

$$=\{(L-\beta)\tilde{W}_{1,\alpha}W_{1,\beta}Pf|z|^{-2\mu}\}^{\sim} \quad \text{by Lemma 5 (i) and (10)}$$

$$=\{(L-\beta)T_{2,\beta,\alpha}\}^{\sim}$$

$$=0 \qquad \text{by the previous result.}$$

(ii) We first compute $(L-\alpha)W_{1,\alpha}Pf|z|^{-2\mu}$. Let $\psi\in\mathscr{D}(\Omega)$. Then, by Lemma 8:

$$\langle (L-\alpha)W_{1,\alpha}Pf|z|^{-2\mu},\psi\rangle=Pf\int_{C_\varrho}\tfrac{1}{2}[\psi,|z|^{-2\mu}W_{1,\alpha}](idx+dy)$$

$=-\pi\cdot$coefficient of $\bar{z}^{-1}$ in the Taylorseries of $[\psi,|z|^{-2\mu}W_{1,\alpha}]$

around 0 (Lemma 7 (i)).

Using (1), (2) and (5), some computation gives this coefficient as

$$\frac{1}{(\mu-1)!(\mu+1)!}\frac{\partial^{2\mu-1}}{\partial z^{\mu}\partial\bar{z}^{\mu-1}}\psi(0)+\sum_{l=0}^{\mu-1}c_l\frac{\partial^{\mu+l-1}}{\partial z^{l}\partial\bar{z}^{\mu-1}}\psi(0)\quad (c_l\in\mathbb{C})$$

so

$$(L-\alpha)W_{1,\alpha}Pf|z|^{-2\mu}=\frac{\pi}{(\mu-1)!(\mu+1)!}\delta^{(\mu,\mu-1)}+\sum_{l=0}^{\mu-1}\pm c_l\delta^{(l,\mu-1)}.$$

Using (5) it is easily checked that for $i,j\geq 0$:

$$\tilde{W}_{1,\beta}\delta^{(i,j)}=\delta^{(i,j)}+\sum_{k=0}^{j-1}d_k\delta^{(i,k)}\quad (d_k\in\mathbb{C}).$$

An appeal to Lemma 5 (vi) proves the first statement in (ii) and the second follows as in (i). ●

LEMMA 11.
(i) *If* $\mu=0$, *then* $(L-\alpha)T_{3,\alpha,\beta}=(\tilde{L}-\beta)T_{3,\alpha,\beta}=0$ *for all* α,β.
(ii) *If* $\mu=1,2,3,\ldots$, *then*

$$(L-\alpha)T_{3,\alpha,\beta}=(-1)^{\mu-1}\frac{\pi\mu\lambda_\alpha}{(\mu-1)!}\tilde{W}_{1,\beta}\delta^{(0,\mu-1)}$$

$$(\tilde{L}-\beta)T_{3,\alpha,\beta}=(-1)^{\mu-1}\frac{\pi\mu\lambda_\beta}{(\mu-1)!}W_{1,\alpha}\delta^{(\mu-1,0)}$$

for all (α,β) *such that* $(\lambda_\alpha,\lambda_\beta)\neq(0,0)$.

PROOF. Let $\psi \in \mathscr{D}(\Omega)$. Then:

$$\langle (L-\alpha)T_{3,\alpha,\beta}, \psi\rangle = Pf \int_{\Omega_\varrho} (\lambda_\alpha \Phi_\alpha \tilde{W}_{1,\beta}\bar{z}^{-\mu} + \lambda_\beta W_{1,\alpha}\tilde{\Phi}_\beta z^{-\mu})(L^t-\alpha)\psi d(x,y)$$

$$+ \int_{\Omega} 2\lambda_\alpha\lambda_\beta \Phi_\alpha \tilde{\Phi}_\beta \log|z|(L^t-\alpha)\psi d(x,y)$$

$$= Pf \int_{C_\varrho} \tfrac{1}{2}[\psi, \lambda_\alpha \Phi_\alpha \tilde{W}_{1,\beta}\bar{z}^{-\mu}](idx+dy) +$$

$$+ Pf \int_{C_\varrho} \tfrac{1}{2}[\psi, \lambda_\beta W_{1,\alpha}\tilde{\Phi}_\beta z^{-\mu}](idx+dy) +$$

$$+ Pf \int_{C_\varrho} \tfrac{1}{2}[\psi, 2\lambda_\alpha\lambda_\beta \Phi_\alpha\tilde{\Phi}_\beta \log|z|](idx+dy)$$

where we have used Lemma 8 and the theorem on dominated convergence. By Lemma 7 (i), the second term is zero and, using $\partial \log|z|/\partial z = 1/2z$, we see that the third term is zero as well.

When $\mu=0$, the first term is bounded, so (i) is proved.

If $\mu = 1, 2, 3, \ldots,$ we have, since $[\psi, \lambda_\alpha \Phi_\alpha \tilde{W}_{1,\beta}\bar{z}^{-\mu}] = \lambda_\alpha \tilde{W}_{1,\beta}\bar{z}^{-\mu}[\psi, \Phi_\alpha]$:

$$\langle (L-\alpha)T_{3,\alpha,\beta}, \psi\rangle = Pf \int_{C_\varrho} \tfrac{1}{2}\lambda_\alpha \tilde{W}_{1,\beta}\bar{z}^{-\mu}[\psi, \Phi_\alpha](idx+dy)$$

$$= \frac{-\pi\lambda_\alpha}{(\mu-1)!}\frac{\partial^{\mu-1}}{\partial\bar{z}^{\mu-1}}\{\tilde{W}_{1,\beta}[\psi,\Phi_\alpha]\}_{z=0}$$

$$= \frac{-\pi\lambda_\alpha}{(\mu-1)!}\sum_{j=0}^{\mu-1}\binom{\mu-1}{j}\frac{\partial^j \tilde{W}_{1,\beta}}{\partial\bar{z}^j}\frac{\partial^{\mu-j-1}}{\partial\bar{z}^{\mu-j-1}}[\psi,\Phi_\alpha]\bigg|_{z=0}$$

$$= \frac{-\pi\lambda_\alpha}{(\mu-1)!}\sum_{j=0}^{\mu-1}\binom{\mu-1}{j}\frac{\partial^j \tilde{W}_{1,\beta}}{\partial\bar{z}^j}\left[\frac{\partial^{\mu-j-1}\psi}{\partial\bar{z}^{\mu-j-1}}, \Phi_\alpha\right]\bigg|_{z=0}$$

$$= \frac{-\pi\lambda_\alpha}{(\mu-1)!}\sum_{j=0}^{\mu-1}\binom{\mu-1}{j}\frac{\partial^j \tilde{W}_{1,\beta}(0)}{\partial\bar{z}^j}(1-(1+\mu))\frac{\partial^{\mu-j-1}}{\partial\bar{z}^{\mu-j-1}}\psi(0)$$

$$= \frac{\pi\mu\lambda_\alpha}{(\mu-1)!}\frac{\partial^{\mu-1}}{\partial\bar{z}^{\mu-1}}(\tilde{W}_{1,\beta}\psi)\bigg|_{z=0}$$

$$= \left\langle (-1)^{\mu-1}\frac{\pi\mu\lambda_\alpha}{(\mu-1)!}\tilde{W}_{1,\beta}\delta^{(0,\mu-1)}, \psi\right\rangle$$

where (1) is used in the fifth step.

The second statement in (ii) follows as usual. •

LEMMA 12. *If* $\mu = 1, 2, 3, \ldots$ *then:*

(i) $(L-\alpha)T_{4,\alpha,\beta} = (-1)^{\mu-1}\dfrac{\pi\mu}{(\mu-1)!}\tilde{W}_{1,\beta}\delta^{(0,\mu-1)}$

$(\tilde{L}-\beta)T_{4,\alpha,\beta} = 0$

(ii) $(L-\alpha)T_{5,\alpha,\beta}=0$

$$(\tilde{L}-\beta)T_{5,\alpha,\beta}=(-1)^{\mu-1}\frac{\pi\mu}{(\mu-1)!}W_{1,\alpha}\delta^{(\mu-1,0)}$$

for all (α,β) *such that* $(\lambda_\alpha,\lambda_\beta)=(0,0)$.

PROOF. The statement concerning $(L-\alpha)T_{4,\alpha,\beta}$ is proved as in Lemma 11 (ii), whereas $(L-\alpha)T_{5,\alpha,\beta}=0$ follows from Lemma 8 and 7 (i).

Since $\tilde{T}_{4,\alpha,\beta}=T_{5,\alpha,\beta}$ and $\tilde{T}_{5,\alpha,\beta}=T_{4,\beta,\alpha}$, the combination of the previous results and Lemma 5 proves the Lemma. ●

LEMMA 13. *If* $\mu=-1,-2,-3,\ldots$, *then*

$$(L-\alpha)T_{7,\alpha,\beta}=(\tilde{L}-\beta)T_{7,\alpha,\beta}=0$$

for all (α,β) *such that* $(\lambda_\alpha,\lambda_\beta)\neq(0,0)$.

PROOF. Let $\psi\in\mathscr{D}(\Omega)$. Using the theorem on dominated convergence and Lemma 8, we have:

$$\langle(L-\alpha)T_{7,\alpha,\beta},\psi\rangle=\lim_{\varrho\downarrow 0}\int_{C_\varrho}\tfrac{1}{2}[\psi,\lambda_\alpha z^{-\mu}\Phi_\alpha\tilde{W}_{1,\beta}+2\lambda_\alpha\lambda_\beta|z|^{-2\mu}\Phi_\alpha\tilde{\Phi}_\beta\log|z|$$
$$+\lambda_\beta W_{1,\alpha}\bar{z}^{-\mu}\tilde{\Phi}_\beta](idx+dy).$$

Since $-\mu\geq 1$, $[\cdot,\cdot]$ is bounded as $\varrho\downarrow 0$, which proves the lemma. ●

We will now study distributions with support in $\{0\}$.

LEMMA 14. *For all* μ, *all* (α,β) *and all* $k,l\geq 0$:

$$(L-\alpha)\delta^{(k,l)}=(\mu-k-1)\delta^{(k+1,l)}+\sum_{j=0}^{k}c_j\delta^{(j,l)}\quad(c_j\in\mathbb{C})$$

$$(\tilde{L}-\beta)\delta^{(k,l)}=(\mu-l-1)\delta^{(k,l+1)}+\sum_{j=0}^{l}d_j\delta^{(k,j)}\quad(d_j\in\mathbb{C}).$$

PROOF. As usual, we only prove the first statement.

Let $\psi\in\mathscr{D}(\Omega)$. Then, using (1) and (2):

$$\langle(L-\alpha)\delta^{(k,0)},\psi\rangle=(-1)^k\frac{\partial^k}{\partial z^k}\left\{\frac{\partial^2}{\partial z^2}(a\psi)-\frac{\partial}{\partial z}(b\psi)-\alpha\psi\right\}\Bigg|_{z=0}$$

$$=(-1)^k\Bigg\{a(0)\frac{\partial^{k+2}\psi}{\partial z^{k+2}}(0)+\binom{k+2}{1}\frac{\partial a}{\partial z}(0)\frac{\partial^{k+1}\psi}{\partial z^{k+1}}(0)+\text{lower order terms}$$

$$-b(0)\frac{\partial^{k+1}\psi}{\partial z^{k+1}}(0)+\text{lower order terms}\Bigg\}$$

$$=\langle(\mu-k-1)\delta^{(k+1,0)},\psi\rangle+\text{lower order terms}.$$

Now use $(L-\alpha)\delta^{(k,l)}=\dfrac{\partial}{\partial\bar{z}^l}\{(L-\alpha)\delta^{(k,0)}\}$. ●

As a next step we find a fundamental solution in the origin.

Define for $\mu=1,2,3,\ldots$, irrespective of $(\lambda_\alpha, \lambda_\beta)$:

$$E_\alpha = \frac{(-1)^{\mu-1}}{(\mu-1)!} W_{1,\alpha}\delta^{(\mu-1,0)} \text{ and } F_\beta = \frac{(-1)^{\mu-1}}{(\mu-1)!} \tilde{W}_{1,\beta}\delta^{(0,\mu-1)}.$$

LEMMA 15. *If* $\mu=1,2,3,\ldots$ *then:*

$$(L-\alpha)E_\alpha = -\mu\lambda_\alpha\delta$$

$$(\tilde{L}-\beta)F_\beta = -\mu\lambda_\beta\delta$$

for all (α,β).

PROOF. Note that $W_{1,\alpha}\delta^{(\mu-1,0)} = \sum_{j=0}^{\mu-1} c_j\delta^{(j,0)}$ for some $c_j \in \mathbb{C}$. By Lemma 14, we see that $(L-\alpha)E_\alpha = \sum_{j=0}^{\mu} d_j\delta^{(j,0)}$ for some $d_j \in \mathbb{C}$. It is therefore sufficient to test $(L-\alpha)E_\alpha$ on all $\psi \in \mathcal{D}(\Omega)$ which are holomorphic on a neighborhood of 0. For the rest of this proof, let ψ be such a function.

Observe that, with W_α as in lemma 1,

$$W_\alpha\psi(z) = \sum_{j=2}^{\mu} d_j z^{-j} + \langle E_\alpha, \psi\rangle z^{-1} + O(|\log z|) \quad (z\to 0)$$

so

$$(12) \qquad W_\alpha(z)L^t\psi(z) - \alpha W_\alpha(z)\psi(z) = \sum_{j=2}^{\mu} d_j' z^{-j} + \langle (L-\alpha)E_\alpha, \psi\rangle z^{-1} + O(|\log z|). \quad (z\to 0)$$

Consider

$$(13) \qquad Pf \int_{-C_\varrho} \frac{-i}{2}(W_\alpha L^t\psi - \alpha W_\alpha\psi)dz$$

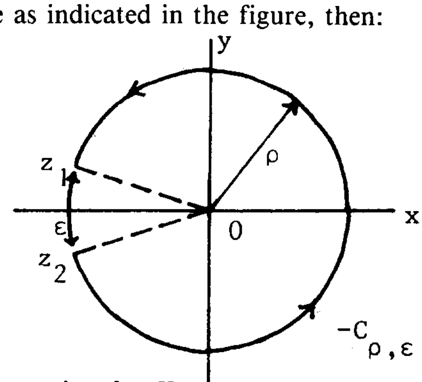
for small ϱ. Using (12) and the residue theorem one sees immediately that (13) equals

$$(14) \qquad \pi\langle (L-\alpha)E_\alpha, \psi\rangle.$$

On the other hand, if $-C_{\varrho,\varepsilon}$, z_1 and z_2 are as indicated in the figure, then:

$$(13) = Pf \lim_{\varepsilon\downarrow 0} \int_{-C_{\varrho,\varepsilon}} \frac{-i}{2}(W_\alpha L^t\psi - \alpha W_\alpha\psi)dz$$

$$= \frac{-i}{2} Pf\{ \lim_{z_1\to -\varrho} [\psi, W_\alpha](z_1) - \lim_{z_2\to -\varrho} [\psi, W_\alpha](z_2)\} = -\mu\lambda_\alpha\pi\psi(0)$$



where we used (1), (2), (5), (11) and the expression for W_α. ●

Finally, define $S_{\alpha,\beta}$ for $\mu=1,2,3,\ldots$ as:

$$S_{\alpha,\beta}=\frac{(-1)^{\mu-1}\pi}{(\mu-1)!}\tilde{W}_{1,\beta}\frac{\partial^{\mu-1}}{\partial\bar{z}^{\mu-1}}E_\alpha=\frac{\pi}{((\mu-1)!)^2}W_{1,\alpha}\tilde{W}_{1,\beta}\delta^{(\mu-1,\mu-1)}.$$

LEMMA 16. *If* $\mu=1,2,3,\ldots,$ *then*

$$(L-\alpha)S_{\alpha,\beta}=(-1)^\mu\frac{\pi\mu\lambda_\alpha}{(\mu-1)!}\tilde{W}_{1,\beta}\delta^{(0,\mu-1)}$$

$$(\tilde{L}-\beta)S_{\alpha,\beta}=(-1)^\mu\frac{\pi\mu\lambda_\beta}{(\mu-1)!}W_{1,\alpha}\delta^{(\mu-1,0)}.$$

PROOF.

$$\begin{aligned}(L-\alpha)S_{\alpha,\beta}&=\frac{(-1)^{\mu-1}\pi}{(\mu-1)!}\tilde{W}_{1,\beta}(L-\alpha)\left\{\frac{\partial^{\mu-1}}{\partial\bar{z}^{\mu-1}}E_\alpha\right\}\\&=\frac{(-1)^{\mu-1}}{(\mu-1)!}\pi\tilde{W}_{1,\beta}\frac{\partial^{\mu-1}}{\partial\bar{z}^{\mu-1}}\{(L-\alpha)E_\alpha\}\\&=(-1)^\mu\frac{\pi\mu\lambda_\alpha}{(\mu-1)!}\tilde{W}_{1,\beta}\delta^{(0,\mu-1)}\text{ by Lemma 15.}\end{aligned}$$ ●

At last, we are able to solve (3) in all cases.

THEOREM 17. *The following distributions are a basis for the solution space of* (3):

(i) *If* $\mu\notin\mathbb{Z}$: $T_{1,\alpha,\beta}, T_{2,\alpha,\beta}$.
(ii) *If* $\mu=1,2,3,\ldots$ *and* $(\lambda_\alpha,\lambda_\beta)\neq(0,0)$: $T_{1,\alpha,\beta}, T_{3,\alpha,\beta}+S_{\alpha,\beta}$.
(iii) *If* $\mu=1,2,3,\ldots$ *and* $(\lambda_\alpha,\lambda_\beta)=(0,0)$: $T_{1,\alpha,\beta}, S_{\alpha,\beta}$.
(iv) *If* $\mu=0$: $T_{1,\alpha,\beta}, T_{3,\alpha,\beta}$.
(v) *If* $\mu=-1,-2,-3,\ldots$ *and* $(\lambda_\alpha,\lambda_\beta)=(0,0)$: $T_{6,\alpha,\beta}, T_{8,\alpha,\beta}, T_{9,\alpha,\beta}, T_{10,\alpha,\beta}$.
(vi) *If* $\mu=-1,-2,-3$ *and* $(\lambda_\alpha,\lambda_\beta)\neq(0,0)$: $T_{6,\alpha,\beta}, T_{7,\alpha,\beta}$.

PROOF. We prove only (ii) since the other cases are treated similarly. Suppose $\lambda_\alpha\neq0$. If $T_{\alpha,\beta}$ is a solution of (3), then there exist, by Lemma 4, $c_1,c_3\in\mathbb{C}$ and $X\in\mathscr{D}'(\Omega)$ with $\operatorname{supp}(X)\subset\{0\}$ such that:

$$T_{\alpha,\beta}=c_1T_{1,\alpha,\beta}+c_3T_{3,\alpha,\beta}+X.$$

The Lemmas 9, 11 (ii) and 16 imply:

$$(L-\alpha)X=-c_3(L-\alpha)T_{3,\alpha,\beta}=c_3(L-\alpha)S_{\alpha,\beta},\text{ so }(L-\alpha)(X-c_3S)=0.$$

We claim that $X=c_3S$. Write $Y=X-c_3S$ and suppose $Y\neq0$. Write

$$Y=\sum_{Y_j\neq0}\frac{\partial^j}{\partial\bar{z}^j}Y_j$$

where the Y_j are of the form

$$Y_j=\sum_{d_n\neq0}d_n\delta^{(n,0)}.$$

Using $(L-\alpha)Y=0$ and Lemma 14 we see that

$$(L-\alpha)Y_j=0 \text{ for all } j.$$

Take any $Y_j \neq 0$. Lemma 14 gives: order $(Y_j)=\mu-1$. Now note that order$(E_\alpha)=\mu-1$, since $W_{1,\alpha}(0)=1$. Therefore, there exists $c_j \in \mathbb{C}$, $c_j \neq 0$, such that $Y_j - c_j E_\alpha$ is either zero or non-zero and has order $\leq \mu-2$. Now

$$(L-\alpha)(Y_j - c_j E) = -\mu\lambda_\alpha c_j \delta \text{ by Lemma 15.}$$

In the first case, this gives $0=\mu\lambda_\alpha c_j\delta$ which contradicts our assumptions concerning μ, λ_α and c_j. In the second case (which can only occur if $\mu \geq 2$),

$$\text{order } [(L-\alpha)(Y_j - c_j E_\alpha)] = \text{order } (Y_j - c_j E_\alpha) + 1 \geq 1 \text{ by Lemma 14}$$

which is again a contradiction.

So $Y=0$ if $\lambda_\alpha \neq 0$. If $\lambda_\alpha = 0$ and $\lambda_\beta \neq 0$ we use F_β instead of E_α. ●

1.5. Remarks

1) Using techniques as the ones above, it should be possible to describe the general solutions of the problem:

$$\begin{cases} \left(a(z)\dfrac{\partial^2}{\partial z^2} + b(z)\dfrac{\partial}{\partial z} + c(z)\right)T = 0 \\ \left(\bar{d}(z)\dfrac{\partial^2}{\partial \bar{z}^2} + \bar{e}(z)\dfrac{\partial}{\partial \bar{z}} + \bar{f}(z)\right)T = 0 \end{cases} \qquad T \in \mathscr{D}'(\Omega)$$

where:

1. $a, \ldots, f \in H(\Omega)$.
2. $a(z)=0 \Leftrightarrow z=0$; $\dfrac{\partial a}{\partial z}(0) \neq 0$

   $d(z)=0 \Leftrightarrow z=0$; $\dfrac{\partial d}{\partial z}(0) \neq 0$.
3. $b(0) \in \mathbb{R}$

   $e(0) \in \mathbb{R}$.

2) In his thesis, [M.T. Kosters, section 2.5] Kosters found all solutions of the one-dimensional problem:

$$(L-\lambda)T=0 \qquad T \in \mathscr{D}'((\sigma,\tau)),\ \lambda \in \mathbb{C}$$

where:

1. $-\infty \leq \sigma < 0 < \tau \leq \infty$
2. $L = a(t)\dfrac{d^2}{dt^2} + b(t)\dfrac{d}{dt}$

3. a, b analytic on (σ, τ)

4. $a(t)=0 \Leftrightarrow t=0$

5. $a'(0)=1$; $b(0)=\mu+1$ ($\mu \in \mathbb{R}$; $\mu \neq -1, -2, -3, \ldots$)

The solution space turned out to be three-dimensional. As 5. shows, Kosters did not consider the case $\mu=-1, -2, -3, \ldots$, but, using the methods in his thesis, one finds easily that the solution space in this case is three-dimensional as well. In fact, we have the following theorem ([Komatsu, p. 18]):

Let $\Omega \subset \mathbb{R}$ be an interval and let

$$P\left(t, \frac{d}{dt}\right)=a_m(t)\frac{d^m}{dt^m}+\cdots+a_1(t)\frac{d}{dt}+a_0(t)$$

be an ordinary differential operator on Ω with analytic coefficients. Then all singular points in Ω are regular if and only if

$$\dim\{T \in \mathscr{D}'(\Omega) \mid PT=0\}=m+\sum_{t \in \Omega} \operatorname{ord}_t a_m(t)$$

where $\operatorname{ord}_t a_m(t)$ denotes the order of zero at t of $a_m(t)$.

Applying this theorem gives a dimension 3, as expected. In fact, knowing this theorem, it would have been possible to take a shorter path to the solutions, since most of the work is done in *excluding* possibilities (as in our case). Our results show that a general theorem as above can not exist in the two-dimensional (partial!) case, since it is easy to write down a version of (3) that has a four-dimensional solution space:

$$\begin{cases} z\dfrac{\partial^2}{\partial z^2}T=0 \\ & T \in \mathscr{D}'(\Omega) \\ \bar{z}\dfrac{\partial^2}{\partial \bar{z}^2}T=0 \end{cases}$$

has base solutions $1, z, \bar{z}$ and $|z|^2$.

2. SOLUTIONS IN OTHER SPACES

Let η be one of the following functions on Ω_1:

$$\eta(z)=|z|^{2\mu} \quad (\mu \in \mathbb{R}, \mu \neq -1, -2, -3, \ldots)$$

or

$$\eta(z)=|z|^{2\mu}\log|z| \quad (\mu=0, 1, 2, \ldots)$$

Define

$$\mathscr{H}_\eta=\mathscr{D}(\Omega)+\eta\mathscr{D}(\Omega)=\{\psi_0+\eta\psi_1 \mid \psi_0, \psi_1 \in \mathscr{D}(\Omega)\}.$$

We will topologize $\mathscr{H}_\eta$ and solve (3) in $\mathscr{H}'_\eta$.

Note that $\psi \to \eta\psi$ is one-to-one on $\mathscr{D}(\Omega)$. Topologize $\eta\mathscr{D}(\Omega)$ by requiring this map to be a homeomorphism. Let $V_\eta = \mathscr{D}(\Omega) \times \eta\mathscr{D}(\Omega)$, endowed with the product topology. V_η is a topological vector space.

Let $L_\eta = \{(\psi_0, \eta\psi_1) \in V_\eta \mid \psi_0 + \eta\psi_1 = 0 \text{ on } \Omega_1\}$.

L_η is a closed linear subspace of V_η, so V_η/L_η is a topological vector space in the quotient topology. $\mathscr{H}_\eta$ has the topology which is obtained by identifying $\mathscr{H}_\eta$ and V_η/L_η.

We will use the following remarks:

1. The inclusion map $\mathscr{D}(\Omega) \to \mathscr{H}_\eta$ is continuous. Thus, if $T \in \mathscr{H}_\eta'$, $T \mid \mathscr{D}(\Omega)$ is a distribution.

2. The map $\mathscr{D}(\Omega) \to \mathscr{H}_\eta$ given by $\psi \to \eta\psi$ is continuous. Suppose $T \in \mathscr{H}_\eta'$ and $T \mid \mathscr{D}(\Omega) = 0$. Then

$$\langle T, \psi_0 + \eta\psi_1 \rangle = \langle \sum_{\text{finite}} c_{kl}\delta^{(k,l)}, \psi_1 \rangle .$$

3. If $\chi \in C^\infty(\Omega)$, then the map $\mathscr{H}_\eta \to \mathscr{H}_\eta$, defined by $\psi \to \chi\psi$ is continuous. Therefore, if $T \in \mathscr{H}_\eta'$, we can define χT as for distributions.

4. Let D be a differential operator with C^∞-coefficients, and let $T \in \mathscr{H}_\eta'$. We would like to define

$$\langle DT, \psi \rangle = \langle T, D^t\psi \rangle \quad (\psi \in \mathscr{H}_\eta)$$

as for distributions. Unfortunately, D^t will not generally leave $\mathscr{H}_\eta$ invariant, but it is easily checked that $\mathscr{H}_\eta$ is invariant under L^t and $(\tilde{L})^t$. In fact, let $\psi_1 \in \mathscr{D}(\Omega)$ and define:

$$(15) \quad \left\{ \begin{aligned} f_{1,\psi_1} &= a\,\frac{\partial^2\psi_1}{\partial z^2} + \left(2\,\frac{\partial a}{\partial z} + 2\mu z^{-1}a - b\right)\frac{\partial \psi_1}{\partial z} + \\ &\quad + \left[\frac{\partial^2 a}{\partial z^2} - \frac{\partial b}{\partial z} + \mu z^{-2}\left\{2z\,\frac{\partial a}{\partial z} - bz + (\mu - 1)a\right\}\right]\psi_1 \end{aligned} \right.$$

and

$$(16) \qquad f_{2,\psi_1} = |z|^{2\mu}\left[\left\{z^{-1}\,\frac{\partial a}{\partial z} + (\mu - \tfrac{1}{2})z^{-2}a - \tfrac{1}{2}bz^{-1}\right\}\psi_1 + az^{-1}\,\frac{\partial \psi_1}{\partial z}\right].$$

Then

$$(17) \qquad \left.\begin{aligned} L^t(\eta\psi_1) &= \eta f_{1,\psi_1} \\ \tilde{L}^t(\eta\psi_1) &= \eta(f_{1,\bar{\psi}_1})^{-} \end{aligned}\right\} \quad (\eta = |z|^{2\mu})$$

$$(18) \qquad \left.\begin{aligned} L^t(\eta\psi_1) &= f_{2,\psi_1} + \eta f_{1,\psi_1} \\ \tilde{L}^t(\eta\psi_1) &= (f_{2,\bar{\psi}_1})^{-} + \eta(f_{1,\bar{\psi}_1})^{-} \end{aligned}\right\} \quad (\eta = |z|^{2\mu}\log|z|).$$

Using (1) and (2) we see that $f_{1,\psi_1}, f_{2,\psi_1} \in \mathscr{D}(\Omega)$ since

$$(19) \qquad 2\frac{\partial a}{\partial z} + 2\mu z^{-1}a - b = \mu + 1 + \cdots$$

$$(20) \qquad 2z\frac{\partial a}{\partial z} - bz + (\mu - 1)a = cz^2 + \cdots \quad (c \in \mathbb{C})$$

$$(21) \qquad z^{-1}\frac{\partial a}{\partial z} + (\mu - \tfrac{1}{2})z^{-2}a - \tfrac{1}{2}bz^{-1} = \tfrac{1}{2}\mu z^{-1} + \cdots .$$

Moreover, the expressions for f_{1,ψ_1} and f_{2,ψ_1} make it clear that $\psi \to L^t\psi$ and $\psi \to \tilde{L}^t\psi$ are continuous maps from $\mathscr{H}_\eta$ to $\mathscr{H}_\eta$, so it makes sense to consider the problem:

$$(22) \qquad \begin{cases} (L-\alpha)T = 0 \\ (\tilde{L}-\beta)T = 0 \end{cases} \quad T \in \mathscr{H}'_\eta.$$

5. If $T \in \mathscr{H}'_\eta$, we can define $\tilde{T}$ as we did for distributions. In Lemma 5, (i), (iv), (v) and (vi) remain true (use $\tilde{\eta} = \eta$).

6. Suppose $\psi = \psi_0 + \eta\psi_1 \in \mathscr{H}_\eta$. On considering the asymptotic expansion around 0, one sees that

$$(23) \qquad \langle A^{(k,l)}, \psi \rangle \equiv \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l}\psi_0(0)$$

and

$$(24) \qquad \langle B^{(k,l)}, \psi \rangle \equiv \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l}\psi_1(0)$$

are well-defined for $k, l \geq 0$ (if $\eta(z) = |z|^{2\mu}$ and $\mu = 1, 2, \ldots$ we must restrict ourselves to $0 \leq k,\ l \leq \mu - 1$ and the $A^{(k,l)}$). Furthermore, the $A^{(k,l)}$ and $B^{(k,l)}$ are continuous on $\mathscr{H}_\eta$ and

$$(25) \qquad (A^{(k,l)})^{\sim} = A^{(l,k)};\ (B^{(k,l)})^{\sim} = B^{(l,k)}.$$

We extend the relevant distributions in Theorem 17 to elements of $\mathscr{H}'_\eta$:

$$\langle T_{1,\alpha,\beta}, \psi \rangle \equiv Pf \int_{\Omega_\varrho} \Phi_\alpha \tilde{\Phi}_\beta \psi \, d(x,y) \quad (\psi \in \mathscr{H}_\eta)$$

$$\langle T_{2,\alpha,\beta}, \psi \rangle \equiv Pf \int_{\Omega_\varrho} W_{1,\alpha}\tilde{W}_{1,\beta}|z|^{-2\mu}\psi \, d(x,y) \quad (\psi \in \mathscr{H}_\eta)$$

$$\langle T_{3,\alpha,\beta}, \psi \rangle \equiv Pf \int_{\Omega_\varrho} (\lambda_\alpha \Phi_\alpha \tilde{W}_{1,\beta}\bar{z}^{-\mu} + 2\lambda_\alpha\lambda_\beta\Phi_\alpha\tilde{\Phi}_\beta \log|z| + $$
$$+ \lambda_\beta W_{1,\alpha}\tilde{\Phi}_\beta z^{-\mu})\psi \, d(x,y) \quad (\psi \in \mathscr{H}_\eta)$$

$$S_{\alpha,\beta} = \frac{\pi}{((\mu-1)!)^2} W_{1,\alpha}\tilde{W}_{1,\beta}A^{(\mu-1,\mu-1)} \quad (\text{only for } \mu = 1, 2, 3, \ldots).$$

A *reduction sequence* of t is a, finite or infinite, sequence of terms $t_0, t_1, \ldots$, such that $t_0 = t$ and $t_i \to t_{i+1}$. A term t is called *weakly normalizable* if at least one reduction sequence of t terminates in a normal form; t is called *strongly normalizable* if all of t's reduction sequences are finite. In the latter case, by König's lemma, the number of reduction steps in a reduction sequence of t is bounded; the maximum is denoted by $h(t)$ (the *height* of the reduction tree).

2.3. THE EXACT VALUATION. Now in order to obtain the expression for the height, terms are evaluated in L starting from an *assignment* υ, which gives a value $\upsilon(x^\alpha)$ in L_α to each variable x^α. As is customary we write $\upsilon(x/f)$ for the assignment which corresponds with υ everywhere except at x: $\upsilon(x/f)(x) = f$, $\upsilon(x/f)(y) = \upsilon(y)$ if $y \neq x$.

2.3.1. DEFINITION. Let t be a term of type α. The *exact valuation* $[t]_\upsilon \in L$ is defined for any assignment υ, by induction on t.

(i) $[x]_\upsilon = \upsilon(x)$

(ii) $[t_0 t_1]_\upsilon = [t_0]_\upsilon [t_1]_\upsilon$

(iii) $[\lambda x^\alpha \cdot t_0]_\upsilon = \langle \Lambda f \in L_\alpha \cdot [t_0]_\upsilon + f* + 1, [t_0]_\upsilon * \rangle$ if $x \notin t_0$, and
$\langle \Lambda f \in L_\alpha \cdot [t_0]_{\upsilon(x/f)} + 1, [t_0]_{\upsilon(x/c_0^\alpha)} * \rangle$ if $x \in t_0$.

Notice that if $x \notin t$, then $[t]_\upsilon = [t]_{\upsilon(x/f)}$, as can easily be verified by induction on t. Let c be the assignment defined by $c(x^\alpha) = c_0^\alpha$ and put $[t] = [t]_c$.

It may be instructive to calculate the following examples:

$$[\lambda x^o \cdot x] = [\lambda x^o \cdot y^o] = \langle \Lambda m \cdot m + 1, 0 \rangle,$$

$$[\lambda x^{(o)o} y^o \cdot x(xy)] = \langle \Lambda f \cdot \langle \Lambda m \cdot f(fm) + 2, f(f0) + 1 \rangle, 0 \rangle,$$

$$[(\lambda x^{(o)o} y^o \cdot x(xy)) \lambda z^o \cdot z] = \langle \Lambda m \cdot m + 4, 3 \rangle.$$

2.3.2. CLAIM. For any term t, $h(t) = [t]*$.

This will be proved in section 4. Here we first comment on the definition of $[t]$ and then call attention to some consequences of 2.3.2.

2.3.3. COMMENTS ON 2.3.1. The functional behaviour of the valuations was already described in the introduction (section 0.1). By that account clause (ii) is sufficiently explained.

ad i. For a variable x we have $[x] = [x]_c = c(x) = c_0$. Observe that if $t_1, \ldots, t_m$ are strongly normalizable (of the appropriate types), then so is $xt_1 \ldots t_m$ and moreover the height is given by the equation $h(xt_1 \ldots t_m) = h(t_1) + \cdots h(t_m)$. This squares with the fact that $(c_0 f_1 \ldots f_m)* = f_1* + \cdots f_m*$ (1.3(iii)).

ad iii. To get a grasp of this clause the reader should try to invent a strategy for constructing a reduction sequence from t which is as long as possible. Clearly if $x \notin t_0$, then in order to spoil no potential reduction steps a redex $(\lambda x \cdot t_0)t_1$ should not be contracted until t_1 is in normal form. On the other hand, if $x \in t_0$, it is better not to perform reductions inside t_1 before con-

The second, third and fourth term are zero, e.g. the fourth equals

$$Pf \int_{C_\varrho} \tfrac{1}{2}\lambda_\beta \bar{z}^\mu \tilde{\Phi}_\beta [z^\mu \log |z|, W_{1,\alpha} z^{-\mu}](idx+dy)$$

which is zero by Lemma 7, since $\mu \geq 1$. The second statement in b(ii) follows from remark 5.

b(iv). Let $\psi = \psi_0 + \eta\psi_1 \in \mathscr{H}_\eta$. Then, using Lemma 16,

$$\langle (L-\alpha)S_{\alpha,\beta}, \psi\rangle = \langle S_{\alpha,\beta}, (L^t-\alpha)\psi_0\rangle + \langle S_{\alpha,\beta}, (L^t-\alpha)(\eta\psi_1)\rangle$$

$$= \langle -\pi\mu\lambda_\alpha \tilde{W}_{1,\beta} A^{(0,\mu-1)}, \psi\rangle + \langle S_{\alpha,\beta}, f_{2,\psi_1}\rangle + \langle S_{\alpha,\beta}, \eta f_{1,\psi_1} - \alpha\eta\psi_1\rangle.$$

The third term is zero by the definition of $S_{\alpha,\beta}$ and the second equals

$$\frac{\pi}{((\mu-1)!)^2} \langle A^{(\mu-1,\mu-1)}, W_{1,\alpha} \tilde{W}_{1,\beta} f_{2,\psi_1}\rangle.$$

Now note that $\dfrac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} (W_{1,\alpha}\tilde{W}_{1,\beta} f_{2,\psi_1})|_{z=0} = 0$ for all (k,l) with $0 \leq l \leq \mu-1$, because of (16). This proves the first statement and the second follows again by remark 5. ●

LEMMA 19.

$$(L-\alpha)B^{(k,l)} = (\mu+k+1)B^{(k+1,l)} + \sum_{i=0}^{k} c_i B^{(i,l)} \quad (c_i \in \mathbb{C})$$

where $\mu \in \mathbb{Z}$ *if* $\eta(z) = |z|^{2\mu}$.

PROOF. Straightforward, using (1), (2) and (15) to (20).

THEOREM 20. *The solutions of* (22) *are precisely the extension of the solutions of* (3) *to* $\mathscr{H}_\eta$.

PROOF. For $\eta(z) = |z|^{2\mu}$ and $\mu = 0, 1, 2, \ldots$, this is the statement of the paragraph preceding Lemma 18.

In the other cases, Lemma 18 and Theorem 17 state that the extensions of solutions of (3) are solutions of (22). Now, if T is a solution of (22), we can write $T = T_0 + R$ where T_0 is an extension of a solution of (3) and $R|\mathscr{D}(\Omega) = 0$, so $\mathrm{supp}(R) \subset \{0\}$. Note that the map $\mathscr{D}(\Omega) \to \mathbb{C}$ defined by $\psi \to R(\eta\psi)$ is a distribution with support in $\{0\}$, so R is a finite linear combination of the $B^{(k,l)}$. Note that R itself must be a solution of (22). Lemma 19 and the restrictions of the values of μ give $R = 0$. ●

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