Non-decoupling of heavy scalars in cosmology

Hardeman, A.R.

Author: Hardeman, Sjoerd Reimer
Title: Non-decoupling of heavy scalars in cosmology
Date: 2012-06-08

APPENDIX A

Commutation relations for quantum multi-fields

In this appendix we show that the commutation relations (eq. 4.61) in section 4.3.2 are fully consistent with the evolution of the $v^I_\alpha(k,\tau)$ dictated by the set of equations of motion (eq. 4.64). To begin with, observe that in order to satisfy the commutation relation (eq. 4.61) the $\mathcal{N}$ mode solutions $v^I_\alpha(k,\tau)$ must satisfy the following conditions:

$$\sum_\alpha \left[v^I_\alpha \frac{\mathcal{D}v^{J*}_\alpha}{d\tau} - \frac{\mathcal{D}v^J_\alpha}{d\tau} v^{I*}_\alpha \right] = i\delta^{IJ} \,, \tag{A.1}$$

$$\sum_\alpha \left[v^I_\alpha v^{J*}_\alpha - v^J_\alpha v^{I*}_\alpha \right] = 0 \,, \tag{A.2}$$

$$\sum_\alpha \left[\frac{\mathcal{D}v^I_\alpha}{d\tau} \frac{\mathcal{D}v^{J*}_\alpha}{d\tau} - \frac{\mathcal{D}v^J_\alpha}{d\tau} \frac{\mathcal{D}v^{I*}_\alpha}{d\tau} \right] = 0 \,. \tag{A.3}$$

To show that these relations are satisfied at any given time t we proceed as follows: first, let us define the tensors

$$A^{IJ} = i\sum_{\alpha}\left[v_{\alpha}^{I}v_{\alpha}^{J*} - v_{\alpha}^{J}v_{\alpha}^{I*}\right], \tag{A.4}$$

$$B^{IJ} = i\sum_{\alpha}\left[\frac{\mathcal{D}v_{\alpha}^{I}}{d\tau}\frac{\mathcal{D}v_{\alpha}^{J*}}{d\tau} - \frac{\mathcal{D}v_{\alpha}^{J}}{d\tau}\frac{\mathcal{D}v_{\alpha}^{I*}}{d\tau}\right], \tag{A.5}$$

$$E^{IJ} = i\sum_{\alpha}\left[v_{\alpha}^{I}\frac{\mathcal{D}v_{\alpha}^{J*}}{d\tau} - \frac{\mathcal{D}v_{\alpha}^{J}}{d\tau}v_{\alpha}^{I*}\right]. \tag{A.6}$$

These tensors satisfy the properties

$$A^{IJ} = A^{IJ*} = -A^{JI}, \tag{A.7}$$

$$B^{IJ} = B^{IJ*} = -B^{JI}, \tag{A.8}$$

$$E^{IJ} = E^{IJ*}. \tag{A.9}$$

In other words, they are real, with A^{IJ} and B^{IJ} antisymmetric while E^{IJ} has no specific symmetries. Because of these properties A^{IJ} and B^{IJ} consist of $\mathcal{N}(\mathcal{N}-1)/2$ independent real components each, whereas E^{IJ} consists of $\mathcal{N}^2$ independent real components. Thus, in order to fix the values of all of these tensors we need to specify $2\mathcal{N}^2 - \mathcal{N}$ independent quantities. These tensors also satisfy the following equations of motion:

$$\frac{\mathcal{D}}{d\tau}A^{IJ} = E^{IJ} - E^{JI}, \tag{A.10}$$

$$\frac{\mathcal{D}}{d\tau}B^{IJ} = \Omega^{I}{}_{K}E^{KJ} - \Omega^{J}{}_{K}E^{KI}, \tag{A.11}$$

$$\frac{\mathcal{D}}{d\tau}E^{IJ} = B^{IJ} + A^{IK}\left(k^2\delta_K^J + \Omega_K{}^J\right). \tag{A.12}$$

Taking the trace to the last equation, we obtain that the trace $E \equiv E^{I}{}_{I}$ satisfies

$$\frac{dE}{d\tau} = 0, \tag{A.13}$$

and therefore E is a constant of motion of the system. Furthermore, observe that the configuration $E^{IJ} = E\delta^{IJ}/\mathcal{N}$ and $A^{IJ} = B^{IJ} = 0$ for which conditions (eq. A.1-A.3) are satisfied corresponds to a fixed point of the set of equations (eq. A.10-A.12). That is, they automatically satisfy

$$\frac{\mathcal{D}}{d\tau}A^{IJ} = \frac{\mathcal{D}}{d\tau}B^{IJ} = \frac{\mathcal{D}}{d\tau}E^{IJ} = 0. \tag{A.14}$$

Therefore, it only remains to verify whether there exist sufficient independent degrees of freedom in order to satisfy the initial conditions $E^{IJ} = E\delta^{IJ}/\mathcal{N}$ and $A^{IJ} = B^{IJ} = 0$ at a given initial time τ_i. As a matter of fact, we have exactly the right number of degrees of freedom. As we have already noticed there exists $\mathcal{N}$ independent solutions $v^I_\alpha(k,\tau)$ to the equations of motion. To fix each solution $v^I_\alpha(k,\tau)$ we therefore need to specify $2\mathcal{N}^2$ independent quantities, corresponding to the addition of $\mathcal{N}^2$ components $v^I_\alpha(\tau_i)$ and $\mathcal{N}^2$ momenta $\mathcal{D}v^I_\alpha/d\tau(\tau_i)$. However, we must notice that the overall phase of each solution $v^I_\alpha(k,\tau)$ plays no roll in setting the initial values for A^{IJ}, B^{IJ} and E^{IJ}. We therefore have precisely $2\mathcal{N}^2 - \mathcal{N}$ free parameters to set $E^{IJ} = E\delta^{IJ}/\mathcal{N}$ and $A^{IJ} = B^{IJ} = 0$. Of course, the value of the trace of E is part of this freedom, and we are free to fix it in such a way that $E/\mathcal{N} = 1$.

To summarise, it is always possible to choose the initial conditions for $v^I_\alpha(k,\tau)$ and $\mathcal{D}v^I_\alpha/d\tau(k,\tau)$ in such a way that conditions (eq. A.1-A.3) are satisfied. These conditions ensure the commutation relation (eq. 4.61). To finish this discussion, recall that one possible choice for the initial conditions for the perturbations allowing (eq. A.1) to (eq. A.2) to be satisfied, are precisely those expressed in (eq. 4.67), with suitable choices for the coefficients $v_\alpha(k)$ and $\pi_\alpha(k)$:

$$v_\alpha(k)\pi^*_\alpha(k) - v^*_\alpha(k)\pi_\alpha(k) = i\,, \tag{A.15}$$

for $\alpha = 1, \cdots \mathcal{N}$. We should emphasise however that this is not the unique choice for initial conditions and, in general, any choice for which $E^{IJ} = E\delta^{IJ}/\mathcal{N}$ and $A^{IJ} = B^{IJ} = 0$ will do just fine.

APPENDIX B

Zeroth-order theory of the background fields

In this appendix we study in detail the dynamics offered by the tree level potential $V(\phi) = V_*(\phi)$ discussed in Section 4.4.1. We shall focus only on potentials V for which the Hessian V_{ab} is positive definite. Let us for a moment independently consider solutions to the equation

$$V^a = 0\,. \tag{B.1}$$

In general, these will correspond to a set of fields parametrising a surface $\mathcal{S}$ in $\mathcal{M}$. The fields lying on this surface correspond to exactly flat directions of the potential V. Let us express this surface by means of the parametrisation

$$\phi_*^a = \phi_*^a(\chi^\alpha)\,, \tag{B.2}$$

where $\alpha = 1, \cdots n_{\mathcal{S}}$, with $n_{\mathcal{S}}$ the number of flat directions of the potential. Then

$$V_a\left[\phi_*(\chi)\right] = 0 \tag{B.3}$$

for any χ. Clearly, $n_{\mathcal{S}}$ is the dimension of the surface. We may now define the induced metric on the surface by making use of the pullbacks $X^a{}_\alpha \equiv \partial_\alpha \phi_*^a$:

$$g_{\alpha\beta} = X^a{}_\alpha X^b{}_\beta \gamma_{ab}\,. \tag{B.4}$$

Let us for a moment disregard the degrees of freedom perpendicular to this surface and consider only those lying on $\mathcal{S}$. This corresponds to truncating the theory by considering only the fields χ^α. The theory for such fields would be deduced from the action

$$S = -\frac{1}{2}\int d^4x\, g_{\alpha\beta}\partial_\mu\chi^\alpha\partial^\mu\chi^\beta\,, \tag{B.5}$$

and the equations of motion would be given by

$$\frac{D}{dt}\dot{\chi}^\alpha = \frac{d^2\chi^\alpha}{dt^2} + \hat{\Gamma}^\alpha_{\beta\gamma}\frac{d\chi^\beta}{dt}\frac{d\chi^\gamma}{dt} = 0\,, \tag{B.6}$$

where

$$\hat{\Gamma}^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}\right) \tag{B.7}$$

is the connection deduced out of the induced metric $g_{\alpha\beta}$. The relation between $\hat{\Gamma}^\alpha_{\beta\gamma}$ and Γ^a_{bc} is given by

$$\hat{\Gamma}^\alpha_{\beta\gamma} = X_a{}^\alpha\left(X^b{}_\beta X^c{}_\gamma\Gamma^a_{bc} + X^a{}_{\beta\gamma}\right)\,, \tag{B.8}$$

where $X^a{}_{\beta\gamma} \equiv \partial_\gamma X^a{}_\beta$. It is convenient here to define $M^a_{\beta\gamma} \equiv X^b{}_\beta X^c{}_\gamma\Gamma^a_{bc} + X^a{}_{\beta\gamma}$, which yields $\hat{\Gamma}^\alpha_{\beta\gamma} = X_a{}^\alpha M^a_{\beta\gamma}$. To review under what conditions the previous truncation is consistent, let us recall how much a solution to (eq. B.6) deviates from the equation of motion of the full theory given by (eq. 4.6). By differentiating with respect to time the solution (eq. B.2) with χ^α satisfying (eq. B.6), we find

$$\frac{D}{dt}\dot{\phi}^a_*(\chi) = X^a{}_\alpha\ddot{\chi}^\alpha + M^a_{\alpha\beta}\dot{\chi}^\alpha\dot{\chi}^\beta \tag{B.9}$$

$$\implies \quad \frac{D}{dt}\dot{\phi}^a_*(\chi) = \left(M^a_{\alpha\beta} - X^a{}_\gamma X_b{}^\gamma M^b_{\alpha\beta}\right)\dot{\chi}^\alpha\dot{\chi}^\beta\,. \tag{B.10}$$

It is useful to define $Q^a_{\alpha\beta} \equiv P^a_b M^b_{\alpha\beta}$, where $P^a{}_b \equiv \delta^a{}_b - X^a{}_\gamma X_b{}^\gamma$ is the projector along the space perpendicular to the surface. $Q^a_{\alpha\beta}$ transforms as a tensor:

$$Q^a_{\alpha\beta} = \partial_\alpha X^a{}_\beta + \Gamma^a_{b\alpha}X^b_\beta - \hat{\Gamma}^\gamma_{\alpha\beta}X^a{}_\gamma = D_\alpha X^a{}_\beta\,, \tag{B.11}$$

where $\Gamma^a_{b\alpha} \equiv \Gamma^a_{bc}X^c{}_\alpha$. The previous notation is consistent as $X^a{}_\alpha$ transforms homogeneously under reparametrisations of ϕ and χ. Thus, finally we are left with

$$\frac{D}{dt}\dot{\phi}^a_*(\chi) = Q^a_{\alpha\beta}\dot{\chi}^\alpha\dot{\chi}^\beta\,. \tag{B.12}$$

Therefore, since $V^a(\phi_*) = 0$ by definition, if $Q^a_{\alpha\beta}\dot{\chi}^\alpha\dot{\chi}^\beta$ is non-vanishing along the trajectory followed by χ^α, then ϕ^a_* does not satisfy the equations of motion for ϕ^a in the full theory. In fact, since we are interested in an arbitrary solution $\chi^\alpha = \chi^\alpha(t)$

of (eq. B.6), in general either $\dot{\chi}^\alpha = 0$ or $Q^a_{\alpha\beta} = 0$. The first case corresponds to a stationary solution, where the background is not evolving. The second case $Q^a_{\alpha\beta} = 0$ is more interesting, as it corresponds to the case in which $\mathcal{S}$ is geodesically generated. To appreciate this, notice first that if $Q^a_{\alpha\beta} = 0$ then $\phi^a_* = \phi^a_*(t)$ satisfies the equation of a geodesic. In second place, it is possible to deduce the following identity

$$\begin{aligned} \mathcal{R}^a{}_{\alpha\beta\gamma} &\equiv P^a{}_b X^c{}_\alpha X^d{}_\beta X^e{}_\gamma \mathcal{R}^B{}_{cde} \\ &= P^a{}_b \left(D_\beta Q^b_{\gamma\alpha} - D_\gamma Q^b_{\beta\alpha} \right) . \end{aligned} \tag{B.13}$$

Thus, if $Q^a_{\alpha\beta} = 0$ then arbitrary vectors, which are tangent to $\mathcal{S}$, will not generate a component normal to $\mathcal{S}$ after being transported around an arbitrary loop in $\mathcal{S}$. Finally, one also has the general relation

$$\hat{\mathcal{R}}^\alpha{}_{\beta\gamma\delta} = X_a{}^\alpha X^b{}_\beta X^c{}_\gamma X^d{}_\delta \mathcal{R}^a{}_{bcd} + \left(Q^a_{\beta\delta}\gamma_{ab} Q^b_{\sigma\gamma} g^{\sigma\alpha} - Q^a_{\beta\gamma}\gamma_{ab} Q^b_{\sigma\delta} g^{\sigma\alpha} \right) , \tag{B.14}$$

meaning that if $Q^a_{\alpha\beta} = 0$ one has that the Riemann tensor $\hat{\mathcal{R}}^\alpha{}_{\beta\gamma\delta}$ characterising $\mathcal{S}$ coincides with the induced Riemann tensor $X_a{}^\alpha X^b{}_\beta X^c{}_\gamma X^d{}_\delta \mathcal{R}^a{}_{bcd}$ to the surface.

It is rather clear that whenever the surface $\mathcal{S}$ is not geodesically generated, the solution $\phi^A = \phi^A(\chi)$ is not a solution of the full set of equations of motion. Let us now ask under what circumstances this might be a good approximation. For this, consider the following notation for the full solution:

$$\phi^a = \phi^a_* + \Delta^a , \tag{B.15}$$

where Δ^a has the purpose of parametrising the displacement of the full solution from ϕ^a_* defining the surface $\mathcal{S}$. To deduce the equation of motion for Δ^a notice that

$$\begin{aligned} \frac{D\dot{\phi}^a}{dt} &= \ddot{\phi}^a + \Gamma^a_{bc}(\phi)\dot{\phi}^b\dot{\phi}^c \\ &= \ddot{\phi}^a_* + \ddot{\Delta}^a + \Gamma^a_{bc}(\phi_* + \Delta)\left(\dot{\phi}_* + \dot{\Delta}\right)^b \left(\dot{\phi}_* + \dot{\Delta}\right)^c \\ &= \frac{D\dot{\phi}^a_*}{dt} + \ddot{\Delta}^a + \Gamma^a_{bc}(\phi_*)\dot{\Delta}^b\dot{\phi}^c_* + \Gamma^a_{bc}(\phi_*)\dot{\phi}^b_*\dot{\Delta}^c + \partial_d\Gamma^a_{bc}(\phi_*)\dot{\phi}^b_*\dot{\phi}^c_*\Delta^d . \end{aligned} \tag{B.16}$$

On the other hand, we have the relation

$$\frac{D^2\Delta^a}{dt^2} = \left[\dot{\Delta}^a + \Gamma^a_{bc}(\phi_*)\Delta^b\dot{\phi}^c_*\right]^{\cdot} + \Gamma^A_{bc}(\phi_*)\left[\dot{\Delta}^b + \Gamma^b_{de}(\phi_*)\Delta^d\dot{\phi}^e_*\right]\dot{\phi}^c_* . \tag{B.17}$$

Putting these two expressions together we find the equation of motion for Δ^a to be given by

$$\frac{D^2\Delta^a}{dt} + Q^a_{\alpha\beta}\dot{\chi}^\alpha\dot{\chi}^\beta + C^a{}_b(\phi_*)\Delta^b = 0 , \tag{B.18}$$

where we are neglecting terms of higher order in Δ. In the previous expression we have defined

$$C^a{}_b(\phi_*) \equiv V^a{}_b(\phi_*) - \mathcal{R}^a{}_{cdb}(\phi_*)\dot{\phi}^c_*\dot{\phi}^d_* \,, \tag{B.19}$$

where $V^a{}_b(\phi_*) \equiv \gamma^{ac}(\phi_*)\nabla_c V_b(\phi_*)$. In deriving this expression we have assumed that $Q^a_{\alpha\beta}\dot{\chi}^\alpha\dot{\chi}^\beta$ is of $O(\Delta)$. This is correct since we need to demand $\Delta = 0$ for the particular case $Q^a_{\alpha\beta}\dot{\chi}^\alpha\dot{\chi}^\beta = 0$. That is to say, we are strictly interested in the inhomogeneous solution of the previous equation. Notice that the effective mass $C^a{}_b$ contains a contribution from the Riemann tensor. However, the direction given by $\dot{\phi}^a_*$ continues to be a flat direction since $\mathcal{R}^a{}_{cdb}(\phi_*)\dot{\phi}^c_*\dot{\phi}^d_*\dot{\phi}^b_* = 0$. In other words,

$$C^a{}_b(\phi_*)\dot{\phi}^b_* = 0 \,. \tag{B.20}$$

Additionally, notice that C_{ab} is symmetric. To proceed, let us define a few more quantities. First, the tangent vector to the trajectory defined by $\phi_*(t)$ on the surface is given by

$$T^a_* = \frac{\dot{\phi}^a_*}{\dot{\phi}_*} \,, \tag{B.21}$$

where $\dot{\phi}^2_* = \gamma_{ab}\dot{\phi}^a_*\dot{\phi}^b_*$. In fact, notice that

$$T^a_* = X^a{}_\alpha T^\alpha_* \,, \tag{B.22}$$

$$T^\alpha_* = \frac{\dot{\chi}^\alpha}{\dot{\phi}_*} \,, \tag{B.23}$$

$$\dot{\phi}^2_* = g_{\alpha\beta}\dot{\chi}^\alpha\dot{\chi}^\beta \,. \tag{B.24}$$

It is a simple matter to show that

$$\frac{DT^a_*}{dt} = \dot{\phi}_* Q^a_{\alpha\beta}T^\alpha_* T^\beta_* \,, \tag{B.25}$$

$$\ddot{\phi}_* = 0 \,. \tag{B.26}$$

It follows that $N^a_* \propto Q^a_{\alpha\beta}T^\alpha_* T^\beta_*$. It should be clear that $T^b_* V^a{}_b(\phi_*) = 0$, as T^a_* is by definition along the flat directions of the potential. It is useful to consider the definition of the radius of curvature κ_* parametrising the deviation of the trajectory in $\mathcal{S}$ with respect to geodesics in $\mathcal{M}$. The radius of curvature κ_* comes defined as

$$\frac{DT^a_*}{d\phi_*} = -\frac{N^a_*}{\kappa_*} \,, \tag{B.27}$$

and therefore one has

$$\frac{1}{\kappa_*} = -N_{*a}Q^a_{\alpha\beta}T^\alpha_* T^\beta_* = \sqrt{\gamma_{ab}Q^a_{\alpha\beta}T^\alpha_* T^\beta_* Q^b_{\gamma\delta}T^\gamma_* T^\delta_*} \,. \tag{B.28}$$

Notice that this quantity depends only on geometrical objects, as it should. Coming back to (eq. B.18), we may now write

$$\frac{D^2\Delta^a}{dt^2} - \dot{\phi}_*^2 N_*^a \kappa_*^{-1} + C^a{}_b(\phi_*)\Delta^b = 0 \,. \tag{B.29}$$

At this point one may argue that there are no good reasons to consider κ_*^{-1} to be a small parameter. In fact, typically, for theories incorporating modular fields, κ should be of $O(1)$ in Planck units. Since $\dot{\phi}_*$ is constant, it is convenient to parametrise the trajectory with ϕ_*. We can in fact write

$$\frac{D\Delta^a}{dt} = \dot{\phi}_* \frac{D\Delta^a}{d\phi_*} \,, \tag{B.30}$$

$$\frac{D^2\Delta^a}{dt^2} = \dot{\phi}_*^2 \frac{D^2\Delta^a}{d\phi_*^2} \,. \tag{B.31}$$

We can therefore re-express the equation of motion for Δ^a in terms of the proper parameter ϕ_* along the curve:

$$\frac{D^2\Delta^a}{d\phi_*^2} + \frac{1}{\dot{\phi}_*^2} C^a{}_b(\phi_*)\Delta^b = N_*^a \kappa_*^{-1} \,. \tag{B.32}$$

To gain experience with this equation, consider the following situation. Suppose we have a trajectory in field space characterised by a constant curvature κ_* and such that $C^a{}_b N^b = M^2 N^a$ with $M^2 > 0$ a constant. That is, N^a is an eigenvector of $C^a{}_b$. Under such conditions, using the results of section 4.2.1 we find that

$$\frac{D^2 N_*^a}{d\phi_*^2} = -\frac{N_*^a}{\kappa_*^2} \,. \tag{B.33}$$

Then, we can see that $\Delta^a = \Delta\, N^a$ with Δ constant is a solution of the equation, with

$$\Delta = \frac{\dot{\phi}_*^2}{\kappa_*} \left(M^2 - \frac{\dot{\phi}_*^2}{\kappa_*^2} \right)^{-1} \,. \tag{B.34}$$

It is entirely reasonable to expect $M^2 \gg \dot{\phi}_*^2/\kappa_*^2$, which corresponds to the case in which the energy scale of the low energy dynamics is much smaller than the energy scale associated to the heavy fields. In such a case we simply have

$$\Delta \simeq \frac{\dot{\phi}_*^2}{M^2 \kappa_*} \,, \tag{B.35}$$

This is the typical deviation from the true minimum of the potential if the surface of this minimum is not a geodesic, the deviation from which is parametrised by κ_*. To be more general, let us focus on a class of background trajectories in which

$$\frac{D\Delta^a}{d\phi_*} \sim \mathcal{O}\left(\frac{\Delta}{\kappa_*}\right). \tag{B.36}$$

This is a very reasonable situation to look into (our previous example is a particular case of this) as it correspond to those cases in which the main scale encoding the geometrical effects in the trajectory is its curvature. Then, if the non-vanishing eigenvalues of $C^a{}_b$ are much larger than $\dot{\phi}_*^2/\kappa^2$ we can neglect the first term in (eq. B.32) and write

$$C^a{}_b(\phi_*)\Delta^b \simeq \frac{\dot{\phi}_*^2 N^a}{\kappa_*}. \tag{B.37}$$

Thus more generally $\Delta \simeq \dot{\phi}_*^2/(M^2\kappa_*)$ is indeed a good measure of the deviation from the true minimum. Notice that in the case of a system with two scalar fields this is precisely the case.