A GREEK MATHEMATICAL PAPYRUS

BY

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The Papyrus Vindobonensis Gr. Inv. No 26740 is a dark brown papyrus which comes from Soknopaiu Nesus and which entered the Vienna Papyrus Collection in 1894. It has now been split into two pieces which are mounted one below the other under a single glass plate, although the two pieces originally formed a single stretch of scroll. The upper piece measures 13.8 × 66.5 cms. At 17 cms from the left and 13 cms from the right κολλήματα are found. The lower part consists of two fragments: frag. 1 measures 13.8 × 24.2 cms (17 cms from the left is a κόλλημα); frag. 2 measures 13.8 × 34 cms (8.5 cms from the left is a κόλλημα). The scribe wrote on both sides of the papyrus. On the one side a demotic text is found, written in various columns against the grain of the fibres; on the other side (parallel with the grain of the fibres) are 5 geometrical problems, a passage of Homer (Iliad Z, 373-410) and two conversion problems. The top and bottom edges of the papyrus are for the most part evenly cut off (at regular intervals along the top edge there are pieces broken out of the papyrus itself. This damage probably occurred when the papyrus was in a rolled-up state.). The left front side of the papyrus is evenly cut off. Partly because of the fact that on the other side the cut runs straight through a column, we may assume that a papyrus that had already been covered with a demotic text was cut through in order to use the other side for the Greek text. It might be thought surprising that a text that was written later was written on the recto side (cf. E. G. Turner, *Recto and Verso*, JEA 40, 1954, pp. 102 ff.), but, before the Greek text was written on the recto side, the original text seems to have been washed out. It is moreover typical for a "school" papyrus to reuse an old papyrus (cf. J. A. Davidson, *The Study of Homer in Greco-Roman Egypt*, Akten des VIII. internationalen Kongresses für Papyrologie (=MPER, NS, V. Folge), Vienna, 1956, pp. 51 ff.). The right-hand back side of the papyrus is badly damaged. The possibility must not be ruled out that the scroll used to carry still more writing. Each mathematical problem has its own column. The Homeric passage begins at a distance of 2 to 2.5 cms from the fifth mathematical problem. The text has been divided over three columns: cols. 1 and 2 are about 10 cms long, col. 3 4 cms. To the left of this last column stand the conversion problems, one below the other.

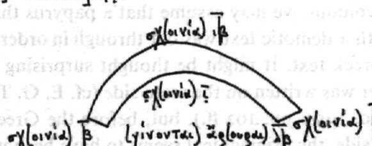
A part of this papyrus (frag. 2 of the lower part) has been known since as early as 1938 (cf. H. Oellacher, *Griechische Literarische Papyri aus der Papyrussammlung Erzherzog Rainer in Wien*, Études de Papyrologie IV, 1938, pp. 133 ff. = R. A. Pack, *The Greek and Latin Literary Texts from the Greco-Roman Egypt*[2], Ann Arbor, 1965, no. 791). The present restorer of the Papyrus collection, Michael Fackelmann, has found the other parts of this papyrus.

The subject dealt with[1]) and the quality of the Greek make it quite possible that we have here a papyrus stemming from a school[2]). We shall publish the Homeric passage elsewhere[3]). Below we give a transcription of and a commentary on the mathematical problems.

We need not wonder, on the contrary it is typical for Egypt, that the measurements are given in σχοίνια and that the schoolboy is asked to determine how many ἄρουραι he obtains.

Problem 1.

Ἔστω μηνίσκος, οὗ ἡ μὲν ἐκτὸς περίμετρος σχοιν[ίων ῑ], ἡ δὲ ἐντὸς σχοινί{νι}ων ιβ, βάσις σχοινίων β̄. Πόσων ἀρουρῶν ἐστιν; Ὡ⟨ς⟩ δεῖ ποι{.}ῆσαι, συνθ[ε]ὶς τὰς δύο περιμέτρους, γείνεται κ{.}β̄. Τούτων λαβὲ τὸ (ἥμισυ). γείν[εται ια. Τ]αῦτα τὰ σ(χοινία) ποίησον ἐπὶ τὰ δύο τῆς βάσεος. γεί[νεται κβ.] Τοσούτων ἀρουρῶν ἐστιν ὁ μηνίσκ[ος] [ὡς ὑπόκει]ται.



2. γίνεται, also l. 5 and l. 6

The schoenion was the side of an aroura and consisted of 100 πήχεις (cf. P. Tebt. I, 13, note to line 13).
2. A clear example of dittography (cf. problem 3, 1-2).
4. Elsewhere the writer uses the plural γείνονται already thinking of the ἄρουραι, but here he uses the singular γείνεται still thinking of the σχοίνια, which is more correct. ἥμισυ has been written by a symbol. In the drawing the schoolboy interchanged the measures of the ἐκτὸς and ἐντὸς περίμετρος

Let be given a crescent of which the outer perimeter is 10 schoenia and the inner one 12 schoenia, the basis 2 schoenia. How many arouras is it? How one has to operate: add the two perimeters, result 22. Take one half of these, result 11. Multiply these schoenia into the 2 of the basis, result 22. So many arouras is the crescent, which was propounded.

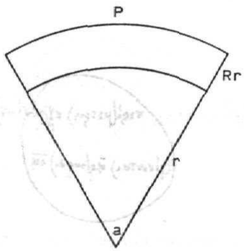


Fig. 1.

Comments.

If the central angle of the circle sectors is a and the radii are R and r, respectively, the area of the crescent is to be computed as the difference of two sectors (fig. 1)

$$^1/_2 aR^2 - {}^1/_2 ar^2 = {}^1/_2\,(aR+ar)\,(R\text{-}r) = {}^1/_2\,(P+p)b,$$

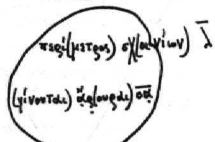
where R-r=b is the basis and aR=P is the outer, ar=p the inner perimeter. The formula which has been applied is therefore exact

$$C = {}^1/_2\,(P+p)b.$$

From the data follows that a=1 as P=12, p=10, b=2, which means that a is one radian or 57°17′.

Problem 2.

1 Ἔστ[ω] κύκλος, οὗ ἡ μὲν πε[ρί]μετρος σχοι-
2 νίων [λ̄]. Πόσων ἀρουρῶν ἐστ[ιν;] Ὡς δεῖ ποιῆ-
3 σαι, ποίησον τὰ σ(χοινία) λ ἐφ' ἑαυτά, γείνεται ϡ̄.
4 Τούτων λαβὲ τὸ (δωδέκατον), γείνονται [οε̄]. Τοσούτων
5 ἀρουρῶν ἐστιν ὁ κύκλος, ὡς ὑπόκειται.



3 l. γίνεται   4 l. γίνονται

4. δωδέκατον: on the papyrus ιβ'.
In the drawing the schoolboy puts wrongly οα in stead of the correct οε.

Let be given a circle of which the perimeter is 30 schoenia. How many arouras does it contain? How one has to operate: multiply 30 schoenia into themselves, result 900. Take one twelfth of those, result 75. So many arouras is the circle, which was propounded.

Comments.

The area of the circle is

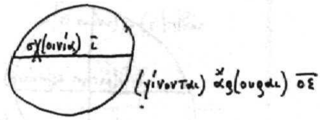
$$C = \pi r^2 = \pi/4 \cdot d^2 = p^2/4\pi,$$

where r is the radius, d the diameter and p the perimeter. Using the approximation $\pi = 3$ the procedure arises. It is identical with the computation of the Old Babylonian Period, which computes, in the sexagesimal system,

$$C = 0;5\ p^2.$$

Problem 3.

1 [*Ε]στω κύκλος, οὗ ἡ διάμετρ[ος σχο]ινίων ῑ. Πόσων ἀρουρων
2 [{Πό]σων ἀρουρῶν} ἐστιν; Ὡς δ[εῖ ποι]ῆσαι, ποίησον τὰ ῑ ἐφ' ἑαυ[τά,]
3 γείνε{ν}ται ρ̅. Τούτων λαβὲ τὸ [(ἥμισυ) (τέταρτον),] γείνονται ο̅ε̅. Τοσού-
τ[ω]ν ἀρου[ρῶν]
4 ἐστιν ὁ κύκ[λ]{κ}ος, ὡς ὑπόκει[ται.]



3 r. γίνεται; l. γίνονται

3. It is possible that the papyrus has γείνονται (the wrong number!) or that the ε has been written over the original ο.
In view of the available space the lacune ἥμισυ must have been written as a symbol and τέταρτον must have been rendered by δ'.

Let be given a circle of which the diameter is 10 schoenia. How many arouras is it? How one has to operate: multiply the 10 schoenia into themselves, result 100. Take of those one half and a quarter, result 75. So many arouras is the circle, as was propounded.

Comments.

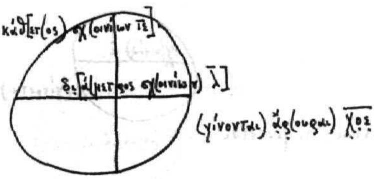
Inserting $\pi = 3$ in

$$C = \pi/4 \cdot d^2 = (1/2 + 1/4) \cdot d^2$$

shows that the computation is exact, but for the approximation of π.

Problem 4.

Ἔστω κύκλος ημ..[.....].ς, οὗ ἡ μὲν κάθετος
σχοινίων ιε̅, ἡ δὲ δ[ιάμετρ]ος σχοι[νίων λ. Πόσων]
ἀρουρ[ῶν] ἐστ[ί]ν; Ὡς δ[εῖ ποιῆσ]αι, συνθὲς τὰ σ(χοινία) [τῆ]ς
καθέτ[ου traces of ink]πόησον [ἐφ' ἑ-
αυτά, γείνεται Βκ[ε. Τ]ούτω]ν λ[α]βὲ τὸ (τρίτον), πόησον [τα χ̅οε̅.
Τοσούτων ἀρουρῶν [ἐστ]ὶν ὁ κύκλος, ὡς ὑπόκειται.



4 l. ποίησον 5 l. γίνεται; l. ποιήσοντα

1. It is possible, although the present state of the papyrus does not allow us to be certain, that the schoolboy wrongly wrote ἡμικύκλιος or the like.
4. We are not able to decipher the ink-traces in the middle of this line.

Let be given a circle (corresponding to a hemicircle) of which the altitude is 15 schoenia and the diameter 30 schoenia. How many arouras is it? How one has to operate: add the (number of) schoenia of the altitude to the (number of) (schoenia of the diameter, result 45). Square! Result 2025. Take one third of these, making 675. So many arouras is the circle as was propounded.

Comments.

The formula used is

$$C = (r+d)^2/3.$$

This leads to the exact result with $\pi=3$ as $d= 2r$ and

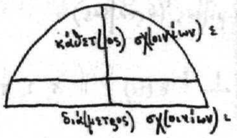
$$C = (3r)^2/3 = 3r^2 = \pi r^2.$$

In the Old Babylonian Period the area of the hemicircle was computed in three different ways from diameter and perimeter:

either $d \cdot p / 4$ —see BM 85210, TMB 98;
or $3dr/4$ —see YBC 5022, list of constants;
or $p^2/6$ —see YBC 5022, list of constants.

Problem 5.

*Ἔστω ἡ[μικύκ]λιον, οὗ ἡ μὲν κάθετος σχοινίων ε̄, ἡ δὲ διάμετρος σχοινίων ῑ. Δυσὶν οὖν πόσων ἀρουρῶν ἐστ[ιν;] Ὡς δεῖ ποιῆσαι, συνθὲς τὰ σ(χοινία) τῆς καθέτου καὶ τὰ σ(χοινία) τῆς διαμέτρου, γείνεται ῑε̄. Τούτων λαβὲ τὸ (ἥμισυ), γείνεται [ζ∠.] Ταῦτα τὰ σ(χοινία) ποίησον ἐπὶ τὰ ε τῆς καθέτου, ποιήσοντα λζ∠. Τοσούτων ἀρουρῶν ἐστιν τὸ ἡμικύκλιον, ὡς ὑπόκειται.



4 r. γίνεται, also l. 5

Let be given a hemicircle of which the altitude is 5 schoenia and the diameter 10 schoenia. From these two (data), how many arouras is it? How one has to operate: add (the number of) schoenia of the altitude and the (number of) schoenia of the diameter, result 15. Take one half of this, result $7^1/_2$. *Multiply these schoenia into the 5 of the altitude.* Result $37^1/_2$. So many arouras is the hemicircle, as propounded.

Comments.

The formula used is

$$S = {}^1/_2\,(k+d)\,k,$$

and with $d = 2k = 2r$ it gives the exact result for $\pi = 3$ as

$$S = 3r^2/2 = {}^1/_2\pi r^2.$$

The seemingly superfluous addition: from these two data — indeed for a hemicircle the data are interdependent! — indicates that a more general formula is applied, that for a segment of a circle of which basis and altitude are given. Heron, Metrica I, 30 indicates that "the Ancients" computed the segment of a circle C very roughly by

$$C = {}^1/_2\,(b+k)k$$

and he adds that for those segments for which the basis b is not greater than three times the altitude k the approximation is satisfactory[4]).

Διάνοια [μ]έτρων
Δρόμωι ἀρτ[άβ]αι ρ̄. Πόσαι[
Ὡς δεῖ ποιῆσα[ι], ⟦ποίησον⟧ λαβὲ τῶν[
τὸ ρ´, γείνεται α̅ ἐπὶ κη̅, γείνε[ται
προκιμένας (ἀρτάβας) [.....] ἔσται[

Δρόμωι [ἀ]ρτάβ[αι   Ὡς δεῖ ποιῆ-
σαι, λαβὲ τῶν[
γείνεται ‾ ἐπὶ κγβ, [γ]είνε[ται
ἐστι τὰς προκιμέ[νας (ἀρτάβας)
traces

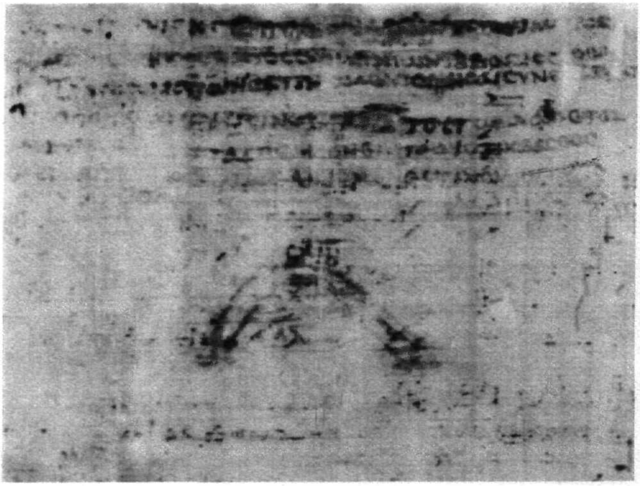
4 l. γίνεται (2x), also l. 8 (1x)   5 l. προκειμένας

Conversion of Measures

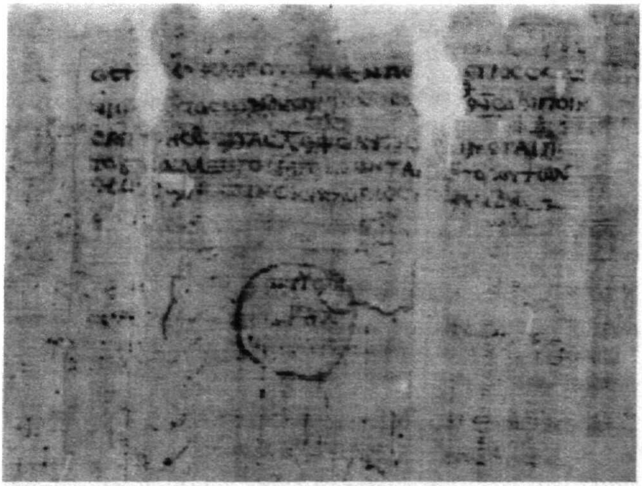
100 dromos-artabas. How many .....
How one has to operate: Take of the (100)
the one hundredth, result 1; into 28, result (28)
The .....
x dromos-artabas .... How one has to operate:
Take of the....
Result ....; into 23 and two thirds, result ...
It is....

Comments.

For practical use approximating conversion tables were computed. An extensive one has been preserved in the British Museum, Papyrus CCLXV.[5]) For modern people the skill with which the ancient computer chooses his for practical use close and sufficiently accurate approximations a not justified verdict of "not working exactly with fractions" sometimes does arise. As in the preserved part of the papyrus under consideration here one of the counterparts is missing we can only state that the constants 28 and $23 + 2/3$ correspond to some sections of the BM analogue quoted, It may be of interest to consider the arithmetical part of this papyrus here too.

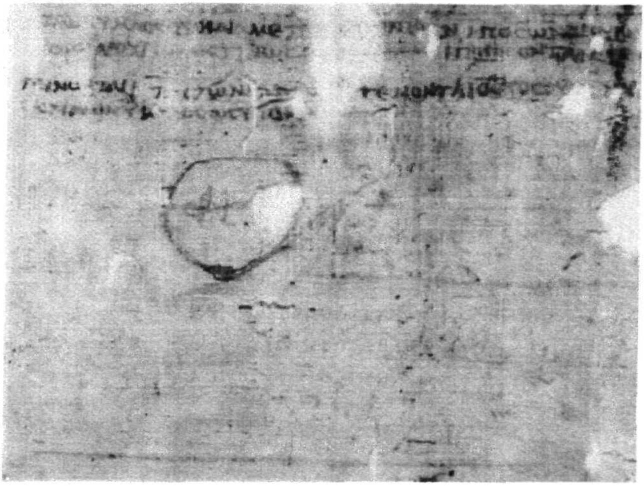


Problem 1

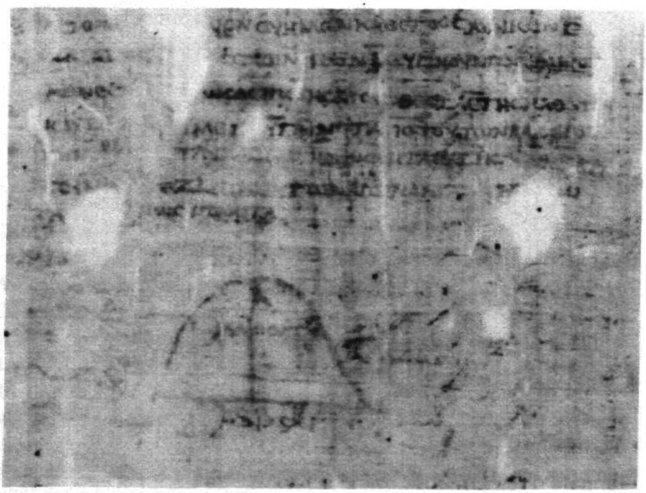


Problem 2

Papyrus Vindobonensis Gr. Inv. 26740

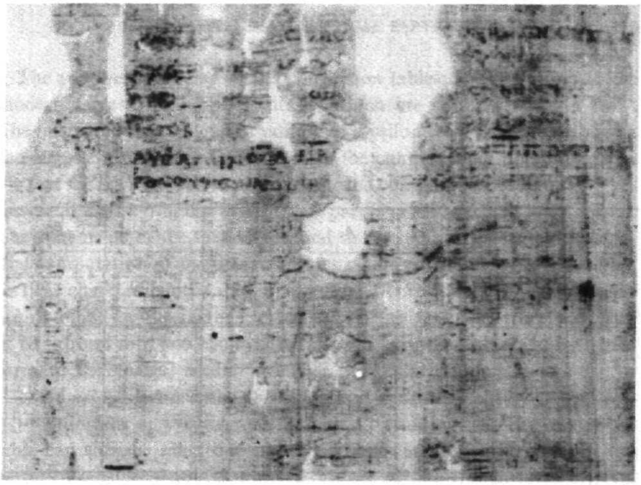


Problem 3



Problem 4

Papyrus Vindobonensis Gr. Inv. 26740



Problem 5

Papyrus Vindobonensis Gr. Inv. 26740

	Δ	X	A	Φ	Γ	E
Δ	1	$1 + \frac{28}{100}$	✱	$1 + \frac{16\frac{2}{3}}{100} = 1 + \frac{1}{6}$	$1 + \frac{23\frac{2}{3}}{100} = \frac{371}{300}$	$1 + \frac{25}{100} = 1 + \frac{1}{4}$
X	$\frac{1}{2}\ \frac{1}{4}\ \frac{1}{32}$	1	$1 + \frac{1}{20}$	$1 - \frac{1}{11}$	$\frac{200}{207}$	$\frac{100}{104}$
A	✱	$1 - \frac{1}{21}$	1	✱	✱	✱
Φ	✱	$1 + \frac{1}{10}$	∘	1	∘	✱
Γ	∘	$\frac{103\frac{1}{2}}{100}$	∘	✱	1	✱
E	∘	$1 + \frac{1}{25}$	✱	✱	∘	1.

The values indicated by ✱ are computed by application of the factors of the second row to the values of the second column. The values indicated by ∘ are destroyed in the text of B. M. Papyrus CCLXV.

The very verbose texts of the conversion tables do correspond to the modern tables with double entries, which are "simply read in full". The misreading in copying or editing of e-psilon, beta, zeta, theta, sigma can suggest errors in arithmetic which the computer did not commit. In the text CCLXV quoted this is evident: the computation of line 126 gives the result $646 + 7/8$, but this integral part is given as 642 in line 129-147. The statements of the text correspond to the table on page 306.

The computation of the first row, lines 1-38, provide the conversion factors for chalki, analotiki, philippi, galli and hermi for dromi. The chalki are obtained by the factor $1 + 28/100$, resulting in the transformation of 625 dromi into 800 chalki. The analotiki are then obtained by transforming the chalki, adding one twentieth; the philippi by the factor $1 + (16 + {}^2/_3)$-hundredths or adding one sixth, the galli by $1 + (23 + {}^2/_3)$-hundredths or $371/300$ and the hermi by adding $1/4$. The rest of the table can then be filled out *and it was*, but for the factor used for the hermi! A lapsus calami occurred in line 29, writing $1/42$ in stead of $1/12$.

Transforming the chalki, in the second row (lines 39-43) the reciprocal value of $128/100$ is indicated by, $100 = (64 + 32 + 4)$, $100/128$ as ${}^1/_2 + {}^1/_4 + {}^1/_{32}$, which correct factor is stated in line 41. The transformation of the 625 chalki should then be obtained by adding $312^1/_2$, $156^1/_4$, $19^1/_2 + {}^1/_{32}$, which neglecting the last fraction gives the result of the text $488^1/_4$ (line 43).

The analotiki are then obtained by adding one twentieth, the philippi by subtracting one eleventh. The last fraction is according to the first row $100/128$ times $7/6$ or $700/768$. This is very near to $700/770$ or $1 - 1/11$. The final results obtained in the text are exact.

Again the galli should be computed by $100/128$ times $371/300$ or 371 by 384. Here we have 371 times 207 resulting in 76797 and 200 times 384 giving 76800. The factor can for computational practical use be replaced by the simple one $200/207$!

Then 200 times 625 gives dividing by 207 the integral part 603 and the fractional part $179/207 = (207 + 138 + 13)/414 = {}^1/_2 + {}^1/_3 + 13/414$, whereas $13/416 = 1/32$. The text gives this result, obvious by miscopying the last fraction as $1/12$ (comp. line 29!).

The factor for the reduction of the hermi $100/104$, used everywhere else, is different from that following from the last figure of the first row. We shall comment on that below. Here we have $62500/104 = 600 + 100/104$ and as $100 = 52 + 26 + 22$ the first fractions are ${}^1/_2$ ${}^1/_4$. Then 22 by

104 = 110/520 and 110=104 +6, yielding the third fraction 1/5 with a remainder of 6/520, simply approximated by 6/600 = 1/100 in the text. *Only 1/650 has been neglected by that, and that intentionally!*

The analotiki are first reduced to chalki by subtracting one twentyfirst part. Here we have 625/21 = 29 + 16/21 and a remainder of 595 + 5/21 and this last fraction is replaced by 5/20 = 1/4, leading to 595+1/4.

From these the dromi are obtained by adding one half, one quarter and one thirty second part, yielding 297 $^1/_2$+$^1/_8$ and 148 $^1/_2$+$^1/_4$+$^1/_{16}$ and 18+$^1/_2$+$^1/_{16}$+$^1/_{32}$+$^1/_{128}$ together 465+$^1/_{32}$+$^1/_{128}$. The reconstruction of the text line 71 is wrong. In order to obtain the philippi one should subtract one eleventh (see line 76). Here the eleventh part is 54 + 1/11 + 1/44 and 5/44 is very near to 5/45 = 1/9. Subtracting this the result is 541 + 5/36 = 541+1/12 + 1/18. Should one add this last fraction at the end of line 77?!

The galli are then obtained by a factor 200/207 leading to 575 + 25/207 and this last fraction is (23+2)/207 = 1/9 + 1/138 + 1/414. We suggest that the last very small fractions were neglected and that the fraction in line 86 is a misreading of sigma for theta.

The value for the hermi taking exactly 100/104 leads to 572 + 1/3 + 1/7 + 1/1560. The result is destroyed in the text. (line 94).

In order to obtain the row for the philippi first the amount of chalki is computed by adding a tenth. Indeed the reciprocal value of 10/11 is 11/10 and this is obtained by adding 1/10 to the unit and the correct result is 687$^1/_2$ (line 99). The dromi are then to be obtained by adding one half, one quarter and one thirtysecond part:

343	1/2	1/4				
171	1/2	1/4	[1/8]			
21		1/4	[1/8]	1/16	1/32	[1/64]

and the result had to be 537 + 1/16 + 1/32 + 1/64. This means that the 1/8 were neglected, having its consequence for the 1/64 disappearing too and the result of the text is 536 + $^1/_2$ + 1/4 + 1/16 + 1/32.

The analotiki are missing in the text; the galli are obtained as 200/207 of the chalki, but the result has not been preserved in the text either. The hermi yield, multiplying into 100/104, the final value 661 + 6/104 and the fractional part is 3/52 to be approximated by 3/51 or 1/17. We suggest that the 1/12 in line 121 is a misreading of 1/17.

In the row of the galli we have first the amount of the chalki by multiplying into 207/200. The result 646 + 7/8 is exactly given in the fractional part as $1/2 + 1/3 + 1/24$ as $21 = 12+8+1$. As has already been remarked the final sigma of the integral value has been read as beta, as would follow from the preserved and reused values in lines 129, 140, 144, 145, 147, The same factors applied to the amount of chalki give the dromi — result not preserved — and the analotiki, for which the whole corresponding section has been destroyed. The result for the philippi had to be $584+1/3 + 1/12 + 1/66$, which leaves open the question whether the last fraction was neglected — as one might expect — or has to be restored in the gap in line 141, at the end.

The hermi follow by the factor 100/104 as $618 + 31/208$. Now, as 7 times $31 = 217$, a good approximation is 1/7. The series of convergents of the continued fraction shows 1/6, 1/7, 3/20, 7/47. One might ask whether the only preserved sign in line 150 is indeed a 6 or should be a 7. Both are numerically possible, but 7 would be somewhat better.

For the row of the hermi very little data have been preserved. The chalki had to be computed by adding one twentyfifth part, using the reciprocal value 104/100. It would lead to the result 650, which must have been obtained by the scribe as a basis for the other computations Indeed only in line 155 this number is preserved! The analotiki should follow adding 1/20, which addendum $32^1/_2$ is indeed preserved in line 154. In the section on the philippi the only preserved number is the fraction 1/11, which should be subtracted. The results for dromi and galli are missing.

We can now easily explain the only discrepancy, the value for dromi stating the factor $1 + 25/100$ or $1 + 1/4$ for hermi. One should use 128/100 times 100/104 or 128/104. Th is yields $1 + 24/104 = 1 + 3/13$. The normal approximation by $3/12 = 1/4$ and neglecting the remainder explains the number, for which the remainder 1/52 is in fact too great to be neglected.

We see, that but for some misreadings of numbers, the text of BM-papyrus CCLXV, are correctly approximated, whereas the only number which is too great, that for conversion of hermi into dromi, was obtained using the normal procedure, leading to too great a difference, which is quite exceptional!

NOTES

1) The combination of mathematics and literature is not rare an occurrence in papyri deriving from schools. Cf. e.g. O. Guéraud et P. Jouguet, *Un Livre d'Ècolier du IIIe Sciècle avant J,-C.*, Le Caire, 1938 (this papyrus too is in the form of a scroll!) and P. J. Parsons, *A School-Book from the Sayce Collection*, ZPE 6, 1970, pp. 133 ff.

2) Cf. G. Zalateo, Papiri scolastici, Aegyptus 41, 1961, pp. 160 ff.

3) In a forthcoming volume of Chronique d'Egypte.

4) Codex constantinopolitanus palatii veteris no 1, III, pp 260-268, ed. E. M. Bruins, Leiden, 1964.

5) F. G. Kenyon, Catalogue of Greek Papyri in the British Museum, Vol. II.

We thank Dr. H. Loebenstein (Vienna) for her permission to publish this papyrus.

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