A recipe for desert : analysis of an extended Klausmeier model

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1 Introduction

Ingredients of a recipe for disaster

Drylands are regions with little precipitation[1] that cover about 41% of earth's land surface, with a human population of over 2 billion [10] (Figure 1.1). Climate change and population growth impose increased amounts of stress on vegetation in drylands. Human population in drylands grew by 18.5% between 1990 and 2000 [10] and is continuing to increase. Together with increased aspirations for raised standards of living this results in, e.g., the proliferation of livestock [10].

The United Nations Convention to Combat Desertification (UNCCD, established in 1994) defines desertification as land degradation in arid, semi-arid and dry sub-humid areas resulting from various factors, including climatic variation and human activities. In the absence of vegetation, nutrients in the soil are swept away in the event of rain: the disappearance of vegetation triggers soil erosion processes, leaving behind an inhospitable environment unable to offer the services to build a livelihood.

Desertification can be expressed as the ineffective use of the scarce water resource [84], which is lost through evaporation and runoff instead of being used by plants (through transpiration) [142]. In case vegetation has disappeared, all of the water is lost (none is used by plants) and the desertification process is complete.

Clearly, the disappearance of vegetation is something to be avoided. Much is at stake to predict and prevent the formation and expansion of bare deserts. For this, the dynamics of the vegetation need to be understood, at least on a qualitative level.

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[1] A measure of aridity is defined by dividing the Precipitation by the Potential Evaporation and Transpiration (P/PET). Drylands can be defined by an aridity index less than 0.65, with a subdivision ranging from hyper-arid (< 0.05) to arid ($0.05 - 0.2$) to semi-arid ($0.2 - 0.5$) to dry subhumid ($0.5 - 0.65$).

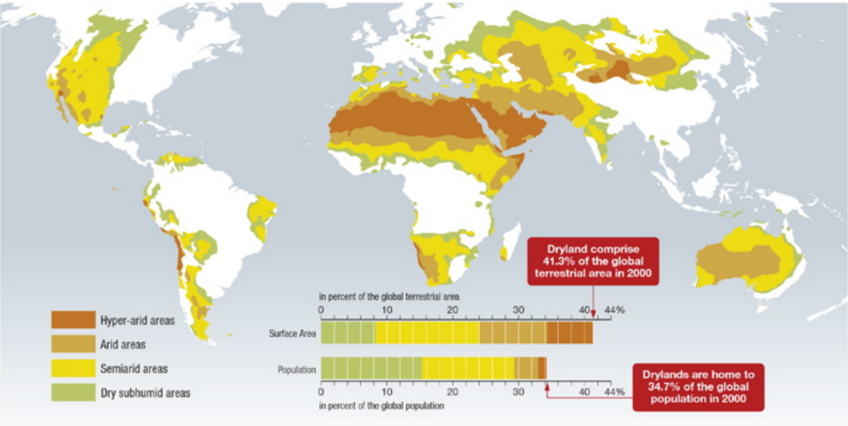

Figure 1.1: Map of the distribution of the four types of dryland over the world [10].

Desertification and climate

In drylands, water is a crucial ingredient for sustenance of vegetation and soil quality. The 2013 report by the Intergovernmental Panel on Climate Change (IPCC) states a “*high confidence* that the contrast of annual mean precipitation between dry and wet regions (...) will increase over most of the globe as temperatures increase” [87], meaning that wet areas become wetter and dry areas become dryer. Under the scenario with a radiative forcing of $7.6\,\mathrm{Wm^{-2}}$, which is an imposed net change in the energy balance of the Earth system, in the year 2100 “many mid-latitude and subtropical arid and semi-arid regions will *likely* experience less precipitation” [87]. A drop in the annual precipitation increases environmental stress and is an important driver of desertification. As a side note, increased levels of CO_2 do lead to a higher water use efficiency by plants through decreased transpiration [32].

Rather than only acting as input, desertification also feeds back to the climate, which is another reason for it having attracted global attention. Desertification changes the albedo (reflectance coefficient of sunlight), both local through changes in plant cover [23] and nonlocal through deposition of desert dust [13]. Additionally, desertification implies reduced carbon sequestration within vegetation and soil [73].

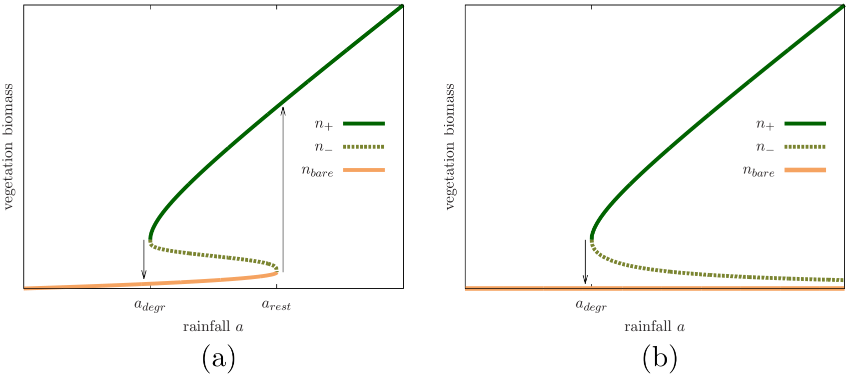

Figure 1.2: Catastrophic transition from a vegetated state n_+ to a degraded bare state n_{bare}, as rainfall reaches the tipping point a_{degr}. The intermediate vegetated state n_- is unstable. (a) The system suffers from hysteresis since the rainfall level required for restoration $a_{rest} > a_{degr}$. (b) In this system the degradation is not reversible by a rainfall increase.

Vegetation on the verge of collapse

Vegetation improves the infiltration characteristics of the soil [150], which increases the water supply to the vegetation itself. So the presence of vegetation reasserts itself. If environmental conditions deteriorate, the amount of vegetation may become critical. Beyond the tipping point, too little vegetation may lead to a smaller water supply: a positive feedback loop that initiates the total collapse of the vegetation (Figure 1.2). Restoration of the ecosystem is difficult because of soil erosion and the infiltration feedback loop, leading to hysteresis or irreversibility.

Brief overview of arid ecosystem models

Various low-dimensional models have been proposed to help explain desertification and vegetation patterns. Early on, in 1997 a scalar (1-component) model for the vegetation was set up [108], with kernel functions with varying ranges built to represent long range inhibition, medium range reproduction and short range toxic interactions. The modeling gained momentum with the publication of the 2-component Klausmeier model [97] in 1999, where a surface water component interacts with the vegetation component. In [202] (2001) the second component is utilized for soil water. Multiple 3-component models have been studied, where both surface and soil water take part, as in [79] (2001) or [148] (2002) and [70] (2004). In [70] (2004) long range inhibition is present due to extended root systems and soil water diffusion.

These models can of course be further expanded, e.g. by explicitly incorporating herbivores [196] or distinguishing functional groups of vegetation with different traits [69]. Incorporation of additional mechanisms makes the analysis more challenging, which often necessitates subsequent simplifying assumptions. In this thesis, attention is mostly restricted to extensions of the Klausmeier model that maintain the 2-component structure of this model. A more complete overview of arid ecosystem models is presented in [19].

Desertification scenarios with increasing complexity

Disregarding differences between plant species and temporarily disregarding spatial heterogeneity, vegetation can be represented by a single scalar. In this case, desertification may be understood as the tipping of a vegetated state. In regions where water is scarce, vegetation growth is limited by water supply and is basically modeled by

$$\begin{aligned}\text{water change} &= \text{rainfall} - \text{evaporation} - \text{uptake},\\ \text{plant change} &= \qquad\qquad\quad -\text{plant death} + \text{uptake}.\end{aligned}$$

The rainfall can change seasonally and tends to be intermittent in time [98, 179], but here we view rainfall as a climatic parameter that may slowly change over time and is constant in the absence of climate change. The water evaporation is modeled by a linear term. Water is regarded as surface water, and the process of infiltration and subsequent uptake of soil water are combined in the surface water uptake terms. Since both the infiltration and the soil water uptake are assumed to be linear in the vegetation, the surface water uptake terms are quadratic in the vegetation [97]. Plant death is, for simplicity, modeled by a linear term.

After rescaling [180], representing water by w and vegetation by n and their change in time by the derivatives w_t and n_t, the model is given by

$$\begin{aligned}w_t &= a - w - wn^2,\\ n_t &= -mn + wn^2,\end{aligned} \tag{1.1}$$

the *nonspatial* Klausmeier model [97]. For $a \geq 2m$ (and $m < 2$) it has a stable vegetated state (w_+, n_+) that disappears in a fold bifurcation at

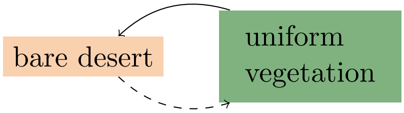

Figure 1.3: Desertification diagram for nonspatial models. The transition from the desert to the vegetated state is represented by a dashed arrow to signify irreversibility.

$a_{degr} = 2m$, as in Figure 1.2(b). If rainfall drops below a_{degr}, the system falls down to the degraded bare desert state and recovery to (w_+, n_+) is troublesome. This is graphically represented by the transition diagram in Figure 1.3.

We will now see that allowing for spatially heterogeneous solutions adds complexity to the desertification scenario. The aforementioned infiltration feedback loop has a short range, since only nearby vegetation benefits from the improved infiltration characteristics of the soil. Overall, vegetation is in competition for water, which may flow relatively long distances over bare soil before infiltrating at a vegetation patch. The short range facilitation and long range competition together drive the formation of periodic vegetation patterns [150].

The widespread appearance of these vegetation patterns in all tropical and subtropical arid areas has been evidenced. They are ubiquitous at the interface between arid and semi-arid regions [36]. These patterns include gapped, labyrinthine, banded and spot patterns (Figure 1.4). The patterns are found on flat or constant-slope terrain without spatial heterogeneities other than those that can be attributed to the vegetation itself [156]. Field observations of vegetation patterns necessitate the development of spatially extended models.

To spatially extend (1.1), we need to take into account processes that displace the surface water and vegetation. We start out in one space dimension x. We allow for the possibility of the terrain having a constant slope, in the Klausmeier model [97] the downslope advection is assumed to be dominant and displacement is modeled by $2cw_x$. We extend the Klausmeier model by adding a (possibly nonlinear) diffusion $d_1 {w^\gamma}_{xx}$ [199]. This form with $\gamma = 2$ can be derived from the shallow water equations [70]. The dispersal of plants is modeled by diffusion $d_2 n_{xx}$, with $d_2 \ll d_1$. This gives

$$\begin{aligned} w_t =& d_1 {w^\gamma}_{xx} + 2cw_x + a - w - wn^2, \\ n_t =& d_2 n_{xx} \qquad\qquad\quad - mn + wn^2. \end{aligned} \tag{1.2}$$

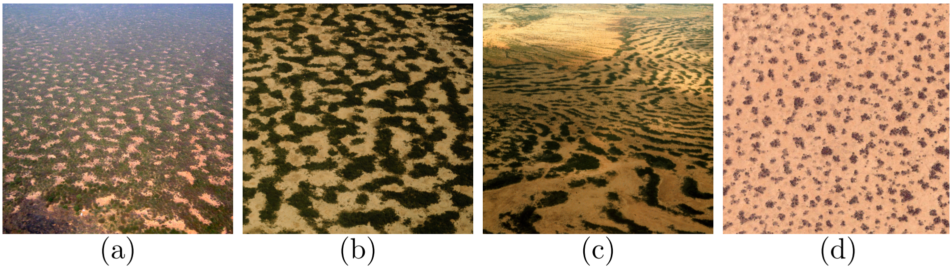

Figure 1.4: Periodic vegetation patterns. Aerial photograph of (a) pattern of gaps, (b) labyrinth, (c) banded pattern [149]. (d) Satellite image of a spot pattern over Sudan.

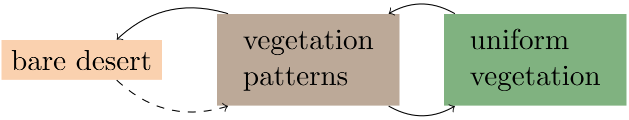

Figure 1.5: Desertification diagram for the extended Klausmeier model in one space dimension.

Because vegetation patterns extend over large areas, the model domain is taken to be unbounded, so without boundary conditions. Since the modeling equations are strikingly similar to the Gray-Scott model [74], this puts vegetation patterns in the same framework as patterns in chemical reactions.

In the spatially extended model, for decreasing rainfall the fold bifurcation of the spatially homogeneous solution (w_+, n_+) is preceded by a Turing bifurcation [190]. At the Turing bifurcation periodic patterns form. In Chapter 2 we show that a further decrease of the rainfall parameter leads to a coarsening cascade of patterns with larger and larger wavelength (or equivalently smaller and smaller wavenumber ($= 2\pi/\text{wavelength}$)). Eventually, for sufficiently small rainfall, the bare desert state is reached (Figure 1.5).

The occurrence of vegetation patterns can thus be regarded as a rough early warning signal for the final transition to the bare desert state, with larger wavelengths signalling a more imminent danger of complete collapse. This is in accordance with one of Ni's conjectures [133], which states that the last patterns to destabilize have large wavelengths.

We now add a second space dimension by replacing the second derivatives to x by second derivatives to both x and y, denoted by the Laplace operator $\Delta = \partial_x^2 + \partial_y^2$. Restricting to the case with linear water diffusion, the extended

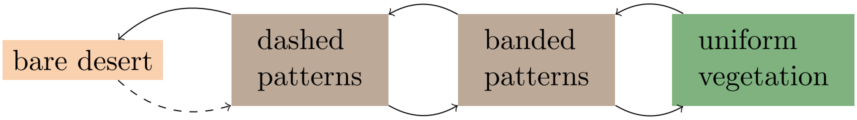

Figure 1.6: Desertification diagram for the extended Klausmeier model in two space dimensions.

Klausmeier model becomes

$$\begin{aligned} w_t =& d_1 \Delta w + 2c w_x + a - w - wn^2, \\ n_t =& d_2 \Delta n \qquad\qquad\quad - mn + wn^2. \end{aligned} \tag{1.3}$$

The slope is assumed to be in the x-direction, so that the advection term remains the same.

The restriction to one space dimension overestimates stability of the vegetation patterns, which correspond to vegetation bands (Figure 1.4(c)) when viewed as solutions of the two-dimensional extended Klausmeier model. Stability of vegetation bands in two space dimensions implies stability of the patterns in one space dimension but not the other way around: stable 1D patterns may be unstable against perturbations with a nontrivial transverse component.

In Chapter 3 we show that vegetation patterns indeed can be transversely unstable. The extent in which this occurs depends strongly on the slope of the terrain, which was modeled by the advective term. The smaller the slope, the more the banded patterns tend to be transversally unstable. Banded vegetation patterns on steeper slopes remain stable for a wider range of values of the rainfall parameter, thus are ecologically more resilient. In this case the desertification process only diverges from the one-dimensional case after the coarsening cascade of banded patterns with longer and longer wavelengths has progressed.

As a result of the breakup of banded vegetation patterns at low rainfall values, dashed patterns are formed. This way we have identified a possible natural next step in the desertification process, as depicted in Figure 1.6. The appearance of dashed patterns could be a warning signal that the transition to the bare desert state is imminent.

Dramatic sudden loss of productivity or gradual decline?

The current consensus is that in the final step of the desertification process, like in Figure 1.2(b), the ecosystem occupies a vegetated state with a reasonable amount of biomass but then undergoes a critical transition where all biomass is lost [149, 163]. This is what happens in the nonspatial Klausmeier model (1.1). How does this relate to the desertification scenarios for the spatially extended Klausmeier model we described?

As mentioned previously, the spatially extended Klausmeier model is in accordance with Ni's conjecture: the last patterns to destabilize have large wavelengths. These patterns form out of patterns with a smaller wavelength through a coarsening cascade. Large wavelength patterns have vegetation patches that are few and far between, summing up to a small amount of biomass. This means that the final transition to the bare desert state comprises only a minor loss of biomass. Thus, adhering to Ni's conjecture opposes the popular view that desertification entails a sudden large final jump in biomass.

In Chapter 4 we present a model refinement for capturing grazing, as an addition to the linear local vegetation death term already present in most arid ecosystem models. For this we employ nonlocal terms, so that the grazing pressure at one location depends on the presence of vegetation elsewhere. Because the grazing pressure on isolated vegetation patches may increase since there are no alternative foraging sites, the large wavelength patterns may become inadmissible in the presence of grazers.

We make a distinction between sustained and natural grazing. In sustained grazing systems, we assume that the number of herbivores is constant. In this case we indeed see dramatic transitions from vegetated states to the bare desert state, since none of the states with low biomass are admissible. Thus the situation is conceptually like Figure 1.2(b). In natural grazing systems, herbivore numbers drop if biomass decreases. If the number of herbivores drops fast enough, a transition from a high productivity (with a large herbivore number) to a low productivity (with a small herbivore number) system state occurs. Since in principle recovery to high productivity states is possible, the situation is similar to Figure 1.2(a).

The spatially extended Klausmeier model without nonlocal grazing adheres to Ni's conjecture and generally doesn't show a dramatic sudden loss of productivity. Only after introducing nonlocal grazing terms, do these dramatic regime shifts occur. In sustained grazing systems, Ni's conjecture does not hold. In systems with natural grazing, a dramatic loss of productivity may lead to a low productivity state, after which Ni's conjecture may still hold.

Outline

In Chapter 2, which is based on [180], we numerically study the extended Klausmeier model in one space dimension (1.2). We show model dynamics with a slowly decreasing rainfall parameter that leads the system from a homogeneously vegetated quasi steady equilibrium state through a Turing instability to a periodic pattern. A further decrease of the rainfall parameter leads to a coarsening cascade of patterns with larger and larger wavelength (or equivalently smaller and smaller wavenumber ($= 2\pi/\text{wavelength}$)). Eventually, for sufficiently small rainfall, the bare desert state is reached (Figure 1.4).

An important conceptual tool is the Busse balloon, first used in fluid mechanics [21], which represents all stable periodic vegetation patterns of the autonomous system (where the parameters are time independent). These patterns are represented by (rainfall, wavenumber)-pairs. By performing simulations with a slowly changing rainfall parameter, the system traces out a trajectory in (parameter, wavenumber)-space when it resides in a periodic system state. For small rates of change the length of the trajectory outside the Busse balloon is small, so destabilization can be predicted accurately. For large rates of change the discrepancy can become considerable, to the point that the system directly transitions from a state with the Turing wavenumber to the bare desert state, omitting the coarsening cascade.

The trajectories also depend on the application of noise. A higher noise level introduces heterogeneity in the population of vegetation patches, so that a larger share of the patches survive destabilization, leading to a coarsening cascade with smaller (and thus more) jumps in wavenumber and smaller trajectories outside the Busse balloon. Period doubling (wavenumber halving) occurs often, except in the regime with high wavenumbers/high rainfall, high noise levels and small rates of change.

Chapter 3 is based on [176]. We begin by presenting a general linear stability analysis of homogeneous steady states of inhibitor-activator type in

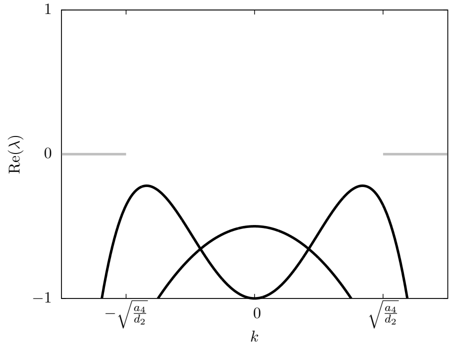

Figure 1.7: Sketch of real parts of curves of spectrum $\mathrm{Re}(\lambda)$ of the linearization about the homogeneous steady state (u_*, v_*), as a function of the linear wavenumber k. The spectrum can only cross the imaginary axis $\mathrm{Re}(\lambda) = 0$ in between the grey line segments.

a reaction-advection-diffusion system. These systems are of the form

$$\begin{aligned} u_t =& d_1 \Delta u + c_1 u_x + f(u, v), \\ v_t =& d_2 \Delta v + c_2 v_x + g(u, v). \end{aligned}$$

A homogeneous steady state (u_*, v_*) is of inhibitor-activator type if both $\frac{\partial f}{\partial u}(u_*, v_*) < 0$ and $a_4 := \frac{\partial g}{\partial v}(u_*, v_*) > 0$. This setup allows for the simultaneous treatment of both a diffusion driven Turing instability and a differential flow driven instability, and mixtures of both, with differential flow being the difference between the advection coefficients $|c_1 - c_2|$. We unravel the role of the differential flow in the selection of striped patterns at pattern formation in two space dimensions.

The analysis starts out in one space dimension. Here a pivotal upper estimate on destabilizing wavenumbers (Figure 1.7) allows for the removal of the wavenumber in the analysis. Thus all requirements can be set on the linearization of the kinetics and the spatiality of the problem is put to the background. This paves the way to results on the:

- monotonicity of the destabilization locus;
- direction of motion of emerging patterns.

Subsequently we employ a Squire transformation, which extends applicability of the previous results to two space dimensions and shows that destabilization of the homogeneous steady state occurs through perturbations perpendicular to the advection. The Squire transformation was, like the Busse balloon, first used in fluid mechanics [183].

All results are shown to apply to the extended Klausmeier model. This means that the Turing instability monotonically moves to larger rainfall values for increasing slope (advection), destabilization occurs by perturbations perpendicular to the slope and the emerging vegetation patterns move uphill.

Numerical analysis for the arid ecosystem model in two space dimensions shows that vegetation bands are (ecologically) more resilient in a regime with a steeper slope (advection). Under increasing environmental stress, banded vegetation eventually breaks up in a dashed vegetation pattern, which identifies dashed vegetation patterns as a next step in the desertification process. Observational studies back up that vegetation bands are rarely found at low precipitation, at low wavenumber, and in the absence of a slope. In accordance with this last finding, we prove, in an asymptotic scaling regime, that long wavelength striped patterns are transversely unstable on flat terrain.

In Chapter 4, which contains unpublished material, we extend the Klausmeier model with nonlocal grazing terms. For this we use well-established theory concerning herbivore distribution and responses to available biomass distributions. We make a distinction between sustained grazing - where the number of herbivores is assumed to be kept constant, and natural grazing - where herbivores are themselves responsible for acquiring a sufficient forage intake.

Through a linear analysis we investigate how the destabilization of the homogeneously vegetated state is affected by the introduction of grazing terms. The formation of vegetation patterns may be suppressed if the effective foraging potential is a superlinear function of the vegetation, meaning that herbivores are strongly attracted to locations with much vegetation.

As already discussed, we show how sustained grazing systems do not adhere to Ni's conjecture, since in this case long wavelength patterns are not the last patterns to destabilize. Whereas natural grazing has the potential to make a divide between high productivity and low productivity system states.

In Chapter 5, based on [127], we change gears and study the question of well-posedness in the context of quasilinear systems of partial differential equations (PDEs). Quasilinear PDEs are a class of PDEs that are 'less linear' than semi-linear but 'more linear' than fully nonlinear PDEs. A PDE is semi-linear if its highest order derivative terms have coefficients that are independent of the dependent variable, it is quasilinear if its highest order derivative terms have coefficients that may depend on the dependent variable but not on its derivatives.

In arid ecosystem modeling, the quasilinearity stems from a water-dependent diffusion coefficient of the water component, that results from the modeling of surface water flow through the shallow water equations [70]. This corresponds to $\gamma = 2$ in (1.2), where ${w^2}_{xx} = ww_{xx} + 2{w_x}^2$, so that the coefficient of the highest order derivative w_{xx} indeed depends on w. These types of differential equations also arise in the modeling of flow through porous media and are therefore often referred to as porous medium equations [200].

For illustration, let's consider a real life example. Since everyone has (had) a mother, it makes sense to ask "What's the name of your mother?" to anyone. But since not everyone has a child, it may be ill-posed to ask "What's the name of your child?". Depending on the context, the question "What's the name of your child?" can be well-posed or ill-posed. For instance, when posed to a man pushing a stroller this question should work well, but not when posed to the child inside. Only after affirming that the person in question has a child, it makes sense to ask for properties of the child.

Likewise, when analyzing a model, the most fundamental question to ask is whether the model makes *any* sense: under what conditions does a solution exist? Thus we look for functional analytic settings where the existence of a solution can be established. Only then it makes sense to wonder if the model makes the *right* sense in relation to the (natural) system that is being modeled: does the model solution behave in accordance with observations or experiments?

For systems of quasilinear PDEs on unbounded domains the issue of well-posedness and stability of nonlinear waves was not straightforward to reconcile based on the existing quite abstract literature. We settle well-posedness by presenting various suitable function spaces in which (unique) solutions to systems of PDEs exist, using various results from the theory of maximal regularity. To assess stability of a nonlinear wave, it is necessary to choose a function space of perturbations in which the evolution problem of the perturbations is well-posed. In addition we prove an orbital instability result for spectrum invading the half plane with positive real part, without assuming a spectral gap or the existence of an unstable eigenvalue.

We apply the results to the GKGS model [199], which is a slight variation of the extended Klausmeier model where $a - w$ is replaced by $a(1 - w)$. In this context the vegetation bands are the nonlinear waves.

Chapter 6 contains a brief outlook on future research possibilities.

Project collaborators

This thesis has been written at the Mathematical Institute, Leiden University, under supervision of Arjen Doelman and Jens Rademacher (University of Bremen). The research has been done in close collaboration with ecologists Max Rietkerk, Maarten Eppinga en Koen Siteur at the Copernicus Institute of Sustainable Development, Utrecht University.

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