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ON SOME CLASSES OF MODULES AND THEIR ENDOMORPHISM RINGS

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To my parents and my wife

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Partial list of notations:

$\mathbb{N}$	the set of non-negative integers.
$\mathbb{Z}$	the ring of integers.
R	an associative ring with $1 \neq 0$.
$Z(R)$	$= \{\, x \in R \mid xy = yx$ for every $y \in R\}$, the center of R.
$J(R)$	the Jacobson radical of a ring R.
M_R	a right module over a ring R.
${}_RM$	a left module over a ring R.
$E(M_R)$	the (an) injective envelope of M_R.
$M_R^{(I)}$	the direct sum $\bigoplus_{i\in I} M_i$ with $M_i \cong M_R$ for every $i \in I$.
$\mathrm{rad}(M_R)$	the radical of M_R.
$\mathrm{soc}(M_R)$	the socle of M_R.
$\mathrm{End}(M_R)$	the endomorphism ring of M_R.
$\mathrm{End}({}_RM)$	the endomorphism ring of ${}_RM$.
$\mathrm{Hom}(N_R, M_R)$	the group of all module homomorphisms of N_R to M_R.
$M \leq N$	M is a submodule of N.
$M < N$	M is a proper submodule of N.
$I \subseteq J$	I is a subset of J.
$I \subset J$	I is a proper subset of J.
$M_n(R)$	the ring of $n \times n$ matrices over a ring R.
$f\vert_A$	the restriction of a mapping $f : B \to C$ to a subset A of B.
1_A or Id_A	the identity mapping $A \to A$, where A is a set.
$(A : B)_R$	$= \{\, r \in R \mid Br \leq A\,\}$, where A, B are submodules of module M_R.
$\mathrm{ann}_R(A)$	$= (0 : A)_R$, where A is a submodule of module M_R.
$\mathrm{ann}_R(x)$	$= (0 : \{x\})_R$, where x is an element of module M_R.

Introduction

Classically, modules were used as a method in the study of representation theory. Since the 1950s, the scope of module theory has become much broader. An effective way to understand the behavior of an arbitrary module is to study its endomorphism ring. The goal of studying endomorphism rings, in our case, is to consider the decompositions of a module, so that the following examples concern direct sums and direct summands.

Let M_R be a right module over an arbitrary ring R and $S = \mathrm{End}(M_R)$ be the endomorphism ring of M_R. There is a classical fact that if S is a local ring, that is, the set of all non-invertible elements of S is a two-sided ideal of S, then M_R is indecomposable. In [15], Facchini collected several properties of M_R that hold when the endomorphism ring S of M_R is semilocal, that is, $S/J(S)$ is a semisimple artinian ring where $J(S)$ is the Jacobson radical of S. In fact, if S is semilocal, then:

1. M_R is a direct sum of finitely many indecomposable modules.
2. M_R is directly finite. That is, if $M_R \oplus N_R \cong M_R \oplus N'_R$, then $N_R \cong N'_R$.
3. If n is a positive integer and $M_R^n \cong N_R^n$, then $M_R \cong N_R$. This property is called *n-th root property.*
4. If N_R is a module isomorphic to a direct summand of M_R, then the endomorphism ring $\mathrm{End}(N_R)$ of N_R is also semilocal.
5. If N_R is an R-module with semilocal endomorphism ring, then the endomorphism ring $\mathrm{End}(M_R \oplus N_R)$ is also semilocal.

Moreover, the study of rings can be aided by the machinery provided by endomorphism rings.

It has now become difficult to establish the border between module theory and the theory of endomorphism rings.

The aim of this thesis is to study some classes of modules such as injective modules, Loewy modules and max modules, and their endomorphism rings. In fact, in Chapter 2, we consider the endomorphism ring of a square-free injective module. Here, an injective module is called *square-free* if M_R has no direct summand isomorphic to $N \oplus N$ for some non-zero direct summand N of M_R. It is well known that an injective module is indecomposable if and only if its endomorphism ring is local [33]. For convenience, some basic notions concerning injective modules are presented in Chapter 1. The main result of Chapter 2 is to prove that an injective module is square-free if and only if its endomorphism ring is quasi-duo.

In Chapter 3, we describe all maximal right (left, two-sided) ideals of the endomorphism ring of an injective module. All maximal right (left, two-sided) ideals of the endomorphism ring of a vector space over a division ring were described completely in [35].

In Chapter 4, we first consider the class of Loewy modules with finite Loewy invariants and their endomorphism rings. It is trivial that every artinian module is a Loewy module with finite Loewy invariants. Facchini proved that if the base ring is commutative then the class of Loewy modules with finite Loewy invariants coincides with the class of artinian modules [12]. In this Chapter, we present an example to show that this is not true for modules over non-commutative rings. In 1993, Camps and Dicks proved that the endomorphism ring of an artinian module is semilocal [9]. We prove that every Loewy module with finite Loewy invariants also has a semilocal endomorphism ring. We then obtain similar results for the dual class of max modules over a semilocal ring. Here, a module M_R is called a *max module* if every non-zero submodule of M_R has a superfluous radical.

Chapter 5 is devoted to considering two questions concerning maximal subfields of a division algebra proposed in [32]. In fact, we prove that for any division algebra D with center F, there exist x, y, a, b in the multiplicative group D^* of D such that $F(xy - yx)$

and $F(aba^{-1}b^{-1})$ are maximal subfields of D.

Chapter 1

Injective modules

In this Chapter, we first recall notions concerning injective modules and injective envelopes. In Section 1.2, the construction of the maximal right ring of quotients of an arbitrary ring will be presented briefly. These rings of quotients will be used not only in this Chapter, but also in Chapter 5. In Section 1.3, as an example, we consider some properties of injective modules over a non-commutative free algebra. It is also proved that the maximal right ring of quotients of a free algebra R is the injective envelope of R_R as an R-module. In Section 4.2.3, we summarize some basic properties of the endomorphism ring of an injective module and an indecomposable injective module.

1.1 Some basic notions on injective modules, free modules and projective modules

In this Section, we briefly present some basic properties of injective modules, free modules and projective modules. These facts are elementary and can be found in any book of module theory such as [30], [14] or [4], so that we will omit the proofs of all these simple properties. In all cases, we indicate the citations (often [30] and [14]). Notice that the properties presented in this Section are just to help the readers review some needed elementary facts.

Injective modules and injective envelopes.

Let R be a ring. A right R-module M_R is called *injective* if, for any right R-modules A_R, B_R, any monomorphism $f\colon A_R \to B_R$ and any homomorphism $g\colon A_R \to M_R$, there exists a homomorphism $h\colon B_R \to M_R$ such that $g = hf$. In a diagram, this definition can be presented as follows:

$$\begin{tikzcd}
0 \arrow[r] & A_R \arrow[r, "f"] \arrow[d, "g"] & B_R \arrow[dl, dotted, "\exists h"] \\
 & M_R &
\end{tikzcd}$$

Informally, in some books, this property is introduced by saying that any homomorphism $g\colon A_R \to M_R$ can be "extended" to $h\colon B_R \to M_R$.

Theorem 1.1.1. [30, Baer's Test] *A right R-module M_R is injective if and only if for any right ideal I of R, any homomorphism $g\colon I_R \to M_R$ can be extended to $h\colon R_R \to M_R$.*

Theorem 1.1.2. [30, Proposition 3.4] *A direct product of a family of injective modules is injective. In particular, a direct sum of finitely many injective modules is injective.*

Notice that a direct sum of an arbitrary family of injective modules is not necessarily injective. A direct sum of an arbitrary family of injective right R-modules is injective if and only if R is right Noetherian [4, Proposition 18.13].

Corollary 1.1.3. *All direct summands of an injective module are injective.*

Here, a submodule A_R of a right R-module M_R is said to be a *direct summand* of M_R if $A_R \oplus B_R = M_R$ for some submodule B_R of M_R.

Lemma 1.1.4. [30, Remarks 3.23] *Let M_R be a right R-module and A_R be a submodule of M_R. The following conditions are equivalent:*

1. *For any non-zero submodule B_R of M_R, $A_R \cap B_R \neq 0$.*
2. *For any non-zero element $x \in M_R$, there exists $r \in R$ such that $0 \neq xr \in A_R$.*

If two modules $A_R \leq M_R$ satisfy any of the properties in Lemma 1.1.4, then M_R is called an *essential extension* of A_R or A_R is said to be *essential* in M_R, denoted by $A_R \leq_e M_R$. For a right ideal I of the ring R, we say that I is an *essential right ideal* of R if $I_R \leq_e R_R$ as R-modules.

The dual notion of essential submodules is that of a superfluous submodules. A submodule A_R of a right module M_R is called *superfluous* (or *small*) in M_R if, for any submodule B_R of M_R with $A_R+B_R = M_R$, one has that $B_R = M_R$ (notation $A_R \leq_s M_R$).

Proposition 1.1.5. *Let M_R, N_R be right R-modules. Assume that $f : N_R \to M_R$ is a homomorphism, A_R, B_R are submodules of M_R and C_R is a submodule of N_R. Then*

1. *If A_R and C_R are essential in respectively M_R and N_R, then $A_R \oplus C_R$ is essential in $M_R \oplus N_R$.*
2. *If A_R and B_R are essential in M_R, then $A_R \cap B_R$ is essential in M_R.*
3. *If A_R is an essential submodule in B_R and B_R is essential in M_R, then A_R is essential in M_R.*
4. *If A_R is essential in M_R, then $f^{-1}(A_R)$ is essential in N_R.*

Moreover, if $M_R = N_R$, $A_R \leq_e M_R$ and f is an idempotent endomorphism of M_R, that is, $f^2 = f$, then $f(A_R) \leq_e f(M_R)$.

The proof is elementary.

Theorem 1.1.6. [30, Theorem 3.30 and Corollary 3.32] *Let $M_R \leq N_R$ be right R-modules. The following conditions are equivalent:*

1. *N_R is a maximal essential extension of M_R. That is, there is no proper extension module N'_R of N_R such that $M_R \leq_e N'_R$.*
2. *N_R is injective and is an essential extension of M_R.*
3. *N_R is minimal injective over M_R. That is, N_R is injective and there is no injective module M'_R such that $M_R \leq M'_R < N_R$.*

Moreover, if N'_R is another essential extension of M_R satisfying any of properties (1), (2) *and* (3), *then there exists an isomorphism $f\colon N_R \to N'_R$ such that $f(m) = m$ for any $m \in M_R$.*

Corollary 1.1.7. *A module M_R is injective if and only if M_R is a direct summand of any module which contains M_R.*

It is proved that every module M_R is contained in an injective module which is an essential extension of M_R, denoted by $E(M_R)$. It is called an *injective envelope* or *injective hull* of M_R, denoted by $E(M_R)$. By Theorem 1.1.6, this injective envelope is unique up to isomorphisms. Moreover, if N_R is an injective module and M_R is a submodule of N_R, then there exists a direct summand of N_R such that it is an injective envelope of M_R.

Theorem 1.1.8. [30, Example 3.38] *For a family $\{ M_i \mid i = 1, 2, \cdots, n \}$ of finitely many right R-modules, $E(\bigoplus_{i=1}^{n} M_i) = \bigoplus_{i=1}^{n} E(M_i)$.*

Projective modules and hereditary rings.

Let R be a ring. A right R-module F_R is called *free* if it is isomorphic to a direct sum of copies of the module R_R. A right R-module P_R is called *projective* if, for any right R-modules A_R, B_R, any epimorphism $f\colon A_R \to B_R$ and any homomorphism $g\colon P_R \to B_R$, there exists a homomorphism $h\colon P_R \to A_R$ such that $g = fh$. In a diagram, this definition can be represented as follows:

$$\begin{tikzcd}
 & P_R \arrow[dl, dotted, "\exists h"'] \arrow[d, "g"] & \\
A_R \arrow[r, "f"] & B_R \arrow[r] & 0
\end{tikzcd}$$

Informally, this property say that any homomorphism $g\colon P_R \to B_R$ can be "lifted" to $h\colon P_R \to A_R$.

Proposition 1.1.9. [30, Corollary 2.6] *A right R-module P_R is projective if and only if it is isomorphic to a direct summand of some free right R-module.*

Corollary 1.1.10. *Every direct summand of a projective module is projective.*

A ring R is called *right hereditary* if every right ideal of R is projective as a right R-module. *Left hereditary* rings are defined similarly. A ring is said to be *hereditary* if it is left and right hereditary. Here are some examples of right (or left) hereditary rings.

Examples 1.1.11. 1. A direct product of two hereditary rings is a hereditary ring. It can be checked easily from the definition.

2. A non-zero module is called *simple* if its proper submodule it the zero submodule 0. A direct sum of a family of simple modules is said to be a *semisimple module.* We say that a ring R is a *semisimple artinian ring* if R_R is semisimple as a right R-module. It is possible to prove that R is semisimple artinian if and only if ${}_RR$ is semisimple as a left R-module, if and only if it is isomorphic to the direct product of finitely many matrix rings over division rings.

Every semisimple artinian ring is hereditary [14, Theorem 1.2]. In particular, the matrix ring $M_n(D)$, for some integer $n \geq 1$ and division ring D, is hereditary.

3. Let K be a field and $\{ x_i \mid \in I \}$ be a set of non-commutative indeterminates. Then every right (or left) ideal of the free algebra $R := K\langle \{ x_i \mid \in I \} \rangle$ is free as an R-module [11, Page 106]. Therefore, R is a hereditary ring.

Proposition 1.1.12. [40, Proposition I.9.5] *A ring R is right hereditary if and only if every factor module of an injective right R-module is injective.*

1.2 The maximal ring of quotients

There are two methods to define the maximal right ring of quotients of a ring R. The first one is to build it through the endomorphism ring $\mathrm{End}(E(R_R))$ (for examples, see [30, Chapter 5]) and the second one is to define as the direct limit over the set of all dense left ideals of R (for example, see [5, Chapter 2]). These are equivalent [30, Theorem 13.21]. In this Section, we use the second method.

Let R be a ring. Let M_R be a right R-module and A_R be a submodule of M_R. We say that A_R is *dense* in M_R if, for any $x_1, x_2 \in M_R$ with $x_1 \neq 0$, there exists $r \in R$ such that $x_1 r \neq 0$ and $x_2 r \in A_R$. A right ideal I of R is called *dense* if I_R is a dense submodule of R_R. The set of all dense right ideals of R is denoted by $\mathcal{D}_r(R)$.

By definition, every dense submodule of a module M_R is essential. However, the inverse is not true in general. It holds if the module M_R is non-singular. Recall a module M_R is called *non-singular* if, for any $m \in M_R$ with $\mathrm{ann}_R(m) = \{ r \in R \mid mr = 0 \} \leq_e R_R$, one has $m = 0$. A ring R is called *right non-singular* (resp. *left non-singular*) if the module R_R (resp. ${}_RR$) is non-singular. We say that R is *non-singular* if R is both right and left non-singular. For instance, every domain is non-singular. Here, a ring is said to be a *domain* if it has no left nor right zero-divisor.

Lemma 1.2.1. [30, Corollary 8.9] *Every essential submodule of a non-singular module is dense.*

Corollary 1.2.2. [30, Corollary 8.9] *In a right non-singular ring, the set of all dense right ideals coincides with the set of all essential right ideals.*

Proposition 1.2.3. *Let I and J be dense right ideals of R and $f\colon I_R \to R_R$ be a homomorphism. Then $I \cap J$ and $f^{-1}(J) = \{\, x \in I \mid f(x) \in J \,\}$ are dense in R.*

The proof is elementary. See [5, Proposition 2.1.1].

From Proposition 1.2.3, $\mathcal{D}_r(R)$ is a directed set with the relation defined by $I \preceq J$ if $J \subseteq I$. For any $I \preceq J$ in $\mathcal{D}_r(R)$, put $\varphi_{I,J}\colon \operatorname{Hom}(I_R, R_R) \to \operatorname{Hom}(J_R, R_R), \varphi(f) = f|_J$, the restriction of f to J. One can check easily that $\varphi_{I,I}\colon \operatorname{Hom}(I_R, R_R) \to \operatorname{Hom}(I_R, R_R)$ is the identity of $\operatorname{Hom}(I_R, R_R)$ and $\varphi_{I_1,I_3} = \varphi_{I_1,I_2}.\varphi_{I_2,I_3}$ for any I, I_1, I_2, I_3 in $\mathcal{D}_r(R)$ with $I_1 \preceq I_2 \preceq I_3$. Thus, $(\operatorname{Hom}(I_R, R_R), \varphi_{I,J}), I, J \in \mathcal{D}_r(R)$, is a direct system over $\mathcal{D}_r(R)$. Hence, we can define the direct limit $Q_{mr}(R) := \varinjlim_{I \in \mathcal{D}_r(R)} \operatorname{Hom}(I_R, R_R)$.

Now, in $Q_{mr}(R)$, we consider the addition and multiplication defined as follows: for $(f, I), (g, J) \in Q_{rm(R)}$, where $I, J \in \mathcal{D}_r(R)$, $(f, I)+(g, J) := (f+g, I\cap J)$, $(f, I)\cdot(g, J) := (fg, g^{-1}(I))$. It is easy to check (or see [5, page 56]) that these operations are well-defined and that $Q_{mr}(R)$ is a ring with zero element $(0, R)$ and identity (Id_R, R). The ring $Q_{mr}(R)$ is called the *maximal right ring of quotients* of R.

Let S be a ring containing R as a subring. Assume that a is an element of S such that the right ideal $(R\colon a)_R = \{\, r \in R \mid ar \in R \,\}$ is dense in R. Then we may consider an element $(\ell_a, (R\colon a)_R)$ of $Q_{mr}(R)$. Here, the homomorphism $\ell_a \in \operatorname{Hom}((R\colon a)_R, R_R)$ is the left multiplication determined by a, that is, $\ell_a(x) = ax$ for any $x \in (R\colon a)_R$.

Proposition 1.2.4. *The map* $\iota\colon R \to Q_{mr}(R)$ *defined by* $\iota(a) = (\ell_a, R)$ *is an injective homomorphism of rings. Thus,* R *may be considered as a subring of* $Q_{mr}(R)$ *via the map* ι*.*

The proof is elementary.

Proposition 1.2.5. *For any element* $x \in Q_{mr}(R)$*,* $(R\colon x)_R$ *is a dense right ideal of* R*. In particular,* R_R *is an essential submodule of* $Q_{mr}(R)_R$*.*

Proof. Let $x = (f, I) \in Q_{mr}(R)$, where $I \in \mathcal{D}_r(R)$ and $f \in \mathrm{Hom}(I_R, R_R)$. For any element $r \in I$, $xr = (f, I)(\ell_r, R) = (\ell_{f(r)}, R) \in R$. It follows that $I \subseteq (R\colon x)_R$. Hence, $(R\colon x)_R$ is dense in R. ■

Proposition 1.2.6. *For any element* $x \in Q_{mr}(R)$*, if* $xJ = 0$ *for some* $J \in \mathcal{D}_r(R)$*, then* $x = 0$*.*

Proof. Assume that $x = (f, I)$ for some $I \in \mathcal{D}_r(R)$ and $f \in \mathrm{Hom}(I_R, R_R)$. Then for any $r = (\ell_r, R) \in J$, $0 = xr = (\ell_{f(r)}, R)$ for all $r \in I \cap J$. Thus $f(I \cap J) = 0$. Therefore $x = (f, I) = (f|_{I\cap J}, I \cap J) = 0$. ■

Theorem 1.2.7. *Let* Q *be an extension ring of* R*. Assume that for any* $a \in Q$*,* $(R\colon a)_R$ *belongs to* $\mathcal{D}_r(R)$ *and if* $aJ = 0$ *for some* $J \in \mathcal{D}_r(R)$*, then* $a = 0$*. Then there exists a ring monomorphism* $\phi\colon Q \to Q_{mr}(R)$ *extending* ι*.*

Proof. Consider $\phi\colon Q \to Q_{mr}(R)$ defined by $\phi(a) = (\ell_a, (R\colon a)_R)$ for any $a \in Q$. Since $(R\colon a)_R$ belongs to $\mathcal{D}_r(R)$ for any $a \in Q$, ϕ is well-defined. It is elementary to check that ϕ is a ring homomorphism. We must show that ϕ extends ι and is injective. Indeed, for $a \in R$, $\phi(a) = (\ell_a, (R\colon a)_R) = (\ell_a, R) = a \in R$. Thus ϕ extends ι. Now if $0 = \phi(a) = (\ell_a, (R\colon a)_R)$, then $a(R\colon a)_R = 0$, which implies $a = 0$ by the hypothesis. ■

By Theorem 1.2.7, $Q_{mr}(R)$ is "maximal" among the rings having the properties in Propositions 1.2.4, 1.2.5 and 1.2.6.

The *maximal left ring of quotients* $Q_{ml}(R)$ of R is defined similarly. See [5, Section 2.2, Chapter 2].

1.3 An example

Let k be a field, n a positive integer. If $x_1, x_2, \dots, x_n$ are n commuting indeterminats, then the polynomial ring $k[x_1, x_2, \cdots, x_n]$ is a noetherian ring. Injective modules over a noetherian ring were described in [33]. In this Section, we give some properties of injective modules over $k\langle x_1, x_2, \cdots, x_n\rangle$ in the case $x_1, x_2, \cdots, x_n$ are non-commuting indeterminates. Throughout this Section, *it is assumed that* $x_1, x_2, \dots, x_n$ *are* n *non-commuting indeterminates,* $n > 1$ *and* $R := k\langle x_1, x_2, \dots, x_n\rangle$ *denotes the free* k*-algebra.* Notice that R is a *free ideal ring*, that is, a ring in which every right (or left) ideal is a free R-module of unique rank (Example 1.1.11). In particular, R is a hereditary ring.

In the following, "countable" means either finite or of cardinality $\aleph_0$, the first infinite cardinal.

A set $\{\, f_\lambda \mid \lambda \in \Lambda \,\}$ of elements of R is called *right linearly independent* over R if, for any $m > 0$, a subset $\{\, f_{\lambda_1}, f_{\lambda_2}, \cdots, f_{\lambda_m} \,\}$ of m elements of $\{\, f \mid \lambda \in \Lambda \,\}$ and $r_1, r_2, \dots, r_n \in R$ with $f_{\lambda_1} r_1 + f_{\lambda_2} r_2 + \cdots + f_{\lambda_m} r_m = 0$, one has $r_1 = r_2 = \cdots = r_m = 0$. Since R is a domain, it is easy to check that $\{\, f_\lambda \mid \lambda \in \Lambda \,\}$ is right linearly independent over R if and only if $\sum_{\lambda\in\Lambda} f_\lambda R = \bigoplus_{\lambda\in\Lambda} f_\lambda R$.

Lemma 1.3.1. *Every right ideal* I *of* R *is generated by a right linearly independent set of countably many elements of* I*.*

Proof. Let I be a right ideal of R. Then I is a free right R-module, so that it has a free set of generators f_λ, $\lambda \in \Lambda$. Thus $I = \bigoplus_{\lambda\in\Lambda} f_\lambda R$, which implies that the f_λ's are right linearly independent. Since they are right linearly independent over R, they are a fortiori linearly independent over k. But R is countably dimensional over k, so that Λ is countable. ■

Proposition 1.3.2. *A right* R*-module* M_R *is injective if and only if for every right linearly independent set* $\{\, f_\lambda \mid \lambda \in \Lambda \,\}$ *over* R*, with* Λ *a countable set, and every family* $\{\, x_\lambda \mid \lambda \in \Lambda \,\}$ *of elements of* M_R*, there exists* $x \in M_R$ *such that* $x f_\lambda = x_\lambda$ *for every* $\lambda \in \Lambda$*.*

PROOF. The module M_R is injective if and only if, for every right ideal I of R, any homomorphism $I_R \to M_R$ extends to R_R. By Lemma 1.3.1, I_R is generated by a right linearly independent set $\{ f_\lambda \mid \lambda \in \Lambda \}$, with Λ countable, that is, $I = \bigoplus_{\lambda \in \Lambda} f_\lambda R$ and $f_\lambda R \cong R_R$. Thus a homomorphism $I_R \to M_R$ is completely determined by the image of the f_λ's, which can be arbitrary elements x_λ of M_R. Thus a homomorphism $\varphi\colon I_R \to M_R$ with $\varphi(f_\lambda) = x_\lambda$ extends to R_R if and only if there exists $x \in M_R$ such that $x f_\lambda = x_\lambda$ for every $\lambda \in \Lambda$. ∎

Proposition 1.3.3. *Every homomorphic image of an injective right R-module is injective.*

PROOF. This follows from Proposition 1.1.12 because R is a hereditary ring. ∎

We recall the following Lemma, that appears in Bergman [6, Lemma 1].

Lemma 1.3.4. [6, Lemma 1] *Let S be a ring, κ an infinite regular cardinal such that every right ideal of S can be generated by less than κ elements, and $\{ M_i \mid i \in I \}$ a family of injective right S-modules. Then the submodule $\prod_{i\in I}^{\kappa} M_i$ of the direct product $\prod_{i\in I} M_i$ whose elements are the $x \in \prod_{i\in I} M_i$ with support of cardinality $< \kappa$ is an injective submodule of $\prod_{i\in I} M_i$.*

Here, an infinite cardinal κ is called a *regular cardinal* if for any set I with cardinal $|I| < \kappa$ and any family of sets $A_i, i \in I$, such that $|A_i| < \kappa$ for every $i \in I$, then the cardinal $|\cup_{i\in I} A_i| < \kappa$.

Proposition 1.3.5. *Let $\aleph_1$ be the second smallest infinite cardinal. For every family $\{ M_i \mid i \in I \}$ of injective right R-modules, the submodule $\prod_{i\in I}^{\aleph_1} M_i$ of the direct product $\prod_{i\in I} M_i$ whose elements are the $x \in \prod_{i\in I} M_i$ with countable support is injective.*

PROOF. The statement is implied directly from Lemma 1.3.4 since $\aleph_1$ is a regular cardinal. ∎

Theorem 1.3.6. *The module $Q_{mr}(R)_R$ is an injective envelope of R_R.*

PROOF. By Proposition 1.2.5, R_R is essential in $Q_{mr}(R)_R$. Now R is non-singular, so that, from [5, Theorem 2.1.15], $Q_{mr}(R)_R$ is injective. Thus, $Q_{mr}(R)_R$ is an injective envelope of R_R. ∎

1.4 The endomorphism ring of an injective module

In this Section, we recall some well known results on the endomorphism ring of an injective module.

Recall that the *Jacobson radical* $J(R)$ of an arbitrary ring R is the intersection of all maximal left ideals of R. It is easily proved that $J(R)$ is equal to the intersection of all maximal right ideals of R and, hence, $J(R)$ is a two-sided ideal of R.

Theorem 1.4.1. [30, Theorem 13.1] *Let M_R be an injective right R-module and $S = \mathrm{End}(M_R)$ be the endomorphism ring of M_R. Then*

1. *The Jacobson radical $J(S)$ of S is $\{\, f \in S \mid \ker f \leq_e M_R\}$.*
2. *The ring $S/J(S)$ is a von Neumann regular ring. That is, for every $\overline{f}$ in $S/J(S)$, there exists $\overline{g}$ in $S/J(S)$ such that $\overline{f} = \overline{fgf}$.*
3. *Idempotents of $S/J(S)$ can be lifted to idempotents of S. That is, for any $\overline{f}$ in $S/J(S)$, if $\overline{f}^2 = \overline{f}$, there exists g in S such that $g^2 = g$ and $\overline{g} = \overline{f}$.*
4. *The quotient ring $S/J(S)$ is a right self-injective ring. That is, $S/J(S)$ is an injective right $S/J(S)$-module.*

A non-zero right R-module M_R is called *indecomposable* if, for two submodules A, B of M_R with $A \oplus B = M_R$, one has $A = 0$ or $B = 0$. A module is said to be *indecomposable injective* if it is injective and indecomposable. The class of indecomposable injective modules and their endomorphism rings were studied in [33].

Theorem 1.4.2. [33, Theorem 2.6] *An injective module is indecomposable if and only if its endomorphism ring is a local ring.*

Here, a ring R is called *local* if R has a unique maximal left ideal, equivalently if R has a unique maximal right ideal, equivalently if the set of non-invertible elements of R is a two-sided ideal, equivalently if the Jacobson radical $J(R)$ of R is the set of all non-invertible elements of R.

Chapter 2

Quasi-duo rings and endomorphism rings of injective modules

Let M_R be an injective module over R. It is well known that M_R is indecomposable if and only if the endomorphism ring $S := \mathrm{End}(M_R)$ of M_R is a local ring (Theorem 1.4.2). In this trivial case, every maximal right ideal and maximal left ideal of S is two-sided. Rings in which all maximal right ideals and maximal left ideals are two-sided are called *quasi-duo*. Hence, the endomorphism ring of an indecomposable injective module is quasi-duo. The aim of this Chapter is to consider injective modules whose endomorphism rings are quasi-duo. Such injective modules will be proved to be *square-free*, that is, they contain no non-zero direct summand isomorphic to $A \oplus A$ for some direct summand A of M_R. Conversely, we also show that if M_R is square-free injective, then S is quasi-duo.

In Sections 2.1 and 2.2, we give some examples of square-free modules and quasi-duo rings. In Section 2.3 we present the main results of this Chapter, which appear in [7].

2.1 Square-free injective modules

In this Section, we recall some notions and properties of square-free injective modules.

A right module M_R over a ring R is called *square-free* if it contains no non-zero submodule isomorphic to $A \oplus A$ for some submodule A of M_R. An injective module which is square-free is said to be *square-free injective.*

Proposition 2.1.1. *An injective right R-module M_R is square-free if and only if it contains no non-zero direct summand isomorphic to $A \oplus A$ for some direct summand A of M_R.*

PROOF. If M_R is square-free, then obviously M_R contains no non-zero direct summand isomorphic to $A \oplus A$ for some direct summand A of M_R.

Conversely, assume that M_R is not square-free. Then there exists a non-zero submodule B of M_R such that N is a submodule of M_R and $N \cong B \oplus B$. Let $E(N)$ and $A := E(B)$ be injective envelopes of N and B respectively in M_R. By Theorem 1.1.8, $E(N) \cong A \oplus A$. Notice that $E(N)$ and A are non-zero direct summands of M_R, so that M_R contains the direct summand $E(N)$ which is isomorphic to $A \oplus A$. ■

Obviously, indecomposable injective modules are square-free injective. Since the injective envelope of a simple module is indecomposable, one has.

Proposition 2.1.2. *The injective envelope of a simple module is square-free injective.*

The following Lemma is trivial (see [8]).

Lemma 2.1.3. *Let A_R, B_R be simple right R-modules and $E(A_R)$, $E(B_R)$ be injective envelopes respectively of A_R and B_R. Then $E(A_R)$ is isomorphic to $E(B_R)$ if and only if A_R is isomorphic to B_R.*

The following Proposition is trivial.

Proposition 2.1.4. *Over a semisimple artinian ring, there are only finitely many square-free (injective) modules up to isomorphisms.*

Proposition 2.1.5. *Let $\{\, M_\lambda \mid \lambda \in \Lambda \,\}$ be a family of simple R-modules and $M_R = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a semisimple module. The injective envelope $E(M_R)$ is square-free if and only if $\{\, M_\lambda \mid \lambda \in \Lambda\}$ is a family of pair-wise non-isomorphic modules.*

Proof. We have $E(M_R) = E(\bigoplus_{\lambda \in \Lambda} M_\lambda) = E(\bigoplus_{\lambda \in \Lambda} E(M_\lambda))$ where $E(M_\lambda)$ is the injective envelope of M_λ. The “if” part is from Lemma 2.1.3 and the other part is from [43, Corollary 4.2] and Lemma 2.1.3. ■

2.2 Quasi-duo rings

A ring R is called *left duo* (resp. *left quasi-duo*) if every left ideal (resp. maximal left ideal) of R is a two-sided ideal. *Right duo* (resp. *right quasi-duo*) rings are defined similarly. If R is both left and right duo (resp. quasi-duo), then R is said to be *duo* (resp. *quasi-duo*). Of course, every duo ring is quasi-duo, commutative rings and division rings are duo.

Proposition 2.2.1. *A ring R is quasi-duo if and only if $R/J(R)$ is quasi-duo.*

Proof. The statement is trivial since all maximal left ideals and maximal right ideals of R contain the Jacobson radical $J(R)$. ■

Proposition 2.2.2. *The matrix ring $R = M_n(D)$ over a division ring D is quasi-duo if and only if $n = 1$.*

Proof. If $n = 1$, then $R = D$ is quasi-duo. If $n > 1$, then the set of all matrices whose first column is 0 is a maximal left ideal of R. On the other hand, R has only two two-sided ideals 0 and R. Hence, R is not quasi-duo. ■

Proposition 2.2.3. *The ring $R = R_1 \times R_2 \times \cdots \times R_n$ is quasi-duo if and only if each ring R_i is quasi-duo.*

Proof. The result is elementary because I_i is a left (resp. right or two-sided) ideal of R_i if and only if $R_1 \times \cdots \times I_i \times \cdots \times R_n$ is a left (resp. right or two-sided) ideal of R. ■

For any positive integer n, a ring R is called *type n* if the quotient ring $R/J(R)$ is isomorphic to $D_1 \times D_2 \times \cdots \times D_n$ where D_i is a division ring. From Proposition 2.2.1, 2.2.2 and 2.2.3, the following Corollary follows immediately.

Corollary 2.2.4. *Every ring of type n is quasi-duo.*

Now let us give some examples of quasi-duo rings.

Example 2.2.5. The following rings are quasi-duo:

1. Local rings. Hence, so are the endomorphism rings of indecomposable injective modules (Theorem 1.4.2).

2. The endomorphism rings of couniform projective modules ([2, Theorem 8.7] and Corollary 2.2.4). Recall that a non-zero right R-module M_R is said to be *couniform* (or *hollow*) if, for any submodules A, B of M_R, $M_R = A + B$ implies $A = M_R$ or $B = M_R$. By the definition of superfluous submodules, M_R is couniform if and only if every proper submodule of M_R is superfluous. A *couniform projective* module is a module which is projective and couniform.

3. The endomorphism rings of non-zero uniserial modules ([14, Theorem 9.1] and Corollary 2.2.4). Here, recall that an *uniserial module* is a module whose set of submodules is totally ordered by inclusion. That is, for any two submodules N_1 and N_2, either $N_1 \leq N_2$ or $N_2 \leq N_1$.

4. The endomorphism rings of cyclically presented modules over a local ring ([2, Theorem 2.1] and Corollary 2.2.4). Notice that a right module over a ring R is *cyclically presented* if it is isomorphic to R/aR for some element $a \in R$.

5. The endomorphism ring of the kernel of a non-zero homomorphism between indecomposable injective modules ([18] and Corollary 2.2.4).

Notice that examples of right quasi-duo rings which are not left quasi-duo are unknown [31, Question 7.7]. Of course, if there exists such a ring R, then the opposite ring R^{op} of R is left quasi-duo and not right quasi-duo.

2.3 The endomorphism ring of a square-free injective module

Let M_R be a right module over a ring R and N be a direct summand of M_R, that is, $M_R = N \oplus N'$ for some submodule N' of M_R. Throughout this section, ι_N denotes the embedding of N into M_R, and π_N denotes the projection of M_R onto N.

Proposition 2.3.1. *Let M_R be an injective R-module. If the endomorphism ring $S = \mathrm{End}(M_R)$ of M_R is either left quasi-duo or right quasi-duo, then M_R is square-free.*

Proof. Assume that S is left quasi-duo and there exist direct summands A, B, C of M_R such that $A \oplus B \oplus C = M_R$ and A is isomorphic to B. Call I a maximal left ideal of S containing $s = \iota_{B\oplus C}\pi_{B\oplus C}$. Then I is a two-sided ideal of S by hypothesis. Let α be an isomorphism from A to B. Define $f := \iota_B\alpha\pi_A$ and $g := \iota_A\alpha^{-1}\pi_B + \iota_C\pi_C$. Then $sf + gs \in I$ and

$$\begin{aligned}\ker(sf + gs) &= \{\, a + b + c \in A \oplus B \oplus C \mid (sf + gs)(a + b + c) = 0 \,\}\\ &= \{\, a + b + c \in A \oplus B \oplus C \mid \alpha(a) + \alpha^{-1}(b) + c = 0 \,\} = 0.\end{aligned}$$

Hence, $sf + gs$ is injective. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \oplus B \oplus C = M_R \arrow[r, "sf+gs"] \arrow[d, "\mathrm{Id}_{M_R}"] & M_R \arrow[dl, dotted, "h"] \\
& M_R &
\end{tikzcd}$$

Since M_R is injective, there exists a homomorphism h of S such that $h(sf+gs) = 1$, which implies $I = S$. Contradiction. Therefore, if S is left quasi-duo, then M_R is square-free.

Similarly to the right side. ■

Lemma 2.3.2. *Let M_R be a square-free injective R-module. If N_1, N_2 are two isomorphic direct summands of M_R, then N_1 and N_2 are two injective envelopes of $N_1 \cap N_2$.*

Proof. Let A_1, A_2 be injective envelopes of $N_1 \cap N_2$ respectively in N_1, N_2 and B_1, B_2 be respectively direct summands of N_1, N_2 such that

$$N_1 = A_1 \oplus B_1, N_2 = A_2 \oplus B_2.$$

Since $N_1 \cong N_2$ and $B_1 \cap A_2 = 0$, B_1 is isomorphic to a direct summand C of N_2 and since $B_1 \cap N_2 = 0$, $B_1 + N_2 = B_1 \oplus N_2$ is injective, which implies that M_R contains $B_1 \oplus C$, with $B_1 \cong C$, as a direct summand. Therefore $B_1 = 0$. Similarly, $B_2 = 0$. Thus, N_1, N_2 are two injective envelopes of $N_1 \cap N_2$. ■

Lemma 2.3.3. *Let M_R be a square-free injective R-module with $M_R = M_1 \oplus M_2$. If N is a direct summand of M_R, then N is an injective envelope of $(N \cap M_1) \oplus (N \cap M_2)$.*

PROOF. Let A be an injective envelope of $N \cap M_1$ in N and B be a direct summand of N such that $N = A \oplus B$. Consider $\pi := \pi_{M_2}$, the projection of $M_1 \oplus M_2$ onto M_2, and the restriction $\pi|_B : B \to M_2$. Because $B \cap M_1 = 0$, $\pi|_B$ is injective. Hence, B is isomorphic to a direct summand C of M_2. By Lemma 2.3.2, B is an injective envelope of $B \cap C$. Therefore, B is an injective envelope of $N \cap M_2 = B \cap M_2 \geq B \cap C$. This implies N is an injective envelope of $(N \cap M_1) \oplus (N \cap M_2)$. ■

Lemma 2.3.4. *Let M_R be an injective R-module and $S = \mathrm{End}(M_R)$ be the endomorphism ring of M_R. For any element $f \in S$, there exist $e_1, e_2, g_1, g_2, h_1, h_2 \in S$ and $i_1, i_2, j_1, j_2 \in J(S)$ such that e_1, e_2 are idempotents and*

$$e_1 = fg_1 + i_1, f = e_1h_1 + j_1,$$

$$e_2 = g_2f + i_2, f = h_2e_2 + j_2.$$

PROOF. Notice that any finitely generated left ideal of a von Neumann regular ring is generated by an idempotent. Now the ring $S/J(S)$ is a Von Neumann regular ring and any idempotent of $S/J(S)$ can be lifted to an idempotent of S (see Theorem 1.4.1). Hence, for any $f \in S$, there exists an idempotent $e_1 \in S$ such that

$$(fS + J(S))/J(S) = (e_1S + J(S))/J(S).$$

Therefore, $e_1 = fg_1 + i_1, f = e_1h_1 + j_1$ for some $g_1, h_1 \in S$ and $i_1, j_1 \in J(S)$. Similarly to $e_2 = g_2f + i_2, f = h_2e_2 + j_2$. ■

Lemma 2.3.5. *Let M_R be an injective R-module, $S = \mathrm{End}(M_R)$ be the endomorphism ring of M_R and e_1, e_2 be idempotents of S. Then $e_1 = e_2s + j$ for some $s \in S, j \in J(S)$ if and only if $e_1(M_R) \cap A \leq e_2(M_R)$ for some essential submodule A of M_R.*

PROOF. Suppose that $e_1 = e_2 s + j$ for some $s \in S$ and $j \in J(S)$. Then $A := \ker j$ is essential in M_R and for any $e_1(x) \in A$, $e_1(x) = (e_1 - j)(e_1(x)) = e_2 s(x) \in e_2(M_R)$. Hence, $e_1(M_R) \cap A \leq e_2(M_R)$. Conversely, assume that $e_1(M_R) \cap A \leq e_2(M_R)$ for some essential submodule A of M_R. If we put $A_1 := e_1(M_R) \cap A$ and $A_2 := (1 - e_1)(M_R) \cap A$, then $A_1 \oplus A_2$ is essential in A. Hence, $A_1 \oplus A_2$ is essential in M_R. Put $\pi := \pi_{e_1(M_R)}$ and $\iota := \iota_{e_1(M_R)}$. By $A_1 \leq e_1(M_R) \cap e_2(M_R)$, for any $a + b \in A_1 \oplus A_2$, $(e_1 - e_2 \iota\pi)(a + b) = 0$. Therefore, $e_1 - e_2\iota\pi \in J(S)$. Hence, $e_1 = e_2 s + j$ with $s = \iota\pi \in S, j = e_1 - e\iota\pi \in J(S)$. ■

Proposition 2.3.6. *Let M_R be an injective R-module and $S = \mathrm{End}(M_R)$ be the endomorphism ring of M_R. If M_R is square-free, then every right ideal of S containing $J(S)$ of S is a two-sided ideal. In particular, if M_R is square-free, then S is right quasi-duo.*

PROOF. Let I be a right ideal of S containing $J(S)$. Let $f \in I$ and $\phi \in S$. We must show that $\phi f \in I$. By Lemma 2.3.4, there exist $e, g, h \in S$ and $i, j \in J(S)$ such that $e = fg + i$ is an idempotent and $f = eh + j$. The element e belongs to I and $\phi f = \phi e h + \phi j$, so that it suffices to show that $\phi e \in I$. Indeed, let N' be a direct summand of M_R such that $M_R = N \oplus N'$ with $N = e(M_R)$, and M_1 be an injective envelope of $\ker\phi$ in M_R. Then there exists a direct summand M_2 of M_R such that $M_R = M_1 \oplus M_2$. Let N_1, N_2 be respectively injective envelopes of $N \cap M_1, N \cap M_2$ in N. By Lemma 2.3.3, we may assume that

$$M_R = N \oplus N' = N_1 \oplus N_2 \oplus N'.$$

Consider the restriction $\phi|_{N_2} \colon N_2 \to N_1 \oplus N_2 \oplus N'$ of ϕ to N_2. It is easy to check that $\phi|_{N_2}$ is injective. Hence, $N_2 \cong \phi|_{N_2}(N_2)$ and by Lemma 2.3.2, $A = N_2 \cap \phi|_{N_2}(N_2)$ is an essential submodule of N_2. Put $B := \phi|_A^{-1}(A)$ with $\phi|_A : A \to N_2$ the restriction of ϕ to A. Then B is also an essential submodule of N_2. Write $\psi \colon N_2 \to N_2$ for a homomorphism extending $\phi|_A$ and let $\psi' = \iota_{N_2}\psi\pi_{N_2}$. One has $\psi'(M_R) \leq N_2 \leq N = e(M_R)$, which implies from Lemma 2.3.5 that $\psi' \in eS \subseteq I$. Moreover, since for any $a + b + c \in (\ker\phi \cap N_1) \oplus B \oplus N'$, $(\phi e - \psi')(a + b + c) = 0$ and $(\ker\phi \cap N_1) \oplus B \oplus N'$ is essential in M_R, $\phi e - \psi'$ belongs to $J(S)$. This shows that $\phi e \in I$. ■

Lemma 2.3.7. *Let M_R be an injective R-module, $S = \operatorname{End}(M_R)$ be the endomorphism ring of M_R. For any two elements f, g of S, $g = hf + j$ for some $h \in S, j \in J(S)$ if and only if* $\ker f \cap A \leq \ker g$ *for some essential submodule A of M_R.*

Proof. Assume that $g = hf + j$ for some $h \in S$ and $j \in J(S)$. Then $\ker g \geq \ker(hf) \cap \ker j \geq \ker f \cap A$ with $A = \ker j \leq_e M_R$.

Conversely, assume that $\ker f \cap A \leq \ker g$ for a submodule $A \leq_e M_R$. Let M_1 be an injective envelope of $\ker g$ in M_R, and N_1 be an injective envelope of $\ker f \cap A$ in M_1. Assume that $M_R = N_1 \oplus N_2 \oplus M_2$ for some direct summands N_2, M_2 respectively of M_1 and M_R. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & N_2 \oplus M_2 \arrow[r, "f|_{N_2 \oplus M_2}"] \arrow[d, "g|_{N_2 \oplus M_2}"] & M_R \arrow[dl, dotted, "h"] \\
 & M_R &
\end{tikzcd}$$

Here, $f|_{N_2 \oplus M_2}$ and $g|_{N_2 \oplus M_2}$ are the restrictions of f and g to $N_1 \oplus M_2$. Since M_R is injective, there exists $h : M_R \to M_R$ such that

$$g|_{N_2 \oplus M_2} = h \; f|_{N_2 \oplus M_2}.$$

Because $(g - hf)(a + b + c) = 0$ for any $a + b + c \in (\ker f \cap A) \oplus N_2 \oplus M_2$ which is essential in M_R, $j = g - hf \in J(S)$. ■

Proposition 2.3.8. *Let M_R be an injective R-module and $S = \operatorname{End}(M_R)$ be the endomorphism ring of M_R. If M_R is square-free, then every left ideal of S containing the Jacobson radical $J(S)$ of S is a two-sided ideal. In particular, if M_R is square-free, then S is left quasi-duo.*

Proof. Let I be a left ideal of S containing $J(S)$. Let f, ϕ be elements of S with $f \in I$. We must show that $f\phi \in I$. By Lemma 2.3.4, there exist $e, g, h \in S$ and $i, j \in J(S)$ such that $e = gf + i$ is an idempotent and $f = he + j$. Hence, $e \in I$ and $f\phi = he\phi + j\phi$. It suffices to show that $e\phi \in I$. Indeed, put $N_1 := \ker e$ and let N_2 be a direct summand of M_R such that $M_R = N_1 \oplus N_2$. Consider $\psi := \pi_{N_2} \phi \iota_{N_1} : N_1 \to N_2$. Let N_1' be an injective envelope of $A := \ker \phi$ in N_1. Then exists a direct summand N_1'' of M_R such that $N_1 = N_1' \oplus N_1''$. Since $\ker \psi|_{N_1''} = N_1'' \cap A = 0$, $\psi|_{N_1''}$ is injective. This implies N_1'' is isomorphic to a direct summand C of N_2. Hence, M_R contains a direct summand isomorphic to $C \oplus C$. By hypothesis, $C = 0$. In other words, A is essential in N_1. Now, one has that ϕ can be written as the matrix $\phi = \begin{pmatrix} \pi_{N_1}\phi\iota_{N_1} & \pi_{N_1}\phi\iota_{N_2} \\ \pi_{N_2}\phi\iota_1 & \pi_{N_2}\phi\iota_{N_2} \end{pmatrix}$. Then $\ker(e\phi) = \phi^{-1}(\ker e) = \phi^{-1}(N_1) \geq \ker(\pi_{N_2}\phi\iota_1) \oplus \ker(\pi_{N_2}\phi\iota_{N_2}) \geq A = (A \oplus N_2) \cap N_1 = (A \oplus N_2) \cap \ker e$. Since $A \oplus N_2$ is essential in M_R, by Lemma 2.3.7, there exist $h \in S, j \in J(S)$ such that $e\phi = he + j$. Thus $e\phi \in I$. ■

Now the following is the main result of this Chapter.

Theorem 2.3.9. *Let M_R be an injective right R-module and $S = End(M_R)$. Then the following conditions are equivalent*

1. *M_R is square-free.*
2. *S is left quasi-duo.*
3. *S is right quasi-duo.*
4. *Every left ideal of S containing $J(S)$ is a two-sided ideal of S.*
5. *Every right ideal of S containing $J(S)$ is a two-sided ideal of S.*

PROOF. (1) ⇒ (4) is from Proposition 2.3.8.

(1) ⇒ (5) is Proposition 2.3.6.

(4) ⇒ (2) and (5) ⇒ (3) are obvious.

Finally, (2) ⇒ (1) and (3) ⇒ (1) is Proposition 2.3.1,

■

Chapter 3

Maximal ideals of the endomorphism ring of an injective module

It is well known that the endomorphism ring of an indecomposable injective module is local (Theorem 1.4.2), hence has a unique maximal right (left, two-sided) ideal. Similarly, the maximal right (left, two-sided) ideals of the endomorphism ring $\mathrm{End}(V_D)$ of a right vector space V_D over a division ring D are easy to describe [35]. In this case, the left ideals of $\mathrm{End}(V_D)$ correspond to the filters of the lattice $\mathcal{L}(V_D)$ of all subspaces of V_D, and the right ideals of $\mathrm{End}(V_D)$ correspond to the antifilters of $\mathcal{L}(V_D)$. In particular, maximal left ideals of $\mathrm{End}(V_D)$ correspond to ultrafilters, and maximal right ideals to ultraantifilters. The description of the maximal two-sided ideals of $\mathrm{End}(V_D)$ is also known [35, Theorem 4.3]. In the first Section we will epitomize these results.

The aim of the three Sections 3.2, 3.3 and 3.4 is to describe the maximal right ideals and maximal left ideals of the endomorphism ring S of an arbitrary injective module M_R over a ring R. In this case also, maximal left ideals of S correspond to ultrafilters of a suitable lattice $\mathcal{O}$, which is the set of orbits of the set of all direct summands of M_R under a suitable group action. In Section 3.2, we will present the construction of the lattice $\mathcal{O}$, which is based on a suitable equivalence between injective envelopes of a module, and then, in Section 3.3, we present the correspondence. In fact, this correspondence works not only for maximal left ideals, but also, more generally, for any left ideal containing the Jacobson radical $J(S)$ of S. Similarly, we have a one-to-one correspondence between the set of all right ideals of S containing $J(S)$ and the set of all antifilters of $\mathcal{O}$, which will be presented in Section 3.4.

Maximal two-sided ideals of S correspond to prime ideals of height one of the commutative monoid $V(M_R)$. Here $V(M_R)$ is the commutative monoid of all direct summands of the modules M_R^n, $n \geq 0$, up to isomorphism, with the operation induced by direct sum. The monoid $V(M_R)$ is canonically isomorphic to the monoid $V(S)$ of all finitely generated projective right S-modules up to isomorphism. If P denotes the prime ideal of height one of $V(S)$ corresponding to the maximal two-sided ideal I of S, then the monoid $V(S/I)$ turns out to be isomorphic to the reduced localization $(V(S)_P)_{\mathrm{red}}$ of $V(S)$ at the prime ideal P. Moreover, there is a one-to-one correspondence between the set of all maximal two-sided ideals of S and the set of all ultracofilters of the lattice $\mathcal{L}_{ds}(M_R)$ of all direct summands of M_R. These results appear in Sections 3.5 and 3.6.

The last two Sections are devoted to considering the case in which M_R is an injective envelope of a direct sum of indecomposable injective modules, that is, an injective envelope of a direct sum $\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)}$, where the A_λ's are pair-wise non-isomorphic indecomposable injective modules and the κ_λ's are cardinals.

The main results in this Chapter have been published in [16].

3.1 Maximal ideals of the endomorphism ring of a vector space over a division ring

In this Section, we epitomize some main results on maximal left, maximal right and maximal two-sided ideals of $\mathrm{End}(V_D)$, where V_D is a right vector space over a division ring D.

It is well known that if V_D is a finite dimensional vector space, then the endomorphism ring $\mathrm{End}(V_D)$ is isomorphic to the matrix ring $M_n(D)$ with $n = \dim V_D$, the dimension of V_D. Hence, in this case, $\mathrm{End}(V_D)$ has the unique maximal two-sided ideal 0.

In the case of infinite dimensional vector spaces, Orsatti and Rodino described maximal left, maximal right and maximal two-sided ideals in [35] as follows.

First we need to recall some notions concerning lattices. Let L be a lattice with order $\preceq$, meet $\wedge$ and join $\vee$. A subset F of L is called a *filter* if

(F1): $F \neq \emptyset$.

(F2): $V_1 \vee V_2 \in F$ for any $V_1, V_2 \in F$.

(F3): if $V \in F$, $U \in L$ and $V \preceq U$, then $U \in F$.

If the filter F of L is different from L, then we say that F is a *proper filter*. If F is proper and maximal among all proper filters of L (ordered by inclusion), then F is said to be an *ultrafilter*. The filter F is called *principal* if there exists an element $a \in F$ such that $F = \{\, x \in L \mid a \preceq x \,\}$. Otherwise, F is called *free*.

The dual notion of a filter is that of an antifilter (or *ideal*). A subset A of L is called an *antifilter* if

(AF1): $A \neq \emptyset$.

(AF2): $V_1 \wedge V_2 \in A$ for any $V_1, V_2 \in A$.

(AF3): if $V \in L$, $U \in A$ and $V \preceq U$, then $V \in A$.

If the antifilter A of L is different from L, then we say A a *proper antifilter*. If the filter A is proper and maximal among all proper antifilters of L, then A is said to be an *ultraantifilter*. The antifilter A is called *principal* if there exists an element $a \in A$ such that $A = \{\, x \in L \mid x \preceq a \,\}$. Otherwise, A is called *free*.

From now to the end of this Section, it is assumed that V_D is a vector space with $\dim(V_D) = d$, an infinite cardinal. Let $S := \operatorname{End}(V_D)$ be the endomorphism ring of V_D. Let $\mathcal{L}(V_D)$ be the set of all subspaces of V_D. Then $\mathcal{L}(V_D)$ is a lattice with the inclusion order, the join $V_1 \vee V_2 = V_1 + V_2$ and the meet $V_1 \wedge V_2 = V_1 \cap V_2$ for any $V_1, V_2 \in \mathcal{L}(V_D)$.

Theorem 3.1.1. [35, Proposition 2.3 and 2.4] *Let S and $\mathcal{L}(V_D)$ be as above.*

1. *For any left ideal I of S, the set $\mathcal{F}(I) := \{\, \ker f \mid f \in I \,\}$ is a filter of $\mathcal{L}(V_D)$.*
2. *For any filter F of $\mathcal{L}(V_D)$, the set $\mathcal{I}(F) := \{\, f \in S \mid \ker f \in F \,\}$ is a left ideal of S.*

Moreover, these associations are mutually inverse one-to-one correspondences between the set of all left ideals of S and the set of filters of $\mathcal{L}(V_D)$.

Corollary 3.1.2. [35, Corollary 2.7] *For any left ideal of I of S, the following conditions are equivalent:*

1. *I is generated by an idempotent element.*
2. *I is generated by an element.*
3. *$\mathcal{F}(I)$ is principal.*

Dually, we have results for the right side.

Theorem 3.1.3. [35, Proposition 3.3 and 3.4] *Let $S, \mathcal{L}(V_D)$ be as above.*

1. *For any right ideal I of S, the set $\mathcal{A}(I) := \{ f(V_D) \mid f \in I \}$ is an antifilter of $\mathcal{L}(V_D)$.*
2. *For any antifilter A of $\mathcal{L}(V_D)$, the set $\mathcal{I}(A) := \{ f \in S \mid f(V_D) \in A\}$ is a right ideal of S.*

Moreover, these associations are two mutually inverse one-to-one correspondences between the set of all right ideals of S and the set of antifilters of $\mathcal{L}(V_D)$.

Corollary 3.1.4. [35, Corollary 3.8] *For any right ideal I of S, the following conditions are equivalent:*

1. *I is generated by an idempotent element.*
2. *I is generated by an element.*
3. *$\mathcal{A}(I)$ is principal.*

Next are some results concerning two-sided ideals of S.

Theorem 3.1.5. [35, Theorem 4.3] *Let S, V_D, d be as above.*

1. *For any infinite cardinal $c \leq d$, the set $L_c := \{ f \in S \mid \dim f(V_D) < c \}$ is a proper two-sided ideal of S.*
2. *For any proper two-sided ideal I of S, there exists an infinite cardinal $c \leq d$ such that $I = L_c$.*

Moreover, these associations are two mutually inverse one-to-one correspondences between the set of all infinite cardinals $c \leq d$ and the set of proper two-sided ideals of S.

Corollary 3.1.6. $L_d = \{ f \in S \mid \dim f(V_D) < d \}$ *is the unique maximal two-sided ideal of S.*

3.2 The lattice $\mathcal{O}$

From now to the end of this Chapter, M_R will always be an injective right module over an arbitrary ring R and S will be the endomorphism ring of M_R.

Lemma 3.2.1. *Let B be a submodule of M_R and N_1, N_2 be two direct summands of M_R with $B \leq_e N_1$ and $B \leq_e N_2$. Then there exist an essential submodule A of M_R containing B and an automorphism f of M_R such that $N_2 = f(N_1)$ and $f(a) = a$ for every $a \in A$.*

PROOF. Suppose $M_R = N_1 \oplus N_1'$ and $M_R = N_2 \oplus N_2'$. Set $B_1 := B \oplus N_1'$, $B_2 := B \oplus N_2'$ and $A := B_1 \cap B_2$. Then B_1 and B_2 are essential in M_R, hence so is A. Since both N_1 and N_2 are injective essential extensions of B, there is an isomorphism $g\colon N_1 \to N_2$ that is the identity on B. Extend g to an isomorphism $h\colon N_1 + A \to N_2 + A$ setting $h(x + a) = g(x) + a$ for every $x \in N_1$ and every $a \in A$. Now extend h to an isomorphism $f\colon M_R \to M_R$. Then $f(N_1) = N_2$ and $f(a) = a$ for every $a \in A$. ■

Let $\mathrm{Aut}(M_R)$ be the group of all automorphisms of M_R, that is, the group of all invertible elements of S. Let G be the subgroup of $\mathrm{Aut}(M_R)$ consisting of the automorphisms φ of M_R for which there exists an essential submodule A of M_R with $\varphi(a) = a$ for every $a \in A$, that is, the automorphisms of M_R that are the identity on some essential submodule of M_R. One has that G is a normal subgroup of $\mathrm{Aut}(M_R)$ [24].

The group G acts on the set $\mathcal{L}_{ds}(M_R)$ of all direct summands of M_R in a natural way. Let $\mathcal{O} := \mathcal{L}_{ds}(M_R)/G$ be the set of all orbits GN, $N \in \mathcal{L}_{ds}(M_R)$. Define a relation $\preceq$ on the set $\mathcal{O}$, setting, for every $M', M'' \in \mathcal{L}_{ds}(M_R)$, $GM' \preceq GM''$ if there exists an essential submodule A of M_R with $M' \cap A \leq M''$.

Lemma 3.2.2. *The relation $\preceq$ is a partial order on the set $\mathcal{O}$.*

PROOF. First of all, one must show that the relation $\preceq$ is well defined. Suppose that $M', M'', N', N'' \in \mathcal{L}_{ds}(M_R)$, $GM' = GN'$, $GM'' = GN''$, and that there exists an essential submodule A of M_R with $M' \cap A \leq M''$. Then there are automorphisms φ and ψ of M_R such that $N' = \varphi(M')$, $N'' = \psi(M'')$, φ is the identity on an essential submodule A' of M_R and ψ is the identity on an essential submodule A''. Then $N' \cap A' \cap A'' \cap A =$

$\varphi(M')\cap A'\cap A''\cap A = M'\cap A'\cap A''\cap A$, and similarly $N''\cap A''\cap A'\cap A = M''\cap A''\cap A'\cap A$. Thus $N'\cap A'\cap A''\cap A = M'\cap A'\cap A''\cap A \leq M''\cap A'\cap A''\cap A = N''\cap A'\cap A''\cap A \leq N''$. This proves that the relation $\preceq$ is well defined.

The relation $\preceq$ is clearly reflexive and transitive. As far as symmetry is concerned, suppose $M', M'' \in \mathcal{L}_{ds}(M_R)$, $GM' \preceq GM''$ and $GM'' \preceq GM'$. Then there exist essential submodules A, A' of M_R with $M'\cap A \leq M''$ and $M''\cap A' \leq M'$. Thus $M'\cap A\cap A' = M''\cap A\cap A'$. Now $A\cap A'$ is an essential submodule of M_R, so that $M'\cap A\cap A'$ is essential in M' and $M''\cap A\cap A'$ is essential in M''. By Lemma 3.2.1, one has that $GM' = GM''$. ■

Lemma 3.2.3. *The partially ordered set $\mathcal{O}$ is a lattice with*

$$GM_1 \wedge GM_2 := G(E(M_1\cap M_2)) \qquad \text{and} \qquad GM_1 \vee GM_2 := G(E(M_1+M_2))$$

for any pair $GM_1, GM_2 \in \mathcal{O}$.

PROOF. The most important thing in this Lemma is to prove that the meet $\wedge$ and the join $\vee$ defined above are well-defined. The proof of the rest is elementary. Let $GM_1, GM_2, GM_1', GM_2' \in \mathcal{O}$ such that $GM_1 = GM_1'$ and $GM_2 = GM_2'$. Then there exist automorphisms φ, ψ and essential submodules A, B of M_R such that $M_1' = \varphi(M_1)$, $M_2' = \psi(M_2)$, $\varphi(a) = a$ for any $a \in A$ and $\psi(a) = a$ for every $a \in B$. Let $C = M_1\cap M_2\oplus N$ where N is a submodule of M_R such that $E(M_1\cap M_2)\oplus N = M_R$. Then C is essential in M_R. Now, $M_1\cap M_2\cap A\cap B\cap C = \varphi(M_1)\cap\psi(M_2)\cap A\cap B\cap C = M_1'\cap M_2'\cap A\cap B\cap C \leq E(M_1'\cap M_2')$. Because $A\cap B\cap C$ is essential in M_R, $GE(M_1\cap M_2) \preceq GE(M_1'\cap M_2')$. Since the symmetry of GM_1, GM_2 and GM_1', GM_2', one has $GE(M_1\cap M_2) = GE(M_1'\cap M_2')$, which means that the meet $\wedge$ is well-defined. Similarily for the join $\vee$. ■

3.3 Maximal left ideals

Theorem 3.3.1. *Associate to any left ideal I of S containing $J(S)$ the subset $\mathcal{F}(I) := \{\, G\ker e \mid e \in I, e^2 = e \,\}$ of $\mathcal{O}$. Conversely, associate to any filter F of $\mathcal{O}$ the subset*

$\mathcal{I}(F) := \{\, he + j \mid G\ker e \in F, h \in S, j \in J(S) \,\}$ of S. Then this defines two mutually inverse one-to-one correspondences between the set of all left ideals of S containing $J(S)$ and the set of all filters of $\mathcal{O}$.

Proof. First of all, we will prove that $\mathcal{F}(I)$ is a filter of $\mathcal{O}$ and $\mathcal{I}(F)$ is a left ideal of S for every left ideal I of S containing $J(S)$ and every filter F in $\mathcal{O}$. Suppose I is a left ideal of S containing $J(S)$. Clearly $GM = G\ker 0 \in \mathcal{F}(I)$, which is therefore non-empty. Now, given $G\ker e \in \mathcal{F}(I)$ where e is an idempotent of I and $GN \in \mathcal{O}$ such that $G\ker e \preceq GN$, we must show that $GN \in \mathcal{F}(I)$. Let e' be an idempotent of S such that $\ker e' = N$. As $G\ker e \preceq GN = G\ker e'$, there exists an essential submodule A of M_R such that $\ker e \cap A \leq \ker e'$. By Lemma 2.3.7, there exist $g \in S, j \in J(S)$ such that $e' = ge + j$. Hence $e' \in I$. Therefore $GN \in \mathcal{F}(I)$, as we wanted to prove. In order to conclude the proof that $\mathcal{F}(I)$ is a filter, fix two elements $GN_1, GN_2 \in \mathcal{F}(I)$. Thus $GN_1 = G\ker e_1$ and $GN_2 = G\ker e_2$ for idempotents $e_1, e_2 \in I$. Since $S/J(S)$ is Von Neumann regular and idempotents of $S/J(S)$ can be lifted to idempotents of S, $Se_1 + Se_2 + J(S) = Se + J(S)$ for some idempotent $e \in S$ (Theorem 1.4.1). Hence, $e \in I$ and $e_1 = s_1 e + j_1, e_2 = s_2 e + j_2$ for some $s_1, s_2 \in S$ and $j_1, j_2 \in J(S)$. Thus, $A = \ker j_1 \cap \ker j_2$ is an essential submodule of M_R and $\ker e \cap A \leq \ker e_1 \cap \ker e_2$. This shows that $G\ker e \preceq GE(\ker e_1 \cap \ker e_2)$. Therefore, $GN_1 \wedge GN_2 = GE(\ker e_1 \cap \ker e_2) \in \mathcal{F}(I)$. This proves that $\mathcal{F}(I)$ is a filter in $\mathcal{O}$.

Similarly, we have, dually, $\mathcal{I}(F)$ is a left ideal of S containing $J(S)$ for every filter F in $\mathcal{O}$.

Now we will prove that $\mathcal{F}(\mathcal{I}(F)) = F$ and $\mathcal{I}(\mathcal{F}(I)) = I$. The inclusion $\mathcal{F}(\mathcal{I}(F)) \subseteq F$ is trivial. For the inclusion $F \subseteq \mathcal{F}(\mathcal{I}(F))$, notice that an arbitrary element of $F \subseteq \mathcal{O}$ can be written in the form $G\ker e$ for some idempotent $e \in S$. Then $e \in \mathcal{I}(F)$. It follows that $G\ker e \in \mathcal{F}(\mathcal{I}(F))$. Thus $\mathcal{F}(\mathcal{I}(F)) = F$. Similarly, the proof of the inclusion $I \subseteq \mathcal{I}(\mathcal{F}(I))$ is easy. For the opposite inclusion, suppose $e \in \mathcal{I}(\mathcal{F}(I))$. Then $G\ker e \in \mathcal{F}(I)$. Thus there exists an idempotent $e' \in I$ such that $G\ker e = G\ker e'$. In particular, $\ker e' \cap A \leq \ker e$. From Lemma 2.3.7, we have that $e = ge' + j$ for suitable $g \in S, j \in J(S)$. This proves that $e \in I$, and $I = \mathcal{I}(\mathcal{F}(I))$. ■

Corollary 3.3.2. *Maximal left ideals of S correspond to ultrafilters of $\mathcal{O}$.*

3.4 Maximal right ideals

In this section, we will turn our attention to maximal right ideals.

Theorem 3.4.1. *Associate to any right ideal I of S containing $J(S)$ the subset $\mathcal{A}(I) := \{\, Ge(M_R) \mid e \in I, e^2 = e \,\}$ of $\mathcal{O}$. Conversely, associate to any antifilter A of $\mathcal{O}$ the subset $\mathcal{I}(A) := \{\, eh + j \mid e, h \in S,\ j \in J(S),\ e^2 = e,\ Ge(M_R) \in A \,\}$ of S. Then these associations are two mutually inverse one-to-one correspondences between the set of all right ideals of S containing $J(S)$ and the set of all antifilters of $\mathcal{O}$.*

Proof. We first show that $\mathcal{A}(I)$ is an antifilter of $\mathcal{O}$ for any right ideal I of S containing $J(S)$. From $0 \in I$, it follows that $G0 = G0(M_R) \in \mathcal{A}(I)$.

Now assume that $Ge'(M_R) \preceq Ge(M_R)$, where e, e' are idempotents in S and $Ge(M_R) \in \mathcal{A}(I)$. Then there exists an essential submodule A of M_R with $e'(M_R) \cap A \leq e(M_R)$. By Lemma 2.3.5 and $e \in I$, $e' \in eS + J(S) \subseteq I$. Hence, $Ge'(M_R) \in \mathcal{A}(I)$. Suppose that $Ge_1(M_R), Ge_2(M_R) \in \mathcal{A}(I)$, with e_1, e_2 idempotents belonging to I. Since Theorem 1.4.1, $S/J(S)$ is Von Neumann regular and idempotents of $S/J(S)$ can be lifted to idempotents of S, it follows that there exists an idempotent $e_3 \in S$ such that $(e_1S+e_2S+J(S))/J(S) = (e_3S + J(S))/J(S)$. In particular, $e_3 \in I$, $e_1 \in e_3S + J(S)$ and $e_2 \in e_3S + J(S)$. Hence, $Ge_3(M_R) \in \mathcal{A}(I)$, and, by Lemma 2.3.5, $e_1(M_R) \cap A_1 \leq e_3(M_R)$ and $e_2(M_R) \cap A_2 \leq e_3(M_R)$ for some essential submodules A_1, A_2 of M_R. Therefore, $Ge_1(M_R), Ge_2(M_R) \preceq Ge_3(M_R)$, which implies $Ge_1(M_R) \vee Ge_2(M_R) \preceq Ge_3(M_R)$. This shows that $Ge_1(M_R) \vee Ge_2(M_R) \in \mathcal{A}(I)$. One has $\mathcal{A}(I)$ is an antifilter of $\mathcal{O}$.

Next thing we will show is the fact that $\mathcal{I}(A)$ is a right ideal of S containing $J(S)$ for any antifilter A of $\mathcal{O}$. The only non-trivial thing to show is that $\mathcal{I}(A)$ is additively closed. For this, it suffices to prove that if $e_1, e_2 \in S$ are idempotents of S with $Ge_1(M_R), Ge_2(M_R) \in A$, then $e_1S + e_2S + J(S)$ is of the form $e_3S + J(S)$ for a suitable idempotent $e_3 \in S$ with $Ge_3(M_R) \in A$. Using the argument in the first paragraph, there exists idempotent $e_3 \in S$ such that $(e_1S + e_2S + J(S))/J(S) = (e_3S + J(S))/J(S)$

and $Ge_1(M_R) \vee Ge_2(M_R) \preceq Ge_3(M_R)$. It suffices to prove $Ge_3(M_R) \in A$. Indeed, we have $e_3 = e_1h_1 + e_2h_2 + j$ for some $h_1, h_2 \in S$ and $j \in J(S)$. Hence if put $B := \ker j$, then B is essential in M_R and $e_3(M_R) \cap B \leq e_1(M_R) + e_2(M_R)$, which implies $Ge_3(M_R) \preceq GE(e_1(M_R) + e_2(M_R)) = Ge_1(M_R) \vee Ge_2(M_R)$. Therefore, $Ge_3(M_R) = Ge_1(M_R) \vee Ge_2(M_R) \in A$.

Now we will show that $\mathcal{A}(\mathcal{I}(A)) = A$ and $\mathcal{I}(\mathcal{A}(I)) = I$. Let $Ge(M_R) \in A$ for some idempotent $e \in S$. Then $e \in \mathcal{I}(A)$, whence $Ge(M_R) \in \mathcal{A}(\mathcal{I}(A))$. Conversely, let $Ge(M_R) \in \mathcal{A}(\mathcal{I}(A))$ for some idempotent $e \in \mathcal{I}(A)$. Then there exist $e', g \in S$ and $j \in J(S)$ such that $e'^2 = e', Ge'(M_R) \in A$ and $e = e'g + j$. Using Lemma 2.3.5, one has that $e(M_R) \cap B \leq e'(M_R)$ for some essential submodule B of M_R. Hence, $Ge(M_R) \preceq Ge'(M_R)$, which implies $Ge(M_R) \in A$. Therefore $A = \mathcal{A}(\mathcal{I}(A))$.

Let $f \in I$. Then there exists an idempotent $e \in S$ such that $(fS + J(S))/J(S) = (eS + J(S))/J(S)$. Hence $e \in I$ and $f = eg + j$ for some $g \in S$ and $j \in J(S)$. We have that $Ge(M_R) \in \mathcal{A}(I)$, so $f \in \mathcal{I}(\mathcal{A}(I))$. Conversely, let $f \in \mathcal{I}(\mathcal{A}(I))$. Then there exists idempotent $e \in S$ such that $Ge(M_R) \in \mathcal{A}(I), f = es + i$ for some $s \in S, i \in J(S)$. Hence, $Ge(M_R) = Ge'(M_R)$ for some idempotent $e' \in I$. In particular, $e(M_R) \cap B \leq e'(M_R)$ for some essential submodule B of M_R. By Lemma 2.3.5, $e \in e'S + J(S) \subseteq I$. Therefore $f \in I$. This proves that $I = \mathcal{I}(\mathcal{A}(I))$. ■

Corollary 3.4.2. *Maximal right ideals of S correspond to ultraantifilters of $\mathcal{O}$.*

3.5 Maximal two-sided ideals

Recall that, for any module M_R, it is possible to define a commutative monoid $V(M_R)$ as follows. Given the set $\mathcal{S}(M_R)$ of all direct summands of the modules M_R^n, $n \geq 0$, fix a complete set $V(M_R)$ of representatives of the modules in the set $\mathcal{S}(M_R)$ up to isomorphism. Thus, for every $A \in \mathcal{S}(M_R)$, there is a unique $\langle A \rangle \in V(M_R)$ with $A \cong \langle A \rangle$. Direct sum induces an addition $+$ on the set $V(M_R)$, defined by setting, for every $\langle A \rangle, \langle B \rangle \in V(M_R)$, $\langle A \rangle + \langle B \rangle := \langle A \oplus B \rangle$. Then $V(M_R)$ with this operation becomes a commutative monoid, which naturally describes the direct sum decompositions in the set

$\mathcal{S}(M_R)$. The algebraic pre-order of $V(M_R)$ is defined as follows: If $\langle X\rangle, \langle Y\rangle \in V(M_R)$ then $\langle X\rangle \leq \langle Y\rangle$ if and only if X is isomorphic to a direct summand of Y

For any ring T, we will denote by $V(T) = V(T_T)$ the commutative monoid of a set of representatives of all finitely generated projective right T-modules up to isomorphism. If S denotes the endomorphism ring of a right R-module M_R, the monoid $V(M_R)$ turns out to be isomorphic to the monoid $V(S)$ [14, Theorem 4.7].

Recall that a commutative additive monoid C is *reduced* if $x, y \in C$ and $x+y = 0$ imply $x = y = 0$, that is, if $U(C) = 0$, where $U(C)$ denotes the group of invertible elements of the monoid C, that is, the set of the elements $a \in C$ with an "additive inverse" $-a$. The commutative monoid $V(M_R)$ is reduced for every module M_R. For every commutative monoid C, the monoid $C_{\text{red}} := C/U(C)$ is a reduced monoid.

A *prime ideal* of a commutative monoid C is a proper subset P of C such that, for any $x, y \in C$, one has $x + y \in P$ if and only if either $x \in P$ or $y \in P$. The complements of the prime ideals of C are exactly the *divisor-closed submonoids* of C, that is, the submonoids D of C such that, for all $x, y \in C$, $x + y \in D$ implies $x \in D$ and $y \in D$. The *spectrum* $\operatorname{Spec}(C)$ of a commutative monoid C is the set of all prime ideals of C.

The following Lemma is well known and easy to prove. A generalized version of it appears in [43, Corollary 4.1].

Lemma 3.5.1. *If X is a direct summand of $M_1 \oplus \cdots \oplus M_n$, where n is a positive integer and $M_1, \dots, M_n$ are injective modules, then $X \cong X_1 \oplus \cdots \oplus X_n$, where X_i is a suitable direct summand of M_i for every $i = 1, \dots, n$.*

For any ring S and any subset X of the commutative monoid $V(S)$, we will denote by $\operatorname{Tr}_S(X)$ the *trace* of X in S, that is, the sum of all images $f(A)$, where A ranges over X and f ranges over $\operatorname{Hom}(A, S_S)$. Observe that $\operatorname{Tr}_S(X)$ is a two-sided ideal of S. A two-sided ideal I of S is called a *trace ideal* if $I = \operatorname{Tr}_S(X)$ for some subset X of $V(S)$. A *maximal trace ideal* is a trace ideal that is maximal in the set of all proper trace ideals.

Lemma 3.5.2. *Let M_R be an injective module over a ring R. Denote by S the endomorphism ring of M_R. Assume that I is a two-sided ideal of S. Then I is a trace ideal if and only if I is generated by its idempotents.*

Proof. By Lemma 3.5.1, every direct summand of M_R^n is isomorphic to $X_1 \oplus \cdots \oplus X_n$, where each X_i is a direct summand of M_R. Applying the functor $\mathrm{Hom}(M_R, -)\colon$ Mod-$R \to$ Mod-S, we see that every finitely generated projective right S-module is isomorphic to a direct sum of finitely many cyclic projective right S-modules, that is, S-modules isomorphic to eS for suitable idempotents $e \in S$ [14, Theorem 4.7]. It follows that the trace of a projective S-module $e_1S \oplus \cdots \oplus e_nS$ is isomorphic to the two-sided ideal $\sum_{i=1}^n Se_iS$. Thus the trace ideals are generated by idempotent elements of S. Conversely, a two-sided ideal I generated by a set of idempotents $\{ e_\lambda \mid \lambda \in \Lambda \}$ is the trace of the set $X = \{ \langle e_\lambda S \rangle \mid \lambda \in \Lambda \}$. ■

A prime ideal P of a commutative monoid C is a prime ideal *of height one* if it is minimal in the set of all prime non-empty ideals of C. From [19, Theorem 2.1(c)], we immediately get that:

Proposition 3.5.3. *Let M_R be an injective module over a ring R and S be the endomorphism ring of M_R. Then there is an order-reversing one-to-one correspondence between the set of all trace ideals of the ring S and the spectrum $\mathrm{Spec}(V(M_R))$ of the commutative monoid $V(M_R)$. In particular, maximal trace ideals correspond to prime ideals of height one of $V(M_R)$.*

Lemma 3.5.4. *Every maximal two-sided ideal of S is generated by its idempotents and $J(S)$.*

Proof. Let I be a maximal two-sided ideal of S. Then $I \supseteq J(S)$ and $S/J(S)$ is a Von Neumann regular ring (Theorem 1.4.1). Hence the right ideal $I/J(S)$ of $S/J(S)$, which is the sum of its finitely generated right ideals, is a sum of two-sided ideals of $S/J(S)$ generated by idempotents of $S/J(S)$. Every idempotent of $S/J(S)$ lifts to an idempotent of S (Theorem 1.4.1), so that I is generated as a right ideal by its idempotents and $J(S)$. Therefore I is generated a fortiori as a two-sided ideal by its idempotents and $J(S)$. ■

Proposition 3.5.5. *If A is a maximal two-sided ideal of S, then the two-sided ideal generated by the idempotents of S is a maximal trace ideal of S. Conversely, if B is a maximal trace ideal of S, then $B + J(S)$ is a maximal two-sided ideal of S.*

PROOF. Let A be a maximal two-sided ideal of S and let A' be the two-sided ideal of S generated by all the idempotents of A. Let $C \supseteq A'$ be any other proper trace ideal of S, that is, a two-sided ideal generated by its idempotents. By Lemma 3.5.4, $A = A' + J(S) \subseteq C + J(S)$. By the maximality of A, it follows that either $C + J(S) = A$ or $C + J(S) = S$. If $C + J(S) = A$, then all idempotents of C are in A, hence in A'. Thus $A' = C$ in this case. If $C + J(S) = S$, then $C = S$, because the right S-module $J(S)$ is a superfluous in the right S-module S by Nakayama's Lemma. This proves that A' is a maximal trace ideal of S.

Conversely, let B be a maximal trace ideal of S. We want to prove that $B + J(S)$ is a maximal two-sided ideal in S. Notice that $B + J(S)$ is proper, otherwise $B + J(S) = S$ implies $B = S$ because $J(S)$ is superfluous right ideal of S. Let B' be a maximal two-sided ideal of S containing $B + J(S)$. Now $B \subseteq B + J(S) \subseteq B'$, so that all idempotents of B are in B'. Let B'' be the two-sided ideal of S generated by all the idempotents of B'. Then $B \subseteq B''$, and these are proper two-sided ideals because B' is a proper two-sided ideal. By the maximality of B, we have that $B = B''$. By Lemma 3.5.4, $B' = B'' + J(S) = B + J(S)$. ■

Lemma 3.5.6. *There is a one-to-one correspondence between the set of all maximal two-sided ideals of the ring S and the set of all maximal trace ideals of S.*

PROOF. The correspondence associates to every maximal two-sided ideal A of S the two-sided ideal of S generated by the idempotents of A. This is a maximal trace ideal by Proposition 3.5.5. The inverse correspondence associates to any maximal trace ideal B of S, the maximal two-sided ideal $B + J(S)$ of S (Proposition 3.5.5). These two correspondences are mutually inverse by Lemma 3.5.4. ■

From Proposition 3.5.3 and Lemma 3.5.6, we can conclude that:

Theorem 3.5.7. *There is a one-to-one correspondence between the set of all maximal two-sided ideals of the ring S and the set of all prime ideals of height one in the commutative monoid $V(M_R)$.*

3.6 Reduced localization

Let P be a prime ideal of a commutative monoid C. In $C \times C\backslash P$, we define a relation $\sim$ as follows: for $(x, s), (x', s') \in C \times C\backslash P$, $(x, s) \sim (x', s')$ if there exists $t \in C$ such that $x + s' + t = x' + s + t$. It is easy to check that $\sim$ is an equivalence relation in $C \times C\backslash P$. Put $C_P := (C \times C\backslash P)/\sim$ and every class $[(x, s)]_\sim$ of C_P is denoted by $x - s$ for any $x \in C, s \in C\backslash P$. One can check that C_P with the addition defined by $(x - s) + (x' - s') = (x + x') - (s + s')$ for $x - s, x' - s' \in C_P$. The monoid C_P is called the *localization* of C at P. Let $U(C_P)$ be the set of additive inverses of C_P. The monoid $(C_P)_{\text{red}} := (C_P)_{U(C_P)}$ is called the *reduced localization* of C at P. Hence, the canonical homomorphism $C \to (C_P)_{\text{red}}$, defined by $x \mapsto x - 0 + U(C_P)$, is surjective.

Theorem 3.6.1. *Let S be the endomorphism ring of an injective module, I a maximal two-sided ideal of S and P the corresponding prime ideal of height one of $V(S)$. Then the reduced localization $(V(S)_P)_{\text{red}}$ and the monoid $V(S/I)$ are canonically isomorphic. The isomorphism maps an element $\langle A_S\rangle - \langle 0_S\rangle + U(V(S))$ of $(V(S)_P)_{\text{red}}$ to the element $\langle A_S/A_SI\rangle$ of $V(S/I)$.*

Proof. We must prove that the monoid homomorphism

$$V(\pi)\colon V(S) \to V(S/I),$$

induced by the canonical projection $\pi\colon S \to S/I$ and defined by $V(\pi)(\langle A_S\rangle) = \langle A_S/A_SI\rangle$, is surjective, and that $V(\pi)(\langle A_S\rangle) = V(\pi)(\langle B_S\rangle)$ if and only if there exist $C_S, D_S \in V(S) \setminus P$ with $A_S \oplus C_S \cong B_S \oplus D_S$.

In order to prove that the monoid homomorphism $V(\pi)\colon V(S) \to V(S/I)$ is surjective, notice that $I \supseteq J(S)$ and $S/J(S)$ is Von Neumann regular, so that S/I is Von Neumann regular. Thus every finitely generated projective S/I-module is a finite direct sum of projective cyclic S/I-modules, that is, principal right ideals $\overline{e}(S/I)$ of S/I, where $\overline{e}$ is an idempotent of S/I [25, Proposition 2.6]. Therefore it suffices to prove that idempotents of S/I lift to idempotents of S. This is asserted by [21, Theorem 19.27 (c)].

It remains to show that for every A_S, B_S finitely generated projective S-modules, $V(\pi)(A_S) = V(\pi)(B_S)$ if and only if there exist $C_S, D_S \in V(S) \setminus P$ with $A_S \oplus C_S \cong B_S \oplus$

D_S. Let I' be the maximal trace ideal generated by all the idempotent elements of I. The canonical projection $\pi\colon S \to S/I$ is the composite mapping of the canonical projections $\pi'\colon S \to S/I'$ and $\pi''\colon S/I' \to S/I$. Thus $V(\pi) = V(\pi'') \circ V(\pi')$. By [37, Theorem 1.5] (also see [19, Theorem 2.1(d)]), for every pair finitely generated projective S-modules A_S, B_S, one has $V(\pi')(A_S) = V(\pi')(B_S)$ if and only if there exist $C_S, D_S \in V(S) \setminus P$ with $A_S \oplus C_S \cong B_S \oplus D_S$. Therefore in order to conclude the proof, it is enough to show that, for every pair A_S, B_S of projective S-modules, $A_S/A_SI \cong B_S/B_SI$ implies $A_S/A_SI' \cong B_S/B_SI'$. We claim that $J(S/I') = I/I'$. To prove the claim, notice that $J(S/I')$ is the intersection of all M/I' where M ranges over the set of all maximal right ideals of S containing I'. But every maximal right ideal of S contains $J(S)$, hence contains I' if and only if it contains $I = I' + J(S)$. Now S/I is a simple ring, so that $J(S/I) = 0$. Thus the intersection of the maximal right ideals M of S containing I is I. This concludes the proof of our claim.

Now assume that A_S, B_S are finitely generated projective S-modules and that A_S/A_SI is isomorphic to B_S/B_SI. Then $A_S/A_SI', B_S/B_SI'$ are finitely generated projective S/I'-modules that are isomorphic modulo $J(S/I')$, because

$$\begin{aligned}(A_S/A_SI')/(A_S/A_SI')J(S/I') = (A_S/A_SI')/(A_S/A_SI')(I/I') = \\ = (A_S/A_SI')/(A_SI/A_SI') \cong A_S/A_SI,\end{aligned}$$

and similarly for B_R. Thus $A_S/A_SI \cong B_S/B_SI$ implies $A_S/A_SI' \cong B_S/B_SI'$, as desired. ■

Lemma 3.6.2. *If M_R is an injective module and M_1, M_2 are two direct summands of M_R with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M_R.*

PROOF. The submodules M_1, M_2 are injective, so that $M_1 + M_2 = M_1 \oplus M_2$ is injective. Hence $M_1 \oplus M_2$ is a direct summand of M_R. ■

Let $L := \mathcal{L}_{ds}(M_R)$ be the set of all direct summands of M_R. Then L with set inclusion is a partially ordered set. We will write $X \perp Y$ if $X, Y \in L$ and $X \cap Y = 0$. In this case, that is, if $X \perp Y$, then the upper bound $X \vee Y$ exists and is $X \oplus Y$. Thus we have a partially defined addition $+$ on the set L, defined by $X + Y := X \oplus Y$ for every $X, Y \in L$ with $X \perp Y$. Notice that there is a canonical mapping can: $L \to V(M_R)$ defined by

$\operatorname{can}(X) = \langle X\rangle$ for every $X \in L$. Its image is the closed interval of $V(M_R)$ consisting of all $\langle X\rangle \in V(M_R)$ with X a direct summand of M_R, that is, the interval of all $\langle X\rangle \in V(M_R)$ with $\langle 0\rangle \leq \langle X\rangle \leq \langle M_R\rangle$.

We will say that a subset C of L is a *cofilter* in L if:

(1) $C \neq \emptyset$.

(2) $X \in L$, $Y \in C$ and $X \cong Y$ imply $X \in C$.

(3) $X \in L$, $Y \in C$ and $X \leq Y$ imply $X \in C$.

(4) $X, Y \in C$ and $X \perp Y$ imply $X + Y \in C$.

Proposition 3.6.3. *There is an order-preserving one-to-one correspondence between the set $\mathcal{C}$ of all cofilters in L and the set $\mathcal{D}$ of all divisor-closed submonoids of $V(M_R)$. It associates to any cofilter C in L the submonoid of $V(M_R)$ generated by* $\operatorname{can}(C)$. *The inverse correspondence associates to any divisor-closed submonoid D of $V(M_R)$ the cofilter* $\operatorname{can}^{-1}(D)$ *of L.*

Proof. We first prove that the submonoid $[\operatorname{can}(C)]$ of $V(M_R)$ generated by $\operatorname{can}(C)$ for some cofilter C of L is a divisor-closed submonoid of $V(M_R)$. If $X \in V(M_R)$, $Y \in [\operatorname{can}(C)]$ and $X \leq Y$, then $Y = \langle Y_1\rangle + \cdots + \langle Y_n\rangle$ for suitable $Y_i \in C$. From Lemma 3.5.1, we have that $X \cong X_1 \oplus \cdots \oplus X_n$ with $X_i \leq Y_i$ for every $i = 1, \ldots, n$. But C is a cofilter, so that $X_i \in C$ for every i. Thus $X = \langle X_1\rangle + \cdots + \langle X_n\rangle \in [\operatorname{can}(C)]$.

Conversely, the proof that $\operatorname{can}^{-1}(D)$ is a cofilter for every divisor-closed submonoid D of $V(M_R)$ is trivial.

In order to prove that the two correspondences are inverse to the other, we must prove that $\operatorname{can}^{-1}([\operatorname{can}(C)]) = C$ for every $C \in \mathcal{C}$ and $[\operatorname{can}(\operatorname{can}^{-1}(D))] = D$ for every $D \in \mathcal{D}$. The inclusion $\operatorname{can}^{-1}([\operatorname{can}(C)]) \supseteq C$ follows from (2). Conversely, assume $X \in \operatorname{can}^{-1}([\operatorname{can}(C)])$. Then $X \in L$ and $\langle X\rangle \in [\operatorname{can}(C)]$, so that $\langle X\rangle = \langle X_1\rangle + \cdots + \langle X_n\rangle$ for some $X_i \in C$. Thus $X_1 \oplus \cdots \oplus X_n$ is isomorphic to a direct summand of M_R, i.e., $M_R = M_1 \oplus \cdots \oplus M_n \oplus M'$ with $M_i \cong X_i$. By (2), each $M_i \in C$, and $M_1 \oplus \cdots \oplus M_n \in C$ by (4). Thus $X \in C$ by (2).

Finally, fix a divisor-closed submonoid $D \in \mathcal{D}$. Then $\operatorname{can}(\operatorname{can}^{-1}(D)) = D \cap \operatorname{can}(L)$, and we must show that $[D \cap \operatorname{can}(L)] = D$. The inclusion $[D \cap \operatorname{can}(L)] \subseteq D$ is obvious. Conversely, if $\langle X\rangle \in D$, then $\langle X\rangle \in V(M_R)$, and we can suppose that X is a direct summand of M_R^n. By Lemma 3.5.1, we can suppose $X = X_1 \oplus \cdots \oplus X_n$ with every $X_i \in L$. As D is divisor-closed, each $\langle X_i \rangle$ is in D. This shows that each $\langle X_i \rangle$ is in $D \cap \mathrm{can}(L)$, so that $\langle X \rangle \in [D \cap \mathrm{can}(L)]$, as we wanted to prove. ■

Every cofilter contains 0. The cofilter L is the improper cofilter. A cofilter maximal among all proper cofilters is an *ultracofilter*. Hence:

Theorem 3.6.4. *There is a one-to-one correspondence between the set of all maximal two-sided ideals of S and the set of all ultracofilters of $L = \mathcal{L}_{ds}(M_R)$.*

A standard characterizations of ultrafilters U on a set X is that a filter U on X is an ultrafilter if and only if for every subset A of X, either $A \in U$ or $X \setminus A \in U$. Here is the analogue of this characterization for our ultracofilters.

Proposition 3.6.5. *A cofilter C of L is an ultracofilter if and only if $M_R \notin C$ and, for every $X \in L \setminus C$, there exist an integer $n \geq 0$, a module $X' \in L$ isomorphic to a submodule of X^n and an element Y of C such that $M_R = X' \oplus Y$.*

Proof. Let C be an ultracofilter. Since C is a proper subset of L, we have that $M_R \notin C$. Assume that $X \in L \setminus C$. Then $M_R = X \oplus Z$ for some $Z \in L$. By Proposition 3.6.3, $[\mathrm{can}(C)]$ is a maximal divisor-closed submonoid of $V(M_R)$, and $\mathrm{can}^{-1}([\mathrm{can}(C)]) = C$. Thus $X \notin \mathrm{can}^{-1}([\mathrm{can}(C)])$, that is, $\langle X \rangle \notin [\mathrm{can}(C)]$. By Lemma 3.5.1, the sum of two divisor-closed submonoids of $V(M_R)$ is a divisor-closed submonoid of $V(M_R)$. Hence if $[\![\langle X \rangle]\!]$ denotes the divisor-closed submonoid of $V(M_R)$ generated by $\langle X \rangle$, then $[\![\langle X \rangle]\!] + [\mathrm{can}(C)]$ is a divisor-closed submonoid of $V(M_R)$ properly containing $[\mathrm{can}(C)]$. From the maximality of $[\mathrm{can}(C)]$, it follows that $[\![\langle X \rangle]\!] + [\mathrm{can}(C)] = V(M_R)$. Thus $M_R = X' \oplus M_1 \oplus \cdots \oplus M_t$, where X' is isomorphic to a direct summand of X^n for some $n \geq 0$, and every M_i is in C. By property (4), we get that $Y := M_1 \oplus \cdots \oplus M_t \in C$.

Conversely, let C be a proper cofilter of L with the property that for every $X \in L \setminus C$ there exist $X' \in L$ isomorphic to a submodule of X^n for some n and $Y \in C$ with $M_R = X' \oplus Y$. Let C' be a cofilter of L properly containing L. Let X be in $C' \setminus C$. By hypothesis, there exist $X' \in L$ isomorphic to a submodule of X^n for some n and $Y \in C$ with $M_R = X' \oplus Y$. Then $X \in C'$ implies $\langle X^n \rangle \in [\mathrm{can}(C')]$, which is a divisor-closed submonoid, so that $\langle X' \rangle \in [\mathrm{can}(C')]$. Thus $X' \in C'$ and $Y \in C \subseteq C'$. It follows that $M_R = X' \oplus Y \in C'$, so $C' = L$. ∎

3.7 Ultrafilters in the case $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$.

The following important result was proved in [43].

Proposition 3.7.1. [43, Corollary 4.2] *Let $M_R = E(\bigoplus_{i \in I} M_i)$ be an injective module that is an injective envelope of a direct sum of indecomposable injective submodules M_i, $i \in I$. Then any two such decompositions of M_R are isomorphic. Furthermore, if N is an injective submodule of M_R, there is a subset $J \subseteq I$ such that $M_R = N \oplus E(\bigoplus_{i \in J} M_i)$ and $N \cong E(\bigoplus_{i \in I \setminus J} M_i)$.*

In the following, $\{ A_\lambda \mid \lambda \in \Lambda \}$ will be a set of representatives of all indecomposable injective right R-modules up to isomorphism, and $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$ will be an injective envelope of a direct sum of a family of indecomposable injective modules. Let N be a direct summand of M_R. By Proposition 3.7.1, one has that $N \cong E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\dim_\lambda(N))})$ for uniquely determined cardinals $\dim_\lambda(N)$. It is easy to check that if N_1, N_2 are two direct summands of M_R and $\lambda \in \Lambda$, then:

1. $\dim_\lambda(M_R) = \kappa_\lambda$;
2. If $N_1 \cong N_2$, then $\dim_\lambda(N_1) = \dim_\lambda(N_2)$;
3. If $N_1 \leq N_2$, then $\dim_\lambda(N_1) \leq \dim_\lambda(N_2)$;
4. $\dim_\lambda(N_1 \oplus N_2) = \dim_\lambda(N_1) + \dim_\lambda(N_2)$.

In the rest of this Section, we will describe the ultrafilters of the lattice $\mathcal{O} = \mathcal{L}_{ds}(M_R)/G$ in the case $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$.

Proposition 3.7.2. *Let $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$ be an injective envelope of a direct sum of a family of indecomposable injective modules. If B is an indecomposable injective submodule of M_R, then $\mathcal{F}_B := \{ GN \in \mathcal{O} \mid \ GB \preceq GN \}$ is an ultrafilter of $\mathcal{O}$.*

Proof. It is easily seen that $\mathcal{F}_B$ is a filter of the lattice $\mathcal{O}$. Let $\mathcal{G}$ be a filter strictly containing $\mathcal{F}_B$. We must show that $\mathcal{G} = \mathcal{O}$. In fact, fix $GN \in \mathcal{G}$ with $GN \notin \mathcal{F}_B$. Consider

$GB \wedge GN = G(E(B \cap N)) \in \mathcal{G}$. If $E(B \cap N) \neq 0$, then $B = E(B \cap N)$, because B is indecomposable injective. Thus $GB = G(E(B \cap N)) \preceq GN$. It follows that $GN \in \mathcal{F}_B$, a contradiction. Therefore $E(B \cap N) = 0$, so that $G0 \in \mathcal{G}$, and we can conclude that $\mathcal{G} = \mathcal{O}$. ■

The following result describes all principal ultrafilters in $\mathcal{O}$.

Proposition 3.7.3. *Let* $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$ *be an injective envelope of a direct sum of a family of indecomposable injective modules. If* $\mathcal{F}$ *is a principal ultrafilter of* $\mathcal{O}$*, then there exists an indecomposable injective submodule* B *of* M_R *such that* $\mathcal{F} = \mathcal{F}_B$*.*

Proof. Let GN be the least element of $\mathcal{F}$. By Proposition 3.7.1, there exists an indecomposable injective submodule B of N. Then $GB \preceq GN$, so that the principal filter $\mathcal{F}_B$ generated by GB is a proper filter of $\mathcal{O}$ that contains $\mathcal{F}$. By the maximality of $\mathcal{F}$, one has that $\mathcal{F} = \mathcal{F}_B$. ■

Corollary 3.7.4. *If* $M_R = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ *where* M_i *is indecomposable injective, then all ultrafilters of* $\mathcal{O}$ *are principal, that is, of the form* $\mathcal{F}_B$ *for some indecomposable injective submodule* B *of* M_R*.*

Proof. We know that an ultrafilter $\mathcal{F}$ of $\mathcal{O}$ is free if and only if for every $x \in \mathcal{F}$, there exists $y \in \mathcal{F}$ such that $y \preceq x$ and $y \neq x$. Since every properly descending chain in $\mathcal{L}_{ds}(M_R)$ is finite, it follows that there is no free ultrafilter in $\mathcal{O}$. Now apply Proposition 3.7.3. ■

Recall that a set $\{ N_i \mid i \in I \}$ of non-zero submodules of a module N_R is said to be *independent* if $N_{i_0} \cap \sum_{i \in I \setminus \{i_0\}} N_i = 0$ for every $i_0 \in I$. That is, $\sum_{i \in I} N_i = \bigoplus_{i \in I} N_i$.

Remark 3.7.5. If M_R is an injective R-module and $\{N_j \mid j \in J\}$ is an independent set of submodules of M_R, then $GE(\bigoplus_{j \in J} N_j) = GE(\bigoplus_{j \in J} E(N_j))$, because $\bigoplus_{j \in J} N_j$ is essential in $E(\bigoplus_{j \in J} E(N_j))$.

Lemma 3.7.6. *Let* I *be a subset of* Λ*, so that* $M_R := E(\bigoplus_{\lambda \in I} A_\lambda)$ *is an injective envelope of a direct sum of a family of pair-wise non-isomorphic indecomposable injective modules. Then, for every* $GN \in \mathcal{O}$*, there exists a unique subset* K_N *of* I *such that*

$GN = GE(\bigoplus_{\lambda \in K_N} A_\lambda)$. *In particular, the lattices* $\mathcal{O}$ *and* $\mathcal{P}(I)$ *of all subsets of* I *are isomorphic.*

Proof. By Proposition 3.7.1, there exists a subset K_N of I such that N is isomorphic to $E(\bigoplus_{\lambda \in K_N} A_\lambda)$. Let T be a set of representatives of all indecomposable injective submodules of N up to isomorphism. It is easy to check that T is an independent set of submodules of N and $N = E(\bigoplus_{B \in T} B)$. By Proposition 3.7.1, for every $B \in T$, there exists a unique $\lambda_B \in I$ such that $B \cong A_{\lambda_B}$. If $B \cap A_{\lambda_B} = 0$, then B is a direct summand of $E(\bigoplus_{\lambda \in I \setminus \{\lambda_B\}} A_\lambda)$. This contradicts Proposition 3.7.1. Thus $B \cap A_{\lambda_B} \neq 0$. By Proposition 6.1, $K_N = \{\lambda_B \mid B \in T\}$. By Remark 3.7.5, one finds that $GN = GE(\bigoplus_{\lambda \in K_N} A_\lambda)$. ■

From the lattice isomorphism $\mathcal{O} \cong \mathcal{P}(I)$ of Lemma 3.7.6, we immediately get the following three results.

Theorem 3.7.7. *Let* I *be a subset of* Λ*, so that* $M_R = E(\bigoplus_{\lambda \in I} A_\lambda)$ *is an injective envelope of a direct sum of a family of pair-wise non-isomorphic indecomposable injective modules. Then there is a one-to-one correspondence between the set of all filters of* $\mathcal{O}$ *and the set of all filters of the lattice* $\mathcal{P}(I)$ *of all subsets of* I*. It associates to any filter* $\mathcal{F}$ *of* $\mathcal{O}$ *the filter* $P(\mathcal{F}) = \{K_N \mid GN \in \mathcal{F}\}$ *of* $\mathcal{P}(I)$*, where* K_N *is defined as in Lemma 3.7.6. The inverse correspondence associates to any set* P *of* $\mathcal{P}(I)$ *the filter* $\mathcal{F}(P) = \{GE(\bigoplus_{\lambda \in K} A_\lambda) \mid K \in P\}$ *of* $\mathcal{O}$*. In particular, there is a one-to-one correspondence between the set of all ultrafilters of* $\mathcal{O}$ *and the set of all ultrafilters of the power set* $\mathcal{P}(I)$ *of* I*.*

Proposition 3.7.8. *Let* $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$ *be an injective envelope of a direct sum of a family of indecomposable injective modules. If* $\mathcal{A}$ *is a principal ultraantifilter of* $\mathcal{O}$*, then there exists a direct-sum decomposition* $M_R = B \oplus B'$ *of* M_R *with* B' *indecomposable and* $\mathcal{A} = \{GN \in \mathcal{O} \mid GN \preceq GB\}$*.*

Theorem 3.7.9. *Let* I *be a subset of* Λ*, so that* $M_R = E(\bigoplus_{\lambda \in I} A_\lambda)$ *is an injective envelope of a direct sum of a family of pair-wise non-isomorphic indecomposable injective* *modules. Then there is a one-to-one correspondence between the set of all ultraantifilters of $\mathcal{O}$ and the set of all ultraantifilters of the power set $\mathcal{P}(I)$ of I.*

The one-to-one correspondence in Theorem 3.7.9 is the same as that in Theorem 3.7.7.

3.8 Ultracofilters of $\mathcal{L}_{ds}(E(\bigoplus_{\lambda\in\Lambda} A_\lambda^{(\kappa_\lambda)}))$

In this Section, we will describe the ultracofilters of $\mathcal{L}_{ds}(E(\bigoplus_{\lambda\in\Lambda} A_\lambda^{(\kappa_\lambda)}))$, where the A_λ's are pair-wise non-isomorphic indecomposable injective modules. As the following Proposition shows, the behavior of the module $M_R = E(A^{(\kappa)})$, where A is an indecomposable injective module, is similar to that of vector spaces over a division ring (Corollary 3.1.6).

Proposition 3.8.1. *Let $A = A_\lambda$ be an indecomposable injective module, κ be a non-zero cardinal and $M_R = E(A^{(\kappa)})$. Then there is a unique ultracofilter $\mathcal{U}$ of $\mathcal{L}_{ds}(M_R)$. Moreover,*

1. *If κ is finite, then $\mathcal{U} = \{0\}$.*
2. *If κ is infinite, then $\mathcal{U} = \{\, N \in \mathcal{L}_{ds}(M_R) \mid \dim_\lambda(N) < \kappa \,\}$.*

Proof. Statement (1) is trivial, because when κ is finite, there are only two cofilters in $\mathcal{L}_{ds}(M_R)$, namely $\{0\}$ and $\mathcal{L}_{ds}(M_R)$.

(2) Suppose that κ is infinite. It is suffices to show that every proper cofilter is contained in $\mathcal{U} := \{\, N \in \mathcal{L}_{ds}(M_R) \mid \dim_\lambda(N) < \kappa \,\}$ and that $\mathcal{U}$ is a proper cofilter of $\mathcal{L}_{ds}(M_R)$. Let $\mathcal{C}$ be a proper cofilter of $\mathcal{L}_{ds}(M_R)$. Let U be an arbitrary element of $\mathcal{C}$. If $\dim_\lambda(U) = \kappa$, then $M_R \cong U$, so that $M_R \in \mathcal{C}$. This implies that $\mathcal{C} = \mathcal{L}_{ds}(M_R)$, a contradiction. Therefore $\dim_\lambda(U) < \kappa$. This shows that $\mathcal{C} \subseteq \mathcal{U}$. To conclude the proof, we must see that $\mathcal{U}$ is a proper cofilter of $\mathcal{L}_{ds}(M_R)$. Obviously, $M_R \notin \mathcal{U}$ and $0 \in \mathcal{U}$, so that $\mathcal{U}$ is a non-empty proper subset of $\mathcal{L}_{ds}(M_R)$. Given N_1, N_2 in $\mathcal{L}_{ds}(M_R)$ with $N_1 \cong N_2$ and $N_2 \in \mathcal{U}$, then $\dim_\lambda(N_1) = \dim_\lambda(N_2) < \kappa$ implies that $N_1 \in \mathcal{U}$. If $N_1 \leq N_2$ and $N_2 \in \mathcal{U}$, then $\dim_\lambda(N_1) \leq \dim_\lambda(N_2) < \kappa$ implies that $N_1 \in \mathcal{U}$. Finally, if $N_1, N_2 \in \mathcal{U}$ and $N_1 \perp N_2$, then $\dim_\lambda(N_1) < \kappa$ and $\dim_\lambda(N_2) < \kappa$, so that

$$\dim_\lambda(N_1 + N_2) = \dim_\lambda(N_1 \oplus N_2) = \dim_\lambda(N_1) + \dim_\lambda(N_2) < \kappa + \kappa = \kappa.$$

Therefore $N_1 + N_2 \in \mathcal{U}$. ∎

As a consequence, the endomorphism ring of $M_R = E(A_\lambda^{(\kappa)})$ has a unique maximal two-sided ideal. More precisely:

When κ is finite, the unique ultracofilter $\mathcal{U}$ of $L = \mathcal{L}_{ds}(M_R)$ is $\{0\}$, the corresponding divisor-closed submonoid of $V(M_R) \cong \mathbb{N}$ is the zero submonoid, the corresponding prime ideal of height one of $V(M_R)$ consists of all non-zero elements of $V(M_R)$, the corresponding trace ideal of $S = \mathrm{End}(M_R)$ is the zero ideal, and the unique maximal two-sided ideal of the endomorphism ring S is $J(S)$, consisting of all endomorphisms of M_R with essential kernel (Theorem 1.4.1).

When κ is infinite, the monoid $V(M_R)$ is isomorphic to the interval $[0, \kappa]$, the additive monoid of all cardinals $\leq \kappa$. The divisor-closed submonoid of $[0, \kappa]$ corresponing to $\mathcal{U}$ is the interval $[0, \kappa[$, the additive monoid of all cardinals $< \kappa$. The corresponding prime ideal of height one of $V(M_R)$ consists only of κ. The corresponding trace ideal of $S = \mathrm{End}(M_R)$ is the two-sided ideal I generated by all idempotent endomorphisms of M_R whose image is isomorphic to $E(A^{(\alpha)})$ for some $\alpha < \kappa$. Equivalently, I is the two-sided ideal of S generated by all endomorphisms of M_R whose image is contained in a submodule of M_R isomorphic to $E(A^{(\alpha)})$ for some $\alpha < \kappa$. The unique maximal two-sided ideal of the endomorphism ring S is $I + J(S)$.

Let $M_R = E(\bigoplus_{\lambda \in \Lambda} A_\lambda^{(\kappa_\lambda)})$ be an injective envelope of a direct sum of a family of indecomposable injective modules. In view of Proposition 3.8.1, we will now introduce a kind of support of the submodules of M_R, which will be called the *maximal support* because we will use it to describe the maximal two-sided ideals of the endomorphism ring S of M_R. From now on, I will be the subset of Λ consisting of all indices $\lambda \in \Lambda$ with $\kappa_\lambda > 0$, so that $M_R = E(\bigoplus_{\lambda \in I} A_\lambda^{(\kappa_\lambda)})$ with $\kappa_\lambda > 0$ for every $\lambda \in I$. We can assume that $I \neq \emptyset$. For each direct summand N of M_R, let $\mathrm{supp}(N)$ (the *maximal support of* N) be the set of all $\lambda \in I$ satisfying the following property: if κ_λ is infinite, then $\dim_\lambda(N) = \kappa_\lambda$; if κ_λ is finite, then $\dim_\lambda(N) > 0$. Let N_1, N_2 be two direct summands of M_R. It is easily seen that:

1. If $N_1 \cong N_2$, then $\mathrm{supp}(N_1) = \mathrm{supp}(N_2)$.

2. If $N_1 \leq N_2$, then $\operatorname{supp}(N_1) \subseteq \operatorname{supp}(N_2)$.
3. If $N_1 \cap N_2 = 0$, then $\operatorname{supp}(N_1 \oplus N_2) = \operatorname{supp}(N_1) \cup \operatorname{supp}(N_2)$.

Lemma 3.8.2. *Let $\mathcal{C}$ be a cofilter of $\mathcal{L}_{ds}(M_R)$ and let N be in $\mathcal{C}$. If J is a subset of I and δ_λ is an ordinal $\leq \dim_\lambda(N)$ for every $\lambda \in J$, then $E(\bigoplus_{\lambda\in J} A_\lambda^{(\delta_\lambda)})$ belongs to $\mathcal{C}$.*

Proof. The module $E(\bigoplus_{\lambda\in J} A_\lambda^{(\delta_\lambda)})$ is a direct summand of $E(\bigoplus_{\lambda\in I} A_\lambda^{(\dim_\lambda(N))})$, and $E(\bigoplus_{\lambda\in I} A_\lambda^{(\dim_\lambda(N))})$ is isomorphic to N (Proposition 3.7.1). It follows that $E(\bigoplus_{\lambda\in J} A_\lambda^{(\delta_\lambda)})$ belongs to $\mathcal{C}$. ■

In order to visualize the maximal support and the meaning of Lemma 3.8.2, notice that the quotient set of $\mathcal{L}_{ds}(M_R)$ modulo isomorphism is in a canonical one-to-one correspondence with the direct product $\prod_{\lambda\in I}[0, \dim_\lambda(M_R)]$ of the intervals of cardinal $[0, \dim_\lambda(M_R)]$ (Proposition 3.7.1).

Lemma 3.8.3. *Let $\mathcal{C}$ be a cofilter of $\mathcal{L}_{ds}(M_R)$ and J be a subset of I. Set $\mathcal{C}^* := \{ V_1 \oplus V_2 \mid V_1 \in \mathcal{C},\ V_2 \in \mathcal{L}_{ds}(M_R),\ V_1 \perp V_2,$ and there exists $V_2' \leq E(\bigoplus_{j\in J} A_j^{(\kappa_j)})$ with $V_2' \cong V_2 \}$. Then $\mathcal{C}^*$ is a cofilter of $\mathcal{L}_{ds}(M_R)$ containing $\mathcal{C}$.*

Proof. Clearly $\mathcal{C}^*$ contains $\mathcal{C}$, so that condition (1) in the Definition of cofilter holds trivially. In order to prove (2), assume $V_1 \oplus V_2 \in \mathcal{C}^*$ and $X \in \mathcal{L}_{ds}(M_R)$ with $X \cong V_1 \oplus V_2$. Then $X = X_1 \oplus X_2$ with $X_1 \cong V_1$ and $X_2 \cong V_2$. Thus $X \in \mathcal{C}^*$. For (3), assume $X \in L_{ds}(M_R)$, $V_1 \oplus V_2 \in \mathcal{C}^*$ and $X \leq V_1 \oplus V_2$. By Lemma 3.5.1, $X \cong X_1 \oplus X_2$ with $X_1 \leq V_1$ and $X_2 \leq V_2$. It easily follows that $X_1 \oplus X_2 \in \mathcal{C}^*$. By (2), $X \in \mathcal{C}^*$ also. As far as (4) is concerned, assume $V_1 \oplus V_2 \perp U_1 \oplus U_2$. Then the sum $V_1 + V_2 + U_1 + U_2$ is direct, and $V_1 + V_2 + U_1 + U_2 = (V_1 \oplus U_1) \oplus (V_2 \oplus U_2) \in \mathcal{C}^*$. ■

Proposition 3.8.4. *Let $M_R = E(\bigoplus_{\lambda\in I} A_\lambda^{(\kappa_\lambda)})$ be an injective envelope of a direct sum of a family of indecomposable injective modules. For each cofilter $\mathcal{C}$ of $\mathcal{L}_{ds}(M_R)$, set*

$$P(\mathcal{C}) := \{ \operatorname{supp}(N) \mid N \in \mathcal{C} \}.$$

Then $P(\mathcal{C})$ is an antifilter of the power set $\mathcal{P}(I)$ of I. Moreover, if $\mathcal{U}$ is an ultracofilter and $P(\mathcal{U})$ is a proper subset of $\mathcal{P}(I)$, then $P(\mathcal{U})$ is an ultraantifilter of $\mathcal{P}(I)$.

PROOF. Obviously $\emptyset \in P(\mathcal{C})$, so $P(\mathcal{C}) \neq \emptyset$. Suppose that $X \subseteq \mathrm{supp}(N)$ for some $N \in \mathcal{C}$. By Lemma 3.8.2, $N' := E(\bigoplus_{\lambda \in X} A_\lambda^{(\dim_\lambda(N))}) \in \mathcal{C}$. Therefore $X = \mathrm{supp}(N') \in P(\mathcal{C})$. Assume $X_1 = \mathrm{supp}(N_1), X_2 = \mathrm{supp}(N_2)$ with $N_1, N_2 \in \mathcal{C}$. Consider

$$U := E(\bigoplus_{\lambda \in X_1} A_\lambda^{(\dim_\lambda(N_1))}) \oplus E(\bigoplus_{\lambda \in X_2 \setminus X_1} A_\lambda^{(\dim_\lambda(N_2))}).$$

By Lemma 3.8.2 again, $E(\bigoplus_{\lambda \in X_1} A_\lambda^{(\dim_\lambda(N_1))})$ and $E(\bigoplus_{\lambda \in X_2 \setminus X_1} A_\lambda^{(\dim_\lambda(N_2))})$ belong to $\mathcal{C}$, so that $U \in \mathcal{C}$. Thus, $X_1 \cup X_2 = X_1 \cup (X_2 \setminus X_1) = \mathrm{supp}(U) \in P(\mathcal{C})$. This proves that $P(\mathcal{C})$ is an antifilter of $\mathcal{P}(I)$.

Finally, we must show that $P(\mathcal{U})$ is maximal in the set of all proper antifilters of $\mathcal{P}(I)$ whenever $\mathcal{U}$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$ and $P(\mathcal{U}) \neq \mathcal{P}(I)$. It suffices to prove that if J is a subset of I and $J \notin P(\mathcal{U})$, then $I \setminus J \in P(\mathcal{U})$. Now $J \notin P(\mathcal{U})$ implies that $E(\bigoplus_{\lambda \in J} A_\lambda^{(\kappa_\lambda)}) \notin \mathcal{U}$. By Lemma 3.8.3, $\mathcal{U}^* := \{\, V_1 \oplus V_2 \mid V_1 \in \mathcal{U}, V_2 \in \mathcal{L}_{ds}(M_R),\ V_1 \perp V_2$, and there exists $V_2' \leq E(\bigoplus_{j \in J} A_j^{(\kappa_j)})$ with $V_2' \cong V_2 \,\}$ is a cofilter of $\mathcal{L}_{ds}(M_R)$ containing $\mathcal{U}$ and $E(\bigoplus_{\lambda \in J} A_\lambda^{(\kappa_\lambda)})$. Because of the maximality of $\mathcal{U}$, one has $\mathcal{U}^* = \mathcal{L}_{ds}(M_R)$. Therefore there exists a direct-sum decomposition $M_R = V_1 \oplus V_2$ with $V_1 \in \mathcal{U}$, a direct summand V_2' of $E(\bigoplus_{\lambda \in J} A_\lambda^{(\kappa_\lambda)})$ and $V_2 \cong V_2'$. Hence $\dim_\lambda(V_1) = \kappa_\lambda$ for every $\lambda \in I \setminus J$. In particular, $\mathrm{supp}(V_1) \supseteq I \setminus J$. But $\mathrm{supp}(V_1) \in P(\mathcal{U})$, so that $I \setminus J \in P(\mathcal{U})$, as we wanted to prove. ■

The proof of the following Proposition is straightforward and elementary.

Proposition 3.8.5. *Let $M_R = E(\bigoplus_{\lambda \in I} A_\lambda^{(\kappa_\lambda)})$ be an injective envelope of a direct sum of a family of indecomposable injective modules. For any proper antifilter P of the power set $\mathcal{P}(I)$ of I, the set $\mathcal{C}(P) := \{\, N \mid \mathrm{supp}(N) \in P \,\}$ is a proper cofilter of $\mathcal{L}_{ds}(M_R)$.*

Remark 3.8.6. We will now prove the following two facts.

1. The antifilter $P(\mathcal{C})$ in Proposition 3.8.4 is not necessarily a proper subset of $\mathcal{P}(I)$ whenever $\mathcal{C}$ is properly contained in $\mathcal{L}_{ds}(M_R)$.

2. The cofilter $\mathcal{C}(P)$ in $\mathcal{L}_{ds}(M_R)$ in Proposition 3.8.5 is proper whenever P is proper in $\mathcal{P}(I)$, but it is not necessarily an ultracofilter of $\mathcal{L}_{ds}(M_R)$ when P is an ultraantifilter in $\mathcal{P}(I)$.

To see this, let A_i, $i \in \mathbb{N}$, be pair-wise non-isomorphic indecomposable injective modules indexed in the set $\mathbb{N}$ of positive integers and $M_R = E(\bigoplus_{i\in\mathbb{N}} A_i^i)$. In order to prove (2), let P be any ultraantifilter of $\mathcal{P}(\mathbb{N})$ containing the antifilter of all finite subsets of $\mathbb{N}$. We will show that $\mathcal{C}(P)$ is not an ultraantifilter of $\mathcal{L}_{ds}(M_R)$ applying Proposition 3.6.5, that is, showing that there exists an element $X \in \mathcal{L}_{ds}(M_R)\backslash\mathcal{C}(P)$ such that $M_R \neq X' \oplus Y$ for every integer $n \geq 0$, every direct summand X' of M_R isomorphic to a submodule of X^n and every element Y of $\mathcal{C}(P)$.

Consider an element $X \cong E(\bigoplus_{i\in\mathbb{N}} A_i)$ of $\mathcal{L}_{ds}(M_R)$. As $\operatorname{supp}(X) = \mathbb{N}$, it follows that $X \notin \mathcal{C}(P)$. For every element Y of $\mathcal{C}(P)$, one has that $\operatorname{supp}(Y) \in P$, so $\mathbb{N} \setminus \operatorname{supp}(Y)$ is an infinite subset of $\mathbb{N}$. Therefore $\{\dim_i M \mid i \in \mathbb{N} \setminus \operatorname{supp}(Y)\}$ is not bounded. On the other hand, for any integer $n \geq 0$, any direct summand X' of X^n satisfying $X' \perp Y$ and any $i \in \mathbb{N} \setminus \operatorname{supp}(Y)$, one has that $\dim_i(X' \oplus Y) = \dim_i(X') + \dim_i(Y) = \dim_i(X') + 0 = \dim_i(X') \leq \dim_i(X^n) = n$. Thus $\{\dim_i(X' \oplus Y) \mid i \in \mathbb{N} \setminus \operatorname{supp}(Y)\}$ is bounded. Hence $X' \oplus Y \neq M_R$. By Proposition 3.6.5, $\mathcal{C}(P)$ is not an ultraantifilter of $\mathcal{L}_{ds}(M_R)$.

In order to prove (1), set

$$\mathcal{C} := \{ N \in \mathcal{L}_{ds}(M_R) \mid \text{there exists } m \in \mathbb{N} \text{ with } \dim_i N < m \text{ for every } i \in \mathbb{N}\}.$$

It is easy to check that $\mathcal{C}$ is a proper cofilter of $\mathcal{L}_{ds}(M_R)$ and $P(\mathcal{C}) = \mathcal{P}(\mathbb{N})$.

Let us go back to our notation of $M_R = E(\bigoplus_{\lambda\in I} A_\lambda^{(\kappa_\lambda)})$ with the A_λ's pair-wise non-isomorphic indecomposable injective modules and $\kappa_\lambda > 0$ for every $\lambda \in I$. We will show in Theorems 3.8.8 and 3.8.10 that the difficulties stated in Remark 3.8.6 do not arise when either $\{\lambda \in I \mid \kappa_\lambda < \infty\}$ is a finite subset of I or there exists a positive integer m such that $\kappa_\lambda < m$ for every $\lambda \in I$. We will show that, in these cases, $\mathcal{C}(P)$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$ when P is an ultraantifilter of $\mathcal{P}(I)$ and $P(\mathcal{U})$ is an ultraantifilter of $\mathcal{P}(I)$ when $\mathcal{U}$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$.

Lemma 3.8.7. *Let $\mathcal{C}$ be a cofilter of $\mathcal{L}_{ds}(M_R)$ and let N be a module in $\mathcal{C}$. If there exists a positive integer m such that $\kappa_\lambda < m$ for every $\lambda \in I$, then $E(\bigoplus\limits_{\lambda\in\operatorname{supp}(N)} A_\lambda^{\kappa_\lambda})$ belongs to $\mathcal{C}$.*

Proof. For every $t = 1, 2, \ldots, m-1$, set $J_t := \{\, \lambda \in \mathrm{supp}(N) \mid \kappa_\lambda \geq t \,\}$, so that $E(\bigoplus_{\lambda \in \mathrm{supp}(N)} A_\lambda^{\kappa_\lambda}) \cong \bigoplus_{t=1}^{m-1} E(\bigoplus_{\lambda \in J_t} A_\lambda)$. By Lemma 3.8.2, if J is a subset of $\mathrm{supp}(N)$ and δ_λ is an ordinal $\leq \dim_\lambda(N)$ for every $\lambda \in J$, then $E(\bigoplus_{\lambda \in J} A_\lambda^{\delta_\lambda})$ belongs to $\mathcal{C}$. Apply this Lemma 3.8.2 to the subset J_t of $\mathrm{supp}(N)$ with $\delta_\lambda = 1$ for every $\lambda \in J_t$, so that $E(\bigoplus_{\lambda \in J_t} A_\lambda) \in \mathcal{C}$. Then $E(\bigoplus_{\lambda \in \mathrm{supp}(N)} A_\lambda^{\kappa_\lambda})$ belongs to $\mathcal{C}$ by properties (2) and (4) in the Definition of cofilter. ■

Theorem 3.8.8. *Let m be a positive integer such that $\kappa_\lambda < m$ for every $\lambda \in I$.*

1. *If $\mathcal{U}$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$, then $P(\mathcal{U})$ is an ultraantifilter of the lattice $\mathcal{P}(I)$.*

2. *Conversely, if P is an ultraantifilter of the lattice $\mathcal{P}(I)$, then $\mathcal{C}(P)$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$.*

Proof. 1. By Proposition 3.8.4, it suffices to show that $P(\mathcal{U})$ is a proper subset of $\mathcal{P}(I)$. Now, if $I \in P(\mathcal{U})$, then there exists $N \in \mathcal{U}$ such that $I = \mathrm{supp}(N)$. By Lemma 3.8.7, $M_R = E(\bigoplus_{\lambda \in I} A_\lambda^{\kappa_\lambda}) = E(\bigoplus_{\lambda \in \mathrm{supp}(N)} A_\lambda^{\kappa_\lambda})$ belongs to $\mathcal{U}$, so that $\mathcal{U} = \mathcal{L}_{ds}(M_R)$ by property (3) in the definition of cofilter. This is a contradiction.

2. We already know that $\mathcal{C}(P)$ is a proper cofilter of $\mathcal{L}_{ds}(M_R)$ by Proposition 3.8.5. Suppose that $\mathcal{C}'$ is a cofilter strictly containing $\mathcal{C}(P)$. Let U be an element of $\mathcal{C}'$ not in $\mathcal{C}(P)$. Since P is an ultraantifilter of $\mathcal{P}(I)$, either $\mathrm{supp}(U)$ belongs to P or $I \setminus \mathrm{supp}(U)$ belongs to P. But $U \notin \mathcal{C}(P)$, so that $I \setminus \mathrm{supp}(U) \in P$. In particular, $E(\bigoplus_{\lambda \in I \setminus \mathrm{supp}(U)} A_\lambda^{\kappa_\lambda}) \in \mathcal{C}(P)$ by the definition of $\mathcal{C}(P)$. Applying Lemma 3.8.7 to the module $U \in \mathcal{C}'$, we find that $E(\bigoplus_{\lambda \in \mathrm{supp}(U)} A_\lambda^{\kappa_\lambda})$ belongs to $\mathcal{C}'$. Thus both $E(\bigoplus_{\lambda \in I \setminus \mathrm{supp}(U)} A_\lambda^{\kappa_\lambda})$ and $E(\bigoplus_{\lambda \in \mathrm{supp}(U)} A_\lambda^{\kappa_\lambda})$ belong to $\mathcal{C}'$, so that their direct sum M_R belongs to $\mathcal{C}'$. It follows that $\mathcal{C}' = \mathcal{L}_{ds}(M_R)$. ■

Let us go back to the general case of the κ_λ's possibly infinite or unbounded.

Lemma 3.8.9. *Let $\mathcal{C}$ be a cofilter of $\mathcal{L}_{ds}(M_R)$ and N be a module in $\mathcal{C}$. If J is a finite subset of* $\mathrm{supp}(N)$*, then $E(\bigoplus_{\lambda \in J} A_\lambda^{(\kappa_\lambda)})$ belongs to $\mathcal{C}$.*

Proof. Set $J_1 := \{\, \lambda \in J \mid \kappa_\lambda < \infty \,\}$ and $J_2 := \{\, \lambda \in J \mid \kappa_\lambda = \infty \,\}$. Then J_1 and J_2 are finite sets and $J_1 \cup J_2 = J$. By Lemma 3.8.2 applied to the set J_1 with $\delta_\lambda = 1$ for all λ, we see that $E(\bigoplus\limits_{\lambda \in J_1} A_\lambda)$ is in $\mathcal{C}$. By Lemma 3.8.2 applied to the set J_2 with $\delta_\lambda = \kappa_\lambda$ for all λ, we see that $E(\bigoplus\limits_{\lambda \in J_2} A_\lambda^{(\kappa_\lambda)})$ belongs to $\mathcal{C}$. By Lemma 3.8.2 applied to the set $\{\lambda\}$ with $\delta_\lambda = 1$, we see that A_λ belongs to $\mathcal{C}$ for every $\lambda \in J$. Since J_1 is finite and $\kappa_\lambda < \infty$ for every $\lambda \in J_1$, $E(\bigoplus\limits_{\lambda \in J_1} A_\lambda^{(\kappa_\lambda)}) = \bigoplus\limits_{\lambda \in J_1} A_\lambda^{\kappa_\lambda}$ belongs to $\mathcal{C}$ by property (4) of the definition of cofilter. Thus $E(\bigoplus\limits_{\lambda \in J} A_\lambda^{(\kappa_\lambda)}) \cong E(\bigoplus\limits_{\lambda \in J_1} A_\lambda^{(\kappa_\lambda)}) \oplus E(\bigoplus\limits_{\lambda \in J_2} A_\lambda^{(\kappa_\lambda)})$ belongs to $\mathcal{C}$ by property (4) again. ∎

Theorem 3.8.10. *Suppose that* $\{\, \lambda \in I \mid \kappa_\lambda < \infty \,\}$ *is finite.*

1. *If* $\mathcal{U}$ *is an ultracofilter of* $\mathcal{L}_{ds}(M_R)$*, then* $P(\mathcal{U})$ *is an ultraantifilter of the lattice* $\mathcal{P}(I)$*.*

2. *Conversely, if* P *is an ultraantifilter of the lattice* $\mathcal{P}(I)$*, then* $\mathcal{C}(P)$ *is an ultracofilter of* $\mathcal{L}_{ds}(M_R)$*.*

Proof. 1. By Proposition 3.8.4, it suffices to show that $P(\mathcal{U})$ is proper in $\mathcal{P}(I)$. If $I \in P(\mathcal{U})$, then there exists $N \in \mathcal{U}$ such that $I = \operatorname{supp}(N)$. We have N is isomorphic to $E(\bigoplus\limits_{\lambda \in \operatorname{supp}(N)} A_\lambda^{(\dim_\lambda(N))}) = E(\bigoplus\limits_{\lambda \in K} A_\lambda^{(\dim_\lambda(N))}) \oplus E(\bigoplus\limits_{\lambda \in \operatorname{supp}(N) \setminus K} A_\lambda^{(\dim_\lambda(N))})$. Here K is the set of all $\lambda \in \operatorname{supp}(N)$ satisfying the dimension $\dim_\lambda(N)$ is finite. One has $E(\bigoplus\limits_{\lambda \in I \setminus K} A_\lambda^{(\dim_\lambda(N))}) = E(\bigoplus\limits_{\lambda \in \operatorname{supp}(N) \setminus K} A_\lambda^{(\dim_\lambda(N))}) \in \mathcal{U}$ because it is a direct summand of N (property (3) in the definition of cofilter). Moreover, by Lemma 3.8.9 applied to the cofilter $\mathcal{U}$ and the finite subset K of $\operatorname{supp}(N)$, we see that $E(\bigoplus\limits_{\lambda \in K} A_\lambda^{(\kappa_\lambda)}) \in \mathcal{U}$. Hence $M_R = E(\bigoplus\limits_{\lambda \in I} A_\lambda^{(\kappa_\lambda)}) = E(\bigoplus\limits_{\lambda \in K} A_\lambda^{(\kappa_\lambda)}) \oplus E(\bigoplus\limits_{\lambda \in I \setminus K} A_\lambda^{(\kappa_\lambda)}) \in \mathcal{U}$. Therefore, $\mathcal{U}$ is not proper, which is a contradiction.

2. Let P be an ultraantifilter of $\mathcal{P}(I)$ and $\mathcal{C}'$ be a cofilter strictly containing $\mathcal{C}(P)$. Then there exists an element U in $\mathcal{C}' \setminus \mathcal{C}(P)$. Since P is an ultraantifilter of $\mathcal{P}(I)$, either $\operatorname{supp}(U)$ belongs to P or $I \setminus \operatorname{supp}(U)$ belongs to P. But $U \notin \mathcal{C}(P)$, so that $I \setminus \operatorname{supp}(U) \in P$.

Hence $E(\bigoplus_{\lambda\in I\setminus\mathrm{supp}(U)} A_\lambda^{(\kappa_\lambda)}) \in \mathcal{C}(P)$. Now $J := \{\,\lambda \in \mathrm{supp}(U) |\ \dim_\lambda(U) < \infty\,\}$ is a finite subset of I. By Lemma 3.8.9, we get that $E(\bigoplus_{\lambda\in J} A_\lambda^{(\kappa_\lambda)}) \in \mathcal{C}'$. By Lemma 3.8.2 applied to the element U of $\mathcal{C}'$ and the subset $\mathrm{supp}(U)\setminus J$ of $\mathrm{supp}(N)$ with $\delta_\lambda = \kappa_\lambda$ for all λ, we obtain that $E(\bigoplus_{\lambda\in\mathrm{supp}(U)\setminus J} A_\lambda^{(\kappa_\lambda)}) \in \mathcal{C}'$. Hence $E(\bigoplus_{\lambda\in J} A_\lambda^{(\kappa_\lambda)}) \oplus E(\bigoplus_{\lambda\in\mathrm{supp}(U)\setminus J} A_\lambda^{(\kappa_\lambda)}) \in \mathcal{C}'$. Thus $M_R \in \mathcal{C}'$ and $\mathcal{C}' = \mathcal{L}_{ds}(M_R)$. ■

Theorem 3.8.11. *If either the subset* $\{\,\lambda \in I \mid \kappa_\lambda < \infty\,\}$ *is finite or there exists a positive integer* m *such that* $\kappa_\lambda < m$ *for every* $\lambda \in I$, *then there is a one-to-one correspondence between the set of all ultraantifilters of* $\mathcal{P}(I)$ *and the set of all ultracofilters of* $\mathcal{L}_{ds}(M_R)$.

Proof. The correspondence is that defined in Propositions 3.8.4 and 3.8.5. By Theorem 3.8.8 and 3.8.10, it suffices to prove that $\mathcal{C}(P(\mathcal{U})) = \mathcal{U}$ and $P(\mathcal{C}(P)) = P$ for any ultraantifilter P of $\mathcal{P}(I)$ and any ultracofilter $\mathcal{U}$ of $\mathcal{L}_{ds}(M_R)$. It is clear that $\mathcal{U} \subseteq \mathcal{C}(P(\mathcal{U}))$. As $\mathcal{C}(P(\mathcal{U})) \neq \mathcal{L}_{ds}(M_R)$ and $\mathcal{U}$ is an ultracofilter of $\mathcal{L}_{ds}(M_R)$, it follows that $\mathcal{U} = \mathcal{C}(P(\mathcal{U}))$. Similarly for the equality $P(\mathcal{C}(P)) = P$. ■

The next two corollaries follow directly from Theorem 3.8.11.

Corollary 3.8.12. *Let* $A_1, A_2, \cdots, A_n$ *be* n *indecomposable injective modules which are pair-wise non-isomorphic, and* $\kappa_1, \kappa_2, \cdots, \kappa_n$ *be* n *non-zero cardinals. Then the endomorphism ring of* $M_R = E(\bigoplus_{i=1}^{n} A_i^{(\kappa_i)})$ *has exactly* n *maximal two-sided ideals.*

Corollary 3.8.13. *Let* $M_R = E(\bigoplus_{i\in I} A_i)$ *be an injective envelope of a direct sum of a family of pair-wise non-isomorphic indecomposable injective modules* A_i. *Then there is a one-to-one correspondence between the set of all maximal two-sided ideals of the endomorphism ring* $S = \mathrm{End}(M_R)$ *of* M_R *and the set of all ultraantifilters of the power set* $\mathcal{P}(I)$ *of* I.

Chapter 4

Loewy modules with finite Loewy invariants and max modules with finite radical invariants

For any left module ${}_RM$ over a ring R and any ordinal α, the Loewy series $\mathrm{soc}_\alpha({}_RM)$ of submodules of ${}_RM$ is defined setting

$$\mathrm{soc}_0({}_RM) := \ 0,$$

$$\mathrm{soc}_{\alpha+1}({}_RM)/\mathrm{soc}_\alpha({}_RM) := \mathrm{soc}(M/\mathrm{soc}_\alpha({}_RM))$$

for every ordinal α, and

$$\mathrm{soc}_\beta({}_RM) := \bigcup_{\alpha<\beta} \mathrm{soc}_\alpha({}_RM)$$

for every limit ordinal β (see [14, Section 2.11]). The members of the Loewy series of ${}_RM$ form a well ordered set of fully invariant submodules of ${}_RM$ and the Loewy series is stationary for every module ${}_RM$, that is, there exists an ordinal λ such that $\mathrm{soc}_\alpha({}_RM) = \mathrm{soc}_\lambda({}_RM)$ for $\alpha \geq \lambda$. For such an ordinal λ, the module $\delta({}_RM) := \mathrm{soc}_\lambda({}_RM)$ is called the *Loewy submodule* of ${}_RM$. If $\delta({}_RM) = {}_RM$, then ${}_RM$ is said to be a *Loewy* *module* or a *semiartinian module* or a *min module.* In this case, the least ordinal λ satisfying this property is called the *Loewy length.* For example, every artinian module is Loewy.

For each ordinal α, the Goldie dimension of $\mathrm{soc}_{\alpha+1}({}_RM)/\mathrm{soc}_{\alpha}({}_RM)$ is denoted by $d_\alpha({}_RM)$ and called the α-th *Loewy invariant* of ${}_RM$.

When the base ring R is commutative, Facchini showed in [12, Theorem 2.7] the following results: (1) Artinian R-modules are exactly Loewy modules with all Loewy invariants finite. (2) For every artinian module ${}_RM$ over R, one has that $d_r({}_RM) \leq \begin{pmatrix} d_1({}_RM)+r-1 \\ r \end{pmatrix}$ for every $r \geq 1$ and the Loewy length of ${}_RM$ is $\leq \omega$. Here, ω is the least infinite ordinal. (3) For every artinian module ${}_RM$, define $P({}_RM,t) = \sum_{n=0}^{\infty} d_n({}_RM)t^n \in \mathbb{Z}[[t]]$. Put $s = d_0({}_RM)d_1({}_RM)$. Then $P({}_RM,t)$ is a rational function in t of the form $f(t)/(1-t)^s$, where $f(t) \in \mathbb{Z}[t]$. If d is the order of the pole of $P({}_RM,t)$ at $t = 1$, then, for all sufficiently large n, the n-th Loewy invariant $d_n({}_RM)$ and the length $l(\mathrm{soc}_n({}_RM))$ are polynomials in n with rational coefficients of degree $d-1$ and d respectively. We will recall these results in the first Section. In Section 4.2, we present examples that show that these properties, which hold for modules over a commutative ring, do not hold for modules over non-commutative rings, in general.

Camps and Dicks [9], answering a question posed by Menal [34], proved that artinian modules have semilocal endomorphism rings. In Section 4.3, it is proved, more generally, that Loewy modules ${}_RM$ with finite Loewy invariants have semilocal endomorphism rings. Some further properties of Loewy modules with finite Loewy invariants also are proved.

The dual notion of Loewy modules is that of max modules, which are defined in the first Section of this Chapter. For max modules, the best setting is that of a semilocal ring R as a base ring. In this case, max modules with finite radical invariants and semilocal endomorphism rings are considered in Section 4.4.

In Section 4.5, we apply our results to the study of modules over a perfect ring.

The main results in this Chapter appear in [17].

4.1 Loewy modules, max modules and semilocal rings

The aim of this Section is to recall some basic concepts of Loewy modules, max modules and perfect rings. For convenience, most modules used in this Chapter are left modules. Homomorphisms will be written on the right.

For Loewy modules and max modules, which are studied in this Chapter, we need the Goldie dimension and the dual Goldie dimension, so that we will review briefly these notions.

The Goldie dimension.

We begin with the Goldie dimension. The concept of "independence" of a set of submodules has been already mentioned in Chapter 3, but it is convenient to recall it again here. Let ${}_RM$ be a left R-module and $\{ M_i \mid i \in I\}$ be a non-empty set of submodules of ${}_RM$. One says that $\{ M_i \mid i \in I \}$ is *independent* if $M_i \neq 0$ and $M_i \cap \sum_{i\neq j\in I} M_j = 0$ for any $i \in I$, that is, $\sum_{i\in I} M_i = \bigoplus_{i\in I} M_i$. By convention, the sum of an empty set (of submodules) is 0. The following Lemma is basic.

Lemma 4.1.1. [14, Section 2.7] *If ${}_RM$ contains no infinite independent set of submodules, then there exists a positive integer n such that every independent set of submodules of ${}_RM$ has cardinality $\leq n$.*

The *Goldie dimension* of ${}_RM$, denoted by $\dim {}_RM$, is *infinite* if ${}_RM$ has an independent infinite set of submodules. Otherwise, $\dim {}_RM$ is the supremum of the set

$$\{ k \in \mathbb{N} \mid \text{ there exists an independent set of } k \text{ submodules of } {}_RM \}.$$

For a ring R, the *left Goldie dimension* of R is defined to be $\dim {}_RR$ and the *right Goldie dimension* of R is defined to be $\dim R_R$.

Recall that a non-zero module whose non-zero submodules are essential is called *uniform*. The following basic Proposition is from [30, 6.2 and Corollary 6.10].

Proposition 4.1.2. *Let ${}_RM, {}_RM'$ be left modules over a ring R such that* $\dim {}_RM$ *and* $\dim {}_RM'$ *are finite, and ${}_RN$ be a submodule of ${}_RM$. Then*

1. *If* $\dim {}_RM = n$, *then there exists an independent set* $\{ U_1, U_2, \cdots, U_n\}$ *of* n *uniform submodules of* ${}_RM$ *such that* $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ *is essential in* ${}_RM$.

2. $\dim {}_RN \leq \dim {}_RM$ *and the equality holds if and only if* ${}_RN$ *is essential in* ${}_RM$.

3. $\dim({}_RM \oplus {}_RM') = \dim {}_RM + \dim {}_RM'$.

The notion of Goldie dimension is "opposite" to that of dual Goldie dimension, which is defined as follows.

The dual Goldie dimension.

Let ${}_RM$ be a left R-module and $\{ M_i \mid i \in I \}$ be a non-empty set of submodules of ${}_RM$. One says that $\{ M_i \mid i \in I\}$ is *coindependent* if $M_i \neq {}_RM$ and $M_i + \bigcap_{j\in F} M_j = {}_RM$ for every $i \in I$ and non-empty finite set $F \subseteq I\backslash\{i\}$. By convention, the intersection of an empty set (of submodules) is ${}_RM$.

Lemma 4.1.3. [14, Theorem 2.40] *If* ${}_RM$ *contains no infinite coindependent set of submodules of* ${}_RM$, *then there exists a positive integer* n *such that every coindependent set of submodules of* ${}_RM$ *has cardinality* $\leq n$.

The *dual Goldie dimension* of ${}_RM$, denoted by codim_RM, is *infinite* if ${}_RM$ has a coindependent infinite set of submodules. Otherwise, codim_RM is the supremum of the set

$$\{ k \in \mathbb{N} \mid \text{ there exists a coindependent set of } k \text{ submodules of } {}_RM \}.$$

The *left dual Goldie dimension* of a ring R is defined to be codim_RR and the *right dual Goldie dimension* of a ring R is defined to be $\text{codim}R_R$.

Proposition 4.1.4. [14, Theorem 2.40] *Let* ${}_RM, {}_RM'$ *be left modules over a ring* R *such that* codim_RM *and* codim_RM *are finite, and* N *be a submodule of* ${}_RM$. *Then*

1. *If* $\text{codim}_RM = n$, *then there exists a coindependent set* $\{ N_1, N_2, \cdots, N_n\}$ *of* n *submodules of* ${}_RM$ *such that* $N_1 \cap N_2 \cap \cdots \cap N_n$ *is superfluous in* ${}_RM$ *and* ${}_RM/N_i$ *is couniform for any* $i \in I$.

2. $\text{codim}_RM/N \leq \text{codim}_RM$ *and the equality holds if and only if* N *is superfluous in* ${}_RM$.

3. $\operatorname{codim}({}_RM \oplus {}_RM') = \operatorname{codim}_RM + \operatorname{codim}_RM'$.

The Krull dimension.

We define a family of classes $\mathcal{K}_\alpha$ of modules for $\alpha = -1$ or an ordinal as follows. The class $\mathcal{K}_{-1}$ contains only the zero module. Assume that the class of $\mathcal{K}_\beta$ is defined for any $\beta < \alpha$. The class $\mathcal{K}_\alpha$ is defined as the class of all modules ${}_RM$ such that

1. ${}_RM \notin \bigcup_{\beta<\alpha} \mathcal{K}_\beta$.

2. For every countable descending chain $A_0 \geq A_1 \geq A_2 \geq \cdots$ of submodules of ${}_RM$, there exists a positive integer n such that the factors A_i/A_{i+1} are in $\bigcup_{\beta<\alpha} \mathcal{K}_\beta$ for any $i \geq n$.

It is clear that all the classes $\mathcal{K}_\alpha$ are pairwise disjoint. If ${}_RM$ is contained in $\mathcal{K}_\alpha$ for some α, then ${}_RM$ is said to *have Krull dimension* α, denoted by $\operatorname{Kdim}_RM = \alpha$. Otherwise, we say that ${}_RM$ *fails to have Krull dimension.* By definition, every artinian module has Krull dimension 0. The following Proposition is from [14, Example 7.10, Propositions 7.11 and 7.13].

Proposition 4.1.5. *Let ${}_RM$ be a left module over a ring R and N be a submodule of ${}_RM$. Then*

1. *If ${}_RM$ is noetherian, then ${}_RM$ has Krull dimension.*

2. *If ${}_RM$ has Krull dimension, then the Goldie dimension* $\dim {}_RM$ *is finite.*

3. *${}_RM$ has Krull dimension if and only if ${}_RM/N$ and N have Krull dimension. In this case,* $\operatorname{Kdim}({}_RM) = \max\{\operatorname{Kdim}_RM/N, \operatorname{Kdim}N\}$.

Semilocal rings.

A *semilocal ring* is a ring R for which $R/J(R)$ is a semisimple artinian ring, that is, $R/J(R)$ is the direct product of finitely many matrix rings over division rings. Let R be a semilocal ring. The semilocal ring R is called *semiprimary* if the Jacobson radical $J(R)$ is nilpotent, that is, there exists a positive integer $n > 0$ such that $J(R)^n = 0$. The ring R is said to be *left perfect* (resp. *right perfect*) if the Jacobson radical $J(R)$ is left (resp. right) T-nilpotent, that is, for any sequence $\{a_1, a_2, \cdots\} \subseteq J(R)$, there exists a positive integer $n > 0$ such that $a_1a_2\cdots a_n = 0$ (resp. $a_na_{n-1}\cdots a_1 = 0$). A ring which is both left and right perfect is called *perfect*. Hence, one has that

$$\text{one-sided artinian} \Rightarrow \text{semiprimary} \Rightarrow \text{perfect} \Rightarrow \text{semilocal}.$$

The following is the well known Theorem of Hopkins-Levitzki. It concerns artinian modules, noetherian modules and the length of a module, over a semiprimary ring. Let ${}_RM$ be a left module over a ring R. A chain of submodules $N_0 < N_1 < \cdots < N_n$ of ${}_RM$ is said to have *length n*. The *length* of ${}_RM$, denoted by $l({}_RM)$, is defined to be the supremum (eventually ∞) of the lengths of the chain of submodules of ${}_RM$.

Theorem 4.1.6. [29, Theorem 4.15] *For a left module ${}_RM$ over a semiprimary ring R, the following conditions are equivalent:*

1. *${}_RM$ is noetherian.*
2. *${}_RM$ is artinian.*
3. *The length $l({}_RM)$ of ${}_RM$ is finite.*

For two arbitrary rings R and S, a ring homomorphism $f\colon R \to S$ is called *local* if, for any $r \in R\backslash R^*$, one has $f(r) \in S\backslash S^*$. Here, R^* and S^* are the sets of invertible elements of R and S respectively. Camps and Dicks proved in [9] the following Theorem.

Theorem 4.1.7. [9, Theorem 1] *For a ring R, the following statements are equivalent:*

1. *S is semilocal.*
2. *There exists a local homomorphism of R into a semilocal ring.*
3. *There exists a local homomorphism of R into a semisimple artinian ring.*

Theorem 4.1.8. [14, Page 59] *Let R be a ring.*

1. *If R is semisimple artinian ring, then* $\dim {}_RR = \dim R_R = \text{codim}_RR = \text{codim}R_R < \infty$*.* For any semisimple artinian ring R, we write $\dim(R)$ or $\text{codim}(R)$ for these dimensions.

2. *If R is semilocal ring, then* $\text{codim}({}_RR) = \text{codim}(R_R) = \text{codim}(R/J(R))$*.* For any semilocal ring R, we write $\text{codim}(R)$ for these dimensions.

3. *If $R = M_n(D)$, the ring of $n \times n$ matrices over a division ring D, then* $\dim(R) = \text{codim}(R) = n$*.*

Loewy modules.

The following properties are basic.

Proposition 4.1.9. [14, Lemma 5.28] *Let $_RM$ be a left module over a ring R. The following statements are equivalent:*

1. *$_RM$ is Loewy.*
2. *Every nonzero homomorphic image of $_RM$ has an essential socle.*

Corollary 4.1.10. *Factor modules of a Loewy module are Loewy.*

Proposition 4.1.11. [4, Theorem 28.4] *Every left module over a right perfect ring is Loewy.*

Proposition 4.1.12. *The Loewy invariants of an artinian module are finite.*

Max modules

Dually, the notion of *max modules* is "opposite" to that of Loewy modules. Recall that the *radical* of a module $_RM$, denoted by $\mathrm{rad}(_RM)$, is the intersection of all maximal submodules of $_RM$. In case $_RM$ has no maximal submodule, then $\mathrm{rad}(_RM) := {}_RM$. The descending chain of submodules

$$\mathrm{rad}_0(_RM) \geq \mathrm{rad}_1(_RM) \geq \cdots \geq \mathrm{rad}_\alpha(_RM) \geq \mathrm{rad}_{\alpha+1}(_RM) \geq \cdots$$

of $_RM$ defined by setting

$$\mathrm{rad}_0(_RM) := {}_RM,$$

$$\mathrm{rad}_\alpha(_RM) := \mathrm{rad}(\mathrm{rad}_{\alpha-1}(_RM))$$

for any successor ordinal α, and

$$\mathrm{rad}_\beta(_RM) := \bigcap_{\alpha<\beta} \mathrm{rad}_\alpha(_RM)$$

for any limit ordinal β. This chain $\mathrm{rad}_\alpha(_RM)$ is called the *radical series* of $_RM$. If there exists an ordinal λ such that $\mathrm{rad}_\lambda(_RM) = 0$, then $_RM$ is said to be a *max module* or a *seminoetherian module.* In this case, the least ordinal λ satisfying this property is called the *max length* of ${}_RM$. Obviously, the class of max modules contains the class of noetherian ones.

Proposition 4.1.13. [38, Proposition 2.2] *Let ${}_RM$ be a left module over a ring R. The following statements are equivalent:*

1. *${}_RM$ is max.*
2. *Every non-zero submodule of ${}_RM$ has a superfluous radical.*

Corollary 4.1.14. *Submodules of a max module are max.*

Proposition 4.1.15. [4, Theorem 28.4] *Every left module over a left perfect ring is max.*

For any non-limit ordinal α, the Goldie dimension of $\mathrm{rad}_\alpha({}_RM)/\mathrm{rad}_{\alpha+1}({}_RM)$ is called the α-the *radical invariant* of ${}_RM$, denoted by $r_\alpha({}_RM)$.

Proposition 4.1.16. *In a noetherian module, every radical invariant is finite.*

Next, we summarize some important results about Loewy modules over a commutative ring. All of them are from [12] or [39].

Loewy modules over a commutative ring.

As we have already seen, from the definitions of Loewy modules and Loewy invariants, it follows that every artinian module is Loewy with finite invariants. Facchini proved that the converse is also true if the base ring is commutative.

Theorem 4.1.17. [12, Theorem 2.7] *Let R be a commutative ring and ${}_RM$ be a left R-module. The following statements are equivalent:*

1. *${}_RM$ is artinian.*
2. *${}_RM$ is Loewy and all its Loewy invariants are finite.*

If ${}_RM$ is a left R-module and S is the endomorphism ring of ${}_RM$, then ${}_RM$ has a natural structure of a right S-module with product given by $ms = (m)s$ for $s \in S, m \in M$. We emphasize this by writing M_S.

Theorem 4.1.18. [12, Theorem 2.8] *Let ${}_RM$ be an artinian left module with simple socle over a commutative ring R. Then $S = \mathrm{End}({}_RM)$ is a local Noetherian complete commutative ring and M_S is the injective envelope of the unique simple S-module.*

Theorem 4.1.19. *Let ${}_RM$ be a Loewy module with finite invariants $d_\alpha({}_RM)$ over a commutative ring R. If $d_1({}_RM) = n$ then $d_r({}_RM) \leq \begin{pmatrix} n+r-1 \\ r \end{pmatrix}$ for every positive integer $r \geq 1$.*

Proof. This is a Corollary of [12, Theorem 3.1]. ∎

Theorem 4.1.20. [12, Theorem 3.2] *Let ${}_RM$ be a Loewy module with finite invariants $d_\alpha({}_RM)$ over a commutative ring R. Then $P({}_RM, t) = \sum_{n=1}^{+\infty} d_n({}_RM)t^n$ is a rational function in t of the form $\frac{f(t)}{(1-t)^s}$, where $f(t) \in \mathbb{Z}[t]$ and $s = d_0({}_RM)d_1({}_RM)$. Let $d({}_RM)$ be the order of the pole of $P({}_RM, t)$ at $t = 1$. Then, for all sufficiently large n, $d_n({}_RM)$ and the length $l(\mathrm{soc}_n({}_RM))$ of $\mathrm{soc}_n({}_RM)$ are polynomials in n with rational coefficients of degree $d({}_RM) - 1$ and $d({}_RM)$ respectively.*

Theorem 4.1.21. [39, Theorem 4.2] *Let R be a commutative ring, ${}_RM$ be a Loewy R-module, α be an ordinal and r be a positive integer. If the α-th and $(\alpha + r)$-th Loewy invariants are finite, then the β-th Loewy invariant is finite for any $\beta > \alpha + r$ and ${}_RM = \mathrm{soc}_{\omega+r}({}_RM)$.*

4.2 Some examples

Recall that a left module over an arbitrary ring is said to be *quotient finite dimensional* (*q.f.d.*) if all its factor modules have finite Goldie dimension [23]. For instance, all uniserial modules are q.f.d., and all modules with Krull dimension are q.f.d. (Proposition 4.1.5). In the next result, we have collected some characterizations of artinian modules related to being Loewy modules.

Proposition 4.2.1. *The following conditions are equivalent for a left module ${}_RM$ over an arbitrary ring R:*

1. ${}_RM$ *is artinian.*
2. ${}_RM$ *is Loewy and has Krull dimension.*
3. ${}_RM$ *is Loewy and q.f.d.*
4. ${}_RM$ *is Loewy and its factor modules have finitely generated socles.*
5. *All factor modules of* ${}_RM$ *are essential extensions of their finitely generated socles.*

PROOF. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial. (2) $\Rightarrow$ (3) follows from Propositions 4.1.5. (5) $\Rightarrow$ (1) is proved in [41, Proposition 2*]. ■

As Section 4.1, when the base ring R is commutative, the artinian R-modules are exactly the Loewy R-modules with finite Loewy invariants. In general, for a non-commutative ring, every artinian R-module is a Loewy R-module with finite Loewy invariants. In the next example, it is shown that *there exists a non-artinian module* ${}_RM$ *over a suitable non-commutative ring* R *that is a Loewy module of Loewy length* ω *with finite Loewy invariants.*

Example 4.2.2. Let k be a field, V_k a vector space over k of countable infinite dimension, and $\{ v_n \mid n \geq 0 \}$ be a basis of V_k. Let R be the set of all the endomorphisms $f \in \mathrm{End}(V_k)$ such that, for every $n \geq 0$, $f(v_{2n})$ belongs to the subspace $\langle v_0, v_2, v_4, \dots, v_{2n} \rangle_k$ of V_k and $f(v_{2n+1})$ belongs to the subspace $\langle v_0, v_2, v_4, v_6, \dots, v_{2n}, v_{2n+1} \rangle_k$. It is easily seen that R is a k-subalgebra of $\mathrm{End}(V_k)$, so that V inherits a left R-module structure. More precisely, ${}_RV$ is a left R-module with scalar multiplication defined by $fv = f(v)$ for every $f \in R$ and $v \in V$. Clearly, $\langle v_0, v_1, \dots, v_n \rangle_k$ is a submodule of ${}_RV$ for every $n \geq 0$.

Lemma 4.2.3. *Let* $v = v_0\alpha_0 + v_1\alpha_1 + \cdots + v_n\alpha_n$ *be an element of* V *with* $\alpha_i \in k$ *for every* $i = 0, 1, 2, \dots, n$ *and* $\alpha_n \neq 0$. *Then* $Rv \geq Rv_n$, *and* $Rv \geq Rv_{2m}$ *for every integer* m *with* $0 \leq 2m \leq n$.

PROOF. Let $f \in \mathrm{End}(V_k)$ be defined by $f(v_n) = v_n\alpha_n^{-1}$ and $f(v_i) = 0$ for every $i \neq n$. Then $f \in R$ and $f(v) = v_n$. This proves that $Rv \geq Rv_n$.

Similarly, for every m with $0 \leq 2m \leq n$, let $g \in \mathrm{End}(V_k)$ be defined by $g(v_n) = v_{2m}$ and $g(v_i) = 0$ for every $i \neq n$. Then $g \in R$ and $g(v_n) = v_{2m}$. Thus $gf(v) = v_{2m}$, and therefore $Rv \geq Rv_{2m}$. ■

We will now describe the n-th Loewy submodule of ${}_RV$ for any $n \geq 0$.

Lemma 4.2.4. *Let $n \geq 1$ be an integer. Then the n-th Loewy submodule $\mathrm{soc}_n({}_RV)$ of ${}_RV$ is $\langle v_0, v_1, v_2, \dots, v_{2n-3}, v_{2n-2}\rangle_k$ and the Loewy invariants of ${}_RV$ are $d_0({}_RV) = 1$ and $d_n({}_RV) = 2$ for every $n \geq 1$.*

Proof. We have $\mathrm{soc}_0({}_RV) = 0$ by definition. Let us first consider the case $n = 1$. We want to show that $\mathrm{soc}({}_RV) = Rv_0 = \langle v_0\rangle_k$. Now v_0 belongs to all non-zero submodules of ${}_RV$ by Lemma 4.2.3. That is, Rv_0 is the intersection of all non-zero submodules of ${}_RV$. It follows that Rv_0 is the unique simple submodule of ${}_RV$. Thus $\mathrm{soc}({}_RV) = Rv_0$.

We now proceed by induction on $n \geq 1$. Suppose that the n-th Loewy submodule $\mathrm{soc}_n({}_RV)$ of ${}_RV$ is ${}_R\langle v_0, v_1, v_2, \dots, v_{2n-2}\rangle = \langle v_0, v_1, v_2, \dots, v_{2n-2}\rangle_k$ for some $n \geq 1$. Consider the factor module $\overline{V} := {}_RV/\mathrm{soc}_n({}_RV)$. We will denote its elements as $\overline{v}$, where $v \in {}_RV$ and $\overline{v} = v + \mathrm{soc}_n({}_RV)$. From the definition of R, one sees that $R\overline{v_{2n-1}}$ is a one-dimensional vector space over k, hence a simple R-submodule of $\overline{V}$. Similarly for $R\overline{v_{2n}}$. Thus $R\overline{v_{2n-1}} \oplus R\overline{v_{2n}}$ is a semisimple submodule of $\overline{V}$ of composition length 2. By Lemma 4.2.3, this semisimple submodule is essential in $\overline{V}$. As the socle is the intersection of all essential submodules, it follows that $\mathrm{soc}(\overline{V}) = R\overline{v_{2n-1}} \oplus R\overline{v_{2n}}$. Thus $\mathrm{soc}_{n+1}({}_RV) = \langle v_0, v_1, \dots, v_{2n}\rangle_k$. ■

By Lemma 4.2.4, the left R-module ${}_RV$ is a Loewy module of Loewy length ω with finite Loewy invariants. We will now show that it is not an artinian R-module.

For every $n \geq 0$, let

$$U_n := \langle v_0, v_2, \dots, v_{2n}, v_{2n+1}, v_{2n+2}, \dots\rangle_k$$

be the subspace of V_k generated by all the v_i with $i \geq 2n$ and all the v_i's with i even, $i < 2n$. It is easily seen that U_n is a submodule of ${}_RV$ and that $U_0 > U_1 > U_2 > \dots$ is a strictly descending chain. Thus ${}_RV$ is not artinian.

It is proved that if ${}_RM$ is a Loewy module over a commutative ring R, α is an ordinal and r is a positive integer such that both $d_\alpha({}_RM)$ and $d_{\alpha+r}({}_RM)$ are finite, then $d_\beta({}_RM)$

is finite for every $\beta > \alpha + r$ and ${}_RM = \mathrm{soc}_{\omega+r}({}_RM)$ (Theorem 4.1.21). We will show that there exist modules ${}_RW$ over non-commutative rings R with completely arbitrary Loewy length and Loewy invariants. More precisely:

Example 4.2.5. Let k be a field, λ be an ordinal and f be a mapping of λ in the class $\mathrm{Card} \setminus \{0\}$ of all non-zero cardinals. Then there exists a local k-algebra R with a Loewy left module ${}_RW$ of Loewy length λ and Loewy invariants $d_\alpha({}_RW) = f(\alpha)$ for every ordinal $\alpha < \lambda$.

The construction is as follows. For every ordinal $\alpha < \lambda$, let W_α be a vector space over k of dimension $f(\alpha)$. Let $W_k := \bigoplus_{\alpha<\lambda} W_\alpha$ be the direct sum of all the vector k-spaces W_α, $\alpha < \lambda$. Let I be the set of all the endomorphisms $f \in \mathrm{End}(W_k)$ such that $f(W_\alpha) \subseteq \bigoplus_{\beta<\alpha} W_\beta$ for every $\alpha < \lambda$. Here, for $\alpha = 0$, we mean that the direct sum $\bigoplus_{\beta<\alpha} W_\beta$ is the direct sum of the empty family of vector subspaces, that is, the condition on f to belong to I is that $f(W_0) = 0$. Set $R := k \oplus I \subseteq \mathrm{End}(W_k)$. Then R turns out to be a k-subalgebra of $\mathrm{End}(W_k)$, the ring R is a local ring with maximal two-sided ideal I, residue division ring $R/I \cong k$, and ${}_RW_k$ has an R-k-bimodule structure.

We will now determine the cyclic non-zero submodules of the module ${}_RW$. Let w be a non-zero element of ${}_RW$, so that w can be written in a unique way as $w = w_1 + \cdots + w_n$ with $w_i \in W_{\alpha_i}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \lambda$ and $w_n \neq 0$. Let f be an element of I such that $f(w_n) = -w_1 - \cdots - w_{n-1}$ and $f(W_\alpha) = 0$ for every $\alpha < \lambda$, $\alpha \neq \alpha_n$. For such an f, we have that $f(w) = -w_1 - \cdots - w_{n-1}$ and $(1 + f)(w) = w_n$. It is now easy to see that the cyclic R-module Rw generated by w is $\langle w_n \rangle_k \oplus \left(\bigoplus_{\alpha<\alpha_n} W_\alpha \right)$. These are therefore all non-zero cyclic submodules of ${}_RW$. Since every module is the sum of its non-zero cyclic submodules, we get that the R-submodules of ${}_RW$ are exactly the improper submodule ${}_RW$ and the modules $U_\beta \oplus \left(\bigoplus_{\alpha<\beta} W_\alpha \right)$, where β ranges over the set of all ordinals $< \lambda$ and U_β ranges over the set of all vector subspaces of W_β. Notice that: (1) the zero submodule of ${}_RW$ is obtained in this way for $\beta = 0$ and $U_\beta = 0$; (2) some submodules can be represented in this form in two different ways, because $W_\beta \oplus \left(\bigoplus_{\alpha<\beta} W_\alpha \right) = 0 \oplus \left(\bigoplus_{\alpha<\beta+1} W_\alpha \right)$ for $\beta + 1 < \alpha$. If we want to represent the submodules of ${}_RW$ only once, we must take the improper submodule ${}_RW$ and all the representations of the form $U_\beta \oplus \left(\bigoplus_{\alpha<\beta} W_\alpha \right)$

with β an ordinal $< \lambda$ and U_β a proper subspace of W_β. Now that we have a complete description of all submodules of ${}_RW$, it is easily seen that $\mathrm{soc}_\alpha({}_RW) = \bigoplus_{\beta<\alpha} W_\beta$ for every $\alpha \leq \lambda$. Moreover, $\mathrm{soc}_{\alpha+1}({}_RW)/\mathrm{soc}_\alpha({}_RW) \cong W_\alpha$ is an R-module of Goldie dimension $\dim_k(W_\alpha) = f(\alpha)$. This shows that ${}_RW$ is our required example of a Loewy module of Loewy length λ and Loewy invariants $d_\alpha({}_RW) = f(\alpha)$ for every $\alpha < \lambda$.

The module ${}_RW$ is artinian when its Loewy invariants are finite. More precisely, we will now prove that *${}_RW$ is an artinian R-module if and only if $f(\alpha)$ is finite for every $\alpha < \lambda$.* Thus, for artinian modules over non-commutative rings, Loewy length and finite Loewy invariants can be completely arbitrary. This contrasts with the behavior of artinian modules over commutative rings R (Theorems 4.1.19, 4.1.20 and 4.1.21).

We already know that all artinian modules are Loewy modules with finite Loewy invariants. Conversely, assume that $f(\alpha)$ is a non-zero finite cardinal for every $\alpha < \lambda$, and take a descending chain $A_0 \geq A_1 \geq A_2 \geq \dots$ of submodules of ${}_RW$. If $A_n = {}_RW$ for every n, the chain is stationary. Similarly, the chain is stationary if $A_n = 0$ for some n. Thus we can suppose that there exists an index n such that $0 \neq A_i < {}_RW$ for every $i \geq n$. Then, for every $i \geq n$, $A_i = U_{\beta_i} \oplus \left(\bigoplus_{\alpha<\beta_i} W_\alpha\right)$ for a suitable ordinal $\beta_i < \lambda$ and a suitable proper vector subspace U_{β_i} of W_{β_i}. Thus we have a descending chain $\lambda > \beta_n \geq \beta_{n+1} \geq \beta_{n+2} \geq \dots$ of ordinals, necessarily stationary. Hence, there exists an index $m \geq n$ such that, if we set $W' := \left(\bigoplus_{\alpha<\beta_m} W_\alpha\right)$, then $A_i = U_{\beta_i} \oplus W'$ for every $i \geq m$, where U_{β_i} is a proper vector subspace of W_{β_m}. As W_{β_m} has finite dimension $f(\beta_m)$, the descending chain of subspaces $U_{\beta_m} \geq U_{\beta_{m+1}} \geq U_{\beta_{m+2}} \geq \dots$ is stationary, so that the chain A_i is stationary, and ${}_RW$ is artinian.

It is shown that if R is a commutative ring, ${}_RM$ is an artinian left module with simple socle, and $H := \mathrm{End}({}_RM)$, then M_H is the injective envelope of a simple right H-module (Theorem 4.1.18). The following example proves that if R is not commutative, then M_H is not necessarily the injective envelope of a simple H-module. In our example, M_H will turn out to be a semisimple H-module.

Example 4.2.6. Let R be the $\mathbb{R}$-algebra $R := \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$ and ${}_RM := \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}$. Firstly, we show that ${}_RM$ is an artinian module with simple socle. The ring R is an $\mathbb{R}$-algebra of dimension 5 and ${}_RM$ is a vector space of dimension 4 over $\mathbb{R}$, so that ${}_RM$ is an artinian module. It is easy to check that $R\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbb{C} \end{pmatrix}$ for every $0 \neq y \in \mathbb{C}$. Thus $\begin{pmatrix} 0 \\ \mathbb{C} \end{pmatrix}$ is a simple submodule of ${}_RM$. Moreover, for every $a, b \in \mathbb{C}$ with $a \neq 0$, one has that $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix} \in \begin{pmatrix} 0 \\ \mathbb{C} \end{pmatrix}$, which shows that $\begin{pmatrix} 0 \\ \mathbb{C} \end{pmatrix}$ is essential in ${}_RM$. Since the socle is the intersection of all essential submodules, $\begin{pmatrix} 0 \\ \mathbb{C} \end{pmatrix}$ turns out to be the socle of ${}_RM$.

Now we will see that the endomorphism ring $H := \mathrm{End}({}_RM)$ of ${}_RM$ is isomorphic to $\mathbb{C}$. Let f be an element of H. Since $\begin{pmatrix} 0 \\ \mathbb{C} \end{pmatrix}$ is the socle of ${}_RM$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} f = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$ for some $\alpha \in \mathbb{C}$. Hence, for every $b \in \mathbb{C}$, we have that

$$\begin{pmatrix} 0 \\ b \end{pmatrix} f = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} f\right) = \begin{pmatrix} 0 \\ b\alpha \end{pmatrix}. \tag{4.2.1}$$

For any $a \in \mathbb{C}$, $\begin{pmatrix} a \\ 0 \end{pmatrix} f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\left(\begin{pmatrix} a \\ 0 \end{pmatrix} f\right) = \begin{pmatrix} \beta_a \\ 0 \end{pmatrix}$ for some $\beta_a \in \mathbb{C}$. From (4.2.1), we get that $\begin{pmatrix} 0 \\ a\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix} f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\left(\begin{pmatrix} a \\ 0 \end{pmatrix} f\right) = \begin{pmatrix} 0 \\ \beta_a \end{pmatrix}$, which implies $\beta_a = \alpha a$. Therefore,

$$\begin{pmatrix} a \\ b \end{pmatrix} f = \begin{pmatrix} a \\ 0 \end{pmatrix} f + \begin{pmatrix} 0 \\ b \end{pmatrix} f = \begin{pmatrix} a\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b\alpha \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \alpha.$$

In other words, for any element $f \in H$, there exists a unique $\alpha \in \mathbb{C}$ such that $(x)f = x\alpha$ for every $x \in {}_RM$. Thus $H \cong \mathbb{C}$ and M_H is a vector space of dimension 2 over $\mathbb{C}$.

4.3 Loewy modules whose Loewy invariants are finite

In this Section, we prove that some properties that are classically known to hold for artinian modules, also hold for Loewy modules with finite Loewy invariants.

Let ${}_RM$ be a left module over a ring R. It is clear that the socle $\mathrm{soc}({}_RM)$ of ${}_RM$ and, more generally, all its Loewy submodules $\mathrm{soc}_\alpha({}_RM)$, are fully invariant submodules of ${}_RM$, that is, $(\mathrm{soc}_\alpha({}_RM))f \leq \mathrm{soc}_\alpha({}_RM)$ for every ordinal α and every endomorphism f of ${}_RM$. Equivalently, $\mathrm{soc}_\alpha({}_RM)$ is a subbimodule of the bimodule ${}_RM_{\mathrm{End}({}_RM)}$ for every ordinal α. Thus we can consider the restrictions $f|_{\mathrm{soc}({}_RM)}$ and $f|_{\mathrm{soc}_\alpha({}_RM)}$, which are endomorphisms of $\mathrm{soc}({}_RM)$ and $\mathrm{soc}_\alpha({}_RM)$, respectively.

Proposition 4.3.1. *Let ${}_RM$ be a Loewy module whose Loewy invariants are all finite. The following conditions are equivalent for an endomorphism $f \in \mathrm{End}({}_RM)$:*

1. *f is an automorphism of ${}_RM$.*
2. *f is an injective endomorphism of ${}_RM$.*
3. *The restriction $f|_{\mathrm{soc}({}_RM)} \colon \mathrm{soc}({}_RM) \to \mathrm{soc}({}_RM)$ is injective.*
4. *The restriction $f|_{\mathrm{soc}({}_RM)} \colon \mathrm{soc}({}_RM) \to \mathrm{soc}({}_RM)$ is an automorphism of* $\mathrm{soc}({}_RM)$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial, so that it suffices to show (4) $\Rightarrow$ (1). Let ${}_RM$ be a Loewy module with Loewy length λ and all Loewy invariants finite. Let f be an endomorphism of ${}_RM$ and assume that $f|_{\mathrm{soc}({}_RM)}$ is an automorphism of $\mathrm{soc}({}_RM)$. We must prove that f is an automorphism of ${}_RM$. Since the socle is essential in ${}_RM$, $f|_{\mathrm{soc}({}_RM)}$ injective implies f injective. We will show that f is surjective by induction on the Loewy length λ of ${}_RM$. The cases $\lambda = 0$ and $\lambda = 1$ are trivial. Suppose that the statement holds for every ordinal $\alpha < \lambda$. That is, every endomorphism of a Loewy module of Loewy length $\alpha < \lambda$ and finite Loewy invariants is surjective if the endomorphism restricted to its socle is an automorphism. If λ is a limit ordinal, then ${}_RM = \mathrm{soc}_\lambda({}_RM) = \cup_{\alpha<\lambda}\mathrm{soc}_\alpha({}_RM)$. For any $\alpha < \lambda$, the restriction of f to $\mathrm{soc}_\alpha({}_RM)$ is an automorphism of $L_\alpha({}_RM)$ by the inductive hypothesis. Hence f is surjective. If λ is a non-limit ordinal, then $\lambda = \alpha + 1$ for some ordinal α. Then f induces a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{soc}_\alpha({}_RM) \arrow[r] \arrow[d, "f|\mathrm{soc}_{\alpha({}_RM)}"] & {}_RM \arrow[r] \arrow[d, "f"] & {}_RM/\mathrm{soc}_\alpha({}_RM) \arrow[r] \arrow[d, "\overline{f}"] & 0 \\
0 \arrow[r] & \mathrm{soc}_\alpha({}_RM) \arrow[r] & {}_RM \arrow[r] & {}_RM/\mathrm{soc}_\alpha({}_RM) \arrow[r] & 0.
\end{tikzcd}$$

Here f is injective and $f|_{\mathrm{soc}_\alpha({}_RM)}$ is an automorphism by the inductive hypothesis. From the Snake Lemma, we get an exact sequence

$$0 \longrightarrow \ker(\overline{f}) \longrightarrow 0 \longrightarrow \operatorname{coker}(f) \longrightarrow \operatorname{coker}(\overline{f}) \longrightarrow 0.$$

In particular, $\overline{f}$ is a monomorphism. But ${}_RM/\mathrm{soc}_\alpha({}_RM)$ is a semisimple module of finite length, so that injective endomorphisms are automorphisms. Thus $\operatorname{coker}(\overline{f}) = 0$, so that $\operatorname{coker}(f) = 0$ and f is surjective. ■

Corollary 4.3.2. *Let ${}_RM$ be a Loewy module with finite Loewy invariants. Then the restriction homomorphism $\rho\colon \mathrm{End}({}_RM) \to \mathrm{End}(\mathrm{soc}({}_RM))$, $f \mapsto f|_{\mathrm{soc}({}_RM)}$, is local. In particular,* $\mathrm{End}({}_RM)$ *is a semilocal ring of dual Goldie dimension $\leq d_0({}_RM)$.*

Proof. The Corollary follows immediately from Proposition 4.3.1 and a result of Herbera and Shamsudding, see [14, Theorem 4.3(a)]. For another elementary proof, notice that the ring homomorphism ρ is local by Proposition 4.3.1. By hypothesis, $\mathrm{soc}({}_RM)$ is the direct sum of finitely many simple submodules of ${}_RM$, hence $\mathrm{End}(\mathrm{soc}({}_RM))$ is a semisimple artinian ring. As ρ is a local homomorphism of $\mathrm{End}({}_RM)$ into a semisimple artinian ring, the ring $\mathrm{End}({}_RM)$ must be semilocal by Theorem 4.1.7. ■

Thus Corollary 4.3.2 generalizes the famous result by Camps and Dicks that artinian modules have semilocal endomorphism rings [9, Corollary 6]. The next Corollary generalizes part of Theorem 4.1.18 to the non-commutative case.

Corollary 4.3.3. *Let ${}_RM$ be a Loewy module with finite Loewy invariants and simple socle. Then* $\mathrm{End}({}_RM)$ *is a local ring.*

Proof. This is the case $d_0({}_RM) = 1$ in Corollary 4.3.2, because semilocal rings of dual Goldie dimension 1 are local rings. ■

Thus if ${}_RM$ is a direct sum of modules satisfying the hypotheses of Corollary 4.3.3, that is, Loewy modules with finite Loewy invariants and simple socle, then there is uniqueness of direct-sum decompositions into indecomposables.

Proposition 4.3.1 and Corollary 4.3.2 have several implications, and we will now present some of them. Recall that two modules ${}_RM, {}_RN$ are said to be in the same *monogeny class* [13] if there exist a monomorphism ${}_RM \to {}_RN$ and a monomorphism ${}_RN \to {}_RM$. The notation for "${}_RM$ and ${}_RN$ are in the same monogeny class" is $[{}_RM]_m = [{}_RN]_m$. A class of modules $\mathcal{C}$ is said to be *mono-correct* [44] if, for every ${}_RM, {}_RN \in \mathcal{C}$, $[{}_RM]_m = [{}_RN]_m$ implies ${}_RM \cong {}_RN$. For instance, the class of all injective left R-modules is mono-correct [8].

Corollary 4.3.4. *The class of all Loewy left R-modules whose Loewy invariants are finite is mono-correct. More generally, any Loewy left R-module ${}_RM$ whose Loewy invariants are finite is* mono-correct*, in the sense that, for any left R-module ${}_RN$, $[{}_RM]_m = [{}_RN]_m$ implies ${}_RM \cong {}_RN$.*

Proof. Let ${}_RM, {}_RN$ be left R-modules and suppose that ${}_RM$ is a Loewy left R-module with finite Loewy invariants. If there exist a monomorphism $\varphi\colon {}_RM \to {}_RN$ and a monomorphism $\psi\colon {}_RN \to {}_RM$, then the injective endomorphism $\varphi\psi$ of ${}_RM$ is an automorphism by Proposition 4.3.1. Thus ψ is surjective, hence an isomorphism. ■

Corollary 4.3.4 generalizes [44, 3.3.1] from artinian modules to Loewy modules with finite Loewy invariants. From [14, Proposition 4.9] and our Corollary 4.3.2, we get that:

Corollary 4.3.5. *Every Loewy left R-module ${}_RM$ whose Loewy invariants are finite has only finitely many direct summands up to isomorphism. More precisely, it has at most $2^{d_0({}_RM)}$ isomorphism classes of direct summands.*

Proof. By [14, Proposition 4.9], it suffices to show that the dual Goldie dimension $\operatorname{codim}(\operatorname{End}({}_RM)) \leq d_0({}_RM)$. Now the homomorphism $\rho\colon \operatorname{End}({}_RM) \to \operatorname{End}(\operatorname{soc}({}_RM))$ is a local homomorphism, so that $\operatorname{codim}(\operatorname{End}({}_RM)) \leq \operatorname{codim}(\operatorname{End}(\operatorname{soc}({}_RM)))$ [9, Corollary 2]. Moreover, $\operatorname{soc}({}_RM)$ is a semisimple module of composition length $d_0({}_RM)$. Thus $\operatorname{codim}(\operatorname{End}(\operatorname{soc}({}_RM)))$ is equal to the Goldie dimension $d_0({}_RM)$ of $\operatorname{soc}({}_RM)$. ■

4.4 Max modules whose radical invariants are finite

We will now dualize the results in the previous Sections. If we try to dualize the previous situation, in which all Loewy factors $\mathrm{soc}_{\alpha+1}({}_RM)/\mathrm{soc}_{\alpha}({}_RM)$ are semisimple modules, the problem is that the radical factors $\mathrm{rad}_{\alpha}({}_RM)/\mathrm{rad}_{\alpha+1}({}_RM)$ are only modules with zero radical, and not semisimple modules in general. The following two results consider the situation in which they are semisimple.

Lemma 4.4.1. *Let R be a ring and ${}_RM$ be a left R-module. Then $J(R)^n{}_RM \leq \mathrm{rad}_n({}_RM)$ for any positive integer n, and the equality holds when R is semilocal.*

PROOF. The first part of the statement follows from [14, Lemma 1.3] by induction on n. For the second, assume that R is semilocal. It suffices to show that $\mathrm{rad}({}_RM) = J(R){}_RM$ for any module ${}_RM$. Now ${}_RM/J(R){}_RM$ is a left $R/J(R)$-module, so that ${}_RM/J(R){}_RM$ is semisimple. Since the radical of a semisimple module is 0 and $J(R){}_RM \leq \mathrm{rad}({}_RM)$, one has that

$$\overline{0} = \mathrm{rad}({}_RM/J(R){}_RM) = \mathrm{rad}({}_RM)/J(R){}_RM.$$

Hence $\mathrm{rad}({}_RM) = J(R){}_RM$. ■

Proposition 4.4.2. *The following conditions are equivalent for a ring R:*

1. *The radical factor* $\mathrm{rad}_{\alpha}({}_RM)/\mathrm{rad}_{\alpha+1}({}_RM)$ *is semisimple for every left R-module ${}_RM$ and every ordinal α.*
2. *The radical factor* ${}_RM/\mathrm{rad}({}_RM)$ *is semisimple for every left R-module ${}_RM$.*
3. *The ring R is semilocal.*

PROOF. For (1) ⇒ (2) take $\alpha = 0$. For (2) ⇒ (3) take ${}_RM = {}_RR$.

(3) ⇒ (1) By Lemma 4.4.1, $\mathrm{rad}_{\alpha+1}({}_RM) = J(R)\mathrm{rad}_{\alpha}({}_RM)$, so that

$$\mathrm{rad}_{\alpha}({}_RM)/\mathrm{rad}_{\alpha+1}({}_RM)$$

is an $R/J(R)$-module. By (3), $R/J(R)$ is semisimple artinian. Therefore

$$\mathrm{rad}_{\alpha}({}_RM)/\mathrm{rad}_{\alpha+1}({}_RM)$$

is a semisimple R-module. ■

Notice that the radical of a module is a fully invariant submodule. More generally, every module homomorphism $f\colon {}_RM \to {}_RN$ restricts to a homomorphism

$$f|_{\mathrm{rad}({}_RM)}\colon \mathrm{rad}({}_RM) \to \mathrm{rad}({}_RN)$$

and induces a homomorphism $\overline{f}\colon {}_RM/\mathrm{rad}({}_RM) \to {}_RN/\mathrm{rad}({}_RN)$. By transfinite induction, we have that f restricts to a homomorphism $f|_{\mathrm{rad}_\alpha({}_RM)}\colon \mathrm{rad}_\alpha({}_RM) \to \mathrm{rad}_\alpha({}_RN)$ for every ordinal α and induces a homomorphism

$$\overline{f_\alpha}\colon {}_RM/\mathrm{rad}_\alpha({}_RM) \to {}_RN/\mathrm{rad}_\alpha({}_RN).$$

Proposition 4.4.3. *Let R be a semilocal ring, ${}_RM$ be a max left R-module with finite radical invariants and $f \in \mathrm{End}({}_RM)$ be an endomorphism of ${}_RM$. The following conditions are equivalent:*

1. *f is an automorphism of ${}_RM$.*
2. *The endomorphism f is surjective.*
3. *The homomorphism $\overline{f}\colon {}_RM/\mathrm{rad}({}_RM) \to {}_RM/\mathrm{rad}({}_RM)$ induced by f is a surjective endomorphism of ${}_RM/\mathrm{rad}({}_RM)$.*
4. *The homomorphism $\overline{f}\colon {}_RM/\mathrm{rad}({}_RM) \to {}_RM/\mathrm{rad}({}_RM)$ induced by f is an automorphism of ${}_RM/\mathrm{rad}({}_RM)$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious. (3) ⇒ (4) follows from the fact that R is semilocal and ${}_RM$ has finite radical invariants, so that ${}_RM/\mathrm{rad}({}_RM)$ is a semisimple module of finite Goldie dimension (Proposition 4.4.2). Thus a surjective endomorphism of ${}_RM/\mathrm{rad}({}_RM)$ is an automorphism.

(4) ⇒ (1). Let ${}_RM$ be a max left module with radical length λ and all its radical invariants finite. Let f be a homomorphism of ${}_RM$ with $\overline{f}\colon {}_RM/\mathrm{rad}({}_RM) \to {}_RM/\mathrm{rad}({}_RM)$ an automorphism of ${}_RM/\mathrm{rad}({}_RM)$. The radical of a max module is a superfluous submodule. Thus $\overline{f}$ surjective implies f surjective. It remains to show that f is injective. We will prove by transfinite induction that $\overline{f_\alpha}\colon {}_RM/\mathrm{rad}_\alpha({}_RM) \to {}_RM/\mathrm{rad}_\alpha({}_RM)$ is injective for every ordinal α, and the case $\alpha = \lambda$ will show that (1) holds.

The case $\alpha = 0$ is trivial, and the case $\alpha = 1$ is condition (4). Assume that our property holds for any ordinal $\beta < \alpha$. That is, $\overline{f_\beta}\colon {}_RM/\mathrm{rad}_\beta({}_RM) \to {}_RN/\mathrm{rad}_\beta({}_RN)$ is injective for every ordinal $\beta < \alpha$. If the ordinal α is limit, $x \in {}_RM$ and $(x)f \in \mathrm{rad}_\alpha({}_RM)$, then $(x)f \in \mathrm{rad}_\beta({}_RM)$ for every $\beta < \alpha$. By the inductive hypothesis, $x \in \mathrm{rad}_\beta({}_RM)$ for every $\beta < \alpha$, that is, $x \in \mathrm{rad}_\alpha({}_RM)$. This proves that $\overline{f_\alpha}$ is injective. If α is a non-limit ordinal, then $\alpha = \beta + 1$ for some ordinal β. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{rad}_\beta(M)/\mathrm{rad}_{\beta+1}(M) \arrow[r] \arrow[d, "f'"] & M/\mathrm{rad}_{\beta+1}(M) \arrow[r] \arrow[d, "\overline{f_{\beta+1}}"] & M/\mathrm{rad}_\beta(M) \arrow[r] \arrow[d, "\overline{f_\beta}"] & 0 \\
0 \arrow[r] & \mathrm{rad}_\beta(M)/\mathrm{rad}_{\beta+1}(M) \arrow[r] & M/\mathrm{rad}_{\beta+1}(M) \arrow[r] & M/\mathrm{rad}_\beta(M) \arrow[r] & 0.
\end{tikzcd}$$

In this diagram, $\overline{f_{\beta+1}}$ is surjective and $\overline{f_\beta}$ is an automorphism by the inductive hypothesis. The Snake Lemma yields an exact sequence

$$0 \longrightarrow \ker(f') \longrightarrow \ker(\overline{f_{\beta+1}}) \longrightarrow 0 \longrightarrow \mathrm{coker}(f') \longrightarrow 0.$$

Thus f' is an epimorphism. Now $\mathrm{rad}_\beta({}_RM)/\mathrm{rad}_{\beta+1}({}_RM)$ is a semisimple module of finite composition length, so that its surjective endomorphisms are automorphisms. Hence $\ker(f') = 0$, so that $\ker(\overline{f_{\beta+1}}) = 0$, and $\overline{f_\alpha} = \overline{f_{\beta+1}}$ is injective. ∎

In Proposition 4.4.3, if the commutativity is added to the ring R, that is, R is a commutative semilocal ring, then we find the same conclusion for any max module ${}_RM$ with only $r_0({}_RM)$ finite. That is:

Proposition 4.4.4. *Let R be a commutative semilocal ring, ${}_RM$ be a max R-module whose 0-th radical invariant is finite and $f \in \mathrm{End}({}_RM)$ be an endomorphism of ${}_RM$. The following conditions are equivalent:*

1. *f is an automorphism of ${}_RM$.*
2. *The endomorphism f is surjective.*
3. *The homomorphism $\overline{f}\colon {}_RM/\mathrm{rad}({}_RM) \to {}_RM/\mathrm{rad}({}_RM)$ induced by f is a surjective endomorphism of ${}_RM/\mathrm{rad}({}_RM)$.*
4. *The homomorphism $\overline{f}\colon {}_RM/\mathrm{rad}({}_RM) \to {}_RM/\mathrm{rad}({}_RM)$ induced by f is an automorphism of ${}_RM/\mathrm{rad}({}_RM)$.*

PROOF. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are like in the previous Proposition. As far as (4) $\Rightarrow$ (1) is concerned, let f be an endomorphism of ${}_RM$ with $\overline{f}$ an automorphism of ${}_RM/\mathrm{rad}({}_RM)$. The submodule $\mathrm{rad}({}_RM)$ is superfluous in ${}_RM$, so that f is a surjective endomorphism of ${}_RM$. Moreover, ${}_RM/\mathrm{rad}({}_RM)$ is finitely generated by hypothesis, so that $\mathrm{rad}({}_RM)$ superfluous in ${}_RM$ implies that ${}_RM$ is finitely generated. Now surjective endomorphisms of finitely generated modules over a commutative ring are isomorphisms, which implies that f is an automorphism of ${}_RM$. ■

Corollary 4.4.5. *Let ${}_RM$ be a max module with finite radical invariants over a semilocal ring R. Then the ring homomorphism*

$$\iota\colon \mathrm{End}({}_RM) \to \mathrm{End}({}_RM/\mathrm{rad}({}_RM)), \quad f \mapsto \overline{f},$$

is a local homomorphism. In particular, $\mathrm{End}({}_RM)$ *is a semilocal ring of dual Goldie dimension $\leq r_0({}_RM)$.*

PROOF. The ring homomorphism $\iota\colon \mathrm{End}({}_RM) \to \mathrm{End}({}_RM/\mathrm{rad}({}_RM))$ is local by Proposition 4.4.3. Now apply [9, Corollary 2]. ■

Compare Corollary 4.4.5 with [20, Theorem 3.3], where it is proved that finitely presented modules over a semilocal ring R have a semilocal endomorphism ring.

The result analogous to Corollary 4.3.3 is:

Corollary 4.4.6. *Let R be a semilocal ring and ${}_RM$ a max module with finite radical invariants and ${}_RM/\mathrm{rad}({}_RM)$ simple. Then* $\mathrm{End}({}_RM)$ *is a local ring.*

4.5 Modules over a perfect ring

By Theorems 4.1.11 and 4.1.15, every (right or left) module over a perfect ring is both Loewy and max.

Proposition 4.5.1. *Every left module over a perfect ring with finite* 0*-th radical invariant and finite* 0*-th Loewy invariant has a semilocal endomorphism ring. In particular,* *every finitely generated left module with finite Goldie dimension over a perfect ring has a semilocal endomorphism ring.*

PROOF. Let R be a perfect ring and ${}_RM$ be a left R-module whose socle $\mathrm{soc}({}_RM)$ and factor ${}_RM/\mathrm{rad}({}_RM)$ are the direct sums of finitely many simple submodules. Since ${}_RM$ is Loewy and max, we know that $\mathrm{soc}({}_RM)$ is essential and $\mathrm{rad}({}_RM)$ is superfluous in ${}_RM$. Moreover, the Goldie dimension $\dim({}_RM) = \dim(\mathrm{soc}({}_RM))$ and the dual Goldie dimension $\mathrm{codim}({}_RM) = \mathrm{codim}({}_RM/\mathrm{rad}({}_RM))$ are finite, so that the endomorphism ring of ${}_RM$ is semilocal [27, Theorem 3.3]. The second statement follows immediately from the first. ■

Theorem 4.5.2. *For a left module ${}_RM$ over a semiprimary ring R, the following conditions are equivalent:*

1. *${}_RM$ is noetherian.*
2. *${}_RM$ is artinian.*
3. *All Loewy invariants of ${}_RM$ are finite.*
4. *All radical invariants of ${}_RM$ are finite.*

PROOF. The equivalence (1) $\Leftrightarrow$ (2) is well known (Theorem 4.1.6). We have that (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (3), so it suffices to prove (4) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2). Let $J = J(R)$ be the Jacobson radical of R and k be the nilpotency index of J, that is, $J^k = 0$ and $J^\ell \neq 0$ for $\ell < k$.

To see that (4) $\Rightarrow$ (1), it suffices to show that there exists a positive integer m such that $\mathrm{rad}_m({}_RM) = 0$. This is trivial because $\mathrm{rad}_k({}_RM) = J^k{}_RM = 0$ by Lemma 4.4.1.

The proof of (3) $\Rightarrow$ (2) is by induction on k. Assume that ${}_RM$ has finite Loewy invariants. If $k = 1$, then R is semisimple, so that ${}_RM = \mathrm{soc}({}_RM)$, the direct sum of finitely many simple submodules, is artinian. Suppose that $k > 1$ and that every module with finite Loewy invariants over a semiprimary ring whose Jacobson radical has nilpotency index strictly less than k is artinian. We must show that ${}_RM$ is artinian. Since $\mathrm{soc}({}_RM)$ is the direct sum of finitely many simple modules, it is enough to show that ${}_RM/\mathrm{soc}({}_RM)$ is artinian. One has $J^k{}_RM = 0$, so $J^{k-1}M$ is annihilated by J, hence is an R/J-module, hence a semisimple R-module. Thus $J^{k-1}M \leq \mathrm{soc}({}_RM)$, and the factor module ${}_RM/\mathrm{soc}({}_RM)$ is a left R/J^{k-1}-module. Since $J(R/J^{k-1}) = J/J^{k-1}$ has nilpotence index $k-1$ and ${}_RM/\mathrm{soc}({}_RM)$ has finite Loewy invariants, ${}_RM/\mathrm{soc}({}_RM)$ is artinian as an R/J^{k-1}-module. Thus ${}_RM/\mathrm{soc}({}_RM)$ is also artinian as an R-module. ■

Example 4.5.3. Let R be a semilocal ring and ${}_RM$ be a max left R-module. From Proposition 4.4.4, if R is commutative and ${}_RM$ has finite dual Goldie dimension, then the endomorphism ring $\mathrm{End}({}_RM)$ of ${}_RM$ is semilocal. In [20, Example 3.5] there is an example of a finitely generated module ${}_RM$ over a semiprimary ring R whose endomorphism ring is not semilocal. As R is semiprimary, every left R-module is max. Finitely generated modules over a semilocal ring have finite dual Goldie dimension. In particular, the 0-th radical invariant $r_0({}_RM)$ is finite. Since the endomorphism ring $\mathrm{End}({}_RM)$ is not semilocal, the module ${}_RM$ is not noetherian.

This example shows that in case ${}_RM$ has finite dual Goldie dimension but R is not commutative, the ring $\mathrm{End}({}_RM)$ is not necessarily semilocal. Moreover, it also proves that in Theorem 4.5.2, we cannot weaken (4) (i.e., that "all radical invariants of ${}_RM$ are finite") to "${}_RM$ has finite 0-th radical invariant". That is, the module ${}_RM$ in the example has finite 0-th radical invariant, but ${}_RM$ is not noetherian.

A module is called a *Shock module* if all its factor modules are max [22]. For example, every noetherian module over an arbitrary ring and all modules over a perfect ring are Shock. The dual of Proposition 4.2.1 is as follows.

Proposition 4.5.4. *The following conditions are equivalent for a left module ${}_RM$ over an arbitrary ring R:*

1. *${}_RM$ is noetherian.*
2. *${}_RM$ is Shock and has Krull dimension.*
3. *${}_RM$ is Shock and q.f.d.*
4. *${}_RM$ is Shock and its factor modules have finitely generated socles.*

PROOF. (1) ⇒ (2) ⇒ (3) ⇒ (4) are elementary and (4) ⇒ (1) is proved in [38, Theorem 3.8]. ■

Since semiprimary rings are perfect rings, the following Theorem generalizes the Hopkins-Levitzki Theorem 4.5.2. Its proof follows from Propositions 4.2.1 and 4.5.4.

Theorem 4.5.5. *Let R be a perfect ring. The following conditions are equivalent for a left R-module ${}_RM$:*

1. *$l({}_RM)$ is finite.*
2. *${}_RM$ is artinian.*
3. *${}_RM$ is noetherian.*
4. *${}_RM$ has Krull dimension.*
5. *${}_RM$ is q.f.d.*
6. *All factor modules of ${}_RM$ have finitely generated socles.*
7. *All factor modules of ${}_RM$ are essential extensions of finitely generated modules.*

From Theorem 4.5.5, we immediately get the following corollaries.

Corollary 4.5.6. *Every uniserial module over a perfect ring has a local endomorphism ring.*

PROOF. Every uniserial module is q.f.d., hence of finite composition length, and indecomposable. Indecomposable modules of finite composition length have a local endomorphism ring. ■

Corollary 4.5.7. *Every module with Krull dimension over a perfect ring has Krull dimension* 0.

Corollary 4.5.8. *The following conditions are equivalent for a ring R:*

1. *R is left artinian.*
2. *R is perfect and left noetherian.*
3. *R is perfect and the left module ${}_RR$ has Krull dimension.*
4. *R is perfect and the left module ${}_RR$ is q.f.d.*
5. *R is perfect and, for any left ideal I of R, the left module ${}_RR/I$ has a finitely generated socle.*
6. *R is perfect and, for any left ideal I of R, the left module ${}_RR/I$ is an essential extension of a finitely generated module.*

Chapter 5

Maximal subfields of a division algebra

Let F be a field. A ring is called a *central F-algebra* if its center is equal to F. A central F-algebra D is called a *division F-algebra* if the dimension of D, as a vector space over F, is finite and D has neither non-zero proper left ideals nor non-zero proper right ideals. In other words, D is a division ring with center F and $\dim_F D < \infty$. In some books and papers, such a D is also called *centrally finite* [29, Definition 14.1]. A *central simple F-algebra* is a ring which is isomorphic to $M_n(D)$ for some positive integer n and division F-algebra D. For any central simple F-algebra A, the integer $\sqrt{\dim_F A}$ is said to be the *degree* of A.

Let D be a division F-algebra. Let $(D,+)$ and D^* be the *additive group* and the *multiplicative group* of non-zero elements of D, respectively. Let $[D,D]$ be the additive subgroup of the additive group $(D,+)$ generated by all commutators $xy-yx$ where x,y range over D, and D' be the multiplicative subgroup of the multiplicative group D^* generated by all commutators $xyx^{-1}y^{-1}$ where x,y range over D^*. It is well known from the Kothe Theorem that there exists a maximal subfield K of D such that the extension of fields K/F is separable [29, Theorem 15.12]. In [1, Theorem 7], it was proved that for any separable extension of fields K/F in D, there exists an element $c \in [D,D]$ such that

$K = F(c)$ unless $\mathrm{Char}(F) = \dim_F K = 2$ and 4 does not divide the degree of D. Hence, if the degree D is different from 2 then there exists $c \in [D, D]$ such that $F(c)$ is a maximal subfield of D. In case the degree of D is equal to 2 then, by [29, Corollary 13.5], there exists $c \in [D, D]$ such that $F(c) \neq F$, which implies $F(c)$ is a maximal subfield of D. Therefore, in both cases, there exists $c \in [D, D]$ such that $F(c)$ is a maximal subfield of D. We answer the following natural question that appear in [32]. Is it true that there exists an additive commutator $ab - ba \in [D, D]$ such that $F(ab - ba)$ is a maximal subfield of D (see [32, Problem 28])? Similarly, for multiplicative structure, there exists an element $d \in D^*$ such that $K = F(d)$ is a maximal subfield of D [32, Theorem 2.26]. Do there exist $x, y \in D^*$ such that $F(xyx^{-1}y^{-1})$ is a maximal subfield of D [32, Problem 29]?

The goal of this Chapter is to answer both questions affirmatively. The main tools used in this paper are rational identities over a central simple algebra. In the first Section, we recall some notions of rational identities of a central simple algebra, and also a specific rational expression which works even over the maximal right ring of quotients of a prime ring.

5.1 Rational identities of central simple algebras

Rational expressions.

The notion of "rational identity" which is a generalization of polynomial identities were attended after Amitsur used it for solving some problems of algebra and geometry (see [3]). A *rational expression* over a central F-algebra R is an expression formed from a set $X = \{ x_i \mid i \in I \}$ of non-commutative indeteminates with coefficients in F by addition, subtraction, multiplication and division. A rational expression f over R is said to be a *rational identity* of R if it vanishes on all permissible substitutions from R. In this case, we say that R *satisfies* f. For instance.

Examples 5.1.1. 1. (Hua's identity) $(x^{-1} + (y^{-1} - x^{-1})^{-1})^{-1} - x + xyx$ is a rational identity of every algebra.

2. It is easy to check $(x+y)^{-1} - y^{-1}(x^{-1}+y^{-1})^{-1}x^{-1}$ is a rational identity of every algebra.

3. It is easy to check $[[x,[y,z]x[y,x]^{-1}]^3, z]$ vanishes on permissible substitutions of $M_3(F)$ for any field F.

A rational identity f of a central F-algebra R is called *non-trivial* if there exist a field K which contains all coefficients of f and a central K-algebra S such that f is not a rational identity over S. Otherwise, f is called *trivial.* In examples 5.1.1, (1) and (2) are trivial, and one can check that (3) is non-trivial.

We denote $\mathcal{I}(R)$ the set of all non-trivial rational identities of a central F-algebra R.

Theorem 5.1.2. [36, Theorem 8.2.11] *A division ring D with infinite center F is a division F-algebra if and only if $\mathcal{I}(D) \neq \emptyset$.*

Theorem 5.1.3. [3, Theorem 11] *Let F be an infinite field and A be a central simple F-algebra of degree n. Assume that L is an extension field of F. Then $\mathcal{I}(A) = \mathcal{I}(M_n(F)) = \mathcal{I}(M_n(L))$.*

From Theorem 5.1.3, over a division algebra, there exists a non-trivial rational identity. Next we consider a special rational identity which may be thought of as a generalisation of characteristic polynomials of matrices of degree n over a field.

A rational expression

We consider the following example of a rational expression which is important in this Chapter. Given an integer $n \geq 1$ and $n+1$ non-commutative indeterminates $x, y_1, \cdots, y_n$, put

$$g_n(x, y_1, y_2, \cdots, y_n) := \sum_{\delta \in S_{n+1}} \operatorname{sign}(\delta) x^{\delta(0)} y_1 x^{\delta(1)} y_2 x^{\delta(2)} \ldots y_n x^{\delta(n)},$$

where S_{n+1} is the symmetric group of $\{ 0, 1, \cdots, n \}$ and $\operatorname{sign}(\delta)$ is the sign of permutation δ. This is a rational expression defined in [5] to connect an algebraic element of degree n and a polynomial of $n+1$ indeterminates.

Remark 5.1.4. *If $f, f_1, f_2, \cdots, f_n$ are rational expressions, then so is $g_n(f, f_1, f_2, \cdots, f_n)$.*

Let R be a ring with center $Z = Z(R)$. Recall that an element a of R is called *algebraic of degree* n over Z if there exists a polynomial $f(x)$ of degree n over Z such that $f(a) = 0$ and there is no polynomial of degree less than n vanishing on a. In general, $f(x)$ is not irreducible even if Z is a field. For example, the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in M_2(F)$, where F is a field, satisfies polynomial $f(x) = (x-1)(x-2)$. Since $A \notin F$, the smallest degree of all the polynomials vanishing on A is 2.

A ring R is called *prime* if, for two-sided ideals I, J of R with $IJ = 0$, then $I = 0$ or $J = 0$.

Lemma 5.1.5. [5, Corollary 2.3.8] *Let R be a prime ring, $Q_{mr}(R)$ be the maximal ring of quotients of R as in Chapter 1 and $Z = Z(Q_{mr}(R))$ be the center of $Q_{mr}(R)$. For any element $a \in Q_{mr}(R)$, the following conditions are equivalent:*

1. *The element a is algebraic over Z of degree less than n.*
2. *$g_n(a, r_1, r_2, \cdots, r_n) = 0$ for any $r_1, r_2, \cdots, r_n \in Q_{mr}(R)$.*

Let A be a central simple algebra. Then A is semisimple, so that every A-module is injective [14, Therem 1.2]. In particular, for every right ideal I of A, there exists a right ideal J of A such that $I_A \oplus J_A = A_A$ as A-modules. It means, A has the unique dense right ideal A as Definitions in Chapter 1. Therefore, $Q_{mr}(A) = A$. We have a corollary of Lemma 5.1.5.

Corollary 5.1.6. *Let F be a field and A be a central simple F-algebra. For any element $a \in A$, the following conditions are equivalent:*

1. *The element a is algebraic over F of degree less than n.*
2. *$g_n(a, r_1, r_2, \cdots, r_n) = 0$ for any $r_1, r_2, \cdots, r_n \in A$.*

5.2 An application

We can apply rational identities to study maximal subfields of a division algebra. The following Lemma is basic.

Lemma 5.2.1. *Let F be a field and D be a division F-algebra of degree n. Assume that K is a subfield of D containing F. Then $\dim_F K \leq n$. The equality holds if and only if K is a maximal subfield of D.*

Proof. See [29, Corollary 15.6 and Proposition 15.7] ■

Lemma 5.2.2. *Let F be an infinite field and $n \geq 2$ be an integer. There exist two matrices $A, B \in M_n(F)$ such that the commutator $ABA^{-1}B^{-1}$ is an algebraic element of degree n over F.*

Proof. Put

$$A := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } B := \begin{pmatrix} b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & b_{n-1} & 0 \\ 0 & 0 & 0 & 0 & b_n \end{pmatrix}, \text{ where } b_j \neq 0.$$

One has
$$ABA^{-1}B^{-1} = \begin{pmatrix} b_n b_1^{-1} & 0 & \cdots & 0 & 0 \\ * & b_1 b_2^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & b_{n-2}b_{n-1}^{-1} & 0 \\ * & * & * & * & b_{n-1}b_n^{-1} \end{pmatrix}.$$

If we choose $b_n b_1^{-1}, b_1 b_2^{-1}, \cdots, b_{n-1}b^{-1}$ all distinct (it is possible since F is infinite), then the characteristic polynomial of $ABA^{-1}B^{-1}$ is a polynomial of smallest degree which vanishes on $ABA^{-1}B^{-1}$. That is, $ABA^{-1}B^{-1}$ is an algebraic element of degree n over F. ■

The following Theorem answers Problem 29 in [32, Page 83].

Theorem 5.2.3. *Let F be a field and D be a division F-algebra. There exist $x, y \in D^*$ such that $F(xyx^{-1}y^{-1})$ is a maximal subfield of D.*

Proof. If F is finite, then D is also finite, so that $D = F$. There is nothing to prove. Suppose that F is infinite and D is of degree n over F. By Lemma 5.2.1, it suffices to show that there exist $x, y \in D^*$ such that $\dim_F F(xyx^{-1}y^{-1}) \geq n$. Indeed, put $\ell := \max\{\, \dim_F F(xyx^{-1}y^{-1}) \mid x, y \in D^* \,\}$. Then from Corollary 5.1.6,

$$g_\ell(rsr^{-1}s^{-1}, r_1, r_2, \cdots, r_\ell) = 0$$

for any $r_1, r_2, \cdots, r_\ell \in D$ and $r, s \in D^*$. Hence, $g_\ell(xyx^{-1}y^{-1}, y_1, y_2, \cdots, y_\ell)$ is a rational idenity of D, so that, by Theorem 5.1.3, $g_\ell(xyx^{-1}y^{-1}, y_1, y_2, \cdots, y_\ell)$ is also a rational idenity of $M_n(F)$. Since $g_\ell(ABA^{-1}B^{-1}, r_1, r_2, \cdots, r_\ell) = 0$, for any $r_i \in M_n(F)$ and A, B are chosen in Lemma 5.2.2. Therefore $n \leq \ell$ because $ABA^{-1}B^{-1}$ is an algebraic element of degree n and by Corollary 5.1.6. ■

Lemma 5.2.4. *Let F be an infinite field and $n > 2$ be an integer. There exist two matrices $A, B \in M_n(F)$ such that $AB - BA$ is an algebraic element of degree n over F.*

Proof. Put

$$A := \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } B := \begin{pmatrix} 0 & b_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

One has $AB - BA = \begin{pmatrix} -b_1 & * & \cdots & * & * \\ 0 & b_1 - b_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-2} - b_{n-1} & * \\ 0 & 0 & \cdots & 0 & b_{n-1} \end{pmatrix}$. Since F is infinite, we can choose $b_1, b_2, \cdots, b_{n-1} \in F$ such that $-b_1, b_1 - b_2, \cdots, b_{n-2} - b_{n-1}, b_{n-1}$ all distinct. Hence, the characteristic polynomial of $AB - BA$ is a polynomial of smallest degree vanishing on $AB - BA$. Therefore, $AB - BA$ is an algebraic element of degree n over F. ■

Similar to the proof of Theorem 5.2.3, we have the following Theorem, which answers Problem 28 in [32, Page 83].

Theorem 5.2.5. *Let F be a field and D be a division F-algebra. There exist $x, y \in D$ such that $F(xy - yx)$ is a maximal subfield of D.*

PROOF. If F is finite, then D is also finite, so that $D = F$. There is nothing to prove. Suppose that F is infinite and D is of degree n. By Lemma 5.2.1, it suffices to show that there exist $x, y \in D$ such that $\dim_F F(xy - yx) \geq n$. Indeed, if $n = 2$, by [29, Corollary 13.5], then there exist $x, y \in D$ such that $xy - yx \notin F$, which implies $F(xy - yx) = 2 = n$. Assume that $n > 2$. Then put $\ell := \max\{ \dim_F F(xy - yx) \mid x, y \in D \}$. By Corollary 5.1.6,

$$g_\ell(rs - sr, r_1, r_2, \cdots, r_\ell) = 0$$

for any $r_1, r_2, \cdots, r_\ell \in D$ and $r, s \in D^*$. It follows $g_\ell(xy - yx, y_1, y_2, \cdots, y_\ell)$ is a rational idenity of D. From Theorem 5.1.3, $g_\ell(xy - yx, y_1, y_2, \cdots, y_\ell)$ is also a rational idenity of $M_n(F)$. But because there exist $A, B \in M_n(F)$ such that $AB - BA$ is algebraic of degree n (Lemma 5.2.4), one has

$$g_\ell(AB - BA, r_1, r_2, \cdots, r_\ell) = 0$$

for any $r_i \in M_n(F)$. Therefore, by Corollary 5.1.6, $n \leq \ell$. ■

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Summary

In [33], it was proved that an injective module is indecomposable if and only if its endomorphism ring is local. The first aim of this thesis is to generalize this result from indecomposable modules to square-free ones by showing that an injective module is square-free if and only if its endomorphism ring is quasi-duo. We then describe all maximal right (left, two-sided) ideals of the endomorphism ring of an arbitrary injective module. We then study two classes of modules: Loewy modules with finite Loewy invariants over an arbitrary ring and max modules with finite radical invariants over a semilocal ring. We prove that the endomorphism rings of these modules are semilocal, generalizing in this way a result that proved by Camps and Dicks for artinian modules [9]. In [12], Facchini proved that, over a commutative ring, a module is artinian if and only if it is a Loewy module with finite Loewy invariants. Here we show that this is not necessarily true for modules over non-commutative rings. Finally, we answer two questions posed in [32]. We prove that, for any division algebra D with center F, there exist x, y, a, b in the multiplicative group D^* of D such that $F(xy - yx)$ and $F(aba^{-1}b^{-1})$ are maximal subfields of D.

Samenvatting

In [33] is bewezen dat een injectief moduul niet te ontbinden is dan en slechts dan als zijn endomorfismenring lokaal is. Het eerste doel van dit proefschrift is om dit resultaat te generaliseren van modulen die niet te ontbinden zijn tot kwadraatvrije modulen door te laten zien dat een injectief moduul kwadraatvrij is dan en slechts dan als zijn endomorfismenring quasi-duo is. Vervolgens beschrijven we alle maximale rechts- (links-, tweezijdige) idealen van de endomorfismenring van een willekeurig injectief moduul. Het volgende doel is om twee klassen modulen te bekijken: Loewy-modulen met eindige Loewy-invarianten over een willekeurige ring en maximale modulen met eindige radicale invarianten over een semilokale ring. We bewijzen dat de endomorfismenringen van deze modulen semilokaal zijn, wat een generalisatie is van een resultaat bewezen door Camps en Dicks voor Artin-modulen [9]. In [12] bewees Facchini dat een moduul over een commutatieve ring een Artin-moduul is dan en slechts dan als het een Loewy-moduul is met eindige Loewy-invarianten. We laten zien dat dit niet noodzakelijk waar is voor modulen over niet-commutatieve ringen. Het laatste hoofd resultaat van dit proefschrift is het beantwoorden van twee vragen die in [32] worden gesteld. We bewijzen dat, voor een delings algebra D met centrum F, er x, y, a, b in multiplicatieve groep D^* van D bestaan zodanig dat $F(xy - yx)$ en $F(aba^{-1}b^{-1})$ maximale deellichamen van D zijn.

Sommario

In [33], viene dimostrato che un modulo iniettivo è indecomponibile se e solo se il suo anello degli endomorfismi è locale. Il primo obiettivo di questa tesi è generalizzare questo risultato passando da moduli indecomponibili a privi di quadrati, mostrando che un modulo iniettivo è privo di quadrati se e solo se il suo anello degli endomorfismi è quasi-duo. Successivamente passiamo a descrivere tutti gli ideali destri (sinistri, bilateri) dell'anello degli endomorfismi di un modulo iniettivo arbitrario. Il secondo obiettivo è considerare due classi di moduli: i moduli di Loewy con invarianti di Loewy finiti su un anello arbitrario e i moduli di max con invarianti di radicali finiti su un anello semilocale. Dimostriamo che gli anelli di endomorfismi di questi moduli sono semilocali, generalizzando così un risultato dimostrato da Camps e Dicks per moduli artiniani [9]. In [12], Facchini dimostra che su un anello commutativo, un modulo è artiniano se e solo se è un modulo di Loewy con invarianti di Loewy finiti. Dimostriamo qui che questo non è necessariamente vero per moduli su un anello non commutativo. Il risultato finale principale di questa tesi è rispondere a due domande proposte in [32]. Dimostriamo che, per ogni algebra di divisone D con centro F, esistono x, y, a, b nel gruppo moltiplicativo D^* di D tali che $F(xy - yx)$ e $F(aba^{-1}b^{-1})$ siano sottocampi massimali di D

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Curriculum Vitae

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