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CHAPTER 4

Heavy physics in the Cosmic Microwave Background

4.1 Introduction

One particularly compelling possibility which will be discussed in the next two chapters is the possibility of features in the spectrum of perturbations that are generated by heavy – relative to the scale of inflation – degrees of freedom which do not necessarily decouple from the dynamics of the inflaton. Although the effects of massive degrees of freedom on the density perturbations are known to quickly dissipate during inflation, there are evidently still a number of contexts where features in the primordial spectrum due to heavy physics can survive. It is well understood, for example, that departures from a Bunch-Davies vacuum as the initial condition for the scalar fluctuations will result in oscillatory features in the power spectrum (see for example Martin and Brandenberger, 2001, Kempf and Niemeyer, 2001, Easther et al., 2001, Danielsson, 2002, Kaloper et al., 2002 and Schalm et al., 2004. For a recent review, see Jackson and Schalm, 2010). Other contexts in which features are generated in the power spectrum involve particle production during brief intervals – much smaller than an e-fold – as the universe inflates. Examples of this include those situations where a massive field coupled to the inflaton suddenly becomes massless at a specific point in field space (Chung et al., 2000, Elgaroy et al., 2003, Mathews et al., 2004, Romano and Sasaki, 2008, Barnaby and Huang, 2009). Here it is the transfer of energy out of the inflaton field and the subsequent backscatter of its fluctuations off the condensate of created quanta that can result in features in the power spectrum, as well as in its higher moments (Barnaby and Huang, 2009, Barnaby, 2010). Yet another context where such features have been shown to arise is in chain inflation, where instead of slowly rolling down a smooth continuous potential, the inflaton field gradually tunnels a succession of many vacua (Chialva and Danielsson, 2009).

The purpose of these chapters is to understand, in the context of inflation embedded in a multi-scalar field theory, the general conditions under which features in the power spectrum are generated (see work by Langlois and Renaux-Petel 2008, Peterson and Tegmark 2010, Cremonini et al. 2010a for other recent discussions on this). For this we consider models of inflation where all of the scalar fields remain heavy except for one, the inflaton, which rolls slowly in some multi-dimensional potential. An effective field theory analysis tells us that in such scenarios, inflation should proceed in exactly the same way as in the single-field case, with subleading corrections suppressed by the masses of the heavy scalar fields, see for example Weinberg (2008). In this framework it is easy to take for granted that a simple truncation of any available heavy degrees of freedom is the same as having integrated them out. However, it can certainly be the case that the adiabatic approximation is no longer valid at some point along the inflaton trajectory, e.g. due to a “sudden” turn that mixes heavy and light directions, and higher derivative operators in the effective theory are no longer negligible even as inflation continues uninterrupted.

In various models of inflation in supergravity and string theory, the inflaton is embedded in a non-linear sigma model with typical field manifold curvatures of the string or Planck scale (Gomez-Reino and Scrucca, 2006a, Covi et al., 2008b,a). In this type of scenario the inflaton traverses a curvilinear trajectory generating derivative interactions between the adiabatic and non-adiabatic modes[1] (Gordon et al., 2001, Groot Nibbelink and van Tent, 2000, 2002). In this context, it is straightforward to appreciate heuristically that a sudden enough turn can excite modes normal to the trajectory and non-trivially modify the evolution of the adiabatic mode. We will see that the net effect of this trajectory will translate into damped oscillatory features superimposed on the power spectrum – the transients after a sudden transfer of energy between the excited heavy modes and the much lighter inflaton mode, and the subsequent rescattering of its perturbations off the condensate of heavy quanta that redshift

[1]Here, by adiabatic mode we refer to the mode which fluctuates along the inflationary trajectory whereas non-adiabatic modes correspond to those whose fluctuations remain orthogonal to the trajectory. We will also frequently denote them as curvature and isocurvature modes in this chapter.

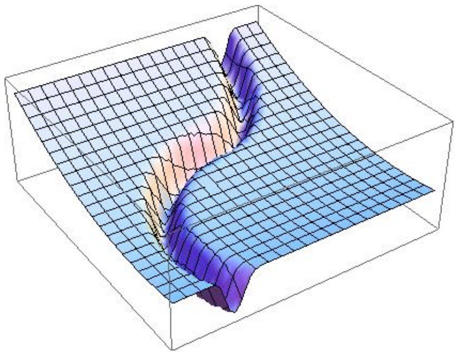

Figure 4.1: *A generic example of a potential where turns happens while one of the fields remain much heavier than the other.*

in short time.[2]

A typical potential exhibiting such a curved trajectory is depicted in figure 4.1. It can be appreciated that there is always a heavy direction transverse to the loci of minima determining the inflaton trajectory. We should emphasise however that the focus of this work is more general and that a curved trajectory in field space is not exclusively due to the shape of the potential, but also depends on the particular sigma model metric defining the scalar field manifold: on a particular curve the two can be transformed into each other by suitable field redefinitions. With this perspective, we will show that curved trajectories appear in any situation where a mismatch exists between the span of geodesics of the scalar field manifold and the actual inflationary trajectory enforced by the scalar potential through the equations of motion. The previously described situation is in fact generic of realisations of inflation in the context of string compactifications, where a large number of scalar fields are expected to remain massive but with their vacuum expectation values depending on the field value of the background inflaton (Blanco-Pillado et al., 2004, Lalak et al., 2007c, Conlon and Quevedo, 2006, Blanco-Pillado et al., 2006b, Simon et al., 2006, Bond et al., 2007, de Carlos et al., 2007, Lalak et al., 2007b, Grimm, 2008, Linde and Westphal, 2008).

Limits of certain cases we wish to study in this chapter have been explored recently in seemingly different, but related contexts. In Chen and Wang (2010a,b), for

[2] We also note the investigations of Tye et al. (2009) and Tye and Xu (2010), where inflation in a putative string landscape is modelled using a random potential. Here, the background inflaton effectively executes a random walk, resulting in features at all scales in the power spectrum.

instance, the effects on density perturbations due to a circular turn with constant curvature in field space was explored within a two-field model. There, it was concluded that such a turn could render non-Gaussian features in the bispectrum but would not generate features in the power spectrum. In another recent publication by Tolley and Wyman (2010), the effects of a sigma model with non-canonical kinetic terms motivated by string theory were explored within inflationary models where one of the fields remained very massive. There, an effective theory was derived describing the multi-field dynamics, characterised by having a speed of sound for the fluctuations smaller than unity (and therefore indicating the possible departure from Gaussianity of the CMB temperature anisotropies). In the framework we are about to discuss, both examples are just different faces of the same coin: while a non-canonical sigma model metric can always be made locally flat along a given trajectory this generally generates contributions to the potential with a curved locus of minima. On the other hand, it is also possible to find a field redefinition which makes the loci of flat directions of the potential look straight at the cost of introducing a non-canonical metric.

We have organised this chapter in the following way. In section 4.2 we present the general setup and the notations used throughout this and the next chapter, extending the work of Groot Nibbelink and van Tent (2000, 2002). There, we will emphasise the need for using a geometric perspective to describe the evolution of the homogeneous background. Then, in section 4.3 we proceed to examine the perturbations of the fields around a time dependent background and consider their quantisation and provide general formulae for the power spectrum. Our formalism allows us to consider situations beyond the regime of applicability of existing methods, such as trajectories with fast, sudden turns (regardless of whether the sigma model metric is canonical or non-canonical), and any other situations in which the masses in the orthogonal direction are changing relatively fast along the trajectory while still remaining much heavier than H^2. Then, in section 4.4 we will first derive the dynamics of turning fields in the Minkowski limit. In this section we will discuss the useful two-field model and its constant turn limit. Additionally, in the limit of large hierarchy we will show that the dynamics can be described by an effective single-field theory for the light field, with a reduced speed of sound. This chapter will be concluded by a discussion on the validity of truncating non-decoupled sectors, as discussed also in chapter 2. In the next chapter we will apply this framework for calculations of features in the inflationary power spectrum.

4.2 Basic considerations

Let us start our study by recalling some of the basic aspects of multi-field inflation and by introducing the notations and conventions that will be used throughout this work. Our starting point is to assume the following effective four dimensional action consisting of gravity and a set of $\mathcal{N}$ scalar fields ϕ^a:

$$S = \int \sqrt{-g}\, d^4x \left[\frac{M_{\rm Pl}^2}{2} R - \frac{1}{2} \gamma_{ab} g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b - V(\phi) \right] . \tag{4.1}$$

Here R denotes the Ricci scalar constructed out of the spacetime metric $g_{\mu\nu}$ with determinant g. Additionally, ϕ^a ($a = 1, \cdots \mathcal{N}$) denotes a set of scalar fields spanning a scalar manifold $\mathcal{M}$ of dimension $\mathcal{N}$, equipped with a scalar metric γ_{ab}. The scalar fields may be thought of as coordinates on $\mathcal{M}$ with Christoffel symbols given by

$$\Gamma^a_{bc} = \frac{1}{2} \gamma^{ad} \left(\partial_b \gamma_{dc} + \partial_c \gamma_{bd} - \partial_d \gamma_{bc} \right) , \tag{4.2}$$

where ∂_a are partial derivatives with respect to the scalar fields ϕ^a. In terms of these, the Riemann tensor associated with $\mathcal{M}$ is given by

$$\mathbb{R}^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce} \Gamma^e_{db} - \Gamma^a_{de} \Gamma^e_{cb} . \tag{4.3}$$

It is also possible to define the Ricci tensor as $\mathbb{R}_{ab} = \mathbb{R}^c{}_{acb}$ and the Ricci scalar $\mathbb{R} = \gamma^{ab} \mathbb{R}_{ab}$. We shall be careful to distinguish geometrical quantities related to the four dimensional spacetime and the $\mathcal{N}$-dimensional abstract manifold $\mathcal{M}$. We should keep in mind that, typically, there will be an energy scale $\Lambda_{\mathcal{M}}$ associated to the curvature of $\mathcal{M}$, and hence, fixing the typical mass scale of the Ricci scalar as $\mathbb{R} \sim \Lambda_{\mathcal{M}}^{-2}$. In many concrete situations, such as the modular sector of string compactifications, the scale $\Lambda_{\mathcal{M}}$ corresponds to the Planck mass $M_{\rm Pl}$. The equations of motion for the scalar fields are given by

$$\Box \phi^a + \Gamma^a_{bc} g^{\mu\nu} \partial_\mu \phi^b \partial_\nu \phi^c = V^a , \tag{4.4}$$

where $V^a \equiv \gamma^{ab} \partial_b V$. In what follows we discuss in detail the homogeneous solutions $\phi^a = \phi^a_0(t)$ to these equations where the scalar fields depend only on time. In this section and section 4.3 we closely follow the formalism of Groot Nibbelink and van Tent (2000, 2002), extended to allow for the possibility of sharp turns before and around horizon exit.

4.2.1 Background solution

We look for background solutions by assuming that all the scalar fields are time dependent $\phi^a = \phi^a_0(t)$, and that spacetime consists of a flat Friedmann-Lemaître-Robertson-Walker (FLRW) geometry (eq. 1.1) of the form

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \ . \tag{4.5}$$

Later on we will also work in conformal time τ, defined through the relation $dt = a\, d\tau$. In this background, the equation of motion (eq. 4.4) describing the evolution of the scalar fields is given by

$$\frac{D}{dt}\dot{\phi}^a_0 + 3H\dot{\phi}^a_0 + V^a = 0 \ , \tag{4.6}$$

where $H = \dot{a}/a$ is the Hubble parameter characterising the expansion rate of spatial slices, and where we have also introduced the convenient notation $DX^a = dX^a + \Gamma^a_{bc}X^b d\phi^c_0$. On the other hand, the Friedmann equation describing the evolution of the scale factor (eq. 1.2) in terms of the scalar field energy density is given by

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\left(\frac{1}{2}\dot{\phi}_0^2 + V\right) \ , \tag{4.7}$$

where $\dot{\phi}_0^2 \equiv \gamma_{ab}\dot{\phi}^a_0\dot{\phi}^b_0$. We thus see that $\dot{\phi}_0$ corresponds to the rate of change of the scalar field vacuum expectation value along the trajectory followed by the background fields, we will assume that $\dot{\phi}_0 > 0$ everywhere. It is also convenient to recall the following equation describing the variation of H,

$$\dot{H} = -\frac{\dot{\phi}_0^2}{2M_{\rm Pl}^2} \ , \tag{4.8}$$

which may be deduced by combining (eq. 4.6) and (eq. 4.7). By specifying the metric γ_{ab} and the scalar potential V, these equations can be solved to obtain the curved trajectory in $\mathcal{M}$ followed by the scalar fields. To discuss several features of this trajectory without explicitly solving the previous equations, it is useful to define unit vectors T^a and N^a distinguishing tangent and normal directions to the trajectory respectively, in such a way that $T^aN_a = 0$. These are defined as

$$\begin{aligned} T^a \equiv& \frac{\dot{\phi}^a_0}{\dot{\phi}_0} \ , \\ N^a \equiv& s_N(t)\left(\gamma_{bc}\frac{DT^b}{dt}\frac{DT^c}{dt}\right)^{-1/2}\frac{DT^a}{dt} \ , \end{aligned} \tag{4.9}$$

where $s_N(t) = \pm 1$, denoting the orientation of N^a with respect to the vector DT^a/dt. That is, if $s_N(t) = +1$ then N^a is pointing in the same direction as DT^a/dt, whereas if $s_N(t) = -1$ then N^a is pointing in the opposite direction. Due to the presence of the square root, it is clear that N^a is only well defined at intervals where $DT^a/dt \neq 0$. However, since DT^a/dt may become zero at finite values of t, we allow $s_N(t)$ to flip signs each time this happens in such a way that both N^a and DT^a/dt remain a continuous function of t. This implies that the sign of s_N may be chosen conventionally at some initial time t_i, but from then on it is subject to the equations of motion respected by the background.[3] In the particular case where $\mathcal{M}$ is two dimensional, the presence of $s_N(t)$ in (eq. 4.9) is sufficient for N^a to have a fixed orientation with respect to T^a (either left-handed or right-handed). This will become particularly useful when we examine two dimensional models in sections 4.4 and 5.2.

Observe that the tangent vector T^a offers an alternative way of defining the total time derivative D/dt along the trajectory followed by the scalar fields,

$$\frac{D}{dt} \equiv \dot{\phi}_0 T^a \nabla_a = \dot{\phi}_0 \nabla_\phi \ . \tag{4.10}$$

Now, taking a total time derivative of T^a, we may use the equation of motion (eq. 4.6) to write

$$\frac{DT^a}{dt} = -\frac{\ddot{\phi}_0}{\dot{\phi}_0} T^a - \frac{1}{\dot{\phi}_0}\left(3H\dot{\phi}_0^a + V^a\right) \ . \tag{4.11}$$

Then, by projecting this equation along the two orthogonal directions T^a and N^a, we obtain the following two independent equations

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V_\phi = 0 \ , \tag{4.12}$$

$$\frac{DT^a}{dt} = -\frac{V_N}{\dot{\phi}_0} N^a \ , \tag{4.13}$$

where we have defined $V_\phi \equiv T^a V_a$ and $V_N \equiv N^a V_a$ to be the projections of $V_a = \partial_a V$ along the tangent and normal directions respectively. It is not difficult to verify that V_a lies entirely along a space spanned by T^a and N^a. That is, we are allowed to write $V_a \equiv V_\phi T_a + V_N N_a$. To anticipate the study of inflation within the present setup, it is

[3] We are assuming here that the background solutions $\phi^a = \phi_0^a(t)$ are analytic functions of time and that $\dot{\phi}_0$ is nonvanishing. Under these conditions this procedure can always be performed.

useful to define the dimensionless quantities[4]

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}_0^2}{2M_{\rm Pl}^2 H^2} , \tag{4.14}$$

$$\eta^a \equiv -\frac{1}{H\dot{\phi}_0}\frac{D\dot{\phi}_0^a}{dt} , \tag{4.15}$$

which are the multi-field equivalents of (eq. 1.5). We will not assume that these parameters are small until much later, where inflation is studied in the slow-roll regime (see section 5.3 in the next chapter). Similarly to the case of V_a, the vector η^a may be decomposed entirely in terms of T^a and N^a as

$$\eta^a = \eta_{||} T^a + \eta_{\perp} N^a , \tag{4.16}$$

$$\eta_{||} \equiv -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} , \tag{4.17}$$

$$\eta_{\perp} \equiv \frac{V_N}{\dot{\phi}_0 H} , \tag{4.18}$$

where we have used (eq. 4.6) to simplify a few expressions. Observe that $\eta_{\perp}$ is directly related to the rate of change of the tangent unit vector T^a, since (eq. 4.13) can be written as

$$\frac{DT^a}{dt} = -H\eta_{\perp} N^a . \tag{4.19}$$

Comparison with (eq. 4.9) shows that $\mathrm{sign}(\eta_{\perp}) = -s_N$. This is one of our main reasons for having introduced $s_N(t)$ in (eq. 4.9): it allows us to keep $\eta_{\perp}$ continuous and avoid some unnecessary difficulties encountered in the definition of isocurvature modes.[5]

Moving on with this discussion, we can relate $\eta_{\perp}$ to the radius of curvature κ characterising the bending of the trajectory followed by the scalar fields. To do so, let us recall that given a curve $\gamma(\phi_0)$ in field space parametrised by $d\phi_0 = \dot{\phi}_0 dt$, we may define the radius of curvature κ associated to that curve through the relation

$$\frac{1}{\kappa} = \left(\gamma_{bc}\frac{DT^b}{d\phi_0}\frac{DT^c}{d\phi_0}\right)^{1/2} . \tag{4.20}$$

[4]Note that our definition of η^a differs from the definition in Groot Nibbelink and van Tent (2000, 2002) by a minus sign.

[5]In Peterson and Tegmark (2010) for instance, a similar parameter $\eta_{\perp}$ is introduced but with a fixed sign. Partly due to this choice their numerical results cannot handle an overshoot that is occurs when a potential turns from one direction to another.

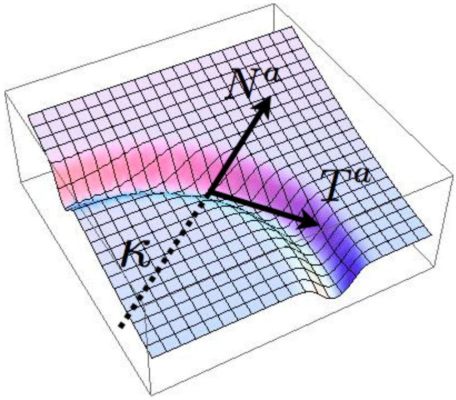

Figure 4.2: *The figure shows schematically the relation between the tangent vector T^a, the normal vector N^a and the radius of curvature κ.*

Here κ stands for the radius of curvature in the scalar manifold $\mathcal{M}$ spanned by the ϕ^a fields, and therefore it has dimension of mass. Figure 4.2 shows the relation between the pair of vectors T^a, N^a and the radius of curvature κ. Using (eq. 4.10) and comparing the last two equations we find that κ and $\eta_\perp$ are related as

$$\kappa^{-1} = \frac{H|\eta_\perp|}{\dot{\phi}_0} \ . \tag{4.21}$$

By definition any autoparallel curve, a curve parallel to a geodesic, $\gamma(\phi_0)$ in $\mathcal{M}$ satisfies the relation $D\dot{\phi}^a/dt \propto \dot{\phi}^a$, which corresponds to the case $\kappa^{-1} = 0$, or alternatively, to the case $\eta_\perp = 0$. Thus, we see that the dimensionless parameter $\eta_\perp$ is a useful quantity that parametrises the bending of the inflationary trajectory with respect to geodesics in $\mathcal{M}$. It is interesting to rewrite the previous relation by replacing $\dot{\phi}_0 = \sqrt{2\epsilon} H M_{\rm Pl}$ coming from the definition of ϵ presented in (eq. 4.14), obtaining

$$|\eta_\perp| = \sqrt{2\epsilon}\frac{M_{\rm Pl}}{\kappa} \ . \tag{4.22}$$

Then, if the radius of curvature is such that $\kappa \ll M_{\rm Pl}$, one already sees that $\eta_\perp^2 \gg 2\epsilon$. We shall come back to this result later when we study curved trajectories in the slow-roll regime $\epsilon \ll 1$. To continue, we may further characterise the variation of N^a as

$$\frac{DN^a}{dt} = H\eta_\perp T^a + \frac{1}{H\eta_\perp} P^{ab}\nabla_\phi V_b \ , \tag{4.23}$$

where we have defined the projector tensor $P^{ab} \equiv \gamma^{ab} - T^a T^b - N^a N^b$ along the space orthogonal to the subspace spanned by the unit vectors T^a and N^a. That is, $P_{ab}N^b = 0$ and $P_{ab}T^b = 0$.

To obtain (eq. 4.23) we proceed as follows: first, by taking a total time derivative to (eq. 4.6) we obtain

$$\frac{1}{\dot{\phi}_0}\frac{D^2\dot{\phi}_0^a}{dt^2} = 3H^2(\epsilon T^a + \eta^a) - \nabla_\phi V^a \,, \tag{4.24}$$

where $\nabla_\phi \equiv T^a\nabla_a$. Recalling that $T^a = \dot{\phi}_0^a/\dot{\phi}_0$, the previous equation can be re-expressed as

$$\frac{D^2T^a}{dt^2} = T^a\nabla_\phi V_\phi - \nabla_\phi V^a - \frac{\left(V_\phi - \ddot{\phi}_0\right)V_N}{\dot{\phi}_0^2}N^a \,. \tag{4.25}$$

On the other hand, taking a total time derivative to (eq. 4.13) we may obtain yet another expression for the second variation D^2T^a/dt^2, given by

$$\frac{D^2T^a}{dt^2} = \left(\frac{V_N\ddot{\phi}_0}{\dot{\phi}_0^2} - \frac{\dot{V}_N}{\dot{\phi}_0}\right)N^a - \frac{V_N}{\dot{\phi}_0}\frac{DN^a}{dt} \,. \tag{4.26}$$

Equating the last two expressions and performing some straightforward algebraic manipulations, we finally obtain (eq. 4.23).

To finish this section, let us state some useful relations that will be used throughout the rest of this work. First, by using the definitions for ϵ and $\eta_\parallel$ in (eqs. 4.14 and 4.17), we may rewrite the background equations (eq. 4.7) and (eq. 4.12) respectively as:

$$3 - \epsilon = \frac{V}{M_{\rm Pl}^2H^2} \,, \tag{4.27}$$

$$3 - \eta_\parallel = -\frac{V_\phi}{\dot{\phi}_0 H} \,. \tag{4.28}$$

With the help of (eq. 4.14) these two relations may be put together to yield:

$$\epsilon = \frac{M_{\rm Pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2\left(\frac{3-\epsilon}{3-\eta_\parallel}\right)^2 \,. \tag{4.29}$$

Next, by deriving (eq. 4.12) with respect to time and using the definitions for ϵ and

$\eta_{\|}$, we deduce[6]

$$3(\epsilon + \eta_{\|}) = M_{\rm Pl}^2 \frac{\nabla_\phi V_\phi}{V}(3-\epsilon) + \xi_{\|}\eta_{\|} \,, \tag{4.30}$$

$$\xi_{\|} \equiv -\frac{1}{H\ddot{\phi}_0}\dddot{\phi}_0. \tag{4.31}$$

Both (eq. 4.29) and (eq. 4.30) are exact equations linking the evolution of background quantities with the scalar potential V. It may be already noticed that if ϵ, $\eta_{\|}$ and $\xi_{\|}$ are all much smaller than unity, then we obtain the usual relations for the slow-roll parameters ϵ and $\eta_{\|}$ in terms of derivatives of the potential (see eqs. 1.5-1.7):

$$\epsilon \approx \frac{M_{\rm Pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2 \,, \tag{4.32}$$

$$\epsilon + \eta_{\|} \approx M_{\rm Pl}^2 \frac{\nabla_\phi V_\phi}{V} \,. \tag{4.33}$$

We shall come back to these relations later, when we consider the evolution of the background in the slow-roll regime.

4.3 Perturbation theory

The notation introduced in the previous section provides a useful tool to analyse perturbations $\delta\phi^a$ about the background solution $\phi^a = \phi_0^a(t)$ by decomposing them into parallel and normal components with respect to the inflaton trajectory. In what follows we proceed to study the evolution and quantisation of these perturbations. First, we consider scalar field perturbations by expanding about the background $\phi^a(t, \boldsymbol{x}) = \phi_0^a(t) + \delta\phi^a(t, \boldsymbol{x})$. It is well known that the equations of motion for the perturbed fields can be cast entirely in terms of the gauge-invariant Sasaki-Mukhanov variables (Sasaki, 1986, Mukhanov, 1988)

$$Q^a \equiv \delta\phi^a + \frac{\dot{\phi}^a}{H}\psi \,, \tag{4.34}$$

where ψ is the curvature perturbation of the spatial metric. The equations of motion for these fields are found to be (Sasaki and Stewart, 1996)

$$\frac{D^2 Q^a}{dt^2} + 3H\frac{DQ^a}{dt} - \frac{\nabla^2}{a^2}Q^a + C^a{}_b Q^b = 0 \,, \tag{4.35}$$

[6]Note that this definition for $\xi_{\|}$ is different from the definition used in Groot Nibbelink and van Tent (2000, 2002).

where $\nabla^2 \equiv \delta^{ij}\partial_i\partial_j$ is the spatial Laplacian and where the tensor $C^a{}_b$ is defined as

$$C^a{}_b \equiv \nabla_b V^a - \dot{\phi}_0^2 \mathbb{R}^a{}_{cdb} T^c T^d + 2\epsilon \frac{H}{\dot{\phi}_0} (T^a V_b + T_b V^a) + 2\epsilon(3-\epsilon) H^2 T^a T_b \; . \tag{4.36}$$

We notice here that $C_{ab} = \gamma_{ac} C^c{}_b$ is symmetric. It is convenient to rewrite the set of equations (eq. 4.35) in terms of perturbations orthogonal to each other. With this in mind, we introduce a complete set of vielbeins $e^I_a = e^I_a(t)$ and work with the following quantities:

$$Q^I(t, \boldsymbol{x}) \equiv e^I_a(t) Q^a(t, \boldsymbol{x}) \; . \tag{4.37}$$

The a-index labels the abstract scalar manifold $\mathcal{M}$ whereas the I-index labels a local orthogonal frame moving along the inflationary trajectory. Recall that vielbeins are defined to satisfy the basic relations $e^I_a e^J_b \gamma^{ab} = \delta^{IJ}$ and $e^I_a e^J_b \delta_{IJ} = \gamma_{ab}$. From these relations one deduces the identities

$$e^I_a \frac{D}{dt} e^a_J = - e^a_J \frac{D}{dt} e^I_a \; , \tag{4.38}$$
$$e^a_I \frac{D}{dt} e^I_b = - e^I_b \frac{D}{dt} e^a_I \; , \tag{4.39}$$

from which it is possible to read

$$\dot{Q}^I = e^I_a \frac{DQ^a}{dt} - Y^I{}_J Q^J \; , \tag{4.40}$$
$$\ddot{Q}^I = e^I_a \frac{D^2 Q^a}{dt^2} - 2Y^I{}_J \dot{Q}^J - \left(Y^I{}_K Y^K{}_J + \dot{Y}^I{}_J \right) Q^J \; , \tag{4.41}$$

where the antisymmetric matrix $Y_{IJ} = -Y_{JI}$ is defined as

$$Y^I{}_J = e^I_a \frac{De^a_J}{dt} \; . \tag{4.42}$$

Before writing down the equations of motion respected by the fields Q^I, it is useful to notice that the matrix Y_{IJ} allows us to define a new covariant derivative $\mathcal{D}/dt$ acting on quantities such as Q^I labelled with the I-index in the following way[7]:

$$\frac{\mathcal{D}}{dt} Q^I \equiv \dot{Q}^I + Y^I{}_J Q^J \; . \tag{4.43}$$

[7] It may be noticed that we can write $Y^I{}_J = \left(e^I_a \partial_b e^a_J + e^I_a \Gamma^a_{bc} e^c_J \right) \dot{\phi}^b_0 = \omega_b{}^I{}_J \dot{\phi}^b_0$ where $\omega_b{}^I{}_J$ are the usual spin connections for non-coordinate basis, hence justifying the definition of the new covariant derivative of (eq. 4.43).

This definition allows us to rearrange (eq. 4.40) and (eq. 4.41) and simply write

$$\frac{\mathcal{D}Q^I}{dt} = e^I_a \frac{DQ^a}{dt} , \tag{4.44}$$

$$\frac{\mathcal{D}^2 Q^I}{dt^2} = e^I_a \frac{D^2 Q^a}{dt^2} . \tag{4.45}$$

Thus, the equations of motion for the perturbations in the new basis become

$$\frac{\mathcal{D}^2 Q^I}{dt^2} + 3H \frac{\mathcal{D}Q^I}{dt} - \frac{\nabla^2}{a^2} Q^I + C^I{}_J Q^J = 0 , \tag{4.46}$$

where $C_{IJ} \equiv e_{Ia} e^b_J C^a{}_b$. To deal with this set of equations, it is convenient to take one last step in simplifying them and rewrite them in terms of conformal time $d\tau = dt/a$, and a new set of perturbations $v^I \equiv aQ^I$. These redefinitions induce a re-scaling of the covariant derivative (eq. 4.43) in the form $\mathcal{D}/d\tau = a\mathcal{D}/dt$, from where we are allowed to write

$$\frac{\mathcal{D}v^I}{d\tau} = \frac{dv^I}{d\tau} + Z^I{}_J v^J , \tag{4.47}$$

where $Z_{IJ} = aY_{IJ}$. Then, the equations of motion for the v^I-perturbations are found to be

$$\frac{\mathcal{D}^2 v^I}{d\tau^2} - \nabla^2 v^I + \Omega^I{}_J v^J = 0 , \tag{4.48}$$

where $\Omega_{IJ} = -a^2 H^2 (2 - \epsilon)\delta_{IJ} + a^2 C_{IJ}$ and we have used the definition of ϵ to write $a''/a = a^2 H^2 (2 - \epsilon)$. For completeness, we notice that the equations of motion (eq. 4.46) may be derived from the action (Groot Nibbelink and van Tent, 2000, 2002)

$$S = \frac{1}{2} \int d\tau d^3x \left[\sum_I \left(\frac{\mathcal{D}v^I}{d\tau} \right)^2 - \sum_I (\nabla v^I)^2 - \Omega_{IJ} v^I v^J \right] , \tag{4.49}$$

which can be alternatively deduced directly from the initial action (eq. 4.1) by considering all of the field redefinitions introduced in the present discussion.

The set of equations (eq. 4.48) contains several non-trivial features. First, notice that the covariant derivative $\mathcal{D}/d\tau$ implies the existence of non-trivial couplings affecting the kinetic term of each field v^I. By the same token, under general circumstances the symmetric matrix Ω_{IJ} does not remain diagonal at all times. In fact, as we are about to see in the next section, it is possible to choose to write this theory either in a frame where the $\mathcal{N}$ scalar fields are canonical (and therefore without non-trivial couplings in the kinetic term), or either in a frame where Ω_{IJ} remains diagonal, but (in general) not both at the same time.

4.3.1 Canonical frame

Observe that by introducing the vielbeins e^I_a in the previous section, we have not specified any alignment of the moving frame. In fact, given an arbitrary frame, characterised by the set e^I_a, it is always possible to find a canonical frame where the scalar field perturbations acquire canonical kinetic terms in the action. To find it, let us introduce a new set of fields u^I defined out of the original fields v^I in the following way:

$$v^I(\tau, \boldsymbol{x}) = R^I{}_J(\tau, \tau_i) u^J(\tau, \boldsymbol{x}) , \tag{4.50}$$

where $R^I{}_J(\tau, \tau_i)$ is an invertible matrix defined to satisfy the first order differential equation

$$\frac{d}{d\tau} R^I{}_J = -Z^I{}_K R^K{}_J , \tag{4.51}$$

with the boundary condition $R^I{}_J(\tau_i, \tau_i) = \delta^I{}_J$ set at some given initial time τ_i. Let us additionally define a new matrix $S^I{}_J$ to be the inverse of $R^I{}_J$, i.e. $S^I{}_K R^K{}_J = R^I{}_K S^K{}_J = \delta^I{}_J$. Then, $S^I{}_J$ satisfies the similar equation

$$\frac{d}{d\tau} S_I{}^J = -Z^J{}_K S_I{}^K , \tag{4.52}$$

where we used the fact that $Z_{IJ} = -Z_{JI}$. Since both solutions to (eq. 4.51) and (eq. 4.52) are unique, then the previous equation tells us that $S_{IJ} = R_{JI}$, that is, S corresponds to R^T the transpose of R. This means that for a fixed time τ, $R_{IJ}(\tau, \tau_i)$ is an element of the orthogonal group O($\mathcal{N}$), the group of matrices R satisfying $RR^T = \mathbb{1}$. The solution to (eq. 4.51) is well known, and may be symbolically written as

$$R(\tau, \tau_i) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\tau_i}^{\tau} \mathcal{T}\left[Z(\tau_1) \cdots Z(\tau_n)\right] d^n\tau = \mathcal{T} \exp\left[- \int_{\tau_i}^{\tau} d\tau \, Z(\tau) \right] , \tag{4.53}$$

where $\mathcal{T}$ stands for the usual time ordering symbol, that is $\mathcal{T}[Z(\tau_1)Z(\tau_2) \cdots Z(\tau_n)]$ corresponds to the product of n matrices $Z(\tau_i)$ for which $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$. Coming back to the u^I-fields, it is possible to see now that, by virtue of (eq. 4.51) one has

$$\frac{\mathcal{D} v^I}{d\tau} = R^I{}_J \frac{du^J}{d\tau} , \tag{4.54}$$

$$\frac{\mathcal{D}^2 v^I}{d\tau^2} = R^I{}_J \frac{d^2 u^J}{d\tau^2} . \tag{4.55}$$

Inserting these relations back into the equation of motion (eq. 4.48) we obtain the following equation of motion for the u^I-fields:

$$\frac{d^2 u^I}{d\tau^2} - \nabla^2 u^I + \left[R^T(\tau)\, \Omega \, R(\tau)\right]^I{}_J u^J = 0 . \tag{4.56}$$

Additionally, it is possible to show that the action (eq. 4.49) is now given by

$$S = \frac{1}{2}\int d\tau d^3x \left\{ \sum_I \left(\frac{du^I}{d\tau}\right)^2 - \left(\nabla u^I\right)^2 - \left[R^T(\tau)\Omega R(\tau)\right]_{IJ} u^I u^J \right\} . \tag{4.57}$$

Thus, we see that the fields u^I correspond to the canonical fields in the usual sense. This result shows, just as we have stated, that it is always possible to find a frame where the perturbations become canonical, but at the cost of having a mass matrix $\left[R^T(\tau)\,\Omega\,R(\tau)\right]_{IJ}$ with non-diagonal entries which are changing continuously in time. Another way to put it is that, while both $R^T(\tau)\,\Omega\,R(\tau)$ and Ω share the same eigenvalues, as long as $R(\tau)$ varies in time, their associated eigenvectors will not remain aligned. To finish, let us notice that by construction, at the initial time τ_i, the canonical fields u^I and the original fields v^I coincide $u^I(\tau_i) = v^I(\tau_i)$. However, it is always possible to redefine a new set of canonical fields by performing an orthogonal transformation of the fields.

4.3.2 Quantisation and initial conditions

Having the canonical frame at hand, we may now quantise the system in the standard way. Starting from the action (eq. 4.57) it is possible to see that the canonical coordinate fields are given by u^I whereas the canonical momentum is given by $\Pi_u^I = du^I/d\tau$. To quantise the system, we demand this pair to satisfy the commutation relation

$$\left[u^I(\tau, \boldsymbol{x}), \Pi_u^J(\tau, \boldsymbol{y})\right] = i\delta^{IJ}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) , \tag{4.58}$$

otherwise zero. With the help of the R transformation introduced in (eq. 4.50) we can rewrite this commutation relation to be valid in an arbitrary moving frame. More precisely, we observe here that we are allowed to define a new pair of fields v^I and Π_v^I given by

$$v^I = R^I{}_J u^J , \tag{4.59}$$

$$\Pi_v^I \equiv \frac{\mathcal{D}}{d\tau} v^I = R^I{}_J(\tau, \tau_i)\Pi_u^J . \tag{4.60}$$

From (eq. 4.58), this new pair is found to satisfy the similar commutation relations

$$\left[v^I(\tau, \boldsymbol{x}), \Pi_v^J(\tau, \boldsymbol{y})\right] = i\delta^{IJ}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) . \tag{4.61}$$

Following convention, it is now possible to obtain an explicit expression for $v^I(\mathrm{x}, \tau)$ in terms of creation and annihilation operators.[8] For this, let us write $v^I(\mathbf{x}, \tau)$ as a sum of Fourier modes:

$$\begin{aligned} v^I(\tau, \boldsymbol{x}) &= \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} v^I(\boldsymbol{k}, \tau) \\ &= \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \sum_\alpha \left[v^I_\alpha(k,\tau)\, a_\alpha(\boldsymbol{k}) + v^{I*}_\alpha(k,\tau)\, a^\dagger_\alpha(-\boldsymbol{k}) \right] . \end{aligned} \tag{4.62}$$

In writing the previous expression we have anticipated the need of expressing the fields $v^I(\tau, \boldsymbol{x})$ as a linear combination of $\mathcal{N}$ time-independent creation and annihilation operators $a^\dagger_\alpha(\boldsymbol{k})$ and $a_\alpha(\boldsymbol{k})$ respectively, with $\alpha = 1, \cdots \mathcal{N}$. These operators are required to satisfy the usual relations

$$\left[a_\alpha(\boldsymbol{k}), a^\dagger_\beta(\boldsymbol{q}) \right] = \delta_{\alpha\beta}\delta^{(3)}(\boldsymbol{k} - \boldsymbol{q}), \tag{4.63}$$

otherwise zero. This set of operators defines the vacuum $|0\rangle$ of the theory by their action $a_\alpha(\boldsymbol{k})|0\rangle = 0$. Since the operators $a^\dagger_\alpha(\boldsymbol{k})$ and $a_\alpha(\boldsymbol{k})$, for different values of α, are taken to be linearly independent, then the time-dependent coefficients $v^I_\alpha(k, \tau)$ appearing in front of them in (eq. 4.62) must satisfy the equation of motion[9]

$$\frac{\mathcal{D}^2}{d\tau^2} v^I_\alpha(k,\tau) + k^2 v^I_\alpha(k,\tau) + \Omega^I{}_J v^J_\alpha(k,\tau) = 0 . \tag{4.64}$$

Observe that there must exist $\mathcal{N}$ independent solutions $v^I_\alpha(k, \tau)$ to this equation (see appendix A for a detailed discussion on the $v^I_\alpha(k, \tau)$-functions).

Of course, a critical issue here is to set the correct initial conditions for the mode amplitudes $v^I_\alpha(k, \tau)$ in such a way that the commutation relations (eq. 4.61) are respected at all times. As a first step towards determining these initial conditions we notice that at a given initial time $\tau = \tau_i$ we may choose each mode $v^I_\alpha(k, \tau)$ to satisfy the following general initial conditions:

$$v^I_\alpha(k, \tau_i) = e^I_\alpha v_\alpha(k) , \tag{4.65}$$

$$\frac{\mathcal{D}v^I_\alpha}{dt}(k, \tau_i) = e^I_\alpha \pi_\alpha(k) , \tag{4.66}$$

[8] From this point on, we continue working with the more general v^I-fields instead of the canonical u^I-fields. Nevertheless, we emphasise that the u^I-fields allowed us to find the correct quantisation prescription for the v^I-fields.

[9] It is crucial to appreciate that the Greek indices α label scalar quantum modes and not directions in field space, as capital Latin indices do. Different α-modes may contribute to the same fluctuation along a given direction I. The quantities linking these two different abstract spaces are the mode functions $v^I_\alpha(k, \tau)$ whose time evolution is dictated by (eq. 4.64). A similar scheme to quantise a coupled multi-scalar field system may be found in Nilles et al. (2001).

where e^I_α is a complete set of unit vectors satisfying $\delta_{IJ} e^I_\alpha e^J_\beta = \delta_{\alpha\beta}$ and $\delta^{\alpha\beta} e^I_\alpha e^J_\beta = \delta^{IJ}$, which should not be confused with the vielbeins defined in (eq. 4.37), and $v_\alpha(k)$ and $\pi_\alpha(k)$ are factors defining the amplitude of the initial conditions. In order for the commutation relations to be fulfilled, these initial conditions must satisfy

$$v_\alpha(k)\pi^*_\alpha(k) - v^*_\alpha(k)\pi_\alpha(k) = i \; , \tag{4.67}$$

which are the analogous relations to the Wronskian condition in single-field slow-roll inflation. Since the operator $\mathcal{D}/d\tau = d/d\tau + Z$ mixes different directions in the v^I-field space and since in general the time-dependent matrix Ω_{IJ} is non-diagonal, then the mode solutions $v^I_\alpha(k,\tau)$ satisfying the initial conditions (eq. 4.67) will not remain pointing in the same direction (nor will they remain orthogonal) at an arbitrary time $\tau \neq \tau_i$. In Appendix A we show that the commutation relations of (eq. 4.61) are consistent with the evolution of the $v^I_\alpha(\tau,k)$ dictated by the set of equations of motion (eq. 4.64).

In the previous expressions the set of unit vectors e^I_α are arbitrary. Moreover, the amplitudes $v_\alpha(k)$ and $\pi_\alpha(k)$ entering (eq. 4.67) are in general not uniquely determined, as there is a family of solutions parametrised by the relative phase between $v_\alpha(k)$ and $\pi_\alpha(k)$. Indeed, notice that without loss of generality we may write

$$\pi_\alpha(k) = \frac{e^{-i\theta_\alpha(k)}}{2v^*_\alpha(k)\sin\theta_\alpha(k)} \; , \tag{4.68}$$

where $\theta_\alpha(k)$ is a set of real phases relating both amplitudes. Any value of $\theta_\alpha(k)$ will satisfy the commutation relations (eq. 4.61), and therefore they specify different choices for the vacuum state $|0\rangle$. Although in general it is not possible to decide among all the possible values for $\theta_\alpha(k)$, fortunately, in the context of inflationary backgrounds $a \to 0$ as $\tau \to -\infty$ and a particular choice for these phases becomes handy. Indeed, observe that in the formal limit $a \to 0$ one has $Z_{IJ} \to 0$ and $\Omega_{IJ} \to 0$, which is made explicit by (eq. 4.47) and (eq. 4.48), and the equations of motion (eq. 4.64) become

$$\left(\frac{d^2}{d\tau^2} + k^2\right) v^I_\alpha(k,\tau) = 0 \; , \qquad (\tau \to -\infty) \; . \tag{4.69}$$

In this limit there is no mixing between different α-modes and perturbations evolve as if they were in Minkowski background.[10] In this case, we are free to choose $e^I_\alpha = \delta^I_\alpha$

[10] To be more rigorous, in inflationary backgrounds this limit is obtained for k-modes such that their wavelength is much smaller than the de Sitter scale $k^2 \gg a^2 H^2$.

and the solutions to (eq. 4.69) satisfying the commutation relations (eq. 4.61) may be chosen as

$$v^I_\alpha(k,\tau) = \delta^I_\alpha \frac{1}{\sqrt{2k}} e^{-ik\tau} \,, \qquad (\tau \to -\infty) \,. \tag{4.70}$$

Thus we see that in the limit $a \to 0$ ($\tau \to -\infty$) we may choose modes in the Bunch-Davies vacuum $\theta_\alpha = \pi/2$. We will come back to these conditions in the next chapter, in section 5.4, where we set initial conditions on a finite initial time surface where (eq. 4.70) cannot be exactly imposed.

4.3.3 Two-point correlation function

To finish this general discussion on multi-field perturbations, we proceed to define the spectrum for the perturbations $v^I(\tau,\boldsymbol{x})$. The power spectrum, the Fourier transform of the two-point correlation function, is defined in terms of the Fourier modes as

$$\left\langle 0 \left| v^I(\boldsymbol{k},\tau) v^{J*}(\boldsymbol{q},\tau) \right| 0 \right\rangle \equiv \delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}) \frac{2\pi^2}{k^3} \mathcal{P}^{IJ}_v(k,\tau) \,. \tag{4.71}$$

In terms of the mode amplitudes $v^I_\alpha(\tau,k)$, this is found to be

$$\mathcal{P}^{IJ}_v(k,\tau) = \frac{k^3}{2\pi^2} \sum_\alpha v^I_\alpha(\tau,k) v^{J*}_\alpha(k,\tau) \,. \tag{4.72}$$

Since the commutation relations require $\sum_\alpha \left[v^I_\alpha(k,\tau) v^{J*}_\alpha(k,\tau) - v^J_\alpha(k,\tau) v^{I*}_\alpha(k,\tau) \right] = 0$ (see Appendix A) we see that the spectrum $\mathcal{P}^{IJ}_v$ is real, as it should be. Additionally, the two point correlation functions in coordinate space may be computed out of $\mathcal{P}^{IJ}_v$ as

$$\left\langle 0 \left| v^I(\tau,\boldsymbol{x}) v^J(\tau,\boldsymbol{y}) \right| 0 \right\rangle = \frac{1}{4\pi} \int \frac{d^3k}{k^3} \mathcal{P}^{IJ}_v(k,\tau) e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} \,. \tag{4.73}$$

We may also define the power spectrum associated to the Q^I fields instead of the v^I fields. Recalling that $Q^I = v^I/a$, the power spectrum for these fields at a given time τ is then given by

$$\mathcal{P}^{IJ}_Q(k,\tau) = \frac{k^3}{2\pi^2 a^2} \sum_\alpha v^I_\alpha(k,\tau) v^{I*}_\alpha(k,\tau) \,. \tag{4.74}$$

This expression will be used to compute the power spectrum of the curvature perturbation produced during inflation. Although, in this section we have chosen to exploit a notation whereby Greek indices α label quantum modes, notice that this formalism is equivalent to the use of stochastic Gaussian variables, as in Tsujikawa et al. (2003) (see also Lalak et al., 2007b).

4.4 Applications in Minkowski space

In this section we will go to the Minkowski limit, meaning that we set the Ricci scalar to zero and replace the action (eq. 4.1) with

$$S = -\int d^4x \left[\frac{1}{2}\gamma_{ab}\partial^\mu\phi^a\partial_\mu\phi^b + V(\phi) \right] , \tag{4.75}$$

Most of the discussion in sections 4.2 and 4.3 remains relevant, where of course all equations are to be understood with $H = \epsilon = \eta = 0$. In particular, this means that the equations for the perturbations (eq. 4.48) reduce to

$$\frac{\mathcal{D}^2}{dt^2}v^I - \nabla^2 v^I + C^I{}_J v^J = 0 , \tag{4.76}$$

where $C^I{}_J \equiv e^I_a e^b_J C^a{}_b$ and

$$C^a{}_b = \nabla_b V^a - \dot{\phi}_0^2 \mathcal{R}^a{}_{cdb} T^c T^d . \tag{4.77}$$

4.4.1 Dynamics in the presence of mass hierarchies

The main quantity determining the dynamics of the present system is the scalar potential $V(\phi)$. Since we are interested in studying the dynamics of multi-scalar field theories in Minkowski space-time, we will assume that it is positive definite, $V(\phi) \geq 0$. From the potential one can define the mass matrix M^2_{ab} associated to the scalar fluctuations around a given vacuum expectation value $\langle\phi^a\rangle = \phi^a_0$ as

$$M^2_{ab}(\phi_0) \equiv \nabla_a\nabla_b V\Big|_{\phi=\phi_0} . \tag{4.78}$$

In general, this definition renders a non-diagonal mass matrix, yet it is always possible to find a “local” frame in which it becomes diagonal and with the entries given by the eigenvalues m_a^2. Now, we take into account the existence of hierarchies among different families of scalar fields, and specifically consider two families, herein referred to as heavy and light fields which are characterised by

$$m_H^2 \gg m_L^2 . \tag{4.79}$$

In the particular case where the vacuum expectation value of the scalar fields remains constant $\dot{\phi}^a_0 = 0$, it is well understood that the heavy fields can be systematically integrated out, providing corrections of $O(k^2/m_H^2)$ with k being the energy scale of interest to the low energy effective Lagrangian describing the remaining light degrees of freedom (Appelquist and Carazzone, 1975). If however the vacuum expectation value ϕ_0^a is allowed to vary with time, new effects start occurring which can be significant at low energies. We focus on scalar potentials $V(\phi)$ for which hierarchies are present, and for which the trajectory followed by the scalar fields is such that

$$T^a T^b M_{ab}^2 \sim m_L^2 \,, \tag{4.80}$$

$$N^a N^b M_{ab}^2 \sim m_H^2 \,. \tag{4.81}$$

Since M_{ab}^2 is in general non-diagonal, for consistency we take $T^a N^b M_{ab}^2$ to be at most of $O(m_L m_H)$. Such trajectories are generic in the following sense: for arbitrary initial conditions, the background field ϕ_0^a typically will start evolving to the minimum of the potential $V(\phi)$ by first quickly minimising the heavy directions. Then the light modes evolve to their minimum much more slowly.

We will continue the present analysis systematically by splitting the potential into two parts,

$$V(\phi) = V_*(\phi) + \delta V(\phi) \,. \tag{4.82}$$

Here, $V_*(\phi) \geq 0$ is the zeroth-order positive definite potential characterised by containing exactly flat directions, and $\delta V(\phi)$ is a correction which breaks this flatness.[11] By construction, $V_*(\phi)$ contains all the information regarding the heavy directions. Therefore, the mass matrix $M_{*\,ab}^2$ obtained out of $V_*(\phi)$ presents eigenvalues which are either zero or $O(m_H^2)$. Consequently, the light masses appear only after including the correction $\delta V(\phi)$. We thus require the second derivatives of $\delta V(\phi)$ to be at most $O(m_L^2)$. It should be clear that such a splitting is not unique, as it is always possible to redefine both contributions by keeping the property $M_{*\,ab}^2 \sim m_H^2$.

It is clear that the solution to the equation

$$V_*^a(\phi) = 0 \tag{4.83}$$

defines a hypersurface $\mathcal{S}$ in $\mathcal{M}$. The dimension of the surface $\mathcal{S}$ corresponds to the number of flat directions present in $V(\phi)$. Let us denote this solution by ϕ_*^a. In appendix B we study in detail the dynamics offered by the zeroth-order theory, in which only the contribution $V_*(\phi)$ to the potential $V(\phi)$ is taken into account. For present purposes we quote here a simple result concerning background solutions offered by potentials of this sort: ϕ_0^a and ϕ_*^a are related by

$$\phi_0^a = \phi_*^a + \Delta^a \,, \tag{4.84}$$

[11] Such a type of splitting happens in the moduli sector of many low energy string compactifications, where V_* appear as a consequence of fluxes (Giddings et al., 2002), and $\delta V(\phi)$ arguably from non-perturbative effects (Kachru et al., 2003a).

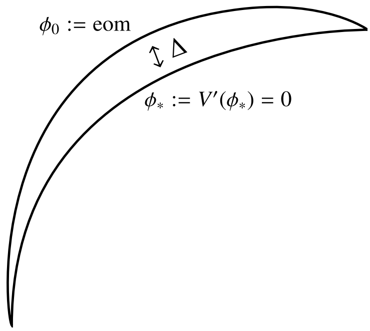

Figure 4.3: *The difference between ϕ_* and ϕ_0.*

where $\Delta^a \simeq N^a \Delta$ parametrises the displacement from $\mathcal{S}$ of the field trajectory obtained by considering the full potential (eq. 4.82), with Δ given by (see figure 4.3)

$$\Delta = \frac{\dot{\phi}_*^2}{m_{*H}^2 \kappa_*} . \tag{4.85}$$

Here, κ_* is the radius of curvature of the projected curve on $\mathcal{S}$. We see that the deviation from the surface $\mathcal{S}$ will be small as long as the dimensionless parameter

$$\frac{\beta}{4} \equiv \frac{\Delta}{\kappa_*} = \frac{\dot{\phi}_*^2}{m_{*H}^2 \kappa_*^2} \tag{4.86}$$

remains small.[12] In what follows, we shall see how this parameter affects the low energy dynamics valid for the light degrees of freedom tangent to $\mathcal{S}$. To simplify our analysis, we focus on two-dimensional models. These results can be easily generalised to an arbitrary number of scalar fields.

4.4.2 Two-field models

For theories with two scalar fields we can always choose the set of vielbeins e_a^I to consists in the following pair:

$$e_1^a = e_T^a = T^a , \tag{4.87}$$

$$e_2^a = e_N^a = N^a . \tag{4.88}$$

[12] As Δ is a function of κ_*, this leads to a lower bound on κ for which this approach is valid. For small Δ we find that we need to require $\kappa_* \approx \kappa \gg \dot{\phi}_*^2/M_{*H}^2$.

With this choice we can write $v^T \equiv T_a\varphi^a$ and $v^N \equiv N_a\varphi^a$ in a similar fashion as is done in (eq. 4.37), which denote the perturbations parallel and normal to the background trajectory, respectively. In the case where $\mathcal{M}$ is two-dimensional, these mutually orthogonal vectors are enough to span all of space. Therefore, the two unit vectors satisfy the relations

$$\frac{DT^a}{dt} = -\frac{\dot{\phi}_0}{\kappa}N^a \; , \tag{4.89}$$

$$\frac{DN^a}{dt} = \frac{\dot{\phi}_0}{\kappa}T^a \; , \tag{4.90}$$

assuming that DT^a/dt is nonvanishing and $s_N(t) < 0$ (this assumes a right turning trajectory, see figure 5.1). In terms of the formalism of the previous sections, these expressions may be written down as $Z_{TN} = -Z_{NT} = \dot{\phi}_0/\kappa$. Further, the entries of the symmetric tensor $C_{IJ} = e^a_I e^b_J C_{ab}$ defined in (eq. 4.77) are given by

$$C_{TT} = T^aT^b\nabla_a V_b \; , \tag{4.91}$$

$$C_{TN} = T^aN^b\nabla_a V_b \; , \tag{4.92}$$

$$C_{NN} = N^aN^b\nabla_a V_b + \frac{\dot{\phi}_0^2}{2}\mathcal{R} \; , \tag{4.93}$$

where $\mathcal{R} = \gamma^{ab}\mathcal{R}^c{}_{acb} = 2\mathcal{R}^T{}_{NTN}$ is the Ricci scalar.[13] Then, we notice that $T^aT^b\nabla_a V_b = T^a\nabla_a\left(T^bV_b\right) - \left(T^a\nabla_a T^b\right)V_b$ and using the fact $T^a\nabla_a \equiv \nabla_\phi = \dot{\phi}_0^{-1}D/dt$, we may rewrite

$$C_{TT} = \nabla_\phi V_\phi + \zeta^2 \; , \tag{4.94}$$

$$C_{TN} = \dot{\zeta} - \frac{2V_\phi}{\kappa} \; , \tag{4.95}$$

where $\zeta \equiv \dot{\phi}_0/\kappa$ and $V_\phi \equiv T^aV_a$. The remaining component C_{NN} cannot be deduced in this way, as it depends on the second variation of V away from the trajectory. Inserting the previous expressions back into (eq. 4.48), the set of equations of motion for the pair of perturbations v^T and v^N is found to be

$$\ddot{v}^T - \nabla^2 v^T + 2\zeta\dot{v}^N + 2\dot{\zeta}v^N + \nabla_\phi V_\phi v^T - 2\frac{V_\phi}{\kappa}v^N = 0 \; , \tag{4.96}$$

$$\ddot{v}^N - \nabla^2 v^N - 2\zeta\dot{v}^T + M^2 v^N - 2\frac{V_\phi}{\kappa}v^T = 0 \; , \tag{4.97}$$

[13]Since $\mathcal{M}$ is two dimensional, $\mathcal{R}^T{}_{NTN} = \mathcal{R}/2$ is the only non-vanishing component of the Riemann tensor.

where $M^2 = C_{NN} - \zeta^2$. The rotation matrix $R^I{}_J$ connecting the perturbations v^I with the canonical counterparts u^I is easily found to be

$$R^I{}_J = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} , \tag{4.98}$$

$$\theta(t) = \int_t dt' \, \zeta(t') \, . \tag{4.99}$$

The convenience of staying in the frame where $e^a_T = T^a$ and $e^a_N = N^a$ is that the matrix C_{IJ} has elements with a well defined physical meaning.

4.4.3 Constant radius of curvature

To gain some insight into the dynamics behind these equations, let us consider the particular case where $V_\phi = T^a V_a = 0$ and $\nabla_\phi V_\phi = 0$. This is the situation in which the background solution consists of a trajectory in field space crossing an exactly flat valley within the landscape. As $V_\phi = 0$ requires $\ddot{\phi}_0 = 0$, we see that $\dot{\phi}_0$ becomes a constant of motion. Additionally, let us assume that the radius of curvature κ remains constant, and that the mass matrix $M^2 = C_{NN} - \zeta^2$ is also constant.[14] Under these conditions ζ is a constant and one has $C_{TT} = \zeta^2$ and $C_{TN} = 0$. Then, the equations of motion for the perturbations become

$$\ddot{v}^T - \nabla^2 v^T + 2\zeta \dot{v}^N = 0 \, , \tag{4.100}$$

$$\ddot{v}^N - \nabla^2 v^N - 2\zeta \dot{v}^T + M^2 v^N = 0 \, . \tag{4.101}$$

We can solve and quantise these perturbations by following the procedure deduced in section 4.3. First, the mode solutions $v^I_\alpha(k)$ must satisfy

$$\ddot{v}^T_\alpha + 2\zeta \dot{v}^N_\alpha + k^2 v^T_\alpha = 0 \, , \tag{4.102}$$

$$\ddot{v}^N_\alpha - 2\zeta \dot{v}^T_\alpha + \left(M^2 + k^2\right) v^N_\alpha = 0 \, . \tag{4.103}$$

To obtain the mode solutions let us try the ansatz

$$v^T_\alpha(k,t) = v^T_\alpha(k) e^{-i\omega_\alpha t}, \tag{4.104}$$

where $\omega_\alpha \geq 0$ ($\alpha = 1, 2$) corresponds to a set of frequencies to be deduced shortly. Notice that the associated operators $a^\dagger_\alpha(\mathbf{k})$ and $a_\alpha(\mathbf{k})$ create and annihilate states char-

[14]Notice that in the particular case where $\mathcal{M}$ is flat and with a trivial topology, these conditions would correspond to an exact circular curve, such as the one that would happen near the bottom of the 'Mexican hat' potential.

acterised by the frequency ω_α and momentum $\mathbf{k}$. With the former ansatz, the equations of motion take the form

$$\left(k^2 - \omega_\alpha^2\right) v_\alpha^T(k) - 2i\omega_\alpha \zeta v_\alpha^N(k) = 0 \,, \tag{4.105}$$

$$\left(M^2 + k^2 - \omega_\alpha^2\right) v_\alpha^N(k) + 2i\omega_\alpha \zeta v_\alpha^T(k) = 0 \,. \tag{4.106}$$

Combining them one finds the equation determining the values of ω_α as

$$\left(k^2 - \omega_\alpha^2\right)\left(M^2 + k^2 - \omega_\alpha^2\right) = 4\zeta^2\omega_\alpha^2 \,. \tag{4.107}$$

The solutions to this equation are

$$\omega_\pm^2 = \frac{1}{2}\left[\left(M^2 + 2k^2 + 4\zeta^2\right) \pm \sqrt{\left(M^2 + 2k^2 + 4\zeta^2\right)^2 - 4k^2\left(M^2 + k^2\right)^2}\right] \,. \tag{4.108}$$

On the other hand, the coefficients $v_\alpha^T(k)$ and $v_\alpha^N(k)$ must be such that the relations (eq. A.1) and (eq. A.2) are satisfied. After straightforward algebra, it is possible to show that these coefficients are given by

$$|v_-^T(k)|^2 = \frac{\left(\omega_+^2 - k^2\right)\omega_-}{2k^2\left(\omega_+^2 - \omega_-^2\right)} \,, \qquad |v_-^N(k)|^2 = \frac{\left(\omega_+^2 - M^2 - k^2\right)\omega_-}{2\left(M^2 + k^2\right)\left(\omega_+^2 - \omega_-^2\right)} \,, \tag{4.109}$$

$$|v_+^T(k)|^2 = \frac{\left(k^2 - \omega_-^2\right)\omega_+}{2k^2\left(\omega_+^2 - \omega_-^2\right)} \,, \qquad |v_+^N(k)|^2 = \frac{\left(M^2 + k^2 - \omega_-^2\right)\omega_+}{2\left(M^2 + k^2\right)\left(\omega_+^2 - \omega_-^2\right)} \,. \tag{4.110}$$

If $\zeta^2 \ll M^2$ we can in fact expand all the relevant quantities in powers of ζ^2. One finds, up to leading order in ζ^2/M^2,

$$\omega_- = k\left(1 - \frac{2\zeta^2}{M^2}\right) \,, \tag{4.111}$$

$$\omega_+ = \sqrt{M^2 + k^2}\left(1 + \frac{2\zeta^2}{M^2}\right) \,, \tag{4.112}$$

$$|v_-^T(k)|^2 = \frac{1 - 2\zeta^2/M^2}{2k} \,, \tag{4.113}$$

$$|v_-^N(k)|^2 = \frac{2\zeta^2 k}{M^4} \,, \tag{4.114}$$

$$|v_+^T(k)|^2 = \frac{2\zeta^2\sqrt{M^2 + k^2}}{M^4} \,, \tag{4.115}$$

$$|v_+^N(k)|^2 = \frac{1 - 2\zeta^2/M^2}{2\sqrt{M^2 + k^2}} \,. \tag{4.116}$$

Thus we see that in the particular case where $\zeta = 0$ at all times (a straight trajectory) one has $|v_-^N(k)|^2 = 0$ and $|v_+^T(k)|^2 = 0$ and one recovers the standard results describing the quantisation of a massless scalar field v_-^T and a massive scalar field v_+^N of mass M. Observe that in this case it was not necessary to choose (eqs. 4.65-4.67) as initial conditions to ensure the quantisation of the system.

4.4.4 Low energy effective theory

Although in general it is not possible to solve (eq. 4.96) and (eq. 4.97) analytically, we may integrate the heavy mode to deduce a reliable low energy effective theory describing the light degree of freedom parallel to the trajectory as long as $\zeta \ll M$ and $k \ll M$. From the discussion of the previous section, it is not difficult to anticipate that the two modes $\alpha = 1, 2$ will be closely related to the light and heavy direction. Let us therefore adopt the notation $\alpha = L, H$ and focus on the light mode v_L^I, which here we express as

$$v_L^I \to \begin{pmatrix} v_L^T \\ v_L^N \end{pmatrix} \equiv \begin{pmatrix} \psi \\ \chi \end{pmatrix} , \tag{4.117}$$

where χ is a contribution satisfying $|\ddot{\chi}| \ll M^2|\chi|$, that is, its time variation is much slower than the time scale M^{-1} characterising the heavy mode. Then, inserting (eq. 4.117) back into the second equation of motion (eq. 4.97) and keeping the leading term in χ, we obtain the result

$$\chi = \frac{2\zeta}{M^2}\dot{\psi} + 2\frac{V_\phi}{M^2\kappa}\psi . \tag{4.118}$$

Of course, we are due to verify that $|\ddot{\chi}| \ll M^2|\chi|$ is a good ansatz for the solution. Inserting (eq. 4.118) back into the first equation of motion (eq. 4.96) we obtain

$$\ddot{\psi} + 4\frac{d}{dt}\left(\frac{\zeta^2}{M^2}\dot{\psi}\right) + \left(k^2 + m_L^2\right)\psi = 0 , \tag{4.119}$$

$$m_L^2 = \nabla_\phi\left[\left(1 + \frac{4\zeta^2}{M^2}\right)V_\phi\right] . \tag{4.120}$$

Simple inspection of this equation shows that indeed $|\ddot{\chi}| \ll M^2|\chi|$ is satisfied. Additionally, from (eq. 4.118) notice that the vector (eq. 4.117) is pointing almost entirely towards the direction (1,0), which corresponds to the direction parallel to the motion of the background field. To deal with the previous equation we define $e^\beta =$

$1 + 4\zeta^2/M^2$.[15] Then, we may write

$$e^{\beta}\left(\ddot{\psi} + \dot{\beta}\dot{\psi}\right) + \left(k^2 + m_L^2\right)\psi = 0\ , \tag{4.121}$$

$$m_L^2 = \nabla_\phi\left(e^{\beta}V_\phi\right)\ . \tag{4.122}$$

We can alternatively rewrite $m_L^2 = \nabla_\phi\left(e^{\beta}V_\phi\right) = e^{\beta}\nabla_\phi V_\phi + e^{\beta}V_\phi\dot{\beta}/\dot{\phi}_0$. It is possible to see that the previous equation of motion can be obtained from the action

$$S = \frac{1}{2}\int dtd^3x\left[e^{\beta}\dot{\psi}^2 - (\nabla\psi)^2 - m_L^2\psi^2\right]\ . \tag{4.123}$$

By performing a field redefinition $\varphi \equiv e^{\beta/2}\psi$, we see that the previous action may be re-expressed as

$$S = \frac{1}{2}\int dtd^3x\left[\dot{\varphi}^2 - e^{-\beta}(\nabla\varphi)^2 - M_L^2\varphi^2\right]\ , \tag{4.124}$$

$$M_L^2 = \nabla_\phi V_\phi + \frac{V_\phi\dot{\beta}}{\dot{\phi}_0} + \frac{\ddot{\beta}}{2} + \frac{\dot{\beta}^2}{4}\ . \tag{4.125}$$

For the particular case in which $\nabla_\phi V_\phi = 0$ and the bending of the trajectory is such that $\dot{\beta} = 0$, then the frequency ω of the light mode reduces to $\omega = ke^{-\beta/2} \simeq k\left(1 - 2\zeta^2/M^2\right)$, which coincides with the previous result (eq. 4.111).

4.5 Discussion

In this chapter, we considered the structure of scalar field theories with a pronounced hierarchy of mass scales. First, we set up a framework for describing a light field moving along a multi-field trajectory in field space. From this, we determined the background equations of motion, around which we can study perturbations. Finally, we deduced the effective theory describing light perturbations for the case in which the background field is following a curved trajectory in field space.

The main manifestation of the non-trivial mixing of the heavy and the light directions is in the appearance of the coefficient $e^{-\beta}$ in front of the term $(\nabla\varphi)^2$ containing spatial derivatives appearing in the action (eq. 4.124). Since $\beta \geq 0$, the net effect of the bending of the background trajectory is to reduce the energy per scalar field quantum. This is due to the fact that during bending the light modes momentarily start exciting heavy modes, therefore transferring energy to them. Yet, since $\beta = 4\zeta^2/M^2$,

[15] Since $\zeta^2/M^2 \ll 1$, then $\beta \simeq 4\zeta^2/M^2$.

there are two effects competing against each other in this process. On one hand one has $\zeta = \dot{\phi}/\kappa$, which may be interpreted as the angular speed of the background field along the curved trajectory. On the other hand, there is the mass of the heavy mode M, which must be excited by the light modes during the bending. In what follows we discuss two applications of our results.

4.5.1 Inflation

Understanding in detail how light and heavy modes remain coupled under more general circumstances could be particularly significant for cosmic inflation (Guth, 1981, Albrecht and Steinhardt, 1982, Linde, 1982). Indeed, although current observations are consistent with the simplest model of single-field inflation, it is rather hard to conceive a realistic model where the inflaton field alone is completely decoupled from UV degrees of freedom. One way of addressing this issue is by studying multi-field scenarios where many scalar fields have the chance to participate in the inflationary dynamics (Starobinsky, 1985), despite of different mass scales among the inflaton candidates. Hence, there could exist certain phenomena related to inflation in which the effects studied in this chapter can be relevant. This has also recently been considered in Tolley and Wyman (2010) and Chen and Wang (2010b).

First, note that the equation of motion deduced out of the action (eq. 4.124) is given by

$$\ddot{\varphi} + e^{-\beta} k^2 \varphi + M_L^2 \varphi = 0 \,. \tag{4.126}$$

For definiteness, let us focus on phenomena characterised by $|\dot{\beta}| \ll k$ and consider the case in which the potential is flat enough so that $M_L^2 \ll k^2$ is satisfied. Then, the time variation of β along the trajectory is small enough to allow us to write the mode solution as

$$\varphi(k,t) = \frac{e^{\beta/4}}{\sqrt{k}} \exp\left[i e^{-\beta/2} k\, t \right] \,, \tag{4.127}$$

where the factor $e^{\beta/4}/\sqrt{k}$ is necessary in order to satisfy the commutation relation $[\varphi, \dot{\varphi}] = i$. This factor coincides with the one found in (eq. 4.113) for the amplitude of light modes in the case where β is a constant. To continue, from (eq. 4.127) we can see that in the vacuum, the two point correlation function of the perturbation $\varphi(\mathbf{x}, t)$, has the form $\langle \varphi(\mathbf{x}, t) \varphi(\mathbf{y}, t) \rangle \propto e^{\beta(t)/2}$. One direct consequence of this result is for inflation, where the amplitudes of scalar fluctuations freeze after crossing the horizon, i.e. when the physical wavelength k^{-1} satisfies the condition $e^{-\beta/2} k = H$. More precisely, if we generalise (eq. 4.126) to include gravity, we would conclude that the speed of sound of adiabatic perturbations is given by

$$c_s^2 = e^{-\beta} \; . \tag{4.128}$$

Such an effect is known to produce sizable levels of nongaussianities (Alishahiha et al., 2004). Additionally it modifies the power spectrum as

$$P(k) \simeq e^{\beta(k)/2} P_*(k) \; , \tag{4.129}$$

where $P_*(k)$ is the conventional power spectrum $P_*(k) \propto k^{n_s - 1}$ deduced in single-field slow-roll inflation, and $\beta(k)$ is the value of $\beta(t)$ at the time t when the mode k crosses the horizon.[16] The more interesting case is a varying β and since β can be as large as ϵ, such an effect may be sizable and observable in the near future. In many scalar field theories, such as supergravity, the masses of heavy degrees of freedom during inflation are typically of order $M \sim H$, leading to the relation

$$\beta \sim 4\epsilon \frac{M_{\rm Pl}^2}{\kappa^2} \; . \tag{4.130}$$

If the bending is such that the radius of curvature becomes of order $\kappa \sim M_{\rm Pl}$ (a rather conservative value) one then obtains effects as large as $\beta \sim \epsilon$. In the case where a turn of the trajectory happens during a few e-folds, one then should be able to observe features in the power spectrum of $\mathcal{O}(\epsilon)$, particularly by modifying the running of the spectral index as $dn_s/d\ln k$, which otherwise would be of $\mathcal{O}(\epsilon^2)$. A detailed computation of this effect is done in the next chapter.

4.5.2 Decoupling of light and heavy modes in supergravity

Let us next point out that our results can be also used to assess when a low energy effective theory, deduced from a multi-scalar field theory containing both heavy directions and light directions, is accurate enough. As discussed in full detail in appendix B, whenever the background fields are evolving (as in inflation) the only way of having a vanishing β-parameter is for a trajectory to correspond to a curve autoparallel to a geodesic in the full scalar field manifold $\mathcal{M}$. It is clear that the only way of achieving this is by having some property relating the shape of the potential $V(\phi)$ with the geometry of $\mathcal{M}$. In the particular case of supergravity such a property is known to exist, and therefore one should expect supergravity theories rendering low

[16]In the constant curvature case $\kappa =$ const., $\beta(\kappa)$ is constant as well and we find an overall modulation of the power spectrum compatible with Chen and Wang (2010b).

energy effective theories for which β vanishes exactly. To be more precise, in $\mathcal{N} = 1$ supergravity the scalar field potential is given by

$$V = e^G \left(G^{i\bar{j}} G_i G_{\bar{j}} - 3 \right) , \tag{4.131}$$

with definitions matching those of section 1.3.2. Consider a supergravity theory in which a set of massive chiral fields ϕ^H satisfy the condition

$$G_H = 0 , \tag{4.132}$$

along a given hypersurface $\mathcal{S}$ in $\mathcal{M}$ parametrised *only* by the light fields L.[17] It is then possible to verify that the scalar fluctuations ϕ^L parallel to $\mathcal{S}$ are decoupled from the fields ϕ^H, rendering $\beta = 0$. To appreciate this, observe first that at any point on the surface $\mathcal{S}$ the Kähler metric $G_{I\bar{J}}$ is diagonal between the two sectors. Indeed, since $G_H = 0$ holds at any point in the surface, then it must be independent of arbitrary displacements $\delta\phi^L$ along $\mathcal{S}$. This implies that

$$\partial_{\bar{L}} G_H = G_{H\bar{L}} = 0. \tag{4.133}$$

This condition automatically ensures that in the absence of a scalar field potential, the trajectory along $\mathcal{S}$ will be on an autoparallel curve. It remains then to verify that the potential does not imply quadratic couplings between both sectors, therefore leaving these autoparallel geodesic trajectories unmodified. Given the first derivative of the scalar potential (eq. 1.37) and using the requirement for supersymmetry (eq. 1.51), one immediately obtains $V_H = 0$ in $\mathcal{S}$. Using (eq. 1.38 and 1.39), it is not difficult to notice that also $\nabla_L V_H = \nabla_{\bar{L}} V_H = \nabla_L V_{\bar{H}} = \nabla_{\bar{L}} V_{\bar{H}} = 0$ on $\mathcal{S}$, which also hinge on $G_{H\bar{L}} = 0$ and $G_H = 0$. Put together, these results imply that the heavy sector will not affect the light sector as long as the background trajectory tracks the geodesically generated surface $\mathcal{S}$.

Conversely, deviations from the condition $G_H = 0$, or a surface $\mathcal{S}$ with $H \neq$ const., will produce interactions leading to the appearance of the coupling β studied in the present work. A particularly interesting example is Gallego and Serone (2009), where $O(\epsilon)$ couplings between heavy and light fields in the superpotential result in suppressed, $O(\epsilon^2)$ terms in the effective action for the light fields. This result was obtained by expanding about a particular $H =$ const. configuration which, for constant light background fields, only deviates at $O(\epsilon)$ from the true solution to the equations of motion. But along an arbitrary background $L(t)$ the deviation will exceed $O(\epsilon)$ for displacements $\Delta L/\kappa > \epsilon$ (due to the $\Gamma^H_{LL}\dot{L}^2$ term in the H equation of motion) and the corrections to the effective action discussed in this chapter become dominant.

[17] This means that a surface $\mathcal{S}$ is defined by $f(H, \bar{H}) = 0$ rather that by a function $f(H, \bar{H}, L, \bar{L}) = 0$ (de Alwis, 2005a and chapter 2).

4.5.3 Consistent decoupling and autoparallel trajectories

The condition (eq. 4.132) leads to $(\partial_L)^n(\partial_{\bar{L}})^m G_H = 0$ for all n and m on the hypersurface $\mathcal{S}$. This means, in particular, that the metric is block diagonal, $G_{L\bar{H}} = 0$, on $\mathcal{S}$. Also, $\Gamma^H_{LL} = G_{LL\bar{H}} G^{H\bar{H}} = 0$ and the heavy direction is at a critical point as $V_H \propto G_H = 0$. Thereby, $V^H = G^{H\bar{H}} V_{\bar{H}} + G^{H\bar{L}} V_{\bar{L}} = 0$ on the hypersurface $\mathcal{S}$.

Inserting this in the equations of motion (eq. 4.6),

$$\ddot{\phi}^L + \Gamma^L_{HH}(\dot{\phi}^H)^2 + 2\Gamma^L_{HL}\dot{\phi}^H\dot{\phi}^L + \Gamma^L_{LL}(\dot{\phi}^L)^2 + 3H\dot{\phi}^L + V^L = 0 \,, \tag{4.134}$$

$$\ddot{\phi}^H + \Gamma^H_{HH}(\dot{\phi}^H)^2 + 2\Gamma^H_{HL}\dot{\phi}^H\dot{\phi}^L + \Gamma^H_{LL}(\dot{\phi}^L)^2 + 3H\dot{\phi}^H + V^H = 0 \,, \tag{4.135}$$

we see that the second one is satisfied identically for $H = \text{const.}$, while the first one becomes a function of the light fields only,

$$\ddot{\phi}^L + \Gamma^L_{LL}(\dot{\phi}^L)^2 + 3H\dot{\phi}^L + V^L = 0 \,. \tag{4.136}$$

Note that this derivation only holds *on* $\mathcal{S}$. Any quantity that depends on a region around this trajectory, such as derivatives, would see the connection in the heavy direction.

Whether the trajectory is autoparallel in terms of the real and imaginary parts of the light fields L depends on the consistently truncated Kähler function. The effective action for the light fields has to be re-expressed to a real field sigma model action which can analysed using the machinery presented in this chapter.