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Chapter 4

Decomposable form in $n$ variables of degree $n + 1$

Recall that Theorem 2.1.4 in Chapter 2 provides an asymptotic formula for the number of solutions of a decomposable form inequality in $n$ variables of degree $d$. Unfortunately in this formula, the error term depends on the coefficients of $F$.

A lot of work on removing the dependence of the error term on $F$ has been done by Thunder. We recall some results of his below. The following notation is needed. Consider the inequality

$$|F(x)| \leq m \text{ in } x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n.$$ 

Define the discriminant $D(F)$ of a decomposable form $F = aL_1, \ldots, L_d \in \mathbb{Z}[X_1, \ldots, X_n]$ to be

$$D(F) = a^{2(d-1)} \prod_{1 \leq i_1 < \cdots < i_n \leq d} (\det(L_{i_1}, \ldots, L_{i_n}))^2.$$ 

Put $A_F(m) = \{x \in \mathbb{R}^n : |F(x)| \leq m\}$ and $A_F = A_F(1)$. Denote the volume of $A_F$ by $\mu^n_{\infty}(A_F)$ and the number of integer solutions in $A_F(m)$ by $N_F(m)$.

**Theorem 4.0.10** (Thunder [18]). Let $F \in \mathbb{Z}[X, Y]$ be a binary cubic form in two var-
ables that is irreducible over \( \mathbb{Q} \). Then

\[
|N_F(m) - m^{2/3} \mu_\infty^2(\mathbb{A}_F)| \leq 9 + \frac{2008m^{3/2}}{|D(F)|^{1/12}} + 3156m^{1/3} \text{ for all } m \geq 1.
\]

Later, Thunder proved a Theorem concerning decomposable forms \( F \in \mathbb{Z}[X_1, \ldots, X_n] \) of degree \( n + 1 \) of finite type (hence \( D(F) \neq 0 \)).

**Theorem 4.0.11** (Thunder [20]).

\[
|N_F(m) - m^{n/(n+1)} \mu_\infty^n(\mathbb{A}_F)| \ll \frac{m^{(n-1)/n}}{|D(F)|^{1/(2n(n+1))}} (1 + \log m)^{n-2} + m^{(n-1)/(n+1)} (1 + \log m)^{n-1}.
\]

where the implicit constant depends only on \( n \).

The goal of this Chapter is to prove a \( p \)-adic generalization of Theorem 4.0.11, removing the dependence on \( F \) of the error term in Theorem 2.1.4. Thunder’s main idea is to find an equivalent form \( G \) of \( F \) such that it is possible to give a upper bound for \( \mathcal{H}(G) \) in terms of its discriminant \( D(F) \). In our proof, we first give the \( p \)-adic generalization of this idea.

### 4.1 Statement of the Theorem

Let \( F(X) \in \mathbb{Z}[X_1, \ldots, X_n] \) be a decomposable form of degree \( n+1 \). Let \( S = \{\infty, p_1, \ldots, p_r\} \) be a finite subset of \( M_\mathbb{Q} \). We consider the inequality

\[
\prod_{p \in S} |F(x)|_p \leq m \quad \text{in} \quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n
\]

with \( \gcd(x_1, x_2, \ldots, x_n, p_1 \cdots p_r) = 1 \). (4.1.1)
Recall the notation

\[ I(F) := \text{the set of all ordered linearly independent } n \text{-tuples among } L_1, \ldots, L_d, \]

\[ a(F) := \max_{(L_{i_1}, \ldots, L_{i_j}) \subset I(F) \ 1 \leq j \leq n-1} \max_j \left| \left\{ L_i \in \text{span} \{L_{i_1}, \ldots, L_{i_j}\} \right\} \right|, \]

\[ A^n_S = \prod_{p \in S} \mathbb{Q}^n_p, \]

\[ A_{F,S}(m) := \left\{ (x_p)_p \in A^n_S : \prod_{p \in S} |F(x_p)|_p \leq m, \ x_p|_p = 1 \text{ for } p \in S_0 \right\}, \]

\[ N_{F,S}(m) := \left| \left\{ x \in \mathbb{Z}^n : \prod_{p \in S} |F(x)|_p \leq m, \ \gcd(x_1, x_2, \ldots, x_n, p_1 \cdots p_r) = 1 \right\} \right|. \]

Recall that \( \mu_\infty \) is the normalized Lebesgue measure on \( \mathbb{R} = \mathbb{Q}_\infty \) such that \( \mu_\infty([0, 1]) = 1 \) and that \( \mu_p \) is the normalized Haar measure on \( \mathbb{Q}_p \) such that \( \mu_p(\mathbb{Z}_p) = 1 \). Define the product measure \( \mu^n = \prod_{p \in S} \mu^n_p \) on \( A^n_S \).

For each \( p \in S \), we can decompose \( F \) as

\[ F = a_p L_{p,1} \cdots L_{p,n+1} \]

where \( a_p \in \mathbb{Q}^*_p \) and \( \{ L_{p,1}, \ldots, L_{p,n+1} \} \) are linear forms in \( \mathbb{Q}_p[X_1, \ldots, X_n] \) such that the decomposition is \( \mathbb{Q}_p \)-symmetric. It means that each element of \( \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \) permutes the linear forms \( L_{p,1}, \ldots, L_{p,n+1} \).

For \( p \in S \), put \( \Delta_F^p = \Delta_{p,1} \cdots \Delta_{p,n+1} \) where

\[ \Delta_{pi} = \det(L_{p,1}, \ldots, \widehat{L}_{pi}, \ldots, L_{p,n+1}) \text{ for } i = 1, \ldots, n + 1. \]

Then

\[ \sum_{j=1}^{n+1} (-1)^j \Delta_{pj} \cdot L_{pj} = 0. \tag{4.1.2} \]

In what follows, the constants implied by the occurring Vinogradov symbols \( \ll \) and \( \gg \) will be effectively computable and depend only on \( n \) and \( S \). We prove the following Theorem.
Theorem 4.1.1. Let \( F \in \mathbb{Z}[X_1, \ldots, X_n] \) be a decomposable form of degree \( n + 1 \). Suppose \( F(x) \neq 0 \) for every non-zero \( x \in \mathbb{Z}^n \). Also suppose \( a(F|_T) < \frac{d}{\dim T} \) for every linear subspace \( T \) of dimension at least 2 of \( \mathbb{Q}^n \). Then we have \( D(F) \neq 0 \) and

\[
|N_{F,S}(m) - \mu^n(\mathbb{A}_{F,S}(m))| \ll \frac{m^{(n-1)/n}(1 + \log m)^{|S|(n+1)}}{(\prod_{p \in S} |D(F)|_p)^2m^{n+1}} + m^{n-1}(1 + \log m)^{|S|(n-1)}. 
\]

4.2 About discriminants of decomposable forms

In this section, we collect some facts about discriminants of decomposable forms. We can be more general by letting \( F \) vary for each \( p \in S \) and \( \mathbb{K} \) be a field with \( \text{char} \mathbb{K} = 0 \).

Definition 4.2.1. Let \( F = aL_1, \ldots, L_d \in \mathbb{K}[X_1, \ldots, X_n] \) be a decomposable form where \( a \in \mathbb{K}^* \) and \( L_1, \ldots, L_d \in \mathbb{K}[X_1, \ldots, X_n] \) are linear forms. We say that \( F \) is in general position if \( \det(L_{i_1}, \ldots, L_{i_n}) \neq 0 \) for each \( \{i_1, \ldots, i_n\} \subseteq \{1, \ldots, d\} \).

Definition 4.2.2. The discriminant \( D(F) \) of a decomposable form \( F = aL_1, \ldots, L_d \in \mathbb{K}[X_1, \ldots, X_n] \) in general position is defined to be

\[
D(F) = a^{2(d-1)} \cdot \prod_{1 \leq i_1 < \cdots < i_n \leq d} \left( \det(L_{i_1}, \ldots, L_{i_n}) \right)^2.
\]

It is easy to check that \( D(F) \) is independent of the choice of \( a, L_1, \ldots, L_d \).

Lemma 4.2.3. Let \( F \in \mathbb{K}[X_1, \ldots, X_n] \) be a decomposable form of degree \( d \). Then

(a) \( D(F) \in \mathbb{K}^* \).

(b) \( D(\lambda F) = (\lambda)^{2\binom{d-1}{n-1}} D(F) \) for \( \lambda \in \mathbb{K}^* \).

(c) \( D(F_T) = (\det T)^2 \binom{d}{n} D(F) \) for \( T \in GL_n(\mathbb{K}) \).

Proof. (b) and (c) are straightforward.
(a) For every \( \sigma \in \text{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \) there is a permutation \( \hat{\sigma} \) of \( \{1, \ldots, d\} \) such that \( \sigma(L_i) = \lambda_i L_{\hat{\sigma}(i)} \) with \( \lambda_i \in \overline{\mathbb{K}}^* \) and \( \lambda_1 \cdots \lambda_d = 1 \). Hence

\[
\sigma(D(F)) = a^{2(\frac{d-1}{n-1})} \prod_{1 \leq i_1 < \cdots < i_n \leq d} (\lambda_{i_1} \cdots \lambda_{i_n})^2 \left( \det(L_{\hat{\sigma}(i_1)}, \ldots, L_{\hat{\sigma}(i_n)}) \right)^2
\]

\[
= (a\lambda_1 \cdots \lambda_d)^{2(\frac{d-1}{n-1})} \prod_{1 \leq i_1 < \cdots < i_n \leq d} \left( \det(L_{\hat{\sigma}(i_1)}, \ldots, L_{\hat{\sigma}(i_n)}) \right)^2
\]

\[
= a^{2(\frac{d-1}{n-1})} \prod_{1 \leq i_1 < \cdots < i_n \leq d} \left( \det(L_{i_1}, \ldots, L_{i_n}) \right)^2 = D(F).
\]

\[\square\]

Let \( (F_p : p \in S) \) be a system of decomposable forms with \( F_p \in \mathbb{Q}_p[X_1, \ldots, X_n] \) of degree \( d \). For each \( T_p \in \text{GL}_n(\mathbb{Q}_p) \), define \( (F_p)_{T_p}(X) = F_p(T_pX) \).

Recall that

\[
\mathcal{A}(F_p : p \in S) := \left\{ (x_p)_p \in \prod_{p \in S} \mathbb{A}^n_S : |F_p(x_p)|_p \leq 1, |x_p|_p = 1 \text{ for } p \in S_0 \right\}.
\]

**Lemma 4.2.4.** Let \( (F_p : p \in S) \) with \( F_p \in \mathbb{Q}_p[X_1, \ldots, X_n] \) for \( p \in S \) be a system of decomposable forms of degree \( d \) in general position. Let \( \lambda_p \in \mathbb{Q}_p^* \) and \( T_p \in \text{GL}_n(\mathbb{Q}_p) \) for \( p \in S \). Then

\[
\left( \prod_{p \in S} |D(\lambda_p F_{T_p})|_p^{\frac{1}{n}} \right) \cdot \mu^n \left( \mathcal{A}(\lambda_p F_{T_p} : p \in S) \right) = \left( \prod_{p \in S} |D(F_p)|_p^{\frac{1}{n}} \right) \cdot \mu^n \left( \mathcal{A}(F_p : p \in S) \right)
\]

(where possibly both sides of the identity are infinite).

**Proof.** This is a combination of Lemmas 1.3.3, 1.3.4 and 4.2.3 \[\square\]

**Lemma 4.2.5.** For \( p \in S \), let \( F_p \in \mathbb{Q}_p[X_1, \ldots, X_n] \) be a homogeneous polynomial of degree \( d \). Assume that \( |F_p|_p = 1 \) for \( p \in S_0 \). Then

\[
\mu^n \left( \left\{ (x_p)_p \in \mathbb{A}^n_S : \prod_{p \in S} |F_p(x_p)|_p \leq 1, |x_p|_p = 1 \text{ for } p \in S_0 \right\} \right) =
\]

\[
\mu^n \left( \left\{ (x_\infty) \in \mathbb{R}^n : |F_\infty(x_\infty)| \leq 1 \right\} \right) \cdot \prod_{p \in S_0} \left( \sum_{r_p=0}^{d-1} p^{-r_p} \cdot \mu_p^n \left( \left\{ y_p \in \mathbb{Q}_p^n : |F_p(y_p)|_p = p^{-r_p} \right\} \right) \right)
\]

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Proof. We can express the set under consideration as a disjoint union
\[
\prod_{k=(k_p) \in S_0 \in (\mathbb{Z}_{\geq 0})^{\mid S_0 \mid}} \left\{ (x_p)_{p \in S} \in \mathbb{A}_S^n : \begin{array}{l}
|F_\infty(x_\infty)| \leq \prod_{p \in S_0} p^{k_p} \\
|F_p(x_p)|_p = p^{-k_p}, |x_p|_p = 1 \text{ for } p \in S_0
\end{array} \right\}.
\]

Thus, the measure to be computed can be expressed as
\[
\sum_{k \in (\mathbb{Z}_{\geq 0})^{\mid S_0 \mid}} \mu_\infty^n \left\{ x_\infty \in \mathbb{R}^n : |F_\infty(x_\infty)| \leq \prod_{p \in S_0} p^{k_p} \right\} \cdot \prod_{p \in S_0} \mu_p^n \left\{ x_p \in \mathbb{Q}_p^n : |F_p(x_p)|_p = p^{-k_p}, |x_p|_p = 1 \right\}
\]

\[
= \mu_\infty^n \left\{ x_\infty \in \mathbb{R}^n : |F_\infty(x_\infty)| \leq 1 \right\} \cdot \prod_{p \in S_0} \sum_{k_p=0}^\infty (p^{k_p})^{n/d} \mu_p^n \left\{ x_p \in \mathbb{Q}_p^n : |F_p(x_p)|_p = p^{-k_p}, |x_p|_p = 1 \right\}
\]

We have to rewrite the sums occurring in the product. Write \( k_p = d l_p + r_p \) with \( 0 \leq r_p \leq d - 1 \). Then
\[
\mu_p^n \left\{ x_p \in \mathbb{Q}_p^n : |F_p(x_p)|_p = p^{-k_p}, |x_p|_p = 1 \right\}
\]

\[
= \sum_{r_p=0}^{d-1} \sum_{l_p=0}^\infty (p^{k_p})^{n/d} p^{-rl_p} \mu_p^n \left\{ y_p \in \mathbb{Q}_p^n : |F_p(y_p)|_p = p^{-r_p}, |y_p|_p = p^{l_p} \right\}
\]

\[
= \sum_{r_p=0}^\infty (p^{k_p})^{n/d} \mu_p^n \left\{ y_p \in \mathbb{Q}_p^n : |F_p(y_p)|_p = p^{-r_p}, |y_p|_p \geq 1 \right\}
\]

since if \( |y_p| \leq 1/p \) then \( |F(y_p)|_p \leq p^{-d} \) contradicting \( |F(y_p)|_p = p^{-r_p} \). This implies the lemma. \( \square \)
Let \( p \in S_0 \). Further, let \( F \in \mathbb{Q}_p[X_1, \ldots, X_n] \) be a decomposable form of degree \( n + 1 \) with \( |F|_p = 1 \) and \( D(F) \neq 0 \). We compare

\[
A_{p,r}(F) := \mu^n_p \left( \{ x_p \in \mathbb{Q}_p^n : |F(x_p)|_p = p^{-r} \} \right)
\]

with \( |D(F)|_p \). Notice that for \( T \in \text{GL}_n(\mathbb{Q}_p) \), we have

\[
A_{p,r}(F T) = |\det T|_p^{-1} A_{p,r}(F).
\]

We prove the following:

**Lemma 4.2.6.** \( A_{p,r}(F)|D(F)|_p \frac{1}{2^{(n+1)!}} \ll 1 \) where the implicit constant is effectively computable and depends only on \( n \) and \( p \).

**Proof.** Let \( \mathbb{E}_p \) be the splitting field of \( F \) over \( \mathbb{Q}_p \). Denote by \( e \) the ramification index of \( \mathbb{E}_p \). Then \( e \) divides \( [\mathbb{E}_p : \mathbb{Q}_p] \), so \( e \leq (n+1)! \).

We factor \( F \) as \( F = L_1 \cdots L_{n+1} \) with \( L_i \) a linear form in \( \mathbb{E}_p[X_1, \ldots, X_n] \).

Let \( \delta_i := \det(L_{i+1}, \ldots, L_{n+1}, L_1, \ldots, L_{i-1}) \) and put \( L'_i = \delta_i L_i \) for \( i = 1, \ldots, n+1 \).

Then

\[
L'_1 + \cdots + L'_{n+1} = 0, \quad (4.2.1)
\]

\[
L'_1 \cdots L'_{n+1} = \pm D(F)^{1/2} F, \quad (4.2.2)
\]

the coefficients of \( L'_1, \ldots, L'_{n+1} \) are integral over \( \mathbb{Z}_p \), \( (4.2.3) \)

\{\( L'_1, \ldots, L'_{n+1} \)\} is up to sign \( \text{Gal}(\mathbb{E}_p/\mathbb{Q}_p) \)-symmetric \( (4.2.4) \)

(see [1.2.9] for definition).

Only (4.2.3) and (4.2.4) require some explanation. As for (4.2.3), by the ultrametric inequality and Gauss Lemma, we have

\[
|L'_i|_p = |\det(L_{i+1}, \ldots, L_{n+1}, L_1, \ldots, L_{i-1})|_p |L_i|_p \leq |F|_p = 1 \quad (i = 1, \ldots, n + 1).
\]

As for (4.2.4), for every \( \sigma \in \text{Gal}(\mathbb{E}_p/\mathbb{Q}_p) \) there is a permutation \( \hat{\sigma} \) of \( \{1, \ldots, n+1\} \) such that \( \sigma(L'_i) = \lambda_{\sigma,i} L'_{\hat{\sigma}(i)} \) for some \( \lambda_{\sigma,1} \cdots \lambda_{\sigma,n+1} = 1 \). Thus,

\[
\sigma L'_i = \det(\lambda_{\sigma,i+1} L_{\hat{\sigma}(i+1)}, \ldots, \lambda_{\sigma,i-1} L_{\hat{\sigma}(i-1)}) \lambda_{\sigma,i} L_{\hat{\sigma}(i)} = \pm L_{\hat{\sigma}(i)} \quad (\text{for } \sigma \in \text{Gal}(\mathbb{E}_p/\mathbb{Q}_p)).
\]
Let $|D(F)|^{1/2} = p^{-s/e}$. Notice that for $x \in \mathbb{Q}_p^n$ with $|F_p(x)|_p = p^{-r}$ we have

$$\max\{|L'_1(x)|_p, \ldots, |L'_{n+1}(x)|_p\} \geq |L_1'(x) \cdots L'_{n+1}(x)|_p^{\frac{1}{n+1}}$$

$$= \left(|D(F)|^{1/2}F(x)\right)^{\frac{1}{n+1}} = p^{\frac{l-er+s}{e(n+1)}}.$$

So in fact,

$$\max\{|L'_1(x)|_p, \ldots, |L'_{n+1}(x)|_p\} = p^{\frac{l-er+s}{e(n+1)}} \text{ with } l \in \mathbb{Z}_{\geq 0}. \quad (4.2.5)$$

Further, we may write

$$|L'_i(x)|_p = p^{\frac{l-er+s-m_i}{e(n+1)}} \text{ with } m_i \in \mathbb{Z}_{\geq 0} \text{ for } i = 1, \ldots, n+1. \quad (4.2.6)$$

We have collected some properties of the integers $m_1, \ldots, m_{n+1}$:

$$m_1 + \cdots + m_{n+1} = (n+1)l \text{ (by (4.2.2))}, \quad (4.2.7)$$

at least two among $m_1, \ldots, m_{n+1}$ are 0 (by (4.2.1), (4.2.5)), \quad (4.2.8)

$$m_i = m_j \text{ if there is } \sigma \in \text{Gal}(\mathbb{E}_p/\mathbb{Q}_p) \text{ with } \hat{\sigma}(i) = j. \quad (4.2.9)$$

We consider the set of $x \in \mathbb{Q}_p^n$ satisfying (4.2.6) for some tuple of integers $m = (m_1, \ldots, m_{n+1})$ with (4.2.7), (4.2.8) and (4.2.9). Since $\{L'_1, \ldots, L'_{n+1}\}$ is up to sign, a $\text{Gal}(\mathbb{E}_p/\mathbb{Q}_p)$-symmetric system, we have

$$\mu^n_p \left( \left\{ x \in \mathbb{Q}_p^n : |L'_i(x_p)| = p^{\frac{l-er+s-m_i}{e(n+1)}} \text{ for } i = 1, \ldots, n+1 \right\} \right)$$

$$\leq \mu^n_p \left( \left\{ x \in \mathbb{Q}_p^n : |L'_i(x_p)| \leq p^{\frac{l-er+s-m_i}{e(n+1)}} \text{ for } i = 1, \ldots, n+1 \right\} \right)$$

$$\ll \min_{1 \leq j \leq n+1} \left| \det(L_{j+1}, \ldots, L_{n+1}, L_1, \ldots, L_{j-1}) \right|_p$$

$$\ll \frac{p^{\frac{n^2}{e(n+1)}}}{|D(F)|^{1/2}} \cdot p^{\frac{-l}{e(n+1)}} = |D(F)|^{\frac{1}{p^{(n+1)}}} \cdot p^{\frac{-l}{e(n+1)}}.$$
Summing over all $l \in \mathbb{Z}_{\geq 0}$ and all tuples $m$ with (4.2.7), (4.2.8) and (4.2.9), we get

$$\mu_p^n(\{x \in \mathbb{Q}_p^n : |F(x)|_p = p^{-r}\}) \ll |D(F)|_p^{-(n+1)/2} \sum_{l=0}^{\infty} \left( \sum_{m \text{ with (4.2.7), (4.2.8), (4.2.9)}} 1 \right) p^{-l/2(n+1)}$$

$$\ll |D(F)|_p^{-(n+1)/2} \sum_{l=0}^{\infty} \binom{n+1}{l} p^{-l/2(n+1)}$$

$$\ll |D(F)|_p^{-(n+1)/2} \sum_{l=0}^{\infty} \binom{n+1}{l} p^{-l/2(n+1)^2} = \left( 1 - p^{-l/2(n+1)^2} \right)^{-(n+1)} |D(F)|_p^{-(n+1)/2}$$

$$\ll |D(F)|_p^{-(n+1)/2}.$$  

Lemma 4.2.7. For $p \in S$, let $F_p \in \mathbb{Q}_p[X_1, \ldots, X_n]$ be a decomposable form of degree $n+1$ with $D(F_p) \neq 0$. Assume that $|F_p|_p = 1$ for $p \in S_0$. Then

$$\left( \prod_{p \in S} |D(F_p)|_p \right)^{1/(2(n+1))} \mu^n(\mathbb{A}(F_p : p \in S)) \leq C$$

where $C$ is an effectively computable number depending only on $n$ and $S$.

Proof. Combine Theorem of Bean and Thunder in [1] (for $p = \infty$) with Lemma 4.2.5 and Lemma 4.2.6 (for $p \in S_0$).

Remark 4.2.8. This is a $p$-adic generalization of the result of Bean and Thunder [1] on $n$-variable decomposable forms of degree $n+1$ with non-zero discriminant. In the case $S = \{\infty\}$, Bean and Thunder [1] proved a more general result: for arbitrary decomposable forms $F \in \mathbb{C}[X_1, \ldots, X_n]$ of degree $d$, we have

$$|D(F)|^{\frac{(d-n)n!}{2d}} \mu^n(\mathbb{A}_F) \leq C$$

where $C$ is an effectively computable number depending only on $n$. It is still open to generalize their result in the $p$-adic setting.
4.3 Auxiliary Lemmas

In this section, let \( F \in \mathbb{Z}[X_1, \ldots, X_n] \) be a decomposable form in \( n \) variables of degree \( n + 1 \) in general position. Assume that \( I(F) \neq \emptyset \) and \( F(x) \neq 0 \) for \( x \in \mathbb{Z}^n \setminus \{0\} \).

Recall that: we say that two decomposable forms \( F, G \in \mathbb{Z}[X_1, \ldots, X_n] \) are \( S \)-equivalent if there exist \( T \in GL(n, \mathbb{Z}_S) \) and \( t \in \mathbb{Z}_S^* \) such that \( G = t \cdot F_T \). For the definition of \( H(G) \), see [1.1.2].

**Lemma 4.3.1.** There exists a decomposable form \( G \in \mathbb{Z}[X_1, \ldots, X_n] \) in the \( S \)-equivalent class of \( F \) such that
\[
H(G) \leq c_1 \cdot (\prod_{p \in S} |D(G)|_p)^{\frac{2}{n+1}}
\]
where \( c_1 \) is an effectively computable constant depending only on \( n \) and \( S \).

**Proof.** For \( p \in S \), we choose a factorization \( F = a_p L_{p,1} \cdots L_{p,n+1} \) where \( a_p \in \mathbb{Q}_p^* \) and \( \{L_{p,1}, \ldots, L_{p,n+1}\} \) is a \( \mathbb{Q}_p^* \)-symmetric system of linear forms.

For \( p = \infty \), we assume that
\[
\begin{align*}
L_{\infty i} &\in \mathbb{C}^n \ (i = 1, \ldots, 2r), \quad L_{\infty i} \in \mathbb{R}^n \ (i = 2r + 1, \ldots, n+1) \\
L_{\infty i} &= L_{\infty,i+r} \ (i = 1, \ldots, r). 
\end{align*}
\]

If \( r \) is even, put
\[
\begin{align*}
M_{\infty 1} &= \text{Re}(\Delta_{\infty 1} L_{\infty 1}), \quad M_{\infty 2} = \text{Im}(\Delta_{\infty 1} L_{\infty 1}), \ldots, \quad M_{\infty,2r-1} = \text{Re}(\Delta_{\infty r} L_{\infty r}), \\
M_{\infty,2r} &= \text{Im}(\Delta_{\infty r} L_{\infty r}), \quad M_{\infty i} = \Delta_{\infty i} L_{\infty i} \ (i = 2r + 1, \ldots, n+1), \\
M_{pi} &= \Delta_{pi} L_{pi} \ (p \in S_0, i = 1, \ldots, n+1). 
\end{align*}
\]

If \( r \) is odd, put
\[
\begin{align*}
M_{\infty 1} &= \text{Im}(\Delta_{\infty 1} L_{\infty 1}), \quad M_{\infty 2} = \text{Re}(\Delta_{\infty 1} L_{\infty 1}), \ldots, \quad M_{\infty,2r-1} = \text{Im}(\Delta_{\infty r} L_{\infty r}), \\
M_{\infty,2r} &= \text{Re}(\Delta_{\infty r} L_{\infty r}), \quad M_{\infty i} = \Delta_{\infty i} L_{\infty i} \ (i = 2r + 1, \ldots, n+1), \\
M_{pi} &= \Delta_{pi} L_{pi} \ (p \in S_0, i = 1, \ldots, n+1). 
\end{align*}
\]
With these choices, we have rank\(\{M_{p2}, \ldots, M_{p,n+1}\} = n\) for \(p \in S\). Consider the following symmetric convex body:

\[
C := \left\{ (x_p)_p \in \mathbb{A}^n_S : \begin{array}{l}
|M_{\infty i}(x_\infty)| \leq 1 \quad (i = 2, \ldots, n + 1), \\
|M_{pi}(x_p)|_p \leq 1 \quad (i = 2, \ldots, n + 1, \, p \in S_0)
\end{array} \right\}.
\]

Let \(\lambda_1, \ldots, \lambda_n\) be the successive minima of \(C\) with respect to \(\mathbb{Z}^n_S\).

By a Theorem of K. Mahler in [11], \(\mathbb{Z}^n_S\) has a basis \(\{a_1, \ldots, a_n\}\) such that

\[
|M_{\infty i}(a_j)| \leq \max\{1, j/2\} \lambda_j \quad \text{for} \quad i = 2, \ldots, n + 1, \quad j = 1, \ldots, n,
\]

\[
|\Delta_{pi} L_{p i}(a_j)|_p \leq 1 \quad \text{for} \quad i = 2, \ldots, n + 1, \quad p \in S_0, \quad j = 1, \ldots, n.
\]

By Lemma 3.3.5 in [4, Chap. 4], there exist a permutation \(\sigma\) of \(\{1, \ldots, n\}\) and another basis \(\{a'_1, \ldots, a'_n\}\) of \(\mathbb{Z}^n_S\) such that

\[
|M_{\infty i,j+1}(a'_j)| \leq n4^n \min\{\lambda_{\sigma(i)}, \lambda_j\} \quad \text{for} \quad i = 1, \ldots, n, \quad j = 1, \ldots, n. \tag{4.3.4}
\]

Further, \(a'_1, \ldots, a'_n\) are \(\mathbb{Z}\)-linear combinations of \(a_1, \ldots, a_n\). As a consequence

\[
|M_{p,i,j+1}(a'_j)|_p \leq 1 \quad \text{for} \quad i = 1, \ldots, n, \quad p \in S_0, \quad j = 1, \ldots, n. \tag{4.3.5}
\]

Denote the matrix with columns \(a'_1, \ldots, a'_n\) by

\[
T := (a'_1, \ldots, a'_n). \tag{4.3.6}
\]

Write \(G = u \cdot F_T\) where \(T \in GL_n(\mathbb{Z}_S)\) and \(u \in \mathbb{Z}^*_S\) such that \(G\) is primitive. Then

\[
D(G) = u^{2n} \cdot \det(T)^2 D(F)
\]

and hence

\[
\prod_{p \in S} |D(G)|_p = \prod_{p \in S} |D(F)|_p = \prod_{p \in S} \left|a'_1\right|_p^{2n} \left|\Delta_p F\right|_p^2. \tag{4.3.7}
\]

Consider again \(C\). By Lemma [1.2.10] we know that

\[
\lambda_1 \cdots \lambda_n \ll \prod_{p \in S} |\det(M_{p2}, \ldots, M_{p,n+1})| \ll \prod_{p \in S} |\Delta_p F|_p \tag{4.3.8}
\]

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and also
\[
\lambda_1 \cdots \lambda_n \gg \prod_{p \in S} |\Delta_p^F|_p. \tag{4.3.9}
\]

We bound \( \lambda_1 \) from below and \( \lambda_n \) from above. There is a non-zero \( x \in \mathbb{Z}_n^S \) such that
\[
|M_{\infty,i+1}(x)| \leq \lambda_1 \quad \text{for } i = 1, \ldots, n,
\]
\[
|M_{p,i+1}(x)|_p \leq 1 \quad \text{for } i = 1, \ldots, n, \ p \in S_0.
\]

By our assumption that \( F(x) \neq 0 \) for \( x \in \mathbb{Q}^n \setminus \{0\} \), we have \( \prod_{p \in S} |F(x)|_p \geq 1 \).

Since
\[
|\Delta_{\infty,i+r}L_{\infty,i+r}(x)| = |\Delta_{\infty}L_{\infty}(x)| = |M_{\infty,2i-1}(x) \pm \sqrt{-1} \cdot M_{\infty,2i}(x)| \ll \lambda_1 \quad \text{for } i = 2, \ldots, r
\]
we have
\[
|\Delta_{\infty}L_{\infty}(x)| \leq \sum_{j=2}^{n+1} |\Delta_{\infty,j}L_{\infty,j}(x)| \ll \lambda_1,
\]
therefore
\[
\prod_{p \in S} |\Delta_p^F|_p \leq \prod_{p \in S} |\Delta_p^F F(x)|_p
\]
\[
= \prod_{p \in S} |a_p|_p \cdot \left( |\Delta_{\infty}L_{\infty}(x)| \prod_{j=2}^{n+1} |\Delta_{\infty,j}L_{\infty,j}(x)| \right) \cdot \prod_{p \in S_0} \left( |\Delta_{p1}L_{p1}(x)|_p \prod_{j=2}^{n+1} |\Delta_{pj}L_{pj}(x)|_p \right)
\]
\[
\ll \prod_{p \in S} |a_p|_p \cdot n \lambda_1 \cdot (\lambda_1)^{2r-1} \cdot (\lambda_1)^{n+1-2r}
\]
\[
\ll \prod_{p \in S} |a_p|_p \cdot (\lambda_1)^{n+1}.
\]

This implies
\[
\lambda_1 \gg \left( \prod_{p \in S} |a_p|_p \right)^{\frac{1}{n+1}} \tag{4.3.10}
\]
and
\[
\lambda_n \ll \prod_{p \in S} |\Delta_p^F|_p \leq \prod_{p \in S} |\Delta_p^F|_p \leq \left( \prod_{p \in S} |\Delta_p^F|_p \right)^{\frac{2}{n+1}} \left( \prod_{p \in S} |a_p|_p \right)^{\frac{n-1}{n+1}}. \tag{4.3.11}
\]

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Since
\[ |M_{\infty, i+1}(a'_j)| \ll \min \{ \lambda_{\sigma(i)}, \lambda_j \} \text{ for } i = 1, \ldots, n, \quad j = 1, \ldots, n, \]
we have
\[ |M_{\infty 1}(a'_j)| \leq \sum_{i=1}^{n} |M_{\infty, i+1}(a'_j)| \ll \lambda_j \text{ for } j = 1, \ldots, n, \]
\[ |M_{\infty, i+1}(a'_j)| \ll \lambda_{\sigma(i)} \text{ for } i = 1, \ldots, n, \quad j = 1, \ldots, n, \]
and hence
\[ |(M_{\infty 1}(a'_1), \ldots, M_{\infty 1}(a'_n))|_{\infty} \ll \lambda_n, \]
\[ |(M_{\infty, i+1}(a'_1), \ldots, M_{\infty, i+1}(a'_n))|_{\infty} \ll \lambda_{\sigma(i)} \text{ for } i = 1, \ldots, n. \]
This leads to
\[
\prod_{p \in S} |\Delta_{p}^F|_p \cdot \mathcal{H}(G) = \prod_{p \in S} |a_{p}|_p \cdot \prod_{p \in S} \left( \prod_{i=1}^{n+1} (|\Delta_{p_{1+1}(a'_1)}, \ldots, \Delta_{p_{1+1}(a'_n)})|_p \right)
\]
\[
\leq \prod_{p \in S} |a_{p}|_p \cdot \prod_{p \in S_0} \prod_{i=1}^{n+1} \max \{ |M_{p_i}(a'_1)|_p, \ldots, |M_{p_i}(a'_n)|_p \}
\]
\[
\cdot \prod_{i=1}^{n+1} \left| \sum_{j=1}^{n} M_{\infty, 2i-1}(a'_j)X_j + \sqrt{-1} \cdot \sum_{j=1}^{n+1} M_{\infty, 2i}(a'_j)X_j \right|_{\infty} \cdot \sum_{i=2r+1}^{n+1} \left| \sum_{j=1}^{n} M_{\infty, i}(a'_j)X_j \right|_{\infty}
\]
\[
\ll \prod_{p \in S} |a_{p}|_p \cdot \lambda_n \cdot \prod_{i=n-r+2}^{n} \lambda_i^2 \cdot \prod_{j=r+1}^{n-r+1} \lambda_j
\]
\[
\ll \prod_{p \in S} |a_{p}|_p \cdot \frac{\lambda_n^2 \prod_{i=1}^{n} \lambda_i^2}{\prod_{i=1}^{n-r+1} \lambda_i} \leq \prod_{p \in S} |a_{p}|_p \cdot \frac{\lambda_n^2 \prod_{i=1}^{n} \lambda_i^2}{\lambda_{n+1}^2} \]
By (4.3.8), (4.3.10) and (4.3.11), the last expression is at most
\[
\ll \prod_{p \in S} |a_{p}|_p \cdot \left( \prod_{p \in S} |\Delta_{p}^F|_p \right)^\frac{4n}{n+1} \cdot \left( \prod_{p \in S} |\Delta_{p}|_p \right)^\frac{2n-2}{n+1} \cdot \left( \prod_{p \in S} |\Delta_{p}^F|_p \right)^2 \cdot \prod_{p \in S} |a_{p}|_p \cdot \frac{\prod_{p \in S} |a_{p}|_p}{\prod_{p \in S} |\Delta_{p}|_p}.
\]
Hence we get
\[ \mathcal{H}(G) \ll \left( \prod_{p \in S} |a_{p}|_p \right)^\frac{4n}{n+1} \cdot \left( \prod_{p \in S} |\Delta_{p}|_p \right)^\frac{4}{n+1} \]
and then an application of (4.3.7) completes the proof. \( \square \)
Lemma 4.3.2. Let \( L \) show that

Proof. Put \( G \) These lead to decompositions by Lemma 4.2.4. So for proving Theorem 4.1.1, we may as well work with \( G \) and \( F, G \) are \( \sigma \) where \( \sigma \) and \( \sigma \) that by our choice of \( \sigma \) and \( \sigma \), where the linear forms \( M \) Define the linear forms \( M \) Further, by (4.3.9) we have

From now on, we will work with the decomposable form \( G \) as in Lemma 4.3.1. Since \( F, G \) are \( \mathbb{Z}_S \)-equivalent, we have \( N_{F,S}(m) = N_{G,S}(m) \) and \( \mu^n(\mathbb{A}_{F,S}(m)) = \mu^n(\mathbb{A}_{G,S}(m)) \) by Lemma 4.2.4. So for proving Theorem 4.1.1, we may as well work with \( G \).

Recall that we have chosen a decomposition \( F = a_p L_{p1} \cdots L_{p,n+1} \) for each \( p \in S \). These lead to decompositions \( G = u F_T = a'_p \cdot L'_{p1} \cdots L'_{p,n+1} \) where \( a'_p = u a_p \in \mathbb{Q}_p^* \) and \( L'_{pi}(X) = \sum_{j=1}^n L_{pi}(a'_j) X_j \). Note that formula (4.1.2) still holds for \( G \).

Lemma 4.3.2. Let \( (x_p)_{p \in S} \in \mathbb{A}_S \) such that \( x_p \neq 0 \) for each \( p \in S \). Then there is a set of indices \( J := \{ j_p \in \{1, \ldots, n+1\} : p \in S \} \) such that

\[
\prod_{p \in S} \prod_{i \neq j_p} |L'_{pi}(x_p)|_p \leq c_2 \prod_{p \in S} \frac{|G(x_p)|_p}{|x_p|_p|D(G)|^{1/2(n+1)}}
\]

where \( c_2 \) is an effectively computable constant depending only on \( n \) and \( S \).

Proof. Put \( y_p = T x_p \) for \( p \in S \) where \( T \) is given by (4.3.6). Using (4.3.7), it suffices to show that

\[
\prod_{p \in S} \prod_{i \neq j_p} |L_{pi}(y_p)|_p \ll \prod_{p \in S} |F(y_p)|_p \ll \prod_{p \in S} |T^{-1} y_p|_p|D(F)|^{1/2(n+1)}. \tag{4.3.12}
\]

Define the linear forms \( M'_{pi} := M_{pi}T = \sum_{j=1}^n M_{pi}(a'_j) X_j \) (\( p \in S, i = 1, \ldots, n+1 \)) where the linear forms \( M_{pi} \) (\( p \in S, i = 1, \ldots, n+1 \)) have been defined by (4.3.2). Recall that by our choice of \( \sigma \) and \( \sigma \), we have \( \lambda_{\sigma(i)} \) (\( i = 1, \ldots, n \)), \( |M'_{\infty,i+1}|_\infty \ll \lambda_{\sigma(i)} \) (\( i = 1, \ldots, n \)), \( |M'_{p,i+1}|_p \ll 1 \) (\( p \in S_0, i = 1, \ldots, n \)). \tag{4.3.13}

Further, by (4.3.9) we have

\[
\lambda_1 \ldots \lambda_n \gg \prod_{p \in S} |\det(M_{p2}, \ldots, M_{pn+1})|_p = \prod_{p \in S} |\det(M'_{p2}, \ldots, M'_{pn+1})|_p. \tag{4.3.14}
\]

hence

\[
\prod_{p \in S} \prod_{i=1}^n |M'_{p,i+1}|_p \ll \prod_{i=1}^n \lambda_{\sigma(i)} \ll \prod_{p \in S} |\det(M'_{p2}, \ldots, M'_{pn+1})|_p.
\]
On the other hand, by Hadamard's inequality we have

\[
\prod_{p \in S} |\det(M'_{p2}, M'_{p3}, \ldots, M'_{p,n+1})|_p \ll \prod_{p \in S} \prod_{i=1}^n |M'_{p,i+1}|_p.
\]

So in fact,

\[
\prod_{p \in S} |\det(M'_{p2}, M'_{p3}, \ldots, M'_{p,n+1})|_p \gg \prod_{p \in S} \prod_{i=1}^n |M'_{p,i+1}|_p. \tag{4.3.15}
\]

By Lemma 2.2.1, there is a set of indices \(\{i_p \in \{2, \ldots, n + 1\} : p \in S\}\) such that

\[
\prod_{p \in S} \frac{|M'_{i_p}(x_p)|_p}{|M'_{i_p}|_p} \gg \prod_{p \in S} \frac{|x_p|_p |\det(M'_{p2}, M'_{p3}, \ldots, M'_{p,n+1})|_p}{\prod_{i=1}^n |M'_{p,i+1}|_p}. \tag{4.3.16}
\]

Thus (4.3.15) and (4.3.16) imply

\[
\prod_{p \in S} |M_{i_p}(y_p)|_p = \prod_{p \in S} |M'_{i_p}(x_p)|_p \gg \prod_{p \in S} |x_p|_p |M'_{i_p}|_p = \prod_{p \in S} |T^{-1}y_p|_p |M'_{i_p}|_p. \tag{4.3.17}
\]

By (4.3.13), we also have

\[
\prod_{p \in S} \prod_{i \neq 1 \atop i+1 \neq p}^n |M'_{p,i+1}|_p \ll \prod_{i=2}^n \lambda_{\sigma(i)} \ll \prod_{i=2}^n \lambda_i
\]

and together with (4.3.14), (4.3.15) this implies

\[
\prod_{p \in S} |M'_{p,i_p}|_p \gg \lambda_1.
\]

Together with (4.3.10), this implies

\[
\prod_{p \in S} |M'_{p,i_p}|_p \gg \left(\frac{\prod_{p \in S} |\Delta_p^F|_p}{\prod_{p \in S} |a_p|_p}\right)^{1/(n+1)}.
\]

Inserting this into (4.3.17), we obtain

\[
\prod_{p \in S} |M_{i_p}(y_p)|_p \gg \left(\frac{\prod_{p \in S} |\Delta_p^F|_p}{\prod_{p \in S} |a_p|_p}\right)^{1/(n+1)} \prod_{p \in S} |T^{-1}y_p|_p.
\]
For each $p \in S$ there is a $j_p \in \{1, \ldots, n + 1\}$ such that $|M_{p,i_p}(y_p)| \leq |\Delta_{p,j_p}L_{p,j_p}(y_p)|_p$. So we have

$$\prod_{p \in S} |\Delta_{p,j_p}L_{p,j_p}(y_p)|_p \gg \left( \prod_{p \in S} |\Delta_p|_p \right)^{1/(n+1)} \prod_{p \in S} |T^{-1}y_p|^p.$$  

By multiplying this on both sides with

$$\prod_{p \in S} (|a_p|_p \prod_{i \neq j_p} |\Delta_{pi}L_{pi}(y_p)|_p)$$

we get

$$\prod_{p \in S} |\Delta_p^F(y_p)|_p \gg \prod_{p \in S} |a_p|_p^{n/(n+1)} \cdot \prod_{p \in S} \prod_{i \neq j_p} |L_{pi}(y_p)|_p \cdot \prod_{p \in S} |\Delta_p|_p^{1/(n+1)} \cdot \prod_{p \in S} |T^{-1}y_p|^p$$

which implies (4.3.12).

\[\Box\]

### 4.4 Proof of Theorem 4.1.1

We separate the proof into two cases: the small discriminant case and the large discriminant case.

#### 4.4.1 The small discriminant case

Assume

$$\prod_{p \in S} |D(G)|_p \leq m^{2(n+1)}.$$  

Put

$$B_0 = \frac{m^{1/n}}{\left( \prod_{p \in S} |D(G)|_p \right)^{1/(2n(n+1))}}.$$  

Note that $B_0 \geq 1$.

For $l \in \mathbb{Z}_{\geq 0}$, let

$$B_l = e^l B_0, \quad C_l = e \cdot B_l, \quad A_l = c_2 \frac{B_{l_0}}{B_l}$$

where $c_2$ is the constant from Lemma 4.3.2.
We recall that

\[ A_{G,S}(m, B_0) = \left\{ (x_p)_p \in A_{G,S}(m) : |x_\infty| \leq B_0 \right\}. \]

By Proposition 1.4.6, we have

\[ \left| A_{G,S}(m, B_0) \cap \mathbb{Z}^n \right| - \mu^n(A_{G,S}(m, B_0)) \ll B_0^{n-1}(1 + \log(H(G)B_0))^{(n+1)|S_0|. \] (4.4.1)

Now by Lemma 4.3.2, for each \((x_p)_p \in A_{G,S}(m)\) with \(x_p \neq 0\) for \(p \in S\) and \(|x_\infty| \geq B_l\), there is a set of indices \(J := \{j_p : p \in S\}\) such that

\[ \prod_{p \in S} \prod_{i \neq j_p} \left| L'_p(x_p)_p \right| p \leq c_2 \prod_{p \in S} \left| G(x_p)_p \right| D(G)_p^{1/2(n+1)} \leq c_2 \frac{m}{\|x_\infty\|} \cdot \prod_{p \in S} \left| D(G)_p \right|^{1/2(n+1)} \leq c_2 \frac{B_0^{n-1}}{B_l} = A_l. \]

Note that

\[ |S|(n - 1) \cdot n^{2|S|} \cdot (\log(n^{n/2}n! \prod_{p = S_0} (pd)^{nd/2} \cdot A_l))^{S(n-1)-1} \leq |S| \cdot n^{2|S|+1} \cdot (\log(B_0e^{2l+(n-1)(l+1)})^{S(n-1)-1} \leq |S| \cdot n^{2|S|+1} \cdot (\log B_0 + (n + 1)(l + 1))^{S(n-1)-1} \]

and

\[ |S| \cdot n^{2|S|+1} \cdot (\log B_0 + (n + 1)(l + 1))^{S(n-1)-1} \geq (n!)^{S(n-1)}. \]

Using Lemma 2.2.12 and counting the possibilities of \(j_p (p \in S)\), we deduce that for every \(l \geq 0\) the set

\[ S_l := \left\{ (x_p)_p \in A^n_S : \prod_{p \in S} \prod_{i \neq j_p} \left| L'_p(x_p)_p \right| p \leq A_l, B_l \leq |x_\infty| \leq C_l \right\} \]

can be covered by at most

\[ (n + 1)^{|S|} \cdot |S| \cdot n^{2|S|+1} \cdot (\log B_0 + (n + 1)(l + 1))^{S(n-1)-1} \]

sets of the form

\[ C := \left\{ (x_p)_p \in A^n_S : \left| N'_p(x_p)_p \right| p \leq a_{pi}, i = 1, \ldots, n, p \in S \right\} \] (4.4.2)
where $N'_{p_1}, N'_{p_2}, \ldots, N'_{p_n}$ are linear forms in $\mathbb{Q}_p[X_1, \ldots, X_n]$ with

$$|\det(N'_{p_1}, N'_{p_2}, \ldots, N'_{p_n})|_p = 1, |N'_{p_1}|_p = \cdots = |N'_{p_n}|_p = 1$$

for $p \in S$ and the $a_{pi}$ ($p \in S, i = 1, \ldots, n$) are reals with

$$\prod_{p \in S} \prod_{i=1}^n a_{pi} < \frac{C_i A_i}{B_i} \cdot n^{n/2} n! \prod_{p \in S_0} (pd)^{nd/2} \cdot e^{|S|(n-1)+1} \ll e^{-l} A_0.$$ 

Further, Lemma 1.2.5 implies

$$\mu^n(C) \ll \prod_{p \in S} \prod_{i=1}^n a_{pi} \ll e^{-l} A_0.$$ 

Hence

$$\sum_{l=0}^\infty \mu^n(S_l) \ll \sum_{l=0}^\infty \left( \log B_0 + (n+1)(l+1) \right)^{|S|(n-1)-1} \cdot e^{-l} A_0$$

$$\ll (\log B_0 + 1)^{|S|(n-1)-1} \cdot A_0.$$ 

Therefore we have

$$|\mu^n(A_{G,S}(m)) - \mu^n(\mathbb{A}_{G,S}(m, B_0))| \ll \sum_{l=0}^\infty \mu^n(S_l) \ll (\log B_0 + 1)^{|S|(n-1)-1} \cdot A_0$$

and

$$\left| \mu^n(A_{G,S}(m)) - \mu^n(\mathbb{A}_{G,S}(m, B_0) \cap \mathbb{Z}^n) \right|$$

$$\leq \left| \mu^n(A_{G,S}(m)) - \mu^n(\mathbb{A}_{G,S}(m, B_0)) \right| + \left| \mathbb{A}_{G,S}(m, B_0) \cap \mathbb{Z}^n - \mu^n(\mathbb{A}_{G,S}(m, B_0)) \right|$$

$$\ll (\log B_0 + 1)^{|S|(n-1)-1} \cdot A_0 + B_0^{n-1} \left( 1 + \log(\mathcal{H}(G)B_0) \right)^{(n+1)|S_0|}$$

$$\ll B_0^{n-1} \left( 1 + \log(\mathcal{H}(G)B_0) \right)^{|S|(n+1)}.$$ \quad (4.4.3)

We next estimate the cardinality of the set

$$\mathcal{L} := \left\{ x \in \mathbb{Z}^n : \prod_{p \in S} |G(x)|_p \leq m, |x|_\infty \geq B_0, |x|_p = 1 \text{ for } p \in S_0 \right\}.$$ 

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Lemma 4.4.1. The set $\mathcal{L}$ can be covered by a union of a finite set $\Omega$ of cardinality

$$|\Omega| \ll (\log B_0 + 1)^{|S|(n-1)} \cdot B_0^{n-1}$$

and

$$\ll (1 + \log m)^{|S|(n-1)}$$

proper linear subspaces of $\mathbb{Q}^n$.

Proof. Similarly as in Lemma 2.4.1, we can estimate the cardinality of $\Omega$ by

$$|\Omega| \ll \sum_{l=0}^{\infty} \mu^n(S_l) \ll (\log B_0 + 1)^{|S|(n-1)} \cdot B_0^{n-1}.$$ 

Let

$$l_0 = \lfloor 2 \log(c_2 \cdot B_0^n) \rfloor, \quad l_1 = l_0 + \lceil \log(c_1 \cdot m^5) \rceil$$

where $c_1$ is the constant from Lemma 4.3.1.

Define

$$\mathcal{L}_1 = \{ x \in \mathbb{Z}^n : \prod_{p \in S} |G(x)|_p \leq m, B_0 \leq |x|_\infty \leq C_l, |x|_p = 1 \text{ for } p \in S_0 \},$$

$$\mathcal{L}_2 = \{ x \in \mathbb{Z}^n : \prod_{p \in S} |G(x)|_p \leq m, |x|_\infty \geq C_l, |x|_p = 1 \text{ for } p \in S_0 \}.$$  \hfill (4.4.4)

For any $x$ in $\mathcal{L}_2$ such that $\prod_{p \in S} |G(x)|_p \neq 0$, we have $\prod_{p \in S} |G(x)|_p \geq 1$. Let $x$ be such a solution and write $x = gx'$ with $x'$ is primitive and $\gcd(g, \prod_{i=1}^{\ell} p_i) = 1$. Then

$$m \geq \prod_{p \in S} |G(x)|_p = g^{n+1} \prod_{p \in S} |G(x')|_p \geq g^{n+1}.$$ 

Thus $g \leq m^{1/(n+1)}$.

By Lemma 4.3.1 we have

$$\mathcal{H}(G) \leq c_1 \cdot (\prod_{p \in S} |D(G)|_p)^{2/\pi} \leq m^4.$$
Hence $|x'|_\infty = g^{-1}|x|_\infty \geq m^{-1/(n+1)}C_l \geq m^{-1/(n+1)}c_1 \cdot m^5C_l \geq \max\{C_l, \mathcal{H}(G)\}$.

Using Lemma 4.3.2, there is a set of indices $\mathcal{J} := \{j_p : p \in S\}$ such that

$$\prod_{p \in S} \frac{\prod_{i \neq j_p} |L'_{p_i}(x')|_p}{\det(L'_{i,p})_{i \neq j_p}} \leq c_2 \cdot \frac{B_0^n}{|x'|_\infty} \leq c_2 \cdot \frac{B_0^n}{|x'|_\infty^{1/2}C_l^{1/2}} \leq |x'|^{-1/2}. \quad (4.4.5)$$

By Lemma 1.1.6, we may assume that each linear form $L'_{p_i}$ occurring here is defined over a number field of degree at most $d$. So we have

$$[\mathbb{Q}(L'_{p_i}) : \mathbb{Q}] \leq d \quad \text{and} \quad H_{\mathbb{Q}(L'_{p_i})}(L'_{p_i}) \leq \mathcal{H}(G) \leq |x'|_\infty$$

where $\mathbb{Q}(L'_{p_i})$ is the extension of $\mathbb{Q}$ generated by the coordinates of $L'_{p_i}$. Therefore we can apply a version of the quantitative Subspace Theorem such as [6, Corollary] which implies that the primitive integer solutions the inequality (4.4.5) with $|x'|_\infty \geq \max\{C_l, \mathcal{H}(G)\}$

lie in the union of $\ll 1$ proper linear subspaces of $\mathbb{Q}^n$. Taking into account of the number of possible tuples $\{j_p : p \in S\}$, we conclude that the elements of $\mathcal{L}_2$ lie in $\ll 1$ proper subspaces.

A similar estimate as that for $\mu^n(S_l)$ gives that the elements $x \in \mathcal{L}_1$ with $B_0 \leq |x|_\infty \leq C_l$ lie in the union of at most

$$(\log B_0 + (n + 1)(l + 1))^{\mathcal{S}(n-1)-1}$$

convex sets $\mathcal{C}$ of the form (4.4.2). The set of integer points in each such kind of set $\mathcal{C}$ is contained in a proper linear subspace of $\mathbb{Q}^n$ that is related to $\mathcal{C}$. Hence the solutions $x$ with $B_0 \leq |x|_\infty \leq C_l$ that are not counted in $\Omega$ lie in the union of

$$\ll \sum_{l=0}^{l_1} \left(\log B_0 + (n + 1)(l + 1)\right)^{\mathcal{S}(n-1)-1} \ll \left(\log B_0 + (n + 1)(l_1 + 1)\right)^{\mathcal{S}(n-1)-1}

\ll (1 + \log m)^{\mathcal{S}(n-1)}$$

proper linear subspaces of $\mathbb{Q}^n$. \qed
By Theorem 2.1.3 we know that the number of integral solutions of (4.1.1) in a proper subspace is \( \ll m^{n+1} \). Hence Lemma 4.4.1 implies

\[
|\mathcal{L}| \ll (\log B_0 + 1)^{|S|(n-1)-1} \cdot B_0^{n-1} + (1 + \log m)^{|S|(n-1)} m^{(n-1)/(n+1)}.
\] (4.4.6)

Combining (4.4.3) and (4.4.6), we conclude our proof of Theorem 4.0.10 for the small discriminant case.

### 4.4.2 The large discriminant case

Fix \( \epsilon \) with \( 0 < \epsilon < 1 \). One may take \( \epsilon = n/(n+1) \).

Assume

\[
\left( \prod_{p \in S} |D(G)|_p \right)^{1-\epsilon} \geq m^2.
\]

Then \( \prod_{p \in S} |D(G)|_p \geq m^2 \).

Choose \( \lambda \) with \( 0 < \lambda < \epsilon/4 \) and let \( B_0 = m^{1/(n+1)} / \mathcal{H}(G)^\lambda \), \( B_l = d^l B_0 \), \( C_l = e B_l \). So \( B_0 \leq m^{1/n+1} \).

By Lemma 4.3.2 for every solution \( x \) of inequality (4.1.1) with \( |x|_\infty \geq B_0 \), there are indices \( J := (j_p)_{p \in S} \) such that

\[
\prod_{p \in S} \frac{\prod_{i \neq j_p} |L_{p_i}'(x)|_p}{|\det(L_{p_i}')_{i \neq j_p}|_p} \leq c_2 \prod_{p \in S} \frac{|G(x)|_p^{1/2(n+1)}}{|x|_\infty \cdot \prod_{p \in S} |D(G)|_p^{1/2(n+1)}} \leq \frac{c_2 m}{|x|_\infty \cdot \prod_{p \in S} |D(G)|_p^{1/2(n+1)}}.
\]

Let

\[
t_1 = \max \left\{ \log \left( \prod_{p \in S} |D(G)|_p^{(2^n m^{n+1})} \right) + 2 \log (c_2 \cdot m), \log \left( c_1 \cdot \prod_{p \in S} |D(G)|_p^{4(1+\lambda)/(n+1)} \right) \right\}.
\]
Then by Lemma 4.3.1,
\[ C_{l_1} \geq B_0 \max\{\mathcal{H}(G)^\lambda (c_2 \cdot m)^2, \mathcal{H}(G)^{1+\lambda}\} = m^{1/(n+1)} \max\{(c_2 \cdot m)^2, \mathcal{H}(G)\}. \]

Define the sets \( \mathcal{L}_1, \mathcal{L}_2 \) as in (4.4.4). We first count the cardinality of \( \mathcal{L}_2 \). Let \( x \in \mathcal{L}_2 \). As before, we write \( x = gx' \) with \( x' \) primitive. Then we have
\[ g \leq m^{1/(n+1)} \text{ and } |x'|_\infty \geq C_{l_1} g \geq \max\{(c_2 \cdot m)^2, \mathcal{H}(G)\}. \]

Again by Lemma 4.3.2, we have
\[
\prod_{p \in S} \frac{\prod_{i \neq j_p} |L'_{i,p}(x')_p|}{\det(L'_{i,p})_{i \neq j_p}} \leq \frac{c_2 \cdot m}{|x'|_\infty} \prod_{p \in S} |D(G)|_p^{1/2(n+1)} \leq \frac{c_2 \cdot m}{|x'|_\infty} < \frac{1}{|x'|_\infty}. \tag{4.4.7}
\]

By the \( p \)-adic Subspace Theorem, the set of primitive integer solutions of (4.4.7) lies in the union of \( \ll 1 \) proper linear subspaces of \( \mathbb{Q}^n \). Taking into account the number of possible tuples \( \{j_p : p \in S\} \), the integer solutions of (4.1.1) with \( |x|_\infty \geq C_{l_1} \) lie in \( \ll 1 \) proper subspaces. By Theorem 2.1.3 each subspace contains \( \ll m^{n+1} \) solutions of (4.1.1), leading to
\[
|\mathcal{L}_2| \ll m^{n+1}. \tag{4.4.8}
\]

We next estimate the cardinality of \( \mathcal{L}_1 \).

Set
\[
A = \frac{c_2 \cdot m^{n+1}}{\prod_{p \in S} |D(G)|_p^{1/2(n+1)}}, \quad B = B_0, \quad C = C_{l_1} \text{ and } D = \prod_{p \in S} |D(G)|_p^{2(n+1)}(|S|^{n+1}).
\]

Using Lemma 2.2.12 and taking into consideration the number of tuples \( \{j_p, p \in S\} \), we deduce that \( \mathcal{L}_1 \) can be covered by at most
\[
\ll \left( \log_D \left( \frac{C^n}{n! \prod_{p \in S_0} (pd)^{nd/2} \cdot A} \right) \right)^{|S|/n-1}
\]
\[
\ll \left( \frac{n(log m + l_1)}{(\epsilon/2 - 2\lambda) \log(\prod_{p \in S} |D(G)|_p)} \right)^{|S|/n-1}
\]
\[
\ll (\epsilon/2 - 2\lambda)^{-|S|/n-1}
\]

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sets of the form

\[ C := \{(x_p)_p \in \mathbb{A}^n_S : |N'_p(x_p)|_p \leq a_{pi} \text{ for } i = 1, \ldots, n, \ p \in S\} \]

where \( N'_p, N'_p, \ldots, N'_p \) are linear forms in \( \mathbb{Q}_p[x_1, \ldots, x_n] \) with

\[ |\det(N'_p, N'_p, \ldots, N'_p)|_p = 1, \ |N'_p|_p = \cdots = |N'_p|_p = 1, \ p \in S \]

and the \( a_{pi} \) are reals with

\[ \prod_{p \in S} \prod_{i=1}^n a_{pi} < A \cdot n! \prod_{p \in S_0} (pd)^{nd/2} \cdot |S|^{-n+1} \ll m^{\frac{n-1}{n+1}}. \]

By Lemma 1.2.5, a convex set \( C \) that contains \( n \) linearly independent integral points contains \( \ll m^{\frac{n-1}{n+1}} \) integral points. For the other convex sets, the number of elements of \( \mathcal{L}_1 \) that are contained in a proper subspace is \( \ll m^{\frac{n-1}{n+1}} \) and the number of such proper subspaces is \( \ll (\epsilon - 4\lambda)^{-\left(|S|n-1\right)} \). So we have

\[ |\mathcal{L}_1| \ll (\epsilon - 4\lambda)^{-\left(|S|n-1\right)} m^{\frac{n-1}{n+1}}. \quad (4.4.9) \]

It remains to bound the cardinality of \( \mathbb{A}_{G,S}(m, B_0) \cap \mathbb{Z}^n \). According to Lemma 4.2.7, for every decomposable form \( G \in \mathbb{Z}[x_1, \ldots, x_n] \) of degree \( n + 1 \) with \( D(G) \neq 0 \), we have

\[ \left( \prod_{p \in S} |D(G)|_p \right)^{1/2(n+1)} \mu^n(A_{G,S}(1)) \ll 1. \]

By Lemma 1.3.1, \( \mu^n(A_{G,S}(m)) = m^{n/(n+1)} \cdot \mu^n(A_{G,S}(1)) \), hence

\[ \mu^n(A_{G,S}(m)) \ll \frac{m^{n/(n+1)}}{\left( \prod_{p \in S} |D(G)|_p \right)^{1/2(n+1)}} \ll \frac{m^{n/(n+1)}}{m^{1/(n+1)}} = m^{\frac{n-1}{n+1}} (m \geq 1). \quad (4.4.10) \]

Using Lemma 1.4.6, we have

\[ \left| \mathbb{A}_{G,S}(m, B_0) \cap \mathbb{Z}^n - \mu^n(\mathbb{A}_{G,S}(m, B_0)) \right| \ll (B_0 + 1)^{n-1}(1 + \log(H(G)B_0))^{(n+1)|S_0|} \ll m^{\frac{n-1}{n+1}}(1 + \log m)^{(n+1)|S_0|}. \]

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and hence

\[ |A_{G,S}(m, B_0) \cap \mathbb{Z}^n| \ll m^{\frac{n-1}{n+1}} (1 + \log m)^{(n+1)|S_0|} + \mu^n(A_{G,S}(m, B_0)) \]
\[ \ll m^{\frac{n-1}{n+1}} (1 + \log m)^{(n+1)|S_0|} + \mu^n(A_{G,S}(m)) \]
\[ \ll m^{\frac{n-1}{n+1}} (1 + \log m)^{(n+1)|S_0|}. \] (4.4.11)

Combining (4.4.8), (4.4.9) and (4.4.11) and choosing appropriately \( \epsilon, \lambda \), we have

\[ N_{G,S}(m) \ll |A_{G,S}(m, B_0) \cap \mathbb{Z}^n| + |L_1| + |L_2| \ll m^{\frac{n-1}{n+1}} (1 + \log m)^{(n+1)|S_0|} \]

and therefore

\[ |N_{G,S}(m) - \mu^n(A_{G,S}(m))| \ll N_{G,S}(m) + \mu^n(A_{G,S}(m)) \ll m^{\frac{n-1}{n+1}} (1 + \log m)^{(n+1)|S_0|}. \]

Together with the result in 4.4.1, this completes our proof of Theorem 4.1.1.