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An Uncertainty Principle for Integral Operators

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The classical uncertainty principle for the Fourier transform has been extended to the spherical transform for Gelfand pairs by Wolf. We sharpen the principle and extend its validity to the context of integral operators with a bounded kernel for which there is a Plancherel theorem.

1. INTRODUCTION

The uncertainty principle for the Fourier transform on $\mathbf{R}$ states, roughly speaking, that a function and its Fourier transform can not *both* be localized with arbitrary precision. Actually, there are at least two versions of the uncertainty principle.

The first one holds for functions f that are sufficiently regular, e.g., Schwartz functions. This form of the principle gives a lower bound for the product of the variance of f and $\hat{f}$; it is this form that can be bound in any book on quantum theory.

The second principle is valid for arbitrary functions in $L_2(\mathbf{R})$. In order to formulate it, we need a definition. Let $\varepsilon > 0$ and let $T \subset \mathbf{R}$ be Lebesgue-measurable, with characteristic function 1_T. We say that $f \in L_2(\mathbf{R})$ is *ε-concentrated on T* if $\|f - 1_T f\|_2 \leqslant \varepsilon \|f\|_2$. Then the second form of the uncertainty principle, due to Donoho and Stark ([DS]), reads as follows:

> *Let $f \in L_2(\mathbf{R})$, $f \neq 0$. Let $\varepsilon, \varepsilon' \geqslant 0$ and let $T, T' \subset \mathbf{R}$ be Lebesgue-measurable sets. Suppose that f is ε-concentrated on T and $\hat{f}$ is ε'-concentrated on T'. Then, with μ denoting Lebesgue-measure, we have $\mu(T)\,\mu(T') \geqslant (1 - \varepsilon - \varepsilon')^2$.*

This second form of the uncertainty principle has first been generalized to the Fourier transform for Abelian groups by Smith ([S]), and later to the spherical transform for Gelfand pairs by Wolf ([W]). The proof in [loc.cit.] is based on the integral equation for the spherical functions, the

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Fubini theorem, and the Hölder inequalities. However, an examination of the structure of the proof reveals that the integral equation is redundant. This observation allows one to prove this form of the uncertainty principle in more general situations where a Plancherel theorem holds for an integral operator with bounded kernel. The possibility of such an extension has been suggested by Katznelson and Diaconis ([DS, p. 926]).

The method of proof is essentially the same as in [W] and [DS], but leads to a sharper form of the principle, which is "obscured" in the case of Gelfand pairs if one uses the integral equation for the spherical functions.

2. The Uncertainty Principle

We now prove this more general form of the uncertainty principle.

Let (X, Λ, μ) and (X', Λ', μ') be σ-finite complete measure spaces.

Let $M(X)$ be the measurable functions on X, where we agree to identify functions that are equal almost everywhere. Let $M(X)^\natural$ be a subspace of $M(X)$, invariant under conjugation. If $V \subset M(X)$, put $V^\natural = V \cap M(X)^\natural$.

Assume that

(1) $L_2(X)^\natural$ is closed in $L_2(X)$;

(2) there exist three projections, $P_p: L_p(X) \mapsto L_p(X)^\natural$ $(p = 1, 2, \infty)$, agreeing on intersection of domains, commuting with conjugation, and such that for $p = 1, 2, \infty$:

(3) $\int_X (P_p f)(x)\, g(x)\, d\mu(x) = \int_X f(x)(P_{p'} g)(x)\, d\mu(x)$ whenever $f \in L_p(X)$ and $g \in L_{p'}(X)$ (where p' denotes the conjugate exponent of p).

Note that these assumptions imply that P_2 is the orthogonal projection of $L_2(X)$ onto $L_2(X)^\natural$. Hence P_2 is continuous; since $L_1(X) \cap L_2(X)$ is dense in $L_2(X)$, we conclude that $L_1(X)^\natural \cap L_2(X)^\natural$ is dense in $L_2(X)^\natural$.

Let $\Psi: X' \times X \mapsto \mathbf{C}$ be measurable with respect to the completed product measure on $X' \times X$ and suppose that $\|\Psi\|_\infty < \infty$.

For $f \in L_1(X)$ and $x' \in X'$, put

$$(Af)(x') = \int_X f(x)\, \Psi(x', x)\, d\mu(x).$$

Note that the map $x \mapsto f(x)\, \Psi(x', x)$ is measurable for μ'-almost all x'; if we define $(Af)(x')$ to be zero for those x' for which this does not hold, then $Af \in L_\infty(X')$. In the sequel we always tacitly assume that such adaptations have been made.

Assume that

(4) (Plancherel theorem, first part) A maps $L_1(X)^\natural \cap L_2(X)^\natural$ into $L_2(X')$, isometrically for the two-norms.

By density of $L_1(X)^\natural \cap L_2(X)^\natural$ in $L_2(X)^\natural$, A extends uniquely to an isometric isomorphism A_e of $L_2(X)^\natural$ into $L_2(X')$. Assume that

(5) (Plancherel theorem, second part) $A_e: L_2(X)^\natural \mapsto L_2(X')$ is surjective.

We define the operator $A': L_1(X') \mapsto L_\infty(X)$ by

$$(A'f')(x) = \int_{X'} f'(x') \overline{\Psi(x', x)} \, d\mu'(x') \qquad (f' \in L_1(X')).$$

Assume that

(6) A' maps $L_1(X')$ into $L_\infty(X)^\natural$.

In many situations, we have $M(X)^\natural = M(X)$, and then the assumptions reduce to the Plancherel theorem and the boundedness of Ψ. In other cases, one may have a Plancherel theorem only for a special subspace of $L_2(X)$, and the assumptions (1), (2), (3), and (6) are designed to let a class of such situations fit into the general framework. In the case of a Gelfand pair (G, K), $\natural$ denotes of course K-invariance, the projections are just averaging on the right and the left over K, and A (resp., A_e) is the spherical transform on $L_1(G)^\natural$ (resp., $L_2(G)^\natural$).

If $f' \in L_1(X') \cap L_2(X')$, then there is an obvious candidate for $(A_e)^{-1} f'$. Let us show that this guess is correct.

LEMMA 2.1. *If $f' \in L_1(X') \cap L_2(X')$, then $(A_e)^{-1} f' = A'f'$.*

Proof. Let $g \in L_1(X) \cap L_2(X)$. Then by (5), (2), and (3) we have

$$(f', A_e P_2 g)_{X'} = ((A_e)^{-1} f', P_2 g)_X = (P_2 (A_e)^{-1} f', g)_X = ((A_e)^{-1} f', g)_X.$$

But the left-hand side equals

$$\begin{aligned} &\int_{X'} f'(x') \overline{(A_e P_1 g)(x')} \, d\mu'(x') \\ &= \int_{X'} f'(x') \overline{(A P_1 g)(x')} \, d\mu'(x') \\ &= \int_{X'} f'(x') \left\{ \int_X \overline{(P_1 g)(x) \Psi(x', x)} \, d\mu(x) \right\} d\mu'(x') \\ &= \int_X \left\{ \int_{X'} f'(x') \overline{\Psi(x', x)} \, d\mu'(x') \right\} (P_1 \bar{g})(x) \, d\mu(x) \\ &= \int_X \left\{ \int_{X'} f'(x') \overline{\Psi(x', x)} \, d\mu'(x') \right\} \overline{g(x)} \, d\mu(x). \end{aligned}$$

We used (3) and (6) in the last equality. Hence

$$\int_X \left\{ \int_{X'} f'(x')\, \overline{\Psi(x', x)}\, d\mu'(x') - ((A_e)^{-1} f')(x) \right\} \overline{g(x)}\, d\mu(x) = 0.$$

Since $L_1(X) \cap L_2(X)$ contains the step-functions, this equation proves the lemma. ∎

It is interesting that the above assumptions imply a weak form of an inversion theorem. This is the content of the following corollary.

COROLLARY 2.2. *If* $f \in L_1(X)^\natural \cap L_2(X)^\natural$ *and* $Af \in L_1(X')$, *then* $f = A'Af$.

Proof. $f = (A_e)^{-1} A_e f = (A_e)^{-1} Af = A'Af$. ∎

Now let $T \in \Lambda$ and $T' \in \Lambda'$ be sets of finite measure with characteristic functions 1_T and $1_{T'}$. If f is a function, let M_f denote pointwise multiplication with f. As in [W] and [DS], the uncertainty principle is derived from the study of the operator $S = M_{1_T} \circ (A_e)^{-1} \circ M_{1_{T'}} \circ A_e$. S is a continuous operator from $L_2(X)^\natural$ into $L_2(X)$. As a first estimate we have $\|S\| \leqslant 1$. We proceed to show that $\|S\| \leqslant \|1_{T \times T'} \Psi\|_2$ (note that this new estimate may be worse). The uncertainty principle follows from this new estimate in exactly the same way as in [W].

PROPOSITION 2.3. *For* $S\colon L_2(X)^\natural \mapsto L_2(X)$ *we have* $\|S\| \leqslant \|1_{T \times T'} \Psi\|_2$.

Proof. Let $f \in L_1(X)^\natural \cap L_2(X)^\natural$. Since $A_e f = Af$ we see that $A_e f \in L_\infty(X')$. Hence $M_{1_{T'}} A_e f \in L_1(X') \cap L_2(X')$, so Lemma 2.1 permits us to conclude that actually $Sf = M_{1_T} A' M_{1_{T'}} Af$. Now let $x \in X$. Then we calculate as follows.

$$\begin{aligned}
(Sf)(x) &= 1_T(x) \int_{X'} 1_{T'}(x') (Af)(x')\, \overline{\Psi(x', x)}\, d\mu'(x') \\
&= 1_T(x) \int_{X'} 1_{T'}(x')\, \overline{\Psi(x', x)} \left\{ \int_X f(y)\, \Psi(x', y)\, d\mu(y) \right\} d\mu'(x') \\
&= 1_T(x) \int_X f(y) \left\{ \int_{X'} 1_{T'}(x')\, \overline{\Psi(x', x)}\, \Psi(x', y)\, d\mu'(x') \right\} d\mu(y) \qquad (7) \\
&= \int_X f(y)\, k_x(y)\, d\mu(y),
\end{aligned}$$

where

$$k_x(y) = 1_T(x) \int_{X'} 1_{T'}(x') \overline{\Psi(x', x)}\, \Psi(x', y)\, d\mu'(x')$$

$$= 1_T(x) \overline{\int_{X'} 1_{T'}(x')\, \Psi(x', x)\, \overline{\Psi(x', y)}\, d\mu'(x')}.$$

Put

$$k'_x(x') = 1_{T'}(x')\, \Psi(x', x).$$

Then

$$k_x = 1_T(x)\, \overline{(A'k'_x)}.$$

Note that $k'_x \in L_1(X') \cap L_2(X')$, so $A'k'_x = (A_e)^{-1} k'_x$. Hence, if we apply the Schwartz inequality to (7), we see that

$$|Sf(x)|^2 \leqslant \|f\|_2^2\, \|k_x\|_2^2$$

$$= 1_T(x)\, \|(A_e)^{-1} k'_x\|_2^2\, \|f\|_2^2$$

$$= 1_T(x)\, \|k'_x\|_2^2\, \|f\|_2^2.$$

So

$$\|Sf\|_2^2 \leqslant \int_T \|f\|_2^2\, \|k'_x\|_2^2\, d\mu(x)$$

$$= \|f\|_2^2 \int_T \left\{ \int_{T'} |\Psi(x', x)|^2\, d\mu'(x') \right\} d\mu(x)$$

$$= \|1_{T \times T'} \Psi\|_2^2\, \|f\|_2^2.$$

Since $L_1(X)^\natural \cap L_2(X)^\natural$ is dense in $L_2(X)^\natural$, the proposition follows. ∎

We can now prove the sharper form of the uncertainty principle. We state it in a slightly different way than was done in [W] and [DS]. To this end, let $L_2(X)_T = \{ f \in L_2(X) \mid M_{1_T} f = f \}$. Then M_{1_T} is nothing but the orthogonal projection of $L_2(X)$ onto $L_2(X)_T$. Similarly, put $L_2(X)^\natural_{T'} = \{ f \in L_2(X)^\natural \mid (A_e)^{-1} M_{1_{T'}} A_e f = f \}$. Then $(A_e)^{-1} M_{1_{T'}} A_e$ is the orthogonal projection of $L_2(X)^\natural$ onto $L_2(X)^\natural_{T'}$.

THEOREM 2.4 (Uncertainty principle).

$$\|1_{T \times T'} \Psi\|_2 \geqslant 1 - \inf_{f \in L_2(X)^\natural,\ \|f\|_2 = 1} \left(d(f, L_2(X)_T) + d(f, L_2(X)^\natural_{T'}) \right).$$

Proof. The proof is mimicked from [W] and [DS]. We include it for the sake of completeness. Let $f \in L_2(X)^\natural$ such that $\|f\|_2 = 1$. Then

$$\begin{aligned} 1 - \|M_{1_T}(A_e)^{-1} M_{1_{T'}} A_e f\|_2 &\leqslant \|f - M_{1_T}(A_e)^{-1} M_{1_{T'}} A_e f\|_2 \\ &\leqslant \|f - M_{1_T} f\|_2 + \|M_{1_T} f - M_{1_T}(A_e)^{-1} M_{1_{T'}} A_e f\|_2 \\ &\leqslant d(f, L_2(X)_T) + \|f - (A_e)^{-1} M_{1_{T'}} A_e f\|_2 \\ &= d(f, L_2(X)_T) + d(f, L_2(X)^\natural_{T'}). \end{aligned}$$

Hence the norm of $S: L_2(X)^\natural \mapsto L_2(X)$ satisfies

$$\|S\| \geqslant 1 - \inf_{f \in L_2(X)^\natural,\ \|f\|_2 = 1} (d(f, L_2(X)_T) + d(f, L_2(X)^\natural_{T'})),$$

and the theorem follows from Proposition 2.3. ∎

The uncertainty principle is phrased in terms of ε-concentration as in the Introduction by the other authors. This formulation is equivalent to ours since the concentration definition is a statement about distance.

3. Closing Remarks

Inspection of the above proofs reveals that the uncertainty principle is basically a consequence of the use of Fubini's theorem in the proof of Proposition 2.3. The application of the theorem is possible since $(A_e)^{-1}$ can be identified with an integral operator on $L_1(X') \cap L_2(X')$. This identification is established in Lemma 2.1, on basis of Assumptions (2), (3), and (6). This is the sole place where these assumptions are used; in fact, they are tailored just to make the proof of Lemma 2.1 work. Hence the method of proof still has a degree of freedom left: if one removes Assumptions (2), (3), and (6) but postulates the statement of Lemma 2.1 instead, then Theorem 2.4 still holds.

We emphasize that Lemma 2.1 is a *consequence* of the Plancherel theorem if $M(X)^\natural = M(X)$. Thus the uncertainty principle for an integral operator as above holds at least in the cases where

— the kernel is bounded;

— there is a Plancherel theorem for $L_2(X)$.

The uncertainty principle is thus seen to be inherent to Plancherel theorems in general, rather than being connected with harmonic analysis in the context of topological groups. This range of validity is rather wide, it contains e.g. some Sturm–Liouville problems and some problems involving operators other than differential operators (see e.g. [J]).

In some situations for given X, X', and Ψ, one *knows* that a Plancherel theorem holds for the integral operator associated to Ψ, but the Plancherel measure is yet to be determined. This occurs e.g. in the case of Gelfand pairs.

It is a "fact of life" that in many of these cases the Plancherel measure is closely connected with the asymptotic behaviour of Ψ (one needs of course additional structure on X and X' to define what asymptotic behaviour is). Now the Plancherel measure and Ψ are related to each other in Proposition 2.3 and Theorem 2.4; it is an intriguing (and ambitious) question whether this theorem (or a refinement of the method to prove it) can be of any help in establishing an explanation of this phenomenon, rigorous enough to determine the Plancherel measure from Ψ or vice versa.

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