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SURVIVAL ASYMPTOTICS FOR BROWNIAN MOTION IN A POISSON FIELD OF DECAYING TRAPS[1]

BY ERWIN BOLTHAUSEN AND FRANK DEN HOLLANDER

Universität Zürich and Universiteit Utrecht

Let $W(t)$ be the Wiener sausage in $\mathbb{R}^d$, that is, the a-neighborhood for some $a > 0$ of the path of Brownian motion up to time t. It is shown that integrals of the type $\int_0^t \nu(s)\,d|W(s)|$, with $t \to \nu(t)$ nonincreasing and $\nu(t) \sim \nu t^{-\gamma}$, $t \to \infty$, have a large deviation behavior similar to that of $|W(t)|$ established by Donsker and Varadhan. Such a result gives information about the survival asymptotics for Brownian motion in a Poisson field of spherical traps of radius a when the traps decay independently with lifetime distribution $\nu(t)/\nu(0)$. There are two critical phenomena: (i) in $d \geq 3$ the exponent of the tail of the survival probability has a crossover at $\gamma = 2/d$; (ii) in $d \geq 1$ the survival strategy changes at time $s = [\gamma/(1+\gamma)]t$, provided $\gamma < 1/2$, $d = 1$, respectively, $\gamma < 2/d$, $d \geq 2$.

1. Introduction.

1.1. *Integrals of the Wiener sausage.* Let $\beta(t)$, $t \geq 0$, be standard Brownian motion in $\mathbb{R}^d$. For $a > 0$ the *Wiener sausage* is the random process $W(t)$, $t \geq 0$, given by

$$W(t) = \bigcup_{0 \leq s \leq t} B_a(\beta(s)), \tag{1.1}$$

where $B_a(x)$ is the closed ball of radius a around x. Donsker and Varadhan (1975) proved that for $\nu > 0$,

$$\lim_{t \to \infty} t^{-d/(d+2)} \log E\big(\exp[-\nu|W(t)|]\big) = -k(d,\nu) \tag{1.2}$$

where $|\cdot|$ is Lebesgue measure and

$$k(d,\nu) = \frac{d+2}{d}\left(\frac{d}{2}\nu\omega_d\right)^{2/(d+2)} \mu_d^{d/(d+2)}. \tag{1.3}$$

Here ω_d and μ_d are the volume of the unit ball $B_1(0)$ the principal Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ on ball $B_1(0)$. Observe that $k(d,\nu)$ does not depend on a.

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Equation (1.2) gives information about the probability that $|W(t)|$ is untypically small.

It is the aim of this paper to extend (1.2) in the following way:

THEOREM 1.4. *Let $\nu(t)$, $t \geq 0$, be any positive nonincreasing function such that $\nu(t) \sim \nu t^{-\gamma}$ as $t \to \infty$. If $\nu > 0$ and either*

$$0 < \gamma < \frac{1}{2} \quad \text{and } d = 1, \text{ or}$$
$$0 < \gamma < \frac{2}{d} \quad \text{and } d \geq 2, \tag{1.5}$$

then for any $a > 0$

$$\lim_{t \to \infty} t^{-(d-2\gamma)/(d+2)} \log E\left(\exp\left[-\int_0^t \nu(s)\, d|W(s)|\right]\right) = -k(d,\nu,\gamma) \tag{1.6}$$

with

$$k(d,\nu,\gamma) = \frac{d+2}{d-2\gamma}\left(\frac{d}{2}\frac{(1+\gamma)^{1+\gamma}}{\gamma^\gamma}\nu\omega_d\right)^{2/(d+2)} \mu_d^{d/(d+2)}. \tag{1.7}$$

We shall obtain (1.7) by deriving the following *variational representation* as an intermediate step.

THEOREM 1.8. *Under the conditions of Theorem 1.4,*

$$k(d,\nu,\gamma) = \inf_{\omega \in \Omega}\left\{\nu\left[\omega(1) + \gamma\int_0^1 \frac{\omega(u)\, du}{u^{1+\gamma}}\right] + \mu_d \omega_d^{2/d} \int_0^1 \frac{du}{\omega^{2/d}(u)}\right\}, \tag{1.9}$$

where the infimum is over the set

$$\Omega = \{\omega\colon [0,1] \to \mathbb{R}_+ \colon \omega(0) = 0,$$
$$0 < \omega(u) < \infty \text{ for } u \in (0,1], \text{ nondecreasing}\}. \tag{1.10}$$

Equation (1.9) has a unique minimizer $\omega^* \in \Omega$ given by

$$\omega^*(u) = l(d,\nu,\gamma)u^{d(1+\gamma)/(d+2)} \quad \text{for } 0 \leq u \leq \frac{\gamma}{1+\gamma}$$
$$= \omega^*\left(\frac{\gamma}{1+\gamma}\right) \quad \text{for } \frac{\gamma}{1+\gamma} < u \leq 1 \tag{1.11}$$

with $l(d,\nu,\gamma) = \omega_d^{2/(d+2)}(2\mu_d/(d\gamma\nu))^{d/(d+2)}$.

Equation (1.9) should be interpreted as an *optimization of the growth profile* of $|W(s)|$, $0 \leq s \leq t$. Here is a heuristic explanation. We shall see that the best strategy for the Wiener sausage is to fill balls of a growing radius, namely,

$$|W(ut)| \approx \left|B_{v(u)t^{(1+\gamma)/(d+2)}}(0)\right|, \qquad 0 \leq u \leq 1, \tag{1.12}$$

for some $v \in \Omega$. We shall prove that as $t \to \infty$ the event in (1.12) has probability approximately

$$\exp\left[-\mu_d \int_0^1 \frac{d(ut)}{\left(v(u)t^{(1+\gamma)/(d+2)}\right)^2}\right]. \tag{1.13}$$

The integral in (1.6) can be written

$$\begin{aligned}\int_0^t \nu(s)\, d|W(s)| &= \nu(t)|W(t)| - \int_0^t |W(s)|\, d\nu(s)\\ &\sim \nu t^{-\gamma}\left[|W(t)| + \gamma \int_0^1 \frac{|W(ut)|\, du}{u^{1+\gamma}}\right]\end{aligned} \tag{1.14}$$

where we substitute $\nu(t) \sim \nu t^{-\gamma}$ and $d\nu(s) \sim -\nu\gamma s^{-1-\gamma}\, ds$ and put $s = ut$. On the event in (1.12) the r.h.s. of (1.14) assumes the value approximately

$$\nu t^{-\gamma} \omega_d t^{d(1+\gamma)/(d+2)}\left[v^d(1) + \gamma \int_0^1 \frac{v^d(u)\, du}{u^{1+\gamma}}\right]. \tag{1.15}$$

If we now combine (1.13) and (1.15), then we see the power $t^{(d-2\gamma)/(d+2)}$ appearing and find that a natural guess for the limit in (1.6) is

$$-\inf_{v \in \Omega}\left\{\nu\omega_d\left[v^d(1) + \gamma\int_0^1 \frac{v^d(u)\, du}{u^{1+\gamma}}\right] + \mu_d \int_0^1 \frac{du}{v^2(u)}\right\}. \tag{1.16}$$

Indeed, this is the same as the r.h.s. of (1.9) with $\omega(u) = \omega_d v^d(u)$.

The above heuristic explanation is made precise by the following result. Let P_t be the Wiener measure on $C_0([0,t]; \mathbb{R}^d)$ (the set of continuous paths in $\mathbb{R}^d$ of length t starting at 0) and define a new measure Q_t by putting

$$\frac{dQ_t}{dP_t}(x_{[0,t]}) = \frac{1}{Z_t}\exp\left[-\int_0^t \nu(s)\, d\left|W(x_{[0,t]}; s)\right|\right] \tag{1.17}$$

where $W(x_{[0,t]}; s) = \bigcup_{0 \le r \le s} B_a(x(r))$ and Z_t is the normalizing constant equal to the expectation appearing in (1.6).

THEOREM 1.18. *Under the conditions of Theorem* 1.4,

$$\lim_{t\to\infty} Q_t\left(\sup_{0 \le u \le 1}\left|t^{-d(1+\gamma)/(d+2)}|W(ut)| - \omega^*(u)\right| > \varepsilon\right) = 0 \tag{1.19}$$

for all $\varepsilon > 0$ *with* ω^* *given by* (1.11).

Equation (1.19) identifies ω^* as the *optimal growth profile*. Note that it only makes a statement about the volume $|W(ut)|$, not about the set $W(ut)$ itself. Probably it is true that $|t^{-(1+\gamma)/(d+2)}W(ut) \triangle B_{v(u)}(z(u))| \to 0$ as $t \to \infty$ for every u and some random $z(u)$. Techniques to handle this question have been developed in Schmock (1990), Bolthausen (1994) and Sznitman (1991) for the case $\gamma = 0$, $d = 1, 2$.

1.2. *Brownian motion and decaying Poisson traps.* The results in Theorems 1.4 and 1.18 have an interpretation in terms of *survival asymptotics*. Let $\{X_i\}$ be a Poisson point process in $\mathbb{R}^d$ with intensity $\nu > 0$. Let τ_i be i.i.d. random variables in $(0, \infty)$, independent of the X_i's, with distribution

$$\xi(t) = P(\tau_i > t), \quad t \geq 0, \tag{1.20}$$

where $\xi(t)$ is any nonincreasing function with $\xi(0) = 1$. Consider now the random process $V(t)$, $t \geq 0$, given by

$$V(t) = \bigcup_{\{i:\, \tau_i > t\}} B_a(X_i). \tag{1.21}$$

Think of $V(t)$ as a collection of spherical traps (with radius a and centered at the X_i's) that decay independently according to the lifetime distribution $\xi(t)$ in (1.20). The *trapping time* of Brownian motion is the first hitting time of a surviving trap, that is,

$$T = \inf\{t \geq 0 \colon \beta(t) \in V(t)\}. \tag{1.22}$$

The following two propositions make the link with subsection 1.1.

PROPOSITION 1.23. *With* $\nu(t) = \nu\xi(t)$,

$$P(T > t) = E\left(\exp\left[-\int_0^t \nu(s)\, d|W(s)|\right]\right). \tag{1.24}$$

PROOF. Condition on $\beta(s)$, $0 \leq s \leq t$, and use the fact that $\beta(r) \in V(r)$ if and only if $X_i \in B_a(\beta(r))$ for some i with $\tau_i > r$, to obtain $P(T > r + dr | T > r;\, \beta(s), 0 \leq s \leq t) = 1 - \nu\xi(r)\, d|W(r)|$. □

PROPOSITION 1.25. *On* $C_0([0, t]; \mathbb{R}^d)$,

$$P(\cdot | T > t) = Q_t(\cdot). \tag{1.26}$$

PROOF. Obvious from (1.17). □

There are two interesting phenomena contained in Theorems 1.4 and 1.18.

(i) For the range of γ-values in (1.5) the exponent $(d - 2\gamma)/(d + 2)$ in (1.6) assumes values in the intervals

$$\left(0, \frac{d}{d+2}\right) \quad d = 1, 2, \qquad \left(\frac{d-2}{d}, \frac{d}{d+2}\right) \quad d \geq 3. \tag{1.27}$$

Thus in $d \geq 3$ we are left with a gap, namely $(0, (d-2)/d]$, which must correspond to $\gamma \geq 2/d$ (i.e., fast decay of traps). At $\gamma = 2/d$ there is a *cross over*. Indeed, from Jensen's inequality applied to (1.6) together with $E|W(t)| \sim c_a t$ we see that the exponent in (1.6) cannot exceed $1 - \gamma$ [recall that $\nu(t) \sim \nu t^{-\gamma}$], and so it cannot stick to the value $(d - 2\gamma)/(d + 2)$ for $\gamma > 2/d$. What happens is that for $\gamma > 2/d$ the strategy in (1.12) breaks down because it would correspond to $|W(t)|$ growing faster than linear instead of slower, that is, in the opposite direction from the typical linear growth. Such large deviations would obviously not be the ones giving the main contribution to the expectation in (1.6). By using our result for $\gamma = 2/d - \varepsilon$ ($\varepsilon > 0$ arbitrary) we can actually show that the exponent, if it exists, must be $1 - \gamma$ for $\gamma \geq 2/d$. However, we do not know how to prove its existence nor how to identify the underlying variational principle.

A discrete time/space version of the trapping model was investigated by den Hollander and Shuler (1992). Here the correct exponent was proved only for $d = 1$, and for $d \geq 2$ it was argued heuristically that the exponent should be $(d - 2\gamma)/(d + 2)$, $0 < \gamma < 2/d$ and $1 - \gamma$, $2/d \leq \gamma < 1$, as above.

In order that $P(T = \infty) = 0$ we must restrict γ to $0 \leq \gamma \leq 1/2$ if $d = 1$, and to $0 \leq \gamma \leq 1$ if $d \geq 2$ [see den Hollander and Shuler (1992)].

(ii) The minimizer ω^* in (1.11) sticks at $\omega^*(u_c)$ for all $u > u_c$, where $u_c = \gamma/(1 + \gamma)$ is a *critical scaled time*. Apparently, it is easier for the Brownian motion to stay inside the ball of radius $v^*(u_c)t^{(1+\gamma)/(d+2)}$ [recall (1.12), and $\omega^*(u) = \omega_d v^{*d}(u)$] than it is for the traps in the annulus around this ball to decay in order to allow the Brownian motion more space. Thus $v^*(u_c)$ is a *critical scaled radius* where the strategy changes.

Note that whereas $v^*(u_c)$ depends on all parameters, remarkably u_c only depends on γ.

2. Proof of the lower bound. For the lower bound it suffices to consider strategies of the type in (1.12). Subsection 2.1 contains a standard technical lemma. The proof comes in subsection 2.2.

2.1. *Confinement to balls.*

LEMMA 2.1. *There exists $c > 0$ such that for all $r \geq 2$ and $t \geq 0$,*

$$\inf_{x \in B_1} P(\beta(s) \in B_r \text{ for } 0 \leq s \leq t, \quad \beta(t) \in B_1 | \beta(0) = x) \geq \frac{c}{r^d} \exp\left[-\mu_d \frac{t}{r^2}\right]. \tag{2.2}$$

PROOF. Let $p_t^r(x, y)$ be the transition kernel of Brownian motion killed outside B_r. Pick $r \geq 2$. There exists $c_1 > 0$ and independent of r such that

$$\inf_{x, y \in B_1} p_1^r(x, y) \geq c_1. \tag{2.3}$$

Pick $t \geq 2$. Use (2.3) to estimate

$$\begin{aligned} \inf_{x, y \in B_1} p_t^r(x, y) &= \inf_{x, y \in B_1} \int_{B_r} da \int_{B_r} db\, p_1^r(x, a) p_{t-2}^r(a, b) p_1^r(b, y) \\ &\geq c_1^2 \int_{B_r} da \int_{B_r} db\, 1_{B_1}(a) p_{t-2}^r(a, b) 1_{B_1}(b) \\ &= c_1^2 \langle 1_{B_1}, p_{t-2}^r 1_{B_1} \rangle. \end{aligned} \tag{2.4}$$

By the spectral theorem,

$$\langle 1_{B_1}, p^r_{t-2} 1_{B_1}\rangle \ge \langle \psi_r, 1_{B_1}\rangle^2 e^{-\lambda_r(t-2)} \tag{2.5}$$

with λ_r, and ψ_r the principal Dirichlet eigenvalue and eigenfunction of $-\frac{1}{2}\Delta$ on B_r ($\int_{B_r}\psi_r^2(x)\,dx = 1$). Hence

$$\inf_{x\in B_1}\int_{B_1} p^r_t(x,y)\,dy \ge |B_1| c_1^2 \langle \psi_r, 1_{B_1}\rangle^2 e^{-\lambda_r(t-2)}. \tag{2.6}$$

The l.h.s. of (2.6) equals the l.h.s. of (2.2). For the r.h.s. of (2.6) use scaling

$$\begin{aligned} \lambda_r &= \frac{\lambda_1}{r^2} \quad \text{with } \lambda_1 = \mu_d, \\ \psi_r(x) &= \frac{1}{r^{d/2}}\psi_1\left(\frac{x}{r}\right). \end{aligned} \tag{2.7}$$

Since there exists $c_2 > 0$ independent of r such that $\inf_{x\in B_{1/r}} \psi_1(x) \ge c_2$, it follows from (2.7) that $\langle \psi_r, 1_{B_1}\rangle \ge |B_1| c_2 / r^{d/2}$. This proves the claim in (2.2) for $t \ge 2$. The case $0 \le t < 2$ is trivial. □

2.2. *Lower bound*. Pick N large. Split the time interval $[0, t]$ into pieces by cutting it at the sequence of times

$$\begin{aligned} \tau_0 &= 0, \\ \tau_i &= 2^{i-1}, \qquad & i &= 1,\dots,i_t, \\ \tau_i &= \frac{1}{N}(i - i_t)t, \qquad & i &= i_t + 1,\dots,i_t + N, \end{aligned} \tag{2.8}$$

where i_t is chosen such that $2^{i_t} = (1/N)t$. The reason for cutting differently below and above $(1/N)t$ has to do with the singularity in (1.9) and (1.16) at $u = 0$. Next, pick $v \in \Omega$ [recall (1.10)] and $\alpha \in (\gamma/d, (1+\gamma)/(d+2))$ and define the sequence of radii

$$\begin{aligned} r_i &= v\left(\frac{1}{N}\right) t^{(1+\gamma)/(d+2)}\left[2^{i-1} N \frac{1}{t}\right]^{\alpha}, \qquad & i &= 1,\dots,i_t, \\ r_i &= v\left(\frac{1}{N}(i - i_t)\right) t^{(1+\gamma)/(d+2)}, \qquad & i &= i_t + 1,\dots,i_t + N. \end{aligned} \tag{2.9}$$

Note that the τ_i's and r_i's are increasing and that $r_i \to \infty$ as $t \to \infty$ for all i.
According to Lemma 2.1,

$$\begin{aligned} &P\left(\bigcap_i \{\beta(s) \in B_{r_i} \text{ for } s \in [\tau_{i-1}, \tau_i], \beta(\tau_i) \in B_1\right) \\ &\qquad \ge \prod_i \frac{c}{r_i^d} \exp\left[-\mu_d \frac{\tau_i - \tau_{i-1}}{r_i^2}\right]. \end{aligned} \tag{2.10}$$

On the event in (2.10) the integral in (1.6) can be estimated by

$$\begin{aligned}\int_0^t \nu(s)\, d|W(s)| &= \nu(t)|W(t)| - \int_0^t |W(s)|\, d\nu(s)\\ &\le \nu(\tau_{i_t+N})|B_{a+r_{i_t+N}}| + \sum_i \big(\nu(\tau_{i-1}) - \nu(\tau_i)\big)|B_{a+r_i}|.\end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11), and substituting (2.8), (2.9) and $\nu(t) \sim \nu t^{-\gamma}$, we obtain

$$\begin{aligned}&\log E\left(\exp\left[-\int_0^t \nu(s)\, d|W(s)|\right]\right)\\ &\quad\ge -(1+o(1))\Bigg\{\nu t^{-\gamma}\omega_d v^d(1) t^{d(1+\gamma)/(d+2)}\\ &\qquad + \sum_{i=1}^{i_t+1} \nu(2^\gamma - 1)2^{-\gamma(i-1)}\omega_d v^d\left(\frac{1}{N}\right) t^{d(1+\gamma)/(d+2)}\\ &\qquad \times\left[2^{i-1}\frac{N}{t}\right]^{\alpha d}\\ &\qquad + \sum_{i=i_t+2}^{i_t+N} \nu\left\{\left[\frac{(i-i_t-1)t}{N}\right]^{-\gamma} - \left[\frac{(i-i_t)t}{N}\right]^{-\gamma}\right\}\\ &\qquad \times \omega_d v^d\left(\frac{i-i_t}{N}\right) t^{d(1+\gamma)/(d+2)}\\ &\qquad + \mu_d \sum_{i=1}^{i_t+1} 2^{i-1} v^{-2}\left(\frac{1}{N}\right) t^{-2(1+\gamma)/(d+2)}\\ &\qquad \times\left[2^{i-1}N\frac{1}{t}\right]^{-2\alpha}\\ &\qquad + \mu_d \sum_{i=i_t+2}^{i_t+N} \frac{1}{N} t v^{-2}\left(\frac{i-i_t}{N}\right) t^{-2(1+\gamma)/(d+2)} + O(i_t \log t)\Bigg\}.\end{aligned} \tag{2.12}$$

In the first sum we have also inserted the asymptotic form of $\nu(\tau_{i-1}) - \nu(\tau_i)$ for small i, thereby making an error that is incorporated in the lead factor $1 + o(1)$. In the third sum we have dropped τ_{i-1} for $i \ge 2$. Both these sums can now be computed and expressed in terms of N and t via the relation $2^{i_t} = (1/N)t$. Collecting powers of t and noting that $\alpha d > \gamma$, $2\alpha > 1$ and

$i_t = O(\log t)$, we get for the r.h.s. of (2.12)

$$\begin{aligned}
&-(1+o(1))t^{(d-2\gamma)/(d+2)} \\
&\times\Bigg\{\nu\omega_d\Bigg[v^d(1)+\frac{2^\gamma-1}{1-2^{\gamma-\alpha d}}\left(\frac{1}{N}\right)^{-\gamma}v^d\left(\frac{1}{N}\right) \\
&\qquad+\sum_{j=2}^{N}\left\{\left(\frac{j-1}{N}\right)^{-\gamma}-\left(\frac{j}{N}\right)^{-\gamma}\right\}v^d\left(\frac{j}{N}\right)\Bigg] \qquad (2.13)\\
&\quad+\mu_d\left[\frac{1}{1-2^{2\alpha-1}}\frac{1}{N}v^{-2}\left(\frac{1}{N}\right)+\sum_{j=2}^{N}\frac{1}{N}v^{-2}\left(\frac{j}{N}\right)\right]\Bigg\}.
\end{aligned}$$

Next let $N\to\infty$. We claim that the term between braces in (2.13) converges to

$$\nu\omega_d\left[v^d(1)+\int_0^1 d(-u^{-\gamma})v^d(u)\right]+\mu_d\int_0^1 du\, v^{-2}(u), \qquad (2.14)$$

provided $v\in\Omega$ is such that the two integrals in (2.14) are finite. The proof uses the fact that v is nondecreasing. Indeed, by this property the second and the fourth term in (2.13) can be bounded by

$$\begin{aligned}
\left(\frac{1}{N}\right)^{-\gamma}v^d\left(\frac{1}{N}\right)&\le\frac{1}{1-2^{-\gamma}}\int_{1/N}^{2/N}d(-u^{-\gamma})v^d(u), \\
\frac{1}{N}v^{-2}\left(\frac{1}{N}\right)&\le\int_0^{1/N}du\, v^{-2}(u)
\end{aligned} \qquad (2.15)$$

and therefore both tend to zero. The fifth term in (2.13) can be estimated by

$$\frac{1}{N}v^{-2}(1)+\int_{2/N}^1 du\, v^{-2}(u)\le\sum_{j=2}^{N}\frac{1}{N}v^{-2}\left(\frac{j}{N}\right)\le\int_{1/N}^1 du\, v^{-2}(u)$$

and therefore converges to the second integral in (2.14). A similar estimate applies to the third term in (2.13) [after noting that $[((j-1)/N)^{-\gamma}-(j/N)^{-\gamma}]/[(j/N)^{-\gamma}-((j+1)/N)^{-\gamma}]\to 1$ as $j\to\infty$ uniformly in N], which therefore converges to the first integral in (2.14).

Finally, take the infimum of (2.14) over $v\in\Omega$ and afterwards remove the restriction on the finiteness of the integrals. The resulting expression is the same as (1.16). □

3. Proof of the upper bound. Subsections 3.1 and 3.2 are preparations. The actual proof is given in subsection 3.3.

3.1. *The shrinking Wiener sausage on the torus.* Write $W^a(t)$ instead of $W(t)$ to display the radius $a>0$ of the Wiener sausage. By scaling we have for arbitrary $\delta>0$

$$(|W^a(ut)|)_{0\le u\le 1}=_D\left(t^{\delta d}|W^{at^{-\delta}}(ut^{1-2\delta})|\right)_{0\le u\le 1} \qquad (3.1)$$

where $=_D$ means equality in distribution. Pick δ such that $\delta d - \gamma = 1 - 2\delta$, that is, $\delta = (1+\gamma)/(d+2)$, and abbreviate $\tau = t^{(d-2\gamma)/(d+2)}$ and $\alpha = (1+\gamma)/(d-2\gamma)$, to get

$$(t^{-\gamma}|W^a(ut)|)_{0\le u\le 1} =_D (\tau|W^{a\tau^{-\alpha}}(u\tau)|)_{0\le u\le 1}. \tag{3.2}$$

Pick N large. The integral in (1.6) can be estimated by

$$\begin{aligned}\int_0^t \nu(s)\, d|W^a(s)| &= \nu(t)|W^a(t)| - \int_0^t |W^a(s)|\, d\nu(s) \\ &\ge \nu(t)|W^a(t)| + \sum_{i=1}^{N-1}\left(\nu\left(\frac{i}{N}t\right) - \nu\left(\frac{i+1}{N}t\right)\right)\left|W^a\left(\frac{i}{N}t\right)\right|.\end{aligned} \tag{3.3}$$

Substitute $\nu(t) \sim \nu t^{-\gamma}$ and use (3.2) to get

$$\begin{aligned}&\limsup_{t\to\infty} t^{-(d-2\gamma)/(d+2)} \log E\left(\exp\left[-\int_0^t \nu(s)\, d|W^a(s)|\right]\right) \\ &\quad\le \limsup_{\tau\to\infty} \tau^{-1} \log E\left(\exp\left[-\tau\nu\sum_{i=1}^{N} b_{i,N}\left|W^{a\tau^{-\alpha}}\left(\frac{i}{N}\tau\right)\right|\right]\right)\end{aligned} \tag{3.4}$$

with the notation

$$b_{i,N} = \left(\frac{i}{N}\right)^{-\gamma} - \left(\frac{i+1}{N}\right)^{-\gamma}, \qquad i = 1,\dots,N-1;\ b_{N,N} = 1. \tag{3.5}$$

The large deviation problem in the r.h.s of (3.4) has two aspects:

(i) the exponent is a functional of the *multivariate* random variable $\{|W^{a\tau^{-\alpha}}((i/N)\tau)|\}_{i=1}^N$;
(ii) the latter involves the *shrinking Wiener sausage* with radius $a\tau^{-\alpha}$.

In order to handle these aspects and to apply some standard large deviation techniques, we go to the torus

$$T_R = [0, R)^d \quad \text{with periodic boundary conditions.} \tag{3.6}$$

Let $\beta_R(t)$, $t \ge 0$, be Brownian motion wound up on T_R and let $W_R^a(t)$, $t \ge 0$, be its Wiener sausage in T_R. Since $|W_R^a(t)| \le |W^a(t)|$ for any t, R and a, we have

$$\begin{aligned}&\limsup_{\tau\to\infty} \tau^{-1} \log E\left(\exp\left[-\tau\nu\sum_{i=1}^{N} b_{i,N}\left|W^{a\tau^{-\alpha}}\left(\frac{i}{N}\tau\right)\right|\right]\right) \\ &\quad\le \limsup_{\tau\to\infty} \tau^{-1} \log E\left(\exp\left[-\tau\nu\sum_{i=1}^{N} b_{i,N}\left|W_R^{a\tau^{-\alpha}}\left(\frac{i}{N}\tau\right)\right|\right]\right).\end{aligned} \tag{3.7}$$

We shall compute the r.h.s. of (3.7) and then let $R \to \infty$, $N \to \infty$.

3.2. *Mollified empirical measures on T_R.* Define the empirical measures

$$(3.8) \qquad L^i_{R,N,t} = \left(\frac{t}{N}\right)^{-1} \int_{[((i-1)/N)t,(i/N)t)} \delta_{\beta_R(s)}\, ds, \qquad i = 1,\dots,N.$$

Let $\chi_a(x) = |B_a|^{-1} 1_{\{x \in B_a\}}$ and define the mollified empirical densities w.r.t. Lebesgue measure:

$$(3.9) \qquad \begin{aligned} l^i_{R,N,t}(x) &= \left(L^i_{R,N,t} * \chi_{a\tau^{-\alpha}}\right)(x) \\ &= \int \chi_{a\tau^{-\alpha}}(x-y) L^i_{R,N,t}(dy), \qquad i = 1,\dots,N. \end{aligned}$$

This is a random element of $\mathscr{D}_1(T_R)$ (the set of probability densities on T_R). Obviously

$$(3.10) \qquad \left| W_R^{a\tau^{-\alpha}}\left(\frac{i}{N}\tau\right)\right| = \left| \bigcup_{1 \le j \le i} \operatorname{supp}\left(l^j_{R,N,t}\right)\right|.$$

We need a large deviation principle (LDP) for $\{l^i_{R,N,t}\}_{i=1}^N$ on $(\mathscr{D}_1(T_R))^N$ in the product of the norm topologies. This is contained in the following lemma, which is a multivariate extension of Theorem 2 in Bolthausen (1990) [see also Sznitman (1990)].

LEMMA 3.11. *Assume $a > 0$ and $0 < \alpha < \infty$ if $d = 1, 2$, or $0 < \alpha < 1/(d-2)$ if $d \ge 3$. Then $\{l^i_{R,N,t}\}_{i=1}^N$ satisfies the LDP on $(\mathscr{D}_1(T_R))^N$ in the product of the norm topologies with rate function I_R^N given by*

$$(3.12) \qquad I_R^N\left((f_i)_{i=1}^N\right) = \frac{1}{N} \sum_{i=1}^N I_R(f_i)$$

with

$$(3.13) \qquad I_R(f) = \tfrac{1}{8} \int_{T_R} \frac{|\nabla f|^2}{f}(x)\, dx,$$

provided the weak gradient ∇f and the integral exist, and otherwise $I_R(f) = \infty$.

PROOF. We start with the result of Donsker and Varadhan that the empirical measure $L_{R,t} = t^{-1}\int_0^t \delta_{\beta_R(s)}\, ds$ satisfies the *uniform* LDP on $\mathscr{M}_1(T_R)$ (the set of probability measures on T_R) in the *weak topology* with rate function $I_R(\mu)$ given by

$$(3.14) \qquad \begin{aligned} I_R(\mu) &= I_R(f) \quad \text{if } \frac{d\mu}{d\lambda} = f, \\ &= \infty \qquad \text{otherwise,} \end{aligned}$$

where λ is Lebesgue measure. That is,

$$
\begin{aligned}
-\inf_{\mu\in\operatorname{int}A} I_R(\mu) &\le \liminf_{t\to\infty}\frac{1}{t}\log\inf_{x\in T_R}P_x(L_{R,t}\in A)\\
&\le \limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in T_R}P_x(L_{R,t}\in A)\\
&\le -\inf_{\mu\in\operatorname{cl}A} I_R(\mu),
\end{aligned}
\tag{3.15}
$$

where P_x is the path space measure conditioned on $\beta_R(0) = x$, and int A and cl A are the weak interior and closure of A [see Deuschel and Stroock (1989), Theorems 4.2.43 and 4.2.58, eqs. (4.2.48) and (4.2.49), page 113]. Because the estimates in (3.15) are uniform in the starting point $x \in T_R$, it immediately follows that for every finite N, $\{L^i_{R,N,t}\}_{i=1}^N$ satisfies the LDP on $(\mathscr{M}_1(T_R))^N$ in the product of the weak topologies with rate function $I_R((\mu_i)_{i=1}^N) = (1/N)\sum_{i=1}^N I_R(\mu_i)$. Namely, by the uniformity, the above $(\mathscr{M}_1(T_R))^N$-valued random element satisfies the same LDP as N independent copies of $L^1_{R,N,t}$. To get the claim in Lemma 3.11 we use the *mollification procedure* in Donsker and Varadhan (1975) and Bolthausen (1990), as follows.

First, since $\mu * \chi_{a\tau^{-\alpha}}$ converges to μ uniformly on compact subsets of $\mathscr{M}_1(T_R)$ as $\tau \to \infty$, it follows from the contraction principle as formulated in Varadhan [(1984) Theorem 2.4] that

$$
\{L^i_{R,N,t} * \chi_{a\tau^{-\alpha}}\}_{i=1}^N = \{l^i_{R,N,t}\}_{i=1}^N
\tag{3.16}
$$

satisfies the LDP in the weak topology. [Here we interpret both sides as elements in $(\mathscr{M}_1(T_R))^N$ given by the density in (3.9).]

Next, let $\chi \in \mathscr{D}_1(T_R)$ be any mollifier. Then, since $\mu \to \mu * \chi$ is continuous from $\mathscr{M}_1(T_R)$ with the weak topology to $\mathscr{D}_1(T_R)$ with the norm topology, it follows, again from Varadhan [(1984), Theorem 2.4], that

$$
\{l^i_{R,N,t} * \chi\}_{i=1}^N
\tag{3.17}
$$

satisfies the LDP in the norm topology.

Next, use Lemma 14.6 in Varadhan (1984), which states that for the empirical measure $L_{R,t}$

$$
\lim_{\chi\to\delta_0}\limsup_{t\to\infty}\frac{1}{t}\log P\big(\|L_{R,t} * \chi_{a\tau^{-\alpha}} * \chi - L_{R,t} * \chi_{a\tau^{-\alpha}}\|_1 > \varepsilon\big) = -\infty
\tag{3.18}
$$

for every $\varepsilon > 0$ and $\alpha = 1/d$. By applying this to each of the components $i = 1, \dots, N$ in (3.17) we obtain that

$$
\{l^i_{R,N,t}\}_{i=1}^N
\tag{3.19}
$$

satisfies the LDP in the norm topology, again for $\alpha = 1/d$.

Finally, use the proposition in Bolthausen (1990), which states that for $L_{R,t}$,

$$\lim_{t\to\infty} \frac{1}{t} \log P(\|L_{R,t} * \chi_{a\tau^{-\alpha}} - L_{R,t} * \chi_{a\tau^{-\alpha'}}\|_1 > \varepsilon) = -\infty \tag{3.20}$$

for every $\varepsilon > 0$ and $0 < \alpha < \alpha' < \infty$ if $d = 1, 2$, or $0 < \alpha < \alpha' < 1/(d-2)$ if $d \geq 3$. By applying this to each of the components we obtain the claim. □

3.3. *Upper bound.* Apply Lemma 3.11 to the r.h.s. of (3.7), recalling the representation in (3.10). Then, since $f \to |\mathrm{supp}(f)|$ is lower semicontinuous in the norm topology, we get [see Varadhan (1984), Theorem 2.3]

$$\begin{aligned}
&\limsup_{\tau\to\infty} \tau^{-1} \log E\left(\exp\left[-\tau\nu \sum_{i=1}^{N} b_{i,N}\left|W_R^{a\tau^{-\alpha}}\left(\frac{i}{N}\tau\right)\right|\right]\right)\\
&\quad \leq -\inf_{f_1,\dots,f_N\in\mathscr{D}_1(T_R)}\left[\nu\sum_{i=1}^{N} b_{i,N}\left|\bigcup_{j=1}^{i}\mathrm{supp}(f_j)\right| + \frac{1}{N}\sum_{i=1}^{N} I_R(f_i)\right]\\
&\quad \equiv -J_{R,N}.
\end{aligned}\tag{3.21}$$

Here we use the fact that, for the range of γ-values in (1.5), the shrinking rate $\alpha = (1+\gamma)/(d-2\gamma)$ in (3.7) [recall (3.2)] exactly matches the range of α-values in Lemma 3.11.

As R is arbitrary, we get via (3.4) and (3.7) that

$$\limsup_{t\to\infty} t^{-(d-2\gamma)/(d+2)} \log E\left(\exp\left[-\int_0^t \nu(s)\, d|W^a(s)|\right]\right) \leq -\sup_{R>0} J_{R,N}. \tag{3.22}$$

By the same approximation argument as in Deuschel and Stroock [(1989) Section 4.3, leading up to (4.3.21)] we have

$$\sup_{R>0} J_{R,N} \geq \inf_{A_1,\dots,A_N\in O_b}\left[\nu\sum_{i=1}^{N} b_{i,N}\left|\bigcup_{j=1}^{i} A_j\right| + \frac{1}{N}\sum_{i=1}^{N}\lambda(A_i)\right], \tag{3.23}$$

where O_b is the set of nonempty bounded open subsets of $\mathbb{R}^d$ and $\lambda(A)$ is the principal Dirichlet eigenvalue of $-\frac{1}{2}\Delta$ on A. The latter comes in because

$$\inf_{f\in\mathscr{D}_1(\mathbb{R}^d):\ \mathrm{supp}(f)\subset A} I(f) = \lambda(A)$$

when $I(f)$ is defined as in (3.13) with T_R is replaced by $\mathbb{R}^d$ [as is seen after substituting $f = g^2$ into the r.h.s. of (3.13)].

Putting $B_i = \bigcup_{j=1}^{i} A_j$ and noting that $\lambda(A)$ is nonincreasing in A, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} t^{-(d-2\gamma)/(d+2)} \log E\left(\exp\left[-\int_0^t \nu(s)\, d|W^a(s)|\right]\right)\\
&\quad \leq -\inf_{B_1\subset\cdots\subset B_N\in O_b}\left[\nu\sum_{i=1}^{N} b_{i,N}|B_i| + \frac{1}{N}\sum_{i=1}^{N}\lambda(B_i)\right].
\end{aligned}\tag{3.24}$$

We are now in a position to reduce the variational problem still further by using the fact that $\lambda(B_i)$ given $|B_i|$ is minimal when B_i is a ball. This implies

$$
\begin{aligned}
&-\inf_{B_1\subset\cdots\subset B_N\in O_b}\left[\nu\sum_{i=1}^{N} b_{i,N}|B_i| + \frac{1}{N}\sum_{i=1}^{N}\lambda(B_i)\right]\\
&\qquad\le -\inf_{0<r_1\le\cdots\le r_N<\infty}\left[\nu\omega_d\sum_{i=1}^{N} b_{i,N}r_i^d + \mu_d\sum_{i=1}^{N}\frac{1}{N}\frac{1}{r_i^2}\right]
\end{aligned}\tag{3.25}
$$

[recall (2.7)].

Next, for every $0<r_1\le\cdots\le r_N<\infty$ the function v given by

$$
\begin{aligned}
v(u) &= uNr_1 \quad \text{for } u\in\left[0,\frac{1}{N}\right),\\
v(u) &= r_i \qquad \text{for } u\in\left[\frac{i}{N},\frac{i+1}{N}\right) \text{ and } i=1,\dots,N-1,\\
v(1) &= r_N
\end{aligned}
$$

is an element of Ω [recall (1.10)]. In terms of this v we can write

$$
\begin{aligned}
&\nu\omega_d\sum_{i=1}^{N} b_{i,N}r_i^d + \mu_d\sum_{i=1}^{N}\frac{1}{N}\frac{1}{r_i^2}\\
&\qquad= \nu\omega_d\left[v^d(1) + \int_{1/N}^{1} d(-u^{-\gamma})v^d(u)\right]\\
&\qquad\quad+\mu_d\left[\frac{1}{N}v^{-2}(1) + \int_{1/N}^{1} du\, v^{-2}(u)\right],
\end{aligned}\tag{3.26}
$$

using (2.9). Hence

$$
\begin{aligned}
&-\inf_{0<r_1\le\cdots\le r_N<\infty}\left[\nu\omega_d\sum_{i=1}^{N} b_{i,N}r_i^d + \mu_d\sum_{i=1}^{N}\frac{1}{N}\frac{1}{r_i^2}\right]\\
&\qquad\le -\inf_{v\in\Omega}\left\{\nu\omega_d\left[v^d(1) + \int_{1/N}^{1} d(-u^{-\gamma})v^d(u)\right]\right.\\
&\qquad\quad\left.+\mu_d\left[\frac{1}{N}v^{-2}(1) + \int_{1/N}^{1} du\, v^{-2}(u)\right]\right\}.
\end{aligned}\tag{3.27}
$$

Finally, drop the term $(1/N)v^{-2}(1)$ and let $N\to\infty$. One easily justifies pulling the limit under the infimum after assuming convergence of the integrals. The resulting expression is the same as (1.16) □

4. Solution of the variational problem. Let $\delta = \nu\mu_d^{-1}\omega_d^{2/d}$. Then (1.9) becomes

$$
k(d,\nu,\gamma) = \nu\inf_{\omega\in\Omega}\left[\omega(1) + \gamma\int_0^1\frac{\omega(u)\,du}{u^{1+\gamma}} + \delta\int_0^1\frac{du}{\omega^{2/d}(u)}\right].\tag{4.1}
$$

We proceed in two steps:

(i) minimize over ω under the restriction $\omega(1) = x$;
(ii) minimize over x.

PROOF. (i) For u fixed define

$$\psi_u(y) = \frac{\gamma}{u^{1+\gamma}} y + \delta \frac{1}{y^{2/d}}. \tag{4.2}$$

The function $y \to \psi_u(y)$ is strictly convex with a unique minimum at

$$y_0 = y_0(u) = \left(\frac{2\delta}{d\gamma} u^{1+\gamma} \right)^{d/(d+2)} \tag{4.3}$$

Hence the sum of the two integrands in (4.1) is minimal when $\omega(u) = y_0(u)$. However, the restriction that $\omega(u) \le \omega(1) = x$ for $u < 1$ [recall (1.10)] forces the minimizer to be

$$\begin{aligned} \omega(u) &= \left(\frac{2\delta}{d\gamma} u^{1+\gamma} \right)^{d/(d+2)}, && \text{for } 0 \le u < u_c(x), \\ &= x, && \text{for } u_c(x) \le u \le 1, \end{aligned} \tag{4.4}$$

with $u_c(x)$ the smallest u for which $\omega(u) = x$, that is,

$$u_c(x) = \left(\frac{d\gamma}{2\delta} x^{(d+2)/d} \right)^{1/(1+\gamma)} \wedge 1. \tag{4.5}$$

[Note that if $x < x_c = (2\delta/d\gamma)^{d/(d+2)}$, then $u_c(x) < 1$ and $\omega(u)$ sticks at the value x for $u > u_c(x)$, which still minimizes $\psi_u(y)$ uniquely because $y \to \psi_u(y)$ is strictly convex.] After substituting (4.4) and (4.5) into (4.1) and performing the integrations, we arrive at the following expression:

$$k(d,\nu,\gamma) = \nu \inf_{0 \le x \le x_c} \left[\frac{d(1+\gamma)^2}{d-2\gamma} \left(\frac{2\delta}{d\gamma} \right)^{\gamma/(1+\gamma)} x^{(d-2\gamma)/d(1+\gamma)} + \frac{\delta}{x^{2/d}} \right]. \tag{4.6}$$

Here we may already put in the restriction $x \le x_c$, because the minimizer in (4.4) is the same for all $x \ge x_c$ (i.e., $u_c(x) = 1$) except at the point $u = 1$. So $x = x_c$ gives a smaller infimum than any $x > x_c$ because of the term $\omega(1)$ in the r.h.s. of (4.1).

(ii) The sum between the square brackets in (4.6) has a unique minimum at

$$x_0 = \left(\frac{2\delta}{d} \frac{\gamma^\gamma}{(1+\gamma)^{1+\gamma}} \right)^{d/(d+2)} < x_c. \tag{4.7}$$

Substitution into (4.6) gives

$$k(d,\nu,\gamma) = \nu \frac{d+2}{d-2\gamma} \left(\frac{d}{2} \frac{(1+\gamma)^{1+\gamma}}{\gamma^\gamma} \right)^{2/(d+2)} \delta^{d/(d+2)}. \tag{4.8}$$

Recalling that $\delta = \mu_d \omega_d^{2/d}/\nu$, we have proved (1.7). Substitution of (4.7) into (4.5) yields

$$u_c(x_0) = \frac{\gamma}{1+\gamma} \tag{4.9}$$

and so via (4.4) we have also proved (1.11). □

5. Proof of Theorem 1.18. Because $|W(ut)|$ is nondecreasing and $\omega^*(u)$ is continuous in u, it suffices to prove that for all N and $\varepsilon > 0$,

$$\lim_{t\to\infty} Q_t\left(\max_{1\le i\le N-1} \left| t^{-d(1+\gamma)/(d+2)} \left|W\left(\frac{i}{N}t\right)\right| - \omega^*\left(\frac{i}{N}\right)\right| > \varepsilon \right) = 0. \tag{5.1}$$

For this, in turn, it suffices to prove that for all $\zeta = i_0/N$, $1 \le i_0 \le N-1$, and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{t\to\infty} t^{-(d-2\gamma)/(d+2)} \log Q_t\left(t^{-d(1+\gamma)/(d+2)}|W(\zeta t)| \ge \omega^*(\zeta) + \varepsilon\right) \le -\delta, \tag{5.2}$$

$$\limsup_{t\to\infty} t^{-(d-2\gamma)/(d+2)} \log Q_t\left(t^{-d(1+\gamma)/(d+2)}|W(\zeta t)| \le \omega^*(\zeta) - \varepsilon\right) \le -\delta. \tag{5.3}$$

The arguments to prove these two inequalities are slightly different.

We start with (5.2). By (1.17) we have

$$\begin{aligned} &Q_t\left(t^{-d(1+\gamma)/(d+2)}|W(\zeta t)| \ge \omega^*(\zeta) + \varepsilon\right) \\ &\quad = \frac{1}{Z_t} E\left(\exp\left[-\int_0^t \nu(s)\, d|W(s)|\right]; |W(\zeta t)| \ge t^{d(1+\gamma)/(d+2)}(\omega^*(\zeta) + \varepsilon)\right) \\ &\quad \le \frac{1}{Z_t} E\left(\exp\left[-\int_0^t \nu(s)\, dV_\zeta(s)\right]\right), \end{aligned} \tag{5.4}$$

where Z_t is the expectation in (1.6) and

$$V_\zeta(s) = \begin{cases} |W(s)| \vee t^{d(1+\gamma)/(d+2)}(\omega^*(\zeta) + \varepsilon), & \text{for } s \ge \zeta t, \\ |W(s)|, & \text{for } s < \zeta t. \end{cases} \tag{5.5}$$

The same argument as in Section 3 gives [recall (1.16)]

$$\begin{aligned} &\limsup_{t\to\infty} t^{-(d-2\gamma)/(d+2)} E\left(\exp\left[-\int_0^t \nu(s)\, dV_\zeta(s)\right]\right) \\ &\quad \le -\inf_{v\in\Omega}\left\{\nu\left[\omega_d v^d(1) \vee (\omega^*(\zeta) + \varepsilon) + \gamma\int_0^\zeta \frac{\omega_d v^d(u)\, du}{u^{1+\gamma}}\right.\right. \\ &\qquad \left.\left. + \gamma\int_\zeta^1 \frac{\omega_d v^d(u) \vee (\omega^*(\zeta) + \varepsilon)}{u^{1+\gamma}}\, du\right] + \mu_d\int_0^1 \frac{du}{v^2(u)}\right\} \\ &\quad < -k(d,\nu,\gamma). \end{aligned} \tag{5.6}$$

The last inequality is strict because ω^* is the unique minimizer of (1.9) [and of (1.16) with $\omega(u) = \omega_d v^d(u)$]. Theorem 1.4 and (5.6) prove (5.2).

We cannot use the same argument for (5.3). Indeed, if we would replace $\omega_d v^d(u)$ in (1.16) by $\omega_d v^d(u) \wedge (\omega^*(\zeta) - \varepsilon)$ on the interval $[0, \zeta]$, then this would increase $-\inf_{v \in \Omega}$, and so the inequality in (5.6) would go in the other direction. To get the desired bound we proceed as follows from the equality in (5.4). Observe that the indicator function of the event

$$\{|W(\zeta t)| \le t^{d(1+\gamma)/(d+2)}(\omega^*(\zeta) - \varepsilon)\}$$

does not decrease when we replace $W(\zeta t)$ by $W_R(\zeta t)$, the torus Wiener sausage. We can therefore argue in the same way as in subsection 3.1, obtaining as in (3.4) and (3.7):

$$\begin{aligned}(5.7)\qquad &\limsup_{t\to\infty} t^{-(d-2\gamma)/(d+2)} \log E\Bigg(\exp\Bigg[-\int_0^t \nu(s)\, d|W(s)|\Bigg];\\ &\qquad\qquad |W(\zeta t)| \le t^{d(1+\gamma)/(d+2)}(\omega^*(\zeta) - \varepsilon)\Bigg)\\ &\le \limsup_{\tau\to\infty} \tau^{-1} \log E\Bigg(\exp\Bigg[-\tau\nu \sum_{i=1}^N b_{i,N}\Bigg|W_R^{a\tau^{-\alpha}}\Bigg(\frac{i}{N}\tau\Bigg)\Bigg|\Bigg];\\ &\qquad\qquad \Big|W_R^{a\tau^{-\alpha}}(\zeta\tau)\Big| \le \omega^*(\zeta) - \varepsilon\Bigg).\end{aligned}$$

Next, since $f \to 1_{\{|\mathrm{supp}(f)| \le a\}}$ is upper semicontinuous on $\mathscr{D}_1(T_R)$ in the norm topology, we can argue as in subsection 3.3, obtaining as in (3.24) and (3.25) (after letting $R \to \infty$)

$$\begin{aligned}&\text{l.h.s. } (5.7) \le -\inf_{0<r_1\le \cdots \le r_N<\infty}\Bigg[\nu\omega_d \sum_{i=1}^N b_{i,N} r_i^d\\ (5.8)\qquad &\qquad + \mu_d \sum_{i=1}^N \frac{1}{N}\frac{1}{r_i^2};\ \omega_d r_{i_0}^d \le \omega^*(\zeta) - \varepsilon\Bigg].\end{aligned}$$

Letting $N \to \infty$ we get

$$\begin{aligned}&\text{l.h.s. } (5.7) \le -\inf_{v\in\Omega}\Bigg\{\nu\omega_d\Bigg[v^d(1) + \gamma\int_0^1 \frac{v^d(u)\, du}{u^{1+\gamma}}\Bigg]\\ (5.9)\qquad &\qquad + \mu_d \int_0^1 \frac{du}{v^2(u)};\ \omega_d v^d(\zeta) \le \omega^*(\zeta) - \varepsilon\Bigg\}\\ &\qquad < -k(d, \nu, \gamma).\end{aligned}$$

This proves (5.3). □

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INSTITUT FÜR ANGEWANDTE MATHEMATIK
UNIVERSITÄT ZÜRICH
WINTERTHURERSTRASSE 190, CH-8057 ZÜRICH
SWITZERLAND

MATHEMATISCH INSTITUUT
UNIVERSITEIT UTRECHT
BUDAPESTLAAN 6, NL-3584CD UTRECHT
THE NETHERLANDS