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MIXING PROPERTIES FOR RANDOM WALK IN RANDOM SCENERY

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Consider the lattice Z^d, $d \geq 1$, together with a stochastic black–white coloring of its points and on it a random walk that is independent of the coloring. A local scenery perceived at a given time is a pattern of colors seen by the walker in a finite box around his current position. Under weak assumptions on the probability distributions governing walk and coloring, we prove asymptotic independence of local sceneries perceived at times 0 and n, in the limit as $n \to \infty$, and at times 0 and T_k, in the limit as $k \to \infty$, where T_k is the random kth hitting time of a black point. An immediate corollary of the latter result is the convergence in distribution of the interarrival times between successive black hits, i.e., of $T_{k+1} - T_k$ as $k \to \infty$. The limit distribution is expressed in terms of the distribution of the first hitting time T_1. The proof uses coupling arguments and ergodic theory.

1. Statement of results. Consider the lattice of d-dimensional integers Z^d, $d \geq 1$, together with two *independent* probabilistic structures: a *stochastic coloring* $(C(z))_{z \in Z^d}$, assigning either of the colors black or white to each point of the lattice; a *random walk* $(W_n)_{n \geq 0}$ on the points of the lattice, starting at the origin ($W_0 = 0$). The formal setup is as follows. Let C be the set of all possible colorings, F_C the σ-algebra generated by the cylinder sets and P_C a probability measure on (C, F_C) having the properties:

(A1) P_C is *stationary* and *ergodic* (w.r.t. translation in Z^d).
(A2) $0 < q := P_C(C(0) = \text{black}) < 1$.

Let W be the set of all possible walks (starting at 0), F_W the σ-algebra generated by the cylinder sets and P_W a probability measure on (W, F_W) such that:

(A3) The increments $W_{n+1} - W_n$, $n \geq 0$, are i.i.d. with density function $p\colon Z^d \to [0,1]$ which is *aperiodic*, i.e., there is no proper sublattice containing 0 and the support of p.

Then the combination of walk and coloring is described by the product probability space (Ω, F, P) given by $\Omega = C \times W$, $F = F_C \times F_W$ and $P = P_C \times P_W$. This is an example of *random walk in random scenery*.

We shall be concerned with the local color patterns the walker sees around himself while stepping through the lattice.

DEFINITIONS. (i) A *local scenery* s consists of a finite set $Q_s \subset Z^d$ and a black–white coloring of the points of Q_s. The color in s of $z \in Q_s$ is denoted by $s(z)$.

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(ii) The local scenery s is *perceived at time* n, an event which will be denoted by $[s]_n$, if $C(W_n + z) = s(z)$ for each $z \in Q_s$.

The following proposition may be inferred from Meilijson (1974). Define

$L_p :=$ smallest sublattice containing the set $\{z - z' \colon p(z)p(z') > 0,\ z, z' \in Z^d\}$.

In view of (A3), L_p either has dimension d or $d - 1$.

PROPOSITION. *Suppose that*

(P1) *P_C is ergodic w.r.t. translation in L_p.*

Then for any local sceneries s and t,

$$(*) \qquad \lim_{n \to \infty} P([s]_0 \cap [t]_n) = P([s]_0)P([t]_0).$$

Conversely, if L_p has dimension d, then $(*)$ for all s and t implies (P1).

Assumption (P1) is satisfied, e.g., when p is *strongly aperiodic* (i.e., $L_p = Z^d$) or when P_C is *strongly mixing*, meaning that for all (cylinder) sets A and B in F_C,

$$\lim_{|z| \to \infty} P_C(A \cap T^z B) = P_C(A)P_C(B),$$

where $T^z B$ is the translate of B over the vector z. (In fact strong mixing implies ergodicity w.r.t. translation in any sublattice.) The proof of the preceding proposition will be given in Section 2; it is essentially a refinement of ideas of Meilijson in the present context and is included for reasons of exposition [see also Berbee (1986)]. Counterexamples to $(*)$ are easily constructed within the class of *periodic* P_C, i.e., color distributions obtained from a given infinite periodic coloring by assigning equal probability to all its distinct translates.

If L_p has dimension $d - 1$, then $(*)$ may hold without (P1), as is seen from Example 1.

EXAMPLE 1. $d = 2$, $p(0,1) = p(1,0) = \frac{1}{2}$, P_C assigns i.i.d. colors to complete diagonals $\{z \in Z^d \colon z^1 + z^2 = k\}$, $k \in Z$.

It is not hard to formulate necessary and sufficient conditions, but we shall not do so and refer the reader to Section 2. Note, for instance, that in $d = 1$ the case $L_p = \{0\}$ corresponds to the "one-sided" random walk with either $p(1) = 1$ or $p(-1) = 1$ and then $(*)$ is obviously equivalent to strong mixing of P_C.

Our main result involves a version of the preceding proposition with a different time scale, viz. one in which time is counted according to the number of visits to black points. Let

$$T_k := \text{random time at which the walker hits a black point for the } k\text{th time}, \qquad k \geq 1.$$

Our basic assumptions (A1)–(A3) imply that the sequence of consecutive colors of the points visited by the walker, $(C(W_n))_{n\geq 0}$, which we shall henceforth refer to as the *color sequence*, is stationary and ergodic [Kasteleyn (1985) and Kakutani (1951), Theorem 3]. Therefore, in particular, $T_k < \infty$ P-a.s. for all $k \geq 1$.

Let F_C^N be the color σ-algebra generated by the cylinder sets outside the box

$$K_N := \{z \in Z^d \colon |z^i| \leq N, 1 \leq i \leq d\}$$

and let $F_C^\infty := \bigcap_N F_C^N$ denote the *color σ-algebra at infinity*. Our main theorem is:

THEOREM. *Suppose that*:

(T1) *There exist $z, z' \in Z^d$ such that* $p(z)p(z')P_C(C(z) \neq C(z')) > 0$.
(T2) F_C^∞ *is trivial.*

Then for any local sceneries s and t such that $0 \notin Q_t$,

$$(**) \qquad \lim_{k\to\infty} P([s]_0 \cap [t]_{T_k}) = P([s]_0)P([t]_0 \mid C(0) = black).$$

Assumption (T2) requires that all elements of F_C^∞ have probability either 0 or 1, or equivalently, for any (cylinder) set A in F_C,

$$P_C(A \mid F_C^\infty) = P_C(A) \quad \text{a.s.}$$

Obviously this is stronger than the strong mixing property mentioned previously, for the latter only requires that A is asymptotically independent of any far away cylinder set and not necessarily of the whole infinite coloring on the outside of a large cube.

An important class of probability measures for which (T2) holds is the class of *Gibbs states* which satisfy (A1) [Ruelle (1978), Theorem 1.11]. Here (A1) implies (T2) because of the assumption, inherent in the definition of Gibbs states, that for each coloring the total "interaction energy" between a given lattice site and its surroundings is finite, which naturally puts a restriction on the correlations.

It is important to note that $(**)$ is a much deeper result than $(*)$. The point to appreciate is that T_k is a random variable depending both on the walk and on the coloring, so a change of the colors anywhere in the lattice may (and will in general) change the distribution of the position the walker occupies at time T_k (for all k). Therefore, $(**)$ is in no way directly related to $(*)$. This may be illustrated by the following examples.

EXAMPLE 2. $d = 1$, $p(1) = 1$, P_C the distribution obtained by first coloring the sites independently and then replacing each black site by a pair of neighboring black sites. In this case $(*)$ holds, but $(**)$ does not.

EXAMPLE 3. $d = 1$, $p(1) = p(-1) = \frac{1}{2}$, P_C the strictly periodic distribution where black and white alternate. In this case $(**)$ holds, but $(*)$ does not.

Keane and den Hollander (1986) recently proved $(**)$ when P_C is the Bernoulli measure and p is transient random walk. The proof of the preceding extension will be given in Section 3 and is based on coupling arguments. We use coupling of colorings and coupling of walks. The role of (T2) is to allow us to couple colorings that differ inside a given finite box around 0 in such a way that they are identical outside a big (random) box. Once this is done, the idea is to exploit the fact that the walker, while progressing, spends more and more of his time outside the big box. Walks are then coupled so as to allow controlling of the random time scale T_k, and this uses (T1). The hardest part is to include recurrent random walk. In particular we will have some trouble with the case $d = 1$, where a special color coupling will be needed to make things work. This special coupling is based on part (a) of the following Tail Theorem, the proof of which will be presented in a separate paper.

TAIL THEOREM. *Let $d = 1$. Let $\Sigma_{[x,y]} :=$ # black points inside $[x, y]$.*

(a) *If* (T2) *holds, then $\bigcap_N \sigma(\Sigma_{[x,y]}: x \le -N, y \ge N)$ is trivial.*

(b) *If* (T2) *holds and if there exists an integer N such that with positive probability the random set $\{k: P_C(\Sigma_{[-N,N]} = k | F_C^N) > 0\}$ is not contained in any proper sublattice of Z, then $\bigcap_N \sigma(\Sigma_{[x,0]}: x \le -N)$ and $\bigcap_N \sigma(\Sigma_{[0,y]: y \ge N})$ are trivial.*

This theorem is of some independent interest because it gives conditions for tail triviality of sums of stationary 0–1 random variables. Note the interesting fact that single-sided sums require stronger assumptions than double-sided sums. Example 2 satisfies (a) but not (b).

An immediate corollary of our main theorem is the convergence in distribution of the *interarrival times* between successive black hits. Indeed, let $n_0 := T_1$ and $n_k := T_{k+1} - T_k$, $k \ge 1$. Then we have:

COROLLARY. *When $(**)$ holds, then for any nonnegative integer m,*

$$\lim_{k \to \infty} P(n_k > m) = q^{-1} P(n_0 = m).$$

To see this, first note that n_k depends only on the local sceneries perceived at time T_k. Hence, by $(**)$, $\lim_{k\to\infty}\{P(n_k > m) - P(n_k > m | C(0) = \text{black})\} = 0$. Then use that $P(n_k > m | C(0) = \text{black})$ is independent of k and is equal to $q^{-1}P(n_0 = m)$, both being a consequence of the stationarity and ergodicity of the color sequence [Kasteleyn (1985) and Kac (1947)].

Assumption (T1) requires that with positive probability the support of p contains points of both colors. The role of this assumption becomes clear from Lemmas 1–3, which we prove in Section 4.

LEMMA 1. *Suppose that $P_W(W_n = 0$ for some $n > 0) > 0$. If* (T1) *fails, then the color sequence is a.s. periodic and so is P_C.*

LEMMA 2. *Let* $L_p \neq \{0\}$. *Then* (T2) *implies* (T1).

Thus, our most general result is that $(**)$ holds under (T2) alone, provided $L_p \neq \{0\}$. When $L_p = \{0\}$, however, it is not enough to assume (T2) as may be seen from Example 2. In Section 4 we prove

LEMMA 3. *Let* $L_p = \{0\}$. *Then* $(**)$ *holds under the conditions of part* (b) *of the Tail Theorem.*

Lemma 3 gives a sufficient condition for convergence in distribution of interarrival times in stationary 0–1 sequences. It generalizes a result of Janson (1984) for m-dependent sequences [see also van den Berg (1986)]. All Gibbs states satisfy (b) of the Tail Theorem.

As will become clear in Section 3 later on, (T1) alone implies that

$$(***) \qquad \lim_{k\to\infty} \{P([s]_0 \cap [t]_{T_k}) - P([s]_0 \cap [t]_{T_{k+1}})\} = 0.$$

Since (A1)–(A3) are easily shown to imply that $(**)$ holds at least in the weaker sense of Cesàro (see the remarks at the end of Section 3), it, thus, is not unlikely that $(**)$ is true under (T1) alone, but stronger techniques are needed to settle this question. [Note, e.g., that (T1) is enough when P_C is periodic. This follows from the easily established fact that when P_C is periodic $P([s]_0 \cap [t]_{T_k})$ is asymptotically periodic in k.] Incidentally, Example 3 shows that $(**)$ may hold even without (T1).

Finally, in the language of ergodic theory $(*)$ and $(**)$ are equivalent to strong mixing of, respectively, the dynamical system associated with the local scenery process and the so-called *induced* dynamical system obtained by conditioning on the sceneries which have a black origin [see Keane and den Hollander (1986); in this paper $(**)$ is called *kasteleyn mixing*].

2. Proof of the proposition. This section uses ideas of Meilijson (1974); the results in Meilijson's paper relate to so-called skew products in $d = 1$, but are easily carried over to $d > 1$.

For each $z \in Z^d$, let $t + z$ be the translate over z of the local scenery t, i.e., $Q_{t+z} = Q_t + z$ and $(t + z)(z' + z) = t(z')$, $z' \in Q_t$. By the independence of walk and coloring,

$$P([s]_0 \cap [t]_n) = \sum_z P_W(W_n = z) P([s]_0 \cap [t + z]_0).$$

First assume that p is strongly aperiodic. [A more commonly adopted definition of strong aperiodicity is the requirement that for all $z \in Z^d$ there is no proper sublattice containing 0 and the set $\{z + z'\colon p(z') > 0, z' \in Z^d\}$. This is equivalent to $L_p = Z^d$.] Then it is known that for each positive integer m,

$$\lim_{n\to\infty} \sum_z \left| P_W(W_n = z) - |K_m|^{-1} \sum_{z' \in K_m} P_W(W_n = z + z') \right| = 0,$$

which says that the random walk spreads locally uniformly over Z^d. The easiest proof of this property is based on a coupling method of Ornstein (1969) in which two walkers that start from any two lattice sites are successfully coupled [see the proof of Ornstein's Theorem 7; see also Liggett (1985), pages 68–70]. Thus,

$$\begin{aligned}\lim_{n\to\infty} P([s]_0 \cap [t]_n) &= \lim_{n\to\infty} \sum_z \Big\{|K_m|^{-1} \sum_{z'\in K_m} P_W(W_n = z + z')\Big\} \\ &\qquad\qquad \times P([s]_0 \cap [t+z]_0) \\ &= \lim_{n\to\infty} \sum_y P_W(W_n = y) \\ &\qquad\qquad \times \Big\{|K_m|^{-1} \sum_{y'\in K_m} P([s]_0 \cap [t+y+y']_0)\Big\}.\end{aligned}$$

Since, by (A1), uniformly in y,

$$\lim_{m\to\infty} |K_m|^{-1} \sum_{y'\in K_m} P([s]_0 \cap [t+y+y']_0) = P([s]_0)P([t]_0),$$

this yields $(*)$ because $\sum_y P_W(W_n = y) = 1$ for all n.

If p is not strongly aperiodic, then the random walk spreads locally uniformly over the sublattice L_p and its translates, and hence it is enough to assume ergodicity w.r.t. translation in L_p (simply replace K_m by $K_m \cap L_p$ in the previous argument).

To prove the second statement in the proposition, let $T^z L_p$ denote the translate of L_p over the vector z. By (A1), we have that for any $z \in Z^d$ and uniformly in $y \in T^z L_p$,

$$\lim_{m\to\infty} |K_m \cap L_p|^{-1} \sum_{y'\in K_m\cap L_p} P([s]_0 \cap [t+y+y']_0) = Q^z(s,t),$$

where the limit is independent of $y \in T^z L_p$. (To get the uniformity, use that a.s. convergence implies uniform convergence on a set of measure arbitrarily close to 1.) Now, if L_p has dimension d, then there exists $j \geq 1$ such that Z^d is the union of j distinct translates $T^z L_p$, $z = z_1, \dots, z_j$ (choose $z_1 = 0$), and the walker moves cyclically between these translates. Hence, $(*)$ implies that $Q^z(s,t) = P([s]_0)P([t]_0)$ for $z = z_1, \dots, z_j$ and for all s and t. Taking $z = z_1 = 0$ and $y = 0$, we see that this implies (P1) because cylinders generate F_C.

The preceding argument breaks down when L_p has dimension $d-1$. In this case the walker moves through an infinite succession of parallel translates of L_p, and so for a fixed z we can get no information about $Q^z(s,t)$ from $(*)$.

3. Proof of the main theorem.

3A. *Outline and coupling.* Abbreviate $S := Z^d$, fix an arbitrary positive integer m, let

$$K_m(z) := \{z' \in S: |(z'-z)^i| \leq m, 1 \leq i \leq d\}$$

and define random variables

$$\begin{aligned} B_n &:= \#\{k \in [0, n]: C(W_k) = \text{black}\}, \\ H_n(z) &:= \#\{k \in [0, n]: W_k = z\}, \\ \tau_n(z) &:= \inf\{k > n: W_k = z\}, \\ u_n &:= \text{local scenery perceived at time } n \text{ inside } K_m(W_n), \qquad z \in S,\ n \geq 0. \end{aligned}$$

Consider two copies of S, denoted by S^1 and S^2, each of which accommodates a stochastic coloring and a random walk, which we shall denote by C^1, C^2 and W^1, W^2, respectively, and which are coupled in a way as is described below. (Upper indices 1 and 2 will always refer to S^1 and S^2.)

For the coupling of the colorings C^1 and C^2 we shall need the following Coupling Lemma:

COUPLING LEMMA. *Suppose that P_C satisfies* (T2). *Let C_m^1 and C_m^2 be arbitrary colorings of $K_m = K_m(0)$, each with positive probability, and condition on C^1 and C^2 having to coincide with C_m^1 and C_m^2 inside K_m. There exists a coupling of C^1 and C^2, described by a probability measure $P_C^{1,2}(\cdot \mid C_m^1, C_m^2)$ on $(C^1 \times C^2, F_C^1 \times F_C^2)$, with the following properties*:

(i) *The marginals are $P_C(\cdot \mid C_m^1)$ and $P_C(\cdot \mid C_m^2)$, respectively.*

(ii) *$P_C^{1,2}$-a.s., there exists a random integer $\rho > m$ such that $C^1(z) = C^2(z)$ for all $z \in S \setminus K_\rho$ (ρ will be the smallest such integer).*

Moreover, when $d = 1$ there exists a coupling which has the additional property:

(iii) *$P_C^{1,2}$-a.s., C^1 and C^2 have an equal number of black points inside K_ρ.*

Parts (i) and (ii) of the Coupling Lemma are corollaries of the following theorem of Goldstein (1979).

MAXIMAL COUPLING THEOREM. *Let $(Z_n^1)_{n \geq 0}$ and $(Z_n^2)_{n \geq 0}$ be arbitrary sequences of random variables taking values in the same Borel space and let P^1 and P^2 be their respective probability measures. There exists a successful coupling $P^{1,2}$, meaning that $P^{1,2}(Z_n^1 = Z_n^2$ for all n sufficiently large$) = 1$, iff P^1 and P^2 agree on the tail σ-algebra $\bigcap_N \sigma(Z_n: n \geq N)$.*

Indeed, from Goldstein's theorem it immediately follows that there exists a coupling of the colorings satisfying (i) and (ii) iff $P_C(\cdot \mid C_m^1)$ and $P_C(\cdot \mid C_m^2)$ agree on F_C^∞ for every choice of C_m^1 and C_m^2. But, of course, this is equivalent to (T2). [Incidentally, this also shows that the Coupling Lemma cannot work without (T2).] The existence of a coupling in $d = 1$ with the additional property (iii) is not immediate. This part depends on part (a) of the Tail Theorem in Section 1. Together with Goldstein's theorem this tells us that not only $C^1(z)$ and $C^2(z)$ but also the sums $\Sigma_{[x, y]}^1$ and $\Sigma_{[x, y]}^2$ can be successfully coupled at both ends.

So the Coupling Lemma gives us a coupling of C^1 and C^2. Next we couple the random walks.

Choose $\varepsilon > 0$ and integer N and let $M = [\varepsilon N]$ ($[\cdot]$ denotes the integer part). The random walks W^1 and W^2 are coupled as:

1. W^1 and W^2 trace the same path according to the rule p, until time N.
2. At time N, W^1 and W^2 are "uncoupled" and they proceed by making a succession of pairs of steps as follows: First W^1 makes two steps, according to the rule p; then W^2 makes two steps, independently but conditioned on having to end up at the same site as W^1; etc.
3. This "uncoupling" continues until either time $N + 2M$ is reached or $B_n^1 = B_n^2$ for some n in $[N, N + 2M]$. After that, W^1 and W^2 are "recoupled" and they again continue in unison, but now forever.

As a result of the uncoupling at time N, the walks W^1 and W^2 can visit different lattice points (and thus hit different colors) at times $N + 1$, $N + 3$, etc., until they are recoupled. The net effect of this uncoupling will be (and this will be seen to be *the crux of the proof*) that the difference between B_n^1 and B_n^2 accumulated at time $n = N$ gets pushed to 0 after time N and remains fixed at 0 for some time afterwards [see (3.1)].

We shall denote by $P^{1,2}(\cdot \mid C_m^1, C_m^2)$, $P(\cdot \mid C_m^1)$ and $P(\cdot \mid C_m^2)$ the probability measures describing the coupled system and its marginals. One easily checks that each of the walks separately is controlled by the same rule p, while according to (i) of the Coupling Lemma each of the colorings separately is controlled by the same probability measure P_C, but with the colorings conditioned on having to coincide with C_m^1 and C_m^2, respectively, inside K_m. Thus the marginals are just the probability measures obtained from P by conditioning on these colorings inside K_m.

Now we are ready to lay out the scheme of the proof. In what follows the reader will note that several steps in the argument could be simplified for transient random walk. The setup is dictated primarily by the recurrent case. Let

$$I = I(\varepsilon, N) := [N, N + 2M],$$
$$J = J(\varepsilon, N) := (N + 2M, N + 4M].$$

The main part of the proof consists in showing that

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} P^{1,2}\big(B_n^1 = B_n^2 \text{ for all } n \in J \mid C_m^1, C_m^2\big) = 1. \tag{3.1}$$

This will be carried out later in Section 3B and will require the use of (T1) and of properties (ii) and (iii) in the Coupling Lemma.

Continuing from (3.1), we may use that $W_n^1 = W_n^2$ for all $n \geq N + 2M$ to obtain

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} P^{1,2}\big(B_n^1 = B_n^2 \text{ and } u_n^1 = u_n^2 \text{ for all } n \in J \mid C_m^1, C_m^2\big) = 1, \tag{3.2}$$

provided we show that

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} P^{1,2}\left(K_\rho \cap \left(\bigcup_{n \in J} K_m\big(W_n^1\big)\right) = \varnothing \mid C_m^1, C_m^2\right) = 1, \tag{3.3}$$

so that the box K_ρ at which S^1 and S^2 differ in coloring falls outside each of the local (box) sceneries perceived by W^1 and W^2 in J. But for any positive integer R,

$$P^{1,2}\Big(K_\rho \cap \Big(\bigcup_{n\in J} K_m(W_n^1)\Big) \neq \varnothing \mid C_m^1, C_m^2\Big)$$
$$\leq P_C^{1,2}(\rho > R \mid C_m^1, C_m^2) + \sum_{z\in K_{R+m}} P_W(\tau_{N+2M}(z) \leq N + 4M),$$

and as part of the proof of (3.1) we shall show that for each $z \in S$,

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} P_W(\tau_N(z) \leq N + 4M) = 0, \tag{3.4}$$

so that (3.3) will follow by letting $R \to \infty$ afterwards.

Next, using that

$$P\Big(\lim_{k\to\infty} k^{-1}T_k = q^{-1}\Big) = 1, \tag{3.5}$$

which is an easy consequence of the stationarity and ergodicity of the color sequence [see, e.g., Breiman (1968), Chapter 6], we know that

$$P^{1,2}\Big(\lim_{k\to\infty} k^{-1}T_k^1 = \lim_{k\to\infty} k^{-1}T_k^2 = q^{-1} \mid C_m^1, C_m^2\Big) = 1.$$

Thus, for any $\varepsilon > 0$,

$$\lim_{N\to\infty} P^{1,2}(T_j^1, T_j^2 \in J \mid C_m^1, C_m^2) = 1$$

with $j := [(1 + 3\varepsilon)Nq]$. Together with (3.2), this gives

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} P^{1,2}\Big(u_{T_j^1}^1 = u_{T_j^2}^2 \mid C_m^1, C_m^2\Big) = 1. \tag{3.6}$$

From (3.6) we proceed as follows. Returning to the notation used in Sections 1 and 2, we let $[t]_n$ be the event that $u_n = t$, where we consider only local sceneries t with $Q_t \subset K_m \setminus \{0\}$. Then

$$\big|P([t]_{T_j} \mid C_m^1) - P([t]_{T_j} \mid C_m^2)\big| = \Big|P^{1,2}\big(u_{T_j^1}^1 = t \mid C_m^1, C_m^2\big) - P^{1,2}\big(u_{T_j^2}^2 = t \mid C_m^1, C_m^2\big)\Big|$$
$$\leq P^{1,2}\big(u_{T_j^1}^1 \neq u_{T_j^2}^2 \mid C_m^1, C_m^2\big).$$

Since the marginals are independent of N and ε and since $(1 + 3\varepsilon)q < 1$ for ε sufficiently small, this combines with (3.6) to give

$$\lim_{k\to\infty} \big\{P([t]_{T_k} \mid C_m^1) - P([t]_{T_k} \mid C_m^2)\big\} = 0. \tag{3.7}$$

We can now complete the proof by observing that for any t the sequence of indicators $1\{[t]_{T_k}\}$, $k \geq 1$, conditioned on the origin being black is stationary and ergodic [Keane and den Hollander (1986) and Kakutani (1943)], so that, in particular, for all $k \geq 1$,

$$P([t]_{T_k} \mid C(0) = \text{black}) = P([t]_0 \mid C(0) = \text{black}).$$

This is the final step, for indeed we can now choose m large enough so that $Q_s, Q_t \subset K_m$, average in (3.7) over those C_m^1 which realize $[s]_0$ and those C_m^2 which realize $\{C(0) = \text{black}\}$ and then we arrive at $(**)$.

3B. *Proofs of* (3.1) *and* (3.4). Let $\Delta B_n^{1,2} := B_n^1 - B_n^2$, $n \geq 0$. Since the uncoupling of the walks in I stops as soon as $\Delta B_n^{1,2} = 0$ for some $n \in I$, we have for any positive integer R,

$$\begin{aligned} &P^{1,2}\big(\Delta B_n^{1,2} \neq 0 \text{ for some } n \in J | C_m^1, C_m^2\big) \\ &\quad \leq P^{1,2}\big(\rho > R | C_m^1, C_m^2\big) \\ &\qquad + P^{1,2}\big(\Delta B_n^{1,2} \neq 0 \text{ for all } n \in I;\ \rho \leq R | C_m^1, C_m^2\big) \\ &\qquad + \sum_{z \in K_R} P^{1,2}\big(\tau_N^1(z) \leq N + 4M | C_m^1, C_m^2\big). \end{aligned} \tag{3.8}$$

We shall show that each of the three terms in the r.h.s. of (3.8) tends to 0 when we take limits in the order $N \to \infty$, $\varepsilon \to 0$ and $R \to \infty$. This will yield (3.1) and (3.4).

The first term is independent of N and ε and tends to 0 as $R \to \infty$ because $\rho < \infty$ $P^{1,2}$-a.s. The summand in the third term equals $P_W(\tau_N(z) \leq N + 4M)$. Fix R. Since for transient random walk, $\lim_{N \to \infty} P_W(\tau_N(z) < \infty) = 0$ for each $z \in S$, we need only worry about the recurrent case. Clearly,

$$\begin{aligned} E_W\big(H_{N+8M}(z) - H_N(z)\big) &\geq E_W\big(H_{N+8M}(z) - H_N(z);\ \tau_N(z) \leq N + 4M\big) \\ &\geq P_W\big(\tau_N(z) \leq N + 4M\big) E_W H_{4M}(0) \end{aligned}$$

and since for recurrent random walk $\lim_{n \to \infty} E_W H_n(0) = \infty$, while $E_W H_n(0) - E_W H_n(z)$ remains bounded for each z [Spitzer (1976), Section 28], it is enough to show that

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} \{E_W H_{N+8M}(0) - E_W H_N(0)\} / E_W H_{8M}(0) = 0.$$

Now there are no recurrent random walks in $d \geq 3$, while in $d = 2$ the numerator is bounded because $P_W(W_n = 0) = O(n^{-1})$, $n \to \infty$ [Spitzer (1976), Section 7]. Thus we need consider only $d = 1$. But it was shown by Kesten and Spitzer [(1963), see the proof of Lemma 4] that in $d = 1$,

$$\lim_{n \to \infty} \big\{E_W H_{(n+1)m}(0) - E_W H_{nm}(0)\big\} / E_W H_m(0) = 0 \quad \text{uniformly in } m$$

and so the preceding follows by setting $m = 8M$ and $n = [N/8M] = [18\varepsilon]$.

Thus we are left to deal with the second term in the r.h.s. of (3.8). For any positive integer K,

$$\begin{aligned} &P^{1,2}\big(\Delta B_n^{1,2} \neq 0 \text{ for all } n \in I;\ \rho \leq R | C_m^1, C_m^2\big) \\ &\quad \leq P^{1,2}\big(|\Delta B_N^{1,2}| > K;\ \rho \leq R | C_m^1, C_m^2\big) \\ &\qquad + P^{1,2}\big(W_n^1 \in K_R \text{ or } W_n^2 \in K_R \text{ for some } n \in I | C_m^1, C_m^2\big) \\ &\qquad + 2P^{1,2}\big(\Delta B_{n;N}^{1,2} > -K,\ \Delta B_n^{1,2} \neq 0 \text{ and } W_n^1, W_n^2 \notin K_\rho \text{ for all } n \in I | C_m^1, C_m^2\big). \end{aligned} \tag{3.9}$$

Here $\Delta B^{1,2}_{n;N} := \Delta B^{1,2}_n - \Delta B^{1,2}_N$, $n \geq N$, and in the last term the symmetry between W^1 and W^2 is used. We shall show that each of the three terms in the r.h.s. of (3.9) tends to 0 in the appropriate limit.

The first term is independent of ε and equals

$$\sum_{U, V \subset K_R} P_C^{1,2}\big(U^{1,2} = U; V^{1,2} = V; \rho \leq R \,|\, C_m^1, C_m^2\big) \tag{3.10}$$
$$\times P_W\left(\left| \sum_{z \in U} H_N(z) - \sum_{z' \in V} H_N(z') \right| > K \right),$$

where we introduce the random sets

$$U^{1,2} = \{z \in K_R: C^1(z) = \text{black}, C^2(z) = \text{white}\},$$

$$V^{1,2} = \{z \in K_R: C^1(z) = \text{white}, C^2(z) = \text{black}\}.$$

We claim that (3.10) tends to 0 as $N \to \infty$, for R fixed, if we let $K \to \infty$ depending on N such that

(I) $K = o(N^{1/2})$, K sufficiently close to $N^{1/2}$.

To see this, first note that (3.10) is bounded above by $K^{-1}|K_R| E_W H_N(0)$. Indeed this tends to 0 under (I), provided $P_W(W_n = 0) = o(n^{-1/2})$, $n \to \infty$. The latter property is shared by all random walks except those that fall in the class $d = 1$, $\Sigma_z |z|^2 p(z) < \infty$ and $\Sigma_z z p(z) = 0$ [Spitzer (1976), Section 7]. For this class, however, we can use property (iii) in the Coupling Lemma, which says that when $d = 1$ the coupling can be chosen in such a way that $|U^{1,2}| = |V^{1,2}|$ $P_C^{1,2}$-a.s. given $\rho \leq R$. We can then bound (3.10) above by

$$P_W\left(\tfrac{1}{2}|K_R| \sup_{z, z' \in K_R} |H_N(z) - H_N(z')| > K \right).$$

But it is known that for all random walks in this class

$$\lim_{n \to \infty} n^{-1/4-\delta} \sup_{z \in S} |H_n(z) - H_n(z+1)| = 0 \quad \text{a.s. for any } \delta > 0$$

[Csáki and Révész (1983), Lemma 5] and so it again follows that the limit of (3.10) is 0 under (I).

The second term in the r.h.s. of (3.9) is bounded above by $2P_W(W_n \in K_R$ for some $n \in I)$, which tends to 0 in the appropriate limit by the argument we just gave for the third term in the r.h.s. of (3.8).

Thus, to finish the proof, we are left to deal with the third term in the r.h.s. of (3.9) and it is here that the uncoupling of W^1 and W^2 in I comes into full play. Because of this uncoupling, $\Delta B^{1,2}_{N+2i;N} =: X_i$, $0 \leq i \leq M$, performs a "random walk" on the integers Z (starting at 0) with the properties:

PROPERTY 1. The absolute increments $|X_{i+1} - X_i|$ take values 0 or 1 and are dependent.

Property 2. Given $X_{i+1} - X_i \neq 0$, the sign of $X_{i+1} - X_i$ is random (i.e., $+$ or $-$, each with probability $\frac{1}{2}$ and independent of previous increments).

Property 2 is a consequence of the symmetry between W^1 and W^2 at each pair of steps taken.

We proceed as follows. Fix $\varepsilon > 0$ and let

$$Y_M := \sum_{i=0}^{M-1} |X_{i+1} - X_i|$$

be the random variable denoting the total number of "actual displacements" by the random walk X_i in the interval I. By Property 2 it is clear that, given any value of Y_M, the total displacement X_M is distributed as the position of a simple random walk on Z with independent increments after Y_M steps and starting at 0. Thus, when we let W_n^{srw} denote the position of simple random walk on Z at time n and P_W^{srw} the corresponding probability measure, then we have, for any positive integer L,

$$\begin{aligned}
&P^{1,2}\big(\Delta B_{n;N}^{1,2} > -K,\ \Delta B_n^{1,2} \neq 0 \text{ and } W_n^1, W_n^2 \notin K_\rho \text{ for all } n \in I \,|\, C_m^1, C_m^2\big)\\
&\quad = \sum_{j \geq 0} P^{1,2}\big(Y_M = j;\ \Delta B_n^{1,2} \neq 0 \text{ and } W_n^1, W_n^2 \notin K_\rho \text{ for all } n \in I \,|\, C_m^1, C_m^2\big)\\
&\qquad\qquad \times P_W^{\mathrm{srw}}\big(W_n^{\mathrm{srw}} > -K \text{ for all } n \in [0, j]\big) \qquad\qquad (3.11)\\
&\quad \leq P^{1,2}\big(Y_M < L;\ \Delta B_n^{1,2} \neq 0 \text{ and } W_n^1, W_n^2 \notin K_\rho \text{ for all } n \in I \,|\, C_m^1, C_m^2\big)\\
&\qquad + P_W^{\mathrm{srw}}\big(W_n^{\mathrm{srw}} > -K \text{ for all } n \in [0, L]\big).
\end{aligned}$$

Now, because the walkers cannot see the difference between C^1 and C^2 when they stay outside the box K_ρ and because the uncoupling in I continues as long as $\Delta B_n^{1,2} \neq 0$, the first term in the r.h.s. of (3.11) is bounded above by

$$P_*^{1,2}\big(Y_M < L \,|\, C_m^1\big). \qquad (3.12)$$

Here we introduce an *auxiliary* coupling measure $P_*^{1,2}$ defined as the probability measure for the coupled system that is obtained by letting C^1 and C^2 be identical over the *whole* lattice and distributed according to P_C and by uncoupling W^1 and W^2 over the *whole* interval I. In (3.12) the condition means that we condition on $C^1 = C^2$ having to coincide with C_m^1 inside K_m, but of course the same bound is true for C_m^2. Now under $P_*^{1,2}$:

Property 3. $|X_{i+1} - X_i|$, $i \geq 0$, is a stationary and ergodic 0–1 sequence.

This is a consequence of (A1) and (A3) in Section 1 [Kakutani (1951), Theorem 3]. (Note that Property 3 does not hold under $P^{1,2}$.) Hence, by the ergodic theorem,

$$P_*^{1,2}\Big(\lim_{M \to \infty} M^{-1} Y_M = E_*^{1,2}|X_1| \,|C_m^1\Big) = 1.$$

By assumption (T1), $E_*^{1,2}|X_1| = P_*^{1,2}(X_1 \neq 0) > 0$ and, therefore, the first term in the r.h.s. of (3.11) tends to 0 if we let $L \to \infty$ depending on N such that

(II) $L = o(N)$

(recall that $M = [\varepsilon N]$ and that ε is kept fixed). The second term also has limit 0 if

(III) $K = o(L^{1/2})$,

because of the well-known space–time scaling property of the simple random walk. Now collecting conditions (I)–(III), we see that K and L can be chosen to depend on N in such a way that all three conditions are met. This completes the proof of (3.1) and (3.4).

REMARK 1. As was mentioned in Section 1, assumption (T1) alone is enough to obtain $(***)$. The proof is easy. Suppose that we couple S^1 and S^2 by letting C^1 and C^2 be identical and distributed according to P_C and by uncoupling W^1 and W^2 at time 0. In this case the marginals both equal P. Then, because of (T1), the "random walk" performed by $\Delta B_n^{1,2}$ will a.s. hit any point of Z in the course of time. When it hits $+1$ we recouple, and $\Delta B_n^{1,2}$ is fixed at $+1$ for all subsequent n. Since $u_n^1 = u_n^2$ after the recoupling, we immediately obtain $(***)$.

REMARK 2. To prove that $(**)$ holds in the weaker sense of Cesàro, as was claimed in Section 1, first note that for any local scenery t the sequence of indicators $1\{[t]_n\}$, $n \geq 0$, is stationary and ergodic [Keane and den Hollander (1986) and Kakutani (1951), Theorem 3]. Hence,

$$\lim_{k\to\infty} k^{-1} \sum_{n=0}^{k-1} 1\{[t]_n\} = P([t]_0) \quad P\text{-a.s.}$$

Next, let t be such that $0 \notin Q_t$ and let t' be the local scenery with $Q_{t'} = Q_t \cup \{0\}$, $t'(z) = t(z)$ for $z \in Q_t$ and $t'(0) = $ black. Then clearly, for any k,

$$\sum_{n=0}^{T_k} 1\{[t']_n\} = \sum_{n=0}^{k} 1\{[t]_{T_n}\},$$

and if we now use (3.5), we get

$$\lim_{k\to\infty} k^{-1} \sum_{n=0}^{k-1} 1\{[t]_{T_n}\} = q^{-1}P([t']_0) = P([t]_0 \,|\, C(0) = \text{black}) \quad P\text{-a.s.}$$

Integration over those colorings which realize $[s]_0$ yields the result. [Incidentally, (3.5) follows from the same argument, by taking for t the local scenery with $Q_t = \varnothing$.]

4. Proof of Lemmas 1–3. For each $n \geq 0$, let

$$V_n = \{z \in Z^d\colon P_W(W_n = z) > 0\}$$

be the set of points the walker can reach at time n. Suppose that (T1) does not hold. Then the points in V_1 are P_C-a.s. identically colored. From stationarity and induction on n, it follows that for each n the points in V_n are P_C-a.s. identically colored. If p is nondegenerate (i.e., $L_p \neq \{0\}$), then for n large the set V_n will contain points arbitrarily far apart. Hence, P_C cannot satisfy (T2) (it is not even strongly mixing). This proves Lemma 2.

Now suppose that $P_W(W_n = 0 \text{ for some } n > 0) > 0$. Then there exists an integer $j \leq 1$ such that $P_W(W_{kj} = 0) > 0$ for all $k \geq 0$. This in turn implies that for each $i = 0, 1, \dots, j-1$ all the points in the union $\bigcup_{k \geq 0} V_{kj+i}$ are P_C-a.s. identically colored, which is the same as saying that the color sequence is P-a.s. periodic. But the color sequence is ergodic and, therefore, it must P-a.s. consist of a single periodic color sequence or one of its translates. But this, in turn, can happen only when P_C is periodic, as is easily seen from (A1) and (A3). This proves Lemma 1.

To prove Lemma 3 for the degenerate random walk in $d = 1$, use Goldstein's theorem together with part (b) of the Tail Theorem. These combine to tell us that there exists a successful coupling of the sums $\Sigma^1_{[0,y]}$ and $\Sigma^2_{[0,y]}$ for all y large when we condition on arbitrary local sceneries C^1_m and C^2_m. This obviously implies $(**)$ via (3.7), because when $p(1) = 1$ then $W^1_n = W^2_n = n$ P_W-a.s.; similarly when $p(-1) = 1$.

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