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Random walks and the contact process

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Citation
Bethuelsen, S. A. (2016, November 22). *Random walks and the contact process*. Retrieved from https://hdl.handle.net/1887/44286

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Author: Bethuelsen, S.A.
Title: Random walks and the contact process
Issue Date: 2016-11-22

Random Walks and the Contact Process

Stein Andreas Bethuelsen

A catalogue record available from the Leiden Universtiy Library.

The research documented in this thesis is supported by the Netherlands Organisation for Scientific Research (NWO) through Mathematics cluster STAR.

Random Walks and the Contact Process

Proefschrift

ter verkrijging van
de graad van Doctor aan de Universiteit Leiden,
op gezag van Rector Magnificus prof. mr. C. J. J. M. Stolker,
volgens besluit van het College voor Promoties
te verdedigen op dinsdag 22 november 2016
klokke 11.15 uur

door

Stein Andreas Bethuelsen
geboren te Stavanger
in 1987

Samenstelling van de promotiecommissie:

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Chapter 1

Introduction

1.1 Background and outline

1.1.1 Background

The main theme of this thesis is the mathematical study of large and complex systems evolving in time according to a certain prescribed rule. The focus is on cases where the evolution of the system is random and correlated with a second system, itself evolving with time and randomly. Our aim is to analyse and quantify the long term behaviour of a large class of such systems, and to determine their key features. To make this setting more concrete, we focus on certain specific mathematical models.

One core model in this thesis is that of a random walk in a random environment. In this model, a "particle", called the random walk, is moving in a "disordered medium", called the random environment. The behaviour of the particle is random and depends on the local properties of the environment close to its location.

The random environment can either be static, that is, stay constant in time, or dynamic, in which case it evolves with time. The static case has been studied intensively in the mathematical and physical literature during the last 40-50 years. This case reflects the situation where changes in the disordered medium take place at a much larger time scale than displacements of the particle. Such models are motivated by various phenomena in physics, biology and chemistry, such as replications of DNA chains (Chernov [45]) and phase transitions in alloys (Temkin [110]).

More recently, there has been much focus on the dynamic case, where the time scale of the particle is comparable to that of the disordered medium. Motivation for such models comes among others from the study of phase transitions in glassy liquids (Benichou et al. [17]) and the diffusive behaviour of a biomolecule affected by other moving macromolecules (Gori et al. [63]).

In this thesis we are mainly interested in the analysis of the asymptotic behaviour of random walks in a dynamic random environment from a mathematical viewpoint. In particular, we

are interested in questions such as how to determine the average velocity of the particle and how to control fluctuations and deviations thereof. This is in general a very challenging problem due to the possibility of strong dependence both in space and time, and many questions are still wide open.

So far there has only been limited progress towards a general theory, unless severe additional assumptions are made. One such assumption is strong mixing properties of the dynamics, which guarantees that the dynamics is sufficiently independent on large time scales.

In this thesis we present new mixing conditions sufficient for analysing the average behaviour of random walks in a dynamic random environment. Our work improves on the existing literature for general models with strongly mixing dynamics and provides new insight for certain models with poorly mixing dynamics.

Another core model in this thesis is the so-called contact process, which is a prototype of a dynamically evolving environment with poor mixing properties. This model originated in the 1970's as a model for the spread of an infection in a population, and has later seen various other applications in e.g. high energy physics, ecology and biology.

Mathematically, much is known by now about the contact process and different tools have been developed for its study. On the other hand, new challenges arise as soon as slight perturbations of the model are considered. One example of such a perturbation is that of the contact process as seen from a random walk. This model plays a central role in this thesis.

1.1.2 Outline

In Section 1.2 we give a formal definition of the random walk in random environment model and, furthermore, discuss the main questions studied in the literature. In Section 1.3 we give an overview of the past progress made for such models in the mathematical literature, first for the static case, then for the dynamic case. In Section 1.4 we turn to the contact process. After a brief introduction, mentioning its main properties, we summarise in more detail the results known about random walks on the contact process. In Section 1.5 we summarise the chapters in this thesis.

1.2 Random walk in random environment

In this section we give a formal definition of the random walk in random environment model (abbreviated by RWRE), both in the static and the dynamic case, and state the main questions and challenges related to this model.

The environment

For $d \geq 1$, we consider the space $\mathbb{Z}^{d+1} := \mathbb{Z}^d \times \mathbb{Z}$. Here, $\mathbb{Z}^d$ should be thought of as the spatial dimension and $\mathbb{Z}$ as the temporal direction.

Let E be a compact Polish space and denote by $\Omega := E^{\mathbb{Z}^{d+1}}$ the space of configurations on $\mathbb{Z}^{d+1}$. To the space Ω we assign the standard product Borel σ-algebra, denoted by $\mathcal{F}$, which is generated by the cylinder functions. For $\Lambda \subset \mathbb{Z}^{d+1}$, we denote by $\mathcal{F}_\Lambda$ the sub-σ-algebra generated by the cylinder functions on E^Λ.

By $\mathcal{M}_1(\Omega)$ we denote the set of probability measures on $(\Omega, \mathcal{F})$. We call $\eta \in \mathbb{Z}^{d+1}$ the *environment* and denote by $\mathbb{P} \in \mathcal{M}_1(\Omega)$ its law.

A particular class of environments contained in our setup are path measures of a stochastic process $(\eta_t)_{t\in\mathbb{Z}}$ whose state space is $\Omega_0 := E^{\mathbb{Z}^d}$. Such environments are said to be *dynamic*. To emphasise this case, for $\eta \in \Omega$ and $(x,t) \in \mathbb{Z}^{d+1}$, we often write $\eta_t(x)$ for the value of η at (x,t).

Another class of environments contained in our setup are those not changing with respect to the time direction, that is, environments whose law concentrates on elements $\eta \in \mathbb{Z}^{d+1}$ for which $\eta_t(x) = \eta_0(x)$ for every $t \in \mathbb{Z}$ and every $x \in \mathbb{Z}^d$. Such environments are said to be *static*. Note that our setup also includes models that are neither static nor dynamic.

We assume throughout that $\mathbb{P}$ is *measure preserving* with respect to translations, that is, for any $x \in \mathbb{Z}^d$, $t \in \mathbb{Z}$,

$$\mathbb{P}(\cdot) = \mathbb{P}(\theta_{x,t}(\cdot)),$$

where $\theta_{x,t}$ denotes the shift operator $\theta_{x,t}\eta_s(y) = \eta_{s+t}(y-x)$. Furthermore, we often assume in addition that $\mathbb{P}$ is *ergodic* with respect to some subclass of the translations $\{\theta_{x,t} \colon (x,t) \in \mathbb{Z}^{d+1}\}$. Recall that a measure $\mathbb{P} \in \mathcal{M}_1(\Omega)$ is ergodic in direction $(x,t) \in \mathbb{Z}^{d+1}$ if $\mathbb{P}(B) \in \{0,1\}$ for every $B \in \mathcal{F}$ for which $\theta_{x,t}B = B$.

The random walk

The *random walk* $(X_t)_{t\in\mathbb{Z}_{\geq 0}}$ is a process on $\mathbb{Z}^d$, where $\mathbb{Z}_{\geq 0} := \{0,1,2,\dots\}$. We always assume (w.l.o.g.) that $X_0 = o$, where $o \in \mathbb{Z}^d$ denotes the origin. The transition probabilities of (X_t) are assumed to depend on the state of the environment as seen from the random walk. That is, given $\eta \in \Omega$, the evolution of (X_t) is governed by

$$\begin{aligned} &P_\eta(X_0 = o) = 1 \\ &P_\eta(X_{t+1} = y + z \mid X_t = y) = \alpha(\theta_{y,t}\eta, z), \end{aligned}$$

where $\alpha \colon \Omega \times \mathbb{Z}^d \to [0,1]$ satisfies $\sum_{z\in\mathbb{Z}^d} \alpha(\eta, z) = 1$ for all $\eta \in \Omega$.

The law of the random walk, $P_\eta \in \mathcal{M}_1((\mathbb{Z}^d)^{\mathbb{Z}_{\geq 0}})$, where the entire environment is fixed once and for all, is called the *quenched* law, and $\mathcal{G}$ denotes its σ-algebra defined by the cylinder functions. Furthermore, for $\mathbb{P} \in \mathcal{M}_1(\Omega)$, we denote by $P_{\mathbb{P}} \in \mathcal{M}_1\left(\Omega \times (\mathbb{Z}^d)^{\mathbb{Z}_{\geq 0}}\right)$ the joint law of (η, X), that is,

$$P_{\mathbb{P}}(B \times A) = \int_B P^\eta(A) d\mathbb{P}(\eta), \quad B \in \mathcal{F}, A \in \mathcal{G}.$$

The marginal law of $P_{\mathbb{P}}$ on the forward half-space $\mathbb{H} := (\mathbb{Z}^d)^{\mathbb{Z}_{\geq 0}}$ is denoted by P and is called the *annealed* (or averaged) law of (X_t).

As mentioned in the beginning, the RWRE model can roughly be divided into two classes depending on whether the environment is static or dynamic. This distinction is governed by the law $\mathbb{P} \in \mathcal{M}_1(\Omega)$. When $\mathbb{P}$ is static, (X_t) is a *random walk in a static random environment* (RWSRE), and if $\mathbb{P}$ is dynamic, then (X_t) is a *random walk in a dynamic random environment* (RWDRE).

Often we pose additional assumptions on the transition probabilities. We say that (X_t) has *finite dependence* if the transition rate only depends on the environment within a finite region around its location. That is, there exist a finite subset $R \subset \mathbb{Z}^{d+1}$ containing the origin such that for all $z \in \mathbb{Z}^d$

$$\alpha(\eta, z) - \alpha(\sigma, z) = 0 \text{ whenever } \sigma \equiv \eta \text{ on } R.$$

Interpreting the last coordinate of $\mathbb{Z}^{d+1}$ as time, it is often natural to assume that $R \subset \mathbb{Z}^d \times \{0\}$. We assume this unless otherwise specified.

Next, (X_t) is said to be of *finite range* if the jump range, denoted by

$$\mathcal{R} := \{y \in \mathbb{Z}^d \colon \sup_{\eta \in \Omega} \alpha(\eta, y) > 0\},$$

is finite. An important class of random walks with finite range are those for which $\mathcal{R} = \{x \in \mathbb{Z}^d \colon \| x \| = 1\}$, which are said to be *nearest neighbour* random walks.

Another important notion is *ellipticity*. We say that (X_t) is *elliptic in direction* $y \in \mathcal{R}$ if

$$\alpha(\eta, y) > 0 \quad \forall\, \eta \in \Omega. \tag{1.2.1}$$

If (1.2.1) holds for all $y \in \mathcal{R}$, then we say that (X_t) is *elliptic*. (X_t) is called *uniformly elliptic* if there is an $\epsilon > 0$ such that (1.2.1) is bounded from below by ϵ for all $y \in \mathcal{R}$.

Lastly, there are two particular classes of RWRE models studied in the literature we wish to mention. The first is the *balanced* RWRE model, where $\alpha \colon \Omega \times \mathbb{Z}^d \to [0, 1]$ is (for simplicity) nearest neighbour and satisfies

$$\alpha(\omega, e) = \alpha(\omega, -e) \text{ for all } \omega \in \Omega \text{ and } e \in \mathcal{R}. \tag{1.2.2}$$

The other is that of a *random conductance model*, where $\alpha\colon \Omega \times \mathbb{Z}^d \to [0,1]$ is (for simplicity) nearest neighbour and satisfies

$$\alpha(\omega, e) = \frac{\omega(e)}{Z(\omega)}, \quad Z(\omega) := \sum_{e \in \mathcal{R}} \omega(e). \tag{1.2.3}$$

The environment seen from the random walk

Observations of the environment along the trajectory of the random walk may often be of interest. Moreover, such observations can be of crucial importance for understanding the asymptotic behaviour of the random walk itself. For this reason, we introduce the so-called *random environment as seen from the random walk* (abbreviated by ESRW-process), which is given by

$$(\eta_t^{EP}) := (\theta_{X_t,t}\eta), \quad t \in \mathbb{Z}_{\geq 0}\,. \tag{1.2.4}$$

This process is also known under the name *environment process* and hence the superscript EP in (1.2.4). Note that (η_t^{EP}) is a Markov process under P with state space Ω and initial law $\mathbb{P}$.

Extension to continuous-time processes

We comment next on how our setup can be extended to include continuous-time models.

First note that continuous-time stochastic processes that are sampled at discrete times are contained in our model. Thus, our model extends to so-called *continuous-time constant speed* random walks, where the random walk waits an exponential amount of time before each jump (instead of a fix deterministic time). Indeed, this is equivalent to a time-delay/acceleration of the environment as long as the waiting time does not depend on the environment (that is, constant speed random walk). Such a time-delay/acceleration of the environment does not change its macroscopic behaviour whenever appropriate ergodicity assumptions are made.

In case the sojourn time of the random walk (that is, the jump rate) depends on the environment, the random walk is said to be of *variable speed*. If the sojourn time is uniformly bounded, say by $\kappa < \infty$, then a similar approach as above applies. Indeed, let the jump rate of the random walk be given by a Poisson process with parameter κ, independent of the environment. Then the same approach applies if for every $\sigma \in \Omega$ we increase $\alpha(\sigma, o)$ such that $\sum_{z \in \mathbb{Z}^d} \alpha(\sigma, z) = \kappa$. Note again that this does not change the behaviour of the random walk or the ESRW-process.

Main questions and challenges

Given the law $\mathbb{P} \in \mathcal{M}_1(\Omega)$ and the transition probabilities of (X_t), we wish to determine the asymptotic behaviour of X_t as $t \to \infty$. That is, we would like to know its average behaviour and to characterise how likely fluctuations and deviations thereof are. Such questions can be asked with respect to both the quenched and the annealed measure. We next outline these questions more in detail, partly following dos Santos [103], Chapter 1.

We say that the random walk X is *transient* if

$$P_\eta \left(\lim_{t\to\infty} \| X_t \| = \infty \right) = 1, \quad \text{for } \mathbb{P} - a.e.\ \eta \in \Omega.$$

and that it is *recurrent* if, for every $x \in \mathbb{Z}^d$,

$$P_\eta \left(\exists t > 0 \colon X_t = x \right) = 1, \quad \text{for } \mathbb{P} - a.e.\ \eta \in \Omega.$$

Furthermore, we say that (X_t) satisfies a *strong law of large numbers* (SLLN) if there exists a $v \in \mathbb{R}^d$, called the velocity of the random walk (or, alternatively, the speed, in one dimension), such that

$$P_\eta \left(\lim_{t\to\infty} t^{-1} X_t = v \right) = 1, \quad \text{for } \mathbb{P} - a.e.\ \eta \in \Omega. \tag{1.2.5}$$

Note that transience, recurrence and the SLLN require no distinction between annealed and quenched laws, since for any $B \in \mathcal{G}$ we have that $P(B) = 1$ if and only if $P_\eta(B) = 1$ for $\mathbb{P}$-almost every η.

We say that (X_t) satisfies the annealed *large deviation principle* (LDP) in $\mathbb{R}^d$ with rate function $H_a \colon \mathbb{R}^d \mapsto [0, \infty]$ if

$$\begin{aligned} &\limsup_{t\to\infty} t^{-1} \log P(t^{-1} X_t \in F) \le - \inf_{x\in F} H_a(x) \quad \forall \quad \text{closed } F \subset \mathbb{R}^d, \\ &\liminf_{t\to\infty} t^{-1} \log P(t^{-1} X_t \in G) \ge - \inf_{x\in G} H_a(x) \quad \forall \quad \text{open } G \subset \mathbb{R}^d . \end{aligned} \tag{1.2.6}$$

Analogously, (X_t) satisfies the quenched LDP if (1.2.6) holds with P_η in place of P and with H_a replaced by a rate function H_q that is deterministic, i.e., does not depend on η, and the statement hold for $\mathbb{P}$-a.e. $\eta \in \Omega$. Note that, in general, H_q is different from H_a, and $H_q \ge H_a$.

The random walk (X_t) satisfies the *central limit theorem* (CLT) if there exists a constant $\sigma^2 \in (0, \infty)$ (the limiting variance) such that for each $t > 0$,

$$\frac{X_{nt} - ntv}{n^{1/2}\sigma} \implies N \quad \text{as } n \to \infty, \tag{1.2.7}$$

where N is a standard normal random variable with mean 0 and variance 1 and $\Longrightarrow$ denotes weak convergence. We say that (X_t) satisfies the functional CLT (FCLT) if

$$\left(\frac{X_{nt} - ntv}{n^{1/2}\sigma}\right)_{t\geq 0} \Longrightarrow B \quad \text{as } n \to \infty,$$

where $B = (B_t)_{t\geq 0}$ is standard Brownian motion and the weak convergence is in Skorohod space.

The CLT (or FCLT) is called quenched or annealed depending on which law is considered for (X_t). In contrast to the LDP (where annealed and quenched rate functions may differ), the quenched CLT with variance σ^2 implies the annealed CLT with the same variance. From an application point of view, a quenched statement is often preferred over an annealed statement, because it gives more quantitative information about the behaviour of the random walk.

Sometimes we consider other scaling limits than the normal distribution and the Brownian motion, in which case the random walk is said to be anomalous. If the scaling in the denominator of (1.2.7) is $n^{\delta}\sigma$ for some $\delta < 1/2$, then the random walk is said to be *sub-diffusive*, for $\delta > 1/2$ *super-diffusive*, and for $\delta = 1/2$ *diffusive*.

We next discuss two of the key features that make RWRE models challenging. The first observation is that under the quenched measure P_η the (X_t) is in general a non-homogeneous Markov process, typically both in space and in time when the law $\mathbb{P}$ is dynamic. On the other hand, under the annealed law P, it is homogeneous, but not Markov. Consequently, many of the standard techniques used for classical random walk models no longer apply.

Furthermore, an additional difficulty, often hidden in the model, is that it may not be *reversible*. This means, informally, that the law of the evolution of (X_t) forward in time is distinct from its evolution viewed backwards in time. Consequently, the model lacks an important symmetry property.

In order to face these challenges it is often convenient to study the ESRW-process, since this is a Markov process under the annealed law, albeit, on a much bigger state space, namely Ω. Many of the questions phrased above for the random walk can be translated into questions about the ESRW-process and new interesting questions can be formulated.

For the ESRW-process one is typically interested in characterising its invariant measures, in particular the ergodic ones, as well as its convergence towards these measures. A central question is whether the ESRW-process is approximately equal to $\mathbb{P}$ far away from the origin. That is, does there exist an invariant measure for (η_t^{EP}), say $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$, that is mutually absolutely continuous with respect to $\mathbb{P}$, potentially when restricted to the forward half-space $\mathbb{H}$? Answering such a question in the affirmative naturally leads to further questions about the corresponding Radon-Nikodym derivative.

For all the questions mentioned above, one seeks both qualitative and quantitative statements. Firstly, we wish to be able to answer the above questions by yes or no. Secondly, we would

like to obtain more quantitative statements, such as the value of v (or at least its sign) in the SLLN or the value of σ in the CLT. Furthermore, it is often of importance to know how the answers vary with respect to the parameters of the model, such as the transition kernel of the random walk, the law of the environment and the dimension. Thus, we are interested in knowing how stable a given model is with respect to small perturbations.

To answer the questions mentioned above, a variety of techniques have been applied, including Fourier analysis and expansion techniques, multi-scale analysis, regeneration times, ergodic theory and martingale theory. Often additional assumptions are needed to ensure that the model satisfies good enough mixing properties or contains sufficient symmetry in order to make it amenable to mathematical analysis. In the next section we give a brief overview of the literature of RWRE models and describe some of these methods.

Other graphs

In this thesis, we restrict to the case where the underlying graph on which (X_t) walks is $\mathbb{Z}^d$, with $d \geq 1$. This is the standard graph for RWRE models. However, it is also interesting to consider other graphs.

One class of graphs different from $\mathbb{Z}^d$ is already contained in the description of the model above. Indeed, by adding extra restrictions to the transition kernels $\alpha(\eta, \cdot)$, $\eta \in \Omega$, we can restrict the trajectory of (X_t) to a certain subspace of $\mathbb{Z}^d$. Since the environment is drawn at random, in general this subspace is itself random and can for instance be the percolation cluster corresponding to a certain percolation model.

Other graphs, not included in Section 1.2, have also been considered in the literature. Since we do not treat such graphs in the forthcoming chapters, we do not provide any further details. Nevertheless, we wish to mention the recent paper by Benjamini, Duminil-Copin, Kozma, and Yadin [18], where a very general framework for studying RWRE models on stationary random graphs is put forward, including the study of the ESRW-process on such graphs.

1.3 Past progress

RWRE models have been studied intensively by the physics and the mathematics community over the last decades. We give next a brief overview with focus on the most recent progress and results relevant for the forthcoming chapters, particularly for the dynamic case. For more information on RWSRE models we refer the reader to the lecture notes of Sznitman [108] and Zeitouni [113] and the recent monograph of Drewitz and Ramírez [54]. For more about RWDRE models, we refer to the introduction of the PhD theses by Avena [5] and dos Santos [103], on which parts of this section are based.

1.3.1 Static random environment

One dimension

The one-dimensional RWSRE model is one of the key models in the field of disordered systems. In particular, it is one of the prime examples of a random walk showing significantly different behaviour when compared to classical simple random walk models. Furthermore, due to its one-dimensional nature it is amenable to a complete mathematical analysis and so it is by now well understood.

The model was first introduced in the mathematical community in Solomon [106], who studied a nearest neighbour random walk on $\mathbb{Z}$ in an i.i.d. environment. For this particular model, he proved an explicit criterion for transience/recurrence, as well as a formula for the speed in the SLLN. An interesting feature of this model is that, depending on the parameters, the random walk can have speed 0, but still be transient, also known as *subballistic* behaviour. Furthermore, the speed of the RWRE is smaller than the speed of a homogeneous random walk whose transition probabilities are given by the average. This is due to the presence of traps, that is, regions in the lattice where the random walk spends a long time because the environment gives it a drift towards the centre of the region.

Further examples of trapping phenomena were found in the work by Kesten, Kozlov, and Spitzer [79] and Sinaĭ [105], who studied the transient case and the recurrent case, respectively. From these works follows a complete characterisation of the scaling limits of (X_t). In particular, depending on the parameters of the model, the random walk can have subdiffusive, superdiffusive as well as diffusive behaviour. Furthermore, in Greven and den Hollander [64] trapping phenomena were also characterised at the level of large deviations.

Results for the one-dimensional RWSRE have since been extended beyond nearest neighbour random walks and i.i.d. environments. Alili [2] extended the work of Solomon [106] to environments that are ergodic with respect to spatial shifts. He also proved a criterion for the convergence of the ESRW-process towards an invariant measure absolutely continuous with respect to the underlying environment. Extension to the non-nearest neighbour setting is contained in Brémont [41]. Precise knowledge about fluctuations (CLTs) has also been extensively studied, in particular, in the transient regime. The LDP results of Greven and den Hollander [64] have been extended and generalised as well, for which we refer to Comets, Gantert, and Zeitouni [48].

Higher dimensions

The one-dimensional model is special and most of the mathematical techniques developed for its study fail completely when going to higher dimensions. The reason for this is that the geometrical structure starts to play a role and the analysis of trapping phenomena is much more delicate.

Even for the nearest neighbour random walk in an i.i.d. environment, basic questions remain open and most of the literature is concentrated around this particular case. Assuming uniform ellipticity, a conditional SLLN was proven in Kalikow [78], showing that the velocity can take at most two values. An (unconditional) SLLN, as in (1.2.5), was proven by Merkl and Zerner [89] for $d = 2$; see Holmes and Salisbury [76] for an extension to non-elliptic random walks. It is noteworthy that the SLLN fails for certain non-i.i.d. environments with a polynomially decaying correlation structure, as shown in Bramson, Zeitouni, and Zerner [40] and Zerner [114]. Further, Berger [23] showed that the conditional SLLN in [78] can take at most one non-zero value when $d \geq 5$. Peres, Popov, and Sousi [93] proved a transience criterion for uniform elliptic random walks, depending only on the dimension and the number of different transition kernels of (X_t).

To overcome the major difficulties encountered for general RWRE in dimensions larger than one, additional assumptions are often made. One such assumption is that of imposing a drift on (X_t) in a given (fixed) direction, implying transience and the SLLN with a non-zero velocity. Sznitman and Zerner [109] proved a SLLN under a drift assumption first introduced by Kalikow. Since then, weaker drift assumptions have been introduced, most notably the T and T' conditions by Sznitman [107] and the recent improvements by Berger, Drewitz, and Ramírez [25] (see also the monograph [54]). We further note the works of Rassoul-Agha and Seppäläinen [97] and Berger and Zeitouni [24], who proved a quenched FCLT, and the recent work by Berger, Cohen, and Rosenthal [22], who improved the quenched FCLT and proved a "local" version under similar drift assumptions (assuming dimension $d \geq 4$).

There also exists literature on RWSRE models where the i.i.d. assumption is weakened and the random walk satisfies some drift condition. By studying the ESRW-process, Rassoul-Agha [94] proved a SLLN under a strong uniform mixing assumption known as the Dobrushin-Shlosman complete analyticity condition. Constructing certain modified regeneration times for the law of the random walk, Comets and Zeitouni [46] weakened the mixing assumptions of Rassoul-Agha [94] slightly, and proved a SLLN. For this, they introduced the so-called cone mixing condition:

Fix $\tau \in (0, \infty)$ and for $\theta \in (0, \frac{1}{2}\pi)$ and $t \geq 0$ let

$$C_t^\theta := \{(x, s) \in \mathbb{Z}^{d+1} \colon \| x \|_1 \leq (s - t) \tan\theta, s \geq t\}$$

be the cone whose tip is at (o, t) and whose wedge opens up with angle θ. Then $\mathbb{P} \in \mathcal{M}_1(\Omega)$ is *cone mixing* if, for all $\theta \in (0, \frac{1}{2}\pi)$,

$$\lim_{t\to\infty} \sup_{\substack{A\in\mathcal{F}_0, B\in\mathcal{F}_t^\theta \\ \mathbb{P}(A)>0}} |\mathbb{P}(B \mid A) - \mathbb{P}(B)| = 0, \tag{1.3.1}$$

where $\mathcal{F}_0$ is the σ-algebra generated by $\{(\eta(x, s) \colon (x, s) \in \mathbb{Z}^{d+1}, s \leq 0\}$ and $\mathcal{F}_t^\theta$ is the σ-algebra generated by $\{\eta(x, s) \colon (x, s) \in C_t^\theta\}$.

The SLLN in Comets and Zeitouni [46] was further strengthened to an annealed FCLT in Comets and Zeitouni [47] under a multiple cone mixing condition. We also mention the

work of Rassoul-Agha [95] and Guo [67], who extended the work of Kalikow [78] and Berger [23].

Other additional assumptions than a drift on the random walk have been studied successfully (mostly assuming i.i.d. environments). We mention the balanced case (e.g. Lawler [80], Guo and Zeitouni [68]), via cut-times (Bolthausen, Sznitman, and Zeitouni [39]), via exit times (e.g. Bricmont and Kupiainen [42], Bolthausen and Zeitouni [38] and Baur [16]), Dirichlet random environments (e.g. Sabot [102]), via lace expansion techniques (e.g. van der Hofstad and Holmes [75]), perturbations of a simple random walk (e.g. Bolthausen and Sznitman [37] and Sabot [101]), and the random conductance model (e.g. Andres, Deuschel, and Slowik [4]).

As for general results about LDP, we refer to Varadhan [111] for an early account of both an annealed and a quenched LDP. Recently, there have been several extensions of the quenched LDP, see e.g. Rassoul-Agha, Seppäläinen, and Yilmaz [98].

1.3.2 Dynamic random environment

In this subsection we discuss the literature on RWDRE. Such models have gained increasing interest in the last two decades. Motivated by the static case, one core question is to which extent trapping phenomena may persist. Typically, in order to show that the random walk is diffusive, one looks for some way to guarantee that the environment is "forgetful" and the walk increments are sufficiently independent on large time scales. For this, the mixing properties of the environment play an important role and general results are known for Markovian environments that are uniformly mixing with respect to the starting configuration.

On the other hand, simulations as well as heuristic arguments indicate that trapping phenomena may persist when the dynamics is slowly mixing (see e.g. Boldrighini, Cosimi, Frigio, and Pellegrinotti [36] and Avena and Thomann [6]). At a mathematical level, models where the dynamical environment has non-uniform mixing properties serve as a major challenge and are still not well understood. Avena, den Hollander, and Redig [7] studied a one-dimensional nearest neighbour random walk on the simple symmetric exclusion process and proved slow-down behaviour at the level of annealed large deviations. Recently, some particular examples yielding diffusive behaviour have also been studied, however, no general theory has so far been developed.

Early works

In the early years of RWDRE the main focus was on models that can be put into one of the following two categories; *independent in time* or *independent in space and Markovian in time*. Possibly the first example of a RWDRE was given by Madras [87], who obtained a recurrence/transience criteria, a SLLN and an annealed FCLT for a very particular model.

The first general model was introduced by Boldrighini, Ignatyuk, Malyshev, and Pellegrinotti [33] and was studied via cluster expansion techniques.

For environments that are independent in time, that is, where the random environment is resampled after each time unit, the random walk is homogeneous under the annealed measure, and the focus is on quenched results. There is now a good understanding of such models. For a general quenched FCLT we refer to Rassoul-Agha and Seppäläinen [96]; see also e.g. Boldrighini, Minlos, and Pellegrinotti [35]. The quenched LDP was treated by Yilmaz [112]. We further mention the recent paper Barraquand and Corwin [15], which brings the scope of KPZ universality to random walks in dynamic random environment.

RWDRE models where the environment is independent in space and Markovian in time are random environments consisting of identically distributed Markov processes, evolving independently at each site of $\mathbb{Z}^d$. The SLLN and both quenched and annealed FCLTs have been considered; see e.g. Boldrighini, Minlos, and Pellegrinotti [34], Bandyopadhyay and Zeitouni [14] and Dolgopyat and Liverani [52].

Space-time dynamic environments

We next focus our discussion on the most challenging class of RWDRE models, namely, those with a non-trivial spatial and temporal correlation structure. By considering the ESRW-process, under a strong space-time mixing assumption, Dolgopyat, Keller, and Liverani [53] obtained a SLLN and a quenched CLT in a perturbative regime with respect to the transition kernel. A quenched FCLT was also obtained by Bricmont and Kupiainen [43] for random walks that depend weakly on the random environment, using renormalisation arguments, under an assumption that implies exponential mixing of the random environment. The latter is not assumed to be Markovian. Avena, den Hollander, and Redig [8] adapted the cone mixing condition (1.3.1) of Comets and Zeitouni [46] to the dynamic case, providing a SLLN for finite range random walks on a general class of uniquely ergodic interacting particle systems. In a perturbative regime, depending both on the mixing properties of the dynamics and the transition kernel of the random walk, they obtained estimates of the velocity by use of expansion methods. In den Hollander, dos Santos, and Sidoravicius [73] the cone mixing approach was further extended to a new class of non-elliptic models.

In Redig and Völlering [100], the SLLN and the annealed FCLT were obtained in a rather general setup, which we describe next. By using coupling methods, they transferred mixing properties of the dynamics to the ESRW-process. That is, for each $\eta, \omega \in \Omega_0$, they assumed the existence of a strong Markov coupling $\widehat{\mathbb{P}}_{\eta,\omega}$ of the dynamics (η_t), started from η and ω respectively, such that

$$\int_0^\infty t^d \phi(t) dt < \infty, \qquad \phi(t) := \sup_{\eta,\omega} \widehat{\mathbb{E}}_{\eta,\omega} \left[\text{dist}(\eta_t^1(o), \eta_t^2(o)) \right]. \tag{1.3.2}$$

Here $\mathrm{dist}(\cdot,\cdot)$ is the distance on the Polish space E and $\widehat{\mathbb{E}}_{\eta,\omega}$ is the expectation corresponding to $\widehat{\mathbb{P}}_{\omega,\eta}$. Under (1.3.2), they showed that the ESRW-process is uniquely ergodic and, moreover, they proved a relation between $\phi(t)$ and the speed of relaxation towards equilibrium for this process. With the help of martingale theory the results for the ESRW-process were used to infer information about (X_t) under minimal assumptions on the transition kernels, in particular a SLLN and an annealed FCLT.

Non-uniformly mixing dynamic environments

RWDRE models with non-uniform correlation structure have been studied in the last decade. Such models have gained much interests the last decade, with 30-40 papers appearing on arXiv in the last 5 years. We comment next on the most recent developments.

RWDRE models where the dynamic random environment is given by the one-dimensional exclusion process have recently been considered in Avena, Franco, Jara, and Völlering [11], Avena, Jara, and Völlering [12] and Huveneers and Simenhaus [77] (see also Avena, den Hollander, and Redig [7], Avena, dos Santos, and Völlering [13] and dos Santos [104]). Focusing on the nearest neighbour model, under a proper space-time rescaling in which the exclusion process is speeded up compared to the random walk, [11] prove a hydrodynamic limit theorem for the corresponding ESRW-process and derive a differential equation describing the macroscopic evolution of the random walk. Expanding on this, [12] prove a joint path LDP. For the same model, assuming the random walk and the exclusion process evolve at the same time-scale, and without taking the hydrodynamic limit, [77] provide an annealed FCLT in regimes where the evolution of the dynamics is either very slow or very fast compared to that of the random walk and the random walk satisfy certain drift assumptions.

A similar model is studied by Hilário, den Hollander, Sidoravicius, dos Santos, and Teixeira [71] (see also den Hollander, Kesten, and Sidoravicius [74]). They consider a nearest neighbour random walk in a one-dimensional dynamic random environment consisting of a collection of independent particles performing simple symmetric random walks. Initialising the dynamic random environment from a Poisson equilibrium with density $\rho \in (0,\infty)$, they prove a SLLN, an annealed FCLT and large deviation bounds, provided ρ is large enough. The proof of this rely on the construction and control of a renewal structure for the random walk trajectory and a multi-scale renormalisation scheme. Firstly, the renormalisation scheme is used to show that the random walk moves at a strictly positive speed. Secondly, as a consequence of this ballistic behaviour, the renewal structure is used to show that the random walk trajectory is sufficiently independent on large time scales. An extension of this work to general dimensions is in preparation.

Recently, a random walk on the East model has been considered in Avena, Blondel, and Faggionato [10]. In a perturbative regime, they study the asymptotic velocity and density profile for the ESRW-process. The perturbation is related to a more general theory developed in Avena, Blondel, and Faggionato [9] for random walks on a dynamic random environment

satisfying a spectral gap. Under certain assumptions, they obtain, among others, the annealed FCLT and information about the invariant measure of the ESRW-process.

There has also been work on more general RWDRE models with poorly mixing dynamics which, however, require other (strong) properties of either the dynamics or the random walk. Besides the work of [9] mentioned above, we note Orenshtein and dos Santos [92], who proved the trichotomy between transience to the right, transience to the left and recurrence for a class of one-dimensional nearest neighbour RWDRE models under fairly general assumptions. We also mention the work of Deuschel, Guo, and Ramirez [50], who treated RWDRE models with balanced transition kernels, and Andres, Chiarini, Deuschel, and Slowik [3], who treated dynamic random conductance models, both proving a quenched FCLT. In both of these works minimal mixing assumptions are required, allowing for general dynamics with slow and non-uniform mixing properties. On the other hand, the strong assumptions on the transition kernels, as seen from (1.2.2) and (1.2.3), make both of these two cases rather special.

In the following section, we introduce a particular example of a dynamic random environment having non-uniform mixing properties, namely, the contact process. This model plays an important role in this thesis and is therefore treated separately.

1.4 The contact process as dynamic random environment

The contact process

The contact process is one of the basic examples of a dynamic random environment with a non-trivial spatial and temporal correlation structure. Given $\lambda \in (0, \infty)$, the contact process (η_t) on $\mathbb{Z}^d$ with parameter λ is the continuous-time interacting particle system on $\{0,1\}^{\mathbb{Z}^d}$ with local transition rates given by

$$\eta \to \eta_x \text{ at rate } \begin{cases} 1, & \text{if } \eta(x) = 1, \\ \lambda \sum_{y \sim x} \eta(y), & \text{if } \eta(x) = 0. \end{cases} \tag{1.4.1}$$

Here, $\sum_{y \sim x}$ denotes summation over nearest neighbours and η_x is defined by $\eta_x(y) = \eta(y)$ for $y \neq x$, and $\eta_x(x) = 1 - \eta(x)$.

The contact process was introduced by Harris [70] in 1974 as a toy model for the spread of an infection in a population. With this interpretation in mind, λ is often referred to as the "infection" parameter and a site is said to be *infected* at time t if $\eta_t(x) = 1$, and otherwise it is said to be *healthy*. Thus, by (1.4.1), a site $x \in \mathbb{Z}^d$ becomes healthy at rate 1, independently of the other sites, and it becomes infected at a rate proportional to the number of infected

neighbours it has. A central question is whether the set of infections "survives" with positive probability or eventually dies out, i.e., all sites become healthy.

The "healthy configuration" where all sites are equal to 0, denoted by $\bar{0}$, is trivially an absorbing state for the contact process. On the other hand, starting from the full configuration where all sites are initially infected, the contact process evolves towards an equilibrium measure $\bar{\nu}_\lambda$, called the *upper invariant measure*.

A basic property of the contact process is that it undergoes a phase transition. That is, there is a critical threshold $\lambda_c \in (0, \infty)$, depending on the dimension d, such that $\bar{\nu}_\lambda = \delta_{\bar{0}}$ whenever $\lambda < \lambda_c$ and $\bar{\nu}_\lambda \neq \delta_{\bar{0}}$ for all $\lambda > \lambda_c$. Here, $\delta_{\bar{0}}$ denotes the measure which concentrates on $\bar{0} \in \Omega$. Since the work of Bezuidenhout and Grimmett [30], it is known that $\bar{\nu}_{\lambda_c} = \delta_{\bar{0}}$. Furthermore, it is known that $\bar{\nu}_{\lambda_c}$ and $\delta_{\bar{0}}$ are the only extremal measures for the contact process and there is a complete understanding of the convergence towards mixtures of them (known as the complete convergence theorem).

For the same reason why the contact process undergoes a phase transition, the contact process is not cone mixing. Indeed, if at time zero there are no infections in the box$[-t^2, t^2] \cap \mathbb{Z}^d$, then, for large t, there will typically be no infections near the tip of C_t^θ as well. In particular, the evolution of the process depends crucially on the starting configuration and does not satisfy the uniform bounds of (1.3.1) and of (1.3.2).

Random walks on the contact process

Random walks on the contact process have been studied by several authors, in particular via the right-most particle process on $\mathbb{Z}$; see e.g. Durrett [56].

Closer to the context considered in this thesis is the work of den Hollander and dos Santos [72], who study a nearest neighbour random walk on the one-dimensional contact process. Combining monotonicity properties of the contact process and the random walk they proved a SLLN throughout the supercritical regime. Furthermore, in a perturbative regime, by constructing a renewal process, they proved an annealed FCLT and large deviation estimates.

The CLT of den Hollander and dos Santos [72] was extended in Mountford and Vares [91] to hold throughout the supercritical regime and for finite range random walks with finite dependence. The approach in [91] is by a multi-scale renormalisation scheme, which is used to approximate certain regeneration times, and appears to be rather general. However, in their paper, they restrict to random walks on $\mathbb{Z}$.

Further properties of random walks on the contact process follow from works by others under additional assumptions. For instance, in [50] and in [3], the random walk is proven to satisfy a quenched FCLT when the random walk is assumed to be balanced or to be given by a dynamic conductance model, respectively. We also mention a related model of a random

walk on the time-reversed (discrete-time) contact process. Motivated by biological applications, this model has recently been studied by Birkner, Černý, and Depperschmidt [32] (see also Birkner, Černý, Depperschmidt, and Gantert [31] and Miller [90]). Under certain assumptions, via a renormalisation argument and the control of a certain regeneration time, they obtain the SLLN and the annealed FCLT for the random walk.

Several chapters of this thesis are devoted to the study of random walks on the contact process, as also briefly described in Section 1.5.

1.5 Overview of the thesis

In this section we give a brief description of each of the forthcoming chapters.

Chapter 2

In this chapter, we study a class of continuous-time random walks on a dynamic random environment on $\mathbb{Z}^d$, $d \geq 1$, having two different transition kernels. By a coupling argument, and assuming certain monotonicity properties of the environment, we are able to prove a SLLN under general assumptions on the transition kernels. In particular, we provide new knowledge for a class of dynamic random environments having a non-trivial spatial and temporal correlation structure.

For this, we assume that the dynamics are given by an interacting particle system with single-spin space $E = \{0, 1\}$ and that the jump probabilities of the random walk only depend on the environment exactly at the location of the random walk.

By a particular coupling construction, we show that the total number of 1's the random walk observes at its jump times is a monotone function of the environment. Combing this with the Markovian property of the environment, by applying the sub-additive ergodic theorem, we obtain a SLLN for the random walk when the environment is additive and attractive and started from a configuration where all sites have the same value. By a different coupling argument, we show how to relax the assumption on the starting configuration under certain mixing conditions. Our approach also yields estimates of large deviations for the random walk, in particular, under a uniform mixing assumption.

As an example, we consider the random walk on the contact process, for which we obtain a law of large numbers in arbitrary dimension throughout the supercritical regime when the process is started from the upper invariant measure. This provides an extension of the SLLN in [72]. For this model, further properties about the speed are derived.

This chapter is based on a paper with Markus Heydenreich [27], to appear in Stoch. Proc. Appl.

Chapter 3

In this chapter, we provide a new approach for determining limiting properties of random walks in a dynamic random environment, in particular, about the invariant law of the ESRW-process. Under general mixing assumptions, we prove the existence of an invariant measure making the ESRW-process ergodic, and show that it is mutually absolute continuous with respect to the law of the random environment in the forward half-space $\mathbb{H}$. Our mixing assumptions are considerably weaker than the uniform mixing conditions present in the literature and do not require the environment to be Markovian.

For instance, we show that the random walk satisfies the SLLN under the assumption that it is weakly elliptic and that the environment satisfies

$$\lim_{t\to\infty} \sup_{\gamma\in\Gamma} \sup_{\substack{A\in\mathcal{F}_\gamma, B\in\mathcal{F}_t^\theta \\ \mathbb{P}(A)>0}} |\mathbb{P}(B \mid A) - \mathbb{P}(B)| = 0.$$

Here Γ contains the set of all paths of finite length, starting at $(o, 0)$ and directed "backwards in time" with jumps bounded by the jump size of the random walk, and $\mathcal{F}_\gamma$ is the σ-algebra generated by the cylinder events along $\gamma \in \Gamma$. This should be compared with the cone mixing condition; see (1.3.1).

Further applications of our theory include a quenched CLT based on [53], which relaxes the mixing requirements.

An important feature of our approach is that it can also be applied to dynamics with non-uniform mixing properties. As examples we include an environment given by Ornstein-Uhlenbeck processes and the contact process.

The key observation in this work is an expansion of the ESRW-process. This expansion enables us to separate the contribution of the random environment to the law of the ESRW-process from that of the transition probabilities of the random walk. Using this expansion, we also give sufficient criteria for stability under perturbations of the environment or the transition kernel of the random walk. Under a strong uniform mixing assumption, we obtain uniform control on the Radon-Nikodym derivative of the law of the ESRW-process with respect to the environment, irrespective of the choice of the jump kernel of the random walk.

This chapter is based on a paper with Florian Völlering [28], which has been submitted.

Chapter 4

In this chapter, we continue the study of the ESRW-process, focusing on a RWDRE model where the dynamics is given by the contact process. We consider two parameters for our model: the infection rate λ of the contact process and the jump rate of the random walk,

denoted by γ. We prove that, when either λ is large or γ is small, the ESRW-process has (at most) two extremal invariant measures. Moreover, the convergence to these extremal measures is characterised by whether the contact process survives or dies out, similar to the complete convergence theorem known for the ordinary contact process. Using this, we furthermore provide a law of large numbers for the random walk.

Our arguments apply to the processes considered on $\mathbb{Z}^d$, $d \geq 1$, and under general assumptions on the jump probabilities of the random walk.

This chapter is based on a paper, [26], which has been submitted.

Chapter 5

In the last chapter, we focus on the contact process in the supercritical regime, in particular, when initialised from the upper invariant measure. Liggett and Steif [85] proved that, for the supercritical contact process on certain graphs, the upper invariant measure stochastically dominates an i.i.d. Bernoulli product measure. In particular, they proved this for $\mathbb{Z}^d$ and (for infection rate sufficiently large) d-ary homogeneous trees T_d.

We prove some space-time versions of their results. We do this by combining their methods with specific properties of the contact process and general correlation inequalities.

One of our main results concerns the contact process on T_d with $d \geq 2$. We show that, for large infection rate, there exists a subset V of the vertices of T_d, containing a "positive fraction" of all the vertices of T_d, such that the following holds: The contact process on T_d observed on V stochastically dominates an independent spin-flip process. (This is known to be false for the contact process on graphs having subexponential growth.)

We further prove that the supercritical contact process on $\mathbb{Z}^d$ observed on certain d-dimensional space-time slabs stochastically dominates an i.i.d. Bernoulli product measure, from which we conclude strong mixing properties important in the study of certain random walks in random environment.

This chapter is based on a paper with Jacob van den Berg, [19], which has been submitted.

Chapter 2

Law of large numbers for random walks on attractive spin-flip dynamics

This chapter is based on a paper with Markus Heydenreich, [27], to appear in Stoch. Proc. Appl.

Abstract

We prove a law of large numbers for certain random walks on certain attractive dynamic random environments when initialised from all sites equal to the same state. This result applies to random walks on $\mathbb{Z}^d$ with $d \geq 1$. We further provide sufficient mixing conditions under which the assumption on the initial state can be relaxed, and obtain estimates on the large deviation behaviour of the random walk.

As prime example we study the random walk on the contact process, for which we obtain a law of large numbers in arbitrary dimension. For this model, further properties about the speed are derived.

2.1 Introduction and main results

2.1.1 Background and outline

Random walks in random environment (RWRE) gained much interest throughout the last decades. Such models serve as natural extensions of the classical random walk model and have broad applications in physics, chemistry and biology.

RWRE models show significantly different behaviours than the simple random walk model. This was already observed in one of the first models studied, Solomon [106], where it was shown that the random walk can behave sub-ballistically. Non-Gaussian scaling limits were established for the same model in Kesten, Kozlov, and Spitzer [79] and Sinaĭ [105]. These characteristics are due to trapping phenomena.

RWRE models on $\mathbb{Z}$ are by now well understood in great generality whenever the environment is static, i.e. it does not change with time. On the other hand, for RWRE models on $\mathbb{Z}^d$, $d \geq 2$, the analysis of trapping phenomena becomes much more delicate and less is known. See for instance Zeitouni [113] or Sznitman [108] for an overview of results, and Drewitz and Ramírez [54] for a monograph with focus on recent developments.

In the last decade, much focus has been devoted to models where the random environment evolves with time, i.e. random walks in *dynamic* random environments (RWDRE). It is believed that the extent to which trapping phenomena occur for RWDRE models depends on the correlation structure of the dynamics.

At a rigorous level, it is known to great generality that RWDRE models scale diffusively when the environment is only weakly correlated in space-time; see for instance Redig and Völlering [100]. These results are not restricted to random walks on $\mathbb{Z}$, but are valid in any dimension. Here weakly correlated essentially means that the environment becomes approximately independent of its starting configuration within a space-time cone, also known as cone mixing environment.

Little is known at a general level when the environment has a non-uniform correlation structure, though trapping phenomena are conjectured to occur for some specific models (Avena and Thomann [6]). Avena, den Hollander, and Redig [7] have shown rigorously that a random walk on the one-dimensional exclusion process exhibits trapping phenomena at the level of large deviations under drift assumptions. On the other hand, several other models with a non-uniform correlation structure have been shown to possess diffusive scaling limits, for example Avena, dos Santos, and Völlering [13], Hilário, den Hollander, Sidoravicius, dos Santos, and Teixeira [71], den Hollander and dos Santos [72], Huveneers and Simenhaus [77] and Mountford and Vares [91].

In this paper we present a strong law of large numbers (SLLN) for random walks on certain attractive (or monotone) interacting particle system (IPS). For this, restrictions on both the

random walk and the IPS are required. In particular, we assume that the sites of the IPS take values 0 or 1 and that the IPS has a graphical representation coupling which is monotone with respect to the initial configuration. One class of IPS satisfying the latter assumption are additive and attractive spin-flip systems.

The SLLN is obtained when the IPS is initialised at time 0 from a configuration where all sites have the same value, assuming in addition that the jump transitions of the random walk only depend on the state of the IPS at the position of the random walk. Under certain mixing conditions, we are able to relax the restriction on the starting configuration.

An important feature of the SLLN is that it does not rely directly on the correlation structure of the environment, but rather assumes monotonicity. In particular, the SLLN applies to a large class of models with non-uniform correlation structure not previously considered in the literature. Furthermore, the SLLN applies to random walks on $\mathbb{Z}^d$ for any $d \geq 1$ and is not restricted to nearest neighbour jumps.

We also provide large deviation estimates for the random walk. In particular, we show that no trapping phenomena occur for our model at the level of large deviations throughout the cone mixing regime.

The supercritical contact process is an example of an IPS satisfying the above requirements and having non-uniform correlation structure. For the random walk on this process, we prove the SLLN throughout the supercritical regime when started from the upper invariant measure (as well as many other initial configurations). Further properties about the speed are also derived. These results extend upon the SLLN obtained in [72] beyond the one-dimensional nearest-neighbour setting.

Outline

The rest of the paper is organised as follows. In the next subsection we give a precise definition of our model and in Subsection 2.1.3 we present our main results. Subsection 2.1.4 contains a discussion of related literature. Section 2.2 is devoted to a particular coupling construction of the environment and the random walk, yielding a monotonicity property, important for our results. In Section 2.3 we present the proofs of our main theorems for general attractive environments. Section 2.4 is devoted to the special case of a random walk on the contact process.

2.1.2 The model

We first introduce the environment. For this, let $d \geq 1$ and denote by $\Omega = \{0,1\}^{\mathbb{Z}^d}$ the configuration space and by $D_\Omega[0,\infty)$ the corresponding path space, that is, the set of càdlàg functions on $[0,\infty)$ taking values in Ω.

As the environment we consider an IPS, $\xi = (\xi_t)_{t\geq 0}$, such that $\xi_t = \{\xi_t(x)\colon x \in \mathbb{Z}^d\}$ is in Ω, $t \in [0,\infty)$, and $\xi \in D_\Omega[0,\infty)$. The process ξ starting from $\xi_0 = \eta$ is denoted by ξ^η and its law is given by P^η. When ξ_0 is drawn from $\mu \in \mathcal{P}(\Omega)$, the set of probability measures on Ω, we write ξ^μ for the corresponding process. Its law is denoted by P^μ and is given by

$$P^\mu(\cdot) = \int_\Omega P^\eta(\cdot)\mu(d\eta).$$

We assume throughout that ξ is translation invariant, that is,

$$P^\eta(\theta_x\xi_t \in \cdot) = P^{\theta_x\eta}(\xi_t \in \cdot) \tag{2.1.1}$$

with θ_x denoting the shift operator $\theta_x\eta(y) = \eta(y-x)$, $\eta \in \Omega$.

Further, to the configuration space Ω we associate the partial ordering such that $\xi \leq \eta$ with $\xi, \eta \in \Omega$ if and only if $\xi(x) \leq \eta(x)$ for all $x \in \mathbb{Z}^d$. A function $f\colon \Omega \to \mathbb{R}$ is called increasing if $\xi \leq \eta$ implies $f(\xi) \leq f(\eta)$. For two measures $\mu_1, \mu_2 \in \mathcal{P}(\Omega)$, μ_2 *stochastically dominates* μ_1, written $\mu_1 \leq \mu_2$, provided that

$$\int_\Omega f d\mu_1 \leq \int_\Omega f d\mu_2$$

for all increasing continuous functions f on Ω. We denote by $\delta_{\bar{0}}, \delta_{\bar{1}} \in \mathcal{P}(\Omega)$ the extremal measures which put all their weight on the configurations $\bar{1}$ and $\bar{0}$, respectively, where $\bar{i}(x) = i$ for all $x \in \mathbb{Z}^d$, $i \in \{0,1\}$. Obviously, it holds that $\delta_{\bar{0}} \leq \delta_{\bar{1}}$.

For a fixed realisation of $(\xi_t)_{t\geq 0}$, let $(W_t)_{t\geq 0}$ be the time-inhomogeneous Markov process on $\mathbb{Z}^d$ that, given $W_t = x$, jumps to $x+z$ at rate $\alpha(\xi_t(x), z)$ for some function $\alpha\colon \{0,1\}\times\mathbb{Z}^d \to [0,\infty)$. We call this process *the random walk*. Further, we assume throughout that

$$\gamma := \max_{i\in\{0,1\}} \left\{ \alpha(i,o) + \sum_{z\in\mathbb{Z}^d} \| z \|_1 \, \alpha(i,z) \right\} < \infty, \tag{2.1.2}$$

where $o \in \mathbb{Z}^d$ denotes the origin. Thus, the speed of the simple random walk seeing only occupied sites ($i = 1$) or only vacant sites ($i = 0$) is given by the local drifts

$$u_i := \sum_{z\in\mathbb{Z}^d} \alpha(i,z)z, \quad i \in \{0,1\}. \tag{2.1.3}$$

We say that (W_t) is *elliptic* if, for the unit vectors $\{\pm e_j\}_{j=1,\dots,d}$, we have

$$\min_{i\in\{0,1\}} \min_{j\in\{1,\dots,d\}} \{\alpha(i,\pm e_j)\} > 0.$$

We also say that (W_t) has *finite second moments* if

$$\max_{i\in\{0,1\}} \left\{ \sum_{z\in\mathbb{Z}^d} \alpha(i,z) \| z \|_1^2 \right\} < \infty$$

and that it has *finite exponential moments* if there exist $\kappa > 0$ such that,

$$\max_{i\in\{0,1\}} \left\{ \sum_{z\in\mathbb{Z}^d} \alpha(i,z) \exp\left(\epsilon \parallel z \parallel_1\right) \right\} < \infty, \quad \text{for all } 0 < \epsilon < \kappa.$$

Lastly, for $\xi \in D_\Omega[0,\infty)$ and $x \in \mathbb{Z}^d$, let $\mathbb{P}_x^\xi$ denote the law of (W_t) starting from $W_0 = x$ in a fixed environment ξ, which is the quenched law of W. The annealed law of W is given by

$$\mathbb{P}_{\mu,x}(\cdot) = \int_{D_\Omega[0,\infty)} \mathbb{P}_x^\xi(\cdot) P^\mu(d\xi).$$

2.1.3 Main results

General IPS can formally be constructed by defining a generator, see Liggett [82, Chapter I.1-3]. Alternatively, one can describe an IPS via a countable set of Poisson processes $\mathcal{I}$, yielding a more probabilistic description, see Durrett [55]. The probabilistic construction has the advantage that it yields a natural coupling, denoted by $\widehat{P}$, of the dynamics starting from any configuration on a joint probability space. For many interacting particle systems this coupling can be constructed explicitly and is known as the graphical representation. Important to our approach is the existence of such a coupling $\widehat{P}$ of the dynamic environment ξ which satisfies the following monotonicity property,

$$\widehat{P}\left(\xi_t^\eta \leq \xi_t^\omega,\ \forall t > 0 \text{ and } \eta, \omega \in \Omega \text{ satisfying } \eta \leq \omega\right) = 1. \tag{2.1.4}$$

A coupling $\widehat{P}$ satisfying (2.1.4) is said to be an *attractive graphical representation coupling*.

Theorem 2.1.1 (Strong law of large numbers)**.** *Assume that ξ has an attractive graphical representation coupling. Then, for each $i \in \{0,1\}$, there exists $\rho_i \in [0,1]$ such that*

$$\lim_{t\to\infty} \frac{1}{t} W_t = \rho_i u_1 + (1-\rho_i) u_0, \quad \mathbb{P}_{\delta_i,o}\text{-a.s. and in } L^1. \tag{2.1.5}$$

Note that Theorem 2.1.1 does not require (W_t) to be elliptic nor set restrictions on finite range jump transitions, technical assumptions often present in the literature. Further, Durrett [55, Theorem 2.5] yields a large class of IPS with spin-flip dynamics having an attractive graphical representation coupling to which Theorem 2.1.1 applies.

The proof of Theorem 2.1.1 makes use of a particular coupling construction of the random walk together with the sub-additive ergodic theorem. The coupling construction (given in Section 2.2) enables us to transfer monotonicity properties of the environment to a functional of (W_t). Informally, this functional counts the number of occupied sites the random walk

observes at its jump times, as a function of time. The monotonicity property of this functional together with the graphical representation of the environment naturally leads to a sub-additive structure which we use to obtain a SLLN by employing the sub-additive ergodic theorem (with limit equal to ρ_i as in (2.1.5)). This is the content of Theorem 2.3.2 below. As a last step, the SLLN in Theorem 2.3.2 is transferred into a SLLN for (W_t), whose proof is given in Section 2.3.1.

The restriction to the extremal starting configurations in Theorem 2.1.1 can in many cases be relaxed. For $m > 0$, let

$$V_m := \{(x,t) \in \mathbb{Z}^d \times [0,\infty) : \| x \|_1 < mt\} \tag{2.1.6}$$

be a cone of inclination m opening upwards in space-time. Let (ξ_t) be an IPS with an attractive graphical representation coupling and, for $\eta \in \Omega$, $i \in \{0,1\}$ and $T \in [0,\infty)$, define

$$\phi_i^{(m)}(\eta, T) := \widehat{P}\left(\exists (x,t) \in V_m \cap \mathbb{Z}^d \times [T,\infty) : \xi_t^{\eta}(x) \neq \xi_t^{\bar{i}}(x)\right).$$

Note that ξ is cone mixing, as defined in [8, Definition 1.1], if

$$\lim_{T\to\infty} \phi_1^{(m)}(\bar{0}, T) = 0, \quad \forall\, m \in [0,\infty). \tag{2.1.7}$$

Further, denote by $\bar{\nu}_0, \bar{\nu}_1 \in \mathcal{P}(\Omega)$ the "lower" and "upper" invariant measures to which (η_t) converges when initialised from $\bar{0}$ and $\bar{1}$ respectively. That is, $\bar{\nu}_0 = \lim_{t\to\infty} P^{\bar{0}}(\xi_t \in \cdot)$ and $\bar{\nu}_1 = \lim_{t\to\infty} P^{\bar{1}}(\xi_t \in \cdot)$. These limits exist and are invariant under (ξ_t) (see [82, Theorem III.2.3]). Lastly, recalling (2.1.3), we denote the convex hull of u_0 and u_1 by

$$U(u_0, u_1) := \text{conv}\,(u_0, u_1)\,.$$

Theorem 2.1.2. *Assume that ξ has an attractive graphical representation coupling. Let $i \in \{0,1\}$ and assume that for some $\epsilon, m > 0$ we have $U(u_0,u_1) \times \{1/\gamma\} \subset V_{m(1-\epsilon)}$ and*

$$\lim_{T\to\infty} \phi_i^{(m)}(\eta, T) = 0, \quad \textit{for } \bar{\nu}_i - a.e.\ \eta \in \Omega. \tag{2.1.8}$$

Then, for all $\nu \in \mathcal{P}(\Omega)$ such that $\nu \leq \bar{\nu}_i$ (if $i = 0$) or $\bar{\nu}_i \leq \nu$ (if $i = 1$),

$$\lim_{t\to\infty} \frac{1}{t} W_t = \rho_i u_1 + (1-\rho_i) u_0 \quad \mathbb{P}_{\nu,o}\textit{-a.s. and in } L^1.$$

Theorem 2.1.2 relaxes the assumption on the starting configuration in Theorem 2.1.1. Note that (2.1.8) is weaker than cone mixing (in the sense of (2.1.7)) and applies to IPS with non-uniform correlation structure. The proof of Theorem 2.1.2 uses a different coupling construction than that needed for the proof of Theorem 2.1.1 and is given in Section 2.3.2.

From the proof of Theorem 2.1.1, we also obtain certain large deviation estimates. These are presented in Section 2.3.3. In particular, we have the following theorem.

Theorem 2.1.3 (Large deviation estimates)**.** *Assume that ξ has an attractive graphical representation coupling. Further, assume $\rho_0 = \rho_1 =: \rho$ (with ρ_0 and ρ_1 as in Theorem 2.1.1). Consequently, (W_t) satisfies* (2.1.5) *$\mathbb{P}_{\mu,o}$-a.s with limiting speed $v := \rho u_1 + (1-\rho)u_0$, irrespectively of $\mu \in \mathcal{P}(\Omega)$. If, in addition, (W_t) has finite exponential moments, then for any $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that*

$$\mathbb{P}_{\mu,o}\left(\| W_t - tv \|_1 > \epsilon t\right) \leq \exp\left(-C(\epsilon)t\right), \quad \text{for all } \mu \in \mathcal{P}(\Omega).$$

The proof of Theorem 2.1.3 is given in Section 2.3.3, where we also prove some additional large deviation properties. The key observation for the proof is that the sub-additive structure obtained for the functional used in the proof of Theorem 2.1.1 yields one-sided large deviations estimates for this functional. The assumption that $\rho_0 = \rho_1$ implies two-sided large deviation estimates, however, for the same functional. An additional argument is needed in order to conclude large deviation estimates for (W_t). For this, we use the assumption that (W_t) has finite exponential moments.

Random walk on the contact process

One classical IPS having an attractive graphical representation coupling is the contact process $\xi = (\xi_t)_{t\geq 0}$. Given $\lambda \in (0,\infty)$, the contact process on $\mathbb{Z}^d$ with "infection rate" λ is defined via its local transition rates, which are given by

$$\eta \to \eta^x \text{ with rate } \begin{cases} 1, & \text{if } \eta(x) = 1, \\ \lambda \sum_{x\sim y} \eta(y), & \text{if } \eta(x) = 0. \end{cases}$$

Here, η^x is defined by $\eta^x(y) := \eta(y)$ for $y \neq x$, and $\eta^x(x) := 1 - \eta(x)$, and $\sum_{x\sim y}$ denotes the sum over nearest neighbours.

Much is known about the contact process; see [83, Chapter 1] for a thorough introduction. In particular, the empty configuration $\bar{0}$ is an absorbing state for the contact process. On the other hand, starting from the full configuration $\bar{1}$, the contact process evolves towards an equilibrium measure $\bar{\nu}_\lambda$, called the "upper invariant measure", which is stationary and ergodic with respect to (ξ_t). Further, there is a critical threshold $\lambda_c(d) \in (0,\infty)$, depending on the dimension d, such that $\bar{\nu}_\lambda = \delta_{\bar{0}}$ for $\lambda \in (0, \lambda_c(d)]$ and, for all $\lambda \in (\lambda_c(d), \infty)$, we have $\bar{\nu}_\lambda(\eta(o) = 1) > 0$. In particular, for $\lambda > \lambda_c(d)$, the contact process does not satisfy (2.1.7).

Theorem 2.1.4. *Consider the contact process on $\mathbb{Z}^d$, $d \geq 1$, and infection rate $\lambda \in (\lambda_c(d), \infty)$.*

a) *There exists $\rho = \rho(\lambda) \in [0,1]$ such that for all $\nu \in \mathcal{P}(\Omega)$ with $\bar{\nu}_\lambda \leq \nu$,*

$$\lim_{t\to\infty} \frac{1}{t} W_t = u_1\rho + (1-\rho)u_0 \quad \mathbb{P}_{\nu,o}\text{-a.s. and in } L^1. \tag{2.1.9}$$

b) *The function $\rho\colon (\lambda_c(d),\infty) \to [0,1]$, $\lambda \mapsto \rho(\lambda)$, is non-decreasing and right-continuous in λ. Moreover, if (W_t) has finite second moments, then*

$$\rho(\lambda) \in (0,1) \quad \textit{and} \quad \lim_{\lambda\to\infty} \rho(\lambda) = 1. \tag{2.1.10}$$

Theorem 2.1.4 extends the law of large numbers of [72], obtained for the nearest neighbour random walk on the supercritical contact process on $\mathbb{Z}$, to higher dimensions and beyond the nearest neighbour assumption.

Concerning the proof of Theorem 2.1.4, note that a) follows immediately from Theorem 2.1.1 and the graphical representation coupling of the contact process when started from $\bar{1}$. To extend this to any measure stochastically dominated by $\bar{\nu}_\lambda$, we prove that the contact process satisfies (2.1.8).

The function $\rho(\cdot)$ in Theorem 2.1.4b) is the same functional as considered in the proof of Theorem 2.1.1. That this is non-decreasing and right-continuous is not difficult to show and follows by monotonicity considerations. Most of the proof of Theorem 2.1.4b) goes about showing that (2.1.10) holds. Note that this result implies that the SLLN in (2.1.9) is non-trivial in the sense that the speed of (W_t) is neither u_0 nor u_1. For the proof of (2.1.10), we treat the two cases $d = 1$ and $d \geq 2$ separately. For $d = 1$ we extend an argument of [72] beyond nearest neighbour jumps. For $d \geq 2$ we use that the supercritical contact process survives in certain (tilted) space-time slabs together with the monotonicity properties of $\rho(\cdot)$. Section 2.4 is dedicated to the proof of Theorem 2.1.4.

2.1.4 Discussion

1. The SLLN for random walks on a 2-state IPS has been proven earlier by Avena, den Hollander, and Redig [8] under strong mixing assumptions on the environment, known as cone mixing. This has been extended to more general IPS by Redig and Völlering [100], however, still under a uniform mixing assumption similar to cone mixing.

   Theorem 2.1.1 in this paper yields an extension of the SLLN in [8] to random walks on IPS which are not cone mixing, but satisfy a monotonicity property. Indeed, instead of cone mixing, we assume that the IPS has an attractive graphical representation coupling and is started from a configuration where all sites are equal. Theorem 2.1.2 present sufficient mixing conditions for relaxing the restriction on the starting configuration.

   Contrary to [8] and [100], it is essential to the proof of Theorem 2.1.1 that the random walk only has two transition kernels. That is, at jump times, the random walk chooses one among two transition kernels, depending on the environment. It is not clear how to extend our argument to random walks having more than two transition kernels. In fact, there are examples showing that the monotonicity property crucial to our proof

(see Lemma 2.2.1) does not always hold for such systems already when the random walk depends on three states; see Holmes and Salisbury [76].

2. Important to the proof of Theorem 2.1.1 is the fact that a certain functional of the environment ξ and (W_t) is monotone in ξ. This functional counts, as a function of time, the number of occupied sites the random walk observes at the jump times of the random walk (see Section 2.2.1 for a definition). The monotonicity property of this functional is proven in Lemma 2.2.1. We note that this property has earlier been exploited by Holmes and Salisbury [76] to study monotonicity properties of random walks on i.i.d. *static* 2-state random environments.

   In several recent works on nearest neighbour RWDRE on $\mathbb{Z}$, e.g. Hilário, den Hollander, Sidoravicius, dos Santos, and Teixeira [71] and Huveneers and Simenhaus [77], monotonicity properties of the random walk have played an important role. Lemma 2.2.1 seems useful in order to extend their results to random walks on $\mathbb{Z}^d$ with more general transition kernels.

3. In Peres, Popov, and Sousi [93], sufficient conditions for general RWRE models to be transient were proven. In particular, [93, Proposition 1.4] implies that (W_t), as studied in this paper, is transient if it is elliptic and $d \geq 5$. If $u_1 = u_0 = 0$, and under weak moment assumptions, ([93, Theorem 1.2]) yields that (W_t) is transient when $d \geq 3$.

4. Theorem 2.1.2 can be extended to hold for measures different from $\bar{\nu}_0$ and $\bar{\nu}_1$, see the remark at the end of Section 2.3.2. In particular, the statement of Theorem 2.1.2 holds if (W_t) is elliptic and the dynamic environment is initialised from a measure $\mu \in \mathcal{P}(\Omega)$ which satisfies (2.1.8) (with μ replacing $\bar{\nu}_i$). As an example, the contact process started from any measure μ stochastically dominating a non-trivial Bernoulli product measure satisfies (2.1.8) with $i = 1$, as follows from the proof of Theorem 2.1.4a).

5. Large deviation estimates, such as those obtained in Theorem 2.1.3, have previously been obtained in Avena, den Hollander, and Redig [7] and Redig and Völlering [99]. In [99], explicit estimates are derived, but under strong mixing assumptions. Closest to this paper is [7], where an annealed as well as a quenched large deviation principle is derived, however, restricted to nearest neighbour random walks on attractive spin-flip dynamics on $\mathbb{Z}$. Note that, Theorem 2.1.3 extends the estimates in [7, Proposition 2.5] to hold throughout the cone mixing regime and for random walks on $\mathbb{Z}^d$ with $d \geq 1$.

6. The proof of Theorem 2.1.4b) can be adapted to more general dynamics such as general additive and attractive spin-flip systems in the supercritical regime. At least when $d \geq 2$, our proof seems to transfer to this case using that such processes also survives in (tilted) space-time slabs, as shown in Bezuidenhout and Gray [29].

   Non-triviality of the speed for RWDRE as in Theorem 2.1.4b) has previously been proven by dos Santos [104] for a random walk on the exclusion process, by employing multi-scale arguments. This argument can perhaps be adapted to yield a different proof of $\rho(\lambda) \in (0, 1)$ for the random walk on the contact process.

2.2 Construction

2.2.1 Coupling construction of the random walk

In this section, we describe a particular coupling construction of the random walk. This construction is at the heart of the argument for the proof of Theorem 2.1.1, as it yields an important monotonicity property; see Lemma 2.2.1 below.

To construct the evolution of the random walk, let (N_t) be a Poisson jump process with jump rate $\gamma \in (0, \infty)$ and with inverse process $(J_k)_{k\geq 0}$. We call these times the jump times of the random walk. Essential to our approach and for the proof of Theorem 2.1.1 is the introduction of two independent sequences of i.i.d. UNIF[0,1] random variables, $O = (O_j)_{j\geq 1}$ and $V = (V_j)_{j\geq 1}$. Here O stands for occupied, whereas V stands for vacant.

Given $\alpha = (\alpha(i,z))_{i\in\{0,1\},z\in\mathbb{Z}^d}$ as introduced in (2.1.2), enumerate $\mathbb{Z}^d = \{z_1, z_2, \ldots\}$ and define $(p_1(m))_{m=0}^{\infty}$ and $(p_0(n))_{n=0}^{\infty}$ by setting $p_1(0) = p_0(0) = 0$, and let

$$p_1(m) = \frac{1}{\gamma}\sum_{j=1}^{m}\alpha(1,z_j) \quad \text{and} \quad p_0(n) = \frac{1}{\gamma}\sum_{j=1}^{n}\alpha(0,z_j), \quad n,m\in\mathbb{N}\,.$$

For convenience, we shall assume that the maximum in (2.1.2) is attained by both $\alpha(1,\cdot)$ and $\alpha(0,\cdot)$ by adapting the values for $\alpha(0,o)$ and $\alpha(1,o)$ appropriately. Note that this does not affect the behaviour of the random walk. Moreover, we can arrange that

$$\lim_{m\to\infty} p_1(m) = \lim_{n\to\infty} p_0(n) = 1.$$

Given a fixed environment $(\xi_t)_{t\geq 0}$ we next define the *discrete-time* random walk $S = (S_k)_{k\in\mathbb{N}}$. For this, we also introduce the functional $(\rho(k,\xi))_{k\in\mathbb{N}}$, taking values in $\mathbb{N}$. Let $S_0 = o$ and $\rho(0,\xi) = 0$ and, given S_k and $\rho(k,\xi)$, define S_{k+1} and $\rho(k+1)$ iteratively by

$$\rho(k+1,\xi) = \rho(k,\xi) + \xi_{J_k}(S_k)$$

and

$$\begin{aligned} S_{k+1} = S_k &+ (1-\xi_{J_k}(S_k))\sum_{n=1}^{\infty} 1_{[p_0(n-1),p_0(n))}(V_{k+1-\rho(k,\xi)})z_n \\ &+ \xi_{J_k}(S_k)\sum_{m=1}^{\infty} 1_{[p_1(m-1),p_1(m))}(O_{\rho(k,\xi)+1})z_m. \end{aligned} \tag{2.2.1}$$

Note that, in (2.2.1), since (ξ_t) is càdlàg and independent of $(J_k)_{k\geq 0}$, we have that $\xi_{J_k^-}(S_k)$ is a.s. equal to $\xi_{J_k}(S_k)$ for all k. Further, we have that

$$\rho(k,\xi) = \sum_{i=0}^{k-1}\xi_{J_i}(S_i), \quad k\in\mathbb{N},$$

counts the number of occupied sites the (discrete-time) random walk (S_k) has observed at the first k jump times. The (continuous-time) random walk (W_t) with transition kernels $\alpha(i,\cdot)$, $i \in \{0,1\}$, is obtained by setting $W_t = S_{N_t}$, as follows by the construction, using that (ξ_t) is right-continuous. The continuous version of the (rescaled) $\rho(n,\xi)$ is given by

$$\rho_t(\xi) := \frac{1}{\gamma}\rho(N_t,\xi), \quad t \in [0,\infty).$$

2.2.2 Generalisation of the coupling construction

In the proof of Theorem 2.1.4b) in Section 2.4.3 we carry out a domination argument. For this purpose, we consider a generalisation of the construction in Section 2.2.1. For any $t \in [0,\infty)$, denote by $\xi_{[0,t]}$ the space-time environment from time 0 to time t. Furthermore, consider a family of Boolean functions (f_k), measurable with respect to the σ-algebra $\sigma(\xi_{[0,J_k]}, N, O, V)$. The construction in Section 2.2.1 can then be generalised in the same manner by setting $S_0 = 0$ and $\rho^{(d)}(0,\xi) = 0$, and iteratively,

$$\begin{aligned}
\rho^{(d)}(k+1,\xi) &= f_{k+1} + \rho^{(d)}(k,\xi)\\
S_{k+1} &= S_k + (1-f_k)\sum_{n=1}^{\infty} 1_{[p_0(n-1),p_0(n))}(V_{k+1-\rho^{(d)}(k,\xi)})z_n\\
&\qquad + f_k \sum_{m=1}^{\infty} 1_{[p_1(m-1),p_1(m))}(O_{\rho^{(d)}(k,\xi)+1})z_m.
\end{aligned}$$

Thus, in this more general setup, $\rho^{(d)}(k,\xi) = \sum_{i=0}^{k-1} f_i$, $k \in \mathbb{N}$. Note that, we recover (2.2.1) when $f_k = \xi_{J_k}(S_k)$. In Section 2.4.3, we consider cases where $f_k = \mathbb{1}_{R_k}\xi_{J_k}(S_k)$ for some event $R_k \in \sigma(\xi_{[0,J_k]}, N, O, V)$, for which we readily see that $\rho^{(d)}(k,\xi) \le \rho(k,\xi)$.

2.2.3 Monotonicity

The construction in the previous two subsections provides us with a coupling that keeps track of the number of occupied sites the random walk has observed at any given time. The key property of the coupling construction in Subsection 2.2.1 is a monotonicity property, which we state next. For this, denote the elements of $D_\Omega[0,\infty)$ by $(\eta_t)_{t\ge0}$ and write $(\eta_t)_{t\ge0} \le (\omega_t)_{t\ge0}$ if $\eta_t(x) \le \omega_t(x)$ for all $x \in \mathbb{Z}^d$ and $t \in [0,\infty)$.

Lemma 2.2.1 (Monotonicity of particle density)**.** *For any $\eta = (\eta_t)_{t\ge0}$ and $\omega = (\omega_t)_{t\ge0}$ contained in $D_\Omega[0,\infty)$ satisfying $(\eta_t)_{t\ge0} \le (\omega_t)_{t\ge0}$;*

$$\rho_s(\eta) \le \rho_s(\omega) \quad \forall\, s \in [0,\infty). \tag{2.2.2}$$

Proof. Consider two (discrete-time) random walkers $(S^1_n)_{n\geq 0}$ and $(S^2_m)_{m\geq 0}$, both constructed as in Subsection 2.2.1 using the same realisation of $((N_t),(O_k),(V_k))$ and having identical transition kernels, seeing environment (η_t) and (ω_t), respectively. We claim that

$$\rho(n,\eta) \leq \rho(n,\omega) \quad \text{for all } n \geq 0. \tag{2.2.3}$$

To see this, we argue by induction. First note that, by definition, we have that $\rho(0,\eta) = \rho(0,\omega)$. As the induction hypothesis, we assume that $\rho(k,\eta) \leq \rho(k,\omega)$ for some $k \geq 0$.

In the case that $\rho(k,\eta) < \rho(k,\omega)$, we have $\rho(k{+}1,\eta) \leq \rho(k,\eta){+}1 \leq \rho(k,\omega) \leq \rho(k{+}1,\omega)$. Thus, the induction step holds in this case. In the other case, where $\rho(k,\eta) = \rho(k,\omega)$, we have by construction that

$$\begin{aligned} S^1_k = &\sum_{i=1}^{\rho(k,\eta)} \big[\sum_{m=1}^{\infty} 1_{[p_1(m-1),p_1(m))}(O_i) z_m \big] \\ &+ \sum_{j=1}^{k-\rho(k,\omega)} \big[\sum_{n=1}^{\infty} 1_{[p_0(n-1),p_0(n))}(V_j) z_n \big] = S^2_k. \end{aligned}$$

Since $\eta \leq \omega$, it holds that $\eta_{J_k}(S^1_k) \leq \omega_{J_k}(S^2_k)$, which in particular implies that $\rho(k+1,\eta) \leq \rho(k+1,\omega)$. Hence, also the second case satisfy the induction step and consequently (2.2.3) holds. Finally, by replacing n in (2.2.3) by N_t and multiplying by γ^{-1}, this proves (2.2.2). □

Remark 2.2.1. For any attractive IPS ξ, Lemma 2.2.1 transfers to an almost sure statement with respect to the annealed measure (and thus also the quenched measure). In this case, for every $\mu,\nu \in \mathcal{P}(\Omega)$ with $\mu \leq \nu$ there exists a coupling $\widehat{\mathbb{P}}$ of $\mathbb{P}_{\mu,o}$, $\mathbb{P}_{\nu,o}$ and ξ such that $\widehat{\mathbb{P}}(\rho_t(\xi^\mu) \leq \rho_t(\xi^\nu)) = 1$. The existence of such a coupling follows by [82, Theorem II.2.4] and the construction above.

Remark 2.2.2. The coupling construction in Section 2.2.1 for random walks on a *dynamic* random environment is to our knowledge new. Apparently the same coupling construction has previously been used to study monotonicity properties for certain specific random walks in *static* random environment by Holmes and Salisbury [76]. Lemma 2.2.1 can be seen as an immediate extension of [76, Theorem 4.1i)].

Remark 2.2.3. Lemma 2.2.1 can be extended to hold in certain cases under the general construction considered in Section 2.2.2. For this, the functions (f_k) need to be monotone in the sense that (for $\eta \leq \xi$ as above), if $\rho^{(d)}(k,\eta) = \rho^{(d)}(k,\xi)$, then $\rho^{(d)}(k{+}1,\eta) \leq \rho^{(d)}(k{+}1,\xi)$.

2.3 Proofs

2.3.1 Proof of Theorem 2.1.1

In this subsection we first present the proof of Theorem 2.1.1 for the case when the environment is started from all sites occupied. Essentially the same proof can be applied to the case where the environment is started from all sites vacant. We comment at the end of this subsection on which changes to the proof are necessary for this case.

The main idea is to show that $\rho_t(\xi)$ is sub-additive, by using that ξ has an attractive graphical representation coupling and Lemma 2.2.1. Subsequently, the subadditive ergodic theorem applied to $\rho_t(\xi)$ yields that $t^{-1}\rho_t(\xi)$ converges towards a deterministic constant. This, in turn, identifies the limiting speed.

Let ξ be an IPS with an attractive graphical representation coupling, $\widehat{P}$, where by $\mathcal{I}$ we denote the corresponding collection of Poisson point processes. In order to formulate the proof, we have to be more specific about $\mathcal{I}$ and write $\mathcal{I}$ as a countable set of Poisson point processes indexed by the lattice $\mathbb{Z}^d$, $\mathcal{I} = \big((\mathcal{X}^1_y)_{y\in\mathbb{Z}^d}, (\mathcal{X}^2_y)_{y\in\mathbb{Z}^d}, \dots\big)$, where every $\mathcal{X}^i_y$ is an (independent) Poisson point process on $[0,\infty)$. Further, for $x \in \mathbb{Z}^d$ and $t \in [0,\infty)$ let $\Theta_{x,t}$ be the space-time shift operator on the realisations of $\mathcal{I}$:

$$\Theta_{x,t}\left((\mathcal{X}^1_{s,y}), (\mathcal{X}^2_{s,y}), \dots\right)_{s\in[0,\infty), y\in\mathbb{Z}^d} = \left((\mathcal{X}^1_{s+t,y+x}), (\mathcal{X}^2_{s+t,y+x}), \dots\right)_{s\in[0,\infty), y\in\mathbb{Z}^d}.$$

For the contact process, the set $\mathcal{I}$ as considered in Section 2.4, consists of the Poisson processes $\big\{H_x, I_{x,e}\colon x, e \in \mathbb{Z}^d, |e| = 1\big\}$, and $\Theta_{x,t}$ shifts crosses and arrows in space by x and in time by t. Further, in [55, Theorem 2.5], $\mathcal{I}$ is the set of birth and death events (which in [55] are denoted by $\big\{T^{y,i}_n\colon n \geq 0, y \in \mathbb{Z}^d, i \in \{0,1\}\big\}$).

To emphasise the graphical representation, we write $\rho_t(\xi) = \rho_t(\eta, \mathcal{I}, N, O, V)$ for $\xi_0 = \eta$ and let $\widehat{\mathbb{P}}$ denote the joint law of the graphical construction coupling and N, O and V. Note that, by Lemma 2.2.1 and (2.1.4), for any $\eta \in \Omega$,

$$\rho_t(\eta, \mathcal{I}, N, O, V) \leq \rho_t(\bar{1}, \mathcal{I}, N, O, V), \quad \widehat{\mathbb{P}} - a.s. \tag{2.3.1}$$

Moreover, let

$$\begin{aligned} N^{(s)} &= (N^{(s)}_t)_{t\geq 0} := (N_{t+s} - N_s)_{t\geq 0}, \\ O^{(s)} &= (O^{(s)}_n)_{n\geq 0} := (O_{n+\gamma\rho_s(\xi)})_{n\geq 0}, \\ V^{(s)} &= (V^{(s)}_n)_{n\geq 0} := (V_{n+N_s-\gamma\rho_s(\xi)})_{n\geq 0}. \end{aligned}$$

Similar to $\Theta_{x,t}$ we introduce the space-time shift $\theta_{x,t}$ on $\Omega^{[0,\infty)}$ by

$$(\theta_{x,t}\xi_s)_{s\geq 0} = (\theta_x \xi_{s+t})_{s\geq 0}$$

with space-shift θ_x introduced in (2.1.1). Next, define the continuous-time process $(X_{t,s})_{0\le t\le s}$ by

$$X_{t,s} := \rho_{s-t}(\bar{1}, \Theta_{W_t,t}\mathcal{I}, N^{(t)}, O^{(t)}, V^{(t)}), \quad \text{for } 0 \le t \le s.$$

Note that, if ξ is such that $\xi_0 = \bar{1}$, then

$$X_{0,s} = \rho_s(\bar{1}, \mathcal{I}, N, O, V) = \rho_s(\xi), \qquad s \in [0,\infty).$$

In the next statement and in the proceedings, for $\mu \in \mathcal{P}(\Omega)$ and $x \in \mathbb{Z}^d$, we write $\widehat{\mathbb{P}}_{\mu,x}$ to emphasise the starting configuration of both ξ and (W_t).

Lemma 2.3.1 (Sub-additivity)**.** *The process $(X_{t,s})_{0\le t\le s}$ has the following properties.*

i) *$X_{0,0} = 0$ and for all $t, s \in [0,\infty)$: $X_{0,t+s} \le X_{0,t} + X_{t,t+s}$.*

ii) *For all $t \in (0,\infty)$, $(X_{t,k+t})_{k\ge1}$ has the same distribution as $(X_{0,k})_{k\ge1}$.*

iii) *For all $t \in [0,\infty)$, $(X_{(k-1)t,kt})_{k\ge1}$ is a sequence of i.i.d. random variables.*

iv) *For all $t \in [0,\infty)$, the expectation $\widehat{\mathbb{E}}_{\delta_{\bar{1}},o}[X_{0,t}]$ is finite and $X_{0,t} \ge 0$.*

Proof. Fix $t, s \in [0,\infty)$ and recall (2.3.1). By the Markov property of the Poisson point process $\mathcal{I}$, we have that

$$\begin{aligned} X_{0,t+s} &= \rho_{t+s}(\bar{1}, \mathcal{I}, N, O, V) \\ &= \rho_t(\bar{1}, \mathcal{I}, N, V, O) + \rho(s, \theta_{W_t,t}\xi, \Theta_{W_t,t}\mathcal{I}, N^{(t)}, O^{(t)}, V^{(t)}) \\ &\le \rho_t(\bar{1}, \mathcal{I}, N, O, V) + \rho(s, \bar{1}, \Theta_{W_t,t}\mathcal{I}, N^{(t)}, O^{(t)}, V^{(t)}) \\ &= X_{0,t} + X_{t,t+s}. \end{aligned}$$

Properties i) and ii) follow from the equality $X_{0,0} = 0$, the translation invariance in (2.1.1) and the equality in distribution $X_{0,s} = X_{t,t+s}$. Moreover, iii) follows by the Markov property of ξ and the graphical representation. Lastly, property iv) holds trivially, since $X_{0,t}$ is non-negative by definition and since $X_{0,t} \le N_t$. □

Lemma 2.3.1 enables us to prove the SLLN for the process $\rho_t(\xi)$ when ξ is initialised at time 0 by $\bar{1}$, by applying the subadditive ergodic theorem.

Theorem 2.3.2 (Law of large numbers for $\rho_t(\xi)$)**.** *Assume that ξ has an attractive graphical representation coupling. There exists $\rho_1 \in [0,1]$ such that*

$$\lim_{t\to\infty} \frac{1}{t}\rho_t(\xi) = \rho_1 \qquad \widehat{\mathbb{P}}_{\delta_{\bar{1}},o}\text{-a.s. and in } L^1. \tag{2.3.2}$$

Moreover, $\rho_1 = \inf_{t\ge1} t^{-1}\widehat{\mathbb{E}}_{\delta_{\bar{1}},o}(\rho_t(\xi))$.

Proof. By Lemma 2.3.1 we know that X satisfies property a)-d) of [82, Theorem VII.2.6]. In particular, by the independence property in Lemma 2.3.1iii), the process is stationary and ergodic. Hence, the conclusion of Theorem 2.3.2 holds when t takes integer values. This can easily be extended to continuous t. Indeed, for any $t \in (0, \infty)$ we have that

$$X_{0,\lfloor t \rfloor} \leq X_{0,t} \leq X_{0,\lceil t \rceil}. \tag{2.3.3}$$

In particular, by dividing by t in (2.3.3) and taking $t \to \infty$ (as in (2.3.2)), we conclude the proof. □

We are now in position to present the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. By the construction in Section 2.2.1, W_t can be written as

$$W_t = \sum_{i=1}^{\rho(N_t,\xi)} \left(\sum_{m=1}^{\infty} 1_{[p_1(m-1),p_1(m))}(O_i) z_m \right) + \sum_{j=1}^{N_t - \rho(N_t,\xi)} \left(\sum_{n=1}^{\infty} 1_{[p_0(n-1),p_0(n))}(V_j) z_n \right).$$

Dividing by $t > 0$ gives

$$\begin{aligned} \frac{W_t}{t} =& \frac{\rho(N_t,\xi)}{t} \frac{1}{\rho(N_t,\xi)} \sum_{i=1}^{\rho(N_t,\xi)} \left(\sum_{m=1}^{\infty} 1_{[p_1(m-1),p_1(m))}(O_i) z_m \right) \\ &+ \frac{N_t - \rho(N_t,\xi)}{t} \frac{1}{N_t - \rho(N_t,\xi)} \sum_{j=1}^{N_t - \rho(N_t,\xi)} \left(\sum_{n=1}^{\infty} 1_{[p_0(n-1),p_0(n))}(V_j) z_n \right). \end{aligned}$$

Taking the limit as $t \to \infty$ and applying Theorem 2.3.2 we obtain

$$\lim_{t\to\infty} \frac{1}{t} W_t = \rho_1 u_1 + (1 - \rho_1) u_0 \quad \mathbb{P}_{\delta_{\bar{1}},o}\text{-a.s. and in } L^1,$$

where ρ_1 is as in Theorem 2.3.2 and $u_0, u_1 \in \mathbb{R}^d$ are as in (2.1.3). This proves Theorem 2.1.1 for the case when the environment is started from all sites equal to 1.

We next comment on the changes necessary in the argument for proving Theorem 2.1.1 when started from all sites equal to 0. For this case we can define the process $(Y_{t,s})_{0 \leq t \leq s}$ given by

$$Y_{t,s} := \rho(s - t, \bar{0}, \Theta_{W_t,t}\mathcal{I}, N^{(t)}, O^{(t)}, V^{(t)}) \text{ for } 0 \leq t \leq s.$$

By the same arguments as in Lemma 2.3.1 we can prove that $-Y$ is a sub-additive process satisfying property ii) and iii) as in Lemma 2.3.1. Moreover, since $Y_{0,t}$ is dominated by N_t it follows that $\widehat{\mathbb{E}}_{\delta_{\bar{0}},o}[Y_{0,t}] \leq \widehat{\mathbb{E}}_{\delta_{\bar{0}},o}[N_t] = t$. This is sufficient in order to apply [82, Theorem VII.2.6]. By a literal adaptation of the proof under $\widehat{\mathbb{P}}_{\delta_{\bar{1}},o}$ above we obtain

$$\lim_{t\to\infty} \frac{1}{t} W_t = \rho_0 u_1 + (1 - \rho_0) u_0 \quad \mathbb{P}_{\delta_{\bar{0}},o}\text{-a.s. and in } L^1,$$

where ρ_0 is the limit in Theorem 2.3.2 when $\widehat{\mathbb{P}}_{\delta_{\bar{1}},o}$ is replaced by $\widehat{\mathbb{P}}_{\delta_{\bar{0}},o}$. This completes the proof of Theorem 2.1.1. □

2.3.2 Proof of Theorem 2.1.2

In this subsection we present the proof of Theorem 2.1.2. The presentation is inspired by the proof of [72, Proposition 3.3] (see also Remark 3.4 therein), and the proof of Theorem 2.1.2 is an extension of their proof to higher dimensions. We only provide the proof when $i = 1$. The proof for the case $i = 0$ is analogous.

Proof of Theorem 2.1.2. We start with the construction of the random walk. Let $U := (U_k)_{k\in\mathbb{N}_0}$ be an i.i.d. sequence of UNIF$[0,1]$ random variables, independent of the jump process $N = (N_t)_{t\geq 0}$. Set $S_0^{(U)} := 0$ and, recursively for $k \geq 0$,

$$S_{k+1}^{(U)} := S_k^{(U)} + (1 - \xi_{J_k}(S_k^{(U)})) \sum_{n=1}^{\infty} 1_{[p_0(n-1),p_0(n))}(U_k) z_n + \xi_{J_k}(S_k^{(U)}) \sum_{m=1}^{\infty} 1_{[p_1(m-1),p_1(m))}(U_k) z_m,$$

and let $W_t^{(U)} := S_{N_t}^{(U)}$ and $\rho_t^{(U)} = \sum_{k=0}^{N_t} \xi_{J_k}(S_k)$. Clearly $W_t^{(U)}$ and W_t (as constructed in Section 2.2.1) are equal in distribution, and similarly for $\rho_t^{(U)}$ and ρ_t, and hence onwards we do not distinguish them and write W_t and ρ_t for both processes.

Let $N^{(\bar\nu_1)}$, $U^{(\bar\nu_1)}$ and $N^{(\delta_{\bar 1})}$, $U^{(\delta_{\bar 1})}$ be independent copies of U and N and denote by $\widehat{\mathbb{P}}$ the joint law of $\widehat{P}, N^{(a)}, U^{(a)}$, $a \in \{\bar\nu_1, \delta_{\bar 1}\}$. Then $W^{(\bar\nu_1)} := W(\xi^{(\bar\nu_1)}, N^{(\bar\nu_1)}, U^{(\bar\nu_1)})$ and $\rho^{(\bar\nu_1)} := \rho(\xi^{(\bar\nu_1)}, N^{(\bar\nu_1)}, U^{(\bar\nu_1)})$ under $\widehat{\mathbb{P}}$ have the same law as W and ρ under $\widehat{\mathbb{P}}_{\bar\nu_1,o}$. Similarly, $W^{(\delta_{\bar 1})} := W(\xi^{(\delta_{\bar 1})}, N^{(\delta_{\bar 1})}, U^{(\delta_{\bar 1})})$ and $\rho^{(\delta_{\bar 1})} := \rho(\xi^{(\delta_{\bar 1})}, N^{(\delta_{\bar 1})}, U^{(\delta_{\bar 1})})$ have the same law as W and ρ under $\mathbb{P}_{\delta_{\bar 1},o}$. Further, for $T > 0$ fixed, let $\hat N = (\hat N_s)_{s\geq 0}$ and $\hat U = (\hat U_n)_{n\in\mathbb{N}}$ be defined by

$$\hat U_n := \begin{cases} U_n^{(\delta_{\bar 1})} & \text{if } n \leq N_T^{(\delta_{\bar 1})}; \\ U_n^{(\bar\nu_1)} & \text{otherwise,} \end{cases}$$

$$\hat N_s := \begin{cases} N_s^{(\delta_{\bar 1})} & \text{if } s \leq T; \\ N_T^{(\delta_{\bar 1})} + N_s^{(\bar\nu_1)} - N_T^{(\bar\nu_1)} & \text{otherwise.} \end{cases}$$

It is clear that $\hat W := W(\xi^{(\delta_{\bar 1})}, \hat N, \hat U)$ and $\hat\rho := \rho(\xi^{(\delta_{\bar 1})}, \hat N, \hat U)$ have the same laws as $W^{(\delta_{\bar 1})}$ and $\rho^{(\delta_{\bar 1})}$. Furthermore, $\hat N$ and $N^{(\bar\nu_1)}$ are independent up to time T, and thus the jump times of $W^{(\bar\nu_1)}$ and $\hat W$ are independent in the time interval $[0, T]$. By construction, for times later than T, the jumping times of $W^{(\bar\nu_1)}$ and $\hat W$ are the same.

Next, let $\epsilon, m > 0$ be such that $U(u_0, u_1) \times \{1/\gamma\} \subset V_{m(1-\epsilon)}$ and

$$\lim_{T\to\infty} \phi_1^{(m)}(\eta, T) = 0, \qquad \text{for } \bar\nu_1\text{-a.e. } \eta \in \Omega. \tag{2.3.4}$$

For $T \geq 0$, define the event

$$D_T := \{(W_t^{(\bar{\nu}_1)}, t) \in V_{m(1-\epsilon)},\ \forall\, t \geq T\}$$

and let

$$\Gamma_T := D_T \cap \{\xi_s^{(\bar{\nu}_1)}(x) = \xi_s^{(\delta_{\bar{1}})}(x),\ \forall\, (x,s) \in V_m \cap \mathbb{Z}^d \times [T,\infty)\}.$$

Note that, since $\widehat{\mathbb{P}}(\Gamma) \geq 1 - \Big(\widehat{\mathbb{P}}(D_T^c) + \int \phi_1^{(m)}(\eta, T)\bar{\nu}_1(d\eta)\Big)$, by (2.3.4) and since $U(u_0,u_1)\times \{1/\gamma\} \subset V_{m(1-\epsilon)}$, it holds that $\lim_{T\to\infty} \widehat{\mathbb{P}}(\Gamma_T) = 1$. Furthermore, by stationarity under $\bar{\nu}_1$, and since $\hat{N}_T$ is independent of $(W_t^{(\bar{\nu}_1)})$ and (ξ_t), we have that

$$\begin{aligned}
\widehat{\mathbb{P}}\left(\lim_{t\to\infty} t^{-1}\rho_t^{(\bar{\nu}_1)} = \rho_1\right) &= \widehat{\mathbb{P}}\left(\lim_{t\to\infty} t^{-1}\rho_t^{(\bar{\nu}_1)} = \rho_1 \mid N_T^{(\bar{\nu}_1)} = \hat{N}_T = 0\right) \\
&\geq \widehat{\mathbb{P}}\left(\left\{\lim_{t\to\infty} t^{-1}\rho_t^{(\bar{\nu}_1)} = \rho_1\right\} \cap \Gamma_T \mid N_T^{(\bar{\nu}_1)} = \hat{N}_T = 0\right) \\
&= \widehat{\mathbb{P}}\left(\left\{\lim_{t\to\infty} t^{-1}\rho_t^{(\delta_{\bar{1}})} = \rho_1\right\} \cap \Gamma_T \mid N_T^{(\bar{\nu}_1)} = \hat{N}_T = 0\right) \\
&= \widehat{\mathbb{P}}\left(\Gamma_T \mid N_T^{(\bar{\nu}_1)} = \hat{N}_T = 0\right) = \widehat{\mathbb{P}}\left(\Gamma_T\right) \xrightarrow{T\to\infty} 1.
\end{aligned}$$

Hence $\widehat{\mathbb{P}}\left(\lim_{t\to\infty} t^{-1}\rho_t^{(\bar{\nu}_1)} = \rho_1\right) = 1$. From this and the monotonicity property obtained in Lemma 2.2.1, and by arguing as in the proof of Theorem 2.1.1, we conclude the proof. □

Remark 2.3.1. The last paragraph of the proof of Theorem 2.1.2 is based on [72, Remark 3.4]. Following the proof of [72, Proposition 3.3], assuming that (W_t) is elliptic, this part can be extended to hold for any measure $\mu \in \mathcal{P}(\Omega)$ for which (2.1.8) holds, with $\bar{\nu}_1$ replaced by μ. For this, since $W_T^{(\mu)} \in [-mT, mT]^d$ on Γ_T, we have that

$$\begin{aligned}
&\widehat{\mathbb{P}}\left(\lim_{t\to\infty} t^{-1}W_t^{(\mu)} = v_1 \mid \Gamma_T\right) \\
&= \sum_{x\in[-mT,mT]^d} \widehat{\mathbb{P}}\left(\lim_{t\to\infty} t^{-1}W_t^{(\mu)} = v_1 \mid \Gamma_T, W_T^{(\mu)} = x\right) \widehat{\mathbb{P}}\left(W_T^{(\mu)} = x \mid \Gamma_T\right).
\end{aligned}$$

In particular, it is sufficient to show that

$$\widehat{\mathbb{P}}\left(\lim_{t\to\infty} t^{-1}W_t^{(\mu)} = v_1 \mid \Gamma_T, W_T^{(\mu)} = x\right) = 1,\ \forall\, x \in [-mT, mT]^d. \tag{2.3.5}$$

To prove the latter equation, use the ellipticity assumption to construct events A_x, $x \in [-mT, mT]^d$, having the following properties: $A_x \subset \{\hat{W}_T = x\}$ and A_x is independent of (ξ_t) and $(W_t^{(\mu)})$. Conclude (2.3.5) by first conditioning on A_x and then noting that, under $\Gamma_T \cap \{W_T^{(\mu)} = x\} \cap A_x$, we can replace $\{\lim_{t\to\infty} t^{-1}W_t^{(\mu)}\}$ by $\{\lim_{t\to\infty} t^{-1}\hat{W}_t\}$.

2.3.3 Large deviations properties and proof of Theorem 2.1.3

We constructed in Lemma 2.3.1 an independent sub-additive process $(X_{t,s})_{0\le t\le s}$. Such processes are well known to satisfy large deviation properties, see e.g. Grimmett [65]. In particular, we have the following large deviation estimates for $\rho_t(\xi)$.

Theorem 2.3.3 (Large deviation estimates for $\rho_t(\xi)$)**.** *Assume that ξ has an attractive graphical representation coupling. Then, for any $\epsilon > 0$, there exists $R_i(\epsilon) > 0$, $i \in \{0,1\}$, such that*

$$\widehat{\mathbb{P}}_{\delta_{\bar{1}},o}\left(\rho_t(\xi) > t(\rho_1+\epsilon)\right) \le \exp\left(-tR_1(\epsilon)\right), \quad \textit{for all } t>0;$$
$$\widehat{\mathbb{P}}_{\delta_{\bar{0}},o}\left(\rho_t(\xi) < t(\rho_0-\epsilon)\right) \le \exp\left(-tR_0(\epsilon)\right), \quad \textit{for all } t>0.$$

Proof of Theorem 2.3.3. We follow the proof of Theorem 3.2 in [66] and give the proof with respect to $\delta_{\bar{1}}$ only. The proof with respect to $\delta_{\bar{0}}$ follows analogously. Let $\epsilon > 0$ and choose $T > 1$ such that

$$g_T := \frac{1}{T}\widehat{\mathbb{E}}_{\delta_{\bar{1}},o}\big[X_{0,T}\big] \le \rho_1 + \epsilon. \tag{2.3.6}$$

We first consider the case when $t = rT$ for some $r \in \mathbb{N}$. Using the properties from Lemma 2.3.1, we have

$$\widehat{\mathbb{P}}_{\delta_{\bar{1}},o}\left(X_{0,t} \ge t(\rho_1+2\epsilon)\right) \le \widehat{\mathbb{P}}_{\delta_{\bar{1}},o}\left(Q_1+\cdots+Q_r \ge t(\rho_1+2\epsilon)\right),$$

where $Q_i = X_{(i-1)T,iT}$. Moreover, the Q_i's are i.i.d., and, since Q_1 is dominated by a Poisson random variable, $\widehat{\mathbb{E}}_{\delta_{\bar{1}},o}\big[e^{zQ_1}\big] < \infty$ for all $z \in \mathbb{R}$.

Next, let $Z_i = Q_i - \widehat{\mathbb{E}}_{\delta_{\bar{1}},o}(Q_i)$, and note that (by (2.3.6))

$$\widehat{\mathbb{P}}_{\delta_{\bar{1}},o}\left(Q_1+\cdots+Q_r \ge t(\rho_1+2\epsilon)\right) \le \widehat{\mathbb{P}}_{\delta_{\bar{1}},o}\left(Z_1+\cdots+Z_r \ge rT\epsilon\right).$$

Further, applying the exponential Chebyshev inequality implies that for each $y \ge 0$,

$$\widehat{\mathbb{P}}_{\delta_{\bar{1}},o}\left(Z_1+\cdots+Z_r \ge rT\epsilon\right) \le e^{-rT\epsilon y}\widehat{\mathbb{E}}_{\delta_{\bar{1}},o}\big[e^{yZ_1}\big]^r.$$

Since $\widehat{\mathbb{E}}_{\delta_{\bar{1}},o}\big[e^{yZ_1}\big] < \infty$ for all $y \le 1$ and $\widehat{\mathbb{E}}_{\delta_{\bar{1}},o}\big[Z_1\big] = 0$, there exists a constant $c = c(1) > 0$ such that $\widehat{\mathbb{E}}_{\delta_{\bar{1}},o}\left[e^{yZ_1}\right] \le 1 + cy^2$ for $y \in [0,1]$. Hence, by setting $y = \frac{\epsilon}{2c}$, for r large,

$$\begin{aligned}\widehat{\mathbb{P}}_{\delta_{\bar{1}},o}\left(Z_1+\cdots+Z_r \ge rT\epsilon\right) &\le \exp\big[-rT\epsilon y + r\log(1+cy^2)\big]\\ &\le \exp\big[-rT\epsilon y + rcy^2\big]\\ &= \exp\big[-t\frac{\epsilon^2}{4Tc}\big].\end{aligned}$$

This completes the proof for the case when t is a multiple of T.

For general values of t, write $t = rT + s$, where $0 \leq s < T$, and note that

$$X_{0,t} \leq X_{0,rT} + X_{rT,t},$$

where the last two variables are independent. Further, notice that we can bound

$$\widehat{\mathbb{P}}_{\delta_{\bar{1}},o}\left(X_{0,t} \geq t(\rho_1+\epsilon)\right) \leq \widehat{\mathbb{P}}_{\delta_{\bar{1}},o}\left(X_{0,rT} \geq t(\rho_1+\epsilon/2)\right) + \widehat{\mathbb{P}}_{\delta_{\bar{1}},o}\left(X_{0,s} \geq t\epsilon/2\right).$$

By using that $X_{0,s} \leq \sum_{k=1}^{\lceil s \rceil} X_{(k-1),k}$ and Markov's inequality, we obtain that

$$\widehat{\mathbb{P}}_{\delta_{\bar{1}},o}(X_{0,s} \geq t\epsilon) \leq e^{-t\epsilon}\widehat{\mathbb{E}}_{\delta_{\bar{1}},o}(e^{X_{0,1}})^{\lceil s \rceil},$$

which completes the proof since $\widehat{\mathbb{E}}_{\delta_{\bar{1}},o}(e^{X_{0,1}})^{\lceil s \rceil} \leq \widehat{\mathbb{E}}_{\delta_{\bar{1}},o}(e^{X_{0,1}})^{\lceil T \rceil} < \infty$. □

We continue with the proof of Theorem 2.1.3.

Proof of Theorem 2.1.3. First note that, by the construction in Section 2.2, we have that

$$W_t = \sum_{i=1}^{\rho(N_t,\xi)} \tilde{O}_i + \sum_{j=1}^{N_t-\rho(N_t,\xi)} \tilde{V}_j,$$

where $\tilde{O}_i = \sum_{m=1}^{\infty} 1_{[p_1(m-1),p_1(m))}(O_i) z_m$ and $\tilde{V}_j = \sum_{n=1}^{\infty} 1_{[p_0(n-1),p_0(n))}(V_j) z_n$. Let $v = \rho u_1 + (1-\rho)u_0$, where $\rho := \rho_1$ ($= \rho_0$ by assumption). Then, for $\epsilon > 0$ and $\mu \in \mathcal{P}(\Omega)$, we have that

$$\begin{aligned}
\mathbb{P}_{\mu,o}\left(\|W_t - tv\|_1 \geq t\epsilon\right) =& \widehat{\mathbb{P}}_{\mu,o}\left(\left\| \sum_{i=1}^{\rho(N_t,\xi)} \tilde{O}_i + \sum_{j=1}^{N_t-\rho(N_t,\xi)} \tilde{V}_j - tv \right\|_1 \geq t\epsilon\right) \\
\leq& \widehat{\mathbb{P}}_{\mu,o}\left(\left\| \sum_{i=1}^{\rho(N_t,\xi)} \tilde{O}_i - t\rho u_1 \right\|_1 \geq t\frac{\epsilon}{2}\right) \\
&+ \widehat{\mathbb{P}}_{\mu,o}\left(\left\| \sum_{j=1}^{N_t-\rho(N_t,\xi)} \tilde{V}_j - t(1-\rho)u_0 \right\|_1 \geq t\frac{\epsilon}{2}\right).
\end{aligned}$$

To conclude the proof it is thus sufficient to show that both the latter terms decay exponentially in t. Since our argument is almost identical for both terms, we only provide the detailed proof for the first one. By Theorem 2.3.3, for any $\delta > 0$,

$$\begin{aligned}
&\widehat{\mathbb{P}}_{\mu,o}\left(\left\| \sum_{i=1}^{\rho(N_t,\xi)} \tilde{O}_i - t\rho u_1 \right\|_1 \geq t\frac{\epsilon}{2}\right) \\
&\leq \sum_{n=\lfloor t\gamma(\rho-\delta)\rfloor}^{\lceil t\gamma(\rho+\delta)\rceil} \widehat{\mathbb{P}}_{\mu,o}\left(\left\| \sum_{i=1}^{n} \tilde{O}_i - t\rho u_1 \right\|_1 \geq t\frac{\epsilon}{2}\right) + \exp(-tR(\delta)),
\end{aligned} \tag{2.3.7}$$

where $R(\delta) := \min\{R_1(\gamma\delta), R_0(\gamma\delta)\} > 0$. Indeed, the estimates in Theorem 2.3.3 applies to the process started from μ, since by Lemma 2.2.1, for any $\mu \in \mathcal{P}(\Omega)$ and $t, s > 0$, we have $\widehat{\mathbb{P}}_{\delta_{\bar{0}},o}(\rho_t(\xi) \geq s) \leq \widehat{\mathbb{P}}_{\mu,o}(\rho_t(\xi) \geq s) \leq \widehat{\mathbb{P}}_{\delta_{\bar{1}},o}(\rho_t(\xi) \geq s)$. Further, for any integer $n \in t\gamma(\rho - \delta, \rho + \delta)$ we have that

$$\widehat{\mathbb{P}}_{\mu,o}\left(\left\| \sum_{i=1}^{n} \tilde{O}_i - t\rho u_1 \right\|_1 \geq t\frac{\epsilon}{2}\right) \leq \widehat{\mathbb{P}}_{\mu,o}\left(\left\| \sum_{i=1}^{n} \tilde{O}_i - nu_1 \right\|_1 \geq t(\frac{\epsilon}{2} - \| u \|_1 \delta)\right) \tag{2.3.8}$$

Observe that, by taking $\delta > 0$ small enough, we can guarantee that $\frac{\epsilon}{2} - \| u \|_1 \delta > 0$. In this case, since (W_t) has finite exponential moments, (2.3.8) is exponentially small (in n, hence in t) by Cramérs Theorem applied to $(\tilde{O}_i)$. Applying this estimate to (2.3.7), taking δ small, yields that for some $C_1, c_1 > 0$,

$$\widehat{\mathbb{P}}_{\mu,o}\left(\left\| \sum_{i=1}^{\rho(N_t,\xi)} \tilde{O}_i - t\rho u_1 \right\|_1 \geq t\frac{\epsilon}{2}\right) \leq C_1 e^{-c_1 t}.$$

Noting that the same argument can be used to yield, for some $C_0, c_0 > 0$,

$$\widehat{\mathbb{P}}_{\mu,o}\left(\left\| \sum_{j=1}^{N_t - \rho(N_t,\xi)} \tilde{V}_j - t(1-\rho)u_0 \right\|_1 \geq t\frac{\epsilon}{2}\right) \leq C_0 e^{-c_0 t},$$

completes the proof of the theorem. □

2.4 Random walk on the contact process

2.4.1 Preliminaries

A càdlàg version of the contact process can be constructed from a graphical representation in the following standard way. For this, let $H := (H(x))_{x\in\mathbb{Z}^d}$ and $I := (I(x,e))_{x,e\in\mathbb{Z}^d:\ \|e\|_1=1}$ be two independent collections of i.i.d Poisson processes with rate 1 and λ, respectively. On $\mathbb{Z}^d \times [0,\infty)$, draw the events of $H(x)$ as *crosses* over x and the events of $I(x,e)$ as *arrows* from x to $x+e$.

For $x, y \in \mathbb{Z}^d$ and $0 \leq s \leq t$, we say that (x,s) and (y,t) are connected, written $(x,s) \leftrightarrow (y,t)$, if and only if there exists a directed path in $\mathbb{Z}^d \times [0,\infty)$ starting at (x,s), ending at (y,t) and going either forward in time without hitting crosses or "sideways" following arrows in the prescribed direction. For $A \subset \mathbb{Z}^d$ and $s \in [0,\infty)$, define the set at time $t > s$ connected to (A,s) in the graphical representation by

$$\mathcal{C}_t(A,s) := \{y \in \mathbb{Z}^d :\ \text{there exist } x \in A \text{ such that } (x,s) \leftrightarrow (y,t)\}.$$

When $A = \{x\}$ for some $x \in \mathbb{Z}^d$, we write $C_t(x, s)$ for simplicity. Note that this construction allows us to couple copies of the contact processes starting from different configurations. For each $A \subset \mathbb{Z}^d$ denote by $(\xi_t^A)_{t\geq 0}$ the process with initial configuration $\xi_0^A(x) = \mathbb{1}_A$ and, for all $y \in \mathbb{Z}^d$ and $t > 0$,

$$\xi_t^A(y) = \mathbb{1}_{\mathcal{C}_t(A,0)}(y).$$

Let $(\xi_t^{A,\lambda})_{t\geq 0}$ denote the contact process with starting configuration $A \subset \mathbb{Z}^d$ and infection parameter $\lambda > 0$.

The following monotonicity property is a direct consequence of the graphical construction.

Lemma 2.4.1 (Monotonicity property)**.** *The contact process $(\xi_t^{A,\lambda})_{t\geq 0}$ has an attractive graphical representation coupling, which is stochastically monotone in A and in λ.*

We next recall the self-duality property which is used in the proof of Theorem 2.1.4a) and b). For this, define the backwards process $(\hat{\xi}_s^{A,t})_{0\leq s\leq t}$ given $A \subset \mathbb{Z}^d$ and $t > 0$ by

$$\hat{\xi}_s^{A,t}(x) = \begin{cases} 1 & \text{if there exists } y \in A \text{ such that } y \in \mathcal{C}_t(x, t-s); \\ 0 & \text{otherwise.} \end{cases}$$

Then, the distribution of $(\hat{\xi}_s^{A,t}(x))_{s\geq 0}$ is the same as that of the contact process with the same initial configuration. Moreover, the backwards process and the contact process satisfy the duality equation. Namely,

$$\xi_t^A \cap B \neq \emptyset \text{ if and only if } A \cap \hat{\xi}_t^{B,t} \neq \emptyset, \quad \text{for any } A, B \subset \mathbb{Z}^d.$$

2.4.2 Proof of Theorem 2.1.4a)

Theorem 2.1.4a) for the contact process started from $\bar{1}$ is an immediate consequence of Theorem 2.1.1 and Lemma 2.4.1. We next show how to extend this to all measures stochastically dominating $\bar{\nu}_\lambda$. The proof goes by showing that (2.1.8) in Theorem 2.1.2 holds. Actually, our proof of (2.1.8) is more general and applies to the contact process started from any measure stochastically dominating a non-trivial Bernoulli-product measure. For this, we first state and prove two lemmas.

Lemma 2.4.2. *For $\lambda > \lambda_c(d)$ there exist constants $a, C, c > 0$ such that for all $t > 0$,*

$$\widehat{P}\left(|\mathcal{C}_t(o,0)| \leq at \mid \mathcal{C}_t(o,0) \neq \emptyset\right) \leq Ce^{-ct},$$

Proof. By [83, Theorem I.2.30], for some constants $C, c_1 > 0$,

$$\widehat{P}\left(\sup_{s\geq 0} |\mathcal{C}_s(o,0)| < \infty \mid \mathcal{C}_t(o,0) \neq \emptyset\right) \leq Ce^{-c_1 t}. \tag{2.4.1}$$

Let $c_2 > 0$ be the number such that $e^{-c_2} = 1/(1+\lambda)$, where $e^{-c_2} < 1$ is the probability that an occupied site becomes vacant before producing any offsprings. The probability on the left hand side of (2.4.1) is clearly bounded from below by the probability that $|\mathcal{C}_t(o,0)| \le at$ and each of the particles in $\mathcal{C}_t(o,0)$ dies before producing further offsprings. Since the contact process is Markovian and attractive,

$$\widehat{P}\left(\sup_{s\ge 0}|\mathcal{C}_s(o,0)| < \infty \mid \mathcal{C}_t(o,0) \neq \emptyset\right) \ge \widehat{P}\left(|\mathcal{C}_t(o,0)| \le at \mid \mathcal{C}_t(o,0) \neq \emptyset\right) e^{-c_2 at}. \quad (2.4.2)$$

Combining the two bounds (2.4.1) and (2.4.2), we obtain the result for $c = c_1 - ac_2$, and $c > 0$ if and only if $a < c_1/c_2$. □

Using the duality property together with Lemma 2.4.2 we obtain the following estimate.

Lemma 2.4.3 (Coupling of the contact process)**.** *Let $\mu_\rho \in \mathcal{P}(\Omega)$ be the Bernoulli product measure with density $\rho > 0$. For the contact process with $\lambda > \lambda_c(d)$, there exist constants $C, c > 0$ such that*

$$\widehat{P}\left(\xi_t^{(\mu_\rho)}(o) \neq \xi_t^{(\delta_{\bar{1}})}(o) \text{ for some } t \in [T, T+1)\right) \le Ce^{-cT}. \quad (2.4.3)$$

Proof. By attractiveness, $\xi_T^{(\mu)}(o) \le \xi_T^{(\bar{1})}(o)$ for all $\mu \in \mathcal{P}(\Omega)$. In particular, we have that $\xi_T^{(\mu_\rho)}(o) \neq \xi_T^{(\bar{1})}(o)$ if and only if the connected set of the dual process at time T started at (o,T), denoted by $\hat{\mathcal{C}}_T(o,T)$, is non-empty and $\xi_0^{(\mu)}(x) = 0$ for all $x \in \hat{\mathcal{C}}_T(o,T)$. That is,

$$\widehat{P}\left(\xi_T^{(\mu_\rho)}(o) \neq \xi_T^{(\delta_{\bar{1}})}(o)\right) = \widehat{P}\left(\{\hat{\mathcal{C}}_T(o,T) \neq \emptyset\} \cap \{\xi_0^{(\mu_\rho)}(x) = 0 \,\forall\, x \in \hat{\mathcal{C}}_T(o,T)\}\right). \quad (2.4.4)$$

By Lemma 2.4.2 and self-duality of the contact process, we can estimate the size of $\hat{\mathcal{C}}_T(o,T)$. In particular, for certain constants $C_1, c_1, C_2, c_2 > 0$,

$$\begin{aligned}
&\widehat{P}\left(\xi_0^{(\mu_\rho)}(x) = 0 \,\forall\, x \in \hat{\mathcal{C}}_T(o,T) \mid \hat{\mathcal{C}}_T(o,T) \neq \emptyset\right) \\
&\le \widehat{P}\left(|\hat{\mathcal{C}}_T(o,T) \cap B(o,rT)| \le aT \mid \hat{\mathcal{C}}_T(o,T) \neq \emptyset\right) \\
&\qquad + \widehat{P}\left(\xi_0^{(\mu_\rho)}(x) = 0 \,\forall\, x \in \hat{\mathcal{C}}_T(o,T) \mid |\hat{\mathcal{C}}_T(o,T) \cap B(o,rT)| \ge aT\right) \\
&\le C_1 e^{-c_1 T} + C_2 e^{-c_2 T}.
\end{aligned}$$

Thus, the l.h.s. of (2.4.4) decays exponentially (in T). To conclude (2.4.3), by the graphical representation, it is sufficient to control the times at which there is an arrow events $I(e,o)$ from $e \in \mathbb{Z}^d$ with $\|e\|_1 = 1$. Note that the number of such events is Poisson distributed with parameter $2d\lambda$. In particular,

$$\widehat{P}\left(\xi_t^{(\mu_\rho)}(o) \neq \xi_t^{(\delta_{\bar{1}})}(o) \text{ for some } t \in [T, T+1)\right) \le (1+2d\lambda)\left(C_1 e^{-c_1 T} + C_2 e^{-c_2 T}\right).$$

□

The proof of Theorem 2.1.4a) follows as a consequence of Lemma 2.4.3 and Theorem 2.1.2.

Proof of Theorem 2.1.4a). Let $\mu_\rho \in \mathcal{P}(\Omega)$ be a non-trivial Bernoulli product measure, let $m, T \in (0,\infty)$ and consider $V_m(T) := V_m \cap \left(\mathbb{Z}^d \times [T, T+1)\right)$ with V_m as defined in (2.1.6). Since the contact process is attractive and translation invariant,

$$\begin{aligned}&\widehat{P}\left(\exists (x,t) \in V_m(T) :\ \xi_t^{(\mu)}(x) \neq \xi_t^{(\delta_{\bar{1}})}(x)\right)\\ \leq &(m(T+1))^d \widehat{P}\left(\xi_t^{(\mu)}(o) \neq \xi_t^{(\delta_{\bar{1}})}(o) \text{ for some } t \in [T, T+1)\right).\end{aligned}$$

Hence, by Lemma 2.4.3, for some constant $C > 0$,

$$\begin{aligned}&\widehat{P}\left(\exists (x,t) \in V_m \cap \left(\mathbb{Z}^d \times [T,\infty)\right) : \xi_t^{(\mu)}(x) \neq \xi_t^{(\delta_{\bar{1}})}(x)\right)\\ \leq &C \sum_{k=0}^{\infty} (T+k)^d e^{-c(T+k)},\end{aligned}$$

and this vanishes as $T \to \infty$. Hence, since m was arbitrary chosen, the contact process started from μ_ρ satisfies equation (2.1.8) for any $m \in [0,\infty)$. Evoking Theorem 2.1.2 this completes the proof of Theorem 2.1.4a), noting that $\bar{\nu}_\lambda$ stochastically dominates a non-trivial Bernoulli product measure, as shown in [85, Corollary 4.1], and by using Lemma 2.4.1. □

2.4.3 Proof of Theorem 2.1.4b)

In the remaining part of this article we present the proof of Theorem 2.1.4b). We start by showing that $\rho(\cdot)$ is non-decreasing and right-continuous.

Proof of monotonicity and right continuity of $\lambda \mapsto \rho(\lambda)$. Monotonicity of $\lambda \mapsto \rho(\lambda)$ follows directly by the coupling construction in Section 2.2.1 and the graphical representation of the contact process, Lemma 2.2.1 and Lemma 2.4.1.

For right-continuity, let $\lambda \in (0,\infty)$ and denote by $\xi(\lambda)$ the corresponding contact process. For $T > 0$, let

$$f(T,\lambda) := \frac{1}{T}\widehat{\mathbb{E}}_{\delta_{\bar{1}},o}\big[\rho_T(\xi(\lambda))\big].$$

By Theorem 2.3.2, it follows that $f(T,\lambda) \downarrow \rho(\lambda)$ as $T \to \infty$. Moreover, by Lemma 2.2.1 and Lemma 2.4.1, $f(T,\lambda)$ is also non-decreasing in λ. Hence, $\lambda \mapsto \rho(\lambda)$ is right-continuous as the decreasing limit of non-decreasing continuous functions provided that $\lambda \mapsto f(T,\lambda)$ is continuous for any fixed $T > 0$.

To see that $f(T,\lambda)$ is continuous in λ, note first that in order to determine the behaviour of W_t for $t \in [0,T]$ we only need to consider the contact process in a finite space-time box. This follows by large deviation estimates on N_T and our restriction on the transition rates, i.e. $\| u_0 \|_1, \| u_1 \|_1 < \infty$. By the weak law of large numbers, this suffices to conclude that the probability of the walker escaping a box of size L within time T converges to 0 as $L \to \infty$. Continuity of $f(T,\lambda)$ now follows by using the graphical representation of the contact process and standard arguments for functions on the contact process in a finite space-time region (see e.g. the discussion on page 40 in [83]). □

We continue with the proof of (2.1.10). It is not difficult to see that $\rho(\lambda) < 1$ for any $\lambda \in (0,\infty)$, since vacant sites appear independently. This was already observed in den Hollander and dos Santos [72] for $d = 1$, and their argument transfers directly to higher dimensions.

However, in order to show that $\rho(\lambda) > 0$, the arguments in [72] do not carry over. In brief, they essentially use that there is a positive density of "waves" of particles moving from the right to the left in space-time. Using the ordering of $\mathbb{Z}$ and monotonicity in the displacement of a nearest neighbour random walk, the random walk cannot escape less than a positive proportion of the waves. Due to the one-dimensional nature of this argument it does not carry over to general dimensions nor does it extend beyond nearest neighbour jumps.

Our proof is based on monotonicity of $\rho_t(\xi)$. We propose here a simple strategy which also generalises to many other monotone dynamics. In particular, by applying the theory in [29], our argument for $d \geq 2$ seems to extend to all super-critical additive spin-flip systems. For dimension $d = 1$, we use an improved version of the corresponding proof in [72]. For both the one-dimensional and higher dimensional cases we make use of the general construction in Section 2.2.2.

Proof of (2.1.10) *for* $d = 1$. Let $\lambda > \lambda_c(1)$ and consider the contact process (ξ_t) on $\mathbb{Z}$ with infection parameter λ and initial configuration drawn according to $\mu \in \mathcal{P}(\Omega)$, where μ is the distribution of

$$\eta = \eta' \cdot \mathbb{1}_{(-\infty,0)}, \text{ where } \eta' \sim \bar{\nu}_\lambda.$$

Further, for $0 \leq s \leq t$ and $z \in \mathbb{Z}$, denote by

$$r_{s,t}(z) := \sup\{y \in \mathbb{Z} \colon \xi_s(x) = 1 \text{ for some } x \leq z \text{ and } (x,s) \leftrightarrow (y,t)\}$$

the rightmost site that is occupied at time t by a particle and connected to a site to the left of z at time s. It is well known that there exists $\alpha > 0$ (depending on λ) such that

$$\lim_{t\to\infty} \frac{1}{t} r_{0,t}(o) = \alpha \quad P^\mu - a.s. \tag{2.4.5}$$

See [82, Theorem VI.2.19] for a proof of (2.4.5) with respect to $P^{\delta_{\bar{1}}}$. The extension to P^μ follows from this statement and standard coupling arguments (e.g. [82, Theorem VI.2.2]).

(Recall the general construction of (W_t) in Section 2.2.2). We next specify the family of Boolean functions $(f_k)_{k\geq 0}$, which is defined by an iterative procedure involving a second process of elements in $\mathbb{Z}$ denoted by (R_k). Assume w.l.o.g. that $u_0 \leq 0$. Let $R_0 = r_{0,J_0}(o)$ be the position of the rightmost particle of the contact process at the first jump time of the random walk. Recall that $S_0 = o$, and define iteratively for $k \geq 0$,

$$f_k := \begin{cases} \xi_{J_k}(S_k) & \text{if } S_k \leq R_k; \\ 0 & \text{otherwise.} \end{cases}$$

$$R_{k+1} := \begin{cases} r_{J_k,J_{k+1}}(S_k) & \text{if } S_k \leq R_k; \\ r_{J_k,J_{k+1}}(R_k) & \text{otherwise.} \end{cases}$$

Hence, if $r_{0,J_0}(o) \geq 0$, then R_1 is assigned the position of the rightmost particle which at time J_1 is connected to an occupied site to the left of o at time J_0, the first jump time. Further, in this case, f_0 is assigned the value of the contact process at the location of the random walk at the 1'st jump time. Thus, the random walk "observes" the environment and chooses its transition kernel accordingly. Otherwise, if $r_{0,J_0}(o) < 0$, R_1 is assigned the position of the rightmost particle which at time J_1 is connected to an occupied site to the left of $r_{0,J_0}(o)$ at time J_0 and f_0 is assigned the value 0. Consequently, for this latter case, the random walk jumps as if it had observed a vacant site.

For arbitrary $k \geq 0$, if $S_k \leq R_k$, then R_{k+1} is assigned the location of the rightmost site at the $(k+2)$'th jump time which is connected to an occupied site to the left of S_k at the $(k+1)$'th jump time, and in this case f_k is assigned the value $\xi_{J_k}(S_k)$. On the other hand, if $S_{k-1} > R_{k-1}$, R_{k+1} is a prolongation of R_k and we set $f_k = 0$. By construction, and using that $\mathbb{Z}$ is ordered and the contact process has nearest neighbour interactions, the following holds: at times k for which $S_k \leq R_k$ there is a connected path $(\omega_t)_{0\leq t\leq J_k}$ such that $\xi_0(\omega_0) = 1, \omega_l \leq S_l$ for all $l \leq k-1$ and $\omega_k \geq S_k$. Furthermore, since f_k is the product of an indicator function and $\xi_{J_k}(S_k)$, we have $\rho^{(1)}(k,\xi) \leq \rho(k,\xi)$, where $\rho^{(1)}(k,\xi) = \sum_{i=0}^{k-1} f_i$ and $\rho(k,\xi)$ is as in Section 2.2.1. By the last observation, in order to prove the first part of (2.1.10), it is sufficient to show that there exists $\rho^{(1)} > 0$, depending on λ, such that

$$\liminf_{k\to\infty} \frac{1}{k}\rho^{(1)}(k,\xi) \geq \rho^{(1)}, \quad \widehat{\mathbb{P}}_{\mu,o} - a.s. \tag{2.4.6}$$

For this, let $T_0 = 0$ and, for $k \geq 1$, let

$$T_k := \inf\{n > T_{k-1} \colon S_{n-1} \leq R_{n-1}\}$$

denote the k'th time that the random walk observes the environment and set

$$\tau_k := T_k - T_{k-1}, \qquad k \geq 1.$$

As noted above, at times T_k, there is a connected path $(\omega_t)_{0\leq t\leq J_k}$ such that $\xi_0(\omega_0) = 1$, $\omega_l \leq S_l$ for all $l \leq k-1$ and $\omega_k \geq S_k$. At such times, the law of $(\xi_{J_{k-1}}(x))_{x\leq z}$ stochastically dominates $\bar{\nu}_\lambda$, as shown in [72, Lemma 4.1]. Consequently, since τ_k is decreasing

with respect to the configuration of the contact process on sites strictly to the left of $S_{T_{k-1}}$, the times $(\tau_k)_{k\geq 1}$ are dominated by an i.i.d. sequence of τ_0 distributed random variables. Furthermore, for the same reason, at times T_k, the random walk has a probability of at least $\bar{\nu}_\lambda(\eta(o)=1)>0$ of observing an occupied site. Thus, the first part of (2.1.10) follows once we have shown that

$$\widehat{\mathbb{E}}_{\nu_{\lambda,o}}(\tau_0)<\infty. \tag{2.4.7}$$

For this, note that, since $\{\tau_0\geq n\}=\{S_{k-1}\geq R_{k-1} \text{ for all } k\leq n\}$,

$$\widehat{\mathbb{P}}\left(\tau_0\geq n\right)\leq \widehat{\mathbb{P}}\left(S_{n-1}\geq \beta(n-1),\tau_0\geq n\right)+\widehat{\mathbb{P}}\left(R_{n-1}\leq \beta(n-1)\right), \tag{2.4.8}$$

for any $\beta>0$. For any $\beta<\alpha$ (with α as in (2.4.5)), the rightmost term in (2.4.8) decays exponentially (in n) due to large deviation estimates for $r_{0,t}(o)$; see [82, Corollary 3.22]. Moreover, for $\beta>0$, the leftmost term of (2.4.8) decays like n^{-2} as $n\to\infty$. This follows by applying Chebyshev's inequality, using that (W_t) has finite second moments, and since, under $\tau_0>n$, the random walk $(S_k)_{0\leq k\leq n}$ behaves as a simple random walk in a 0-homogeneous environment. Consequently, by setting $0<\beta<\alpha$, (2.4.7) holds and this concludes the first part of (2.1.10).

To conclude the second part of (2.1.10), we note that $\alpha=\alpha(\lambda)$ in (2.4.5) diverges to ∞ as $\lambda\to\infty$. In particular, by reasoning as above and choosing $\beta=\alpha/2$, we have that $\lim_{\lambda\to\infty}\widehat{\mathbb{P}}(\tau_0\geq 2)=0$. Moreover, since $\lim_{\lambda\to\infty}\bar{\nu}_\lambda(\eta(o)=1)=1$, it follows that $\rho^{(1)}$ in (2.4.6) approaches 1 as $\lambda\to\infty$ from which, by the remark above (2.4.6), we conclude the proof. □

We proceed with the proof of (2.1.10) *for the case* $d\geq 2$. For this, we make use of the fact that the supercritical contact process survives in certain space-time slabs, as first shown in Bezuidenhout and Grimmett [30]. For the cases when $u_0\neq o$ their result suffices. However, in order to also treat the special case when $u_0=o$, we use an extension of their theorem to certain tilted slabs.

For $K\in\mathbb{N}$, $L\in\mathbb{R}$ and $A\subset\mathbb{Z}^d$, denote by $\left({}^{L}_{K}\xi^A_t\right)_t$ the *truncated* contact process defined via the graphical representation by

$${}^{L}_{K}\xi^A_t(x)=\begin{cases}1 & \text{if } \{(y,0)\colon y\in A\} \text{ is connected to } (x,t) \text{ within } \mathcal{S}_{K,L};\\ 0 & \text{otherwise.}\end{cases}$$

Here,

$$\mathcal{S}_{K,L}:=\left\{(x,t)\in\mathbb{Z}^d\times[0,\infty)\colon\ x\in[-K,K]\times\mathbb{Z}^{d-1}+[Lt,0,\dots,0]\right\},$$

and $(A,0)$ is connected to (x,t) within $\mathcal{S}_{K,L}$ if $(A,0)$ is connected to (x,t) in the graphical representation without using arrows outside $\mathcal{S}_{K,L}$. We say that $\mathcal{S}_{K,L}$ is a tilted slab if $L\neq 0$, and that it is not tilted if $L=0$.

Proposition 2.4.4 (Survival in tilted slabs)**.** *Let $d \geq 2$.*

i) *For $\lambda > \lambda_c(d)$, there exist $K(\lambda) \in \mathbb{N}$, $L(\lambda) > 0$ such that for all $K > K(\lambda)$ and $L \in (-L(\lambda), L(\lambda))$,*

$$\widehat{P}\left({}^{L}_{K}\xi_t^{\{o\}} \neq \emptyset \quad \forall\, t \geq 0\right) > 0. \tag{2.4.9}$$

ii) *There exists $K \in \mathbb{N}$ such that for all $L > 0$,*

$$\lim_{\lambda\to\infty} \widehat{P}\left({}^{L}_{K}\xi_t^{\{o\}} \neq \emptyset \quad \forall\, t \geq 0\right) = 1. \tag{2.4.10}$$

The proof of survival in non-tilted slabs proceeds via a block argument and comparison with a certain (dependent) oriented percolation model. As pointed out by Bezuidenhout and Grimmett [30], there is a certain freedom in the spatial location of these blocks. The proof of Proposition 2.4.4 is achieved by adapting the proof of [30] in a way where the blocks are organised in a tilted way. A sketch of the proof of Proposition 2.4.4 is given at the end of this section.

Using that infections can spread fast in a small time interval, we note that also the following corollary of Proposition 2.4.4 holds.

Corollary 2.4.5. *Let $\lambda > \lambda_c(d)$ and consider the same parameters as in Proposition 2.4.4. Then, for any $\delta > 0$ small enough there is an $\epsilon > 0$, depending on all parameters, such that for all $(x,t) \in S_{K,L}$ with $(x, t-\delta) \in S_{K,L}$, we have that $\widehat{P}\left({}^{L}_{K}\xi_t^{[-K,K]\times\mathbb{Z}^{d-1}}(x) = 1\right) > \epsilon$, and $\epsilon = \epsilon(\lambda)$ approaches 1 as $\lambda \to \infty$. If $L = 0$, then the claim also holds for $\delta = 0$.*

Proof. First note that, since the process is started from all sites in $[-K, K] \times \mathbb{Z}^{d-1}$ occupied and the space-time slab $\mathcal{S}_{K,L}$ is translation invariant in all coordinate directions besides the first one, we may assume w.l.o.g. that $x = (x_1, 0, \ldots, 0)$. Further, by Proposition 2.4.4i), we have that, for some $y \in [-K + L(t-\delta), K + L(t-\delta)] \times \{0, \ldots, 0\}$,

$$\widehat{P}\left({}^{L}_{K}\xi_{t-\delta}^{[-K,K]\times\mathbb{Z}^{d-1}}(y) = 1\right) > 0.$$

The claim thus follows by the Markov property and since an infection at $(y, t-\delta)$ can spread to each (x,t) with $x \in [-K+Lt, K+Lt]\times\{0,\ldots,0\}$ with positive probability. In particular, since x is such that also $(x, t-\delta) \in S_{K,L}$ and the set $[-K + Lt, K + Lt] \times \{0, \ldots, 0\}$ is finite, this probability is uniformly bounded away from 0. □

We next present the proof of (2.1.10) in Theorem 2.1.4 for the case $d \geq 2$, assuming Proposition 2.4.4 to be true.

Proof of (2.1.10) *for* $d \geq 2$. We consider first the case where $u_0 \neq o$, for which we do not need the notion of tilted slabs in order to prove the l.h.s. of (2.1.10). Moreover, in this case, by translation invariance of the contact process, we assume w.l.o.g. that $u_0 \cdot e_1 < 0$. Let $\lambda > \lambda_c(d)$, and let $K \in \mathbb{N}$ be such that Proposition 2.4.4i) is satisfied with $L = 0$. Partition $\mathbb{Z}^d \times [0, \infty)$ into slabs $\Pi_i = 2Ki + \mathcal{S}_{K,0}$, $i \in \mathbb{Z}$, and consider $(\zeta_t^{(i)})_{i \in \mathbb{Z}}$ consisting of independent copies of the process $({}^0_K\xi_t^{[-K,K] \times \mathbb{Z}^{d-1}})$. Further, denote by (ζ_t) the process on Ω where, for $(x,t) \in \Pi_i$, we set $\zeta_t(x) = \zeta_t^{(i)}(\theta_{2Ki \cdot e_1} x)$.

We next specify the family of Boolean functions $(f_k)_{k \geq 0}$. Let $R_0 = 1$, recall that $S_0 = o$, and define iteratively for $k \geq 0$,

$$f_k := \begin{cases} \zeta_{J_k}(S_k) & \text{if } (S_k, J_k) \in \Pi_i \text{ for some } i < R_k; \\ 0 & \text{otherwise.} \end{cases}$$

$$R_{k+1} := \begin{cases} i & \text{if } (S_k, J_k) \in \Pi_i \text{ for some } i < R_k; \\ R_k & \text{otherwise.} \end{cases}$$

That is, $f_0 = \zeta_{J_0}(o)$ and $R_1 = 0$. Further, for arbitrary k, R_k records the label of the leftmost slab the random walk has "observed" at jump times $J_0, \ldots, J_k$. If, at a jump time, the random walk finds itself inside a slab which is at the left of all the slabs it previously has observed, then, by the definition of f_k, the random walk "observes" the environment. Otherwise, $f_k = 0$, and the random walk acts as if it had seen a vacant site. In particular, we have that $\rho^{(d)}(k, \zeta) \leq \rho(k, \zeta)$, where $\rho^{(d)}(k, \zeta) = \sum_{i=0}^{k-1} f_i$ and $\rho(k, \zeta)$ is as in Section 2.2.1. As in the $d = 1$ case, and since $(\zeta_t) \leq (\xi_t)$, it is sufficient to show that there exists $\rho^{(d)} > 0$ such that

$$\liminf_{k \to \infty} \frac{1}{k} \rho^{(d)}(k, \zeta) \geq \rho^{(d)} \quad \widehat{\mathbb{P}}_{\delta_1, o} - a.s. \tag{2.4.11}$$

For this, let $T_0 = 0$ and, for $k \geq 1$, let

$$T_k := \inf\{n > T_{k-1} \colon R_n < R_{n-1}\}$$

denote the k'th jump time at which the random walk is in a slab to the left of the origin which it previously has not observed, and let

$$\tau_k := T_k - T_{k-1}, \quad k \geq 1,$$

be the number of jumps it takes before the random walk observes a new slab. By Corollary 2.4.5 (which holds with $\delta = 0$ since $L = 0$), at the times (τ_k), the random walk has a positive probability to observe an occupied site, uniform in k. We conclude (2.4.11), since the times τ_k have finite mean, uniformly in k. Indeed, the latter follows since $u_0 \cdot e_1 < 0$ and by Chebyshev's inequality, using that (W_t) has finite second moments.

We continue with the case $u_0 = o$, for which we need to make certain modifications to the approach above. Choose $L > 0$ and $K \in \mathbb{N}$ such that Proposition 2.4.4i) holds and partition

$\mathbb{Z}^d \times [0,\infty)$ into tilted slabs $\tilde{\Pi}_i = 2Ki + \mathcal{S}_{K,L}$. For $i \in \mathbb{Z}$, denote by $(\tilde{\zeta}_t^{(i)})_{i\in\mathbb{Z}}$ independent copies of the process $({}^{L}_{K}\xi_t^{[-K,K]\times\mathbb{Z}^{d-1}})$. Further, denote by $(\tilde{\zeta}_t)$ the process on Ω where, for $(x,t) \in \tilde{\Pi}_i$, we set $\tilde{\zeta}_t(x) = \tilde{\zeta}_t^{(i)}(\theta_{2Ki\cdot e_1}x)$. Next, let $\tilde{R}_0 = 1$ and recall that $S_0 = o$. Fix $\delta > 0$ small such that Corollary 2.4.5 holds, and define iteratively (for $k \geq 0$),

$$f_k := \begin{cases} \tilde{\zeta}_{J_k}(S_k), & \text{if } \{S_k\} \times [J_k - \delta, J_k) \in \tilde{\Pi}_i \text{ for some } i < \tilde{R}_k; \\ 0, & \text{otherwise.} \end{cases}$$

$$\tilde{R}_{k+1} := \begin{cases} i & \text{if } (S_k, J_k) \in \tilde{\Pi}_i \text{ for some } i < \tilde{R}_k; \\ R_k & \text{otherwise.} \end{cases}$$

Next, define the variables $\tilde{T}_k$ and $\tilde{\tau}_k$ similar to the non-tilted case, by replacing R by $\tilde{R}$ and T by $\tilde{T}$. Note that, by our choice of $\delta > 0$, with strictly positive probability (uniformly in k), it holds that $\tilde{\zeta}_{J_{\tau_k}}(S_k) = 1$. It hence suffices to show that the times τ_k have finite mean, which implies that (2.4.11) holds and thus the l.h.s. of (2.1.10). To see that this indeed is the case, first note that, since the jump times (N_t) are continuous, each time the random walk enters a new slabs there is the possibility that it satisfies $\{S_k\} \times [J_k - \delta, J_k) \in \Pi_i$. Lastly, the number of new slabs that (W_t) observes is a positive fraction of its jumping times. This follows similarly as for the case that $u_0 \neq o$, by using that now $u_0 \cdot e_1 = 0$ and that $L > 0$. In particular, by again using Chebyshev's inequality we obtain sufficient estimates on the time it takes until a new slab is observed. By this we conclude that the times τ_k have finite mean, and thus the l.h.s. of (2.1.10).

Note that the approach with tilted slabs also applies in the case when $u_0 \cdot e_1 < 0$ (with the same K and L). In order to conclude $\rho(\lambda) \to 1$ as $\lambda \to \infty$ for both cases, we argue based on this approach. First note that, by Proposition 2.4.4ii), we may take L large (keeping K fixed) by choosing λ large enough. Consequently, we may chose L large and $\delta > 0$ small so that the times $(\tilde{\tau}_k)$ have mean bounded from above by $1 + c$, uniformly in k and for any fixed $c > 0$. Subsequently, for each such K, L and δ, the probability $\tilde{\zeta}_{J_{\tau_k}}(S_k) = 1$ can be made arbitrary close to 1, uniformly in k, by again tuning λ large. This yields the proof of (2.1.10) for $d \geq 2$ and thus concludes the proof of Theorem 2.1.4. □

We end with a sketch of the proof of Proposition 2.4.4.

Sketch of the proof of Proposition 2.4.4. For the proof of Proposition 2.4.4 we adapt the proof of [83, Theorem 1.2.30]. The idea is to proceed in the same manner, however, instead of comparing the survival of the process with the ordinary oriented percolation structure, we compare it to a certain tilted oriented percolation model.

To make this more precise, consider the following sub-graph of $\mathbb{Z}^d \times \mathbb{Z}$. Fix $l \in \mathbb{N}$. Next, set $V_0 = \{(0,0,\dots,0)\}$ and, for $n \in \mathbb{N}$, iteratively define

$$V_n = \begin{cases} V_{n-1} + \{(0,l,0,\dots 0,l)\} & \text{if } n \text{ is odd,} \\ V_{n-1} + \{(1,0,\dots,0,1),(-1,0,\dots,0,1\} & \text{if } n \text{ is even.} \end{cases}$$

Let the directed graph $G = (V, E)$ be given by $V = \bigcup_{i=1}^{\infty} V_i$, and, for any pair $x, y \in V$, the (directed) edge $(x, y) \in E$ if and only if $x \in V_{n-1}$ and $y \in V_n$ for some $n \in \mathbb{N}$ with

$$y = \begin{cases} x + (0, l, 0, \ldots, 0, l) & \text{if } n \text{ is odd,} \\ x + (\pm 1, 0, \ldots, 0, 1) & \text{if } n \text{ is even.} \end{cases}$$

This produces a tilted oriented percolation graph where the tilting in the second coordinate depends on the ratio $\frac{l}{l+1} < 1$. As for the ordinary oriented percolation model, let each edge be open with probability p independently, otherwise closed. By [83, Theorem B26], when p is large enough (depending on l), the origin lies in an infinite connected component of open edges with positive probability.

The proof of the first part of Proposition 2.4.4 now follows similar as the proof of Theorem 2.23 in [83], by constructing a coupling between the contact process and the oriented percolation model on the above defined graph G. Thus, by choosing $\epsilon > 0$ in [83, Proposition 2.22] small enough, with positive probability G percolates when edges are open with probability $p = 1 - \epsilon$. By the coupling construction in [83], this also holds for the contact process, depending only on the graphical representation within $\mathcal{S}_{K,L}$, with $K = 5a + 2a\frac{l}{l+1}$ and $L = \frac{l}{l+1}\frac{2a}{5b}$. This completes the proof of the first part.

The second part follows by monotonicity in λ. By a standard coupling procedure we may couple the systems with infection rates λ and 3λ such that, if [83, Proposition 2.22] is true for λ and constants a and b, then it also holds for the system with infection rate 3λ and constants a and $b/3$. Hence, given the constants a and b for a fixed λ, by letting λ converge towards infinity we may take b as small as we wish. In particular, we may choose L large and still satisfy Equation (2.4.9). On the other hand, for fixed $K, L > 0$, the probability in [83, Proposition 2.22] converges to 1 as $\lambda \to \infty$, and therefore (2.4.10) follows. □

Chapter 3

Absolute continuity and weak mixing of random walks in dynamic random environment

This chapter is based on a paper with Florian Völlering, [28], which has been submitted.

Abstract

We prove results for random walks in dynamic random environments which do not require the strong uniform mixing assumptions present in the literature. We focus on the “environment seen from the walker"-process and in particular its invariant law. Under general conditions it exists and is mutually absolutely continuous to the environment law. With stronger assumptions we obtain for example uniform control on the density or a quenched CLT. The general conditions are made more explicit by looking at hidden Markov models or Markov chains as environment and by providing simple examples.

3.1 Introduction and main results

3.1.1 Background and motivation

We study the asymptotic behaviour of a class of random walks (X_t) on $\mathbb{Z}^d$ whose transition probabilities depend on another process, the random environment. Such models play an important role in the understanding of disordered systems and serve as natural generalisations of the classical simple random walk model for describing transport processes in inhomogeneous media.

These types of random walks, which are called random walks in random environment, can be split into two broad areas, static and dynamic environments. In static environments the environment is created initially and then stays fixed in time. In dynamic environments the environment instead evolves over time. Note that a dynamic environment in $\mathbb{Z}^d$ can always be reinterpreted as a static environment in $\mathbb{Z}^{d+1}$ by turning time into an additional space dimension.

A major interest in dynamic environments are their often complicated space-time dependency structure. Typically, in order to show that the random walk is diffusive, one looks for some way to guarantee that the environment is "forgetful" and random walk increments are sufficiently independent on large time scales.

One approach to this are various types of mixing assumptions on the environment. By now, general results are known for Markovian environments which are uniformly mixing with respect to the starting configuration (see Avena, den Hollander, and Redig [8] and Redig and Völlering [100]). For this, the rate at which the dynamic environment converges towards its equilibrium state plays an important role.

On the other hand, models where the dynamical environment has non-uniform mixing properties serve as a major challenge and are still not well understood. Opposite to the diffusive behaviour known for uniform mixing environments, it has been conjectured that (X_t) may be sub- or super-diffusive for certain non-uniform mixing environments, see Avena and Thomann [6]. Though some particular examples yielding diffusive behaviour have recently been studied by rigorous methods, e.g. Deuschel, Guo, and Ramirez [50], Hilário, den Hollander, Sidoravicius, dos Santos, and Teixeira [71], Huveneers and Simenhaus [77] and Mountford and Vares [91], these results are model specific and/or perturbative in nature. No general theory has so far been developed.

In this article we provide a new approach for determining limiting properties of random walks in dynamic random environment, in particular about the invariant law of the "environment as seen from the walker"-process. Under general mixing assumptions, we prove the existence of an invariant measure mutually absolutely continuous with respect to the random environment (Theorems 3.1.1 and 3.1.3). Our mixing assumptions are considerably weaker than the

uniform mixing conditions present in the literature (e.g. cone mixing) and do not require the environment to be Markovian.

An important feature of our approach is that it can also be applied to dynamics with non-uniform mixing properties. Examples include an environment given by Ornstein-Uhlenbeck processes and the supercritical contact process.

Knowledge about the invariant measures for the "environment as seen from the walker"-process yield limit laws for the random walk itself. One immediate application of our approach is a strong law of large numbers for the random walk. Further applications include a quenched CLT based on Dolgopyat, Keller, and Liverani [53], Theorem 1, considerably relaxing its requirements.

Our key observation is an expansion of the "environment as seen from the walker"-process (Theorem 3.3.1). This expansion enables us to separate the contribution of the random environment to the law of the "environment as seen from the walker"-process from that of the transition probabilities of the random walk.

We also show stability under perturbations of the environment or of the jump kernel of the random walk. Under a strong uniform mixing assumption, we obtain uniform control on the Radon-Nikodym derivative of the law of the "environment as seen from the walker"-process with respect to the environment, irrespective of the choice of the jump kernel of the random walker.

Outline

In the next two subsections we give a precise definition of our model and present our main results, Theorem 3.1.1 and Theorem 3.1.3. Section 3.2 is devoted to examples and applications thereof. In Section 3.3 we derive the aforementioned expansion, and present results on stability and control on the Radon-Nikodym derivative. Proofs are postponed until Section 3.4.

3.1.2 The model

In this subsection we give a formal definition of our model. In short, (X_t) is a random walk in a translation invariant random field with a deterministic drift in a fixed coordinate direction.

The environment

Let $d \in \mathbb{N}$ and let $\Omega := E^{\mathbb{Z}^{d+1}}$ where E is assumed to be a finite set. We assign to the space Ω the standard product σ-algebra $\mathcal{F}$ generated by the cylinder events. For $\Lambda \subset \mathbb{Z}^{d+1}$, we

denote by $\mathcal{F}_\Lambda$ the sub-σ-algebra generated by the cylinders of Λ. For the forward half-space $\mathbb{H} := \mathbb{Z}^d \times \mathbb{Z}_{\geq 0}$ we write $\mathcal{F}_{\geq 0}$ for $\mathcal{F}_{\mathbb{H}}$.

By $\mathcal{M}_1(\Omega)$ we denote the set of probability measures on $(\Omega, \mathcal{F})$. We call $\eta \in \Omega$ the *environment* and denote by $\mathbb{P} \in \mathcal{M}_1(\Omega)$ its law. A particular class of environments contained in our setup are path measures of a stochastic process (η_t) whose state space is $\Omega_0 := E^{\mathbb{Z}^d}$. To emphasise this, for $\eta \in \Omega$ and $(x,t) \in \mathbb{Z}^d \times \mathbb{Z}$, we often write $\eta_t(x)$ for the value of η at (x,t).

We assume throughout that $\mathbb{P}$ is measure preserving with respect to translations, that is, for any $x \in \mathbb{Z}^d$, $t \in \mathbb{Z}$,

$$\mathbb{P}(\cdot) = \mathbb{P}(\theta_{x,t}\cdot),$$

where $\theta_{x,t}$ denotes the shift operator $\theta_{x,t}\eta_s(y) = \eta_{s+t}(y+x)$. Furthermore, we assume that $\mathbb{P}$ is *ergodic in the time direction*, that is, all events $B \in \mathcal{F}$ for which $\theta_{o,1}B := \{\omega \in \Omega \colon \theta_{o,-1}\omega \in B\} = B$ are assumed to satisfy $\mathbb{P}(B) \in \{0,1\}$. Here, $o \in \mathbb{Z}^d$ denotes the origin.

Remark 3.1.1. We present in Subsection 3.2.2-3.2.4 an approach where E is allowed to be a general Polish space, by considering the environment as a hidden Markov model.

The random walk

The *random walk* (X_t) is a process on $\mathbb{Z}^d$. We assume w.l.o.g. that $X_0 = o$. The transition probabilities of (X_t) is assumed to depend on the state of the environment as seen from the random walk. That is, given $\eta \in \Omega$, then the evolution of (X_t) is given by

$$\begin{aligned} &P_\eta(X_0 = o) = 1 \\ &P_\eta(X_{t+1} = y + z \mid X_t = y) = \alpha(\theta_{y,t}\eta, z), \end{aligned}$$

where $\alpha \colon \Omega \times \mathbb{Z}^d \to [0,1]$ satisfies $\sum_{z \in \mathbb{Z}^d} \alpha(\eta, z) = 1$ for all $\eta \in \Omega$. The law of the random walk, $P_\eta \in \mathcal{M}_1((\mathbb{Z}^d)^{\mathbb{Z}_{\geq 0}})$, where we have conditioned on the entire environment, is called the *quenched* law. We denote its σ-algebra by $\mathcal{G}$. Further, for $\mathbb{P} \in \mathcal{M}_1(\Omega)$, we denote by $P_{\mathbb{P}} \in \mathcal{M}_1\left(\Omega \times (\mathbb{Z}^d)^{\mathbb{Z}_{\geq 0}}\right)$ the joint law of (η, X), that is,

$$P_{\mathbb{P}}(B \times A) = \int_B P_\eta(A) d\mathbb{P}(\eta), \quad B \in \mathcal{F}, A \in \mathcal{G}.$$

The marginal law of $P_{\mathbb{P}}$ on $(\mathbb{Z}^d)^{\mathbb{Z}_{\geq 0}}$ is the *annealed* (or averaged) law of (X_t).

We assume that the transition probabilities of (X_t) only depend on the environment within a finite region around its location. That is, there exist $R \in \mathbb{N}$ such that for all $z \in \mathbb{Z}^d$

$$\alpha(\eta, z) - \alpha(\sigma, z) = 0 \text{ whenever } \sigma \equiv \eta \text{ on } [-R, R]^d \times \{0\}.$$

Further, define

$$\mathcal{R} := \{y \in \mathbb{Z}^d \colon \sup_{\eta \in \Omega} \alpha(\eta, y) > 0\} \tag{3.1.1}$$

as the jump range of the random walker, which we assume to be finite and to contain o. By possibly enlarging R we can guarantee that

$$\sup_{y \in \mathbb{Z}^d} \{\| y \|_1 : y \in \mathcal{R}\} \leq R.$$

Lastly, we say that (X_t) is *elliptic in the time direction* if

$$\alpha(\eta, o) > 0, \quad \forall\, \eta \in \Omega. \tag{3.1.2}$$

If, after replacing o with $y \in \mathcal{R}$, (3.1.2) holds for all $y \in \mathcal{R}$, then we say that (X_t) is *elliptic*.

The environment process

"The environment as seen from the walker"-process is of importance for understanding the asymptotic behaviour of the random walk itself, but it is also of independent interest. This process, which is given by

$$(\eta_t^{EP}) := (\theta_{X_t, t}\eta), \quad t \in \mathbb{Z}_{\geq 0},$$

is called the *environment process*. Note that (η_t^{EP}) is a Markov process on Ω under P_η, $\eta \in \Omega$, with initial distribution $\mathbb{P}$.

3.1.3 Main results

In this subsection we present our main results about the asymptotic behaviour of (X_t) and (η_t^{EP}). However, before stating our first theorem we need to introduce some more notation.

Recall (3.1.1) and let

$$\Gamma_k := \left\{(\gamma_{-k}, \gamma_{-k+1}, ..., \gamma_0) \colon \gamma_i \in \mathbb{Z}^d, \gamma_i - \gamma_{i-1} \in \mathcal{R}, -k \leq i < 0, \gamma_0 = o\right\}$$

be the set of all possible backwards trajectories from $(o, 0)$ of length k. For $\gamma \in \Gamma_k$ and $\sigma \in \Omega$, denote by

$$A_{-k}^{-m}(\gamma, \sigma) := \bigcap_{i=-k}^{-m} \left\{\theta_{\gamma_i, -i}\eta \equiv \sigma_i \text{ on } [-R, R]^d \times \{0\}\right\}, \quad 1 \leq m \leq k,$$

the event that an element $\eta \in \Omega$ equals σ in the R-neighbourhood along the path $(\gamma_k, \dots, \gamma_{-m})$. $A_{-k}^{-1}(\gamma, \sigma)$ is the event that the path of the environment observed by the random walk equals σ if the random walk moves along the path γ. Given $\gamma \in \Gamma_k$, denote by

$$\mathcal{A}_{-k}^{-m}(\gamma) := \left\{ A_{-k}^{-m}(\gamma, \sigma) \colon \sigma \in \Omega \text{ and } \mathbb{P}(A_{-k}^{-m}(\gamma, \sigma)) > 0 \right\}$$

the set of all possible observations along the path $(\gamma_k, \dots, \gamma_{-m})$. We write $\mathcal{A}_{-\infty}^{-m}$ for the set of events $\bigcup_{k \geq m, \gamma \in \Gamma_k} \mathcal{A}_{-k}^{-m}(\gamma)$. If $m = 1$ we simply write $\mathcal{A}_{-\infty}$.

Further, denote by $\mathcal{C} := \{(x,t) \in \mathbb{H} \colon \| x \|_1 \leq (R+1)t\}$ the forward cone with centre at $(o, 0)$ and slope proportional to $R + 1$. For $j \in \mathbb{Z}$, denote by $\mathcal{C}(j) := \mathcal{C} \cap \theta_{o,j}\mathbb{H}$ and let $\mathcal{F}_\infty^\infty := \bigcap_{j \in \mathbb{N}} \mathcal{F}_{\mathcal{C}(j)}$ be the tail-σ-algebra with respect to $\mathcal{F}_\mathcal{C}$.

Theorem 3.1.1 (Existence of an ergodic measure for the environment process)**.** *Assume that $\mathbb{P} \in \mathcal{M}_1(\Omega)$ satisfies*

$$\lim_{l \to \infty} \sup_{B \in \mathcal{F}_{\mathcal{C}(l)}} \sup_{A \in \mathcal{A}_{-\infty}} |\mathbb{P}(B \mid A) - \mathbb{P}(B)| = 0. \tag{3.1.3}$$

Then there exists $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$ invariant under (η_t^{EP}) satisfying $\mathbb{P}^{EP} = \mathbb{P}$ on $\mathcal{F}_\infty^\infty$.

If (X_t) is elliptic in the time direction and $\mathbb{P}$ is ergodic in the time direction, then $\mathbb{P}^{EP}$ is ergodic with respect to (η_t^{EP}). Moreover, for any $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{F}_\infty^\infty$,

$$\frac{1}{t} \sum_{s=0}^{t-1} P_\mathbb{Q}(\eta_s^{EP} \in \cdot) \text{ converges weakly towards } \mathbb{P}^{EP} \text{ as } t \to \infty. \tag{3.1.4}$$

Remark 3.1.2. There is a certain freedom in the ellipticity and the ergodicity assumptions in Theorem 3.1.1. For instance, the statement still holds if, for some $k \in \mathbb{N}$, the walker has a positive probability to return to o after k time steps, uniformly in the environment. The definitions can also be modified to require ellipticity and ergodicity with respect other directions $(y, 1) \in \mathbb{L}$, with $y \in \mathcal{R}$ (instead of in the direction $(o, 1)$. On the other hand, both ellipticity and ergodicity in the time direction are natural assumptions if $\mathbb{P}$ is the path measure of some stochastic process.

Corollary 3.1.2 (Law of large numbers)**.** *Assume that $\mathbb{P} \in \mathcal{M}_1(\Omega)$ is ergodic in the time direction and satisfies* (3.1.3)*, and that (X_t) is elliptic in the time direction. Then there exists $v \in \mathbb{R}^d$ such that $\lim_{t \to \infty} \frac{1}{t} X_t = v$, $P_\mathbb{P} - a.s.$*

Condition (3.1.3) is a considerably weaker mixing assumption than the cone mixing condition introduced by Comets and Zeitouni [46] (see Condition $\mathcal{A}_1$ therein) and used in Avena, den Hollander, and Redig [8] in the context of random walks in dynamic random environment. For comparison, note that cone mixing is equivalent to taking the supremum over events $A \in \mathcal{F}_{<0} := \mathcal{F}_{\mathbb{Z}^{d+1} \setminus \mathbb{H}}$ in (3.1.3). That Condition (3.1.3) is strictly weaker can already be seen in the case where $\mathbb{P}$ is i.i.d. with respect to space; see Theorem 3.2.1. Further examples where Condition (3.1.3) improve on the classical cone mixing condition are given in Section 3.2 and include dynamic random environments with non-uniform mixing properties.

Under a slightly stronger mixing assumption on the environment we obtain more information about $\mathbb{P}^{EP}$. For this, denote by $\Lambda(l) := \{x \in \mathbb{H}\colon \|x\|_1 \geq l\}$, $l \in \mathbb{N}$, where $\|\cdot\|_1$ denotes the l_1 distance from $(o,0)$, and let $\mathcal{F}_{\geq 0}^{\infty} := \bigcap_{l\in\mathbb{N}} \mathcal{F}_{\Lambda(l)}$ be the tail-σ-algebra with respect to $\mathcal{F}_{\geq 0}$.

Theorem 3.1.3 (Absolute continuity)**.** *Let $\phi\colon \mathbb{N} \to [0,1]$ be such that*

$$\sup_{B\in\mathcal{F}_{\Lambda(l)}} \sup_{A\in\mathcal{A}_{-\infty}} |\mathbb{P}(B \mid A) - \mathbb{P}(B)| \leq \phi(l), \tag{3.1.5}$$

with $\lim_{l\to\infty} \phi(l) = 0$. Then $\mathbb{P}^{EP} = \mathbb{P}$ on $\mathcal{F}_{\geq 0}^{\infty}$ (with $\mathbb{P}^{EP}$ as in Theorem 3.1.1) and

$$\sup_{B\in\mathcal{F}_{\Lambda(l)}} |\mathbb{P}(B) - \mathbb{P}^{EP}(B)| \leq \phi(l). \tag{3.1.6}$$

Furthermore, if (X_t) in addition is elliptic, then $\mathbb{P}$ and $\mathbb{P}^{EP}$ are mutually absolutely continuous on $(\Omega, \mathcal{F}_{\geq 0})$.

Knowing that the environment process converges toward an ergodic measure, it is well known how to apply martingale technics in order to deduce an annealed functional central limit theorem. However, it may happen that the covariance matrix is trivial. In Redig and Völlering [100] it was shown that the covariance matrix is non-trivial in a rather general setting when the environment is given by a Markov process satisfying a certain uniform mixing assumption. It is an interesting question whether (X_t) satisfies an annealed functional central limit theorem with non-trivial covariance matrix under the weaker mixing assumption of (3.1.3).

To obtain a quenched central limit theorem is a much harder problem and is only known in a few cases for random walks in dynamic random environment, see e.g. Bricmont and Kupiainen [43], Deuschel, Guo, and Ramirez [50], Dolgopyat and Liverani [52] and Dolgopyat, Keller, and Liverani [53]. In [53], Theorem 1, a quenched central limit theorem was proven under technical conditions on both the environment and the environment process. One important condition there was that the environment process has an invariant measure mutually continuous with respect to the invariant measure of the environment. By Theorem 3.1.3 above this condition is fulfilled. Combining this result with rate of convergence estimates obtained in [100], we conclude a quenched central limit theorem for a large class of uniformly mixing environments.

Corollary 3.1.4 (Quenched central limit theorem)**.** *Assume that (η_t) is a Markov chain on $E^{\mathbb{Z}^d}$. For $\sigma, \omega \in \Omega_0$ let $\widehat{P}_{\sigma,\omega}$ be a coupling of (η_t) started from $\sigma, \omega \in \Omega_0$ respectively and such that, for some $c, C > 0$,*

$$\sup_{\sigma,\omega\in\Omega} \widehat{P}_{\sigma,\omega}(\eta_t^{(1)}(o) \neq \eta_t^{(2)}(o)) \leq Ce^{-ct}. \tag{3.1.7}$$

Furthermore, assume that (η_t) satisfies Conditions (A3)-(A4) in [53] and that (X_t) is elliptic. Then, there is a non-trivial $d \times d$ matrix Σ such that for P_μ-a.e. environment history (η_t)

$$\frac{X_N - Nv}{\sqrt{N}} \text{ converges weakly towards } \mathcal{N}(0,\Sigma) \quad \mathbb{P}_{(\eta_t)} - a.s.,$$

where $\mu \in \mathcal{M}_1(\Omega_0)$ is the unique ergodic measure with respect to (η_t).

Conditions (A3)-(A4) in [53] are mixing assumptions on the dynamic random environment (η_t). Condition (A3) is a (weak) mixing assumption on μ, whereas Condition (A4) ensures that (η_t) is "local". For the precise definitions we refer to [53], page 1681.

In [53], Theorem 2, the statement of Corollary 3.1.4 was proven in a perturbative regime. Corollary 3.1.4 extends their result as there are no restrictions (other than ellipticity) on the transition probabilities of the random walk. We expect that Corollary 3.1.4 can be further improved to a functional CLT assuming only a polynomial decay in (3.1.7).

3.2 Examples and applications

In this section we present examples of environments which satisfy the conditions of Theorem 3.1.1 and Theorem 3.1.3. Particular emphasise is put on environments associated to a hidden Markov model for which we can improve on the necessary mixing assumptions.

3.2.1 Environments i.i.d. in space

The influence of the dimension on required mixing speeds is somewhat subtle. On the one hand, the random walk observes only a local area, and, in the case of conservative particle systems like the exclusion process, one can expect that in high dimensions information about observed particles in the past diffuses away. On the other hand, the higher dimension, the more sites the random walk can potentially visit in a fixed time. Furthermore, a comparison with a contact process or directed percolation gives an argument that information can spread easier in higher dimensions, hence observations along the path of the random walk could have more influence on future observations if the dimension increases.

This problem becomes significantly easier when the environment is assumed to be i.i.d. in space, that is $\mathbb{P} = \bigtimes_{x\in\mathbb{Z}^d} \mathbb{P}_o$, and $\mathbb{P}_o \in \mathcal{M}_1(E^{\mathbb{Z}})$ is the law of $(\eta_t(x))_{t\in\mathbb{Z}}$ for any $x \in \mathbb{Z}^d$.

Theorem 3.2.1. *Assume that $\mathbb{P} = \bigtimes_{x\in\mathbb{Z}^d} \mathbb{P}_o$ and that*

$$\sum_{t\geq 1} \sup_{B\in\mathcal{G}_{\geq t}, A\in\mathcal{G}_{<0}} |\mathbb{P}_o(B \mid A) - \mathbb{P}_o(B)| < \infty, \tag{3.2.1}$$

where $\mathcal{G}_{\geq t}$ ($\mathcal{G}_{<0}$) is the σ-algebra of $E^{\mathbb{Z}}$ generated by the values after time t (before time 0) with respect to $\mathbb{P}_o$. Then (3.1.5) *holds.*

Observe that (3.2.1) does not depend on the dimension. This is in contrast to the cone mixing condition of Comets and Zeitouni [46], where an additional factor t^d inside the sum of (3.2.1) is required. In Subsection 3.2.3 we present a class of environments which have arbitrary slow polynomial mixing, thus showing that Theorem 3.2.1 yields an essential improvement.

3.2.2 Hidden Markov models

When $\mathbb{P}$ is the path measure of a stochastic process (η_t) evolving on Ω_0, the results of Subsection 3.1.3 can be improved. In this subsection we discuss in detail the case where the random environment is governed by a hidden Markov model.

The environment (η_t) is a hidden Markov model if it is given via a function of a Markov chain (ξ_t). To be more precise, let $\widetilde{E}$ be a Polish space, $\widetilde{\Omega}_0 = \widetilde{E}^{\mathbb{Z}^{\widetilde{d}}}$ with $\widetilde{d} \geq d$, and $\widetilde{\Omega} = \widetilde{\Omega}_0^{\mathbb{Z}}$. Denote by $\tilde{\mathcal{F}}$ the corresponding σ-algebra. We assume that the Markov chain (ξ_t) is defined on $\widetilde{\Omega}$ with law $\widetilde{\mathbb{P}}_\xi$ and is ergodic with law $\widetilde{\mu} \in \mathcal{M}_1(\widetilde{\Omega}_0)$. Here $\xi \in \widetilde{\Omega}_0$ denotes the starting configuration. Let $\Phi : \widetilde{\Omega}_0 \to \Omega_0 = E^{\mathbb{Z}^d}$ be a translation invariant map and let $\eta_t = \Phi(\xi_t)$. We call (η_t) a *hidden Markov model*, which has μ as the induced measure on Ω_0 as invariant measure. We assume throughout that Φ is of finite range, that is, the function $\Phi(\cdot)(o)$ is $\tilde{\mathcal{F}}_\Lambda$-measurable for some $\Lambda \subset \mathbb{Z}^{\tilde{d}}$ finite.

Remark 3.2.1. When $\tilde{E}$ is finite, the canonical choice of Φ is the identity map. However, our setup opens for more sophisticated choices. One example is the projection map. For instance, if $\tilde{d} > 1$ and $d = 1$, one can consider the hidden Markov model given by $\eta_t(x) = \xi_t(x, 0, \dots, 0)$. In other words, the random walk only observes the environment in one coordinate.

Condition (3.1.5) in Theorem 3.1.3 is an infinite volume condition which can be hard to verify by direct computation. The next result yields a sufficient condition which only needs to be checked for single site events. For its statement, we first introduce the concept of $\mathbb{P} \in \mathcal{M}_1(\Omega)$ having *finite speed of propagation*.

Definition 3.2.1. *We say that $\mathbb{P} \in \mathcal{M}_1(\Omega)$ has finite speed of propagation if the following holds: for some $\alpha > 0$, and for each $A \in \mathcal{F}_{<0}$ and $A' \in \mathcal{F}_{\Lambda(\alpha t,t)}$, where $\Lambda(\alpha t, t) := \{(x, s) \in \mathbb{H} \colon \| x \|_1 \geq \alpha t, 0 < s \leq t\}$, there is a coupling $\widehat{\mathbb{P}}_{A,A'}$ of $\mathbb{P}(\cdot \mid A, A')$ and $\mathbb{P}(\cdot \mid A)$ such that*

$$\sum_{t \geq 1} t^d \sup_{A \in \mathcal{F}_{<0}, A' \in \mathcal{F}_{\Lambda(\alpha t,t)}} \widehat{\mathbb{P}}_{A,A'} \left(\eta_t^1(o) \neq \eta_t^2(o) \right) < \infty. \tag{3.2.2}$$

Furthermore, any such coupling satisfies $\widehat{\mathbb{P}}_{A,A'}(\cdot) = \widehat{\mathbb{P}}_{\theta_{x,s}A, \theta_{x,s}A'}(\theta_{x,s}\cdot)$ for all $(x, s) \in \mathbb{Z}^{d+1}$, where $\omega \in \theta_{x,s}A$ if and only if $\theta_{-x,-s}\omega \in A$.

Finite speed of propagation is a natural assumption for many physical applications. Note that, for many interacting particle systems there is a canonical coupling given by the so-called graphical representation coupling.

Corollary 3.2.2. *Assume that (ξ_t) has finite speed of propagation and that*

$$\sum_{t \geq 1} t^d \sup_{A \in \mathcal{A}_{-\infty}^{-t}} \widehat{\mathbb{P}}_{\Omega,A} \left(\Phi(\xi^1)_0(o) \neq \Phi(\xi^2)_0(o) \right) < \infty. \tag{3.2.3}$$

Then (3.1.5) *holds for* $(\eta_t) = \Phi(\xi_t)$.

Remark 3.2.2. The measure $\widehat{\mathbb{P}}_{\Omega,A}$ denotes the coupling of $\mathbb{P}(\cdot)$ and $\mathbb{P}(\cdot \mid A)$.

Corollary 3.2.2 follows by a slightly more general statement, see Theorem 3.4.4. This approach can also be used in cases where (ξ_t) does not have finite speed of propagation. In such cases, (3.2.3) is sufficient for (3.1.3) to hold. Observe also that, by applying the projection map introduced in Remark 3.2.1, the dimensionality dependence in Condition (3.2.3) can be replaced by the dimensionality of the range of the random walk.

Markovian environment

If $\mathbb{P}$ is the path measure of a Markov chain (η_t), we can weaken the mixing assumption. In such cases, we consider α as a function from $\Omega_0 \times \mathbb{Z}^d$. Because the Markov property allows us to look at the invariant measure of the environment process just at a time 0 instead of in the entire upper half-space $\mathbb{H}$, we have the following mixing condition. Here we denote by $\mathcal{F}_{=0}^{\infty}$ the tail-σ-algebra of $\mathcal{F}_{=0} := \mathcal{F}_{\mathbb{Z}^d \times \{0\}}$.

Theorem 3.2.3. *Assume that* (η_t) *is a Markov chain with ergodic invariant measure* $\mu \in \mathcal{M}_1(\Omega_0)$. *Further, assume that the path measure* $\mathbb{P}_\mu \in \mathcal{M}_1(\Omega)$ *has finite speed of propagation and that*

$$\sum_{t \geq 1} t^{d-1} \sup_{A \in \mathcal{A}_{-\infty}^{-t}} \widehat{\mathbb{P}}_{\Omega,A} \left(\eta_0^1(o) \neq \eta_0^2(o) \right) < \infty. \tag{3.2.4}$$

Then there exists $\mu^{EP} \in \mathcal{M}_1(\Omega_0)$ *invariant for the Markov chain* (η_t^{EP}) *such that* μ^{EP} *agrees with* μ *on* $\mathcal{F}_{=0}^{\infty}$. *If in addition* (X_t) *is elliptic, then* μ^{EP} *and* μ *are mutually absolutely continuous and* μ^{EP} *is ergodic with respect to* (η_t^{EP}).

It is important to note that (3.2.4) (as well as (3.2.3)) does not require (η_t) to be uniquely ergodic. However, if for every $\sigma, \xi \in \Omega_0$, there is a coupling $\widehat{\mathbb{P}}_{\sigma,\xi}$ of $\mathbb{P}_\sigma$ and $\mathbb{P}_\xi$ which satisfies the finite speed of propagation property and

$$\sum_{t=1}^{\infty} t^{d-1} \sup_{\sigma,\xi \in \widetilde{\Omega}_0} \widehat{\mathbb{P}}_{\sigma,\xi}(\eta_t^1(o) \neq \eta_t^2(o)) < \infty, \tag{3.2.5}$$

then it follows, under the assumptions of Theorem 3.2.3, that (η_t^{EP}) is uniquely ergodic.

Equation (3.2.5) should be compared with Assumption 1a in [100], that is;

$$\int_0^{\infty} t^{(d)} \sup_{\eta,\xi} \hat{\mathbb{E}}_{\eta,\xi} \rho(\eta_t^1(o), \eta_t^2(o)) dt < \infty, \tag{3.2.6}$$

where $\rho \colon E \times E \to [0,1]$ is the distance function. Their assumption was used to show (among others) the existence of $\mu^{EP} \in \mathcal{M}_1(\Omega_0)$ invariant and ergodic for the Markov chain (η_t^{EP}), see Lemma 3.2 therein. Note in particular that Assumption (3.2.6) has $t^{(d)}$ inside the integral, whereas (3.2.5) only requires $t^{(d-1)}$.

3.2.3 Polynomially mixing environments

As example of environments which fully utilise the polynomial mixing assumption of Theorem 3.2.1 and Corollary 3.2.2, we consider layered environments. These were already considered in [100] for the same purpose, but since we are in a different setting we use the setting of hidden Markov models.

The idea of layered environments is that, given a summable sequence $(b_n) \subset (0,1)$, for each layer n, the process $(\xi_t(\cdot,n))_{t\in\mathbb{Z}_{\geq 0}}$ is an uniform exponentially mixing Markov chain on $[-1,1]$ with an exponential relaxation rate b_n, and independent layers. For simplicity, in this example, we choose $\xi_t(\cdot,n)$ to be i.i.d. spin flips, that is, for each $x \in \mathbb{Z}^{\tilde{d}}$,

$$\xi_{t+1}(x,n) = \begin{cases} \xi_t(x,n), & \text{with probability} \quad 1-b_n; \\ \text{Unif}[-1,1], & \text{with probability} \quad b_n; \end{cases}$$

independent for all x, n, t. In other words, at each time step the spin retains its old value with probability $1-b_n$ and chooses uniformly on $[-1,1]$ with probability b_n.

In the context of the previous subsection we thus have $\widetilde{E} := [-1,1]^{\mathbb{N}}$. We further choose $\widetilde{d} = d \geq 1$, $E = \{0,1\}$ and set, for a summable sequence $(a_n) \subset (0,1)$,

$$\Phi(\xi)(x) = \mathbb{1}_{\sum_{n=1}^{\infty} a_n \xi(x,n) > 0}.$$

The behaviour of this kind of processes is then determined by the two sequences (a_n) and (b_n). When $a_n = \frac{1}{2}n^{-\alpha}$, $b_n = \frac{1}{2}n^{-\beta}$ for some $\alpha, \beta > 1$, we have the following bound on the mixing of (η_t).

Theorem 3.2.4. *There are constants $0 < c_1 < c_2 < \infty$ so that*

$$c_1 t^{\frac{-\alpha+1}{\beta}} \leq \sup_{\xi,\sigma} \| \mathbb{P}_\xi(\eta_t(0) \in \cdot) - \mathbb{P}_\sigma(\eta_t(0) \in \cdot) \|_{TV} \leq c_2 t^{\frac{-\alpha+1}{\beta}} (\log t)^{\frac{\alpha-1}{\beta}}.$$

Here $\|\cdot\|_{TV}$ is the total variation distance between the two distributions. In particular, if $\alpha > \beta + 1$, then (3.1.5) *holds.*

3.2.4 Independent Ornstein-Uhlenbeck processes

With the approach of environments as hidden Markov models, we can also allow for unbounded state spaces where the environment does not mix uniformly, as long as the random walk transition function is simple enough. Here we choose an underlying environment of independent Ornstein-Uhlenbeck processes $(\xi_t^x)_{t\in\mathbb{R}}$ for each site $x \in \mathbb{Z}^d$, and the jump rates depend only on the signs, that is,

$$\eta_t(x) = \text{sign}(\xi_t^x) := 1 - 2\mathbb{1}_{\xi_t^x < 0}, \quad t \in \mathbb{Z}.$$

To state the example more formally, we have $\tilde{E} = \mathbb{R}$ and $E = \{-1, 1\}$, and

$$d\xi_t^x = -\xi_t^x dt + dW_t^x,$$

where $(W_t^x)_{t\in\mathbb{R}}$, $x \in \mathbb{Z}^d$, are independent two-sided Brownian motions. The stationary measure of ξ_t^x is a normal distribution, and $\tilde{\mu}$ is the product measure of normal distributions.

Theorem 3.2.5. *Let $(\xi_t)_{t\in\mathbb{R}}$ be an Ornstein-Uhlenbeck process and $\mathbb{P}$ the two-sided path measure in stationarity. There are constants $c, C > 0$ so that*

$$\| \mathbb{P}(\xi_t \in \cdot \mid A) - \mathbb{P}(\xi_t \in \cdot) \|_{TV} \leq Ce^{-ct}$$

for all $t \geq 0$ and any A of the form $A = \{\mathrm{sign}(\xi_{-t_k}) = a_k, 1 \leq k \leq n\}$, (t_k) increasing sequence with $t_1 = 0$ and $a_k \in \{-1, 1\}$, n arbitrary. In particular, (3.1.5) *holds for (η_t).*

3.2.5 The contact process

As a second example of an environment with non-uniform space-time correlations and which do not satisfy the cone mixing property of [46], we consider the contact process (η_t) on $\{0, 1\}^{\mathbb{Z}^d}$ with infection parameter $\lambda \in (0, \infty)$.

The contact process is one of the simplest interacting particle systems exhibiting a phase transition. That is, there is a critical threshold $\lambda_c(d) \in (0, \infty)$, depending on the dimension d, such that the following holds: if $\lambda \leq \lambda_c(d)$, then the contact process is uniquely ergodic with the measure concentrating on the configuration where all sites equal to 0 as invariant measure. On the other hand, for all $\lambda > \lambda_c(d)$, the contact process is not uniquely ergodic. In particular, it has a non-trivial ergodic invariant measure, denoted here by $\bar{\nu}_\lambda$, also known as the upper invariant measure. As a general reference, and for a precise description of the contact process, we refer to Liggett [83].

Random walks on the contact process have recently been studied by den Hollander and dos Santos [72] and Mountford and Vares [91], where the one-dimensional random walk (i.e. on $\mathbb{Z}$) was shown to behave diffusively for all $\lambda > \lambda_c(1)$. See also Bethuelsen and Heydenreich [27] for some results in general dimensions.

The next theorem sheds new light on the behaviour of the environment process and the random walk for this model on $\mathbb{Z}^d$ with $d \geq 2$. In the theorem we make use of the projection map, as introduced in Remark 3.2.1. That is, we assume $\tilde{d} \geq 2$ and denote by $(\eta_t) = \phi((\xi_t))$ the projection of (ξ_t) onto the 1-dimensional lattice such that, for $x \in \mathbb{Z}$ and $t \in \mathbb{Z}$, we set $\eta_t(x) = \xi_t(x, 0, \ldots, 0)$.

Theorem 3.2.6. *Let $\tilde{d} \geq 2$ and let (ξ_t) be the contact process with parameter $\lambda > \lambda_c(\tilde{d})$ started from $\bar{\nu}_\lambda$. Further, let (η_t^{EP}) be the environment process corresponding to the process $(\eta_t) = \phi((\xi_t))$. Then* (3.1.5) *holds for (η_t).*

Theorem 3.2.6 can be extended to higher dimensional projections by following the same approach. The proof strategy of Theorem 3.2.6 also applies to a larger class of models which satisfy the so-called *downward FKG* property; see Theorem 3.4.5.

3.3 Understanding the environment process

3.3.1 Expansion of the environment process

In this subsection, we present a key observation for understanding the environment process and for the proofs of Theorem 3.1.1 and Theorem 3.1.3.

Intuitively, the distribution of (η_t^{EP}) should converge to an invariant measure, say $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$, which describes asymptotic properties. To obtain $\mathbb{P}^{EP}$ and show that it is absolutely continuous with respect to $\mathbb{P}$, we start by interpreting the law of η_t^{EP}, $P_{\mathbb{P}}(\eta_t^{EP} \in \cdot)$, as an approximation. With this point of view, t becomes the present time. Going from t to $t+1$ thus means that we look one step further into the past. To reinforce this point of view, we denote by $\mathbb{P}^{-k} := \mathbb{P}_{\mathbb{P}}^{-k} \in \mathcal{M}_1(\Omega \times (\mathbb{Z}^d)^{\mathbb{Z}_{\geq -k}})$ the joint law of the environment $\mathbb{P}$ and random walk $(X_t)_{t \geq -k}$ so that $X_0 = o$. That is, for $k \in \mathbb{N}$,

$$\mathbb{P}^{-k}\left((\eta, X) \in (B_1, B_2)\right) := P_{\mathbb{P}}\left(\eta_{k+\cdot}^{EP} \in B_1, (X_{k+\cdot} - X_k) \in B_2\right).$$

For events $B \in \mathcal{F}$, we use the shorthand notation $\mathbb{P}^{-k}(B)$ for the probability that $\mathbb{P}^{-k}((\eta, X) \in (B, (\mathbb{Z}^d)^{\mathbb{Z}_{\geq -k}}))$.

Theorem 3.3.1 (Expansion of the environment process)**.** *For any $k \geq 1$ and $B \in \mathcal{F}$,*

$$\mathbb{P}^{-k}(B) = \sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}\left(B, A_{-k}^{-1}\right) P\left(X_{-k,\ldots,0} = \gamma \mid A_{-k}^{-1}\right), \tag{3.3.1}$$

where $P\left(X_{-k,\ldots,0} = \gamma \mid A_{-k}^{-1}\right) := \prod_{i=-k}^{-1} \alpha(\hat{\sigma}_i, \gamma_{i+1} - \gamma_i)$, and $\hat{\sigma} \in \Omega$ is any environment so that $A_{-k}^{-1}(\gamma, \hat{\sigma}) = A_{-k}^{-1}$.

Proof. We can rewrite $\mathbb{P}^{-k}(B)$ as follows;

$$\begin{aligned}
\mathbb{P}^{-k}(B) &= \sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}^{-k}(B,\ X_{-k,\ldots,0} = \gamma, A_{-k}^{-1}) \\
&= \sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \big[\mathbb{P}^{-k}(B \mid A_{-k}^{-1}, X_{-k,\ldots,0} = \gamma) \\
&\qquad\qquad \mathbb{P}^{-k}(X_{-k,\ldots,0} = \gamma, A_{-k}^{-1})\big].
\end{aligned}$$

By definition,

$$\begin{aligned}\mathbb{P}^{-k}(X_{-k,\dots,0}=\gamma, A_{-k}^{-1}) &= P_{\mathbb{P}}(X_{0,\dots,k}=\gamma-\gamma_{-k}, \eta_k^{EP}\in A_{-k}^{-1})\\ &= P_{\mathbb{P}}(X_{0,\dots,k}=\gamma-\gamma_{-k}, \theta_{-\gamma_{-k},k}\eta\in A_{-k}^{-1})\\ &= P(X_{-k,\dots,0}=\gamma \mid A_{-k}^{-1})\mathbb{P}(\theta_{-\gamma_{-k},k}\eta\in A_{-k}^{-1})\\ &= P(X_{-k,\dots,0}=\gamma \mid A_{-k}^{-1})\mathbb{P}(A_{-k}^{-1}),\end{aligned}$$

where the last equality holds since first the law of the environment is translation invariant. Similarly,

$$\mathbb{P}^{-k}(B \mid A_{-k}^{-1}, X_{-k,\dots,0}=\gamma) = \mathbb{P}(B \mid A_{-k}^{-1}).$$

□

The sum in Expansion (3.3.1) represents all the possible pasts of the random walk and the corresponding observed environments from time $-k$ to -1. There are two key features with this expansion.

First, it separates the contribution to (η_t^{EP}) of the random walk from that of the random environment. Indeed, the rightmost term in the sum, i.e. $P\left(X_{-k,\dots,0}=\gamma \mid A_{-k}^{-1}\right)$, can be calculated directly from the transition probabilities of (X_t). On the other hand, the leftmost term in the sum, i.e. $\mathbb{P}\left(B, A_{-k}^{-1}\right)$, only involves the random environment $\mathbb{P}$.

A second key feature of (3.3.1) is that it serves as a (formal) expression for the Radon-Nikodym derivate of $P_{\mathbb{P}}(\eta_k^{EP}\in\cdot)$ with respect to $\mathbb{P}$. Indeed, (3.3.1) yields that for any $B\in\mathcal{F}$ with $\mathbb{P}(B)>0$,

$$\frac{P_{\mathbb{P}}(\eta_k^{EP}\in B)}{\mathbb{P}(B)} = \sum_{\gamma\in\Gamma_k}\sum_{A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)} \frac{\mathbb{P}\left(B, A_{-k}^{-1}\right)}{\mathbb{P}(B)} P\left(X_{-k,\dots,0}=\gamma \mid A_{-k}^{-1}\right).$$

3.3.2 Stability

It is also of interest to compare the effect of changing the environment $\mathbb{P}$ or the transition probabilities $\alpha\colon \Omega\times\mathbb{Z}^d\to[0,1]$ on the behaviour of the environment process. Our next result gives sufficient conditions for the environment process to be stable with respect to perturbations of both these parameters. This result follows as another consequence of the expansion in Theorem 3.3.1.

To state the theorem precisely, denote by $(\mathbb{P}^n)_{n\geq1}$ a family of measures on $\mathcal{M}_1(\Omega)$ and let $\left(\alpha_n\colon \Omega\times\mathbb{Z}^d\to[0,1]\right)_{n\geq1}$ be a collection of transition probabilities. Consider for each $n\in\mathbb{N}$ the corresponding environment process, $(\eta_t^{EP(n)})$, and let $\mathbb{P}^{EP(n)}\in\mathcal{M}_1(\Omega)$ be a measure invariant under $(\eta_t^{EP(n)})$.

Theorem 3.3.2. *Assume that the following holds.*

a) $\epsilon(n) = \sup_{m>n} \sup_{\eta\in\Omega, y\in\mathbb{Z}^d} |\alpha_m(\eta, y) - \alpha_n(\eta, y)| \downarrow 0$ *as* $n \to \infty$.

b) $\mathbb{P}^n \implies \mathbb{P} \in \mathcal{M}_1(\Omega)$ *weakly as* $n \to \infty$.

c) $P_{\mathbb{P}^n}(\eta_t^{EP(n)} \in \cdot) \implies \mathbb{P}^{EP(n)}$ *weakly as* $t \to \infty$, *uniformly in* n.

Then both $\mathbb{P}^{EP(n)}$ *and* $P_{\mathbb{P}}(\eta_t^{EP})$ *converge weakly towards* $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$. *In particular,* $\mathbb{P}^{EP}$ *is invariant with respect to* (η_t^{EP}).

Condition c) in Theorem 3.3.2 is a strong uniform assumption. If the $\mathbb{P}^n$'s are path measures of Markov chains $(\eta_t^{(n)})$, this condition can be replaced by the assumption that the environment process (η_t^{EP}), i.e., after taking $n \to \infty$), is uniquely ergodic. For this, recall notation from Section 3.2.2 and let $\mu^{EP(n)} \in \mathcal{M}_1(\Omega_0)$ be an invariant measure with respect to $(\eta_t^{EP(n)})$.

Theorem 3.3.3. *Let* (η_t) *be a Markov chain and assume that the following holds.*

a) $\epsilon(n) = \sup_{m>n} \sup_{\eta\in\Omega, y\in\mathbb{Z}^d} |\alpha_m(\eta, y) - \alpha_n(\eta, y)| \downarrow 0$ *as* $n \to \infty$.

b') $\mathbb{P}_\sigma^n \implies \mathbb{P}_\sigma \in \mathcal{M}_1(\Omega)$ *for every starting configuration* $\sigma \in \Omega_0$.

c') (η_t^{EP}) *is uniquely ergodic with invariant measure* $\mu^{EP} \in \mathcal{M}_1(\Omega_0)$.

Then $\mu^{EP(n)} \implies \mu^{EP}$ *weakly.*

Remark 3.3.1. Theorem 3.3.3 does only require that the limiting process (η_t^{EP}) is uniquely ergodic. In particular, the processes $(\eta_t^{EP(n)})$ do not need to be uniquely ergodic. As an example of the latter, one can consider the case where $(\eta_t^{(n)})$ is the contact process with parameter $\lambda(n) \downarrow \lambda_c$ and $\inf_{\eta\in\Omega} \alpha_n(\eta, o) \uparrow 1$.

Theorem 3.3.3 gives a generalisation of Theorem 3.3 in [100]. There they showed continuity for the environment process with respect to changes of the transition probabilities of the random walk, assuming that Assumption 1a therein to hold (which we also stated in (3.2.6)). Theorem 3.3.3 yields a similar continuity result which in addition allow for changes in the dynamics of the environment (η_t). Moreover, unique ergodicity is a weaker assumption than the mixing assumption given by (3.2.5), as we have already seen in Subsection 3.2.1 and Subsection 3.2.2.

3.3.3 Estimating the Radon-Nikodym derivative

We end this section with an alternative route for proving the existence of an invariant measure for the environment process which is absolutely continuous with respect to the underlying

environment. An advantage of this approach is that it implies bounds on the Radon-Nikodym derivative.

Theorem 3.3.4. *Assume that, for some $M_1 \in (0, \infty)$,*

$$\sup_{B \in \mathcal{F}_{\geq 0}} \sup_{A_l \in \mathcal{A}_{-l}^{-1}} \left| \frac{\mathbb{P}(B \mid A_l)}{\mathbb{P}(B)} - 1 \right| \leq M_1, \quad \forall\, l \in \mathbb{N}\,. \tag{3.3.2}$$

Then there is a $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$, invariant under (η_t^{EP}), and $\mathbb{P}^{EP} \ll \mathbb{P}$ on $(\Omega, \mathcal{F}_{\geq 0})$. Moreover, the corresponding Radon-Nikodym derivative is bounded by M_1 in the L_∞-norm. Furthermore, if for some $M_2 \in (0, \infty)$,

$$\sup_{B \in \mathcal{F}_{\geq 0}} \sup_{A_l \in \mathcal{A}_{-l}^{-1}} \left| \frac{\mathbb{P}(B)}{\mathbb{P}(B \mid A_l)} - 1 \right| \leq M_2, \quad \forall\, l \in \mathbb{N}\,. \tag{3.3.3}$$

Then $\mathbb{P} \ll \mathbb{P}^{EP}$ and the corresponding Radon-Nikodym derivative is bounded by M_2 in the L_∞-norm.

Remark 3.3.2. Mutually absolute continuity can also be shown without requiring (3.3.3) to hold. In particular, if (3.3.2) holds and (X_t) is elliptic in the time direction, it can be shown that $\mathbb{P} \ll \mathbb{P}^{EP}$. Under these assumptions it can moreover be shown that $\mathbb{P}^{EP}$ is ergodic and that $\left(t^{-1} \sum_{k=1}^{t} P_{\mathbb{P}}(\eta_k^{EP} \in \cdot)\right)_{t \geq 1}$ converges weakly towards $\mathbb{P}^{EP}$.

Mixing assumption of the type (3.3.2) and (3.3.3) are typically much stronger than mixing assumptions as in Theorems 3.1.1 and 3.1.3. Nevertheless, we believe that Theorem 3.3.4 is applicable to a wide range of models and is not restricted to the uniform mixing case. However, it seems difficult to verify (3.3.2) and (3.3.3) for concrete examples unless strong mixing assumptions are made.

One class of examples to which Theorem 3.3.4 applies are Gibbs measures in the high-temperature regime satisfying the Dobrushin-Shlosman strong mixing condition (as considered in Rassoul-Agha [94] for RWRE models); see Theorem 1.1 (in particular, Condition IIId) in Dobrushin and Shlosman [51]. Another class of environments are certain monotone Gibbs measures for which Alexander [1] proved (see Theorems 3.3 and 3.4 therein) that weak mixing implies ratio mixing. In particular, the models considered there satisfy (3.3.2) and (3.3.3) throughout the uniqueness regime. We also mention the method of disagreement percolation, which is particularly useful for models with hard-core constraints, see van den Berg and Maes [21].

In the case of dynamic random environments which in addition are reversible with respect to time, typically, the methods described above for random fields can be adapted to yield similar bounds. In Section 3.4.5 we introduce a new class of dynamic random environments satisfying (3.3.2), allowing for non-reversible dynamics. We comment next on the scope of this approach.

Our approach is by means of disagreement percolation and applies to *discrete-time finite-range Markov chain* (η_t). In fact, we shall need more than subcriticality of the ordinary disagreement process. For what we believe to be technical reasons, we will introduce what we call the *strong disagreement percolation coupling*. This is a triple $(\eta_t^1, \eta_t^2, \xi_t)$ where (η_t^1, η_t^2) is a coupling of $\mathbb{P}_{\eta_0^1}$ and $\mathbb{P}_{\eta_o^2}$, $\xi_t(x) = 0$ implies $\eta_t^1(x) = \eta_t^2(x)$, and η^1 and ξ are independent. That is, the disagreement process ξ and the process η^1 are independent. This independence is a stronger assumption than regular disagreement percolation and the strong disagreement percolation process is subcritical for models at "very high-temperature". We refer to Section 3.4.5 for a precise construction of the strong disagreement percolation coupling and a proof of the following theorem.

Theorem 3.3.5 (Strong disagreement percolation)**.** *Suppose the strong disagreement percolation process is subcritical. Then there exists $\delta < 1$ and $C > 0$ so that for any $B \in \mathcal{F}_{=0}$,*

$$\sup_{\substack{A_{-(k+1)}^{-1} \in \mathcal{A}_{-(k+1)}^{-1}, A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1} \\ A_{-k}^{-1} \subset A_{-(k+1)}^{-1}}} \sup_{B \in \mathcal{F}_{=0}} \left| \left[\frac{\mathbb{P}\left(B \mid A_{-k-1}^{-1}\right)}{\mathbb{P}\left(B \mid A_{-k}^{-1}\right)} \right]^{\pm 1} - 1 \right| \leq C\delta^k.$$

Theorem 3.3.5 implies that the environment process (η_t^{EP}) has a unique invariant distribution, $\mu^{EP} \in \mathcal{M}_1(\Omega_0)$. In particular, μ^{EP} is absolutely continuous with respect to the (necessarily unique) invariant measure of (η_t), denoted by $\mu \in \mathcal{M}_1(\Omega_0)$. As a further consequence, we obtain uniform control on the Radon-Nikodym derivative.

Corollary 3.3.6 (Uniform control on the Radon-Nikodym derivative)**.** *Assume that the environment (η_t) has a strong disagreement percolation coupling which is subcritical. Then μ^{EP} and μ are mutually absolutely continuous. Moreover, there exists a constant $M \in (0, \infty)$, depending only on the environment, such that $\left\| \frac{d\mu^{EP}}{d\mu} \right\|_\infty \leq M$ and $\left\| \frac{d\mu}{d\mu^{EP}} \right\|_\infty \leq M$.*

Subcriticality of the strong disagreement coupling is a much stronger assumption than the uniform mixing assumption in (3.2.5). For comparison with other coupling methods, consider for concreteness the stochastic Ising model with inverse temperature $\beta > 0$ (see e.g. [44] for a definition). This model satisfies (3.2.5) for all $\beta < \beta_c$, where β_c is the critical inverse temperature. On the other hand, it has a subcritical strong disagreement coupling whenever

$$\beta < \frac{1}{4d} \log\left(\frac{2d}{2d-1}\right) < \beta_c. \tag{3.3.4}$$

For comparison, this condition is better (with a factor 2) compared with the disagreement percolation coupling introduced in [44] (see Equation (11) therein).

Remark 3.3.3. The estimate in (3.3.4) is valid for antiferromagnetic models and models with a magnetic field, as also considered in [44]. In particular, the strong disagreement percolation method is not restricted to monotone environments.

3.4 Proofs

In this section, we present the proofs of the theorems given in the previous sections. In Subsection 3.4.1 we give the proofs of theorems in Section 3.1.3. Proofs of theorems in Section 3.2 are given in Subsection 3.4.2. In the remaining subsections we present proofs of theorems from Section 3.3. In particular, Subsection 3.4.5 introduces the strong disagreement coupling and contains the proof Theorem 3.3.5.

3.4.1 Proof of main results

The main application of the expansion in Theorem 3.3.1 for the proofs of Theorems 3.1.1 and 3.1.3 is the following lemma.

Lemma 3.4.1. *Let $\Lambda \subset \mathbb{Z}^{d+1}$. For $B \in \mathcal{F}_\Lambda$ and $k \in \mathbb{N}$ we have that*

$$|P_{\mathbb{P}}(\eta_k^{EP} \in B) - \mathbb{P}(B)| \leq \sup_{A \in \mathcal{A}_{-\infty}} |\mathbb{P}(B \mid A) - \mathbb{P}(B)|.$$

Proof. Let $l \in \mathbb{N}$ and consider any $B \in \mathcal{F}_\Lambda$. From Theorem 3.3.1 we have that for every $k \in \mathbb{N}$,

$$\begin{aligned}
&\left|P_{\mathbb{P}}\left(\eta_k^{EP} \in B\right) - \mathbb{P}\left(B\right)\right| \\
=&\Big|\sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}\left(B, A_{-k}^{-1}\right) P\left(X_{-k,\ldots,0} = \gamma \mid A_{-k}^{-1}\right) - \mathbb{P}\left(B\right)\Big| \\
=&\Big|\sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}\left(B \mid A_{-k}^{-1}\right) \mathbb{P}^{-k}\left(X_{-k,\ldots,0} = \gamma, A_{-k}^{-1}\right) - \mathbb{P}\left(B\right)\Big| \\
\leq& \sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \left|\mathbb{P}\left(B \mid A_{-k}^{-1}\right) - \mathbb{P}\left(B\right)\right| \mathbb{P}^{-k}\left(X_{-k,\ldots,0} = \gamma, A_{-k}^{-1}\right) \\
\leq& \sup_{A \in \mathcal{A}_{-\infty}} \left|\mathbb{P}\left(B \mid A\right) - \mathbb{P}\left(B\right)\right|,
\end{aligned}$$

where in the second last and the last inequality we used the fact that

$$\sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}^{-k}\left(X_{-k,\ldots,0} = \gamma, A_{-k}^{-1}(\gamma, \sigma)\right) = 1.$$

□

Proof of Theorem 3.1.1. Consider a sequence (t_k) and a sequence of measures given by $\mathbb{Q}_k := \frac{1}{t_k} \sum_{t=0}^{t_k - 1} P_{\mathbb{P}}(\eta_t^{EP} \in \cdot)$ that converges weakly to $\mathbb{Q} \in \mathcal{M}_1(\Omega)$. By standard compactness arguments such a sequence exists and, moreover, any such limiting measure $\mathbb{Q}$ is

invariant for (η_t^{EP}). A proof of the last claim is e.g. given in [94]; see page 1457 in the proof of Theorem 3 therein.

Since (3.1.3) is assumed to hold, it follows by Lemma 3.4.1 that for any $l \in \mathbb{N}$ and $B \in \mathcal{F}_{\mathcal{C}(l)}$,

$$|\mathbb{Q}_k(B) - \mathbb{P}(B)| \leq \phi(l), \tag{3.4.1}$$

for some $\phi\colon\ \mathbb{N} \to [0,1]$ such that $\lim_{l\to\infty} \phi(l) = 0$. As this estimate is uniform in k, we claim that (3.4.1) also holds when $\mathbb{Q}_k$ is replaced by $\mathbb{Q}$. To see this, consider the space of measures measurable with respect to $(\Omega, \mathcal{F}_{\mathcal{C}(l)})$. The ball of radius $\phi(l)$ around $\mathbb{P}$ (in the total variation sense) is compact in the topology of weak convergence by the Banach-Alaoglu-Theorem. Here we use that the space is compact, the dual of the continuous bounded functions are finite signed measures equipped with the total variation norm, and the weak convergence of measures is the weak-* convergence in this functional-analytic setting. Since the ball is compact it is closed, and any limit point $\mathbb{Q}$ of the sequence $\mathbb{Q}_k$ is also inside the ball. Hence $|\mathbb{Q}(B) - \mathbb{P}(B)| \leq \phi(l)$ for any $B \in \mathcal{F}_{\mathcal{C}(l)}$ and consequently, since $\lim_{l\to\infty} \phi(l) = 0$, we have $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_\infty^\infty$.

We continue with the proof that $\mathbb{Q}$ is ergodic with respect to (η_t^{EP}), by following the proof of Theorem 2ii) in [94]. Denote by $\mathcal{I} \subset \mathcal{F}$ the σ-algebra consisting of those events invariant under the evolution of (η_t^{EP}). Further, let f be any local bounded function on Ω and define $g = \mathbb{E}_{\mathbb{Q}}(f \mid \mathcal{I})$. Birkhoffs ergodic theorem implies that,

$$P_\eta\left(\lim_{n\to\infty} n^{-1} \sum_{m=1}^{n} f(\eta_m^{EP}) = g(\eta)\right) = 1, \quad \text{for } \mathbb{Q}\text{-a.e. } \eta \in \Omega. \tag{3.4.2}$$

Using that $\mathbb{Q}$ is invariant and that g is harmonic, we have

$$\begin{aligned}
&\sum_{|x|\leq R} \int \alpha(\eta, x)(g(\eta) - g(\theta_{x,1}\eta))^2 \mathbb{Q}(d\eta) \\
=&\int g^2(\eta)\mathbb{Q}(d\eta) - 2\int g(\eta) \sum_{|x|\leq R} \alpha(\eta, x) g(\theta_{x,1}\eta)\mathbb{Q}(d\eta) \\
&\qquad + \int \sum_{|x|\leq R} \alpha(\eta, x)(g(\theta_{x,1}\eta))^2 \mathbb{Q}(d\eta) \\
=&0.
\end{aligned}$$

In particular, since (X_t) is elliptic in the time direction, $g = g \circ \theta_{o,1}$, $\mathbb{Q}$-a.s.

Next, for each $t \in \mathbb{N}$, denote by $B_t \subset \{X_i = o \text{ for all } i \in \{0, \dots, t\}\}$ the event that the random walk does not move in the time-interval $[0, t]$, irrespectively of the environment. Since (X_t) is elliptic in the time direction, B_t has strictly positive probability and can be taken independently of the environment. Further, define

$$\bar{g}(\eta) := \limsup_{n\to\infty} \frac{1}{\mathbb{Q}(B_t)} \int_{B_t} n^{-1} \sum_{m=1}^{n} f(\eta_m^{EP}) dP_\eta.$$

Then, because of (3.4.2), we know that $g = \bar{g}$, $\mathbb{Q}$-a.s. Further, using the above mentioned independence property, and by possibly taking t large, we note that $\bar{g}$ is $\mathcal{C}(k)$-measurable for any $k \in \mathbb{N}$. Consequently, the same holds for g, and hence g is $\mathcal{F}_\infty^\infty$-measurable. Furthermore, since $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_\infty^\infty$, this implies that (3.4.2) holds $\mathbb{P}$-a.s., and that $g = g \circ \theta_{o,1}$, $\mathbb{P}$-a.s. As $\mathbb{P}$ is ergodic with respect to $\theta_{o,1}$, it moreover follows that g is constant $\mathbb{P}$-a.s., and hence also $\mathbb{Q}$-a.s. Since f was an arbitrary local bounded function, we conclude from this that $\mathcal{I}$ is trivial and thus that $\mathbb{Q}$ is ergodic with respect to (η_t^{EP}).

To conclude the proof we also note that (3.1.4) holds. Indeed, since $\mathbb{Q}$ was an arbitrary (sub) sequence of $(\mathbb{Q}_k)$, all the estimates above are valid for any such limiting measure. In particular, each of these limiting measures equal $\mathbb{P}$ on $\mathcal{F}_\infty^\infty$, and consequently, they are all ergodic and equal on $\mathcal{I}$. Thus, they are the same, and we conclude that (3.1.4) holds with respect to $\mathbb{P}$, where we call the limiting measure $\mathbb{P}^{EP}$. Initialising (η_t^{EP}) with any other probability measure, absolute continuous with respect to $\mathbb{P}$ on $\mathcal{F}_\infty^\infty$, the exact same argument as outlined above applies, from which we conclude (3.1.4) and the proof of Theorem 3.1.1. □

Proof of Corollary 3.1.2. The claim is an (almost direct) application of ergodicity and that $\mathbb{P} = \mathbb{P}^{EP}$ on $\mathcal{F}_\infty^\infty$. Indeed, let $D(\eta) := \sum_{z \in \mathbb{Z}^d} z\alpha(\eta, z)$ be the local drift of the random walker in environment η. A direct consequence of Theorem 3.1.1 is that

$$P_{\mathbb{P}^{EP}}\left(\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} D(\theta_{X_k,k}\eta) = v\right) = 1, \tag{3.4.3}$$

where $v = \int D(\eta) P_{\mathbb{P}^{EP}}(d\eta)$. By using that $\mathbb{P} = \mathbb{P}^{EP}$ on $\mathcal{F}_\infty^\infty$, it follows that this also holds with respect to $P_{\mathbb{P}}$. Now, note that $M_n = X_n - \sum_{m=0}^{n-1} D(\theta_{X_k,k}\eta)$ is a martingale with bounded increments under P_η. Therefore $P_\eta(\lim_{n\to\infty} n^{-1} M_n = 0) = 1$ which together with (3.4.3) implies the law of large numbers. □

We next turn to the proof of Theorem 3.1.3. The following lemma is essentially copied from [22].

Lemma 3.4.2. *Assume $\mathbb{P}^{EP}$ is invariant with respect to (η_t^{EP}) and that $\mathbb{P}^{EP}$ and $\mathbb{P}$ restricted to $(\Omega, \mathcal{F}_{\geq 0})$ are not singular. Assume (X_t) is elliptic. Then there exists $\mathbb{P}_c^{EP} \in \mathcal{M}_1(\Omega)$, invariant for (η_t^{EP}) and mutually absolutely continuous to $\mathbb{P}$ on $(\Omega, \mathcal{F}_{\geq 0})$.*

Proof of Lemma 3.4.2. Consider the (unique) Lebesgue decomposition of $\mathbb{P}^{EP}$ with respect to $\mathbb{P}$ restricted to $(\Omega, \mathcal{F}_{\geq 0})$. That is, let

$$\mathbb{P}^{EP}(B) = \alpha \mathbb{P}_c^{EP}(B) + (1-\alpha)\mathbb{P}_s^{EP}(B), \quad \forall\, B \in \mathcal{F}_{\geq 0},$$

where $\mathbb{P}_c^{EP} \ll \mathbb{P}$ and $\mathbb{P}_s^{EP} \perp \mathbb{P}$ on $(\Omega, \mathcal{F}_{\geq 0})$. By assumption, we know that $\alpha > 0$. If $\alpha = 1$, the statement is immediate. Thus, assume $\alpha \in (0, 1)$. In a first step, observe that

$(\theta_{y,1} \circ \mathbb{P}^{EP})_c = \theta_{y,1} \circ \mathbb{P}_c^{EP}$ for every $y \in \mathcal{R}$. This follows from translation invariance of $\mathbb{P}$ which implies that taking the continuous part with respect to $\mathbb{P}$ is the same as taking the continuous part with respect to $\theta_{y,1} \circ \mathbb{P}$. The same is true for the singular part $\mathbb{P}_s^{EP}$.

Note that, since E is finite and (X_t) is finite range, we have that ellipticity in fact implies uniform ellipticity. That is, there is an $\epsilon > 0$ such that

$$\inf_{y \in \mathcal{R}} \inf_{\eta \in \Omega} \alpha(\eta, y) \geq \epsilon > 0.$$

In particular, by invariance of $\mathbb{P}^{EP}$,

$$\mathbb{P}^{EP} = \sum_{y \in \mathcal{R}} \alpha(\cdot, y) \theta_{y,1} \circ \mathbb{P}^{EP} \geq \epsilon \sum_{y \in \mathcal{R}} \theta_{y,1} \circ \mathbb{P}^{EP}$$

and therefore $\theta_{y,1} \circ \mathbb{P}^{EP} \ll \mathbb{P}^{EP}$ for every $y \in \mathcal{R}$. By using first ellipticity and then $(\theta_{y,1} \circ \mathbb{P}^{EP})_c = \theta_{y,1} \circ \mathbb{P}_c^{EP}$ we have

$$\begin{aligned} \mathbb{P}_c^{EP} &= \left(\sum_{y \in \mathcal{R}} \alpha(\cdot, y) \theta_{y,1} \circ \mathbb{P}^{EP} \right)_c = \sum_{y \in \mathcal{R}} \alpha(\cdot, y) \left(\theta_{y,1} \circ \mathbb{P}^{EP} \right)_c \\ &= \sum_{y \in \mathcal{R}} \alpha(\cdot, y) \theta_{y,1} \circ \mathbb{P}_c^{EP}, \end{aligned} \tag{3.4.4}$$

which means that $\mathbb{P}_c^{EP}$ is invariant for (η_t^{EP}).

Let $f = \frac{d\mathbb{P}_c^{EP}}{d\mathbb{P}}$ and define $B = \{\eta \in \Omega \colon f(\eta) > 0\}$. As a consequence of (3.4.4) we have

$$f(\eta) \geq \epsilon \sum_{y \in \mathcal{R}} f(\theta_{y,1} \eta),$$

and, in particular, $\eta \in B$ implies $\theta_{y,1}\eta \in B$, $y \in \mathcal{R}$. In particular, B is invariant under $\theta_{o,1}$, and by ergodicity of $\mathbb{P}$ this is a $0-1$ event. Since by assumption $\alpha > 0$ we have $\mathbb{P}(B) = 1$ and therefore $\mathbb{P} \ll \mathbb{P}_c^{EP}$ on $(\Omega, \mathcal{F}_{\geq 0})$. □

Lemma 3.4.3. *Let $\Lambda \subset \mathbb{Z}^{d+1}$ finite and fix $\sigma \in E^{\Lambda^c}$. Let $\mathbb{P}(\cdot \mid \sigma)$ and $\mathbb{P}^{-k}(\cdot \mid \sigma)$ be the regular conditional probabilities of $\mathbb{P}$ and $\mathbb{P}^{-k}$ on E^{Λ} given σ. Then, for $B \in \mathcal{F}_\Lambda$ and $k \geq 1$,*

$$\mathbb{P}^{-k}(B \mid \sigma) = \sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}\left(B, A_{-k}^{-1} \mid \sigma\right) P\left(X_{-k,\ldots,0} = \gamma \mid A_{-k}^{-1}\right).$$

Proof. The proof is mostly as for the unconditional expansion. Additionally we use the following equalities:

$$\begin{aligned} \mathbb{P}^{-k}(B \mid A_{-k}^{-1}, X_{-k,\ldots,0} = \gamma, \sigma) &= P_{\mathbb{P}}(\theta_{-\gamma_{-k},k} B \mid \theta_{-\gamma_{-k},k} A_{-k}^{-1}, \theta_{-\gamma_{-k},k}\sigma) \\ &= \mathbb{P}(B \mid A_{-k}^{-1}, \sigma) \end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}^{-k}(X_{-k,\dots,0} = \gamma, A_{-k}^{-1} \mid \sigma)\\
=&P_{\mathbb{P}}(\eta_k^{EP} \in A_{-k}^{-1}, X_{0,\dots,k} = \gamma - \gamma_{-k} \mid \theta_{X_k,k}\sigma)\\
=&P(X_{0,\dots,k} = \gamma - \gamma_{-k} \mid A_{-k}^{-1})\mathbb{P}(\theta_{-\gamma_{-k,k}}\eta \in A_{-k}^{-1} \mid \theta_{-\gamma_{-k,k}}\sigma)\\
=&P(X_{0,\dots,k} = \gamma - \gamma_{-k} \mid A_{-k}^{-1})\mathbb{P}(\eta \in A_{-k}^{-1} \mid \sigma).
\end{aligned}$$

Take also note that summation should only include events A_{-k}^{-1} which have positive probability with respect to the conditional law given σ. □

Proof of Theorem 3.1.3. By applying the same line of reasoning as in the proof of Theorem 3.1.1, it easy to see that (3.1.6) holds as a consequence of (3.1.5) and Lemma 3.4.1. In particular, there is a measure $\mathbb{Q}$ invariant under (η_t^{EP}) such that (3.1.6) holds, and consequently $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_{\geq 0}^{\infty}$. We focus on the proof that $\mathbb{Q}$ and $\mathbb{P}$ are mutually absolutely continuous under the additional assumption that (X_t) is elliptic.

Since (3.1.6) holds, there is an $l \in \mathbb{N}$ such that $\sup_{B \in \mathcal{F}_{\Lambda(l)}} |\mathbb{Q}(B) - \mathbb{P}(B)| < 1$. In particular, $\mathbb{P}$ and $\mathbb{Q}$ are not singular on $\mathcal{F}_{\Lambda(l)}$. In order to conclude that $\mathbb{Q}$ and $\mathbb{P}$ are not singular on $\mathcal{F}_{\geq 0}$ we make use of Lemma 3.4.3 and the assumption that $|E| < \infty$. Indeed, for any $\sigma \in E^{\Lambda(l)}$ we have by Lemma 3.4.3 that $\mathbb{P}^{-k}(\cdot \mid \sigma) \ll \mathbb{P}(\cdot \mid \sigma)$. Further, since E is finite any local function is continuous and hence we also have $\mathbb{Q}(\cdot \mid \sigma) \ll \mathbb{P}(\cdot \mid \sigma)$. And since $\mathbb{Q}$ is non-singular on $\Lambda(l)$ with respect to $\mathbb{P}$ it has non-trivial continuous part and corresponding density on $\Lambda(l)$. Thus, we now also have shown that conditioned on $\Lambda(l)$ the measure $\mathbb{Q}$ has a density inside $\mathbb{H} \setminus \Lambda(l)$. It hence follows that $\mathbb{Q}$ is not singular with respect to $\mathbb{P}$ on $\mathbb{H}$. As a consequence of Lemma 3.4.2 and that $\mathbb{Q}$ and $\mathbb{P}$ are not singular on $\mathbb{H}$ we conclude that, when (X_t) is elliptic, there is a measure $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$ invariant under (η_t^{EP}) such that $\mathbb{P}^{EP}$ and $\mathbb{P}$ are mutually absolutely continuous on $\mathcal{F}_{\geq 0}$.

□

Proof of Corollary 3.1.4. Corollary 3.1.4 is an application of Theorem 1 in [53]. In order to fulfil the requirements of their theorem six conditions needs to be satisfied, i.e. (A0)-(A5) therein. Our main contribution is that Condition (A1) is satisfied when (3.1.7) holds. This follows as a direct consequence of Theorem 3.1.3. Furthermore, that Conditions (A0) holds under (3.1.7) follows partly by the mixing assumptions on μ. Moreover, that also μ^{EP} and μ are mixing is a consequence of Theorem 3.4 in [100] which yields exponential rate of convergence for environment process under the assumption that (3.1.7) holds. By the same reasoning, Theorem 3.4 in [100] also implies that Condition (A2) holds true. Lastly, we note that Conditions (A3) and (A4) are true by assumption and Condition (A5) is satisfied since (X_t) is elliptic. □

3.4.2 Proof of examples

Proof of Theorem 3.2.1

Proof. For $k \in \mathbb{N}$, let $\gamma \in \Gamma_k$ and consider $A \in \mathcal{A}_{-k}^{-1}(\gamma)$. Since A consists of a fixed observation of the environment along the path γ we can write $A = \bigcap_{x\in\mathbb{Z}^d} A_x$, where A_x is the observation on the line $\{(x,s)\colon s \in \mathbb{Z}\}$. Without change of notation we also treat A_x as an event on the space $E^{\mathbb{Z}}$. Denote by $\widehat{\mathbb{P}}_x$ the optimal coupling (in the sense of total variation distance) of $\mathbb{P}_o(\cdot \mid A_x)$ and $\mathbb{P}_o(\cdot)$, and by $\widehat{\mathbb{P}} = \bigtimes_{x\in\mathbb{Z}^d} \widehat{\mathbb{P}}_x$. The product structure of $\mathbb{P}$ plus the fact that A is given by the intersection of the events A_x gives us that $\widehat{\mathbb{P}}$ is a coupling of $\mathbb{P}(\cdot \mid A)$ and $\mathbb{P}(\cdot)$.

For $l \in \mathbb{N}$, let $B \in \mathcal{F}_{\Lambda(l)}$. We have that

$$\begin{aligned}
|\mathbb{P}(B \mid A) - \mathbb{P}(B)| &\le \widehat{\mathbb{P}}\left(\exists (x,s) \in \Lambda(l)\colon \eta_s^1(x) \neq \eta_s^2(x)\right) \\
&\le \sum_{x\in\mathbb{Z}^d} \widehat{\mathbb{P}}\left(\exists s \ge 0\colon (x,s) \in \Lambda(l) \text{ and } \eta_s^1(x) \neq \eta_s^2(x)\right) \\
&\le \sum_{t<0} \sum_{(x,t)\in\gamma+[-R,R]^d\times\{0\}} \sup_{B\in\mathcal{G}_{\ge 0\vee(l-|x|)}} |\mathbb{P}_o(B|A_x) - \mathbb{P}(B)|\,.
\end{aligned}$$

The last line follows from the fact that $\eta^1(x)$ and $\eta^2(x)$ can only differ if the site x is part of the observation A, since otherwise $A_x = E^{\mathbb{Z}}$. Condition (3.2.1) thus ensures that the sum in the last line is finite. In particular, the sum converges to 0 as $l \to \infty$. This shows that $\mathbb{P}$ satisfies (3.1.5). □

Proof of Corollary 3.2.2

Corollary 3.2.2 follows by a slightly stronger statement, which we state and prove first.

Theorem 3.4.4. *Assume that $\mathbb{P} \in \mathcal{M}_1(\Omega)$ has finite speed of propagation and that*

$$\sum_{t\ge 1} t^d \sup_{A'\in\mathcal{A}_{-\infty}^{-t}} \widehat{\mathbb{P}}_{\Omega,A'}\left(\eta_0^1(o) \neq \eta_0^2(o)\right) < \infty. \tag{3.4.5}$$

Then $\mathbb{P}$ satisfies the conditions of Theorem 3.1.3.

Proof of Theorem 3.4.4. Let $l \in \mathbb{N}$ and consider $B \in \mathcal{F}_{\Lambda(l)}$. By Theorem 3.1.3, it is sufficient to obtain uniform estimates of the form $|\mathbb{P}(B \mid A) - \mathbb{P}(B)| \le \phi(l)$, where $A \in \mathcal{A}_{-\infty}$, and where $\phi(l)$ approaches 0 as $l \to \infty$. For this, we first note that

$$\begin{aligned}
|\mathbb{P}\,(B \mid A) - \mathbb{P}\,(B)| &\le \widehat{\mathbb{P}}_{A,\Omega}\left(\eta_t^1(x) \neq \eta_t^2(x) \text{ for some } (x,t) \in \Lambda(l)\right) \\
&\le \sum_{(x,t)\in\Lambda(l)} \widehat{\mathbb{P}}_{A,\Omega}\left(\eta_t^1(x) \neq \eta_t^2(x)\right).
\end{aligned}$$

Thus, it suffices to control $\widehat{\mathbb{P}}_{A,\Omega}\left(\eta_t^1(x) \neq \eta_t^2(x)\right)$ for each $(x,t) \in \Lambda(l)$. For this, fix $(x,t) \in \Lambda(l)$ such that $\|(x,t)\|_1 \geq \alpha s$ for some $s \geq 0$. Further, let $A' \in \mathcal{A}_{-\infty}^{-s}$ be such that $A' \cap A = A$, and denote by $\tilde{\mathbb{P}}_{A,A',\Omega}$ a measure on $\Omega \times \Omega \times \Omega$ such that

$$\begin{aligned}
&\tilde{\mathbb{P}}_{A,A',\Omega}(\eta^1 \in \cdot, \eta^2 \in \cdot, \Omega) = \widehat{\mathbb{P}}_{A,A'}(\eta^1 \in \cdot, \eta^2 \in \cdot);\\
&\tilde{\mathbb{P}}_{A,A',\Omega}(\eta^1 \in \cdot, \Omega, \eta^3 \in \cdot) = \widehat{\mathbb{P}}_{A,\Omega}(\eta^1 \in \cdot, \eta^3 \in \cdot);\\
&\tilde{\mathbb{P}}_{A,A',\Omega}(\Omega, \eta^2 \in \cdot, \eta^3 \in \cdot) = \widehat{\mathbb{P}}_{A',\Omega}(\eta^2 \in \cdot, \eta^3 \in \cdot).
\end{aligned}$$

We then have that

$$\begin{aligned}
\widehat{\mathbb{P}}_{A,\Omega}\left(\eta_t^1(x) \neq \eta_t^2(x)\right) =& \tilde{\mathbb{P}}_{A,A',\Omega}(\eta_t^1(x) \neq \eta_t^3(x))\\
\leq& \tilde{\mathbb{P}}_{A,A',\Omega}(\eta_t^1(x) \neq \eta_t^2(x) \text{ or } \eta_t^2(x) \neq \eta_t^3(x))\\
\leq& \tilde{\mathbb{P}}_{A,A',\Omega}(\eta_t^1(x) \neq \eta_t^2(x)) + \tilde{\mathbb{P}}_{A,A',\Omega}(\eta_t^2(x) \neq \eta_t^3(x))\\
=& \widehat{\mathbb{P}}_{A,A'}\left(\eta_t^1(x) \neq \eta_t^2(x)\right) + \widehat{\mathbb{P}}_{A',\Omega}\left(\eta_t^1(x) \neq \eta_t^2(x)\right),
\end{aligned}$$

Furthermore, it holds that

$$\begin{aligned}
&\widehat{\mathbb{P}}_{A,A'}\left(\eta_t^1(x) \neq \eta_t^2(x)\right) + \widehat{\mathbb{P}}_{\Omega,A'}\left(\eta_t^1(x) \neq \eta_t^2(x)\right)\\
\leq& \sup_{A \in \mathcal{F}_{<0}, A' \in \mathcal{F}_{\Lambda(\alpha s, s)}} \widehat{\mathbb{P}}_{A,A'}\left(\eta_s^1(o) \neq \eta_s^2(o)\right) + \sup_{A'' \in \mathcal{A}_{-\infty}^{-\alpha s}} \widehat{\mathbb{P}}_{\Omega,A''}\left(\eta_0^1(o) \neq \eta_0^2(o)\right),
\end{aligned}$$

since the finite speed of propagation coupling is invariant with respect to translations of the conditioning and the argument. Thus, by the analysis above, we obtain that

$$\begin{aligned}
|\mathbb{P}(B \mid A) - \mathbb{P}(B)| \leq& \left(\sum_{\substack{(x,t) \in \Lambda(l)\\ \|(x,t)\|_1 = \alpha s}} \sup_{A \in \mathcal{F}_{<0}, A' \in \mathcal{F}_{\Lambda(\alpha s, s)}} \widehat{\mathbb{P}}_{A,A'}\left(\eta_s^1(o) \neq \eta_s^2(o)\right) \right)\\
&+ \left(\sum_{\substack{(x,t) \in \Lambda(l)\\ \|(x,t)\|_1 = \alpha s}} \sup_{A'' \in \mathcal{A}_{-\infty}^{-\alpha s}} \widehat{\mathbb{P}}_{A'',\Omega}\left(\eta_0^1(x) \neq \eta_0^2(x)\right) \right). \qquad (3.4.6)
\end{aligned}$$

To conclude the proof, we note that the number of site in $\mathbb{H}$ at distance αs from the origin is of order s^d. Thus, due to (3.2.2) the first sum on the r.h.s. of (3.4.6) converges towards 0 as l approaches ∞. Similarly, by applying (3.4.5), also the second sum on the r.h.s. of (3.4.6) converges towards 0 as l approaches ∞. From this we conclude the proof. □

Proof of Corollary 3.2.2. The proof of Corollary 3.2.2 follows along the lines of the proof of Theorem 3.4.4, by making use of the finite speed of propagation property and (3.2.3). First

note that, for any $B \in \mathcal{F}_{\Lambda(l)}$ and $A \in \mathcal{A}_{-\infty}$,

$$\begin{aligned}|\mathbb{P}(B \mid A) - \mathbb{P}(B)| &= \left|\widetilde{\mathbb{P}}(\Phi(\xi) \in B \mid \xi \in \Phi^{-1}A) - \widetilde{\mathbb{P}}(\Phi(\xi) \in B)\right| \\ &\leq \widehat{\mathbb{P}}_{\Omega,\Phi^{-1}A}\left(\eta_t^1(x) \neq \eta_t^2(x) \text{ for some } (x,t) \in \Lambda(l)\right) \\ &\leq \sum_{(x,t)\in\Lambda(l)} \widehat{\mathbb{P}}_{\Omega,\Phi^{-1}A}\left(\eta_t^1(x) \neq \eta_t^2(x)\right).\end{aligned}$$

Thus, it suffices control $\widehat{\mathbb{P}}_{\Omega,\Phi^{-1}A}\left(\eta_t^1(x) \neq \eta_t^2(x)\right)$ for each $(x,t) \in \Lambda(l)$. and to show that the latter term above approaches 0 as $l \to \infty$. For this, since Φ is assumed to be finite range, we note that the finite speed of propagation property (ξ_t) transfers to events of the form $\Phi^{-1}A$. Thus, by considering a coupling $\widetilde{\mathbb{P}}_{\Phi^{-1}A,\Phi^{-1}A',\Omega}$, similar to the coupling in the proof of Theorem 3.4.4, and where $A' \in \mathcal{A}_{-\infty}^{-s}$ and $A' \cap A = A$. Further, by applying the estimates (3.2.2) and (3.2.3), we may proceed by the same line of reasoning as in the proof of Theorem 3.4.4, from which we conclude the proof. □

Proof of Theorem 3.2.3

Proof of Theorem 3.2.3. Denote by μ^{EP} any limiting measure of the Cesaro means $\mu_n^{EP} := \frac{1}{n}\sum_{k=1}^n \mathbb{P}_\mu(\eta_k^{EP} \in \cdot) \in \mathcal{M}_1(\Omega_0)$ (by possibly taking subsequential limits) and note that μ^{EP} is invariant with respect to (η_t^{EP}).

We first show that μ^{EP} agrees with μ on $\mathcal{F}_{=0}^{\infty}$. Let $l \in \mathbb{N}$ and consider any $B \in \mathcal{F}_{\Lambda_0(l)}$ with $\Lambda_0(l) := \{(x,0) \colon \| (x,0) \|_1 \geq l\}$. Similar to the proof of Theorem 3.4.4, it follows that, for any $A \in \mathcal{A}_{-\infty}$,

$$\begin{aligned}|\mathbb{P}(B \mid A) - \mathbb{P}(B)| \leq &\left(\sum_{x\in\Lambda_0(l)} \sup_{A\in\mathcal{F}_{<0}, A'\in\mathcal{F}_{\Lambda(\alpha s,s)}} \widehat{\mathbb{P}}_{A,A'}\left(\eta_{\alpha s}^1(o) \neq \eta_{\alpha s}^2(o)\right)\right) \\ &+ \left(\sum_{x\in\Lambda_0(l)} \sup_{A''\in\mathcal{A}_{-\infty}^{-\alpha s}} \widehat{\mathbb{P}}_{A'',\Omega}\left(\eta_0^1(o) \neq \eta_0^2(o)\right)\right).\end{aligned}$$

Since $\mathbb{P}$ has finite speed of propagation, the first term converges to 0 as l approaches ∞. For the second term, note that the number of sites in $\mathbb{Z}^d$ at distance t from the origin is of order t^{d-1}. Thus, by (3.2.4), also the second term converges to 0 as $l \to \infty$. This yields that μ^{EP} agrees with μ on $\mathcal{F}_{=0}^{\infty}$, and that μ^{EP} and μ are non-singular on $(\Omega, \mathcal{F}_{\Lambda_0(l)})$ for all $l \in \mathbb{N}$ sufficiently large.

Next, assume in addition that (X_t) is elliptic. By Lemma 3.4.3 and an argument as in the proof of Theorem 3.1.3, we conclude that μ and μ^{EP} are non-singular on $(\Omega, \mathcal{F}_{=0})$. From this, we conclude that there is probability measure $\hat{\mu}^{EP}$, invariant under μ and such that μ

and $\hat{\mu}^{EP}$ are mutually absolutely continuous. This follows analogous to the proof of Theorem 3.1.3 by making use of (a slight adaptation of) Lemma 3.4.2 and the assumption that (X_t) is elliptic. Consequently, the path measure of (η_t^{EP}) initialised from μ^{EP}, denoted by $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega, \mathcal{F})$, is mutually absolutely continuous to $\mathbb{P}$ on $(\Omega, \mathcal{F}_{\geq 0})$. Thus, since ellipticity implies ellipticity in the time direction, and since μ is ergodic under (η_t) we conclude that μ^{EP} is ergodic under (η_t^{EP}), as follows similar to the proof of ergodicity in Theorem 3.1.1.

□

Proof of Theorem 3.2.4

We next prove that the environments constructed in Subsection 3.2.3 have arbitrary slow polynomial mixing.

Proof of Theorem 3.2.4. First we will show the upper bound, by choosing a particular coupling. The natural coupling of $\mathbb{P}_\xi$ and $\mathbb{P}_\sigma$ is that $\xi_t^1(x,n)$ and $\xi_t^2(x,n)$ share the resampling events of probability b_n, so that after the first resampling, the spins are identical. Note that this coupling can naturally be extended to an arbitrary number of initial configurations. If we denote by ξ_t^σ the configuration at time t when started in σ, we have under this coupling

$$\xi_t^{-\mathbf{1}}(x,n) \leq \xi_t^\sigma(x,n) \leq \xi_t^{+\mathbf{1}}(x,n)$$

and hence $\eta_t^{-\mathbf{1}}(x) \leq \eta_t^\sigma(x) \leq \eta_t^{+\mathbf{1}}(x)$ for all t, x, n, σ. In particular it follows that

$$\widehat{\mathbb{P}}_{\xi,\sigma}(\eta_t^1(0) \neq \eta_t^2(0)) \leq \widehat{\mathbb{P}}_{\mathbf{1},-\mathbf{1}}(\eta_t^1(0) = 1) - \widehat{\mathbb{P}}_{\mathbf{1},-\mathbf{1}}(\eta_t^2(0) = 1).$$

Let $R_t := \{n \in \mathbb{N} : \xi_t^1(0,n) = \xi_t^2(0,n)\}$. We have

$$\mathbb{P}_{\pm\mathbf{1}}(\eta_t(0) = 1) = \widehat{\mathbb{P}}_{\mathbf{1},-\mathbf{1}}\left(\sum_{n\in R_t} a_n \xi_t^1(0,n) > \mp \sum_{n\in R_t^c} a_n\right), \tag{3.4.7}$$

and hence,

$$\widehat{\mathbb{P}}_{\xi,\sigma}(\eta_t^1(0) \neq \eta_t^2(0)) \leq \widehat{\mathbb{E}} \int_{-\sum_{n\in R_t^c} a_n}^{\sum_{n\in R_t^c} a_n} f_{R_t}(x)dx = \widehat{\mathbb{E}} \sum_{n\in R_t^c} a_n \int_{-1}^{1} f_{R_t}(\sum_{n\in R_t^c} a_n x)dx, \tag{3.4.8}$$

where f_R is the density of $\sum_{n\in R} a_n Y_n$ and $(Y_n)_n$ are i.i.d. uniform $[-1,1]$ distributed. A simple convolution of the individual densities shows that $f_R \leq \min_{n\in R}(2a_n)^{-1}$, hence the

above is bounded by

$$\widehat{\mathbb{E}}\left((\min_{n\in R_t} a_n)^{-1}\sum_{n\in R_t^c} a_n\right) = \sum_{k\in\mathbb{N}} \widehat{\mathbb{P}}(\min R_t = k) a_k^{-1}\left(\sum_{n<k} a_n\right)\widehat{\mathbb{E}}\left(\sum_{n>k} \mathbb{1}_{n\in R_t^c} a_n\right)$$

$$= \sum_{k\in\mathbb{N}}\left(\prod_{n<k}(1-b_n)^t\right)\left(1-(1-b_k)^t\right)a_k^{-1}\left(\sum_{n<k} a_n\right)\left(\sum_{n>k} a_n(1-b_n)^t\right). \quad (3.4.9)$$

To obtain polynomial decay, we choose $a_n = \frac{1}{2}n^{-\alpha}$ and $b_n = \frac{1}{2}n^{-\beta}$. Then we can find some constant $C > 0$ so that

$$\sum_{k\in\mathbb{N}}\left(\prod_{n<k}(1-b_n)^t\right)\left(1-(1-b_k)^t\right)a_k^{-1}\left(\sum_{n<k} a_n\right) \le C.$$

With this and $1 - b_n \le e^{-b_n}$,

$$\begin{aligned}(3.4.9) \le C\sum_{n\in\mathbb{N}} a_n(1-b_n)^t &\le \sum_{n\le(t/\log t^2)^{\frac{1}{\beta}}} a_n e^{-b_n t} + \sum_{n>(t/\log t^2)^{\frac{1}{\beta}}} a_n e^{-b_n t}\\ &\le \sum_{n\le(t/\log t^2)^{\frac{1}{\beta}}} a_n t^{-\log t} + \sum_{n>(t/\log t^2)^{\frac{1}{\beta}}} a_n\\ &\le c_2 t^{\frac{-\alpha+1}{\beta}}(\log t)^{\frac{\alpha-1}{\beta}}. \quad (3.4.10)\end{aligned}$$

For a lower bound, we use (3.4.7) plus the fact that (3.4.8) is an equality for $\xi = +\mathbf{1}$ and $\sigma = -\mathbf{1}$, so that we have

$$\sup_{\xi,\sigma} \| \mathbb{P}_\xi(\eta_t(0)\in\cdot) - \mathbb{P}_\sigma(\eta_t(0)\in\cdot)\|_{TV} = \widehat{\mathbb{E}}\sum_{n\in R_t^c} a_n \int_{-1}^{1} f_{R_t}(\sum_{n\in R_t^c} a_n x)dx.$$

The density f_R has a unique local maximum at 0 and its support is $[-\sum_{n\in R} a_n, \sum_{n\in R} a_n]$, so that we can lower bound the integral by replacing f_{R_t} with $(2\sum_{n\in R_t} a_n)^{-1}$:

$$\begin{aligned}(3.4.10) &\ge \widehat{\mathbb{E}}_{\mathbf{1},-\mathbf{1}}(\sum_{n\in R_t} a_n)^{-1}\sum_{n\in R_t^c} a_n \ge (\sum_{n\in\mathbb{N}} a_n)^{-1}\sum_{n\in\mathbb{N}} a_n(1-b_n)^t\\ &\ge (\sum_{n\in\mathbb{N}} a_n)^{-1}\sum_{n\ge t^{-\beta}} a_n(1-\frac{1}{2}t^{-1})^t \ge c_1 t^{\frac{-\alpha+1}{\beta}}.\square\end{aligned}$$

Proof of Theorem 3.2.5

We continue with the proof of Theorem 3.2.5 and study random walks on an Ornstein-Uhlenbeck process.

Proof of Theorem 3.2.5. Fix n, a sequence t_k and $a \in \{-1,1\}^n$. Define the additional events $A_1 = \{\text{sign}(\xi_{-t_k}) = 1, 1 \le k \le n\}$ and $\overline{A} = \{\text{sign}(\xi_t) = 1, t \le 0\}$.

We will use the following sequence of stochastic domination:

$$\mathbb{P}(\xi_0 \in \cdot \mid A) \preccurlyeq \mathbb{P}(\xi_0 \in \cdot \mid A_1) \preccurlyeq \mathbb{P}(\xi_0 \in \cdot \mid \overline{A}). \tag{3.4.11}$$

Here $\mathbb{P}(\xi_0 \in \cdot \mid \overline{A})$ is the limit of $\mathbb{P}(\xi_0 \in \cdot \mid \text{sign}(\xi_s) = 1, -T \le s \le 0)$ as $T \to \infty$, which exists and has Lebesgue-density $x \exp(-\frac{1}{2}x^2)$ on $[0,\infty)$ (see [88]). The argument for the stochastic domination in (3.4.11) is based on the following fact: Let Y^1 and Y^2 be two diffusions given by $dY_t^i = b_t^i(Y_t^i)dt + \sigma dW_t$. If $b_t^1 \le b_t^2$ and $\mathcal{L}(Y_0^1) \preccurlyeq \mathcal{L}(Y_0^2)$, then

$$\mathcal{L}(Y_t^1) \preccurlyeq \mathcal{L}(Y_t^2) \qquad \forall\, t \ge 0. \tag{3.4.12}$$

To apply this to the first stochastic domination in (3.4.11) holds, let $-t_l$ is the biggest time point with $a_l = -1$. Clearly $\mathbb{P}(\xi_{-t_l} \in \cdot | A) \preccurlyeq \mathbb{P}(\xi_{-t_l} \in \cdot | A_1)$. Furthermore, after $-t_l$ the events A and A_1 agree past $-t_l$, that means that after t_l we condition on the same event. This conditioning changes the drift to some new and time-inhomogeneous drift, for which only the initial law varies, and by (3.4.12) we obtain the stochastic domination.

For the second stochastic domination, we use (3.4.12) and the fact that conditioning the Ornstein-Uhlenbeck process on $\overline{A}$ further increases the drift compared to condition on A_1 (with the convention that the drift is $+\infty$ for $x \le 0$ when conditioning on $\overline{A}$).

An analogous bound to (3.4.11) holds in the other direction when we condition the process to be negative, and $\mathbb{P}(\xi_0 \in \cdot | \underline{A}) = \mathbb{P}(-\xi_0 \in \cdot | \overline{A})$. Together this implies

$$\mathbb{P}(|\,\xi_0\,| \in \cdot \mid A) \preccurlyeq \mathbb{P}(\xi_0 \in \cdot \mid \overline{A}). \tag{3.4.13}$$

A bound on the total variation is then given by a coupling:

$$\| \mathbb{P}(\xi_t \in \cdot \mid A) - \mathbb{P}(\xi_t \in \cdot) \|_{TV} \le \int \widehat{\mathbb{P}}_{x,y}(\tau > t)\pi_A(dx,dy),$$

where $\widehat{\mathbb{P}}_{x,y}$ is a coupling of two OU-processes ξ_t^1 and ξ_t^2 starting in x and y and π_A is any coupling of $\mathbb{P}(\xi_0 \in \cdot \mid A)$ with a normal distribution, and τ is the coupling time.

We take $\widehat{\mathbb{P}}_{x,y}$ to be the coupling where the driving Brownian motions are perfectly negatively correlated until the processes are coupled. Then the difference D_t is an OU-process satisfying

$$dD_t = -D_t dt + 2dW_t \quad \text{and} \quad D_0 = x - y.$$

The coupling time τ is τ_0, the first hitting time of 0 of D_t. Note that the coupling time increases if $|\,x-y\,|$ increases, in particular when we replace $|\,x-y\,|$ by $|\,x\,| + |\,y\,|$. With this fact, choosing π_A to be the independent coupling, and (3.4.13) we get

$$\int \widehat{\mathbb{P}}_{x,y}(\tau > t)\pi_A(dx,dy) \le \int_0^\infty \int_0^\infty \mathbb{P}_{x+y}(\tau_0 > t) x e^{-\frac{x^2}{2}} \frac{2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dxdy.$$

To conclude the proof we use the fact that that $\mathbb{P}_{x+y}(\tau_0 > r + \log(x+y))$ is exponentially small in r.

The claim that this example satisfies the conditions of Theorem 3.1.3 is now a simple computation by telescoping over all sites in B and using the fact that the last time a site $x \in \mathbb{Z}^d$ could be observed is $-|x|/R$, where R is the interaction range of the jump kernel α. □

Proof of Theorem 3.2.6

In this subsection we present the proof of Theorem 3.2.6. Before doing so, we first introduce some definitions and prove a general theorem, Theorem 3.4.5, from which Theorem 3.2.6 follows.

Let $E = \{0,1\}$ and associate to the space Ω the partial ordering such that $\xi \leq \eta$ if and only if $\xi(x) \leq \eta(x)$ for all $x \in \mathbb{Z}^{d+1}$. An event $B \in \mathcal{F}$ is said to be *increasing* if $\xi \leq \eta$ implies $1_B(\xi) \leq 1_B(\eta)$. If $\xi \leq \eta$ implies $1_B(\xi) \geq 1_B(\eta)$ then B is called *decreasing*. For $\mathbb{P}, \mathbb{Q} \in \mathcal{M}_1(\Omega)$, we say that $\mathbb{P}$ *stochastically dominates* $\mathbb{Q}$ if $\mathbb{Q}(B) \leq \mathbb{P}(B)$ for all $B \in \mathcal{F}$ increasing. Furthermore, a measure $\mathbb{P} \in \mathcal{M}_1(\Omega)$ is *positively associated* if it satisfies $\mathbb{P}(B_1 \cap B_2) \geq \mathbb{P}(B_1)\mathbb{P}(B_2)$ for any two increasing events $B_1, B_2 \in \mathcal{F}$. Following [85], we say that $\mathbb{P}$ is *downward FKG* if, for every finite $\Lambda \subset \mathbb{Z}^{d+1}$, the measure $\mathbb{P}(\cdot \mid \eta \equiv 0 \text{ on } \Lambda)$ is positively associated.

Theorem 3.4.5. *Let $\mathbb{P} \in \mathcal{M}_1(\Omega)$ be downward FKG and assume that there exists $\phi\colon \mathbb{N} \to [0,1]$ such that for all $(x,s) \in \Lambda(l)$ and all $\gamma \in \bigcup_{k\geq 1} \Gamma_k$,*

$$\mathbb{P}\left(\eta_s(x) = 1 \mid \eta \equiv 0 \text{ along } \gamma\right) \geq \mathbb{P}\left(\eta_0(o) = 1\right) - \phi(l), \tag{3.4.14}$$

$$\mathbb{P}(\eta_s(x) = 1 \mid \eta \equiv 1 \text{ along } \gamma) \leq \mathbb{P}(\eta_0(o) = 1) + \phi(l). \tag{3.4.15}$$

If $\sum_{l\geq 1} l^d \phi(l) < \infty$, then the conditions of Theorem 3.1.3 are satisfied.

Remark 3.4.1. In the above theorem, and throughout this section, we write "$\eta \equiv i$ along γ", where $i \in \{0,1\}$ and $\gamma \in \Gamma := \bigcup_{k\geq 1} \Gamma_k$, for the event that $\{\eta_s(x) = i \; \forall \; (x,s) \in \gamma + [-R,R]^d \times \{0\}\}$.

Proof of Theorem 3.4.5. Let $B \in \mathcal{F}$. For any $k \in \mathbb{N}$, we have similar to the proof of Lemma 3.4.1 that

$$\begin{aligned}
&\left|P_{\mathbb{P}}\left(\eta_k^{EP} \in B\right) - \mathbb{P}\left(B\right)\right| \\
&\leq \sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \Bigg[\left|\mathbb{P}\left(B \mid A_{-k}^{-1}\right) - \mathbb{P}\left(B \mid \eta \equiv 0 \text{ along } \gamma\right)\right| \\
&+ \left|\mathbb{P}\left(B\right) - \mathbb{P}\left(B \mid \eta \equiv 0 \text{ along } \gamma\right)\right| \Bigg] \mathbb{P}^{-k}\left(X_{-k,\ldots,0} = \gamma, A_{-k}^{-1}\right).
\end{aligned}$$

We next show that, under (3.4.14) and (3.4.15),

$$\sup_{B\in\Lambda(l)} \left|\mathbb{P}\left(B \mid A_{-k}^{-1}\right) - \mathbb{P}\left(B \mid \eta \equiv 0 \text{ along } \gamma\right)\right| \to 0, \quad \text{as } l \to \infty.$$

Fix $\gamma \in \Gamma_k$ and $A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)$. Since $\mathbb{P}$ is downward FKG, it is the case that $\mathbb{P}(\cdot \mid \eta \equiv 0 \text{ along } \gamma)$ is stochastically dominated by $\mathbb{P}(\cdot \mid A_{-k}^{-1})$. Hence, by Strassens Theorem, there exists a coupling $\widehat{\mathbb{P}}_{0,1}$ of $\mathbb{P}(\cdot \mid \eta \equiv 0 \text{ along } \gamma)$ and $\mathbb{P}(\cdot \mid A_{-k}^{-1})$ such that $\widehat{\mathbb{P}}_{0,1}\left(\eta^1 \leq \eta^2\right) = 1$. We moreover have that, for all $B \in \mathcal{F}_{\Lambda(l)}$, $l \in \mathbb{N}$,

$$\begin{aligned}
&|\mathbb{P}(B \mid \eta \equiv 0 \text{ along } \gamma) - \mathbb{P}(B \mid A_{-k}^{-1})| \\
\leq &\widehat{\mathbb{P}}_{0,1}\left(\eta^1 \neq \eta^2 \text{ on } \Lambda(l)\right) \\
\leq &\sum_{(x,s)\in\Lambda(l)} \widehat{\mathbb{P}}_{0,1}\left(\eta_s^1(x) \neq \eta_s^2(x)\right) \\
= &\sum_{(x,s)\in\Lambda(l)} \widehat{\mathbb{P}}_{0,1}\left(\eta_s^1(x) = 0, \eta_s^2(x) = 1\right) \\
= &\sum_{(x,s)\in\Lambda(l)} \left(\widehat{\mathbb{P}}_{0,1}(\eta_s^1(x) = 0) - \widehat{\mathbb{P}}_{0,1}(\eta_s^2(x) = 0)\right).
\end{aligned}$$

Furthermore, since $\mathbb{P}$ is downward FKG, we know that

$$\begin{aligned}
&\widehat{\mathbb{P}}_{0,1}\left(\eta_s^1(x) = 0\right) - \widehat{\mathbb{P}}_{0,1}\left(\eta_s^2(x) = 0\right) \\
=&\mathbb{P}\left(\eta_s(x) = 0 \mid \eta \equiv 0 \text{ along } \gamma\right) - \mathbb{P}\left(\eta_s(x) = 0 \mid A_{-k}^{-1}\right) \\
\leq&\mathbb{P}\left(\eta_s(x) = 0 \mid \eta \equiv 0 \text{ along } \gamma\right) - \mathbb{P}\left(\eta_s(x) = 0 \mid \eta \equiv 1 \text{ along } \gamma\right)
\end{aligned}$$

As a consequence, by using (3.4.14) and (3.4.15), we obtain by the derivations above that

$$\sup_{B\in\mathcal{F}_{\Lambda(l)}} |\mathbb{P}(B \mid \eta \equiv 0 \text{ along } \gamma) - \mathbb{P}(B \mid A_{-k}^{-1})| \leq C \sum_{t\geq l} t^d \phi(t), \tag{3.4.16}$$

for some constant $C \in (0, \infty)$. By a word by word adaptation of this argument, replacing $\mathbb{P}\left(B \mid A_{-k}^{-1}\right)$ by $\mathbb{P}\left(B\right)$, it can similarly be shown that

$$\sup_{B\in\Lambda(l)} \left|\mathbb{P}\left(B\right) - \mathbb{P}\left(B \mid \eta \equiv 0 \text{ along } \gamma\right)\right| \leq C \sum_{t\geq l} t^d \phi(t). \tag{3.4.17}$$

Substituting the estimates from (3.4.16) and (3.4.17) into the first inequality of this proof, and using that $\lim_{l\to\infty} \sum_{t\geq l} t^d \phi(t) = 0$, we obtain that the conditions of Theorem 3.1.3 are satisfied. □

We continue with the proof of Theorem 3.2.6.

Proof of Theorem 3.2.6. Let (ξ_t) be the contact process on $\mathbb{Z}^{\tilde{d}}$ with $\tilde{d} \geq 1$ and $\lambda > \lambda_c(\tilde{d})$. This process is known to satisfy the downward FKG property, as shown by [20], Theorem 3.3 (see also Lemma 2.1 in [19]). Thus, for the proof of Theorem 3.2.6, it is sufficient to show that (3.4.14) and (3.4.15) holds. In fact, it is sufficient to show that the estimates of Theorem 3.4.5 hold for sites (o, s) with $s \in \mathbb{Z}_{\geq 0}$. To see this, recall the graphical representation of the contact process (see p. 32-34 in [83]). Since the spread of information is bounded by a Poisson process with rate $2d\lambda$, it is evident that the finite speed of propagation property holds, and thus that Corollary 3.2.2 applies.

That (3.4.15) holds for the contact process is now a simple application of the graphical representation and the fact that the contact process started from all sites equal to 1 converges exponentially fast towards the upper invariant measure. See [83], Theorem 1.2.30, and the remark directly after for estimates of the latter. In particular, (3.4.15) holds with $\phi(l)$ exponentially decaying in l. Note that, this estimate holds for (ξ_t), that is, without applying the projection map.

In order to conclude a similar estimate for (3.4.14), on the other hand, we restrict to the projection of (ξ_t) onto the one dimensional lattice. In this case, (3.4.14), again with $\phi(l)$ exponentially decaying in l, is a direct application of [19], Theorem 1.7. Thus, by Theorem 3.4.5, we conclude that the conditions of Theorem 3.1.3 are satisfied. □

Remark 3.4.2. The statement of Theorem 3.2.6 can be extended to projection maps from $\mathbb{Z}^{\tilde{d}}$ to $\mathbb{Z}^{\tilde{d}}_{\tilde{d}-1} := \mathbb{Z}^{\tilde{d}-1} \times \{0\}$ for any $\tilde{d} \geq 2$ and $\lambda > \lambda_c(\tilde{d})$. Indeed, Theorem 1.7 in [19] still holds in this generality.

3.4.3 Proof of Theorem 3.3.2 and Theorem 3.3.3

Proof of Theorem 3.3.2. We first show continuity with respect to $\left(\mathbb{P}^{EP(n)}\right)$. Let $\epsilon > 0$, and let $m \leq n$ with $n, m \in \mathbb{N}$. For $\Lambda \subset \mathbb{L}$ finite and $B \in \mathcal{F}_\Lambda$ we have that, for every $t \in \mathbb{N}$,

$$\begin{aligned}|\mathbb{P}^{EP(m)}(B) - \mathbb{P}^{EP(n)}(B)| \leq\ & |\mathbb{P}^{EP(m)}(B) - P_{\mathbb{P}_m}(\eta_t^{EP(m)} \in B)| \\ & + |\mathbb{P}^{EP(n)}(B) - P_{\mathbb{P}_n}(\eta_t^{EP(n)} \in B)| \\ & + |P_{\mathbb{P}_n}(\eta_t^{EP(n)} \in B) - P_{\mathbb{P}_m}(\eta_t^{EP(m)} \in B)|.\end{aligned}$$

By Assumption c) we can fix t such that the sum of the first two terms is less than $\epsilon/2$. By the uniformity assumption, this bound holds irrespectively of m and n. It thus remains to show that also the third term can be made smaller than $\epsilon/2$ by possibly taking m large. To

this end, we use the expansion in (3.3.1), and note that

$$\begin{aligned}
&P_{\mathbb{P}_m}(\eta_t^{EP(m)} \in B) \\
&= \sum_{\substack{\gamma \in \Gamma_t \\ A_{-t}^{-1} \in \mathcal{A}_{-t}^{-1}(\gamma)}} \mathbb{P}_m\left(B, A_{-t}^{-1}\right) P_m\left(X_{-t,\dots,0} = \gamma \mid A_{-t}^{-1}\right) \\
&= \sum_{\substack{\gamma \in \Gamma_t \\ A_{-t}^{-1} \in \mathcal{A}_{-t}^{-1}(\gamma)}} \left(\mathbb{P}_n(B, A_{-t}^{-1}) \pm \delta_{1,m}(t)\right)\left(P_n(X_{-t,\dots,0} = \gamma \mid A_{-t}^{-1}) \pm \delta_{2,m}(t)\right),
\end{aligned}$$

where, due to a) and b), the error terms $\delta_{1,m}(t)$ and $\delta_{2,m}(t)$ approaches 0 as $m \to \infty$. In particular, again since

$$P_{\mathbb{P}_n}(\eta_t^{EP(n)} \in B) = \sum_{\substack{\gamma \in \Gamma_t \\ A_{-t}^{-1} \in \mathcal{A}_{-t}^{-1}(\gamma)}} \mathbb{P}_n\left(B, A_{-t}^{-1}\right) P_n\left(X_{-t,\dots,0} = \gamma \mid A_{-t}^{-1}\right),$$

by taking m large enough we can guarantee that

$$|P_{\mathbb{P}_n}(\eta_t^{EP(n)} \in B) - P_{\mathbb{P}_m}(\eta_t^{EP(n)} \in B)| < \epsilon/2.$$

Since this bound holds for all $n \geq m$ it follows that $(\mathbb{P}^{EP(m)}(B))$ is a Cauchy sequence and hence converges to a limit. Moreover, since B and Λ were arbitrary, this is true for any local local event $B \in \mathcal{F}$. This implies that $\mathbb{P}^{EP(m)}$ converges weakly to $\mathbb{P}^{EP}$ for some $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$.

We next proceed with the proof of $P_{\mathbb{P}}(\eta_t^{EP} \cdot) \implies \mathbb{P}^{EP}$, where $\mathbb{P}^{EP}$ is the limiting measure above. Let $\epsilon > 0$ and $B \in \mathcal{F}$ local. For any $n \in \mathbb{N}$, we have that

$$\begin{aligned}
|\mathbb{P}^{EP}(B) - P_{\mathbb{P}}(\eta_t^{EP} \in B)| \leq\ & |\mathbb{P}^{EP}(B) - \mathbb{P}^{EP(n)}(B)| \\
&+ |\mathbb{P}^{EP(n)}(B) - P_{\mathbb{P}_n}(\eta_t^{EP(n)} \in B)| \\
&+ |P_{\mathbb{P}_n}(\eta_t^{EP(n)} \in B) - P_{\mathbb{P}}(\eta_t^{EP} \in B)|.
\end{aligned}$$

Fix t such that the second term is smaller than $\epsilon/3$. This we can do by applying Assumption c). Next, by taking n large the first term can be made smaller then $\epsilon/3$ as well since $\mathbb{P}^{EP(n)} \implies \mathbb{P}^{EP}$, as we have shown above. For the third term we can proceed as in for the proof of $\mathbb{P}^{EP(n)} \implies \mathbb{P}^{EP}$ above. Indeed, since t is fixed, we can use that $\mathbb{P}_n \implies \mathbb{P}$ and that $\epsilon(n) \downarrow 0$ together with the finite range assumption of the random walk. Hence we may take n so large that also the third term is less that $\epsilon/3$. Since $\epsilon > 0$ was taken arbitrary, this shows that $P_{\mathbb{P}}(\eta_t^{EP} \in B) \to \mathbb{P}^{EP}(B)$ as $t \to \infty$. Since $B \in \mathcal{F}$ was an arbitrary local event, we conclude that $P_{\mathbb{P}}(\eta_t^{EP} \in \cdot)$ converges weakly towards $\mathbb{P}^{EP}(\cdot)$. As a necessary consequence, it also follows that by standard arguments that $\mathbb{P}^{EP}$ is invariant with respect to (η_t^{EP}).

□

Proof of Theorem 3.3.3. Let $\mathbb{P}_\sigma$ be the path measure of (η_t) when started from $\sigma \in \Omega_0$ and assume that (η_t^{EP}) is uniquely ergodic with invariant measure $\mu^{EP} \in \mathcal{M}_1(\Omega)$. We have that, for any $B \in \mathcal{F}_\Lambda$, $\Lambda \subset \mathbb{Z}^d \times \{0\}$ finite, and any $t \in \mathbb{N}$,

$$\begin{aligned}\left|\mu^{EP}(B) - \mu^{EP(n)}(B)\right| &= \left|t^{-1}\sum_{k=1}^{t}\left[P_{\mu^{EP}}(\eta_k^{EP} \in B) - P_{\mu^{EP(n)}}(\eta_k^{EP(n)} \in B)\right]\right| \\ &\leq \left|t^{-1}\sum_{k=1}^{t}\left[P_{\mu^{EP}}(\eta_k^{EP} \in B) - P_{\mu^{EP(n)}}(\eta_k^{EP(n)} \in B)\right]\right| \\ &+ \left|t^{-1}\sum_{k=1}^{t}\left[P_{\mu^{EP(n)}}(\eta_k^{EP} \in B) - P_{\mu^{EP(n)}}(\eta_k^{EP(n)} \in B)\right]\right|. \end{aligned} \tag{3.4.18}$$

Since (η_t^{EP}) is uniquely ergodic, it follows by classical arguments that

$$\sup_{\sigma,\omega\in\Omega_0}\left|t^{-1}\sum_{k=1}^{t}\left[P_\sigma(\eta_k^{EP} \in B) - P_\omega(\eta_k^{EP} \in B)\right]\right| \to 0$$

as t approaches ∞ (see e.g. Theorem 4.10 in [59]). Hence, by taking t large we can assure that the first term of the r.h.s. of (3.4.18) is less than $\epsilon/2$. Next, for the second term, we have that, for any fixed $t > 0$,

$$t^{-1}\sum_{k=1}^{t}\left|P_{\mu^{EP(n)}}(\eta_k^{EP} \in B) - P_{\mu^{EP(n)}}(\eta_k^{EP(n)} \in B)\right| \to 0 \text{ as } n \to \infty.$$

This follows similarly as in the proof of Theorem 3.3.2. Indeed, for each $k \in \{1, \ldots, t\}$, we have that

$$\begin{aligned} & P_{\mu^{EP(n)}}(\eta_k^{EP} \in B) \\ = & \sum_{\substack{\gamma\in\Gamma_k \\ A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)}} \mathbb{P}_{\mu^{EP(n)}}\left(B, A_{-k}^{-1}\right) P_n\left(X_{-k,\ldots,0} = \gamma \mid A_{-k}^{-1}\right) \\ = & \sum_{\substack{\gamma\in\Gamma_k \\ A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)}} \left(\mathbb{P}_{\mu^{EP}}(B, A_{-t}^{-1}) \pm \delta_{1,n}(t)\right) \\ & \qquad \cdot \left(P(X_{-k,\ldots,0} = \gamma \mid A_{-k}^{-1}) \pm \delta_{2,n}(t)\right), \end{aligned}$$

where both the error terms $\delta_{1,n}(t)$ and $\delta_{2,n}(t)$ approaches 0 as $n \to \infty$. Thus, by taking n sufficiently large we can assure that the second term on the r.h.s. of (3.4.18) is less than $\epsilon/2$. From this we conclude that $\left|\mu^{EP}(B) - \mu^{EP(n)}(B)\right| < \epsilon$ for all n large. Since B and Λ were arbitrary chosen, we hence conclude the proof. □

3.4.4 Proof of Theorem 3.3.4

Proof of Theorem 3.3.4. The main part of the proof goes along the same lines as the proof of Theorem 3.1.1. The main difference is an estimate which is similar to Lemma 3.4.1 and which we present next. Let $B \in \mathcal{F}_{\geq 0}$. We have that, for any $k \in \mathbb{N}$,

$$\begin{aligned}
&\left|P_{\mathbb{P}}\left(\eta_k^{EP} \in B\right) - \mathbb{P}\left(B\right)\right| \\
=&\Big|\sum_{\gamma\in\Gamma_k}\sum_{A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}\left(B, A_{-k}^{-1}\right) P\left(X_{-k,\ldots,0} = \gamma \mid A_{-k}^{-1}\right) - \mathbb{P}\left(B\right)\Big| \\
\leq &\sum_{\gamma\in\Gamma_k}\sum_{A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)} \left|\mathbb{P}\left(B \mid A_{-k}^{-1}\right) - \mathbb{P}\left(B\right)\right| \mathbb{P}^{-k}\left(X_{-k,\ldots,0} = \gamma, A_{-k}^{-1}\right),
\end{aligned}$$

where in the last equality we used the fact that,

$$\sum_{\gamma\in\Gamma_k}\sum_{A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}^{-k}\left(X_{-k,\ldots,0} = \gamma, A_{-k}^{-1}(\gamma,\sigma)\right) = 1.$$

Consequently, by the bound in (3.3.2), we conclude that, for any $k \in \mathbb{N}$,

$$\left|P_{\mathbb{P}}\left(\eta_k^{EP} \in B\right) - \mathbb{P}\left(B\right)\right| \leq M_1\mathbb{P}(B), \quad \forall\, B \in \mathcal{F}_{\geq 0}. \tag{3.4.19}$$

In particular, $P_{\mathbb{P}}\left(\eta_k^{EP} \in \cdot\right) \ll \mathbb{P}$ on $\mathcal{F}_{\geq 0}$ and $\frac{dP_{\mathbb{P}}\left(\eta_k^{EP}\in\cdot\right)|_{\mathcal{F}_{\geq 0}}}{d\mathbb{P}(\cdot)|_{\mathcal{F}_{\geq 0}}} \leq M_1$.

Let $\mathbb{Q} \in \mathcal{M}_1(\Omega)$ be a limiting measure of the sequence $\left(t^{-1}\sum_{k=1}^{t} P_{\mathbb{P}}(\eta_k^{EP} \in \cdot)\right)_{t>0}$, by possibly taking sub-sequential limits. Then, by means of weak convergence, since the space of M_1-bounded functions on a compact space form a compact space, and the limit of bounded measurable functions is measurable, (3.4.19) immediately transfers to $\mathbb{Q}$. Consequently, we have $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{F}_{\geq 0}$ and $\frac{d\mathbb{Q}|_{\mathcal{F}_{\geq 0}}}{d\mathbb{P}|_{\mathcal{F}_{\geq 0}}} \leq M_1$. This concludes the first part.

Next, assume that (3.3.3) holds from which it follows that, for every $B \in \mathcal{F}_{\geq 0}$,

$$|\mathbb{P}(B) - \mathbb{P}(B \mid A_k)| \leq M_2\mathbb{P}(B \mid A_k), \quad \forall\, A_k \in \mathcal{A}_{-kl}^{-1},\, k \in \mathbb{N},$$

Similarly to how we obtained (3.4.19), we hence conclude that, for any $k \in \mathbb{N}$,

$$\left|P_{\mathbb{P}}\left(\eta_k^{EP} \in B\right) - \mathbb{P}\left(B\right)\right| \leq M_2 P_{\mathbb{P}}\left(\eta_k^{EP} \in B\right), \quad \forall\, B \in \mathcal{F}_{\geq 0}.$$

From this estimate, and using the same argument as for the proof of the first part, we hence conclude that $\mathbb{P} \ll \mathbb{Q}$ and that $\frac{d\mathbb{P}|_{\mathcal{F}_{\geq 0}}}{d\mathbb{Q}|_{\mathcal{F}_{\geq 0}}} \leq M_2$. □

3.4.5 Strong disagreement percolation

Basic disagreement percolation

For simplicity we assume that $E = \{0, 1\}$ and that the environment (η_t) is a translation invariant nearest neighbour probabilistic cellular automaton (PCA). Further, let $c_i(\eta) := \mathbb{P}_\eta(\eta_1(o) = i), i = 0, 1$. By the nearest neighbour property, $c_i(\eta) = c_i(\xi)$ if $\eta(x) = \xi(x)$ for all $| x | \leq 1$.

The evolution of the PCA can be constructed by a sequence $(U_t(x))_{x\in\mathbb{Z}^d, t\geq 1}$ of i.i.d. $[0, 1]$-uniform variables in an iterative way: given η_t, $\eta_{t+1}(x) := \mathbb{1}_{U_{t+1}(x)\leq c_1(\theta_x\eta_t)}$, $x \in \mathbb{Z}^d$. Here θ_x is the shift on $\mathbb{Z}^d$, that is, for $\eta \in \Omega_0$ we have $(\theta_x\eta)(y) = \eta(y + x)$, $y \in \mathbb{Z}^d$

This construction allows for coupling of $\mathbb{P}_{\eta^1}$ and $\mathbb{P}_{\eta^2}$, the graphical construction coupling, by using the same set of $[0, 1]$-uniform i.i.d. variables $(U_t(x))$. The starting point of disagreement percolation is the observation that the value of $\eta_{t+1}(x)$ is sometimes independent of η_t, namely if either $U_{t+1}(x) < c_- := \inf_{\eta\in\Omega} c_1(\eta)$ or $U_{t+1}(x) > c_+ := \sup_{\eta\in\Omega} c_1(\eta)$. This allows the environment to forget information, which can be encoded in the coupling. The disagreement percolation is then the triple $(\eta_t^1, \eta_t^2, \xi_t)_{t\geq 0}$, where η_t^i is constructed from the initial configuration η^i and the $(U_t(x))$, and ξ_t is given by $\xi_0(x) = \mathbb{1}_{\eta^1(x)\neq\eta^2(x)}$ and for $t > 0$;

$$\xi_t(x) = \begin{cases} 1, & U_t(x) \in [c_-, c_+] \text{ and } \exists\, y, | y - x | \leq 1 : \xi_{t-1}(y) = 1; \\ 0, & otherwise. \end{cases}$$

The name disagreement percolation comes from the fact that $\xi_t(x) = 0$ implies $\eta_t^1(x) = \eta_t^2(x)$ and (ξ_t) is a directed site percolation process with percolation parameter $p = c_+ - c_-$. We denote the law of this so constructed triple $(\eta_t^1, \eta_t^2, \xi_t)_{t\geq 0}$ by $\widehat{\mathbb{P}}_{\eta^1,\eta^2}$.

Definition 3.4.1. *If $p = c_+ - c_- < p_c$, where p_c is the critical value of directed site percolation in $\mathbb{Z}^d$, then we say that the disagreement percolation $\widehat{\mathbb{P}}$ is subcritical.*

Remark 3.4.3. This coupling can be improved by looking at more information. For example the site percolation model does not use the total number of neighbours which satisfy $\xi_{t-1}(y) = 1$, only that the indicator that this number is positive. By taking this information into account when deciding based on whether $\xi_t(x)$ should be 1 or 0 the range of PCA where the disagreement percolation is subcritical can be extended.

Remark 3.4.4. If the disagreement percolation coupling is subcritical, then necessarily there is a uniquely ergodic measure for the process (η_t), as follows by standard coupling arguments and comparison with subcritical directed site percolation.

Disagreement percolation and backward cones

The disagreement percolation we introduced in the previous subsection is a way to control the influence of the initial configuration on the future, by giving an upper bound on the space-

time points which depend on differences in the initial configurations. In the context of this article we want something slightly different, namely to control the influence of a backwards cone. With this in mind we construct a different version of the disagreement percolation coupling.

Denote by $\mathbb{P}_\mu^{-\infty}$ the law of $(\eta_t)_{t\in\mathbb{Z}}$ under the stationary law μ and by $(U_t(x))_{t\in\mathbb{Z},x\in\mathbb{Z}^d}$ the i.i.d. uniform $[0,1]$ variables of the corresponding graphical construction. Denote by $C_b := \{(x,t) \in \mathbb{Z}^d \times \{...,-1,0\} : |\, x\,| \leq |\, t\,|\}$ the infinite backward cone with tip at $(0,0)$ and by $\mathcal{C}_b := \sigma(\eta_t(x) : (x,t) \in C_b) = \sigma(U_t(x) : (x,t) \in C_b)$ the σ-algebra generated by the sites which lie in the cone C_b.

Let $A, B \in \mathcal{C}_b$. We now construct the disagreement percolation process $(\eta_t^1, \eta_t^2, \xi_t)_{t\in\mathbb{Z}}$ with law $\widehat{\mathbb{P}}_{A,B}$, where η^1 has law $\mathbb{P}_\mu^{-\infty}(\cdot|A)$ and η^2 has law $\mathbb{P}_\mu^{-\infty}(\cdot|B)$. The idea is almost the same as in Subsection 3.4.5, the only difference is on the cone C_b. On C_b, we draw $(\eta_t^1(x))_{(x,t)\in C_b}$ from $\mathbb{P}_\mu^{-\infty}(\cdot|A)$, independently $(\eta_t^2(x))_{(x,t)\in C_b}$ from $\mathbb{P}_\mu^{-\infty}(\cdot|B)$, and set $\xi_t(x) = 1$ for $(x,t) \in C_b$. Outside C_b, (η_t^1, η^2, ξ) evolves like the basic disagreement percolation coupling by using the same $(U_t(x))_{(x,t)\in C_b^c}$. As the evolution outside C_b is the same as the basic disagreement percolation, the definition of subcriticality remains the same.

Lemma 3.4.6. *Suppose the disagreement percolation is subcritical. Then the environment satisfies* (3.1.5)*.*

Proof. Let $A \in \mathcal{C}_b$ be arbitrary and let $\widehat{\mathbb{P}}_A$ be the disagreement percolation coupling of $\mathbb{P}_\mu^{-\infty}(\cdot|A)$ and $\mathbb{P}_\mu^{-\infty}$. We then have for any $B \in \mathcal{F}_{\Lambda(l)}$, $l \geq 1$,

$$\left|\, \mathbb{P}_\mu^{-\infty}(B|A) - \mathbb{P}_\mu^{-\infty}(B) \,\right| \leq \widehat{\mathbb{P}}_A(\exists (x,t) \in \Lambda(l) \colon \xi_t(x) = 1),$$

which is exponentially small in l and independent of the choice of B and A. □

Strong disagreement percolation

We say that $(\eta_t^1, \eta_t^2, \xi_t)$ is a strong disagreement percolation coupling if $\xi_t(x) = 0$ implies $\eta_t^1(x) = \eta_t^2(x)$ and η^1 and ξ are independent. This independence is a stronger assumption than regular disagreement percolation.

Lemma 3.4.7. *Suppose* $p^* := \max((c_+ - c_-)/c_+, (c_+ - c_-)/(1 - c_-)) < p_c$*. Then there exist strong versions of the disagreement percolation couplings in Sections 3.4.5 and 3.4.5.*

Proof. The basic concept of the construction is similar to the regular disagreement percolation. The difference is that we no longer use a single $U_t(x)$ to build the processes

$(\eta_t^1(x), \eta_t^2(x), \xi_t(x))$ from $(\eta_{t-1}^1, \eta_{t-1}^2, \xi_{t-1})$. Instead we take three $[0,1]$-uniform i.i.d random variables $(U_t(x)^1, U_t(x)^2, U_t(x)^3)$. We then set

$$\begin{aligned}
\eta_t^1(x) &= \mathbb{1}_{U_t^1(x) \le c_1(\theta_{-x}\eta_{t-1}^1)};\\
\xi_t(x) &= \mathbb{1}_{U_t^3(x) \le p^*}\mathbb{1}_{\exists y, |\, y-x\,| \le 1 : \xi_{t-1}(y)=1};\\
\eta_t^2(x) &= \begin{cases} \eta_t^1(x), & \xi_t(x) = 0;\\ 1, & U_t^2(x) \le \frac{c_1(\theta_{-x}\eta_{t-1}^2) - (1-p^*)c_1(\theta_{-x}\eta_{t-1}^1)}{p^*} \text{ and } \xi_t(x) = 1;\\ 0, & otherwise. \end{cases}
\end{aligned} \tag{3.4.20}$$

The choice of p^* guarantees that $\frac{c_1(\theta_{-x}\eta_{t-1}^2) - (1-p^*)c_1(\theta_{-x}\eta_{t-1}^1)}{p^*} \in [0,1]$, and a direct computation shows that the probability that $\eta_t^2(x) = 1$ is $c_1(\theta_{-x}\eta_{t-1}^2)$. □

Proof of Theorem 3.3.5 and Corollary 3.3.6

Proof of Theorem 3.3.5. The proof is based on a coupling argument. Let

$$C_{-k} := \{(x,t) \in \mathbb{Z}^d \times \{..., -k-1, -k\} : |\, x - \gamma_{-k}\,| \le |\, t - k\,|\}$$

be the infinite backwards cone with tip at $(\gamma_{-k}, -k)$. We construct iteratively the random variables $(\eta_t(x)^{1,m}, \eta_t(x)^{2,m}, \xi_t(x)^m) \in E \times E \times \{0,1\}, (x,t) \in C_{-k-1+m}$, and $H^m \in \mathbb{N}$, and denote their law by $\widetilde{\mathbb{P}}^m$. We start with $\eta_t(x)^{1,0}$ and $\eta_t(x)^{2,0}, \xi_t(x)^0$ chosen independently from $\mathbb{P}_\mu^{-\infty}(\cdot)$ and $\mathbb{P}_\mu^{-\infty}\left(\cdot \,\middle|\, A_{-k-1}^{-k-1}\right)$ restricted to the cone C_{-k-1+m} and set $\xi_t^0(x) = 1$ for $(x,t) \in C_{-k-1}$, and $H^0 = 0$.

Given $\widetilde{\mathbb{P}}^m$, let $\widetilde{\mathbb{P}}^{m,*}$ be the extension of $\widetilde{\mathbb{P}}^m$ to the cone C_{-k+m} based on the strong disagreement percolation coupling, that is

$$(\eta_t^{1,m}(x), \eta_t^{2,m}(x), \xi_t^m(x))_{(x,t) \in C_{-k+m} \setminus C_{-k-1+m}}$$

are distributed according to the evolution described in (3.4.20).

The general strategy is as follows: We want to condition the measure $\widetilde{\mathbb{P}}^{m,*}$ on the event $\{\eta^{1,m} \in A_{-k}^{-k+m}, \eta^{2,m} \in A_{-k-1}^{-k+m}\}$. Observe that, on $\xi_{-k+m}(\gamma_{-k+m}) = 0$, the events $\eta^{1,m} \in A_{-k+m}^{-k+m}$ and $\eta^{2,m} \in A_{-k+m}^{-k+m}$ are equivalent. This is the good case. The bad case is when $\xi_{-k}(\gamma_{-k}) = 1$. In this event, we simply restart the coupling procedure by coupling $\mathbb{P}_\mu^{-\infty}\left(\cdot \,\middle|\, A_{-k-1}^{-k}\right)$ and $\mathbb{P}_\mu^{-\infty}\left(\cdot \,\middle|\, A_{-k}^{-k}\right)$ independently. The role of H^m is to keep track of the number of iterations since the last time we had to reset and try again.

Define

$$\begin{aligned}
q_m(h) :=& \widetilde{\mathbb{P}}^{m,*}\left(\xi^m_{-k+m}(\gamma_{-k+m}) = 0 \;\middle|\; \eta^{1,m} \in A^{-k+m}_{-k}, H^m = h\right) \\
&\wedge \widetilde{\mathbb{P}}^{m,*}\left(\xi^m_{-k+m}(\gamma_{-k+m}) = 0 \;\middle|\; \eta^{2,m} \in A^{-k+m}_{-k-1}, H^m = h\right); \\
Q^1_m :=& \widetilde{\mathbb{P}}^{m,*}\left(\xi^m_{-k+m}(\gamma_{-k+m}) = 0 \;\middle|\; \eta^{1,m} \in A^{-k+m}_{-k}\right) - \sum_{h\geq 0} q_m(h); \\
Q^2_m :=& \widetilde{\mathbb{P}}^{m,*}\left(\xi^m_{-k+m}(\gamma_{-k+m}) = 0 \;\middle|\; \eta^{2,m} \in A^{-k+m}_{-k-1}\right) - \sum_{h\geq 0} q_m(h).
\end{aligned}$$

Let $B^1, B^2, D \in \sigma(C_{-k+m})$. We now define $\widetilde{\mathbb{P}}^{m+1}$ based on $\widetilde{\mathbb{P}}^m$ by

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{m+1}\left(\eta^{1,m+1} \in B^1, \eta^{2,m+1} \in B^2, \xi^{m+1} \in D, H^{m+1} = h+1\right) \\
&:= q_m(h)\widetilde{\mathbb{P}}^{m,*}\left(\eta^{1,m} \in B^1, \eta^{2,m} \in B^2, \xi^m \in D, H^m = h \;\middle|\right. \\
&\qquad\qquad \left.\eta^{1,m} \in A^{-k+m}_{-k+m}, \eta^{2,m} \in A^{-k+m}_{-k+m}, \xi^m_{-k+m}(\gamma_{-k+m}) = 0\right)
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{m+1}\left(\eta^{1,m+1} \in B^1, \eta^{2,m+1} \in B^2, \xi^{m+1} \in D, H^{m+1} = 0\right) \\
&:= \left[\widetilde{\mathbb{P}}^{m,*}\left(\eta^{1,m} \in B^1 \;\middle|\; \eta^1_{-k+m} \in A^{-k+m}_{-k}, \xi^m_{-k+m}(\gamma_{-k+m}) = 0\right) Q^1_m\right. \\
&\left. + \widetilde{\mathbb{P}}^{m,*}\left(\eta^{1,m} \in B^1, \xi^m_{-k+m}(\gamma_{-k+m}) = 1 \;\middle|\; \eta^{1,m} \in A^{-k+m}_{-k}\right)\right] \frac{1}{1 - \sum_{h\geq 0} q_m(h)} \\
&\cdot \left[\widetilde{\mathbb{P}}^{m,*}\left(\eta^{2,m} \in B^2 \;\middle|\; \eta^{1,m}_{-k+m} \in A^{-k+m}_{-k}, \xi^m_{-k+m}(\gamma_{-k+m}) = 0\right) Q^2_m\right. \\
&\left. + \widetilde{\mathbb{P}}^{m,*}\left(\eta^{2,m} \in B^2, \xi^m_{-k+m}(\gamma_{-k+m}) = 1 \;\middle|\; \eta^{2,m} \in A^{-k+m}_{-k}\right)\right] \mathbb{1}_{1 \text{ on } C_{-k+m} \in D}.
\end{aligned}$$

A direct computation shows that $\widetilde{\mathbb{P}}^{m+1}(\eta^{1,m+1} \in B^1) = \mathbb{P}^{-\infty}_{\mu}(B^1 \mid A^{-k+m}_{-k})$ and $\widetilde{\mathbb{P}}^{m+1}(\eta^{2,m+1} \in B^2) = \mathbb{P}^{-\infty}_{\mu}(B^2 \mid A^{-k+m}_{-k})$ assuming that $\widetilde{\mathbb{P}}^m$ satisfies the corresponding properties. Therefore $\widetilde{\mathbb{P}}^k$ extended to all space-time points using the strong disagreement percolation construction (3.4.20) is a coupling of $\mathbb{P}^{-\infty}_{\mu}(\cdot|A^{-1}_{-k})$ and $\mathbb{P}^{-\infty}_{\mu}(\cdot|A^{-1}_{-k-1})$. We call this coupling $\widetilde{\mathbb{P}}^*$ and drop the super-index k from the random variables. By construction of the coupling,

$$\widetilde{\mathbb{P}}^*(\cdot|H = h) = \widehat{\mathbb{P}}_{A^{-1-h}_{-k}, A^{-1-h}_{-k-1}}\left(\cdot \;\middle|\; \eta^1 \in A^{-1}_{-h}, \xi_{-i}(\gamma_{-i}) = 0, i = 1, ..., h\right)$$

where $\widehat{\mathbb{P}}_{A^{-1-h}_{-k}, A^{-1-h}_{-k-1}}$ is the strong disagreement percolation coupling starting from the cone C_{-1-h}. In particular, η^1 and ξ are independent. Denote by $G := \{\xi_0(x) = 0 \; \forall \; x \in \mathbb{Z}^d\}$ the good event that the disagreement process has become extinct by time 0. We have

$$\frac{\mathbb{P}^{-(k+1)}_{\mu}\left(B \mid A^{-1}_{-k-1}\right)}{\mathbb{P}^{-k}_{\mu}\left(B \mid A^{-1}_{-k}\right)} \geq \frac{\widetilde{\mathbb{P}}^*\left(\eta^2_0 \in B, G\right)}{\widetilde{\mathbb{P}}^*\left(\eta^1_0 \in B\right)} = \frac{\widetilde{\mathbb{P}}^*\left(\eta^1_0 \in B, G\right)}{\widetilde{\mathbb{P}}^*\left(\eta^1_0 \in B\right)} = \widetilde{\mathbb{P}}^*(G).$$

Reversing the roles of η^1 and η^2, we also have

$$\frac{\mathbb{P}_\mu^{-(k+1)}\left(B \mid A_{-k-1}^{-1}\right)}{\mathbb{P}_\mu^{-k}\left(B \mid A_{-k}^{-1}\right)} \leq \widetilde{\mathbb{P}}^*(G)^{-1}.$$

Since $\widetilde{\mathbb{P}}(G^c|H=h)$ is exponentially small in h, we have completed the proof once we show that $\widetilde{\mathbb{P}}^*(H \leq h)$ is exponentially small in k for a fixed h. To see this, we look at H^m in more detail. Since H^m either increases by one or is reset to 0, (H^m) is a time-inhomogeneous house-of-cards process with transition probability $\mathbb{P}(H^{m+1}=h+1|H^m=h)=q_m(h)$. We have that $q_m(h)$ equals

$$\begin{aligned}
&\frac{\widetilde{\mathbb{P}}^{m,*}\left(\xi_{-k+m}^m(\gamma_{-k+m})=0, \eta^{1,m} \in A_{-k+m}^{-k+m} \;\middle|\; H^m=h\right)}{\max\left(\widetilde{\mathbb{P}}^{m,*}(\eta^{1,m} \in A_{-k+m}^{-k+m} \;\middle|\; H^m=h), \widetilde{\mathbb{P}}^{m,*}(\eta^{2,m} \in A_{-k+m}^{-k+m} \;\middle|\; H^m=h)\right)} \\
=&\frac{\widetilde{\mathbb{P}}^{m,*}\left(\xi_{-k+m}^m(\gamma_{-k+m})=0 \mid H^m=h\right)}{\max\left(1, \widetilde{\mathbb{P}}^{m,*}(\eta^{2,m} \in A_{-k+m}^{-k+m} \;\middle|\; H^m=h)(\widetilde{\mathbb{P}}^{m,*}(\eta^{1,m} \in A_{-k+m}^{-k+m} \;\middle|\; H^m=h))^{-1}\right)} \\
=&\frac{\widetilde{\mathbb{P}}^{m,*}\left(\xi_{-k+m}^m(\gamma_{-k+m})=0 \mid H^m=h\right)}{\max\left(1, \widetilde{\mathbb{P}}^{m,*}(\xi_{-k+m}^m(\gamma_{-k+m})=0 \;\middle|\; H^m=h) + \widetilde{\mathbb{P}}^{m,*}(\eta^{2,m}/\eta^{1,m})\right)} \\
\geq&\frac{\widetilde{\mathbb{P}}^{m,*}\left(\xi_{-k+m}^m(\gamma_{-k+m})=0 \mid H^m=h\right)}{1+(\inf_{i,\eta} c_i(\eta))^{-1}\left(1-\widetilde{\mathbb{P}}^{m,*}\left(\xi_{-k+m}(\gamma_{-k+m})=0 \mid H^m=h\right)\right)},
\end{aligned}$$

where in the last equality before the inequality we denoted by

$$\widetilde{\mathbb{P}}^{m,*}\left(\eta^{2,m}/\eta^{1,m}\right) := \frac{\widetilde{\mathbb{P}}^{m,*}\left(\eta^{2,m} \in A_{-k+m}^{-k+m}, \xi_{-k+m}^m(\gamma_{-k+m})=1 \;\middle|\; H^m=h\right)}{\widetilde{\mathbb{P}}^{m,*}\left(\eta^{1,m} \in A_{-k+m}^{-k+m} \;\middle|\; H^m=h\right)}.$$

Note that $\inf_{i,\eta} c_i(\eta) > 0$, since $\inf_{i,\eta} c_i(\eta) = 0$ implies $p^* = 1 > p_c$. Conditioned on $H^m = h$ the probability of $\xi_{-k+m}^m(\gamma_{-k+m}) = 0$ is larger than the probability that there is no percolation path from $C_{-k+m-h-1}$ to $(\gamma_{-k+m}, -k+m)$, which converges exponentially fast to 1 in h. Therefore there are constants $c_1, c_2 > 0$ so that $1 - q_m(h) \leq c_1 e^{-c_2 h}$. We also have $q_m(h) \geq (1-p^*)\inf_{i,\eta} c_i(\eta)$. Those two facts imply that $\widetilde{\mathbb{P}}^*(H \leq h) \leq c_3 e^{-c_4(k-c_5 h)}$ for some constants $c_3, c_4, c_5 > 0$. □

Corollary 3.3.6 follows as a direct consequence of Theorems 3.3.4 and 3.3.5.

Proof of Corollary 3.3.6. Let $l \in \mathbb{N}$ and consider $A_l \in \mathcal{A}_{-l}^{-1}$. By telescoping, for any $B \in \mathcal{F}_{=0}$, we have by Theorem 3.3.5 that

$$\left| \frac{\mathbb{P}_\mu(B \mid A_l)^{\pm}}{\mu(B)} - 1 \right| \leq \left[\prod_{i=1}^{k} (1 + C\delta^i) \right].$$

Setting $M := \prod_{i=1}^{\infty}(1 + C\delta^i)$, and noting that the statement (and proof) of Theorem 3.3.4 holds when $\mathcal{F}_{\geq 0}$ is replaced by $\mathcal{F}_{=0}$, we conclude the proof. □

Chapter 4

The contact process as seen from a random walk

This chapter is based on the paper [26], which has been submitted.

Abstract

We consider a random walk on top of the contact process. In particular, we focus on the "contact process as seen from the random walk". Under the assumption that the infection rate of the contact process is large or the jump rate of the random walk is small, we show that this process has at most two extremal measures. Moreover, the convergence to these extremal measures is characterised by whether the contact process survives or dies out, similar to the complete convergence theorem known for the ordinary contact process. Using this, we furthermore provide a law of large numbers for the random walk. Our arguments apply to the processes considered on $\mathbb{Z}^d$, $d \geq 1$, and under general assumptions on the jump probabilities of the random walk.

4.1 Introduction and main results

4.1.1 Motivation, background and outline

In this chapter we study a random walk on top of the contact process on $\mathbb{Z}^d$ with $d \geq 1$. That is, we assume that the transition kernel of the random walk depends on the contact process in a local neighbourhood around the position of the random walk. This is an example of a random walk in a dynamic random environment (abbreviated by RWDRE), a class of models that have recently been the subject of intensive studies in the mathematical literature (see e.g. [3], [9], [32], [50], [71], [77], [100]).

The contact process is a classical interacting particle system. This model was first introduced by Harris [70] in the 1970's as a model for the spread of an infection in a population. In this model, "infections" spread to nearest neighbour sites at a constant rate λ and a site become "healthy" at a constant rate 1.

On the one hand, the contact process is a model to which many of the mathematical tools developed for studying disordered systems apply, such as monotonicity, duality, renormalisation and coupling, and by now much is known. For instance, a full understanding of its dependence on the initial state is known by the complete convergence theorem. As a general reference about the contact process, we refer to Liggett [83, Chapter 1].

On the other hand, the contact process is a complicated model. Indeed, since infections spread in space and time, it has a non-trivial spatial and temporal correlation structure. Moreover, the contact process has a phase transition. For infection rate λ sufficiently small, the whole population eventually becomes healthy, irrespectively of the initial configuration. Interestingly, for infection rate above a certain threshold, infections may spread for all times with positive probability. In particular, in this regime (called the supercritical regime), the evolution of the contact process depends strongly on the initial configuration.

Furthermore, the contact process is an example of a model which, in the supercritical regime, does not fall into the class of well behaved models characterised by the cone-mixing condition. In particular, the general results obtained by Avena, den Hollander, and Redig [8] and Redig and Völlering [100] do not apply to random walks on the supercritical contact process. Despite much progress in the last years, no general theory has so far been developed for RWDRE models when the dynamic random environment is not cone mixing.

For the above reasons, the study of a random walk on the supercritical contact process in the context considered in this chapter was initiated by den Hollander and dos Santos [72]. They considered a class of nearest neighbour random walks on the one dimensional contact process. Combining monotonicity properties of the contact process and the random walk, they proved a law of large numbers, valid throughout the supercritical regime, and, assuming large enough infection rate, a central limit theorem. Since then, the model has been studied in several papers. We mention in particular Bethuelsen and Heydenreich [27], who proved a

law of large numbers for a version of the model on $\mathbb{Z}^d$ with $d \geq 1$, and Mountford and Vares [91], who improved the central limit theorem of den Hollander and dos Santos [72] and proved that it holds throughout the supercritical regime. See also Bethuelsen and Völlering [28] and Birkner, Černý, and Depperschmidt [32] for related results.

In contrast to [27], [72] and [91], who studied the evolution of the random walk directly, the focus of this chapter is on the "contact process as seen from the random walk"-process (abbreviated by CPSRW). That is, we study the shift-perturbed version of the ordinary contact process on $\mathbb{Z}^d$, $d \geq 1$, such that the random walk always remain at the origin. In particular, we are interested in the set of invariant measures for the CPSRW process and its convergence towards the extremal ones.

Our main contribution is that, when the infection rate of the contact process is large or the jump rate of the random walk is small, then the CPSRW process satisfies a complete convergence theorem similar to what is known for the ordinary contact process. That is, the CPSRW process has (at most) two extremal invariant measures making the process ergodic and it converges towards a mixture of these states (in the Cesàro sense) depending only on whether the underlying contact process survives or dies out. For this, we allow for very general transitions kernels of the random walk.

As a consequence of this result about the CPSRW process, we also derive limiting properties about the random walk itself. In particular, we show that it satisfies a law of large numbers under rather general assumptions on the transition kernel.

Outline

In the next subsection we give a more precise definition of our model and in particular the CPSRW process. Our main results are presented in Subsection 4.1.3. As preparations for the proofs, we provide in Section 4.2 some preliminary results about the contact process and in Section 4.3 we provide a particular coupling construction of our model. Section 4.4 contains the proofs of our main results.

4.1.2 The model

The contact process

Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$. For $\eta \in \Omega$ and $x \in \mathbb{Z}^d$, we denote by η_x the configuration which is identical to η except at site x, where a 1 is replaced by a 0 and vice versa. We also denote by $s(\eta, x) := \sum_{y \sim x} \eta(y)$, where $\sum_{y \sim x}$ is the summation over nearest neighbours of x.

The contact process $(\eta_t)_{t \geq 0}$ on $\mathbb{Z}^d$ with "infection" rate $\lambda > 0$ and "recovery" rate 1 is the Markov process on Ω with generator $L \colon \mathcal{C}(\Omega; \mathbb{R}) \mapsto \mathcal{C}(\Omega; \mathbb{R})$, where $\mathcal{C}(\Omega; \mathbb{R})$ denotes the

space of bounded continuous functions from Ω to $\mathbb{R}$, and L is given by

$$Lf(\eta) = \sum_{x\in\mathbb{Z}^d} \Big[\eta(x)\,[f(\eta_x) - f(\eta)] + \lambda s(\eta, x)\,(1 - \eta(x))\,[f(\eta_x) - f(\eta)]\Big].$$

We denote the semi-group generated by L by $(S_t)_{t\geq 0}$, also considered on the space $\mathcal{C}(\Omega, \mathbb{R})$. Note that the contact process is translation invariant, that is,

$$\mathbb{P}_{\eta,\lambda}(\theta_x\eta_t \in \cdot) = \mathbb{P}_{\theta_x\eta,\lambda}(\eta_t \in \cdot)$$

with θ_x denoting the shift operator $\theta_x\eta(y) = \eta(y - x)$ and $\mathbb{P}_{\eta,\lambda}$ the path-space measure of the contact process on $D_\Omega[0, \infty)$, the set of càdlàg functions on $[0, \infty)$ taking values on Ω, starting from $\eta_0 = \eta$. Further, denote by $\mathcal{F}$ the product σ-algebra corresponding to Ω and let $\mathcal{M}_1(\Omega)$ denote the set of probability measures on $(\Omega, \mathcal{F})$. By $\delta_\eta \in \mathcal{M}_1(\Omega)$ we denote the measures which concentrates on $\eta \in \Omega$. For $\mu \in \mathcal{M}_1(\Omega)$ we denote by $\mathbb{P}_{\mu,\lambda}$ the path-space measure of $(\eta_t)_{t\geq 0}$ when the contact process is initialised from μ, that is, $\mathbb{P}_{\mu,\lambda}(\cdot) := \int \mathbb{P}_{\eta,\lambda}(\cdot)\mu(d\eta)$.

The empty configuration where all sites equal to 0, denoted by $\bar{0}$, is an absorbing state for the contact process since $s(\bar{0}, x) = 0$ for all $x \in \mathbb{Z}^d$. On the other hand, when initialised with all sites equal to 1, the contact process is known to evolve towards an equilibrium measure called the *upper invariant measure*. We denote this measure by $\bar{\nu}_\lambda$.

As already mentioned, the contact process has a phase transition. That is, there is a critical threshold $\lambda_c \in (0, \infty)$, where λ_c depends on the dimension, such that $\nu_\lambda = \delta_{\bar{0}}$ for $\lambda \in (0, \lambda_c]$ and, for all $\lambda \in (\lambda_c, \infty)$, it holds that $\bar{\nu}_\lambda(\eta(x) = 1) > 0$ for any $x \in \mathbb{Z}^d$. Further, the two measures, $\delta_{\bar{0}}$ and $\bar{\nu}_\lambda$, are the only extremal measures for the contact process on $\mathbb{Z}^d$. A complete description of the convergence towards any mixture of them is known by the complete convergence theorem, which for later reference we state next. (For a proof we refer to [83, Theorem 1.2.27]).

Theorem 4.1.1 (Complete Convergence for (η_t))**.** *Let* $\tau := \inf\{t \geq 0\colon \eta_t = 1_{\bar{0}}\}$. *Then, for* $\lambda > 0$ *and* $\eta \in \Omega$*;*

$$\delta_\eta S_t \implies \mathbb{P}_{\eta,\lambda}(\tau < \infty)\delta_{\bar{0}} + \mathbb{P}_{\eta,\lambda}(\tau = \infty)\bar{\nu}_\lambda \quad \text{as } t \to \infty,$$

where $\implies$ *denotes weak convergence.*

The random walk

The random walk (X_t) is a process on $\mathbb{Z}^d$ whose transition probabilities depend on the state of the contact process in a neighbourhood around the random walk. More precisely, we assume (w.l.o.g.) that $X_0 = o$, where $o \in \mathbb{Z}^d$ denotes the origin. Further, at any time $t > 0$, the rate to jump from site x to site $x + z$, given that the contact process is in state η at time t, is given by $\gamma\alpha(\theta_x\eta, z) \in [0, \infty)$. Here, $\gamma \in [0, \infty)$ is a parameter of the model.

In order for the above process to be well defined, we need to pose some regularity assumptions. For this purpose, we assume throughout this chapter that

$$\| \alpha \|_1 := \sum_{z \in \mathbb{Z}^d} \| z \| \sup_{\eta \in \Omega} |\alpha(\eta, z)| < \infty, \tag{4.1.1}$$

and that for some $R \in \mathbb{N}$ and every $z \in \mathbb{Z}^d$;

$$\alpha(\eta, z) - \alpha(\omega, z) = 0 \text{ whenever } \eta \equiv \omega \text{ on } [-R, R]^d. \tag{4.1.2}$$

Assumption (4.1.1) assures that the position of (X_t) has a first moment, whereas Assumption (4.1.2) says that the random walk only depends on the contact process within a finite region around its location. Note that, by (4.1.1), the jump rate of (X_t) is bounded by $\gamma \| \alpha \|_1$.

Further, we say that the random walk is *elliptic* if there is a finite subset $E = \{e_1, \dots, e_n\}$ of $\mathbb{Z}^d$ such that

$$\alpha(\eta, e_i) > 0 \quad \forall\, \eta \in \Omega \text{ and } i \in \{1, \dots, n\},$$

and such that $\alpha(\eta, y) > 0$ for some $\eta \in \Omega$ and $y \in \mathbb{Z}^d$ if and only if $y = \sum_{i=1}^n a_i e_i$ with $a_i \in \{0, 1, 2, \dots\}$, $i = 1, \dots, n$.

Lastly, for $\eta = (\eta_t) \in D_\Omega[0, \infty)$, let P^η denote the *quenched* law of (X_t) in environment η. For $\mu \in \mathcal{M}_1(\Omega)$, the *annealed* law of (X_t) is given by

$$P^\mu(\cdot) := \int_{D_\Omega[0,\infty)} P^\eta(\cdot) \mathbb{P}_{\mu,\lambda}(d\eta).$$

The contact process as seen from a random walk

"The contact process seen from the random walk" (that is, the CPSRW process) is the key object of this chapter. This process, which is also useful for understanding the asymptotic behaviour of the random walk itself, is the Markov process on Ω with generator

$$L^{EP} f(\eta) := Lf(\eta) + \sum_{z \in \mathbb{Z}^d} \alpha(\eta, z) \left[f(\theta_{-z}\eta) - f(\eta) \right],$$

corresponding semigroup (S_t^{EP}), both acting on $\mathcal{C}(\Omega; \mathbb{R})$, and with path-space measure denoted by $\mathbb{P}^{EP}_{\eta,\lambda}$. Here, the superscript EP is an abbreviation for *environment process* and is used to distinguish it from $\mathbb{P}_{\eta,\lambda}$, the path-space measure of the contact process.

4.1.3 Main theorems

As for the ordinary contact process, it is clear that $\bar{0}$ is an absorbing state for the CPSRW process as well. If $\lambda < \lambda_c$ it is not difficult to show that $\delta_{\bar{0}}$ is the only stationary distribution

for (η_t^{EP}). This follows for instance from the methods developed in Redig and Völlering [100] together with well known convergence estimates towards $\bar{0}$ for the subcritical contact process, see Theorem 1.2.48 in [83].

On the other hand, when $\lambda > \lambda_c$, one can often show that there exists more than one stationary distribution for the CPSRW process. For this, it is sufficient to show that there is a site $x \in \mathbb{Z}^d$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \eta_t^{EP}(x)dt > 0. \tag{4.1.3}$$

That (4.1.3) holds when $\lambda > \lambda_c$ can been shown by several methods. For instance, [27, Theorem 1.4] and [72, Theorem 1], both proven via monotonicity arguments and particular properties of the contact process, imply that (4.1.3) holds for the class of models studied in these papers. In dos Santos [104] another method is put forward, by use of multiscale analysis, and applied to a random walk on the exclusion process. This method can presumably be applied to random walks on the contact process as well.

Ideally we would like to describe the entire class of stationary distributions corresponding to (η_t^{EP}), given the transition kernel of (X_t) and the infection parameter λ. As we saw in Theorem 4.1.1, a complete description is at hand for the ordinary contact process, i.e. when not perturbed by the random walk. Our main theorem shows that a similar statement holds for (η_t^{EP}) when either λ is sufficiently large or γ is sufficiently small.

Theorem 4.1.2 (Complete convergence for (η_t^{EP}))**.** *Assume that (X_t) satisfies Assumptions* (4.1.1) *and* (4.1.2) *and that it is elliptic.*

a) *Let $\lambda \in (\lambda_c, \infty)$. Then there is a $\gamma_0 \in (0,\infty)$ such that for all $\gamma < \gamma_0$ there exists $\bar{\nu}_\lambda^{EP} \in \mathcal{M}_1(\Omega)$ making $\mathbb{P}^{EP}_{\bar{\nu}_\lambda^{EP},\lambda}$ stationary and ergodic with respect to time-shifts. Furthermore, for any $\eta \in \Omega$,*

$$t^{-1}\int_0^t \delta_\eta S_s^{EP} ds \implies \mathbb{P}_{\eta,\lambda}(\tau = \infty)\bar{\nu}_\lambda^{EP} + \mathbb{P}_{\eta,\lambda}(\tau < \infty)\delta_{\bar{0}}.$$

b) *Let $\gamma \in (0,\infty)$. Then there is a $\lambda_0 \in (0,\infty)$ such that for all $\lambda > \lambda_0$ there exists $\bar{\nu}_\lambda^{EP} \in \mathcal{M}_1(\Omega)$ making $\mathbb{P}^{EP}_{\bar{\nu}_\lambda^{EP},\lambda}$ stationary and ergodic with respect to time-shifts. Furthermore, for any $\eta \in \Omega$,*

$$t^{-1}\int_0^t \delta_\eta S_s^{EP} \implies \mathbb{P}_{\eta,\lambda}(\tau = \infty)\bar{\nu}_\lambda^{EP} + \mathbb{P}_{\eta,\lambda}(\tau < \infty)\delta_{\bar{0}}.$$

The choice of γ_0 and λ_0 in Theorem 4.1.2 is related to the asymptotic speed at which an infection spreads. That is, we require the random walk trajectory to eventually be contained inside a forward space-time cone in which, for any starting configuration $\eta \in \Omega \setminus \{\bar{0}\}$, the

contact process conditioned on survival is approximately in equilibrium. This is similar in spirit to the assumption on λ in [72, Theorem 2].

Similar perturbative regimes have recently been studied for several other RWDRE models with non-uniform dependence on the initial configuration, in particular by Avena, Blondel, and Faggionato [9], Hilário, den Hollander, Sidoravicius, dos Santos, and Teixeira [71] and Huveneers and Simenhaus [77]. These very interesting works do not overlap with that of this chapter and are furthermore based on very different methods.

The proof of Theorem 4.1.2 follows by a coupling argument and uses known mixing properties of the supercritical contact process together with basic ergodic theory, and does not (directly) rely on the monotonicity properties of the contact process. For what appears to be only due to technical matters, we restrict to convergence in the sense of Cesàro.

We further note that the strategy of the proof can be applied to other models with similar mixing properties as the contact process. For instance, Theorem 4.1.2 can be shown to hold for certain extensions of our model where the random walk is allowed to interact with the medium, i.e. the contact process, by locally adding/removing infections. Such extensions may be natural from an application point of view.

The ellipticity assumption in Theorem 4.1.2 seems necessary for the theorem to hold in general. Indeed, an example of an non-elliptic random walk for which there exists three extremal invariant measures for (η_t^{EP}) can be constructed by making the random walk resemble the behaviour of the rightmost particle process of the contact process on $\mathbb{Z}$.

One way to achieve this is by considering a random walk that jumps deterministically to the right at a rate $\gamma \geq \lambda$ when on an infected site and otherwise as a simple random walk with jump rate 1. Considering the corresponding CPSRW process started from $\bar{0}$, $\bar{1}$ or the configuration where all sites on the negative integers are infected and the remaining sites are healthy, it is not difficult to show (using results about the distribution of the contact process seen from the rightmost particle, e.g. Galves and Presutti [60]) that this process has three invariant measures, all singular with respect to the other two.

Presumably, a similar reasoning can be made rigorous when λ is close to the critical value or γ is close to 0, even in cases where (X_t) is elliptic. On the other hand, for the case considered in Theorem 4.1.2, we do not think the ellipticity assumption is really necessary. We prove this rigorously in the case the contact process is started from $\bar{\nu}_\lambda$, as stated next.

Theorem 4.1.3 (Convergence of the upper invariant measure)**.** *Assume that (X_t) satisfies Assumptions* (4.1.1) *and* (4.1.2)*. Furthermore, let λ and γ be as in Theorem 4.1.2. Then there exists $\bar{\nu}_\lambda^{EP} \in \mathcal{M}_1(\Omega)$ making $\mathbb{P}^{EP}_{\bar{\nu}_\lambda^{EP}}$ stationary and ergodic with respect to time-shifts and such that*

$$t^{-1} \int_0^t \bar{\nu}_\lambda S_s^{EP} ds \implies \bar{\nu}_\lambda^{EP} \in \mathcal{M}_1(\Omega).$$

As mentioned above, under fairly general assumptions on the random walk and assuming that either λ is large or γ is small, we believe that the CPSRW-process has exactly two extremal measures. That is, in Theorems 4.1.2 and 4.1.3, we have $\bar{\nu}_\lambda^{EP} \neq \delta_{\bar{0}}$. Although we do not provide a proof of this, nevertheless, from the ergodic property of $\bar{\nu}_\lambda^{EP}$ only, we infer information about the random walk.

Theorem 4.1.4 (Law of large numbers)**.** *Under the assumptions of Theorem 4.1.2 there exists $v_0, v_1 \in \mathbb{R}^d$ such that for all $\eta \in \Omega$,*

$$\lim_{t\to\infty} t^{-1} X_t = \mathbb{P}_{\eta,\lambda}(\tau < \infty) v_0 + \mathbb{P}_{\eta,\lambda}(\tau = \infty) v_1, \quad P^\eta - a.s. \tag{4.1.4}$$

Relaxing the ellipticity assumption on (X_t), (4.1.4) *holds $P^{\bar{\nu}_\lambda}$-a.s.*

Remark 4.1.1. Note that $\mathbb{P}_\eta(\tau = \infty) = 1$ if and only if η has infinitely many infections, as follows by [83, Theorem 2.30]. In particular, the limit in (4.1.4) equals $v_1 \in \mathbb{R}^d$ when the contact process is started from the upper invariant measure $\bar{\nu}_\lambda$.

Presumably, the law of large numbers in Theorem 4.1.4 can be extended to a functional central limit theorem under the annealed law. For this, from the existence of $\bar{\nu}_\lambda^{EP}$ that is ergodic under (η_t^{EP}), martingale methods (as used e.g. in [100]) seem useful.

To this end, a remark about the critical case (i.e., when $\lambda = \lambda_c$) is in place. In this case, $\bar{\nu}_{\lambda_c} = \delta_{\bar{0}}$, as was proven by Bezuidenhout and Grimmett [30]. This result has since been extended to several models. On the other hand, still for $\lambda = \lambda_c$, the contact process process as seen from the rightmost particle is known to have a non-trivial invariant measure, as shown in Cox, Durrett and Schinazi [49].

It is not difficult to show that, by using Theorem 4.5 in [28] and monotonicity of the contact process, for $\lambda = \lambda_c$, any invariant measure for the CPSRW process concentrates on configurations having 0 asymptotic density. We believe that, under reasonable (ellipticity) assumptions, the CPSRW process with $\lambda = \lambda_c$ has no non-trivial invariant measure. However, to show this rigorously seems challenging since the critical contact process has slowly decaying space-time correlation structure.

4.2 Preliminaries about the contact process

Important to our approach is the existence of a coupling $\widehat{\mathbb{P}}^\lambda_{\eta,\omega}$ of the contact process started from any two $\eta, \omega \in \Omega$. The canonical choice is the graphical construction coupling, see p. 32-34 in [83], however, any other coupling satisfying (4.2.1) and (4.2.2) below will do just as fine.

For $\eta, \omega \in \Omega$, the coupled pair $(\eta_t^1, \eta_t^2)_{t\geq 0}$ denotes two copies of the contact process, started from $\eta_0^1 = \eta$ and $\eta_0^2 = \omega$ respectively. Recall that, by definition, a coupling has the marginals

$$\widehat{\mathbb{P}}^\lambda_{\eta,\omega}(\eta_t^1 \in \cdot) = \mathbb{P}_{\eta,\lambda}(\eta_t \in \cdot) \text{ and } \widehat{\mathbb{P}}^\lambda_{\eta,\omega}(\eta_t^2 \in \cdot) = \mathbb{P}_{\omega,\lambda}(\eta_t \in \cdot).$$

We are in this chapter mainly interested in the contact process with $\lambda > \lambda_c$ for which $\bar{\nu}_\lambda$ is non-trivial. In this regime a more global description of the contact process is at hand and known as the shape theorem. For this, denote by (η_t^o) and $(\eta_t^{\bar{1}})$ the contact process started from only the origin initially infected and the entire lattice initially infected respectively and define for $t \geq 0$,

$$H_t := \{x \in \mathbb{Z}^d \colon \eta_s^o(x) = 1 \text{ for some } 0 \leq s \leq t\};$$
$$K_t := \{x \in \mathbb{Z}^d \colon \eta_s^o(x) = \eta_s^{\bar{1}}(x) \,\forall\, s \geq t\}.$$

H_t is the set of sites which have been visited by an infection by time t when the contact process is started with only the origin infected at time 0. K_t is the subset of $\mathbb{Z}^d$ where (η_t^o) and $(\eta_t^{\mathbb{Z}^d})$ remain coupled for all time after time t. The next result shows that when $\lambda > \lambda_c$, then $t^{-1}(H_t \cap K_t)$ has an asymptotic shape. To state the result it is convenient to consider

$$\bar{H}_t := \bigcup_{x \in H_t} (x + Q) \text{ and } \bar{K}_t := \bigcup_{x \in K_t} (x + Q), \quad Q = [-\frac{1}{2}, \frac{1}{2}]^d.$$

Lastly, for $\omega \in \Omega$, denote by $\tau^\omega := \inf\{t \geq 0 \colon \eta_t^\omega = 1_{\bar{0}}\}$ the time until the contact process started from ω "dies out". We write $\tau^o := \inf\{t \geq 0 \colon \eta_t^o = 1_{\bar{0}}\}$ for the case when $\omega(x) = 1$ for $x = o$ only.

We are now prepared to state a version of the shape theorem, which in this generality is due to Garet and Marchand [61]; see Theorem 3 therein.

Theorem 4.2.1 (The shape theorem)**.** *Suppose $\lambda > \lambda_c$. There exists a convex set $D = D(\lambda) \subset \mathbb{R}^d$ such that, for any $\epsilon > 0$, on the event $\{\tau^o = \infty\}$,*

$$\lim_{T \to \infty} \widehat{\mathbb{P}}^\lambda_{o,\bar{1}} \left((1-\epsilon)D \subset \frac{1}{t}(\bar{H}_t \cap \bar{K}_t) \subset \frac{1}{t}\bar{H}_t \subset (1+\epsilon)D \,\forall t \geq T \right) = 1.$$

Moreover, there is a function $f \colon (\lambda_c, \infty) \to (0, \infty)$, non-decreasing, and such that $\{x \in \mathbb{R}^d \colon \| x \|_1 \leq f(\lambda)\} \subset D(\lambda)$ and $\lim_{\lambda \to \infty} f(\lambda) = \infty$.

Theorem 4.2.1 implies mixing properties for the contact process when started from other configurations than only the origin initially infected, as we show next.

Lemma 4.2.2. *Let $\lambda > \lambda_c$ and consider the contact processes (η_t^η) and $(\eta_t^{\bar{1}})$, initialised from η and $\bar{1}$ respectively, where $\eta \in \Omega \setminus \{\bar{0}\}$. Then, for any $\epsilon > 0$ and with D as in Theorem 4.2.1,*

$$\lim_{T \to \infty} \widehat{\mathbb{P}}^\lambda_{\eta,\bar{1}} \left(\eta_t^\eta(x) = \eta_t^{\bar{1}}(x) \,\forall\, x \in t(1-\epsilon)D \,\forall\, t \geq T \mid \tau^\eta = \infty \right) = 1. \tag{4.2.1}$$

Proof. Since the path measure of the contact process is translation invariant with respect to spatial shifts, (4.2.1) holds in the case when $\eta(x) = 1$ for only one site $x \in \mathbb{Z}^d$, as follows

immediately from Theorem 4.2.1. Indeed, for any $0 < \epsilon_1 < \epsilon_2$, it holds that $t(1-\epsilon_2)D \subset \theta_x t(1-\epsilon_1)D$ for all t sufficiently large.

For $T \in (0,\infty)$, let $A_T = \{\eta_t^{\eta}(x) \neq \eta_t^{\bar{1}}(x) \text{ for some } x \in t(1-\epsilon)D \text{ and } t \geq T\}$. Fix $x_1, \ldots, x_n \in \mathbb{Z}^d$ and assume that $\eta \in \Omega$ is such that $\eta(y) = 1$ only when $y = x_1, \ldots, x_n$. By using that the contact process is monotone and additive (in particular, that $\{\tau^{\eta} = \infty\} = \cup_{i=1}^n \{\tau^{x_i} = \infty\}$), we have

$$\begin{aligned}
\widehat{\mathbb{P}}^{\lambda}_{\eta,\bar{1}}\left(A_T \mid \tau^{\eta} = \infty\right) =& \widehat{\mathbb{P}}^{\lambda}_{\eta,\bar{1}}\left(\tau^{\eta} = \infty\right)^{-1} \widehat{\mathbb{P}}^{\lambda}_{\eta,\bar{1}}\left(A_T, \tau^{\eta} = \infty\right) \\
=& \widehat{\mathbb{P}}^{\lambda}_{\eta,\bar{1}}\left(\tau^{\eta} = \infty\right)^{-1} \widehat{\mathbb{P}}^{\lambda}_{\eta,\bar{1}}\left(A_T \cap (\cup_{i=1}^n \{\tau^{x_i} = \infty\})\right) \\
\leq& \widehat{\mathbb{P}}^{\lambda}_{\eta,\bar{1}}\left(\tau^{\eta} = \infty\right)^{-1} \sum_{i=1}^n \widehat{\mathbb{P}}^{\lambda}_{\sigma_i,\bar{1}}\left(A_T \cap \{\tau^{x_i} = \infty\}\right) \\
\leq& \sum_{i=1}^n \widehat{\mathbb{P}}^{\lambda}_{\sigma_i,\bar{1}}\left(A_T \mid \{\tau^{x_i} = \infty\}\right),
\end{aligned}$$

where $\sigma_i(y) = 1$ only if $y = x_i$. Consequently, (4.2.1) holds when η has finitely many 1's, since each term inside the sum of the last equation satisfies (4.2.1).

What remains to be shown is that (4.2.1) holds when the contact process is started from a configuration with infinitely many sites infected. In this case, $\tau^{\eta} = \infty$ a.s. (see Remark 4.1.1) and so

$$\widehat{\mathbb{P}}^{\lambda}_{\eta,\bar{1}}\left(A_T \mid \tau^{\eta} = \infty\right) = \widehat{\mathbb{P}}^{\lambda}_{\eta,\bar{1}}\left(A_T\right).$$

Let $(N_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence such that $\sum_{x\in[-N_n,N_n]^d} \eta(x) \geq n$ and denote by $\sigma_n \in \Omega$ the configuration which equals η on $[-N_n, N_n]^d$ and equals 0 outside $[-N_n, N_n]^d$. By the monotonicity property of the graphical construction coupling we have

$$\begin{aligned}
\widehat{\mathbb{P}}^{\lambda}_{\eta,\bar{1}}\left(A_T\right) &\geq \widehat{\mathbb{P}}^{\lambda}_{\sigma_n,\bar{1}}\left(A_T\right) \geq \widehat{\mathbb{P}}^{\lambda}_{\sigma_n,\bar{1}}\left(A_T \cap \{\tau^{\sigma_n} = \infty\}\right) \\
&= \widehat{\mathbb{P}}^{\lambda}_{\sigma_n,\bar{1}}\left(\tau^{\sigma_n} = \infty\right) \widehat{\mathbb{P}}^{\lambda}_{\sigma_n,\bar{1}}\left(A_T \mid \tau^{\sigma_n} = \infty\right).
\end{aligned}$$

Hence, taking $T \to \infty$ yields that $\lim_{T\to\infty} \widehat{\mathbb{P}}^{\lambda}_{\eta,\bar{1}}\left(A_T\right) \geq \widehat{\mathbb{P}}^{\lambda}_{\eta,\bar{1}}\left(\tau^{\sigma_n} = \infty\right)$. Taking $n \to \infty$, by [83, Theorem 1.2.30], we conclude the proof. □

We also need to control the contact process started from a finite number of 1's and conditioned on dying out, for which we have the following lemma.

Lemma 4.2.3. *Let $\lambda > \lambda_c$ and consider the contact processes (η_t^{η}) and $(\eta_t^{\bar{0}})$, initialised from η and $\bar{0}$ respectively, where η satisfies $\sum_{x\in\mathbb{Z}^d} \eta(x) < \infty$. Then, for any $\Delta \subset \mathbb{Z}^d$,*

$$\lim_{T\to\infty} \widehat{\mathbb{P}}^{\lambda}_{\eta,\bar{0}}\left(\eta_t^{\eta}(x) = \eta_t^{\bar{0}}(x)\ \forall\, x \in t\Delta\ \forall\, t \geq T \mid \tau^{\eta} < \infty\right) = 1. \tag{4.2.2}$$

4.3 Coupling construction

Given the coupling $\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega}$ of the contact process from the previous section, for each $T \in [0,\infty)$, we show in the following lemma how to extend it to a coupling $\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}$ also containing the evolution of two random walks $(X^1_t, X^2_t)_{t\geq 0}$ on $(\eta^1_t, \eta^2_t)_{t\geq 0}$. The coupling construction is motivated by the coupling used in [72], Section 3, and can be seen as a generalisation of their approach to general dimensions and general transition kernels.

Before stating the lemma we need to introduce some notation. For $\gamma \in (0,\infty)$, denote by $\mathcal{R}(\gamma) \subset \mathbb{R}^d$ the convex hull of the transition kernels of (X_t), that is,

$$\mathcal{R}(\gamma) := \gamma \cdot \operatorname{conv}\left(\sum_{z\in\mathbb{Z}^d} z\alpha(\eta, z), \eta \in \Omega \right). \tag{4.3.1}$$

Further, for $\eta, \omega \in \Omega$, $T, \epsilon > 0$, let

$$C_{T,R,\epsilon}(\eta,\omega) := \{\eta^{\eta}_s(x) = \eta^{\omega}_s(x) \,\forall\, x \in s(1+\epsilon)\mathcal{R}(\gamma) + [-R,R]^d, s \in [T,\infty)\} \tag{4.3.2}$$

denote the event that the contact processes started from η and ω respectively are perfectly coupled inside $s(1+\epsilon)\mathcal{R}(\gamma) + [-R,R]^d \subset \mathbb{Z}^d$ for all $s \geq T$, and let

$$D_{T,\epsilon} := \{X^1_t, X^2_t \in t(1+\epsilon)\mathcal{R}(\gamma) \,\forall\, t \geq T\}. \tag{4.3.3}$$

Lemma 4.3.1. *Let $T \in [0,\infty)$ and let $\eta, \omega \in \Omega$. There exists a coupling $\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}$ with the following properties:*

(a) *(Marginals) The coupling supports two contact processes and corresponding random walks:*

1. $\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}((\eta^1_t, X^1_t) \in \cdot) = \tilde{\mathbb{P}}_{\eta,\lambda}((\eta_t, X_t) \in \cdot)$;
2. $\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}((\eta^2_t, X^2_t) \in \cdot) = \tilde{\mathbb{P}}_{\eta,\lambda}((\eta_t, X_t) \in \cdot)$;

where $\tilde{\mathbb{P}}_{\eta,\lambda}$ is the path measure of the joint process (η_t, X_t).

(b) *(Extension of $\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega}$) The contact processes behave as under $\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega}$,*

$$\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}\big((\eta^1_t, \eta^2_t) \in \cdot\big) = \widehat{\mathbb{P}}^{\lambda}_{\eta,\omega}\big((\eta^1_t, \eta^2_t) \in \cdot\big).$$

(c) *(Coupling of the walkers) The jumping times of X^1_t and X^2_t are independent up to time T and identical after time T. Furthermore, for any $\epsilon > 0$,*

$$\lim_{T\to\infty} \widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}\left(X^1_t = X^2_t \,\forall\, t \geq T \mid X^1_T = X^2_T, C_{T,R,\epsilon}(\eta,\omega), D_{T,\epsilon}\right) = 1.$$

Proof. To obtain the properties listed above, we extend the original coupling $\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega}$ to contain three Poisson processes N^i with $i \in \{1,2,3\}$, all with rates $\lambda_i := \gamma \| \alpha \|_1$, as well as a sufficient supply of independent uniform $[0,1]$ variables for each $i \in \{1,2,3\}$, denoted by U^i. The Poisson processes are chosen independent of $\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega}$ and thus property b) is immediate.

To obtain the properties described in a) and c), we chose the Poisson processes N^1 and N^3 independent from each other, as well as the corresponding variables U^1 and U^3. Furthermore, the process N^2 is given by

$$N^2_t := \begin{cases} N^3_t & \text{if } t \le T \\ N^3_T + N^1_t - N^1_T & \text{if } t > T, \end{cases}$$

and the variables U^2 are given by

$$U^2_n := \begin{cases} U^3_n & \text{if } n \le N^3_T \\ U^1_{n+N^1_T-N^3_T} & \text{otherwise.} \end{cases}$$

Now, for $j \in \{1,2\}$, the random walk X^j starts from o and exclusively (but not necessarily) jump when the Poisson clocks N^j rings. To make this precise, enumerate $\mathbb{Z}^d = \{z_1, z_2, z_3, \dots\}$ and let for each $\eta \in \Omega$ and $m \in \mathbb{N}$, $p(\eta, m) := \sum_{i=1}^{m} \alpha(\eta, z_i)$. When the clock N^j rings for the k'th time, the random walk jumps from X^j_t to $X^j_t + z_i$ only if the uniform $[0,1]$ variable U^j_k satisfies

$$\| \alpha \|_1^{-1} p(\theta_{X^j_t}\eta^j_t, z_{m-1}) \le U^j_k < \| \alpha \|_1^{-1} p(\theta_{X^j_t}\eta^j_t, z_m).$$

Clearly this yields property a). Furthermore, note that both the random walks use independent Poisson clocks and U's up to time T, and share the same Poisson clocks and U's after time T. Property c) follows as a consequence of this and since (X_t) satisfies (4.1.2). □

4.4 Proofs

4.4.1 Coupling argument

In this subsection we present the coupling argument essential for the proofs of Theorems 4.1.2 and 4.1.3. For this, we first note that, as a simple consequence of our assumptions on the transition kernels of the random walk (recall Assumptions (4.1.1), (4.1.2) and Definition (4.3.1)), the following lemma holds.

Lemma 4.4.1. *For any $\epsilon > 0$ and any $\eta, \omega \in \Omega$, it holds that*

$$\lim_{t\to\infty} \widehat{\mathbb{P}}_{\eta,\omega,T}\left(X^1_s, X^2_s \in s(1+\epsilon)\mathcal{R}(\gamma)\ \forall\ s \ge t\right) = 1, \quad \textit{for any } T > 0.$$

With the help of the coupling construction in the previous section, together with Lemma 4.2.2 and Lemma 4.4.1, we next present a generalisation of Proposition 3.3 in [72] which allows us to compare possible limiting measures of (η^{EP}_t).

Proposition 4.4.2. *Assume that* (X_t) *satisfies Assumptions* (4.1.1) *and* (4.1.2) *and is elliptic. Furthermore, assume there exists* $\mu^{EP} \in \mathcal{M}_1(\Omega)$ *making* $\mathbb{P}^{EP}_{\mu^{EP},\lambda}$ *stationary and ergodic with respect to time-shifts and such that* $\mu^{EP} \neq \delta_{\bar{0}}$*. If, for some* $\epsilon > 0$*,* $\mathcal{R}(\gamma)(1+\epsilon) \subset D$*, then for every bounded and continuous function* $f\colon \Omega \to \mathbb{R}$*;*

$$\mathbb{P}^{EP}_{\omega,\lambda}\left(\lim_{t\to\infty} t^{-1}\int_0^t f(\eta^{EP}_s)ds = \mu^{EP}(f) \mid \tau^{\omega} = \infty\right) = 1.$$

Proof. Let $\mu^{EP} \in \mathcal{M}_1(\Omega)$ be such that $\mathbb{P}^{EP}_{\mu^{EP},\lambda}$ is ergodic with respect to time-shifts and $\mu^{EP} \neq \delta_{\bar{0}}$. By ergodicity, we know that there exist a set $B \in \mathcal{F}$ of full μ^{EP}-measure such that for any $f \in \mathcal{C}(\Omega;\mathbb{R})$, the set of bounded and continuous functions from Ω to $\mathbb{R}$, and $\eta \in B$;

$$\mathbb{P}^{EP}_{\eta,\lambda}\left(\lim_{t\to\infty} t^{-1}\int_0^t f(\eta^{EP}_s)ds = \mu^{EP}(f)\right) = 1. \tag{4.4.1}$$

Furthermore, since $\mu^{EP} \neq \delta_{\bar{0}}$ both are ergodic they are necessarily singular. Thus, μ^{EP} assigns 0 probability to the class of configurations with only finitely many 1's. To see this, note that a configuration of finitely many 1's may die out in finite time with positive probability and, since μ^{EP} and $\delta_{\bar{0}}$ are singular, this would lead to a contradiction. Consequently, by Remark 4.1.1, the ordinary contact process started from μ^{EP} survives almost surely.

Fix $\eta \in \Omega$ such that (4.4.1) is satisfied. Let $\Lambda \subset [-n,n]^d$ for some $n \in \mathbb{N}$ and consider a function $f \in \mathcal{C}(\Omega;\mathbb{R})$ only depending on the configuration inside Λ. In order to prove Proposition 4.4.2 we will show that for any $\omega \neq \bar{0}$;

$$\mathbb{P}^{EP}_{\omega,\lambda}\left(\lim_{t\to\infty} t^{-1}\int_0^t f(\eta^{EP}_s)ds = \mu^{EP}(f) \mid \tau = \infty\right) = 1 \tag{4.4.2}$$

irrespectively of the choice of Λ and f. This readily implies the statement of Proposition 4.4.2 by standard arguments since $\mathcal{C}(\Omega;\mathbb{R})$ is generated by the set of all local functions.

Now, to prove (4.4.2) and complete the proof, consider the coupling $\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}$ as constructed in Lemma 4.3.1. Let $M = \max(n,R)$, recall the definitions (4.3.2) and (4.3.3), and let

$$\Gamma_T := \{D_{T,\epsilon} \cap C_{T,M,\epsilon}(\eta,\omega) \text{ for some } \epsilon > 0\}.$$

Note that, by Lemma 4.2.2 and Lemma 4.4.1, it holds that, for all $T > 0$,

$$\lim_{t\to\infty} \widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}(\Gamma_t \mid \tau^{\omega} = \infty) = 1,$$

since by assumption $\mathcal{R}(\gamma)(1+\epsilon) \subset D$ for some $\epsilon > 0$. By the law of total expectation, by taking conditional expectation, (4.4.2) thus follows if we can show that

$$\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}\left(\lim_{t\to\infty} t^{-1}\int_0^t f(\theta_{X_s}\eta^2_s)ds = \mu^{EP}(f) \mid \Gamma_T, \tau^{\omega} = \infty\right) = 1, \quad \forall\, T > 0.$$

Moreover, let $\epsilon > 0$ be such that Γ_T holds. Consequently, on Γ_T, $X_T^2 = x$ for some $x \in \mathcal{R}(\gamma)(1+\epsilon)T$. Thus, by property c) of the coupling construction in Lemma 4.3.1, it suffices to show that

$$\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}\left(\lim_{t\to\infty} t^{-1}\int_0^t f(\theta_{X_s}\eta_s^2)ds = \mu^{EP}(f) \mid \Gamma_T, \tau^{\omega} = \infty, X_T^2 = x\right) = 1, \quad (4.4.3)$$

for all $T > 0$ and $x \in \mathcal{R}(\gamma)(1+\epsilon)T$.

To this end, we employ the ellipticity assumption. For each $x \in \mathbb{Z}^d$ fixed, there exists an event B_x generated by $(N^1_{[0,T]}, U_{[1,N_{[0,T]}]})$ which has positive probability and such that $X_T^1 = x$ on B_x. By property c) of the coupling construction and due to the ellipticity assumption, B_x can be chosen independent of the evolution of (η_t^1, η_t^2) and (X_t^2). Using this property, we thus have that

$$\begin{aligned}
&\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}\left(\lim_{t\to\infty} t^{-1}\int_0^t f(\theta_{X_s^2}\eta_s^2)ds = \mu^{EP}(f) \mid \Gamma_T, \tau^{\omega} = \infty, X_T^2 = x\right)\\
=&\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}\left(\lim_{t\to\infty} t^{-1}\int_0^t f(\theta_{X_s^2}\eta_s^2)ds = \mu^{EP}(f) \mid \Gamma_T, \tau^{\omega} = \infty, X_T^2 = x, B_x\right)\\
=&\widehat{\mathbb{P}}^{\lambda}_{\eta,\omega,T}\left(\lim_{t\to\infty} t^{-1}\int_0^t f(\theta_{X_s^1}\eta_s^1)ds = \mu^{EP}(f) \mid \Gamma_T, \tau^{\omega} = \infty, X_T^2 = x\right) \qquad (4.4.4)
\end{aligned}$$

Here, the first equality holds since B_x is independent of all the other variables. To see that the second equality holds, note that Γ_T ensures that the contact processes are perfectly coupled inside the space-time region defined by $C_{T,M,\epsilon}(\eta,\omega)$. Furthermore, since $x \in \mathcal{R}(\gamma)(1+\epsilon)T$ and $X_T^1 = X_T^2$ property c) of the coupling construction apply. To conclude (4.4.2) and hence the proof of Proposition 4.4.2, we note that (4.4.4) equals 1 as a consequence of (4.4.1). □

By a straightforward adaptation of the proof of Proposition 4.4.2, replacing μ^{EP} by $\delta_{\bar{0}}$ and using Lemma 4.2.3 instead of Lemma 4.2.2, we have the following statement.

Proposition 4.4.3. *Assume that (X_t) satisfies Assumptions* (4.1.1) *and* (4.1.2) *and is elliptic. Let $\omega \in \Omega$ be such that $\sum_{x\in\mathbb{Z}^d}\omega(x) < \infty$. Then, for every $f \in \mathcal{C}(\Omega;\mathbb{R})$;*

$$\mathbb{P}^{EP}_{\omega,\lambda}\left(\lim_{t\to\infty} t^{-1}\int_0^t f(\eta_s^{EP})ds = f(\bar{0}) \mid \tau^{\omega} < \infty\right) = 1.$$

Following [72, Remark 3.4], the ellipticity assumption in the above argument is not necessary in the case when the contact process is started from the upper invariant measure. The following proposition is essential for the proof of Theorem 4.1.3.

Proposition 4.4.4. *Assume that (X_t) satisfies Assumptions* (4.1.1) *and* (4.1.2). *Furthermore, assume there exists $\mu^{EP} \in \mathcal{M}_1(\Omega)$ making $\mathbb{P}^{EP}_{\mu^{EP},\lambda}$ ergodic with respect to time-shifts and such that $\mu^{EP} \neq \delta_{\bar{0}}$. If, for some $\epsilon > 0$, $\mathcal{R}(\gamma)(1+\epsilon) \subset D$, then for every $f \in \mathcal{C}(\Omega;\mathbb{R})$;*

$$\mathbb{P}^{EP}_{\bar{\nu}_\lambda,\lambda}\left(\lim_{t\to\infty} t^{-1}\int_0^t f(\eta_s^{EP})ds = \mu^{EP}(f) \mid \tau = \infty\right) = 1.$$

Proof. The statement follows as in the proof of Proposition 4.4.2, only with minor modifications which we highlight next. To adapt the proof, replace the conditioning on $X_T^2 = x$ in (4.4.3) (and the proceeding derivations) by the event $\{N_T^2 = 0\}$, which implies $X_T^2 = 0$. Then, by stationarity of the contact process under $\bar{\nu}_\lambda$, $(X_{t+T}^2 - X_T^2)_{t\geq 0}$ under $\widehat{\mathbb{P}}^\lambda_{\eta,\bar{\nu}_\lambda,T}(\cdot \mid N_T^2 = 0)$ has the same distribution as $(X_t^2)_{t\geq 0}$ under $\widehat{\mathbb{P}}^\lambda_{\eta,\bar{\nu}_\lambda,T}(\cdot)$. Since $N_T^1 = 0$ implies $X_T^1 = 0$ and has positive probability, the claim follows as in the proof of Proposition 4.4.2 by replacing B_x by $\{N_T^1 = 0\}$. □

4.4.2 Proof of Theorems 4.1.2 and 4.1.3

Proof of Theorem 4.1.2. As mentioned in the introduction, the measure $\delta_{\bar{0}}$ is trivially an invariant measure for the CPSRW process. Furthermore, it clearly makes $\mathbb{P}^{EP}_{\delta_{\bar{0}}}$ ergodic with respect to time-shifts and is hence extremal. Thus, in the (unlikely) scenario that $\delta_{\bar{0}}$ is the unique invariant measure for the CPSRW process, Theorem 4.1.2 follows by classical ergodic theory with $\bar{\nu}_\lambda^{EP} = \delta_{\bar{0}}$.

To complete the argument of Theorem 4.1.2, we next consider the (more likely) scenario that there exist a measure $\mu^{EP} \in \mathcal{M}_1(\Omega)$ invariant under (η_t^{EP}) and such that $\mu^{EP} \neq \delta_{\bar{0}}$. Without loss of generality, assume that μ^{EP} is extremal and hence singular with respect to $\delta_{\bar{0}}$. By Proposition 4.4.2 together with Remark 4.1.1, the statement of Theorem 4.1.2 follows immediately when starting the CPSRW process with a configuration having infinitely many 1's in the case that $\mathcal{R}(\gamma)(1+\epsilon) \subset D$. In this case the CPSRW process convergence towards $\bar{\nu}_\lambda^{EP}$. For fixed $\lambda > \lambda_c$, the statement of Theorem 4.1.2a) thus follows by taking γ sufficiently small. Similarly, for fixed $\gamma \in (0,\infty)$, the statement of Theorem 4.1.2b) follows by taking λ sufficiently large, since $D = D(\lambda)$ is growing towards the whole lattice as λ increases.

Similarly, if the starting configuration $\eta \neq \bar{0}$ has only finitely many 1's, the CPSRW process converges towards $\bar{\nu}_\lambda^{EP}$ on the event that $\{\tau = \infty\}$. This follows again by applying Proposition 4.4.2. On the other hand, on the event $\{\tau < \infty\}$, by Proposition 4.4.3, the CPSRW process converges towards $\delta_{\bar{0}}$. This concludes the proof. □

Proof of Theorem 4.1.3. This follows analogous to the proof of Theorem 4.1.2 for the case when $\eta \in \Omega$ has infinitely many 1's, by applying Proposition 4.4.4 instead of Proposition 4.4.2. □

4.4.3 Proof of Theorem 4.1.4

Proof of Theorem 4.1.4. For the first part of the proof, we follow the proof of [99], Theorem 4.1. Consider (η_t^{EP}) started from $\bar{\nu}_\lambda^{EP}$. Let $F_z : D([0,1],\Omega) \mapsto \mathbb{R}$, $z \in \mathbb{Z}^d$ count the number

of shifts of size z a piece of trajectory performs in the interval $[0,1]$, i.e.

$$F_z(\eta^{EP}_{[t,t+1]}) = \sum_{s\in]0,1]} \mathbb{1}_{\{\theta_z \eta^{EP}_{t+s} = \eta^{EP}_{t+s-}\}}.$$

With $F := \sum_{z\in\mathbb{Z}^d} zF_z$, which is well-defined and in $L^1(\mu^{EP})$ because of the rate condition $\| \alpha \|_1 < \infty$, we then have, for any integer $T > 0$,

$$X_T - X_0 = \sum_{n=1}^{T} F(\eta^{EP}_{[n-1,n]}).$$

The ergodic theorem then implies

$$\lim_{T\to\infty} \frac{X_T - X_0}{T} = \lim_{T\to\infty} \frac{1}{T} \sum_{n=1}^{T} F(\eta^{EP}_{[n-1,n]}) = \bar{\nu}^{EP}_\lambda(F). \tag{4.4.5}$$

The same is true for non-integer T, by using the fact that $X_T - X_{\lfloor T \rfloor}$ has bounded expectation. Since

$$\begin{aligned}\bar{\nu}^{EP}_\lambda(F) &= \int \int_0^1 \mathbb{E}_{\eta,\lambda} \sum_{z\in\mathbb{Z}^d} z\alpha(\theta_{X_t}\eta_t, z)\, dt\, \mu^{EP}(d\eta) \\ &= \int \sum_{z\in\mathbb{Z}^d} z\alpha(\eta, z)\, \mu^{EP}(d\eta),\end{aligned}$$

the claim is proven for $\bar{\nu}^{EP}_\lambda$.

To extend the result to an arbitrary probability measure ν we use Theorem 4.1.2. Firstly, if $\bar{\nu}^{EP}_\lambda = \delta_{\bar{0}}$, then it follows that (η^{EP}_t) converges towards $\delta_{\bar{0}}$ when started from any $\eta_0 \in \Omega$. Consequently, the left hand side of (4.4.5) converges towards $F(\bar{0})$ irrespectively of $\eta_0 \in \Omega$.

Secondly, if $\bar{\nu}^{EP}_\lambda \neq \delta_{\bar{0}}$, we concluded in the proof of Theorem 4.1.2 that $\bar{\nu}^{EP}_\lambda$ concentrates on configurations which have infinitely many infections. For any such configuration we showed in Lemma 4.2.2 that (4.2.1) holds. Thus, for any $\eta \in \Omega$, the left hand side of (4.4.5) converges towards $\bar{\nu}^{EP}_\lambda(F)$ when conditioned on $\tau^\eta = \infty$. Similarly, on $\tau^\eta < \infty$, (4.4.5) converges towards $F(\bar{0})$, and this concludes the proof. □

Chapter 5

Stochastic domination in space-time for the contact process

This chapter is based on a paper with Jacob van den Berg, [19], which has been submitted.

Abstract

Liggett and Steif (2006) proved that, for the supercritical contact process on certain graphs, the upper invariant measure stochastically dominates an i.i.d. Bernoulli product measure. In particular, they proved this for $\mathbb{Z}^d$ and (for infection rate sufficiently large) d-ary homogeneous trees T_d.

In this paper we prove some space-time versions of their results. We do this by combining their methods with specific properties of the contact process and general correlation inequalities.

One of our main results concerns the contact process on T_d with $d \geq 2$. We show that, for large infection rate, there exists a subset Δ of the vertices of T_d, containing a "positive fraction" of all the vertices of T_d, such that the following holds: The contact process on T_d observed on Δ stochastically dominates an independent spin-flip process. (This is known to be false for the contact process on graphs having subexponential growth.)

We further prove that the supercritical contact process on $\mathbb{Z}^d$ observed on certain d-dimensional space-time slabs stochastically dominates an i.i.d. Bernoulli product measure, from which we conclude strong mixing properties important in the study of certain random walks in random environment.

5.1 Introduction and main results

5.1.1 Background and outline of this paper

Let $G = (V, E)$ be a connected graph of bounded degree, and let $\lambda \in (0, \infty)$. The contact process (η_t) on G with parameter λ is the continuous-time interacting particle system on $\{0,1\}^V$ with local transition rates given by

$$\eta \to \eta_x \text{ at rate } \begin{cases} 1, & \text{if } \eta(x) = 1; \\ \lambda \sum_{\{y:\ \{x,y\} \in E\}} \eta(y), & \text{if } \eta(x) = 0, \end{cases}$$

where η_x is defined by $\eta_x(y) := \eta(y)$ for $y \neq x$, and $\eta_x(x) := 1 - \eta(x)$.

The contact process was introduced by Harris [70] in 1974 as a toy model for the spread of an infection in a population. With this interpretation in mind, λ is often referred to as the "infection" parameter and a site $x \in V$ is said to be *infected* at time t if $\eta_t(x) = 1$, otherwise it is said to be *healthy*. A central question is whether the infections "survive" with positive probability or eventually dies out, i.e. all sites become healthy. As general references on contact processes we mention the books [82] and [83] by Liggett.

The "healthy" configuration where all sites are equal to 0, denoted by $\bar{0}$, is clearly an absorbing state for the contact process. On the other hand, starting from the full configuration where all sites are initially infected, the contact process evolves towards an equilibrium measure $\bar{\nu}_\lambda$. This state is often called the *upper invariant measure*. Throughout this text we write *upper stationary contact process* to denote the contact process whose law at an arbitrary time equals $\bar{\nu}_\lambda$.

A basic property of the contact process is that, if G is countable infinite, it undergoes a phase transition: there is a critical threshold $\lambda_c \in [0, \infty)$, depending on G, such that, for all $\lambda < \lambda_c$, $\bar{\nu}_\lambda = \delta_{\bar{0}}$, and for all $\lambda > \lambda_c$, $\bar{\nu}_\lambda \neq \delta_{\bar{0}}$. Here, $\delta_{\bar{0}}$ denotes the measure that concentrates on $\bar{0}$. In this paper we focus on the supercritical phase, i.e., the case $\lambda > \lambda_c$.

van den Berg, Häggström and Kahn [20] proved that the upper invariant measure satisfies the following property (called downward FKG in [85]): for any finite $\Delta \subset V$, the conditional measure $\bar{\nu}_\lambda(\cdot \mid \eta \equiv 0 \text{ on } \Delta)$ is positively associated. (See Liggett [84] for a slightly stronger property). Liggett and Steif [85] used this result to show that, for the supercritical contact process on $\mathbb{Z}^d$, $d \geq 1$, the upper invariant measure stochastically dominates a non-trivial Bernoulli product measure (see Corollary 4.1 therein). For the d-ary homogeneous tree T_d, where each site has $d+1$ neighbouring sites, they showed such a domination result for $\lambda > 4$.

In this paper, we investigate to what extent the upper stationary contact process stochastically dominates a non-trivial Bernoulli product measure when observed in space-time.

One of our main results concerns the upper stationary contact process on T_d with $d \geq 2$. We show that, for $\lambda > \lambda_c(\mathbb{Z})$, there exists a subset V of the vertices of T_d, containing a "positive fraction" of all the vertices of T_d, such that the following holds: the contact process on T_d observed on V stochastically dominates an non-trivial independent spin-flip process. This is the content of Theorem 5.1.2 below. Interestingly, this cannot happen for the upper stationary contact process on graphs having subexponential growth (such as $\mathbb{Z}^d$), as shown in Proposition 5.1.1.

We furthermore prove that the upper stationary contact process on $\mathbb{Z}^d$ with $d \geq 1$ and $\lambda > \lambda_c$, observed on certain (discrete-time) d-dimensional space-time slabs, stochastically dominates a non-trivial Bernoulli product measure. This is the content of Theorem 5.1.5 below. Using this, we conclude in Theorem 5.1.7 that the contact process projected onto a thin space-time slab satisfies a strong mixing property known as cone-mixing.

The projection of the contact process onto a sub-lattice can be interpreted as a hidden Markov model and is motivated for instance by the study of phase transition phenomenas in nonlinear filtering (see van Handel and Rebeschini [69]) as well as the study of a random walk in a dynamic random environment (see Bethuelsen and Völlering [28]).

A key observation for our arguments is that the results in [20] imply that the above mentioned downward FKG property extends to the contact process observed in space-time. Our proofs are based on this observation, together with specific properties of the contact process and results and techniques from [85].

Outline of this paper

In the next subsection we recall some basic definitions before we present our main results for the contact process in Subsection 5.1.3. In Subsection 5.1.4 we discuss certain mixing properties which follow from our main results. Section 5.2 is devoted to some preliminary results. Proofs of our main results are provided in Section 5.3. In Section 5.4 we present some open questions.

5.1.2 Stochastic domination and Bernoulli product measures

Besides the contact process there are two key concepts in the presentation of our main results, namely stochastic domination and Bernoulli product measures. For the convenience of the reader, we briefly recall their definitions.

Given a countable set V, we are interested in probability measures on $\Omega := \{0,1\}^V$ and $D_\Omega[0,\infty)$, the set of càdlàg functions on $[0,\infty)$ taking values in Ω. For this, denote by $\mathcal{F}$

the product σ-algebra corresponding to Ω and let $\mathcal{M}_1(\Omega)$ be the set of probability measures on $(\Omega, \mathcal{F})$, and similarly, let $\mathcal{M}_1(D_\Omega[0,\infty))$ be the set of measures on $D_\Omega[0,\infty)$.

For $\rho \in [0,1]$, we denote by $\mu_\rho \in \mathcal{M}_1(\Omega)$ the *Bernoulli product measure* with density ρ. That is, for any finite $\Delta, \Lambda \subset V$ such that $\Delta \cap \Lambda = \emptyset$, the measure μ_ρ has cylinder probabilities given by

$$\mu_\rho\left(\eta \in \Omega \colon \eta(x) = 1 \,\forall x \in \Delta, \eta(x) = 0 \,\forall x \in \Lambda\right) = \rho^{|\Delta|}(1-\rho)^{|\Lambda|}.$$

A related object is the following continuous-time process. Given $\alpha \geq 0$, the *independent spin-flip process* (ξ_t) with parameter α is the continuous-time Markov process on $\{0,1\}^V$ with local transition rates given by

$$\eta \to \eta_x \text{ at rate } \begin{cases} 1, & \text{if } \eta(x) = 1; \\ \alpha, & \text{if } \eta(x) = 0. \end{cases}$$

Note that (ξ_t) is ergodic with unique invariant measure μ_ρ, where $\rho = \rho(\alpha) = \alpha/(\alpha+1)$.

We next introduce the concept of stochastic domination. For this, we associate to Ω the partial ordering such that $\xi \leq \eta$ if and only if $\xi(x) \leq \eta(x)$ for all $x \in V$. An event $B \in \mathcal{F}$ is said to be *increasing* if $\xi \leq \eta$ implies $1_B(\xi) \leq 1_B(\eta)$. If $\xi \leq \eta$ implies $1_B(\xi) \geq 1_B(\eta)$ then B is called *decreasing*. For $\mu_1, \mu_2 \in \mathcal{M}_1(\Omega)$ we say that μ_1 *stochastically dominates* μ_2 if $\mu_2(B) \leq \mu_1(B)$ for all increasing events $B \in \mathcal{F}$. Recall that, by Strassens theorem (see [82], p. 72), μ_1 *stochastically dominates* μ_2 is equivalent to the existence of a coupling (η, ξ) so that η has distribution μ_2 and ξ has distribution μ_1, and $\eta \leq \xi$ a.s. The definition of stochastic domination readily translates to measures on $D_\Omega[0,\infty)$, by extending the partial ordering for elements in Ω to $\Omega^{[0,\infty)}$, requiring that $\xi_t(x) \leq \eta_t(x)$ for all $(x,t) \in V \times [0,\infty)$.

Another key concept used in the proof of the following theorems is that of *downward FKG*, to which we return to in Section 5.2.

5.1.3 Main results

As shown in Liggett and Steif [85], the upper stationary contact process on $\mathbb{Z}^d$, $d \geq 1$, with $\lambda > \lambda_c$ stochastically dominates a non-trivial Bernoulli product measure when observed at a fixed time t. That is, $\bar{\nu}_\lambda$ stochastically dominates μ_ρ for some $\rho \in (0,1)$.

On the other hand, as also shown in [85], stochastic domination of a non-trivial Bernoulli product measure does not hold in general for the entire space-time evolution. This can be extended to the contact process on graphs having subexponential growth.

Let $G = (V, E)$ be a connected graph of bounded degree and denote by $d\colon V \times V \to \mathbb{Z}_{\geq 0}$ the graph distance on G. Following [86, p. 181], the graph G is said to have *subexponential growth* (of balls) if

$$\liminf_{n\to\infty} |\{x \in V\colon d(o,x) \leq n\}|^{1/n} = 1, \quad \text{for some } o \in V, \tag{5.1.1}$$

where $|\cdot|$ denotes the cardinality. Otherwise G is said to have *exponential growth*. Further, we say that $\Delta \subset V$ has *positive density* if

$$\liminf_{n\to\infty} \frac{|\{x \in \Delta\colon d(o,x) \leq n\}|}{|\{y \in V\colon d(o,y) \leq n\}|} > 0, \quad \text{for some } o \in V. \tag{5.1.2}$$

Remark 5.1.1. Since we assume that G is connected and has bounded degree we may in (5.1.1) and (5.1.2) replace "for some $o \in V$" by "for all $o \in V$".

Proposition 5.1.1. *Let (η_t) be the upper stationary contact process on a connected graph $G = (V, E)$ having subexponential growth and bounded degree with $\lambda > 0$. Consider $\Delta \subset V$ having positive density. Then, for **no** parameter value except $\alpha = 0$ can (η_t) and (ξ_t) be coupled so that, when initialised from $\bar{\nu}_\lambda$ and $\mu_{\rho(\alpha)}$ respectively, it holds that*

$$\widehat{\mathbb{P}}\,(\eta_t(x) \geq \xi_t(x) \text{ for all } (x,t) \in \Delta \times [0,\infty)) = 1.$$

The proof of Proposition 5.1.1 follows by an almost direct extension of the proof of [85], Proposition 1.1, and is given in Section 5.3.1. In fact, the proof also works if Condition (5.1.1) is replaced by the following condition,

$$\liminf_{n\to\infty} \frac{|\{(x,y) \in E\colon d(o,x) = n = d(o,y) - 1\}|}{|\{x \in V\colon d(o,x) \leq n\}|} = 0, \quad \text{for some } o \in V, \tag{5.1.3}$$

which is easily seen to be weaker than (5.1.1). A natural question is whether Proposition 5.1.1 also holds if (5.1.3) does not hold. Theorem 5.1.2 below states that this is not the case for homogeneous trees.

Theorem 5.1.2. *Let (η_t) be the upper stationary contact process on T_d, $d \geq 2$, with $\lambda > \lambda_c(\mathbb{Z})$. Let V be the set of vertices of T_d. Then there is a $\Delta \subset V$ having positive density together with an $\alpha = \alpha(\lambda) > 0$ and a coupling $\widehat{\mathbb{P}}$ of (η_t) and (ξ_t), initialised from $\bar{\nu}_\lambda$ and $\mu_{\rho(\alpha)}$ respectively, such that*

$$\widehat{\mathbb{P}}\,(\eta_t(x) \geq \xi_t(x) \text{ for all } (x,t) \in \Delta \times [0,\infty)) = 1. \tag{5.1.4}$$

Thus, (5.1.4) in Theorem 5.1.2 concerns the contact process on T_d *projected onto* a subset of $V \times [0,\infty)$ (a terminology we often refer to later). Theorem 5.1.2 says that the contact process projected onto $\Delta \times [0,\infty)$ stochastically dominates an independent spin-flip process.

To prove Theorem 5.1.2, we first show that the contact process on $\{0, 1, \dots\}$ observed at the vertex 0 stochastically dominates an independent spin-flip process (in fact, we show a generalisation of this). Once this is obtained, Theorem 5.1.2 follows by a monotonicity argument. From the precise argument, given in Section 5.3.4, it moreover follows that the set Δ in Theorem 5.1.2 can be chosen such that the l.h.s. of (5.1.2) equals $\frac{d-1}{d}$.

Denote by

$$\tau^x := \inf\{t \geq 0 \colon \eta_t^x \equiv \bar{0}\}, \quad x \in V, \tag{5.1.5}$$

the extinction time for the contact process (η_t^x) started with only x initially infected.

Theorem 5.1.3. *Let (η_t) be the upper stationary contact process on a connected graph $G = (V, E)$ having bounded degree with $\lambda > 0$. Let $x \in V$ for which there exist $C, c > 0$ such that,*

$$\mathbb{P}(\tau^x = \infty) > 0; \tag{5.1.6}$$

$$\mathbb{P}(s < \tau^x < \infty) \leq Ce^{-cs}, \quad \textit{for all } s \geq 0. \tag{5.1.7}$$

Then there exist $\alpha = \alpha(\lambda) > 0$ and a coupling $\widehat{\mathbb{P}}$ of (η_t) and (ξ_t) initialised from $\bar{\nu}_\lambda$ and $\mu_{\rho(\alpha)}$ respectively, such that

$$\widehat{\mathbb{P}}\left(\eta_t(x) \geq \xi_t(x) \textit{ for all } t \in [0, \infty)\right) = 1.$$

Note that (5.1.6) and (5.1.7) are known to hold for all vertices throughout the supercritical phase for the contact process on $\mathbb{Z}^d$, $d \geq 1$ (see [83], Theorem 1.2.30), and on $\{0, 1, \dots\}$ (see [57], p. 546 and [58]).

For the proof of Theorem 5.1.3 (in Section 5.3.2) we use that the contact process satisfies the downward FKG property in space-time (see Section 5.2 for a proper definition). Combining this with large deviation estimates of the probability that there are no infections at the site x in the time interval $[0, t]$ and a general theorem in [85] (which we state in Lemma 5.2.3) yields the statement of Theorem 5.1.3.

It seems natural that Theorem 5.1.3 can be extended to the case where instead of observing the contact process at a single site, we observe it on a finite subset $\Delta \subset V$. Apart from some special cases, we are not able to show this in general. On the other hand, interestingly, we are able to extend Theorem 5.1.3 when restricting to observations at discrete times. For this, denote by

$$\mathbb{Z}_T := \{0, \pm T, \pm 2T, \dots\}, \quad T \in (0, \infty). \tag{5.1.8}$$

Theorem 5.1.4. *Let (η_t), λ and G be as in Theorem 5.1.3. Let $\Delta \subset V$ be finite and let $x \in \Delta$ be such that* (5.1.6) *and* (5.1.7) *hold. Then, for each $T \in (0, \infty)$, there exist $\rho = \rho(\lambda, T, \Delta) > 0$ such that (η_t) projected onto $\Delta \times \mathbb{Z}_T$ stochastically dominates a Bernoulli product measure with parameter ρ.*

We end this subsection with a result, Theorem 5.1.5 below, for the supercritical contact process on $\mathbb{Z}^d$. As seen in Proposition 5.1.1, this process cannot stochastically dominate a non-trivial independent spin-flip process, not even when projected onto a subset Δ of positive density. This naturally leads to the question what happens for subsets $\Delta \subset \mathbb{Z}^d$ for which the l.h.s. of (5.1.2) equals 0. Theorem 5.1.5 concerns one such case, namely, the contact process projected onto certain (discrete-time) space-time slabs.

For $m \in \mathbb{N}$, let

$$\mathbb{Z}^d_{d-1}(m) := \big\{(x_1, \dots, x_d) \in \mathbb{Z}^d \colon x_d \in \{0, \dots, m-1\}\big\},$$

be the $(d-1)$-dimensional sublattice of $\mathbb{Z}^d$ of width m. When $m = 1$ we simply write $\mathbb{Z}^d_{d-1}$.

Theorem 5.1.5. *Let (η_t) be the upper stationary contact process on $\mathbb{Z}^d$, $d \geq 1$, with $\lambda > \lambda_c$. Let $T \in (0, \infty)$ and $m \in \mathbb{N}$. Then there exists $\rho = \rho(\lambda, T, m) > 0$ such that (η_t) projected onto $\mathbb{Z}^d_{d-1}(m) \times \mathbb{Z}_T$ stochastically dominates a Bernoulli product measure with parameter ρ.*

5.1.4 Mixing properties

The purpose of this subsection is to show that the domination results we have obtained so far are useful in order to conclude mixing properties for the contact process, in particular when observed in a subspace.

We first note that, from the statement of Theorem 5.1.5 with $m = 1$, we obtain a stronger notion of domination, which we present next. For $t \in (0, \infty)$ and $T \in (0, \infty)$, let $\mathbb{Z}_T(t) := \{s \in \mathbb{Z}_T \colon s < tT\}$ and denote by $\mathcal{P}^{\text{slab}}_\lambda(\cdot)$ the law of the projection of (η_t) onto $\mathbb{Z}^d_{d-1} \times \mathbb{Z}_T$.

Corollary 5.1.6. *Let (η_t) be the upper stationary contact process on $\mathbb{Z}^d$, $d \geq 1$, with $\lambda > \lambda_c$. Let $T \in (0, \infty)$. Then, with $\rho = \rho(\lambda, T, 1)$ as in Theorem 5.1.5, for every finite $\Delta \subset \mathbb{Z}^d_{d-1} \times \mathbb{Z}_T(0)$, the measure $\mathcal{P}^{slab}_\lambda(\cdot \mid \eta \equiv 0 \text{ on } \Delta)$ stochastically dominates a Bernoulli product measure with density ρ on $\mathbb{Z}^d_{d-1} \times (\mathbb{Z}_T \setminus \mathbb{Z}_T(0))$.*

Corollary 5.1.6 implies that the contact process projected on $\mathbb{Z}^d_{d-1} \times \mathbb{Z}_T$ has strong mixing properties. We next make precise what we mean by strong mixing properties.

Fix $T \in (0, \infty)$ and let, for $\theta \in (0, \frac{1}{2}\pi)$ and $t \geq 0$,

$$C^\theta_t := \big\{(x, s) \in \mathbb{Z}^d_{d-1} \times \mathbb{Z}_T \colon \| x \| \leq (s - t) \tan\theta\big\}$$

be the cone whose tip is at (o, t) and whose wedge opens up with angle θ, where $o \in \mathbb{Z}^d$ denotes the origin. A process $(\xi_t)_{t \in \mathbb{Z}_T}$ on $\{0,1\}^{\mathbb{Z}^d_{d-1}}$ is said to be *cone-mixing* if, for all

$\theta \in (0, \frac{1}{2}\pi)$,

$$\lim_{t\to\infty} \sup_{\substack{A\in\mathcal{F}_{<0}, B\in\mathcal{F}_t^\theta \\ \mathbb{P}(A)>0}} |\mathbb{P}(B \mid A) - \mathbb{P}(B)| = 0,$$

where $\mathcal{F}_{<0}$ is the σ-algebra generated by the lower half-space $\{\xi_s(x)\colon (x,s) \in \mathbb{Z}^d_{d-1} \times \mathbb{Z}_T(0)\}$ and $\mathcal{F}_t^\theta$ is the σ-algebra generated by $\{\xi_s(x)\colon (x,s) \in C_t^\theta\}$.

Theorem 5.1.7. *Let $T \in (0,\infty)$. The upper stationary contact process on $\mathbb{Z}^d$, $d \geq 1$, with $\lambda > \lambda_c$, projected onto $\mathbb{Z}^d_{d-1} \times \mathbb{Z}_T$, is cone-mixing.*

Cone-mixing was introduced in Comets and Zeitouni [46] and used there to prove limiting properties for certain random walks in mixing random environment. More recently, the cone-mixing condition has been adapted to random walks in dynamically evolving random environments, see Avena, den Hollander and Redig [8]. For such models, a standing challenge is to prove limit properties for the random walk when the dynamic environment does not converge towards a unique stationary distribution, uniformly with respect to the initial state.

Theorem 5.1.7 gives one way to overcome this challenge for the particular case where the random environment is the contact process and the random walk stays inside $\mathbb{Z}^d_{d-1}$. Our result has recently been applied in Bethuelsen and Völlering [28] (see Theorem 2.6 therein) to prove (among other things) a law of large numbers for such random walks.

5.2 Preliminaries

In this section we provide some preliminary results which are important for the proofs of our theorems.

5.2.1 Downward FKG and related properties

As already mentioned, the concept of downward FKG (from now on abbreviated by dFKG) plays a key role in the proof of our main theorems. We next provide a definition of this and some related properties.

Definition 5.2.1. *Let $\mu \in \mathcal{M}_1(\Omega)$. We say that μ is*

a) *positively associated if $\mu(B_1 \cap B_2) \geq \mu(B_1)\mu(B_2)$ for any two increasing $B_1, B_2 \in \mathcal{F}$.*

b) *dFKG if for every finite events $\Lambda \subset V$, the conditional measure $\mu(\cdot \mid \eta \equiv 0 \text{ on } \Lambda)$ is positively associated.*

c) *FKG if for every finite $\Lambda \subset V$ and $\sigma \in \Omega$, the conditional measure $\mu(\cdot \mid \eta \equiv \sigma \text{ on } \Lambda)$ is positively associated.*

It is immediate that FKG implies dFKG, which again implies positive association. The Bernoulli product measures μ_ρ, $\rho \in [0,1]$, are examples of measures which clearly satisfies the FKG property, whereas $\bar{\nu}_\lambda$ is a measure which is known to be dFKG, but not always FKG, as shown in [81].

As proven in [20], the upper stationary contact process satisfies the dFKG property when observed at a single time t (see Theorem 3.3 and Equation (20) therein). With the same arguments this result can be extended to the following lemma.

Lemma 5.2.1. *Consider the upper stationary contact process (η_t) on $G = (V, E)$ with $\lambda > 0$. For any $t_1 < t_2 < \cdots < t_n$ the joint distribution of $(\eta_{t_1}, \ldots, \eta_{t_n})$, which is a probability measure on Ω^n, satisfies the dFKG property.*

Proof. The proof is exactly the same as the proof of Theorem 3.3 in [20]. □

The following lemma gives a useful property, used in the proof of Theorem 5.1.4.

Lemma 5.2.2. *Let V be countable and assume that the random variables $(X_i)_{i \in V}$ are dFKG. Let $P = (P_j)_{j \geq 1}$ be a partitioning of V into disjoint subsets. Then the random variables $(Y_j)_{j \geq 1}$ where $Y_j = \max\{X_i,\ i \in P_j\}$ are dFKG.*

Proof. This follows easily from the dFKG property of (X_i). (Use that the Y_j's are increasing functions of (X_i) and that $\{Y_j = 0\} = \{X_i = 0, i \in P_j\}$). □

The dFKG property was used in [85] to give a sufficient and necessary condition for a translation invariant measure μ on $\{0,1\}^{\mathbb{Z}}$ to dominate a Bernoulli product measure with density $\rho \in [0,1]$. Since their result plays an important role for our proofs, we recall the precise statement.

Lemma 5.2.3 (Theorem 1.2 in [85])**.** *Let $V = \mathbb{Z}$ and let $\mu \in \mathcal{M}_1(\Omega)$ be a translation invariant measure on $\{0,1\}^{\mathbb{Z}}$ which is dFKG. Then the following are equivalent.*

1. *μ stochastically dominates μ_ρ.*
2. *$\mu(\eta \equiv 0 \text{ on } \{1, 2, \ldots, n\}) \leq (1-\rho)^n$ for all n.*
3. *For all disjoint, finite subsets Λ and Δ of $\{1, 2, 3, \ldots\}$, we have*

$$\mu\left(\eta(0) = 1 \mid \eta \equiv 0 \text{ on } \Lambda, \eta \equiv 1 \text{ on } \Delta\right) \geq \rho.$$

In [85] also a generalisation of Lemma 5.2.3 to measures on $\{0,1\}^{\mathbb{Z}^d}$ with $d \geq 2$ is presented. Though most of our arguments only use Lemma 5.2.3, for the proof of Corollary 5.1.6 we need the higher dimensional version, which we state below. We use the notation

$$\mathcal{D} := \left\{(x_1, \ldots, x_d) \in \mathbb{Z}^d \colon \exists m \text{ such that } x_i = 0 \, \forall i < m \text{ and } x_m < 0\right\}.$$

Lemma 5.2.4 (Theorem 4.1 in [85])**.** *Let $V = \mathbb{Z}^d$ with $d \geq 2$ and let $\mu \in \mathcal{M}_1(\Omega)$ be a translation invariant measure on $\{0,1\}^{\mathbb{Z}^d}$ which is dFKG. Then the following are equivalent.*

1. *μ stochastically dominates μ_ρ.*
2. *$\mu(\eta \equiv 0 \text{ on } [1,n]^d) \leq (1-\rho)^{n^d}$ for all n.*
3. *For all disjoint, finite subsets Λ and Δ of $\mathcal{D}$, we have*

$$\mu\left(\eta(o) = 1 \mid \eta \equiv 0 \text{ on } \Lambda, \eta \equiv 1 \text{ on } \Delta\right) \geq \rho.$$

Remark 5.2.1. Lemma 5.2.4 was stated (and proven) in [85] for $d = 2$. However, the extension of their argument to general dimensions is immediate and yields Lemma 5.2.4 (as also commented directly before the proof in [85], see p. 232 therein).

5.2.2 The contact process

We next give a brief and somewhat informal construction of the contact process via the so-called graphical representation. For a more thorough description we refer to [83], p. 32-34.

Let $G = (V, E)$ be a connected graph having bounded degree and fix $\lambda \in (0, \infty)$. Let $H := (H(x))_{x \in V}$ and $I := (I(x,y))_{\{x,y\} \in E}$ be two independent collections of (doubly-infinite) i.i.d Poisson processes with rate 1 and λ, respectively. On $V \times \mathbb{R}$, draw the events of $H(x)$ as *crosses* over x and the events of $I(x,y)$ as *arrows* from x to y.

For $x, y \in V$ and $s \leq t$, we say that (y,t) is connected to (x,s) by a backwards path, written $(x,s) \leftarrow (y,t)$, if and only if there exists a directed path in $V \times \mathbb{R}$ starting at (y,t), ending at (x,s) and going either backwards in time without hitting crosses or "sideways" following arrows in the opposite direction of the prescribed direction. Otherwise we write $(x,s) \nleftarrow (y,t)$. In general, for $\Lambda, \Delta \subset V \times \mathbb{R}$, we write $\Delta \leftarrow \Lambda$ ($\Delta \nleftarrow \Lambda$) if there is a (there is no) backwards-path from Λ to Δ. Next, define the process $(\tilde{\eta}_t)$ on Ω by

$$\tilde{\eta}_t(x) := \begin{cases} 1, & \text{if } V \times \{-\infty\} \leftarrow (x,t); \\ 0, & \text{otherwise,} \end{cases}$$

where $V \times \{-\infty\} \leftarrow (x,t)$ denotes the event that there exists a backwards-path from (x,t) to $V \times \{s\}$ for all $s \leq t$. It is well known that $(\tilde{\eta}_t)$ has the same distribution as the upper

stationary the contact process (η_t) with infection parameter $\lambda > 0$. In the following we use the notation (η_t) for either representations of the contact process and denote by $\mathcal{P}_\lambda$ the corresponding path measure.

We next state a lemma which is useful for most of our proofs. The proof and the statement is inspired by [32], Lemma 2.11. For its statement, recall (5.1.5) and note that, as follows from the graphical representation,

$$\mathcal{P}_\lambda\left(s < \tau^x < \infty\right) = \mathcal{P}_\lambda\left(V \times \{-s\} \leftarrow (x,0) \text{ but } V \times \{-\infty\} \nleftarrow (x,0)\right).$$

Lemma 5.2.5. *Consider the upper stationary contact process on a connected graph $G = (V,E)$ of bounded degree with $\lambda > 0$. Let $\Delta \subset V$ and assume that there exist $\epsilon, C, c > 0$ such that for all $x \in \Delta$,*

$$\mathcal{P}_\lambda\left(\tau^x = \infty\right) > \epsilon; \tag{5.2.1}$$

$$\mathcal{P}_\lambda\left(s < \tau^x < \infty\right) \le Ce^{-cs}, \quad s \ge 0. \tag{5.2.2}$$

Then, for any $T \in (0,\infty)$, there exists $\rho = \rho(T) > 0$ such that for all n and all $x_1, \dots, x_n \in \Delta$;

$$\mathcal{P}_\lambda\left(\eta_{Ti}(x_i) = 0, i = 1,2,\dots,n\right) \le (1-\rho)^n.$$

Proof. Fix $T \in (0,\infty)$ and let $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ be an infinite sequence of elements $x_i \in \Delta$. For $i \in \mathbb{Z}$, denote by

$$D_i := \inf\{l \in \mathbb{N}\colon V \times T(i-l) \nleftarrow (x_i, Ti)\},$$

and note that D_iT yields an approximation (up to an error of at most T) on how far backwards in time (x_i, Ti) is connected to another space-time point. In particular, $\eta_{Ti}(x_i) = 0$ if and only if $D_i < \infty$.

Define $\mathcal{T}_0 = 0$ and, iteratively,

$$\mathcal{T}_{i+1} := \mathcal{T}_i + D_{n-\mathcal{T}_i}, \quad i \ge 0.$$

Let $K := \sup\{i\colon \mathcal{T}_i < \infty\}$. We have the following relation (easy to check) between events:

$$\begin{aligned}\{\eta_{iT}(x_i) = 0 \text{ for } i = 1,\dots,n\} &= \{D_1,\dots,D_n < \infty\}\\ &\subset \{\mathcal{T}_K \ge n\}.\end{aligned}$$

Finally, $\mathcal{P}_\lambda(\mathcal{T}_K \ge n)$ is exponentially small in n. This follows by standard arguments from the following consequences of (5.2.1) and (5.2.2) (using the independence properties of the graphical representation): for all i, all positive integers $t_i > t_{i-1} > \cdots > t_1 \ge 1$, and all $s \ge 1$, we have

$$\mathcal{P}_\lambda\left(D_{n-\mathcal{T}_i} = \infty \mid \mathcal{T}_1 = t_1, \dots, \mathcal{T}_i = t_i\right) > \epsilon;$$

$$\mathcal{P}_\lambda\left(s \le D_{n-\mathcal{T}_i} < \infty \mid \mathcal{T}_1 = t_1, \dots, \mathcal{T}_i = t_i\right) \le Ce^{-c(s-1)}.$$

□

5.3 Proofs

5.3.1 Proof of Proposition 5.1.1

For the proof of Proposition 5.1.1 we follow that of [85], Proposition 1.1, which we extend to graphs having subexponential growth.

Proof of Proposition 5.1.1. Let $G = (V, E)$ be a graph as in the statement of the proposition and let $\lambda \in (0, \infty)$. Fix $o \in V$, and consider $\Delta \subset V$ having positive density. Hence, there is a $\gamma > 0$ and a $N \in \mathbb{N}$ such that, for all $n \geq N$, we have that $|\Delta \cap B(n)| > \gamma |B(n)|$, where $B(n) := \{x \in V \colon d(o, x) \leq n\}$.

Next, assume that the contact process on G with infection parameter $\lambda > 0$, projected onto Δ, stochastically dominates a non-trivial independent spin-flip process with parameter $\alpha > 0$. Consequently, for every $T > 0$ and $n \geq N$, we have that

$$\begin{aligned} &\mathcal{P}_\lambda \left(\eta_t(x) = 0 \text{ for all } (x,t) \in B(n) \times [0,T]\right) \\ \leq &\mathcal{P}_\lambda \left(\eta_t(x) = 0 \text{ for all } (x,t) \in (\Delta \cap B(n)) \times [0,T]\right) \\ \leq &e^{-\gamma\alpha |B(n)| T} = e^{-c_1 |B(n)| T}, \qquad \text{where } c_1 = \gamma\alpha. \end{aligned} \tag{5.3.1}$$

Thus, the probability in (5.3.1) decays exponentially at a rate proportional to the volume of $B(n) \times [0, T]$.

To conclude the statement of Proposition 5.1.1 for $\lambda > 0$, we show that this estimate cannot hold and thus argue by means of contradiction. In doing so, we make use of the graphical representation of the contact process.

Let $A_{n,T}$ denote the event that there are no arrows in the graphical representation from sites outside $B(n)$ to any site in $B(n)$ during the time period $[0, T]$. Note that the l.h.s. of (5.3.1) is bounded below by

$$\mathcal{P}_\lambda \left(\{\eta_0(x) = 0 \text{ for } x \in B(n)\} \cap A_{n,T}\right).$$

Moreover, this is again bounded below by

$$\begin{aligned} &\mathcal{P}_\lambda \left(\{\eta_0(x) = 0 \text{ for } x \in B(n)\}\right) e^{-\lambda d |B(n+1) \setminus B(n)| T} \\ \geq &\left[\prod_{x \in B(n)} \bar{\nu}_\lambda(\eta_0(x) = 0)\right] e^{-\lambda d |B(n+1) \setminus B(n)| T}, \end{aligned} \tag{5.3.2}$$

where d denotes the maximum degree of G, and where we used that the contact process is positively associated.

Next, since G has subexponential growth (and hence satisfies (5.1.3)), we can find n large such that $\lambda d |B(n+1) \setminus B(n)| < c_1 |B(n)|$. For such n, by taking T sufficiently large, the expression (5.3.2) is larger than the r.h.s. of (5.3.1): a contradiction. □

5.3.2 Proof of Theorem 5.1.3

Proof of Theorem 5.1.3. Consider the contact process (η_t) on a connected graph $G = (V, E)$ having bounded degree and with $\lambda > 0$. Fix $x \in V$ such that (5.1.6) and (5.1.7) hold and define, for $t, s \in \mathbb{R}$ with $t < s$, the event $A_{t,s} := \{\eta_u(x) = 0 \colon u \in [t, s)\}$. Further, let $f \colon [0, \infty] \times [0, \infty) \to [0, 1]$ denote the function

$$f(t, u) = \mathcal{P}_\lambda \left(A_{0,t} \mid A_{-u,0} \right).$$

Clearly, $f(t, u)$ is non-increasing in t.

By the paper [20] we have that, for each n, the collection of random variables $\left(\eta_t(y), y \in V, t \in \mathbb{Z}_{1/n}\right)$ is dFKG (recall from (5.1.8) that $\mathbb{Z}_{1/n}$ denotes $\{k/n \colon k \in \mathbb{Z}\}$). Further, it is standard (and easy to see) that, for $t < s$,

$$\mathcal{P}_\lambda(A_{t,s}) = \lim_{n \to \infty} \mathcal{P}_\lambda \left(\eta_u(x) = 0 \text{ for all } u \in [t, s) \cap \mathbb{Z}_{1/n} \right).$$

Using this approximation, the above mentioned dFKG property, and general results for measures satisfying dFKG (see Section 5.2.1), it follows that

$$f(t, u) \text{ is non-decreasing in } u, \tag{5.3.3}$$

so $f(t) := \lim_{u \to \infty} f(t, u)$ exists (and is > 0) and $\mathcal{P}_\lambda(A_{0,t} \mid B) \leq f(t)$ for all events B that are measurable with respect to $(\eta_s(x), s \leq 0)$. Further, since,

$$\begin{aligned} f(t + s, u) &= \mathcal{P}_\lambda(A_{0,t+s} \mid A_{-u,0}) \\ &= \mathcal{P}_\lambda(A_{0,t} \mid A_{-u,0}) \mathcal{P}_\lambda(A_{t,t+s} \mid A_{-u,t}) \\ &= f(t, u) f(s, t + u), \end{aligned}$$

we get, by letting $u \to \infty$, $f(t + s) = f(t) f(s)$, from which we obtain that there is a $c \geq 0$ such that

$$f(t) = e^{-ct}, \quad \text{for all } t \geq 0. \tag{5.3.4}$$

By Lemma 5.2.5 (with $T = 1$), there is an $\alpha > 0$ such that

$$\mathcal{P}_\lambda(A_{0,t}) \leq e^{-\alpha t}, \quad t \geq 1. \tag{5.3.5}$$

We claim that $c \geq \alpha$ (and hence $c > 0$). The proof of this claim uses some of the arguments in the proof of Lemma 5.2.3 in [85]. For completeness, we include it here.

Suppose $c < \alpha$. Let $\alpha' \in (c, \alpha)$. Fix $t > 1$ and take an integer l so large that $f(t, lt)$ is 'very close' to $f(t)$ (and hence, by (5.3.4), to e^{-ct}). More precisely, we take l sufficiently large so that

$$f(t, lt) > e^{-\alpha' t}. \tag{5.3.6}$$

For all integers $k \geq 0$ we have that, on the one hand (by (5.3.5)),

$$\mathcal{P}_\lambda(A_{0,klt}) \leq e^{-\alpha klt}, \tag{5.3.7}$$

while on the other hand

$$\begin{aligned}\mathcal{P}_\lambda(A_{0,klt}) &= \mathcal{P}_\lambda(A_{0,lt}) \prod_{i=l}^{kl-1} \mathcal{P}_\lambda\left(A_{it,(i+1)t} \mid A_{0,it}\right) \\ &\geq \mathcal{P}_\lambda(A_{0,lt})(f(t,lt))^{kl} \\ &> \mathcal{P}_\lambda(A_{0,lt})e^{-\alpha' tkl},\end{aligned}$$

where the first inequality uses (5.3.3) and stationarity, and the second inequality comes from (5.3.6). Since $\alpha' < \alpha$ (and $\mathcal{P}_\lambda(A_{0,lt}) > 0$) this violates (5.3.7) if k is sufficiently large, and yields a contradiction. This proves the claim.

By the claim, and the inequality one line below (5.3.3), we have that $\mathcal{P}_\lambda(A_{0,t} \mid B) \leq e^{-\alpha t}$ for all events B that are measurable with respect to $(\eta_s(x), s \leq 0)$.

Finally, we also clearly have (by the contact process dynamics) that the conditional probability of the event $\{\eta_s(x) = 1 \text{ for all } s \in (0,t)\}$, given that $\eta_0(x) = 1$ and any additional information about the process before time 0, is exactly e^{-t}. We conclude that the process $(\eta_s(x))$ dominates a spin-flip process which goes from state 0 to 1 at rate α and from 1 to 0 at rate 1.

□

5.3.3 Proof of Theorem 5.1.4

Proof of Theorem 5.1.4. Fix $T \in (0,\infty)$ and let $\Delta \subset V$ be finite with $x \in \Delta$ such that (5.1.6) and (5.1.7) hold. Furthermore, consider the doubly infinite sequence $(Y_i)_{i\in\mathbb{Z}}$, where Y_i is given by

$$Y_i := \max\{\eta_{Ti}(y) \colon y \in \Delta\}. \tag{5.3.8}$$

By Lemma 5.2.1 and Lemma 5.2.2 we note that (Y_i) is dFKG, and, since the upper stationary contact process is invariant under temporal shift, the sequence is also translation invariant. By Lemma 5.2.5, there is a $\rho > 0$ such that

$$\mathcal{P}_\lambda\left(Y_j = 0, j = 1, \dots n\right) \leq (1-\rho)^n.$$

Hence, by Lemma 5.2.3, we get

$$\mathcal{P}_\lambda\left(Y_1 = 1 \mid Y_{-j} = 0, j = 0, \dots, n\right) \geq \rho. \tag{5.3.9}$$

It is not difficult to see that (5.3.9) yields the following: for some $0 < \tilde{\rho} \leq \rho$,

$$\mathcal{P}_\lambda \left(\eta_T(x) = 1 \text{ for all } x \in \Delta \mid Y_{-j} = 0, j = 0, \dots, n\right) \geq \tilde{\rho}, \tag{5.3.10}$$

for all $n \in \mathbb{N}$. Indeed, since the contact process evolves in continuous-time and the graph is connected, infections can spread with positive probability from any point in Δ to all other points in Δ in a small time interval.

To make this more formal one can first consider a sequence defined similar to (Y_i), only replacing T by $T/2$ in (5.3.8). By the same argument as above, using again the dFKG property, we have that for some $\delta > 0$,

$$\mathcal{P}_\lambda \left(\max\{\eta_{T/2}(y) \colon y \in \Delta\} = 1 \mid Y_{-j} = 0, j = 0, \dots, n\right) > \delta.$$

Furthermore, since Δ is finite, and G is connected, there is an $\epsilon > 0$ such that (with the notation introduced below (5.1.5))

$$\inf_{z \in \Delta} \mathcal{P}_\lambda \left(\eta^z_{\frac{T}{2}}(y) = 1 \text{ for all } y \in \Delta\right) > \epsilon.$$

Thus, using the fact that the contact process is a Markov process, we conclude (5.3.10) with $\tilde{\rho} \geq \epsilon\delta > 0$.

Finally, using again the dFKG property of the collection $(\eta_{Ti}(y), y \in \Delta, i \in \mathbb{Z})$, we obtain that (5.3.10) still holds if the conditioning $\{Y_{-j} = 0, j = 0, \dots, n\}$ is replaced by any event measurable with respect to $(\eta_{-Ti}(y), y \in \Delta, i \geq 0)$. This concludes the proof of the theorem. □

5.3.4 Proof of Theorem 5.1.2

To prove Theorem 5.1.2 we first prove that the contact process on $\{0, 1, \dots\}$ observed at the vertex $\{0\}$ stochastically dominates an independent spin-flip process. Indeed, the required estimates (5.1.6) and (5.1.7) for this context is provided by the following result in [57], see Equation (21) on page 546 therein.

Lemma 5.3.1 ([57], Equation (21), and [58])**.** *Consider the contact process on $V = \{0, 1, \dots\}$ with $\lambda > \lambda_c$. Then there exists constants $\epsilon, C, c > 0$ such that* (5.1.6) *and* (5.1.7) *hold.*

Proof of Theorem 5.1.2. Firstly, by Lemma 5.3.1 applied to Theorem 5.1.3, we have that the contact process on $\{0, 1, 2, \dots\}$ with $\lambda > \lambda_c$ observed at the vertex $\{0\}$ stochastically dominates an independent spin flip process with $\alpha > 0$.

From the above observation, the statement of Theorem 5.1.2 follows by a monotonicity argument using again the graphical construction of the contact process.

To make this last argument precise, fix an arbitrary point $o \in T_d$ and call it the root. Denote by $u(o) = 0$ its label. Furthermore, label the remaining sites according to their distance with respect to o in a unique way. That is, each $x \in T_d$ with $\| x - o \| = 1$ has a label $u(x) = (0, i)$ for some $i \in \{1, \dots, d+1\}$ and for $y \in T_d$ satisfying $\| y - o \| = \| z - o \| + 1 = n$ and $\| y - z \| < n$, $n \geq 2$, set $u(y) = (u(z), i)$, $i \in \{1, \dots, d\}$. Thus, for each $x \in T_d \setminus \{o\}$, we have that

$$u(x) \in \bigcup_{n \geq 0} \left[\{0\} \times \{1, \dots, d+1\} \times \{1, \dots, d\}^n \right].$$

Denote by $\Delta \subset T_d$ the set of vertices having as last entry of its label a number different from 1. Using the graphical representation of the contact process, consider the process (ξ_t) on T_d where for each $(x, t) \in \Delta \times \mathbb{R}$ we set $\xi_t(x) = 1$ if and only if there is an infinite backwards path from (x, t) constrained to infection arrows between the sites with label $\{u(x), (u(x), 1), (u(x), 1, 1), \dots\}$. Moreover, for $x \in \Delta^c$, let $\xi_t(x) = 0$ for all $t \in \mathbb{R}$.

By construction, the evolution of (ξ_t) on T_d is dominated by that of the contact process. Furthermore, the evolution at site $x \in \Delta$ is in one-to-one correspondence with the contact process on $\{0, 1, 2, \dots\}$, and the evolution at different sites $x, y \in \Delta$ is independent. Thus, on the set Δ the process (ξ_t) stochastically dominates a non-trivial independent spin-flip process, and consequently, so does also the contact process. Lastly, we note that, from the above construction, it holds that Δ has positive density and that the l.h.s. of (5.1.2) equals $\gamma = \frac{d-1}{d} > 0$. This concludes the proof.

□

5.3.5 Proof of Theorem 5.1.5

In order to prove Theorem 5.1.5, we make use of the well known fact that the supercritical contact process on $\mathbb{Z}^d$ with $d \geq 2$ survives in 2-dimensional space-time slabs (see [30]). More precisely, let, for $k \in \mathbb{N}$,

$$S_k := \left\{ x \in \mathbb{Z}^d \colon x_i \in \{0, 1, \dots, k-1\}, i = 1, \dots d-1 \right\},$$

and denote by $({}_k\eta_t)$ the contact process on S_k and by $\mathcal{P}_{\lambda,k}$ its path measure. This process on S_k with $\lambda > \lambda_c(\mathbb{Z}^d)$ survives with positive probability if the width k is large enough. The proof of this proceeds via a block argument and comparison with a certain 2-dimensional (dependent) directed percolation model. This argument also gives a form of exponential decay, more precisely, the following lemma holds.

Lemma 5.3.2. *Let $\lambda > \lambda_c(\mathbb{Z}^d)$ and $d \geq 2$. Then there exists $k \in \mathbb{N}$ and $\epsilon, C, c \in (0, \infty)$, such that for all $x \in S_k$,*

$$\mathcal{P}_{\lambda,k}\left(\tau^x = \infty\right) > \epsilon; \tag{5.3.11}$$

$$\mathcal{P}_{\lambda,k}\left(s < \tau^x < \infty\right) \leq Ce^{-cs}, \quad \text{for all } s > 0. \tag{5.3.12}$$

Proof. This follows again by comparison with a 2-dimensional directed percolation model and a renormalization arguments. For a proof we refer the reader to the proof of Theorem 1.2.30a) in [83], where such an argument is explained in detail. Though proved there for the unrestricted contact process (η_t) the argument works, mutatis mutandis, for $({}_k\eta_t)$ as soon as k is taken sufficiently large. □

Proof of Theorem 5.1.5. Fix $T \in (0, \infty)$ and note that the case $d = 1$ is an immediate consequence of Theorem 5.1.4. Indeed, the estimates (5.1.6) and (5.1.7) for that case are known to hold due to [58], Theorem 5.

For the case $d \geq 2$ we use a slightly more involved argument, by partitioning $\mathbb{Z}^d \times \mathbb{R}$ into slabs. Fix k such that (5.3.11) and (5.3.12) hold. For $\mathbf{i} = (i_1, \ldots, i_{d-1}) \in \mathbb{Z}^{d-1}$, let $P_{\mathbf{i}} = (S_k + k \cdot (i_1, \ldots, i_{d-1}, 0)) \times \mathbb{R}$. Note that $P_{\mathbf{i}} \cap P_{\mathbf{j}} = \emptyset$ whenever $\mathbf{i} \neq \mathbf{j}$ and that $\bigcup_{\mathbf{i} \in \mathbb{Z}^{d-1}} P_{\mathbf{i}} = \mathbb{Z}^d \times \mathbb{R}$.

Next, consider the process (ζ_t) which is obtained from the graphical representation of the contact process on $\mathbb{Z}^d$ by suppressing all infection arrows between slabs $P_{\mathbf{j}}$. Trivially the evolution of (ζ_t) is dominated by that of (η_t). Moreover, the evolution of (ζ_t) in each slab is independent of the others and has the same law as $({}_k\eta_t)$.

Let $\mathbf{i} \in \mathbb{Z}^{d-1}$. By applying Theorem 5.1.4 with $\Delta = \mathbb{Z}^d_{d-1}(m) \cap (S_k + k \cdot (\mathbf{i}, o))$, it follows that the process (ζ_t) observed on the vertices Δ at times that are multiples of T stochastically dominates a non-trivial Bernoulli product measure with density $\rho > 0$. By the above mentioned independence, this implies the statement of Theorem 5.1.5 for (ζ_t). Since (ζ_t) is stochastically dominated by (η_t), we conclude the proof. □

5.3.6 Proof of Corollary 5.1.6

Proof of Corollary 5.1.6. Fix $T \in (0, \infty)$ and recall the definition of $\mathcal{P}^{\text{slab}}_\lambda$ in Section 5.1.4. Note that $\mathcal{P}^{\text{slab}}_\lambda$ is translation invariant and that, due to Lemma 5.2.1, it is also dFKG. In particular, we may apply Lemma 5.2.4 to $\mathcal{P}^{\text{slab}}_\lambda$.

A direct consequence of Theorem 5.1.5 with $m = 1$ is that whenever $\lambda > \lambda_c$, there is a $\rho > 0$ such that

$$\mathcal{P}^{\text{slab}}_\lambda \left(\eta_s(x) = 0, (x, s) \in \mathbb{Z}^d_{d-1} \times \mathbb{Z}_T \cap [1, n]^d \times [T, nT] \right) \leq (1 - \rho)^{n^d}.$$

Hence, the measure $\mathcal{P}^{\text{slab}}_\lambda$ satisfies Property 2 in Lemma 5.2.4. Consequently, $\mathcal{P}^{\text{slab}}_\lambda$ also satisfies Property 3 in Lemma 5.2.4, from which the statement of Corollary 5.1.6 follows. □

5.3.7 Proof of Theorem 5.1.7

Theorem 5.1.7 follows from Corollary 5.1.6 and a standard coupling argument, together with classical properties of the contact process.

Proof of Theorem 5.1.7. Fix $T \in (0,\infty)$ and let $\rho > 0$ be such that the statement of Corollary 5.1.6 holds. Next, denote by $\mu \in \mathcal{M}_1(\Omega)$ the probability measure under which all vertices outside $\mathbb{Z}^d_{d-1}$ have value 0 a.s., and those in $\mathbb{Z}^d_{d-1}$ correspond with independent Bernoulli random variables with parameter ρ. Further, for $\eta \in \Omega$, denote by $\delta_\eta \in \mathcal{M}_1(\Omega)$ the probability measure which concentrates on η, and write $\bar{1} \in \Omega$ for the configuration where all sites are equal to 1. Then, by Corollary 5.1.6, and since $\bar{\nu}_\lambda \leq \delta_{\bar{1}}$, we have, for $\theta \in (0,\pi/2)$, $t > 0$ and $B \in \mathcal{F}^\theta_t$ increasing, and for any $A \in \mathcal{F}_{<0}$ with $\mathcal{P}^{\text{slab}}_\lambda(A) > 0$, that

$$\left|\mathcal{P}^{\text{slab}}_\lambda(B \mid A) - \mathcal{P}^{\text{slab}}_\lambda(B)\right| \leq \widehat{\mathcal{P}}_{\mu,\delta_{\bar{1}}}\left(\eta^1 \neq \eta^2 \text{ on } C^\theta_t\right), \tag{5.3.13}$$

where $\widehat{\mathcal{P}}_{\mu,\delta_{\bar{1}}}$ is the standard graphical construction coupling of the contact processes on $\mathbb{Z}^d$ started at time 0 from a configuration drawn according to μ and $\delta_{\bar{1}}$, respectively.

Furthermore, we have that

$$\begin{aligned}\widehat{\mathcal{P}}_{\mu,\delta_{\bar{1}}}\left(\eta^1 \neq \eta^2 \text{ on } C^\theta_t\right) &\leq \sum_{(x,s)\in C^\theta_t} \widehat{\mathcal{P}}_{\mu,\delta_{\bar{1}}}\left(\eta^1_s(x) \neq \eta^2_s(x)\right) \\ &= \sum_{(x,s)\in C^\theta_t} \widehat{\mathcal{P}}_{\mu,\delta_{\bar{1}}}\left(\eta^1_s(o) \neq \eta^2_s(o)\right),\end{aligned} \tag{5.3.14}$$

where the last equation holds due to translation invariance in the first $(d-1)$ spatial directions.

Since the set of increasing events in $\mathcal{F}^\theta_t$ generates $\mathcal{F}^\theta_t$, in order to conclude the argument, it is sufficient to show that, for some $C, c \in (0,\infty)$, we have

$$\widehat{\mathcal{P}}_{\mu,\delta_{\bar{1}}}\left(\eta^1_s(o) \neq \eta^2_s(o)\right) \leq Ce^{-cs}. \tag{5.3.15}$$

This can be shown using known estimates for the supercritical contact process on $\mathbb{Z}^d$. For completeness we present the details.

Let $N := \inf\{\| x \| : \eta^1_0(x) = 1 \text{ and } \tau^x = \infty\}$. Then, for any $a > 0$, we have that

$$\begin{aligned}\widehat{\mathcal{P}}_{\mu,\delta_{\bar{1}}}\left(\eta^1_s(o) \neq \eta^2_s(o)\right) \leq\ & \widehat{\mathcal{P}}_{\mu_\rho,\delta_{\bar{1}}}\left(\{N > as\}\right) \\ &+ \widehat{\mathcal{P}}_{\mu_\rho,\delta_{\bar{1}}}\left(\{\eta^1_s(o) \neq \eta^2_s(o)\} \cap \{N \leq as\}\right).\end{aligned} \tag{5.3.16}$$

That the first term on the righthand side decays exponentially (in as) follows from [83], Theorem 1.2.30. For the other term, we have that

$$\begin{aligned}&\widehat{\mathcal{P}}_{\mu_\rho,\delta_{\bar{1}}}\left(\{\eta^1_s(o) \neq \eta^2_s(o)\} \cap \{N \leq as\}\right) \\ &\leq \sum_{y\in[-as,as]^d} \widehat{\mathcal{P}}_{\delta_{\bar{0}_y},\delta_{\bar{1}}}\left(\{\eta^1_s(o) \neq \eta^2_s(o)\} \cap \{\tau^y = \infty\}\right) \\ &\leq \sum_{y\in[-as,as]^d} \widehat{\mathcal{P}}_{\delta_{\bar{0}_y},\delta_{\bar{1}}}\left(\eta^1_s(o) \neq \eta^2_s(o) \mid \tau^y = \infty\right),\end{aligned} \tag{5.3.17}$$

where $\bar{0}_y$ is the configuration given by $\bar{0}_y(x) = \bar{0}(x) = 0$ for all $y \neq x$ and $\bar{0}_y(y) = 1 - \bar{0}(y) = 1$. From the large deviation estimates obtained in [62], Theorem 1.4, by choosing $a > 0$ in (5.3.17) sufficiently small, the term inside the sum of (5.3.17) decays exponentially (in s), uniformly for $y \in [-as, as]^d$. Hence, since the sum only contains polynomially many terms, we have that (5.3.16) decays exponentially with respect to s.

In conclusion, there exist $C, c > 0$ such that (5.3.15) holds, from which, by (5.3.13) and (5.3.14), we conclude the proof.

□

5.4 Open questions

We expect that the statement of Theorem 5.1.2 can be improved.

Question 5.4.1. Can the condition $\lambda > \lambda_c(\mathbb{Z})$ in Theorem 5.1.2 be replaced by $\lambda > \lambda_c(T_d)$?

Question 5.4.2. Does Theorem 5.1.2 hold with $\Delta = T_d$?

Motivated by Theorem 5.1.5 and Lemma 5.2.5, the following questions seem natural.

Question 5.4.3. Consider the upper stationary contact process on $\mathbb{Z}^d$, $d \geq 1$, with $\lambda > \lambda_c$, and let $\mathbf{x} = (x_i)_{i \in \mathbb{Z}}$ be an infinite sequence of elements in $\mathbb{Z}^d$. Does the contact process projected onto $\{(x_i, i) \colon i \in \mathbb{Z}\}$ stochastically dominate a non-trivial Bernoulli product measure?

Question 5.4.4. Consider the upper stationary contact process (η_t) on $\mathbb{Z}^d$, $d \geq 1$, with $\lambda > \lambda_c$, and let $\mathbf{X} = (X_i)_{i \geq 0}$ be a simple random walk on $\mathbb{Z}^d$ started at $X_0 = o$. Does the sequence $(\eta_i(X_i))_{i \geq 0}$ dominate a non-trivial Bernoulli sequence?

Remark 5.4.1. A positive answer to Question 5.4.3 with a uniform bound on the density $\rho > 0$ would imply a positive answer to Question 5.4.4.

Lastly, motivated by Proposition 5.1.1 and Theorem 5.1.5, we state the following question.

Question 5.4.5. Consider the upper stationary contact process (η_t) on $\mathbb{Z}^d$, $d \geq 1$, with $\lambda > \lambda_c$. For which $\Delta \subset \mathbb{Z}^d$ having "zero density" (that is, the l.h.s. of (5.1.2) equals 0) does (η_t) projected onto $\Delta \times [0, \infty)$ dominate a non-trivial independent spin-flip process?

Bibliography

[1] K. S. Alexander. On weak mixing in lattice models. *Probab. Theory Related Fields*, 110(4): 441–471, 1998.

[2] S. Alili. Asymptotic behaviour for random walks in random environments. *J. Appl. Probab.*, 36(2): 334–349, 1999.

[3] S. Andres, A. Chiarini, J.-D. Deuschel, and M. Slowik. Quenched invariance principle for random walks with time-dependent ergodic degenerate weights. *ArXiv e-prints*, February 2016. http://arxiv.org/abs/1602.01760.

[4] S. Andres, J.-D. Deuschel, and M. Slowik. Invariance principle for the random conductance model in a degenerate ergodic environment. *Ann. Probab.*, 43(4):1866–1891, 2015.

[5] L. Avena. Random walks in dynamic random environments. *PhD-thesis, Leiden University*, 2010.

[6] L. Avena and P. Thomann. Continuity and anomalous fluctuations in random walks in dynamic random environments: numerics, phase diagrams and conjectures. *J. Stat. Phys.*, 147(6):1041–1067, 2012.

[7] L. Avena, F. den Hollander, and F. Redig. Large deviation principle for one-dimensional random walk in dynamic random environment: attractive spin-flips and simple symmetric exclusion. *Markov Process. Related Fields*, 16(1):139–168, 2010.

[8] L. Avena, F. den Hollander, and F. Redig. Law of large numbers for a class of random walks in dynamic random environments. *Electron. J. Probab.*, 16(21): 587–617, 2011.

[9] L. Avena, O. Blondel, and A. Faggionato. L^2-Perturbed Markov processes and applications to random walks in dynamic random environments. *ArXiv e-prints*, February 2016. http://arxiv.org/abs/1602.06322.

[10] L. Avena, O. Blondel, and A. Faggionato. A class of random walks in reversible dynamic environment: antisymmetry and applications to the East model. *ArXiv e-prints*, May 2016. http://arxiv.org/abs/1605.04816.

[11] L. Avena, T. Franco, M. Jara, and F. Völlering. Symmetric exclusion as a random environment: hydrodynamic limits. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(3): 901–916, 2015.

[12] L. Avena, M. Jara, and F. Völlering. Explicit LDP for a slowed RW driven by a symmetric exclusion process. *ArXiv e-prints*, September 2014. http://arxiv.org/abs/1409.3013.

[13] L. Avena, R. S. dos Santos, and F. Völlering. Transient random walk in symmetric exclusion: limit theorems and an Einstein relation. *ALEA Lat. Am. J. Probab. Math. Stat.*, 10(2): 693–709, 2013.

[14] A. Bandyopadhyay and O. Zeitouni. Random walk in dynamic Markovian random environment. *ALEA Lat. Am. J. Probab. Math. Stat.*, 1: 205–224, 2006.

[15] G. Barraquand and I. Corwin. Random-walk in beta-distributed random environment. *Probability Theory and Related Fields*, 1–60, 2016.

[16] E. Baur. An invariance principle for a class of non-ballistic random walks in random environment. *Probability Theory and Related Fields*, 1–52, 2015.

[17] O. Benichou, A. Bodrova, D. Chakraborty, P. Illien, A. Law, C. Mejía-Monasterio, G. Oshanin and R. Voituriez. Geometry-induced superdiffusion in driven crowded systems. *Physical Review Letters*, 111, 2013.

[18] I. Benjamini, H. Duminil-Copin, G. Kozma, and A. Yadin. Disorder, entropy and harmonic functions. *Ann. Probab.*, 43(5): 2332–2373, 2015.

[19] J. van den Berg and S. A. Bethuelsen. Stochastic domination in space-time for the contact process. *ArXiv e-prints*, October 2016. http://arxiv.org/abs/1606.08024

[20] J. van den Berg, O. Häggström, and J. Kahn. Some conditional correlation inequalities for percolation and related processes. *Random Structures Algorithms*, 29(4): 417–435, 2006.

[21] J. van den Berg and C. Maes. Disagreement percolation in the study of Markov fields. *Ann, Probab.*, 22(2): 749–763, 1994.

[22] N. Berger, M. Cohen, and R. Rosenthal. Local limit theorem and equivalence of dynamic and static points of view for certain ballistic random walks in i.i.d environments. *Ann. Probab.*, 44(4): 1889-1979, 2016.

[23] N. Berger. Limiting velocity of high-dimensional random walk in random environment. *Ann. Probab.*, 36(2): 728–738, 2008.

[24] N. Berger and O. Zeitouni. A quenched invariance principle for certain ballistic random walks in i.i.d. environments. In *In and out of equilibrium. 2*, volume 60 of *Progr. Probab.*, Birkhäuser, Basel, 137–160, 2008.

[25] N. Berger, A. Drewitz, and A. F. Ramírez. Effective polynomial ballisticity conditions

for random walk in random environment. *Comm. Pure Appl. Math.*, 67(12): 1947–1973, 2014.

[26] S. A. Bethuelsen. The contact process as seen from a random walk. *ArXiv e-print*, July 2016. http://arxiv.org/abs/1607.03410.

[27] S. A. Bethuelsen and M. Heydenreich. Law of large numbers for random walks on attractive spin-flip dynamics. *ArXiv e-print*. September 2016. https://arxiv.org/abs/1411.3581.

[28] S. A. Bethuelsen and F. Völlering. Absolute continuity and weak uniform mixing of random walk in dynamic random environment. *ArXiv e-prints*, October 2016. http://arxiv.org/abs/1601.07710.

[29] C. Bezuidenhout and L. Gray. Critical attractive spin systems. *Ann. Probab.*, 22(3): 1160–1194, 1994.

[30] C. Bezuidenhout and G. Grimmett. The critical contact process dies out. *Ann. Probab.*, 18(4): 1462–1482, 1990.

[31] M. Birkner, J. Černý, A. Depperschmidt, and N. Gantert. Directed random walk on the backbone of an oriented percolation cluster. *Electron. J. Probab.*, 18(80): 1–35, 2013.

[32] M. Birkner, J. Černý, and A. Depperschmidt. Random walks in dynamic random environments and ancestry under local population regulation. *Electron. J. Probab.*, 21: 1–43, 2016.

[33] C. Boldrighini, I. A. Ignatyuk, V. A. Malyshev, and A. Pellegrinotti. Random walk in dynamic environment with mutual influence. *Stochastic Process. Appl.*, 41(1): 157–177, 1992.

[34] C. Boldrighini, R. A. Minlos, and A. Pellegrinotti. Random walk in a fluctuating random environment with Markov evolution. In *On Dobrushin's way. From probability theory to statistical physics*, volume 198 of *Amer. Math. Soc. Transl. Ser. 2*, 13–35, 2000.

[35] C. Boldrighini, R. A. Minlos, and A. Pellegrinotti. Random walks in quenched i.i.d. space-time random environment are always a.s. diffusive. *Probab. Theory Related Fields*, 129(1): 133–156, 2004.

[36] C. Boldrighini, G. Cosimi, S. Frigio, and A. Pellegrinotti. Computer simulations for some one-dimensional models of random walks in fluctuating random environment. *J. Stat. Phys.*, 121(3-4): 361–372, 2005.

[37] E. Bolthausen and A.-S. Sznitman. On the static and dynamic points of view for certain random walks in random environment. *Methods Appl. Anal.*, 9(3): 345–375, 2002.

[38] E. Bolthausen and O. Zeitouni. Multiscale analysis of exit distributions for random walks in random environments. *Probab. Theory Related Fields*, 138(3-4): 581–645, 2007.

[39] E. Bolthausen, A.-S. Sznitman, and O. Zeitouni. Cut points and diffusive random walks in random environment. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(3): 527–555, 2003.

[40] M. Bramson, O. Zeitouni, and M. P. W. Zerner. Shortest spanning trees and a counterexample for random walks in random environments. *Ann. Probab.*, 34(3): 821–856, 2006.

[41] J. Brémont. One-dimensional finite range random walk in random medium and invariant measure equation. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(1): 70–103, 2009.

[42] J. Bricmont and A. Kupiainen. Random walks in asymmetric random environments. *Comm. Math. Phys.*, 142(2): 345–420, 1991.

[43] J. Bricmont and A. Kupiainen. Random walks in space time mixing environments. *J. Stat. Phys.*, 134(5-6): 979–1004, 2009.

[44] J.-R. Chazottes, F. Redig, and F. Völlering. The Poincaré inequality for Markov random fields proved via disagreement percolation. *Indag. Math. (N.S.)*, 22(3-4): 149–164, 2011.

[45] A. A. Chernov. Replication of a multicomponent chain by the "lightning mechanicsm". *Biophysics*, 12(2): 336–341, 1967.

[46] F. Comets and O. Zeitouni. A law of large numbers for random walks in random mixing environments. *Ann. Probab.*, 32(1B): 880–914, 01 2004.

[47] F. Comets and O. Zeitouni. Gaussian fluctuations for random walks in random mixing environments. *Israel J. Math.*, 148: 87–113, 2005.

[48] F. Comets, N. Gantert, and O. Zeitouni. Quenched, annealed and functional large deviations for one-dimensional random walk in random environment. *Probab. Theory Related Fields*, 118(1): 65–114, 2000.

[49] J. T. Cox, R. Durrett and R. Schinazi. The critical contact process seen from the right edge. *Probab. Theory Related Fields*, 83(3): 325–332, 1991.

[50] J.-D. Deuschel, X. Guo, and A. F. Ramirez. Quenched invariance principle for random walk in time-dependent balanced random environment. *ArXiv e-prints*, March 2015. http://arxiv.org/abs/1503.01964.

[51] R. L. Dobrushin and S. B. Shlosman. Completely analytical Gibbs fields. In *Statistical*

physics and dynamical systems (Köszeg, 1984), volume 10 of *Progr. Phys.*, Birkhäuser Boston, Boston, MA, 371–403, 1985.

[52] D. Dolgopyat and C. Liverani. Non-perturbative approach to random walk in Markovian environment. *Electron. Commun. Probab.*, 14: 245–251, 2009.

[53] D. Dolgopyat, G. Keller, and C. Liverani. Random walk in Markovian environment. *Ann. Probab.*, 36(5):1676–1710, 2008.

[54] A. Drewitz and A. F. Ramírez. Selected topics in random walks in random environment. In *Topics in percolative and disordered systems*, volume 69 of *Springer Proc. Math. Stat.*, Springer, New York, 23–83. 2014.

[55] R. Durrett. Ten lectures on particle systems. In *Lectures on Probability Theory*, Springer, 97–201, 1995.

[56] R. Durrett. On the growth of one-dimensional contact processes. *Ann. Probab.*, 8(5): 890–907, 1980.

[57] R. Durrett and D. Griffeath. Contact processes in several dimensions. *Z. Wahrsch. Verw. Gebiete*, 59(4): 535–552, 1982.

[58] R. Durrett and D. Griffeath. Supercritical contact processes on $\mathbb{Z}$. *Ann. Probab.*, 11 (1): 1–15, 1983.

[59] M. Einsiedler and T. Ward, Ergodic theory with a view towards number theory, Springer-Verlag, London, 2011.

[60] A. Galves and E. Presutti. Edge fluctuations for the one-dimensional supercritical contact process. *Ann. Probab.*, 15(3): 1131–1145, 1987.

[61] O. Garet and R. Marchand. Asymptotic shape for the contact process in random environment. *Ann. of Applied Probab.*, 22(4): 1362–1410, 2012.

[62] O. Garet and R. Marchand. Large deviations for the contact process in random environment. *Ann. Probab.* 42(4): 1438–1479, 2014.

[63] M. Gori, I. Donato, E. Floriani, I. Nardecchia, and M. Pettini. Random walk of passive tracers among randomly moving obstacles. *ArXiv e-prints*, January 2016. http://arxiv.org/abs/1601.03626.

[64] A. Greven and F. den Hollander. Large deviations for a random walk in random environment. *Ann. Probab.*, 22(3): 1381–1428, 1994.

[65] G. Grimmett. Large deviations in subadditive processes and first-passage percolation. In *Particle systems, random media and large deviations (Brunswick, Maine, 1984)*, volume 41 of *Contemp. Math.*, Amer. Math. Soc., Providence, RI, 175–194, 1985.

[66] G. Grimmett and H. Kesten. First-passage percolation, network flows and electrical resistances. *Z. Wahrsch. Verw. Gebiete*, 66(3): 335–366, 1984.

[67] X. Guo. On the limiting velocity of random walks in mixing random environment. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(2): 375–402, 2014.

[68] X. Guo and O. Zeitouni. Quenched invariance principle for random walks in balanced random environment. *Probab. Theory Related Fields*, 152(1-2): 207–230, 2012.

[69] R. van Handel and P. Rebeschini. Phase transitions in nonlinear filtering. *Electron. J. Probab.*, 20:(7), 1–46, 2015.

[70] T. E. Harris. Contact interactions on a lattice. *Ann. Probability*, 2: 969–988, 1974.

[71] M. Hilário, F. den Hollander, V. Sidoravicius, R. S. dos Santos, and A. Teixeira. Random walk on random walks. *Electron. J. Probab.*, 20:(95): 1–35, 2015.

[72] F. den Hollander and R. S. dos Santos. Scaling of a random walk on a supercritical contact process. *Ann. Inst. H. Poincaré Probab. Statist.*, 50(4): 1276–1300, 2014.

[73] F. den Hollander, R. dos Santos, and V. Sidoravicius. Law of large numbers for non-elliptic random walks in dynamic random environments. *Stochastic Process. Appl.*, 123(1): 156–190, 2013.

[74] F. den Hollander, H. Kesten, and V. Sidoravicius. Random walk in a high density dynamic random environment. *Indag. Math. (N.S.)*, 25(4): 785–799, 2014.

[75] R. van der Hofstad and M. Holmes. An expansion for self-interacting random walks. *Braz. J. Probab. Stat.*, 26(1): 1–55, 2012.

[76] M. Holmes and T. S. Salisbury. Random walks in degenerate random environments. *Canad. J. Math.*, 66(5): 1050–1077, 2014.

[77] F. Huveneers and F. Simenhaus. Random walk driven by simple exclusion process. *Electron. J. Probab.*, 20(105): 1–42, 2015.

[78] S. A. Kalikow. Generalized random walk in a random environment. *Ann. Probab.*, 9 (5): 753–768, 1981.

[79] H. Kesten, M. V. Kozlov, and F. Spitzer. A limit law for random walk in a random environment. *Compositio Math.*, 30: 145–168, 1975.

[80] G. F. Lawler. Weak convergence of a random walk in a random environment. *Comm. Math. Phys.*, 87(1): 81–87, 1982.

[81] T. M. Liggett. Survival and coexistence in interacting particle systems. In *Probability and phase transition (Cambridge, 1993)*, volume 420 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, Kluwer Acad. Publ., Dordrecht, 209–226,1994.

[82] T. M. Liggett. *Interacting particle systems*, volume 276 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[83] T. M. Liggett. *Stochastic interacting systems: contact, voter and exclusion processes*, volume 324 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[84] T. M. Liggett. Conditional association and spin systems. *ALEA Lat. Am. J. Probab. Math. Stat.*, 1: 1–19, 2006.

[85] T. M. Liggett and J. E. Steif. Stochastic domination: the contact process, Ising models and FKG measures. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(2): 223–243, 2006.

[86] R. Lyons and Y. Peres, Probability on Trees and Networks. Cambridge University Press, 2016. Available at http://pages.iu.edu/~rdlyons/

[87] N. Madras. A process in a randomly fluctuating environment. *Ann. Probab.*, 14(1): 119–135, 1986.

[88] P. Mandl. Spectral theory of semi-groups connected with diffusion processes and its application. *Czechoslovak Math. J.*, 11 (86): 558–569, 1961.

[89] Franz Merkl and M. P. W. Zerner. A zero-one law for planar random walks in random environment. *Ann. Probab.*, 29(4): 1716–1732, 2001.

[90] K. Miller. Random walks on weighted, oriented percolation clusters. *ALEA Lat. Am. J. Probab. Math. Stat.*, 13: 53–77, 2016.

[91] T. Mountford and M. E. Vares. Random walks generated by equilibrium contact processes. *Electron. J. Probab.*, 20:(3): 1–17, 2015.

[92] T. Orenshtein and R. S. dos Santos. Zero-one law for directional transience of one-dimensional random walks in dynamic random environments. *Electron. Commun. Probab.*, 21: 1–11 , 2016.

[93] Y. Peres, S. Popov, and P. Sousi. On recurrence and transience of self-interacting random walks. *Bulletin of the Brazilian Mathematical Society, New Series*, 44(4): 841–867, 2013.

[94] F. Rassoul-Agha. The point of view of the particle on the law of large numbers for random walks in a mixing random environment. *Ann. Probab.*, 31(3): 1441–1463, 2003.

[95] F. Rassoul-Agha. On the zero-one law and the law of large numbers for random walk in mixing random environment. *Electron. Comm. Probab.*, 10: 36–44, 2005.

[96] F. Rassoul-Agha and T. Seppäläinen. An almost sure invariance principle for random walks in a space-time random environment. *Probab. Theory Related Fields*, 133(3): 299–314, 2005.

[97] F. Rassoul-Agha and T. Seppäläinen. Almost sure functional central limit theorem for ballistic random walk in random environment. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(2): 373–420, 2009.

[98] F. Rassoul-Agha, T. Seppäläinen, and Atilla Yilmaz. Quenched free energy and large deviations for random walks in random potentials. *Comm. Pure Appl. Math.*, 66(2): 202–244, 2013.

[99] F. Redig and F. Völlering. Limit theorems for random walks in dynamic random environment. *ArXiv e-prints*, 2011. http://arxiv.org/abs/1106.4181.

[100] F. Redig and F. Völlering. Random walks in dynamic random environments: a transference principle. *Ann. Probab.*, 41(5): 3157–3180, 2013.

[101] C. Sabot. Ballistic random walks in random environment at low disorder. *Ann. Probab.*, 32(4): 2996–3023, 2004.

[102] C. Sabot. Random Dirichlet environment viewed from the particle in dimension $d \geq 3$. *Ann. Probab.*, 41(2): 722–743, 2013.

[103] R. S. dos Santos. Some case studies of random walks in dynamic random environments. *PhD-thesis, Leiden University*, 2012.

[104] R. S. dos Santos. Non-trivial linear bounds for a random walk driven by a simple symmetric exclusion process. *Electron. J. Probab.*, 19(49), 1–18, 2014.

[105] Ya. G. Sinaĭ. The limit behavior of a one-dimensional random walk in a random environment. *Teor. Veroyatnost. i Primenen.*, 27(2): 247–258, 1982.

[106] F. Solomon. Random walks in a random environment. *Ann. Probability*, 3: 1–31, 1975.

[107] A.-S. Sznitman. An effective criterion for ballistic behavior of random walks in random environment. *Probab. Theory Related Fields*, 122(4): 509–544, 2002.

[108] A.-S. Sznitman. Topics in random walks in random environment. In *School and Conference on Probability Theory*, ICTP Lect. Notes, XVII, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 203–266, 2004.

[109] A.-S. Sznitman and M. Zerner. A law of large numbers for random walks in random environment. *Ann. Probab.*, 27(4): 1851–1869, 1999.

[110] D. E. Temkin. One-dimensional random walks in a two-component chain. *Dokl. Akad. Nauk SSSR*, 206: 27–30, 1972.

[111] S. R. S. Varadhan. Large deviations for random walks in a random environment. *Comm. Pure Appl. Math.*, 56(8): 1222–1245, 2003.

[112] A. Yilmaz. Large deviations for random walk in a space-time product environment. *Ann. Probab.*, 37(1): 189–205, 2009.

[113] O. Zeitouni. Random walks in random environment. In *Lectures on probability theory and statistics*, volume 1837 of *Lecture Notes in Math.*, Springer, Berlin, 189–312, 2004.

[114] M. P. W. Zerner. The zero-one law for planar random walks in i.i.d. random environments revisited. *Electron. Comm. Probab.*, 12: 326–335 2007.

Samenvatting

Een belangrijk model in dit proefschrift is dat van een stochastische wandeling in een stochastische omgeving. In dit model beweegt zich een deeltje, de zogenaamde stochastische wandeling, in een wanordelijk medium, de zogenaamde stochastische omgeving. De wandeling is afhankelijk van de eigenschappen van de omgeving in de buurt van de wandeling.

De stochastische omgeving kan ofwel statisch zijn, dat wil zeggen, constant zijn in de loopvan de tijd, ofwel dynamisch, waarbij het evolueert met de tijd. Het statische model weerspiegelt de situatie waarin veranderingen in het wanordelijk medium plaatsvinden op een veel grotere tijdschaal dan bewegingen van het deeltje. Zulke modellen worden gemotiveerd door verschillende verschijnselen in de natuurkunde, de biologie en de scheikunde, zoals replicatie van DNA-ketens (Chernov [45]) en faseovergangen in legeringen (Temkin [110]). De laatste tijd is er veel nadruk gelegd op het dynamische geval, waar de tijdschaal van het deeltje vergelijkbaar is met die van het wanordelijk medium. De motivatie voor deze modellen komt onder meer vanuit de studie van faseovergangen in glasachtige vloeistoffen (Benichou et al. [17]) en het gedrag van een biomolecuul dat wordt beïnvloed door andere bewegende macromoleculen (Gori et al. [63]).

Wiskundig gezien zijn stochastische wandelingen goed begrepen in het geval van statische een-dimensionale omgevingen. Criteria voor terugkerend of voorbijgaand gedrag, wetten van grote aantallen, grote afwijkingen principes en schalingslimieten zijn bekend. Dit model is een belangrijk voorbeeld waaruit blijkt dat stochastische wandelingen in stochastische omgevingen hele andere eigenschappen kunnen hebben dan die van homogene stochastische wandelingen. De stochastische wandeling kan bijvoorbeeld voorbijgaand zijn met snelheid nul, en kan niet-diffusieve schalingslimiten hebben. Dit komt door "vertrappingsverschijnselen", dat wil zeggen, gebieden waar de stochastische wandeling een zeer lange tijd doorbrengt. Er is ook veel bekend in meer dimensies, hoewel het beeld daar veel minder volledig is voor algemene klassen van stochastische omgevingen. Dit komt doordat de geometrische structuur een grotere rol spelt, en de analyse van vertrappingsverschijnselen veel gevoeliger is.

In dit proefschrift zijn we vooral geïnteresseerd in de analyse van de asymptotische eigenschappen van stochastische wandelingen in een dynamische stochastische omgeving vanuit een wiskundig perspectief. Dit is in zijn algemeenheid een zeer uitdagend probleem vanwege de mogelijkheid van sterke stochastische afhankelijkheid zowel in ruimte als in tijd. Veel vragen blijven open, zelfs voor het een-dimensionale model.

Gemotiveerd door het statische geval is een centrale vraag in welke mate vertrappingsverschijnselen kunnen voorkomen wanneer de stochastische omgeving dynamisch is. Om te laten zien dat de stochastische wandeling geen vertrappingsverschijnselen heeft, wordt er vaak gekeken naar manieren om ervoor te zorgen dat de omgeving "vergeetachtig" is en stappen van de wandeling voldoende onafhankelijk zijn op grote tijdschalen. Hiervoor spelen de mengeigenschappen van de omgeving een belangrijke rol. Algemene resultaten zijn bekend voor Markov omgevingen die gelijkmatig mengen ten opzichte van de startconfiguratie (zie bijvoorbeeld Redig and Völlering [100]). Anderzijds geven simulaties en heuristische argumenten aan dat vertrappingsverschijnselen kunnen blijven bestaan wanneer de dynamiek langzaam mengt (zie bijvoorbeeld Avena and Thomann [6]). Wiskundig gezien zijn dynamische stochastische omgevingen met niet-gelijkmatige mengeigenschappen nog niet goed begrepen en vormen een belangrijke uitdaging.

In dit proefschrift presenteren we nieuwe mengcondities die voldoende zijn voor het analyseren van de asymptotische eigenschappen van stochastische wandelingen in een dynamische stochastische omgeving. Ons werk verbetert de bestaande literatuur voor algemene modellen, die zich beperkt tot sterk mengende dynamica, en geeft nieuwe inzichten voor bepaalde modellen met een niet-gelijkmatige mengende dynamica.

Hoofdstuk 1 bevat een inleiding waarin het model en de literatuur beschreven worden. Hoofdstuk 2 beschrijft de wet van grote aantallen voor bepaalde klassen van stochastische wandelingen in dynamische stochastische omgevingen in $\mathbb{Z}^d$ met $d \geq 1$. Hiervoor nemen we aan dat de dynamica monotoon is en geïnitialiseerd wordt in een homogene configuratie waarbij alle hoekpunten dezelfde toestand hebben. Een belangrijk kenmerk is dat de wet van grote aantallen in dit geval niet afhangt van de mengeigenschappen van de omgeving, en dus geldt voor een grote klasse van modellen met niet-gelijkmatige mengeigenschappen die niet eerder beschouwd werden in de literatuur. De aannamen op de initiële configuratie kunnen verder afgezwakt worden onder voldoend sterke mengcondities. Als een belangrijk voorbeeld bestuderen we de stochastische wandeling op het zogenaamde contactprocess, waarvoor we verdere eigenschappen over de snelheid bewijzen.

In Hoofdstuk 3 presenteren we een nieuwe aanpak voor het bepalen van asymptotische eigenschappen van algemene stochastische wandelingen in een dynamische stochastische omgeving. In het bijzonder bestuderen we de invariante maat van de omgeving gezien vanuit de wandelaar. Onder algemene mengcondities bewijzen we het bestaan van een invariante maat waarvoor dit proces ergodisch is en bewijzen we wederzijdse absolute continuiteit met betrekking tot de verdeling van de stochastische omgeving. Onze mengcondities zijn aanzienlijk zwakker dan de gelijkmatige mengcondities in de literatuur en zijn niet beperkt tot Markov processen. Een belangrijk kenmerk van onze aanpak is dat die ook kan worden toegepast op dynamica met niet-gelijkmatige mengeigenschappen. Als voorbeelden noemen we omgevingen die door Ornstein-Uhlenbeck processen en het contact proces gegeven worden.

Een ander belangrijk model in dit proefschrift is het contactproces. Dit model ontstond

in de jaren 1970 als een model voor de verspreiding van een infectie in een populatie. In dit model is een vertex ofwel geïnfecteerd ofwel gezond. Na een exponentiële wachttijd met parameter $\lambda \in (0, \infty)$ wordt een infectie op $x \in \mathbb{Z}^d$ verspreid naar een willekeurige buurpunt. Tegelijkertijd wordt x gezond na een exponentiële wachttijd met parameter 1. Dit gebeurt voor elk hoekpunt $x \in \mathbb{Z}^d$ onafhankelijk van de andere hoekpunten. Als λ groter is dan een bepaalde drempelwaarde, dan is er een positive waarschijnlijkheid dat er altijd infecties aanwesig zijn en dat de populatie nooit geheel gezond wordt. In deze context is het contactproces een prototype van een dynamische stochastische omgeving met niet-gelijkmatige mengeigenschappen.

De focus in Hoofdstuk 4 en Hoofdstuk 5 ligt op het contactproces. In Hoofdstuk 4 beschouwen we een stochastisch wandeling op het contact proces op $\mathbb{Z}^d$ met $d \geq 1$. In het bijzonder concentreren we ons het contactproces gezien vanuit de stochastische wandelaar. Onder de veronderstelling dat de infectieparameter λ groot is of de sprong snelheid van de stochastische wandeling klein is, laten we zien dat dit proces hoogstens twee extremale maten heeft. Bovendien wordt de convergentie naar deze extremale maten gekenmerkt door de vraag of het contactproces overleeft of uitsterft, vergelijkbaar met de volledige convergentiestelling die bekend is voor het gewone contactproces. Met behulp hiervan bewijzen we ook een wet van grote getallen voor de stochastische wandeling, die geldt onder algemene condities op de sprongwaarschijnlijkheid van de stochastische wandeling.

In Hoofdstuk 5 richten we ons op het contactproces in het superkritische regime waar λ groter is dan de drempelwaarde. Liggett and Steif [85] bewezen dat in dit geval de bovenste invariante maat een i.i.d. Bernoulli productmaat stochastisch domineert voor het contactproces op bepaalde grafen. In het bijzonder voor $\mathbb{Z}^d$ en (voor infectieparameter groot genoeg) d-ary homogene bomen T_d. Deze resultaten zeggen iets over het contactproces waargenomen op een bepaald tijdstip. We bewijzen specifieke ruimte-tijd varianten van hun resultaten. Een van onze resultaten betreft het contact proces op T_d met $d \geq 2$. We laten zien dat er een deelverzameling V van de hoekpunten van T_d bestaat die een “positieve fractie" van alle hoekpunten van T_d heeft, zodat het contactproces op T_d waargenomen op V een onafhankelijk spin-flip proces voor grote infectieparameter stochastisch domineert. We bewijzen verder dat het superkritische contactproces op $\mathbb{Z}^d$ waargenomen op bepaalde d-dimensionale ruimte-tijd deelverzamelingen een i.i.d. Bernoulli productmaat stochastisch domineert. Hieruit concluderen we sterke mengeigenschappen die van belang zijn bij de studie van bepaalde stochastische wandelingen in stochastische omgeving.

Acknowledgements

The completion of my Ph.D. could not have been accomplished without the support of many people. First, I want to express my gratitude to my supervisor Prof. Dr. Markus Heydenreich for allowing me to experience an exciting and enriching Ph.D.. His constant support, encouragement and guidance throughout these four years were essential for the preparation of this thesis.

I am just as grateful to my supervisor Prof. Dr. Frank den Hollander. I have benefited a lot from his support and guidance. Being part of his group in Leiden has been a wonderful experience. Furthermore, he has been a great role model as the chair of the probability group in Leiden and as a scientist.

A sincere 'thank you' also goes to Prof. Dr. Jacob van den Berg, who has been acting as a third supervisor. Our many meetings in Amsterdam and elsewhere have greatly influenced the preparation of this thesis, and he has played an important role in my development as a scientist.

I would also like to thank Dr. Florian Völlering, with whom I have collaborated in the writing of Chapter 3 of this thesis, and from whom I have learned a lot. I further wish to thank Prof. Dr. Roberto Fernandez, who inspired me during my master studies towards research in probability theory and statistical mechanics and with whom, together with Dr. Siamak Taati, I have had many fruitful discussions.

Many thanks to the promotion committee: Prof. Dr. Tom Mountford, Prof. Dr. Matthias Birkner and Dr. Daniel Valesin. They were very kind to take time to read the thesis manuscript and provided many helpful comments.

I further wish to thank all the scientific and support staff at the Mathematical Institute and the Science Faculty at Leiden University. The atmosphere at the Mathematical Institute is in my opinion unique and it has been a privilege to work in such an inspiring international environment with friends as colleagues.

I would also like to thank the staff at CWI Amsterdam, in particular, the Stochastics group, for providing a stimulating environment. Many thanks also to the staff at LMU Munich and the probability group in Munich, for hosting me as a guest during my final year.

Last but not least, an important part of the path towards this Ph.D. took place outside the academic community. Many thanks to all the people who kept me occupied with other things

than mathematics and who have accompanied me during various activities such as rowing, cycling and fishing. A special ‘thank you’ goes to my family for their constant support and their encouragement in times when it was needed the most. Finally, a heartfelt ‘thank you’ goes to my dear Hedda, who has been immensely supportive and has always been there for me. Your belief in me means more than anything.

Curriculum Vitae

Stein Andreas Bethuelsen was born in Stavanger, Norway, on October 3, 1987. He completed his high-school studies at *St. Olav videregående skole* in Stavanger in June 2006. After that he carried out one year of military service, working at *Fellesoperativt hovedkvarter* in Stavanger. In August 2007 he enrolled in the 'Bachelor's Degree Program in Mathematics' at the *University of Stavanger*, where he graduated in June 2010. The same year he moved to Utrecht, The Netherlands, where he enrolled in the master program 'Mathematical Sciences' at *Utrecht University*, and from which he graduated 'with distinction' in July 2012. He wrote his Master's thesis, entitled "Uniqueness of Gibbs measures", under the supervision of Prof. Dr. R. Fernandez. In September 2012 he moved to Leiden, The Netherlands, where he started his Ph.D. research at *Leiden University*. Under the supervision of Prof. Dr. M.O. Heydenreich and Prof. Dr. W.Th.F. den Hollander, his Ph.D. research focused on the subject of random walks in dynamic random environments. During his time as a Ph.D. student he was also part-time affiliated with *Centrum Wiskunde & Informatica* in Amsterdam under the supervision of Prof. Dr. J. van den Berg. In August 2015 he moved to Munich, Germany, where he stayed as a guest researcher at *Ludwig-Maximilians-Universität München* until the end of his Ph.D. In September 2016, he started as a post-doctoral researcher at *Technische Universität München* under the guidance of Prof. Dr. N. Gantert.