Effects of spin-orbit coupling on quantum transport
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Chapter 5

Mesoscopic Spin Hall Effect

5.1 Introduction

The novel and rapidly expanding field of spintronics is interested in the creation, manipulation, and detection of polarized or pure spin currents [90]. The conventional methods of doing spintronics are to use magnetic fields and/or ferromagnets as parts of the creation-manipulation-detection cycle, and to use the Zeeman coupling and the ferromagnetic-exchange interactions to induce the spin dependency of transport. More recently, ways to generate spin accumulations and spin currents based on the coupling of spin and orbital degrees of freedom have been explored. Among these proposals, much attention has been focused on the spin Hall effect (SHE), where pure spin currents are generated by applied electric currents on spin-orbit (SO) coupled systems. Originally proposed by Dyakonov and Perel [29, 91], the idea was resurrected by Hirsch [30] and extended to crystal SO field (the intrinsic SHE) by Sinova *et al.* [31] and Murakami [92]. The current agreement is that the SHE vanishes for bulk, k-linear SO coupling for diffusive two-dimensional electrons [32, 93, 94]. This result is however specific to these systems [95], and the SHE does not vanish for impurity-generated SO coupling, two-dimensional hole systems with either Rashba or Dresselhaus SO coupling, and for finite-sized electronic systems [93, 95]. These predictions have been, to some extent, confirmed by experimental observations of edge spin accumulations in electron [96, 97] and hole [98] systems, and electrical detection of spin currents via ferromagnetic leads [99–101].

Most investigations of the SHE to date focused on disordered conductors with spin-orbit interaction, where the disorder-averaged spin Hall conductivity was calculated using either the Kubo formalism or a diffusion equation approach [30, 31, 94, 102, 32, 93, 103, 92, 95]. Few numerical works alternatively used the scattering approach to transport [104] to calculate the average spin Hall conductance of explicitly finite-sized samples connected to external electrodes. These investigations were however restricted to tight-binding Hamiltonians with no or weak disorder in simple geometries [105–107]. The data of Ref. 108 in particular suggest that diffusive samples with large enough SO coupling exhibit universal fluctuations of the spin Hall conductance $G_{\rm sH}$ with $\mathrm{rms}[G_{\rm sH}] \approx 0.18e/4\pi$. These numerical investigations call for an analytical theory of the SHE in mesoscopic systems, which we provide here.

We analytically investigate the DC spin Hall effect in mesoscopic cavities with SO coupling. We calculate both the ensemble-average and the fluctuations of the transverse spin current generated by a longitudinal charge current. Our approach is based on random matrix theory (RMT) [60], and is valid for ballistic chaotic and mesoscopic diffusive systems at low temperature, in the limit when the spin-orbit coupling time is much shorter than the mean dwell time of the electrons in the cavity, $\tau_{\rm so} \ll \tau_{\rm dwell}$. We show that while the transverse spin current is generically nonzero for a typical sample, its sign and amplitude fluctuate universally, from sample to sample or upon variation of the chemical potential with a vanishing average. We find that for a typical ballistic chaotic quantum dot, the transverse spin current corresponds to slightly less than one excess open channel for one of the two spin species. These analytical results are confirmed by numerical simulations for a stroboscopic model of a ballistic chaotic cavity.

In the ballistic regime, contributions to the SO coupling arise from the crystal field and confinement potentials. In analogy with diffusive systems, the SHE originating from the crystal field as well as the asymmetry of the confinement potential in the out of plane direction (i.e. the Rashba term) can be thought of as the intrinsic effect, while in plane confinement potentials generate extrinsic contributions to the SHE. Although the bal-

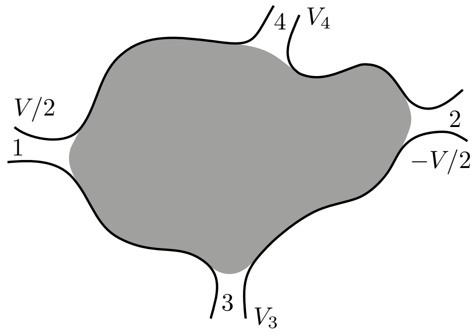

Figure 5.1. Ballistic quantum dot connected to four electrodes. The longitudinal bias V induces a charge current through terminals 1 and 2, while the voltages $V_{3,4}$ are adjusted such that no charge current flows through the transverse leads 3 and 4. Spin-orbit coupling is active only in the gray region.

ance between the two effects modifies nonuniversal properties such as the spin-orbit time, it does not affect the universal features described in this Letter.

5.2 Scattering Approach

We consider a ballistic chaotic quantum dot coupled to four external electrodes via ideal point contacts, each with N_i open channels ($i = 1, \dots 4$). The geometry is sketched in Fig. 5.1. Spin-orbit coupling exists only inside the dot, and the electrochemical potentials in the electrodes are spin-independent. A bias voltage V is applied between the longitudinal electrodes labeled 1 and 2. The voltages V_3 and V_4 are set such that no net charge current flows through the transverse electrodes 3 and 4. We will focus on the magnitude of the spin current through electrodes 3 and 4, in the limit when the openings to the electrodes are small enough, and the spin-orbit coupling strong enough that $\tau_{\rm so} \ll \tau_{\rm dwell}$.

We write the spin-resolved current through the i-th electrode as [104]

$$I_i^\sigma = \frac{e^2}{h} \sum_{j,\sigma'} T_{ij}^{\sigma,\sigma'} (V_i - V_j). \tag{5.1}$$

The spin-dependent transmission coefficients are obtained by summing over electrode channels

$$T_{i,j}^{\sigma,\sigma'} = \sum_{m\in i}\sum_{n\in j}|t_{m,\sigma;n,\sigma'}|^2, \tag{5.2}$$

i.e. $t_{m,\sigma;n,\sigma'}$ is the transmission amplitude for an electron initially in a spin state σ' in channel n of electrode j to a spin state σ in channel m of electrode i. The transmission amplitudes t are the elements of the $2N_T \times 2N_T$ scattering matrix S, with $N_T = \sum_{i=1}^{4} N_i$.

We are interested in the transverse spin currents $I_i^{(z)} = I_i^{\uparrow} - I_i^{\downarrow}$, $i = 3, 4$, under the two constraints that (i) charge current vanishes in the transverse leads, $I_i^{\uparrow} + I_i^{\downarrow} = 0$, $i = 3, 4$ and (ii) the charge current is conserved, $I_1 = -I_2 = I$. From Eq. (5.1), transport through the system is then described by the following equation

$$\begin{pmatrix} 2J \\ J_3^{(z)} \\ J_4^{(z)} \end{pmatrix} = \mathcal{G} \begin{pmatrix} 1/2 \\ \tilde{V}_3 \\ \tilde{V}_4 \end{pmatrix}, \tag{5.3}$$

where

$$\mathcal{G} = \begin{pmatrix} 2N_1 - \mathcal{T}_{11}^{(0)} + 2N_2 - \mathcal{T}_{22}^{(0)} + \mathcal{T}_{12}^{(0)} + \mathcal{T}_{21}^{(0)} & \mathcal{T}_{23}^{(0)} - \mathcal{T}_{13}^{(0)} & \mathcal{T}_{24}^{(0)} - \mathcal{T}_{14}^{(0)} \\ \mathcal{T}_{32}^{(z)} - \mathcal{T}_{31}^{(z)} & -\mathcal{T}_{33}^{(z)} & -\mathcal{T}_{34}^{(z)} \\ \mathcal{T}_{42}^{(z)} - \mathcal{T}_{41}^{(z)} & -\mathcal{T}_{43}^{(z)} & -\mathcal{T}_{44}^{(z)} \end{pmatrix} \tag{5.4}$$

and the transverse voltages (in units of V) read

$$\tilde{V}_3 = \frac{1}{2}\frac{\mathcal{T}_{34}^{(0)}(\mathcal{T}_{42}^{(0)} - \mathcal{T}_{41}^{(0)}) + (2N_4 - \mathcal{T}_{44}^{(0)})(\mathcal{T}_{32}^{(0)} - \mathcal{T}_{31}^{(0)})}{\mathcal{T}_{34}^{(0)}\mathcal{T}_{43}^{(0)} - (2N_3 - \mathcal{T}_{33}^{(0)})(2N_4 - \mathcal{T}_{34}^{(0)})}, \tag{5.5a}$$

$$\tilde{V}_4 = \frac{1}{2}\frac{\mathcal{T}_{43}^{(0)}(\mathcal{T}_{32}^{(0)} - \mathcal{T}_{31}^{(0)}) + (2N_3 - \mathcal{T}_{33}^{(0)})(\mathcal{T}_{42}^{(0)} - \mathcal{T}_{41}^{(0)})}{\mathcal{T}_{34}^{(0)}\mathcal{T}_{43}^{(0)} - (2N_3 - \mathcal{T}_{33}^{(0)})(2N_4 - \mathcal{T}_{34}^{(0)})}, \tag{5.5b}$$

and we defined the dimensionless currents $I = e^2VJ/h$. We introduced generalized transmission probabilities

$$\mathcal{T}_{ij}^{(\mu)} = \sum_{m\in i, n\in j} \mathrm{Tr}[(t_{mn})^{\dagger}\sigma^{(\mu)}t_{mn}], \quad \mu = 0, x, y, z, \tag{5.6}$$

where $\sigma^{(\mu)}$ are Pauli matrices ($\sigma^{(0)}$ is the identity matrix) and one traces over the spin degree of freedom.

5.3 Random Matrix Theory

We calculate the average and fluctuations of the transverse spin currents $J_i^{(\mu)}$, $\mu = x, y, z$ within the framework of RMT. Accordingly, we replace the scattering matrix S by a random unitary matrix, which, in our case of a system with time reversal symmetry (absence of magnetic field) and totally broken spin rotational symmetry (strong spin-orbit coupling), has to be taken from the circular symplectic ensemble[1] (CSE) [60, 41]. We rewrite the generalized transmission probabilities $\mathcal{T}_{ij}^{(\mu)}$ as a trace over S

$$\begin{aligned} \mathcal{T}_{ij}^{(\mu)} &= \mathrm{Tr}\,[Q_i^{(\mu)} S Q_j^{(0)} S^{\dagger}], \\ [Q_i^{(\mu)}]_{m\alpha,n\beta} &= \begin{cases} \delta_{mn}\ \sigma_{\alpha\beta}^{(\mu)}, & \sum_{j=1}^{i-1} N_j < m \le \sum_{j=1}^{i} N_j, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{5.7}$$

Here, m and n are channel indices, while α and β are spin indices. The trace is taken over both set of indices.

Averages, variances, and covariances of the generalized transmission probabilities (5.7) over the CSE can be calculated using the method of Ref. 17. For the average transmission probabilities, we find

$$\langle \mathcal{T}_{ij}^{(\mu)} \rangle = \frac{2\delta_{\mu 0}}{N_T - 1/2}\left(N_i N_j - \frac{1}{2} N_i \delta_{ij}\right), \tag{5.8}$$

[1] We assume that the SO coupling parameters are sufficiently nonuniform, so that SO cannot be removed from the Hamiltonian by a gauge transformation, see Ref. 41.

while variances and covariances are given by

$$\begin{aligned}
&\langle \delta \mathcal{T}_{ij}^{(\mu)} \delta \mathcal{T}_{kl}^{(\nu)} \rangle = \\
&\frac{4\delta_{\mu\nu}}{N_T(2N_T-1)^2(2N_T-3)} \Big\{ N_i N_j (N_T-1)(2N_T-1)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\delta_{\mu 0}) \\
&+ (N_i N_k \delta_{ij}\delta_{kl} - 2N_i N_k N_l \delta_{ij} - 2N_i N_j N_k \delta_{kl} + 4N_i N_j N_k N_l)\delta_{\mu 0} \\
&- N_i N_T (2N_T-1)\delta_{ijkl} + (2N_T-1)\Big[N_i N_l \delta_{ijk} + N_i N_k \delta_{ijl}\delta_{\mu 0} \\
&+ N_i N_j (\delta_{ikl} + \delta_{jkl}\delta_{\mu 0}) - N_i N_j N_l (\delta_{ik} + \delta_{jk}\delta_{\mu 0}) - N_i N_j N_k \delta_{\mu 0}(\delta_{il} + \delta_{jl}) \Big] \Big\},
\end{aligned} \tag{5.9}$$

where $\delta \mathcal{T}_{ij}^{(\mu)} = \mathcal{T}_{ij}^{(\mu)} - \langle \mathcal{T}_{ij}^{(\mu)} \rangle$.

Because the transverse potentials $\tilde{V}_{3,4}$ are spin-independent, they are not correlated with $\mathcal{T}_{ij}^{(\mu)}$. Additionally taking Eq. (5.8) into account, one concludes that the average transverse spin current vanishes ($i = 3, 4$),

$$\langle J_i^{(\mu)} \rangle = \frac{1}{2} \langle \mathcal{T}_{i2}^{(\mu)} - \mathcal{T}_{i1}^{(\mu)} \rangle - \sum_{j=3,4} \langle \mathcal{T}_{ij}^{(\mu)} \rangle \langle \tilde{V}_j \rangle = 0. \tag{5.10}$$

However, for a given sample at a fixed chemical potential $J_i^{(\mu)}$ will in general be finite. We thus calculate var $[J_i^{(\mu)}]$. We first note that $\langle \tilde{V}_{3,4} \rangle = (N_1 - N_2)/2(N_1 + N_2)$, and that var $[\tilde{V}_{3,4}]$ vanishes to leading order in the inverse number of channels. One thus has

$$\begin{aligned}
\text{var}\,[J_i^{(\mu)}] &= \frac{1}{4} \sum_{j=1,2} \text{var}[\mathcal{T}_{ij}^{(\mu)}] - \frac{1}{2} \,\text{covar}[\mathcal{T}_{i1}^{(\mu)}, \mathcal{T}_{i2}^{(\mu)}] \\
&+ \sum_{j=3,4} \Big\{ \text{var}[\mathcal{T}_{ij}^{(\mu)}] \langle \tilde{V}_j \rangle^2 + \text{covar}[\mathcal{T}_{i1}^{(\mu)} - \mathcal{T}_{i2}^{(\mu)}, \mathcal{T}_{ij}^{(\mu)}] \langle \tilde{V}_j \rangle \Big\} \\
&+ 2\,\text{covar}[\mathcal{T}_{i3}^{(\mu)}, \mathcal{T}_{i4}^{(\mu)}] \langle \tilde{V}_3 \rangle \langle \tilde{V}_4 \rangle.
\end{aligned} \tag{5.11}$$

From Eqs. (5.9) and (5.11) it follows that

$$\text{var}\,[J_i^{(\mu)}] = \frac{4N_i N_1 N_2 (N_T - 1)}{N_T(2N_T-1)(2N_T-3)(N_1+N_2)}. \tag{5.12}$$

Eqs. (5.10) and (5.12) are our main results. They show that, while the average transverse spin current vanishes, it exhibits universal sample-to-sample fluctuations. The origin of this universality is the same as for charge transport [60], and relies on the fact expressed in Eq. (5.9) that to leading order, spin-dependent transmission correlators do not scale with the number of channels. The spin current carried by a single typical sample is given by $\mathrm{rms}[J_i^{(\mu)}] \times e^2V/h$, and is thus of order e^2V/h in the limit of large number of channels. In other words, for a given sample, one spin species has of order one more open transport channel than the other one. For a fully symmetric configuration, $N_i \equiv N$, the spin current fluctuates universally for large N, with $\mathrm{rms}[I_3^z] \simeq (e^2V/h)/\sqrt{32}$. This translates into universal fluctuations of the transverse spin conductance with $\mathrm{rms}[G_{\mathrm{sH}}] = (e/4\pi\sqrt{32}) \approx 0.18(e/4\pi)$ in agreement with Ref. 108.

5.4 Numerical Simulation

In the setup of Ref. 108 the universal regime is not very large and thus it is difficult to unambiguously identify it. Moreover, in the same setup all four sides of a square lattice are completely connected to the external leads (see inset to Fig. 1 in Ref. 108). Because of this geometry, there are paths connecting longitudinal to transverse leads that are much shorter than the elastic mean free path. It is well known that such paths contribute nonuniversally to the average conductance. We therefore present numerical simulations in chaotic cavities to further illustrate our analytical predictions (5.10) and (5.12).

We model the electronic dynamics inside a chaotic ballistic cavity by the spin kicked rotator of chapter 2. Averages were performed over 35 values of K in the range $41 < K < 48$, 25 values of ε uniformly distributed in $0 < \varepsilon < 2\pi$, and 10 different lead positions $l^{(k)}$. We set the strength of K_{so} such that $\tau_{\mathrm{so}} = \tau_{\mathrm{dwell}}/1250$, and fixed values of $M = 640$ and $l_0 = 0.2$.

Our numerical results are presented in Fig. 5.2. Two cases were considered, the longitudinally symmetric ($N_1 = N_2$) and asymmetric ($N_1 \neq N_2$) configurations. In both cases, the numerical data fully confirm our predictions that the average spin current vanishes and that the variance of the transverse spin current is universal, i.e. it does not depend on N for large enough value of N. In the asymmetric case $N_4 = 2N_3$, the variance of the spin current in lead 4 is twice as big as in lead 3, giving further

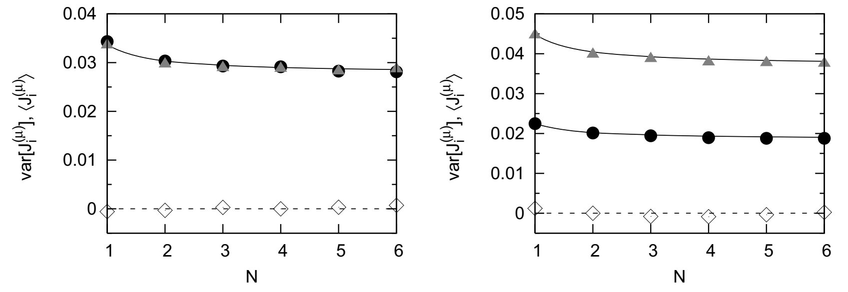

Figure 5.2. Average and variance of the transverse spin current vs. the number of modes. Left panel: longitudinally symmetric configuration with $N_1 = N_2 = 2N_3 = 2N_4 = 2N$; right panel: longitudinally asymmetric configuration with $N_2 = N_4 = 2N_1 = 2N_3 = 2N$. In both cases the total number of modes $N_T = 6N$. The solid (dashed) lines give the analytical prediction (5.10) [(5.12)] for the mean (variance) of the spin currents. Empty diamonds correspond to $\langle J_i^{(\mu)} \rangle$, circles to var $[J_3^{(\mu)}]$ and triangles to var $[J_4^{(\mu)}]$.

confirmation to Eq. (5.12).

5.5 Conclusion

We have calculated the average and mesoscopic fluctuations of the transverse spin current generated by a charge current through a chaotic quantum dot with SO coupling. We find that, from sample to sample, the spin current fluctuates universally around zero average. In particular, for a fully symmetric configuration $N_i \equiv N$, this translates into universal fluctuations of the spin conductance with rms$[G_{\rm sH}] = (e/4\pi\sqrt{32}) \approx 0.18(e/4\pi)$. This universal value is in agreement with the universality observed in the recent simulations in the diffusive regime [108].