Images of Galois representations

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Chapter 10

Irreducible representations

In this chapter we study irreducible 2-dimensional residual modular Galois representations with exceptional image. This means that projective image is either isomorphic to a dihedral group or projectively exceptional. Indeed, if the representation does not have projective image isomorphic to a dihedral group or projectively exceptional then by Dickson's Theorem it has "big" image, i.e. the image contains $\mathrm{SL}_2(\mathbb{F})$, where $\mathbb{F}$, finite extension of $\mathbb{F}_\ell$, is the Dickson's field of the representation.

In the first section we study the case of projective image isomorphic to a dihedral group. Analogously, in the second section we describe the case of image projectively exceptional. Let ℓ be a prime larger than 3 (respectively different from 5), we link modular forms mod ℓ with projectively exceptional image to modular forms in characteristic 3 (respectively 2 and 5) with octahedral and tetrahedral (respectively icosahedral) projective image.

In addition, in the tetrahedral and octahedral case, let f be a form with projectively exceptional image in characteristic 3 (respectively 5 in the icosahedral case). We describe a construction that gives a form in characteristic zero whose reduction mod 3 (respectively mod 5) is the form f and such that its reduction modulo any other odd prime is projectively exceptional. We also study the image of the reduction mod 2 of this characteristic zero form.

The results of the first section are essentially known in the literature, and are due to Serre, Wiese and Dieulefait-Billerey. We give a different argument from the one presented in [BD12] about the ramification of the quadratic character. On the other hand, the results of the second section give a new contribution towards a better comprehension of projectively exceptional modular Galois representations.

10.1 Dihedral case

Let us suppose that the representation ρ_f, defined as in Corollary 4.0.5, is an irreducible representation. If the projective image of ρ_f is isomorphic to a dihedral group D_{2n}, with ℓ not dividing n, then by Dickson's Theorem there exists a Cartan subgroup $C \subseteq \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, that is the group of points of a split or non-split maximal torus, such that $G := \rho_f(G_{\mathbb{Q}})$ is contained in the

normalizer N of C, but not in C. Let $\alpha : G_{\mathbb{Q}} \to \mathbb{Z}/2\mathbb{Z}$ be the composition:

$$\begin{tikzcd} G_{\mathbb{Q}} \arrow[r] \arrow[rrr, bend right, "\alpha"'] & G \subseteq N \arrow[r, two heads] & N/C \cong \mathbb{Z}/2\mathbb{Z}. \end{tikzcd}$$

The kernel of α is an open subgroup of $G_{\mathbb{Q}}$ of index 2, so its fixed field K is quadratic and unramified outside $n\ell$, where $n = \mathrm{cond}(\rho_f)$ by construction.

Let $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K) \subset G_{\mathbb{Q}}$, then $\mathbb{P}\rho_f(G_K) = C$ and ρ_f restricted to G_K is reducible:

$$\rho_f|_{G_K} = \chi \oplus \chi',$$

where χ, χ' are characters such that, denoted by σ the non trivial element of $\mathrm{Gal}(K/\mathbb{Q})$, the character $\chi' = \chi^\sigma$ where for all $\gamma \in G_K$ we have $\chi^\sigma(\gamma) = \chi(\sigma\gamma\sigma^{-1})$. Hence, ρ_f is the representation of $G_{\mathbb{Q}}$ induced by χ.

Moreover, the order of a maximal cyclic subgroup contained in the projective image is given by the order of the character $\chi^{-1}\chi^\sigma$, according to [Ser77b, (7.2.1)(d)], and the character χ can be computed using the statements in [Ser77b, (7.2.1)] if the quadratic character α is known.

Proposition 10.1.1. *Let n and k be positive integers, let ℓ be a prime not dividing n and such that $2 \leq k \leq \ell + 1$. Let $\epsilon\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a character and $f\colon \mathbb{T}_\epsilon(n,k) \to \overline{\mathbb{F}}_\ell$ be a morphism of rings. Let us suppose that the representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, given in Corollary 4.0.5, does not arise from lower level or weight, it is realized over $\mathbb{F}$ and it is irreducible. Then $\mathbb{P}\rho_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ is dihedral if and only if there exists a quadratic character $\alpha\colon (\mathbb{Z}/n\mathbb{Z})^* \to \{\pm 1\}$ such that $\rho_f \cong \alpha \otimes \rho_f$.*

Proof. If $\mathbb{P}\rho_f(G_{\mathbb{Q}})$ is dihedral then the existence of the character follows from the description above. Vice versa let suppose there exists a character $\alpha\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{F}^*$ such that $\rho_f \cong \alpha \otimes \rho_f$. Then for all unramified prime p the traces of the two representations are equal: $f(T_p) = \alpha(p)f(T_p)$, and also the determinant $f(\epsilon(p))p^{k-1} = \alpha(p)^2 f(\epsilon(p))p^{k-1}$. Hence by the Chebotarev density Theorem the character α has to be a quadratic character. Moreover, by [Ser77b, (7.2.1), p.337], we have that the projective image is dihedral. □

Corollary 10.1.2. *In the same hypotheses of Proposition 10.1.1, if the character α is ramified at a prime p dividing n, then p^2 divides n. If the character α ramifies at ℓ then either $f(T_\ell) \neq 0$ and $k = (\ell+1)/2$ or $f(T_\ell) = 0$ and $k = (\ell+3)/2$.*

Proof. The character α in unramified outside $n\ell$ because the representation ρ_f is in unramified outside $n\ell$. Let p be a prime dividing n. Since $\rho_f \cong \alpha \otimes \rho_f$ then $\mathrm{N}_p(\rho_f) = \mathrm{N}_p(\alpha \otimes \rho_f)$. Suppose that α is ramified at p. The result follows from [BD12, Proposition 3.1, 1].

Let us suppose that α is ramified at ℓ.

If $f(T_\ell) \neq 0$ then by Theorem 6.3.1 we have that ρ_f restricted to the inertia subgroup at ℓ is reducible. By [Dia97, Proposition 2.2], it is decomposable. Since α is a quadratic character ramified at ℓ, then, once restricted to the inertia at ℓ, it is equivalent to $\chi_\ell^{(\ell-1)/2}$. Given that $\rho_f \cong \alpha \otimes \rho_f$, we have that:

$$\rho_f|_{I_\ell} \cong \begin{pmatrix} \chi_\ell^{k-1} & 0 \\ 0 & 1 \end{pmatrix} \cong (\alpha \otimes \rho_f)|_{I_\ell} \cong \begin{pmatrix} \chi_\ell^{(\ell-1)/2}\chi_\ell^{k-1} & 0 \\ 0 & \chi_\ell^{(\ell-1)/2} \end{pmatrix},$$

and this is possible only if $k = (\ell+1)/2$.

If $f(T_\ell) = 0$ then by Theorem 6.3.2 we have that $\rho_f|_{I_\ell}$ is reducible and equivalent to $\varphi'^{k-1} \oplus \varphi^{k-1}$, where φ', φ are the two fundamental characters of level 2. Since $\rho_f \cong \alpha \otimes \rho_f$, then using that $\varphi\varphi' = \chi_\ell|_{I_\ell}$ and that $\chi_\ell|_{I_\ell} = \varphi^{\ell+1}$ we have that

$$\ell(k-1) = (k-1) + \frac{\ell^2-1}{2} \quad \text{in } \mathbb{Z}/(\ell^2-1)\mathbb{Z}$$

this is impossible for $\ell = 2$ and if ℓ is odd then it is equivalent to

$$k-1 \equiv \frac{\ell+1}{2} \bmod \ell+1.$$

Therefore, this is possible if and only if $k = (\ell+3)/2$. □

Algorithm 10.1.3 (Check dihedral projective image)**.** Let n, k be positive integers. Let ℓ be a prime not dividing n such that $2 \leq k \leq \ell + 1$. Let $\epsilon\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a character. Let $f\colon \mathbb{T}_\epsilon(n,k) \to \overline{\mathbb{F}}_\ell$ be a morphism of rings and let ρ_f be the representation attached to f in Corollary 4.0.5. Suppose that ρ_f is irreducible and $\mathrm{N}(\rho_f) = n$: Algorithm 6.3.7 and Algorithm 7.2.4 perform the respective checks.

Input: n, k, ℓ, $f(T_p)$ for p prime $p \leq B(n,k)$ where $B(n,k)$ is the Sturm bound for cusp forms for $\Gamma_0(n)$ and weight k;
Output: if $\mathbb{P}\rho_f(G_\mathbb{Q})$ is dihedral return α such that $\rho_f \cong \rho_f \otimes \alpha$; otherwise return 0.

$D \leftarrow [\ ]$;
if $k \notin \{(\ell+1)/2, (\ell+3)/2\}$ **then**
  Set D to be the list of all the possible quadratic characters α of $(\mathbb{Z}/q\mathbb{Z})^*$, where q is the product of all primes dividing n such that their square divides n;
else

Set D to be the list of all the possible quadratic characters α of $(\mathbb{Z}/q'\mathbb{Z})^*$, where $q' = \ell q$ where q is the product of all primes dividing n such that their square divides n;

for $\alpha \in D$ **do**

  if $f(T_p) = \alpha(p)f(T_p)$ for all primes p with $p \leq B(n,k)$ **then**

    return α and **stop**

return 0 and **stop**

Theorem 10.1.4. *Algorithm* 10.1.3 *is correct.*

Proof. Proposition 10.1.1 states that the projective image is dihedral if and only if there exists a quadratic character α such that $\rho_f \cong \alpha \otimes \rho_f$. The set of characters is described in Corollary 10.1.2. Since $\rho_f \cong \alpha \otimes \rho_f$ then their conductor are the same, so the representation $\alpha \otimes \rho_f$ has conductor n by hypothesis on ρ_f. The representation $\alpha \otimes \rho_f$ arises from a form at level n and weight k: this follows from the hypotheses and the equivalence of the representations. Such form has q-expansion given by $\sum \alpha(n)f(T_n)q^n$. Hence, it is enough to check equalities up to $B(n,k)$, the Sturm bound for cusp forms for $\Gamma_0(n)$ and weight k since ρ_f is irreducible. □

10.2 Exceptional groups case

Let the representation ρ_f, defined as in Corollary 4.0.5, be irreducible and projectively exceptional.

Let us recall that the following isomorphisms hold: $\mathfrak{A}_4 \cong \mathrm{SL}_2(\mathbb{F}_3)/\{\pm 1\}$ and $\mathfrak{S}_4 \cong \mathrm{PGL}_2(\mathbb{F}_3)$, while $\mathfrak{A}_5 \cong \mathrm{SL}_2(\mathbb{F}_4)$ and also $\mathfrak{A}_5 \cong \mathrm{SL}_2(\mathbb{F}_5)/\{\pm 1\}$.

This implies that in characteristic 3 if a representation has projective image isomorphic to $\mathfrak{A}_4$ or $\mathfrak{S}_4$ then its image contains $\mathrm{SL}_2(\mathbb{F}_3)$ so it is not exceptional. Analogously, in characteristic 5 and 2 if the projective image is isomorphic to $\mathfrak{A}_5$ then the image is not exceptional.

Let ℓ be a prime greater than 3 (respectively different from 5), in the following proposition we link modular forms mod ℓ with projectively exceptional image to modular forms in characteristic 3 (respectively 2 and 5) with octahedral and tetrahedral (respectively icosahedral) projective image.

Proposition 10.2.1. *Let n and k be two positive integers. Let ℓ be a prime not dividing n such that $2 \leq k \leq \ell + 1$. Let $\epsilon\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a character and let $f\colon \mathbb{T}_\epsilon(n,k) \to \overline{\mathbb{F}}_\ell$ be a morphism of rings. Let us suppose that the representation $\rho_f : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, attached to f in Corollary* 4.0.5*, does not arise from lower level or weight, it is irreducible, minimal up to twisting*

and projectively exceptional. Let us assume that $\ell > 3$ if the projective image is octahedral or tetrahedral otherwise let us assume $\ell \neq 5$.

Then there exists a characteristic zero, weight one cuspidal eigenform such that the reduction mod ℓ of the associated Artin representation is equivalent to ρ_f.

Moreover, if the projective image is octahedral or tetrahedral then the reduction modulo 3 of the Artin representation is not exceptional. Analogously, if the projective image is icosahedral then the reductions modulo 2 and 5 of the Artin representation are not exceptional.

Proof. Let $G = \rho_f(G_{\mathbb{Q}})$ and H be its projective image. Let us suppose that H is an icosahedral group. The case of octahedral and tetrahedral projective image are completely analogous and, hence, we will not discuss them.

Since the representation ρ_f is irreducible and minimal up to twisting then its determinant is a character of 2-power order, otherwise the twist of ρ_f with a power of the determinant would have lower conductor.

The group G is a finite group and it surjects to a group isomorphic to $\mathfrak{A}_5$, hence to $\mathrm{SL}_2(\mathbb{F}_5)/\{\pm 1\}$. The kernel of the projection is given by a cyclic group of 2-power order since the determinant is a character of 2-power order. Hence, G does admit a faithful representation $\phi\colon G \to \mathbb{F}' \cdot \mathrm{GL}_2(\mathbb{F}_5)$, where $\mathbb{F}'$ is a finite extension of $\mathbb{F}_5$. Indeed the map ϕ is injective because $\mathfrak{A}_5$ is a simple group and only elements of 5-power order can be in the kernel, while the kernel is cyclic of 2-power order.

The group G is by construction contained in $\mathrm{GL}_2(\mathbb{F})$, where $\mathbb{F}$ is the field of definition of the representation ρ_f. Let W be the ring of Witt vectors over $\mathbb{F}$, see [Ser95, Section I.6], and let us denote by $q : W \to \mathbb{F}$ the quotient map. Let i denote the inclusion, then there exist, up to conjugation, a unique map $\tilde{i}$ which lifts the representation i of G to $\mathrm{GL}_2(W)$ by the Schur-Zassenhaus Theorem, see [Rob96, (9.1.2)]. The following diagram summarizes the setting:

$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[r, two heads, "\rho_g"] & G \arrow[r, "i"] \arrow[d, "\phi"] \arrow[dr, "\tilde{i}"] & \mathrm{GL}_2(\mathbb{F}) \\
& \mathbb{F}' \cdot \mathrm{GL}_2(\mathbb{F}_5) & \mathrm{GL}_2(W). \arrow[u, "q"']
\end{tikzcd}$$

Since $G_{\mathbb{Q}}$ is compact and ρ_f is continuous, $\tilde{i}(G)$ is a compact subgroup of $\mathrm{GL}_2(W)$. Moreover, G is a finite group, hence $\tilde{i}(G)$ is finite. Therefore by representation theory, see [Ser78, Section 12], there exist a number field such that the representation is realized over the ring of integer $\mathcal{O}$ herein. The ring

$\mathcal{O}$ admits a residue field at 5, hence we have the following:

$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[r, two heads, "\rho_f"] \arrow[dr] & G \arrow[r, "i"] \arrow[d, "\phi"] \arrow[dr, "\tilde{i}"] & \mathrm{GL}_2(\mathbb{F}) \\
& \mathrm{GL}_2(\overline{\mathbb{F}}_5) & \mathrm{GL}_2(\mathcal{O}) \arrow[l] \arrow[u, two heads, "q"']
\end{tikzcd}$$

where the representation ϕ is lifted from $\mathrm{GL}_2(\mathbb{F}')$ to $\mathrm{GL}_2(\overline{\mathbb{F}}_5)$ according to [EC11, Lemma 7.2.3 and Theorem 7.2.2].

Then by [KW09a, Corollary 10.2] there exists a cuspidal eigenform with coefficients in $\mathcal{O}$ and of weight one whose associated representation is equivalent to $\tilde{i} \circ \rho_f$. So its reduction mod ℓ is equivalent to ρ_f. Moreover, this form can be reduced modulo 5. Since the group G has a faithful representation in $\mathrm{GL}_2(\overline{\mathbb{F}}_5)$ and projective image isomorphic to $\mathfrak{A}_5$ then the representation given by composition with the reduction map has not exceptional image. This means that, by Khare-Wintenberger Theorem, there exists a mod 5 modular form h with minimal level and weight such that the associated representation ρ_h is equivalent to the given one and and it has big image.

Since $\mathfrak{A}_5 \cong \mathrm{SL}_2(\mathbb{F}_5)/\{\pm 1\}$ and $\mathfrak{A}_5 \cong \mathrm{SL}_2(\mathbb{F}_4)$, then we repeat the same argument to prove the existence of a mod 2 modular form with non-exceptional image using that $\mathrm{SL}_2(\mathbb{F}_4) \cong \mathfrak{A}_5$. $\square$

Remark 10.2.2. Icosahedral representations in characteristic 2 have a specific characterization. Indeed, up to $\mathbb{F}_4/\mathbb{F}_2$-automorphisms, the trace of an element in $\mathrm{SL}_2(\mathbb{F}_4)$ depends only on its order. The possible orders of elements in $\mathrm{SL}_2(\mathbb{F}_4)$ are 2, 3 and 5. If an element has order 2, it must have characteristic polynomial $x^2 - 1 = (x-1)^2$ and so has trace 0. Similarly, every nontrivial element of order 3 has characteristic polynomial $x^2 + x + 1$ and so has trace 1. Finally, the nontrivial elements of order 5 have as characteristic polynomial a degree 2, monic polynomial over $\mathbb{F}_4$ dividing $x^4+x^3+x^2+x+1$, and the unique two such polynomials are $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$-conjugate as claimed. Therefore, if f is a mod 2 modular form of odd level such that the representation ρ_f given in Corollary 4.0.5 is irreducible and with icosahedral projective image, then

$$f(T_p) = \begin{cases} 0 & \text{if } \mathrm{ord}(\mathrm{Frob}_p) = 1 \text{ or } 2 \\ 1 & \text{if } \mathrm{ord}(\mathrm{Frob}_p) = 3 \\ \omega \text{ or } \omega^2 & \text{if } \mathrm{ord}(\mathrm{Frob}_p) = 5 \end{cases}$$

where ω, ω^2 are the distinct roots of $x^2 + x + 1$ in $\mathbb{F}_4$. Moreover, the form f^σ, where $\langle \sigma \rangle \cong \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$, is a mod 2 modular form with the same level and weight of f and the associated representation has icosahedral projective image.

Proposition 10.2.1 is a statement about existence of a characteristic zero modular form with certain properties and it is not constructive. In the tetrahedral and octahedral case, let f be a form with projectively exceptional image in characteristic 3 (respectively 5 in the icosahedral case). In the next section we describe a construction that gives a form in characteristic zero whose reduction mod 3 (respectively mod 5) is the form f and such that the residual representation modulo any other odd prime is projectively exceptional. We also study the image of the reduction mod 2 of this characteristic zero form.

10.3 Construction for the exceptional cases

– Projective image isomorphic to $\mathfrak{S}_4$ in characteristic 3

Let f be a mod 3 cuspidal eigenform of level n, not divisible by 3, weight k, with $2 \leq k \leq 4$, and let $\overline{\epsilon} : (\mathbb{Z}/n\mathbb{Z})^* \to \overline{\mathbb{F}}_\ell^*$ be the unique character such that $f \in S(n, k, \overline{\epsilon})_{\overline{\mathbb{F}}_\ell}$.

Let us suppose that the representation ρ_f, attached to f as in Corollary 4.0.5, is irreducible with projective image isomorphic to $\mathrm{PGL}_2(\mathbb{F}_3) \cong \mathfrak{S}_4$, and it is minimal with respect to level, weight and twisting.

Let $\mathbb{F}$ be the field of definition of the representation. The field of definition of the projective representation is $\mathbb{F}_3$ since all octahedral groups are conjugated by [Fab11, Proposition 4.17]. By Proposition 5.2.2 the image of the representation: $\rho_f(G_\mathbb{Q}) \subseteq \mathbb{F}^* \cdot \mathrm{GL}_2(\mathbb{F}_3)$.

Let ρ_2 be the 2-dimensional faithful representation of $\mathrm{GL}_2(\mathbb{F}_3)$ in Table 5.2. The intersection $\mathbb{F}^* \cap \mathrm{GL}_2(\mathbb{F}_3)$ is given by $\{\pm 1\}$, hence, a representation of $\mathbb{F}^* \cdot \mathrm{GL}_2(\mathbb{F}_3)$, with $\mathbb{F}^*$ acting by scalars, is given by a character of $\mathbb{F}^*$ and a representation of $\mathrm{GL}_2(\mathbb{F}_3)$ such that the two representations are compatible, i.e. they have the same image for -1.

Let $\beta : \mathbb{F}^* \cdot \mathrm{GL}_2(\mathbb{F}_3) \to \mathrm{GL}_2(\mathcal{O}_K)$ be the irreducible 2-dimensional representation given by the choice of ρ_2 as representation of $\mathrm{GL}_2(\mathbb{F}_3)$ and a compatible character $\tau : \mathbb{F}^* \to \mathbb{Z}[\zeta]^*$, where $\zeta \in \mathbb{C}$ is a root of unity of order equal to the cardinality of $\mathbb{F}^*$. The representation β is realized over $\mathcal{O}_K$, which is the ring of integers of a number field which contains the field of definition of ρ_2, which is $\mathbb{Q}(\sqrt{2})$, as shown in Table 5.2, and the cyclotomic extension of $\mathbb{Q}$ given by the image of τ. Let us remark that the field K is completely determined from the choice of β.

Let ρ be the composition:

$$\begin{tikzcd} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r, "\rho_f"'] \arrow[rr, bend left, "\rho"] & \mathbb{F}^* \mathrm{GL}_2(\mathbb{F}_3) \arrow[r, "\beta"'] & \mathrm{GL}_2(\mathcal{O}_K). \end{tikzcd}$$

By Langlands-Tunnell theorem, or equivalently by [KW09a, Corollary 10.2], there exist a weight one cuspidal eigenform of minimal level and weight, that we will denote as f_β, such that ρ_{f_β} is equivalent to ρ. Moreover, this form has octahedral projective image by construction.

Can we determine the level of f_β?

The representation ρ_f is unramified outside $3n$ by Corollary 4.0.5, hence also the representation ρ_{f_β} is unramified outside $3n$. The level of the form f_β is the conductor of the representation ρ_{f_β} by Khare-Wintenberger Theorem.

Let p be a prime dividing n (let us recall that n is not divisible by 3 by hypothesis), and assume that $f(T_p) \neq 0$. Proposition 8.1.5 gives us a criterion to decide whether the representation restricted to the decomposition group at p is decomposable or not. The representation restricted to the decomposition group at p is such that:

$$\rho_f|_{G_p} \cong \begin{pmatrix} \epsilon_1 \chi_3^{k-1} & * \\ 0 & \epsilon_2 \end{pmatrix} \cong \epsilon_2 \begin{pmatrix} \theta & * \\ 0 & 1 \end{pmatrix},$$

where ϵ_1 and ϵ_2 are characters of G_p with values in $\overline{\mathbb{F}}_3^*$ with ϵ_2 unramified as in Theorem 6.3.3, and θ is a characters of G_p such that $\epsilon_1 \, \chi_\ell^{k-1} = \epsilon_2 \, \theta$. Let us recall that in this case $\epsilon_2(\mathrm{Frob}_p) = f(T_p)$. Since $\overline{\epsilon}|_{G_p} = \epsilon_1 \, \epsilon_2$, then

$$\overline{\epsilon}|_{G_p} \, \chi_3^{1-k} = \epsilon_2^2 \, \theta,$$

hence the character θ is explicitly determined in terms of $\overline{\epsilon}$ and ϵ_2.

The image of the decomposition group at p, for p odd prime, can be either dihedral or cyclic since the group G_p is solvable. For $p = 2$ the image of the decomposition group is either cyclic, either dihedral, either octahedral or tetrahedral (icosahedral images are not possible since $\mathfrak{A}_5$ is not solvable).

The representation ρ_{f_β}, which is equivalent to $\beta \circ \rho_f$, restricted to the decomposition group at p is such that:

$$\rho_{f_\beta}|_{G_p} \cong \beta \left(\epsilon_2 \begin{pmatrix} \theta & * \\ 0 & 1 \end{pmatrix} \right) = \tau(\epsilon_2) \cdot \rho_2 \begin{pmatrix} \theta & * \\ 0 & 1 \end{pmatrix}.$$

If the image of ρ_f restricted to the decomposition group at p is not abelian, then the representation ρ_{f_β} restricted to the decomposition group at p does not admit any stable line with unramified quotient since the representation ρ_2 is faithful and irreducible. Therefore, in this case $f_\beta(T_p) = 0$.

If ρ_f restricted to the decomposition group at p is reducible and decomposable, it is abelian and $f_\beta(T_p)$ is equal to the eigenvalue of Frob_p which acts on the quotient given by a stable line, so it is equal to $\tau(\epsilon_2(\mathrm{Frob}_p))$. Therefore, $f_\beta(T_p) = \tau(f(T_p))$.

Since $f(T_p) \neq 0$ and p divides n, if ρ_f restricted to the decomposition group at p is decomposable then the character θ is ramified and the conductor of the representation ρ_{f_β} at p is the same as the conductor of the representation ρ_f at p.

Meanwhile, if ρ_f restricted to the decomposition group at p is indecomposable then two different cases can occur. If the character θ is trivial then

$$\rho_{f_\beta}|_{G_p} \cong \tau(\epsilon_2) \cdot \rho_2 \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \cong \tau(\epsilon_2) \cdot \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

where $*$ is a ramified character and α is a character of the decomposition group at p of order 3: this follows applying the representation ρ_2. Hence, we have that $\mathrm{N}_p(\rho_{f_\beta}) = 2$ while $\mathrm{N}_p(\rho_f) = 1$. If the character θ is not trivial then the conductor of the representation ρ_{f_β} at p is the same as the conductor of the representation ρ_f at p.

Let p be a prime dividing n and assume that $f(T_p) = 0$. Then ρ_f restricted to the decomposition group at p is irreducible, so also ρ_{f_β} restricted to the decomposition group at p is irreducible. The image of the representation is not abelian, hence $f_\beta(T_p) = 0$. The conductor of the representation ρ_{f_β} at p is the same as the conductor of the representation ρ_f at p: the stable subspaces for the ramification groups can be either 0-dimensional or 2-dimensional and the representation ρ_2 is faithful.

The only case left is $p = 3$.

If ρ_f restricted to the decomposition group at 3 is unramified then ρ_{f_β} restricted to the decomposition group at 3 is unramified. In this case $f(T_3) \neq 0$ by Theorem 6.3.1 and $\rho_f(\mathrm{Frob}_3)$ has eigenvalues $f(T_3)$ and $\overline{\epsilon}(3)f(T_3)^{-1}$ so

$$f_\beta(T_3) = \tau(f(T_3)) + \tau(\overline{\epsilon}(3)f(T_3)^{-1}),$$

since ρ_{f_β} is unramified at 3. Let us remark that if $f(T_3)$ does not belong to $\mathbb{F}$, the field of definition of the representation, then ρ_f is unramified at 3.

If ρ_f restricted to the decomposition group at 3 is reducible and tamely ramified, that is equivalent to say that ρ_f restricted to G_3 is ramified and decomposable, then $f(T_3) \neq 0$ by Theorem 6.3.1 and

$$\rho_f|_{G_3} \cong \begin{pmatrix} \epsilon_2\theta & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$

with ϵ_2 and θ defined as before. The "peu ramifié" case, see [Ser87, p.186], cannot occur since ρ_f is suppose to be minimal with respect to weight. The character θ is ramified and the weight can only be 2: in this case, in fact, there exists a companion form for f, see [Gro90, Proposition 13.2, 3)]. As in the previous case, we conclude that $f_\beta(T_3) = \tau(f(T_3))$. The valuation of

the conductor of ρ_{f_β} at 3 is then equal to 1 because there exists a stable line for the action of the inertia with unramified quotient.

If ρ_f restricted to the decomposition group at 3 is reducible and wildly ramified, then $f(T_3) \neq 0$ by Theorem 6.3.1 and

$$\rho_f|_{G_3} \cong \epsilon_2 \cdot \begin{pmatrix} \theta & * \\ 0 & 1 \end{pmatrix}$$

with ϵ_2 and θ defined as before. Define $b \in \{1, 2\}$ be the integer such that $\theta|_{I_3} = \chi_3^b$, then by [Edi92, Defiition 4.3, 2(b)] we have that $b = 1$ if $k = 4$ and $b = 2$ if $k = 3$. The image of the representation ρ_f restricted to the decomposition group at 3 is not abelian, then $f_\beta(T_3) = 0$. Moreover, the conductor of the representation ρ_{f_β} at 3 is such that $\mathrm{N}_p(\rho_{f_\beta}) = 2$.

If ρ_f restricted to the decomposition group at 3 is irreducible then $f(T_3) = 0$ by Theorem 6.3.2. The image of the representation ρ_f restricted to the decomposition group at 3 is not abelian, then $f_\beta(T_3) = 0$. The valuation of conductor of the representation ρ_{f_β} at 3 is equal to 2.

From what we have described so far we deduce the following proposition:

Proposition 10.3.1. *Let n and k be two positive integers such that n is not divisible by 3 and $2 \leq k \leq 4$. Let $\epsilon\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a character and let $f\colon \mathbb{T}_\epsilon(n,k) \to \overline{\mathbb{F}}_3$ be a morphism of rings. Let us suppose that the representation $\rho_f : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_3)$, attached to f in Corollary 4.0.5, does not arise from lower level or weight, it is irreducible, minimal up to twisting and with octahedral projective image.*

Then there exists a characteristic zero, weight one cuspidal eigenform of level n' such that the reduction mod 3 of the associated Artin representation is equivalent to ρ_f, where n' divides

$$3^i \cdot n \cdot \prod p$$

where the product is taken over the primes p dividing n, such that $f(T_p) \neq 0$ and $\mathrm{N}_p(n) = 1$, and if $f(T_3) \neq 0$ then $i = 0, 1$ or 2 according to $\rho_f|_{G_3}$ being unramified, tamely ramified or wildly ramified otherwise, if $f(T_3) = 0$ then $i = 2$.

Until now we have studied the traces of the representation ρ_{f_β} at the ramified primes. For the unramified primes, we cannot a priori determine uniquely the coefficient, we cannot distinguish elements in $\mathrm{GL}_2(\mathbb{F}_3)$ using only traces and determinants: for example, we cannot distinguish the identity from an element of order 3.

To solve this problem we will list all modular forms with weight one and level given in Proposition 10.3.1 over $\mathcal{O}_K$, the ring of integers of the number fields

where the representation β is defined, and check which one has reduction equal to f. This is possible using [Sch12, Algorithm 7.2.6] which allow us to list a basis for the space of weight one modular forms over $\mathcal{O}_K$ of a given level. To check whether the reduction is the correct one it is enough to check equalities between $f(T_p)$ and the reduced coefficients up to the Sturm bound for cusp forms on $\Gamma_0(n')$ where n' is the level of the characteristic zero form: in this case the reduction is a mod 3 modular form by [Edi97, Lemma 1.9].

The characteristic zero form we obtain in this way can be reduced modulo any odd prime and the image of the Galois representation associated is projectively octahedral.

Let us study the reduction in characteristic 2: we will give a method that can be used instead of [Sch12, Algorithm 7.2.6].

Let π be the reduction map from $\mathcal{O}_K$ to the residue field at 2. By [KW09a, Corollary 10.2], we have that the representation obtained composing ρ_{f_β} with the reduction map π is modular since it is an irreducible continuous representation. Hence, there exist a mod 2 Katz cuspidal eigenform $f_{\pi\beta}$ with associated Galois representation equivalent to $\pi \circ \rho_{f_\beta}$. The following diagram summarizes the setting:

$$\begin{tikzcd}
G_{\mathbb{Q}} \arrow[r, "\rho_f"'] \arrow[rr, bend left, "\rho_{f_\beta}"] \arrow[drr, bend right, "\rho_{f_{\pi\beta}}"'] & \mathbb{F}^* \operatorname{GL}_2(\mathbb{F}_3) \arrow[r, "\beta"'] & \operatorname{GL}_2(\mathcal{O}_K) \arrow[d, "\pi"] \\
& & \operatorname{GL}_2(\overline{\mathbb{F}}_2).
\end{tikzcd}$$

The representation is irreducible, the projective image of ρ_{f_β} is isomorphic to $\mathfrak{S}_4$ and so the kernel of its reduction modulo 2 can only be a 2-power order normal subgroup of $\mathfrak{S}_4$. Therefore, the projective image of $\rho_{f_{\pi\beta}}$ is such that $\mathbb{P}\rho_{f_{\pi\beta}}(G_{\mathbb{Q}}) \cong \mathfrak{S}_3$. Since $\mathfrak{S}_3 \cong \operatorname{GL}_2(\mathbb{F}_2) \cong \operatorname{PGL}_2(\mathbb{F}_2)$, we have that the representation $\rho_{f_{\pi\beta}}$ has not exceptional image, so by Proposition 5.2.2 the image of this representation is a subgroup of $\mathbb{F}'^* \times \operatorname{GL}_2(\mathbb{F}_2)$ where $\mathbb{F}'$ is the field of definition of the representation. By construction it is the field containing the image of the reduction of the character τ through π. Let us remark that, since $\mathcal{O}_K$ is completely explicit once fixed β, also $\mathbb{F}'$ is explicitly computable.

The level of the form $f_{\pi\beta}$ divides the level of the form f_β, see [Car89]. Hence, we can list all forms with the given image and check if the characteristic zero form we have found in the previous step has the desired reduction modulo 2.

In particular, the representation in characteristic 2 distinguishes the conjugacy classes for which it is not possible to decide in characteristic 3: the identity and the Jordan block are sent respectively to a matrix with trace

zero and a matrix with non-zero trace. Since we do not know what is the right choice, we list all possibilities. For each one we get a q-power series expansion in characteristic zero: only one of those is a modular form. We list all systems of eigenvalues in the smallest characteristic co-prime with the level of the form and we exclude all q-power series expansion which do not give by reduction a system of eigenvalues. If the process do not give a unique q-power series expansion then we proceed to the next prime and repeat. We claim that this process concludes. If we are not able to decide then we apply [Sch12, Algorithm 7.2.6] and proceed as explained before.

The following algorithms summarize the construction we have described in this section.

Algorithm 10.3.2 (Companion form for Algorithm 10.3.4)**.** Let n, k be positive integers with 3 not dividing n and $2 \leq k \leq 4$. Let $\epsilon\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a character. Let $f\colon \mathbb{T}_\epsilon(n,k) \to \overline{\mathbb{F}}_3$ be a morphism of rings and let $\rho_f : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_3)$ be the representation attached to f in Corollary 4.0.5. Suppose that ρ_f is irreducible, $\mathrm{N}(\rho_f) = n$, minimal with respect to weight and twisting and with octahedral projective image. Let $\overline{\epsilon} : (\mathbb{Z}/n\mathbb{Z})^* \to \overline{\mathbb{F}}_\ell^*$ be the character defined by $\overline{\epsilon}(a) = f(\langle a \rangle)$ for all $a \in \mathbb{Z}/n\mathbb{Z}^*$.

Input: n, k, $f(T_p)$ for p prime $p \leq B(n,6)$ where B is the Sturm bound for cusp forms for $\Gamma_0(n)$ and weight 6;
Output: 1 if f has a companion form, 0 otherwise.

$v \leftarrow 0$;
if $k \neq 2$ **then**
  return v;
else
  for $g \in S(n,2,\overline{\epsilon})_{\overline{\mathbb{F}}_3}$ **do**
    if $g(T_p) = pf(T_p)$ for all primes p, different from 3 up to $B(n,6)$ **then**
      Set $v \leftarrow 1$ and **stop**
return v.

Theorem 10.3.3. *Algorithm* 10.3.2 *is correct.*

Proof. The proof of correctness follows from [Gro90, Theorem 13.10]: we are in fact interested only into tamely ramified representations at 3, hence the case $k = 3$ is excluded. Also the case $k = 4$ is excluded because the representation is assumed to be minimal with respect to weight and so it cannot be peu ramifiée. The bound used is the Sturm bound for cusp forms

for $\Gamma_0(n)$ and weight 6 since we are checking the equality between $\theta_\ell g$ and $\theta_\ell^2 f$ for $g \in S(n, 2, \overline{\epsilon})_{\overline{\mathbb{F}}_3}$. $\square$

Algorithm 10.3.4 (Octahedral projective image)**.** Let n, k be positive integers with 3 not dividing n and $2 \leq k \leq 4$. Let $\epsilon\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a character. Let $f\colon \mathbb{T}_\epsilon(n,k) \to \overline{\mathbb{F}}_3$ be a morphism of rings and let $\rho_f : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_3)$ be the representation attached to f in Corollary 4.0.5. Suppose that ρ_f is irreducible, $\mathrm{N}(\rho_f) = n$, minimal with respect to weight and twisting and with octahedral projective image. Let $\overline{\epsilon} : (\mathbb{Z}/n\mathbb{Z})^* \to \overline{\mathbb{F}}_\ell^*$ be the character defined by $\overline{\epsilon}(a) = f(\langle a \rangle)$ for all $a \in \mathbb{Z}/n\mathbb{Z}^*$.

Let $\beta : \mathbb{F}^* \cdot \mathrm{GL}_2(\mathbb{F}_3) \to \mathrm{GL}_2(\mathcal{O}_K)$ be a representation given by a character $\tau : \mathbb{F}^* \to \mathbb{Z}[\zeta]^*$, where $\zeta \in \mathbb{C}$ is a root of unity of order equal to the cardinality of $\mathbb{F}^*$, and a faithful representation representation of $\mathrm{GL}_2(\mathbb{F}_3)$, see Table 5.2, which are compatible. The ring $\mathcal{O}_K$ is the ring of integers of the number field over which β is realized.

Input: n, k, β, $f(T_p)$ for p prime $p \leq B(n,k)$ where B is the Sturm bound for cusp forms for $\Gamma_0(n)$ and weight k, $\mathbb{F}$ field of definition of ρ_f;
Output: f_β

$v \leftarrow [\,]$;$N \leftarrow n$;
if $f(T_3) \neq 0$ **then**
  if $f(T_3) \notin \mathbb{F}$ **then**
    $N \leftarrow N$;
  else if Algorithm 10.3.2 return 1 **then**
    $N \leftarrow 3N$;
  else
    $N \leftarrow 3^2 N$;
else
  $N \leftarrow 3^2 N$;
STEP 0:
List all mod 2 modular forms of level N and image contained in $\mathbb{F}'^* \times \mathrm{GL}_2(\mathbb{F}_2)$ where $\mathbb{F}'$ is field containing the image of the reduction of the character τ, i.e. the residue field of $\mathcal{O}_K$ at 2.
List all q-power series expansion in characteristic zero corresponding to the different choices of conjugacy classes corresponding to the different mod 2 forms in the previous list.
List all systems of eigenvalues in the smallest characteristic co-prime with the level of the form and we delete all q-power series expansion which do not give by reduction a system of eigenvalues.
Repeat twice.
if the process give a unique q-power series expansion g **then**
  Add g to v

```
else
    Run [Sch12, Algorithm 7.2.6] with input O_K and level N
    Add g to v
for g in v do
    if N = n then
        if g(T_3) ≠ τ(f(T_3)) + τ(ε̄(3)f(T_3)^{-1}) then
            Remove g from v
    else if N = 3n then
        if g(T_3) ≠ τ(f(T_3)) then
            Remove g from v
    else
        if g(T_3) ≠ 0 then
            Remove g from v
    for p prime such that p ∤ n do
        if f(T_p) ≠ 0 then
            if N_p(ε̄) = N_p(n) then
                if g(T_p) ≠ τ(f(T_p)) then
                    Remove g from v
            else
                if g(T_p) ≠ 0 then
                    Remove g from v
        else
            if g(T_p) ≠ 0 then
                Remove g from v
for g in v do
    for λ maximal ideal in O_K dividing 3 do
        Let ḡ be the reduction of g mod λ
        for p prime such that p ∤ 3n do
            if f(T_p) ≠ ḡ(T_p) then
                Go to the next λ in the cycle and repeat the check,
                if for all λ we are in this case then remove g from v
if v = [] ; then
    for p prime dividing n do
        if f(T_p) ≠ 0 and N_p(n) = 1 then
            N ← Np
            Repeat from STEP 0
return  f_β ← v
```

Theorem 10.3.5. *Algorithm* 10.3.4 *is correct.*

Proof. The Algorithm follows the construction given in the previous section where all the equalities for the coefficients are proved. The form f_β is unique once $\beta \circ \rho_f$ is determined. □

Remark 10.3.6. The construction described depends on the choice of the representation β. In particular, if instead of the representation ρ_2 we choose the representation ρ_3 we get a form which is Galois conjugate to the form we obtain through the construction.

– **Projective image isomorphic to $\mathfrak{A}_4$ in characteristic 3**

There is a completely analogous construction to the previous for the tetrahedral case, starting with a mod 3 cuspidal eigenform with tetrahedral projective image. Hence, we will not repeat the construction and the algorithm. Let us remark that in this case, the field of definition of the image contains $\mathbb{F}_9$ since the image has to contain a matrix corresponding to the image of a complex conjugation.

Let $\alpha : \mathbb{F}^* \cdot \mathrm{SL}_2(\mathbb{F}_3) \to \mathrm{GL}_2(\mathcal{O}_K)$ be the irreducible 2-dimensional representation given by the choice of τ_2 as representation of $\mathrm{SL}_2(\mathbb{F}_3)$, where τ_1 is given in Table 5.1, and a compatible character $\tau : \mathbb{F}^* \to \mathbb{Z}[\zeta]^*$, defined as in the previous case. The representation α is realized over $\mathcal{O}_K$, which is the ring of integers of a number field which contains the field of definition of τ_2, which is $\mathbb{Q}(\zeta)$ by Table 5.1, and the cyclotomic extension of $\mathbb{Q}$ given by the image of τ. Let us remark that the field K is completely determined from the choice of α.

Proposition 10.3.7. *Let n and k be two positive integers such that n is not divisible by 3 and $2 \leq k \leq 4$. Let $\epsilon\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a character and let $f\colon \mathbb{T}_\epsilon(n,k) \to \overline{\mathbb{F}}_3$ be a morphism of rings. Let us suppose that the representation $\rho_f : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_3)$, attached to f in Corollary 4.0.5, does not arise from lower level or weight, it is irreducible, minimal up to twisting and with tetrahedral projective image.*

Then there exists a characteristic zero, weight one cuspidal eigenform of level n' such that the reduction mod 3 of the associated Artin representation is equivalent to ρ_f, where n' divides

$$3^i \cdot n \cdot \prod p$$

where the product is taken over the primes p dividing n, such that $f(T_p) \neq 0$ and $\mathrm{N}_p(n) = 1$, and if $f(T_3) \neq 0$, then $i = 0, 1$ or 2 according to $\rho_f|_{G_3}$ being unramified, tamely ramified or wildly ramified; otherwise, if $f(T_3) = 0$, then $i = 2$.

Using [Sch12, Algorithm 7.2.6], we list all the modular forms with weight one and level given in the previous proposition over $\mathcal{O}_K$, the ring of integers of the number fields where the representation α is defined. Then we check if the reduction is equal to f. It is enough to check equalities between $f(T_p)$ and the reduced coefficients up to the Sturm bound for cusp forms on $\Gamma_0(n')$

where n' is the level of the characteristic zero form: in this case the reduction is a mod 3 modular form by [Edi97, Lemma 1.9].

The characteristic zero form we obtain in this way can be reduced modulo any odd prime and the image of the Galois representation associated is projectively tetrahedral.

Let us study the reduction in characteristic 2. The representation obtained composing ρ_{f_β} with the reduction map is a reducible representation. This follows from the fact that the projective image is isomorphic to a cyclic group of order 3. Hence, the reduction mod 2 of the form obtained has to be reducible. Again we can use the data in characteristic 2 instead of using [Sch12, Algorithm 7.2.6], and use that algorithm only if we cannot conclude.

The construction described depends on the choice of the representation α. If instead of the representation τ_2 we choose the faithful representation τ_3 of $\mathrm{SL}_2(\mathbb{F}_3)$ then we get a form which is Galois conjugate to the form we obtain through the construction since these two representation are Galois conjugate.

– **Projective image isomorphic to $\mathfrak{A}_5$ in characteristic 5**

There is a completely analogous construction and algorithm also for the icosahedral case, starting with a mod 5 cuspidal eigenform with icosahedral projective image. Hence, we will not repeat it.

Let $\gamma : \mathbb{F}^* \cdot \mathrm{SL}_2(\mathbb{F}_5) \to \mathrm{GL}_2(\mathcal{O}_K)$ be the irreducible 2-dimensional representation given by the choice of ι_1 as representation of $\mathrm{SL}_2(\mathbb{F}_5)$, where ι_1 is given in Table 5.3, and a compatible character τ. The representation γ is realized over $\mathcal{O}_K$, which is the ring of integers of a number field which contains the field of definition of ι_1 (which is $\mathbb{Q}(\eta)$ by Table 5.1) and the cyclotomic extension of $\mathbb{Q}$ given by the image of τ.

Proposition 10.3.8. *Let n and k be two positive integers such that n is not divisible by 5 and $2 \leq k \leq 6$. Let $\epsilon\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a character and let $f\colon \mathbb{T}_\epsilon(n,k) \to \overline{\mathbb{F}}_5$ be a morphism of rings. Let us suppose that the representation $\rho_f : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_5)$, attached to f in Corollary 4.0.5, does not arise from lower level or weight, it is irreducible, minimal up to twisting and with icosahedral projective image.*

Then there exists a characteristic zero, weight one cuspidal eigenform of level n' such that the reduction mod 5 of the associated Artin representation is equivalent to ρ_f, where n' where n' divides

$$5^i \cdot n \cdot \prod p$$

where the product is taken over the primes p dividing n, such that $f(T_p) \neq 0$ and $\mathrm{N}_p(n) = 1$, and and if $f(T_5) \neq 0$, then $i = 0, 1$ or 2 according to $\rho_f|_{G_5}$ being unramified, tamely ramified or wildly ramified; otherwise, $f(T_p) = 0$ then $i = 2$.

In this case, since $\mathfrak{A}_5 \cong \mathrm{SL}_2(\mathbb{F}_4)$, we have that:

Corollary 10.3.9. *In the same hypotheses of Proposition* 10.3.8*, the reduction mod* 2 *of the Artin representation associated to the the characteristic zero, weight one cuspidal eigenform is equivalent to the Galois representation of a mod* 2 *modular form with projective image isomorphic to* $\mathrm{SL}_2(\mathbb{F}_4)$.

Moreover, if h is the mod 2 modular form in the Corollary, the $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$-conjugate form has icosahedral projective image.

Let us remark that, also in this case, the construction depends on the choice of the representation γ. More precisely, choosing the representation ι_2 we get a Galois conjugate of the form we obtain through the construction which uses ι_1.