Images of Galois representations

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Chapter 5

Image

Let n and k be positive integers, let ϵ be a character of $(\mathbb{Z}/n\mathbb{Z})^*$ with values in $\mathbb{C}^*$ and let $f : \mathbb{T}_\epsilon(n,k) \to \overline{\mathbb{F}}_\ell$ be a morphism of rings, where ℓ is a prime not dividing n and $2 \leq k \leq \ell + 1$. The aim of this part of the thesis is to give an algorithm which determines the image of the associated Galois representation ρ_f, as in Corollary 4.0.5, up to conjugacy as subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$.

In this chapter we prove that the image of the representation in $\mathrm{GL}_2(\mathbb{F})$, where $\mathbb{F}$ is the field of definition of the representation, is determined, up to conjugation in $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, once we have computed the set of determinants of the representation and the projective image $\mathbb{P}\rho_f(G_\mathbb{Q}) \subset \mathrm{PGL}_2(\mathbb{F}')$ of the representation, where $\mathbb{F}'$ is the field of definition of the projective representation. In Chapter 9 we will show that it is possible to compute such fields.

In the first section of this chapter we recall the classification of the subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$ given in Dickson's Theorem, this is a key element for the algorithm we want to outline. In the second section, we describe the output of the algorithm. We explain how to express the image of the Galois representation ρ_f, up to conjugacy as subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, using the projective image and the set of determinants, under the hypothesis of having already determined the definition field for the representation and the projective representation.

5.1 Projective image

Let n and k be positive integers, let $\epsilon\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a character and let $f\colon \mathbb{T}_\epsilon(n,k) \to \overline{\mathbb{F}}_\ell$ be a morphism of rings from the Hecke algebra of level n, weight k and character $\epsilon : (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ to an algebraic closure of $\mathbb{F}_\ell$, where ℓ is a prime not dividing n. Let ρ_f be the Galois representation associated to f in Corollary 4.0.5, and let $\mathbb{F}$ be the field of definition for the representation.

The image of the representation ρ_f is a conjugate of a subgroup of $\mathrm{GL}_2(\mathbb{F})$.

In the following Theorem, due to Dickson, see [Dic58] and [Lan76], are listed all finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$, for $\ell \geq 3$, up to conjugation:

Dickson's Theorem. *Let $\ell \geq 3$ be a prime and H a finite subgroup of $\mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$. Then a conjugate of H is one of the following groups:*

– *a finite subgroup of the upper triangular matrices;*
– $\mathrm{SL}_2(\mathbb{F}_{\ell^r})/\{\pm 1\}$ *or* $\mathrm{PGL}_2(\mathbb{F}_{\ell^r})$ *for* $r \in \mathbb{Z}_{>0}$*;*
– *a dihedral group* D_{2n} *with* $n \in \mathbb{Z}_{>1}$ *and* $(\ell, n) = 1$*;*
– *a subgroup isomorphic to either* $\mathfrak{A}_4$*, or* $\mathfrak{S}_4$ *or* $\mathfrak{A}_5$*.*

In the last case, i.e. when H is conjugate to a subgroup of $\mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$ isomorphic to either $\mathfrak{A}_4$, or $\mathfrak{S}_4$ or $\mathfrak{A}_5$, we give the following definition:

Definition 5.1.1. Let $\ell \geq 3$ be a prime and let G be a subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$. If the projective image of G is conjugate to a subgroup of $\mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$ isomorphic to either $\mathfrak{A}_4$, or $\mathfrak{S}_4$ or $\mathfrak{A}_5$, we call G *projectively exceptional.*

In characteristic 2, there is a similar classification to the one presented in Dickson's Theorem:

Theorem 5.1.2 ([KW09a, Lemma 6.1])**.** *Let* H *be a finite subgroup of* $\mathrm{PGL}_2(\overline{\mathbb{F}}_2)$*. Then a conjugate of* H *is one of the following groups:*
– *a finite subgroup of the upper triangular matrices;*
– $\mathrm{SL}_2(\mathbb{F}_{2^r})/\{\pm 1\}$ *or* $\mathrm{PGL}_2(\mathbb{F}_{2^r})$ *for* $r \in \mathbb{Z}_{>0}$*;*
– *a dihedral group* D_{2n} *with* $n \in \mathbb{Z}_{>1}$ *and* $(n, 2) = 1$*.*

In particular, if $\ell > 5$ and the order of G is divisible by ℓ, then either G is contained in a Borel subgroup of $\mathrm{GL}_2(\mathbb{F})$, where $\mathbb{F}$ is the field of definition of the representation, or G contains $\mathrm{SL}_2(\mathbb{F}')$, where $\mathbb{F}'$ is the field of definition of the projective representation. If the order of G is prime to the characteristic of the field, then its projective image is:
– either cyclic and G is contained in a Cartan subgroup;
– or dihedral, and G is contained in the normalizer of a Cartan subgroup but not in the Cartan subgroup itself;
– or conjugate to a subgroup isomorphic to one of the following groups: $\mathfrak{A}_4$, $\mathfrak{S}_4$, or $\mathfrak{A}_5$.
Let us recall that a Cartan subgroup C is a semi-simple maximal abelian subgroup of $\mathrm{GL}_2(\mathbb{F})$ and a Borel subgroup is a maximal closed connected solvable subgroup of $\mathrm{GL}_2(\mathbb{F})$. Hence, the subgroup of upper-triangular matrices is a Borel subgroup in $\mathrm{GL}_2(\mathbb{F})$. Since the representation ρ_f is semi-simple, if the projective image is cyclic and the order of G is prime to the characteristic, then the representation ρ_f is reducible: its image is an abelian group, while in the other cases it is irreducible.

What we have just discussed motivates the following definition:

Definition 5.1.3. Let n and k be two positive integers, let ℓ be a prime such that $(n, \ell) = 1$ and $2 \leq k \leq \ell + 1$, and let $\epsilon\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a character. Let $f\colon \mathbb{T}_\epsilon(n, k) \to \overline{\mathbb{F}}_\ell$ be a morphism of rings and let $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be the representation attached to f in Corollary 4.0.5. If $G := \rho_f(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ has order prime to ℓ we call the image *exceptional.* Moreover, we refer to the field of definition of the projective representation $\mathbb{P}\rho$ as the *Dickson's field* for the representation.

We will discuss about reducible representations in Chapter 7 and we will study irreducible representation with exceptional images in Chapter 10. In order to determine the projective image we need to determine the Dickson field of the representation, which is the field of definition of the projective representation and this is done in Chapter 9.

Remark 5.1.4. In characteristic 2, 3 and 5 we give a more explicit statement of Dickson's Theorem, since it will be useful in the following discussion.

For $\ell = 2$, let us underline that in Theorem 5.1.2 groups with projective image isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$ are not explicitly listed, this follows because:
- $\mathfrak{S}_4$ cannot occur as irreducible representation: any element in $\mathrm{GL}_2(\overline{\mathbb{F}}_2)$ of order a power of 2 is forced to be of order 1 or 2.
- $\mathfrak{A}_4$ has a normal subgroup of order 4, hence, any group with projective image $\mathfrak{A}_4$ is conjugate to a subgroup of the upper triangular matrices of $\mathrm{GL}_2(\overline{\mathbb{F}}_2)$: for any finite extension $\mathbb{F}$ of $\mathbb{F}_2$, a Sylow 2-subgroup of $\mathrm{GL}_2(\mathbb{F})$ is given by the unipotent matrices. For more details and a proof see [Fab11, Proposition 4.13].
- $\mathfrak{A}_5$ is isomorphic to $\mathrm{SL}_2(\mathbb{F}_4)$. Since all icosahedral groups are conjugated, see [Fab11, Proposition 4.23], the Dickson's field of the representation is $\mathbb{F}_4$.

For $\ell = 3$, Dickson's Theorem can be stated in the following way: let H a finite subgroup of $\mathrm{PGL}_2(\overline{\mathbb{F}}_3)$, then a conjugate of H is one of the following groups:
- a finite subgroup of the upper triangular matrices;
- $\mathrm{SL}_2(\mathbb{F}_{3^r})/\{\pm 1\}$ or $\mathrm{PGL}_2(\mathbb{F}_{3^r})$ for $r \in \mathbb{Z}_{>0}$;
- a dihedral group D_{2n} with $n \in \mathbb{Z}_{>1}$ and $(n, 3) = 1$;
- a subgroup isomorphic to $\mathfrak{A}_5$.

In any odd characteristic all octahedral (respectively tetrahedral) groups, i.e. groups isomorphic to $\mathfrak{A}_4$ (respectively $\mathfrak{S}_4$), are conjugate, see [Fab11]. In particular, we have that $\mathfrak{S}_4$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_3)$ and applying [Fab11, Proposition 4.17] we have that the Dickson's field of the representation is $\mathbb{F}_3$. Similarly, $\mathfrak{A}_4$ is isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)/\{\pm 1\}$ and by [Fab11, Proposition 4.14] the Dickson's field of the representation is $\mathbb{F}_3$.

Let us remark that icosahedral groups, i.e. groups isomorphic to $\mathfrak{A}_5$, are all conjugate in characteristic different from 5 and that they can occur in characteristic 3 only over extensions of $\mathbb{F}_9$ by [Fab11, Theorem A (4)]. Hence, in this last case $\mathbb{F}_9$ is a subfield of the field of definition of the projective representation since it is the Dickson's field of the representation.

For $\ell = 5$, we have that $\mathfrak{A}_5$ is isomorphic to $\mathrm{SL}_2(\mathbb{F}_5)/\{\pm 1\}$ and that all icosahedral groups are conjugate to $\mathrm{SL}_2(\mathbb{F}_5)/\{\pm 1\}$ by [Fab11, Proposition 4.23].

Applying [Fab11, Proposition 4.13] and [Fab11, Proposition 4.17], we conclude that the Dickson's field for projectively exceptional groups is $\mathbb{F}_5$.

In the following remark we give a criterion to decide if the image of a modular semi-simple 2-dimensional irreducible Galois representation is exceptional.

Remark 5.1.5. For any field k, let $\varphi : \mathrm{PGL}_2(k) \to k$ be the function defined by $\varphi(\gamma) = \mathrm{Trace}(\gamma)^2/\det(\gamma)$ with $\gamma \in \mathrm{PGL}_2(k)$. The following statement holds:

Proposition 5.1.6 ([Bos11, Proposition 1]). *Let $q \geq 4$ be a prime power and let $\varphi : \mathrm{PGL}_2(\mathbb{F}_q) \to \mathbb{F}_q$. Let G be a subgroup of $\mathrm{SL}_2(\mathbb{F}_q)/\{\pm 1\}$. Then we have $G = \mathrm{SL}_2(\mathbb{F}_q)/\{\pm 1\}$ if and only if $\varphi(G) = \mathbb{F}_q$.*

This gives a test to control beforehand if the representation is not exceptional, and we will see that this will be relevant for projectively exceptional images.

5.2 Image

Let n and k be positive integers, let $\epsilon \colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a character and let $f \colon \mathbb{T}_\epsilon(n,k) \to \overline{\mathbb{F}}_\ell$ be a morphism of rings, where ℓ is a prime not dividing n. Let ρ_f be the associated semi-simple Galois representation, as in Corollary 4.0.5. In this section we will use the following notation: $G = \rho_f(G_\mathbb{Q}) \subset \mathrm{GL}_2(\mathbb{F})$ is the image of the representation, where $\mathbb{F}$ is the field of definition of ρ_f; the image of the projective representation is denoted by $H = \pi(G)$, where $\mathrm{GL}_2(\mathbb{F}) \xrightarrow{\pi} \mathrm{PGL}_2(\mathbb{F})$ is the quotient map, and $D := \{\det(g) \in \mathbb{F}^*, \forall g \in G\} \subseteq \mathbb{F}^*$ is the set of determinants, i.e. the image of the map $\det : G \to \mathbb{F}^*$.

The representation is semi-simple, therefore if it is reducible, then it is decomposable. Hence, if we determine the characters in which it splits, then we have a complete description of the image. This problem is addressed in Chapter 7

If the representation is irreducible and projectively dihedral, i.e. H is a dihedral subgroup of $\mathrm{PGL}_2(\mathbb{F})$, then there exist a character, corresponding to a quadratic extension of $\mathbb{Q}$, such that the representation ρ_f is the induced representation of $G_\mathbb{Q}$ by this character, in Chapter 10 we will prove this statement. We will also show that there exists a quadratic character such that the representation and its twist by this character are equivalent. By Dickson's Theorem, in this case the projective image contains a maximal cyclic subgroups of order d not divisible by ℓ. Therefore, the projective image is isomorphic, up to conjugation, to $\mu_d(\overline{\mathbb{F}}_\ell) \rtimes \mathbb{Z}/2\mathbb{Z}$, with the action given by $z \mapsto z^{-1}$. Conjugation corresponds to the choice of an embedding for $\mu_d(\overline{\mathbb{F}}_\ell)$. As stated in [Ser72, Section 2.6, ii)] and [Ser72, Proposition 17], the image G is then contained in the normalizer of a Cartan subgroup. Let $\mathbb{F}'$ be the field of definition of the projective representation, the following two cases can occur:

1. d divides the cardinality of $\mathbb{F}'^*$. In this case there exists an embedding $\zeta_d \mapsto \left(\begin{smallmatrix} \overline{\zeta}_d & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $\overline{\zeta}_d$ is an element of order d. Since all normalizers of split Cartan subgroups are Galois conjugated, see [Lan76, Chapter XI], we have that the image is given by
$$G \cong \left\{ A \begin{pmatrix} \overline{\zeta}_d^i & 0 \\ 0 & 1 \end{pmatrix}, A \begin{pmatrix} 0 & \overline{\zeta}_d^j \\ 1 & 0 \end{pmatrix} \text{ for } i, j \in \mathbb{Z}/d\mathbb{Z} \text{ and } A \in D \right\}$$
2. d does not divide the cardinality of $\mathbb{F}'^*$, but d divides the cardinality of $\mathbb{F}'(\sqrt{\delta})^*$, where $\delta \in \mathbb{F}'^*$ is not a square. In this case there is an embedding $\zeta_d \mapsto \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{smallmatrix}\right)$ where $\alpha \in \mathbb{F}'(\sqrt{\delta})$ is an element of order d. Hence, G is a Normalizer of a non-split Cartan, see [Ser72, Section 2.1, b)].

In the remaining cases, i.e. when the representation is not reducible and the projective image is not dihedral, we derive the description of the image from Goursat's Lemma, see [Rib76a, Section V]:

Goursat's lemma. *Let A, A' be groups, and let B be a subgroup of $A \times A'$ such that the two projections $p_1 : B \to A$ and $p_2 : B \to A'$ are surjective. Let N be the kernel of p_2 and N' the kernel of p_1. Then the image of B in $A/N \times A'/N'$ is the graph of an isomorphism $A/N \approx A'/N'$.*

If the field of definition of the projective image is $\mathbb{F}_2$, then the representation is reducible or dihedral by Theorem 5.1.2: let us recall that
$$\mathrm{PGL}_2(\mathbb{F}_2) \cong \mathrm{GL}_2(\mathbb{F}_2) \cong D_6 \cong \mathfrak{S}_3,$$
and in both cases we are not interested in using this approach.

Let $\mathbb{F}$ be a finite field, such that $\mathbb{F}_2 \not\subset \mathbb{F}$, then the following sequence is exact:
$$1 \to \mu_2(\mathbb{F}) \to \mathrm{GL}_2(\mathbb{F}) \overset{(\pi,\det)}{\longrightarrow} \mathrm{PGL}_2(\mathbb{F}) \times \mathbb{F}^* \to \mathbb{F}^*/(\mathbb{F}^*)^2 \to 1 \qquad (\dagger)$$
where $\mu_2(\mathbb{F})$ and $\mathbb{F}^*/(\mathbb{F}^*)^2$ are respectively the kernel and the co-kernel of the map $(\pi, \det)$. For $\mathbb{F}_2$ we have that $\mathrm{PGL}_2(\mathbb{F}_2) \cong \mathrm{GL}_2(\mathbb{F}_2)$.

Let $\mathbb{F}'$ be a subfield of $\mathbb{F}$, let us recall that if $\mathbb{F} \neq \mathbb{F}_2$, then we have the following sequence:
$$\begin{tikzcd}
1 \arrow[r] & \mu_2(\mathbb{F}) \arrow[r] \arrow[dr] & \mathrm{GL}_2(\mathbb{F}) \arrow[r, "(\pi,\det)"] & \mathrm{PGL}_2(\mathbb{F}) \times \mathbb{F}^* \arrow[r] & \mathbb{F}^*/(\mathbb{F}^*)^2 \arrow[r] & 1 \\
& & \mathrm{GL}_2(\mathbb{F}') \cdot \mathbb{F}^* \arrow[u] \arrow[r, "(\pi,\det)"] & \mathrm{PGL}_2(\mathbb{F}') \times \mathbb{F}^* \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

where the group $\mathrm{GL}_2(\mathbb{F}') \cdot \mathbb{F}^*$ is the subgroup of $\mathrm{GL}_2(\mathbb{F})$ given by matrices of $\mathrm{GL}_2(\mathbb{F}')$ multiplied by scalar matrices in $\mathbb{F}^*$. We will call such a group a *scalar extension* of $\mathrm{GL}_2(\mathbb{F}')$ by $\mathbb{F}$.

Let us denote by q the quotient map $\mathrm{GL}_2(\mathbb{F}) \to \mathrm{GL}_2(\mathbb{F})/\mu_2(\mathbb{F})$. The map $(\pi, \det)$ in sequence (†) is injective on $\mathrm{GL}_2(\mathbb{F})/\mu_2(\mathbb{F})$. Hence, we have the following diagram:

$$
\begin{tikzcd}
\mathbb{F}^*/\mu_2(\mathbb{F}) \arrow[dr] & & \mathbb{F}^* \arrow[r, two heads] & \{\pm 1\} \arrow[dd, bend left=60, "\cong"] \\
& \mathrm{GL}_2(\mathbb{F})/\mu_2(\mathbb{F}) \arrow[ur, two heads, "\det"] \arrow[r, tail, "(\pi,\det)"] \arrow[dr, two heads, "\pi"'] & \mathrm{PGL}_2(\mathbb{F}) \times \mathbb{F}^* \arrow[u, "p_2"'] \arrow[d, "p_1"] \arrow[r, two heads] & \{\pm 1\} \times \{\pm 1\} \arrow[u, "p_2"'] \arrow[d, "p_1"] \\
\mathrm{SL}_2(\mathbb{F})/\{\pm 1\} \arrow[ur] & & \mathrm{PGL}_2(\mathbb{F}) \arrow[r, two heads] & \{\pm 1\}
\end{tikzcd}
$$

and by Goursat's Lemma, we conclude that the image of $\mathrm{GL}_2(\mathbb{F})/\mu_2(\mathbb{F})$ in the quotient is uniquely determined since the isomorphism is unique.

Let $\mathbb{F}$ and $\mathbb{F}'$ be respectively the field of definition of the representation and of the projective representation and let $\overline{G} := q(G)$. Let us assume that the representation is irreducible and not dihedral, since these cases are treated in a different way. We have the following diagram:

$$
\begin{tikzcd}
N' \trianglelefteq D \arrow[dr] & & D \arrow[r, two heads] & \overline{D} := D/N' \arrow[dd, bend left=60, "\cong"] \\
G \arrow[r, two heads, "q"] & \overline{G} \arrow[ur, two heads, "\det"] \arrow[r, tail, "(\pi,\det)"] \arrow[dr, two heads, "\pi"'] & H \times D \arrow[u, "p_2"'] \arrow[d, "p_1"'] \arrow[r, two heads] & \overline{H} \times \overline{D} \arrow[u, "p_2"'] \arrow[d, "p_1"'] \\
N \trianglelefteq H \arrow[ur] & & H \arrow[r, two heads] & \overline{H} := H/N
\end{tikzcd}
$$

where: $\overline{G} \subseteq \mathrm{GL}_2(\mathbb{F})/\mu_2(\mathbb{F})$, by definition of the map q; the group $H \times D$ is a subgroup of $\mathrm{PGL}_2(\mathbb{F}') \times \mathbb{F}^*$ since $\mathbb{F}$ and $\mathbb{F}'$ are the fields of definitions of the image and the projective image respectively; and N, N' are respectively normal subgroups in H and D. Let us remark that N contains all matrices with determinant one of $\overline{G}$, since it is the intersection of $\overline{G}$ with $\mathrm{SL}_2(\mathbb{F})/\{\pm 1\}$, and, similarly, N' contains the scalar matrices of $\overline{G}$: it is the centre of $\overline{G}$. Since $D \subseteq \mathbb{F}^*$, then $\overline{D}$ is a cyclic subgroup. Applying Goursat's lemma it follows that $\overline{H}$ is a cyclic group too since the image of $\overline{G}$ in $\overline{H} \times \overline{D}$ is the graph of an isomorphism between $\overline{H}$ and $\overline{D}$. Then $N \trianglelefteq H$ is a normal subgroups with cyclic quotient.

The representation is irreducible and its projective image is not dihedral, therefore we have the following list of possible projective images H by Dickson's Theorem:

- $H \supseteq \mathrm{SL}_2(\mathbb{F}')/\{\pm 1\}$. If $\mathbb{F}'$ is neither $\mathbb{F}_2$ nor $\mathbb{F}_3$, then $N = \mathrm{SL}_2(\mathbb{F}')/\{\pm 1\}$ or $N = H$, hence, $\overline{H} \subseteq \{\pm 1\}$. This follows because $\mathrm{SL}_2(\mathbb{F}')/\{\pm 1\}$ is a simple group under these hypotheses. The case $\mathbb{F}' = \mathbb{F}_2$ is excluded by our assumptions, and the possibility $\mathbb{F}' = \mathbb{F}_3$ is treated in the cases of octahedral and tetrahedral projective image since we have respectively $\mathrm{SL}_2(\mathbb{F}_3)/\{\pm 1\} \cong \mathfrak{A}_4$ and $\mathrm{PGL}_2(\mathbb{F}_3) \cong \mathfrak{S}_4$.
- $H \cong \mathfrak{S}_4$, then either $N = \mathfrak{A}_4$ and so $\overline{H} = \{\pm 1\}$, or $N = H$, therefore $\overline{H} = \{1\}$.
- $H \cong \mathfrak{A}_4$, then either $N \cong V_4$ is the subgroup given by double transpositions, then $\overline{H} \cong C_3$ the cyclic group of 3 elements, or $N = H$, then $\overline{H} = \{1\}$.
- $H \cong \mathfrak{A}_5$, then $N = \mathfrak{A}_5$ so $\overline{H} = \{1\}$, since $\mathfrak{A}_5$ is simple.

From this list we deduce that if H is not isomorphic to $\mathfrak{A}_4$, Goursat's lemma implies that the group $\overline{G}$ is uniquely determined. In the remaining case, if $\overline{H} \cong C_3$ there are be two possible isomorphism which are twist of each other. We will see that they correspond to the choice of a character of C_3 and, in particular, we will show that also in this case we can determine $\overline{G}$ uniquely. Moreover, if -1 belongs to G then $G = q^{-1}(\overline{G})$, hence it is possible determine G, the image of the representation, up to conjugacy as subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$.

Now we will proceed to describe G, up to conjugacy as subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, for all possible H, listed according to Dickson Theorem. We will not consider the reducible and the dihedral case. We will also show, case by case, that -1 belongs to G.

– $H \supseteq \mathrm{SL}_2(\mathbb{F}')/\{\pm 1\}$ where $\mathbb{F}'$ is the field of definition of the projective representation.

Proposition 5.2.1. *Let ℓ be a prime and let $\mathbb{F}$ be a finite extension of $\mathbb{F}_\ell$. Let G be a subgroup of $\mathrm{GL}_2(\mathbb{F})$ with projective image $H = \mathrm{SL}_2(\mathbb{F}')/\{\pm 1\}$ where $\mathbb{F}' \subseteq \mathbb{F}$. Then, up to conjugation as subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$,*

$$G = (\mathrm{SL}_2(\mathbb{F}') \cdot \mathbb{F}^* \xrightarrow{\det} (\mathbb{F}^*)^2)^{-1}(\det G).$$

Proof. Since $H = \mathrm{SL}_2(\mathbb{F}')/\{\pm 1\}$, then by [EC11, Lemma 2.5.1] we have that G contains $\mathrm{SL}_2(\mathbb{F}')$. Moreover, since $G \subseteq \mathrm{GL}_2(\mathbb{F})$ we have also that $G \subset \mathrm{SL}_2(\mathbb{F}') \cdot \mathbb{F}^*$ because H is $\mathrm{SL}_2(\mathbb{F}')/\{\pm 1\}$. In this case -1 belongs to the image G since it belongs to $\mathrm{SL}_2(\mathbb{F}')$. The following diagram resume the hypotheses:

$$\begin{tikzcd}
\mathrm{SL}_2(\mathbb{F}) \cap G = \mathrm{SL}_2(\mathbb{F}') \arrow[r, tail] \arrow[d, equal] & G \arrow[r, "\det"] \arrow[d] & \det G \arrow[d] \\
\mathrm{SL}_2(\mathbb{F}') \arrow[r, tail, "\triangleleft"] & \mathrm{SL}_2(\mathbb{F}') \cdot \mathbb{F}^* \arrow[r, "\det"] & (\mathbb{F}^*)^2 \\
V \arrow[r, tail] & \mathbb{F}^* \arrow[u, "\triangleleft"] \arrow[ur, "s"] &
\end{tikzcd}$$

where $V = (\mathbb{F}^* \xrightarrow{s} (\mathbb{F}^*)^2)^{-1}(\det G)$ and s is the map sending x to x^2. Since $\mathrm{SL}_2(\mathbb{F}') \subseteq G \subseteq \mathrm{SL}_2(\mathbb{F}') \cdot \mathbb{F}^*$ and $G = V \cdot \mathrm{SL}_2(\mathbb{F}')$ then the statement follows. □

Proposition 5.2.2. *Let ℓ be a prime and let $\mathbb{F}$ be a finite extension of $\mathbb{F}_\ell$. Let G be a subgroup of $\mathrm{GL}_2(\mathbb{F})$ with projective image $H = \mathrm{PGL}_2(\mathbb{F}')$ where $\mathbb{F}' \subseteq \mathbb{F}$. Then, up to conjugation as subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$,*

$$G = (\mathrm{GL}_2(\mathbb{F}') \cdot \mathbb{F}^* \xrightarrow{\det} \mathbb{F}^*)^{-1}(\det G).$$

Proof. The proof is analogous to the proof of Proposition 5.2.1, but in this case we have the following inclusions $\mathrm{SL}_2(\mathbb{F}') \subseteq G \subseteq \mathrm{GL}_2(\mathbb{F}') \cdot \mathbb{F}^*$. □

Remark 5.2.3. If the projective image is isomorphic to $\mathrm{SL}_2(\mathbb{F}')/\{\pm 1\}$, then the determinants of the matrices in G belong to the set of squares $(\mathbb{F}^*)^2$, while if the projective image is isomorphic to $\mathrm{PGL}_2(\mathbb{F}')$, then they belong to $\mathbb{F}^*$.

– $H \cong \mathfrak{A}_4$

Let us recall that the group $\mathfrak{A}_4$ has no non-trivial 2-dimensional irreducible linear representation. Moreover, let r be a positive integer, the second cohomology group $\mathrm{H}^2(\mathfrak{A}_4, \mathbb{Z}/2^r\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ by [Que95, Proposition 2.1 (i)]. Therefore, since this group classifies the central extensions of $\mathfrak{A}_4$

with kernel $\mathbb{Z}/2^r\mathbb{Z}$, we have that there exists only one non-trivial extension up to isomorphism. For $r = 1$, this extension is isomorphic to the group $\mathrm{SL}_2(\mathbb{F}_3)$ since $\mathrm{SL}_2(\mathbb{F}_3)/\{\pm 1\} \cong \mathfrak{A}_4$, and this group does admit 2-dimensional irreducible representations.

The complex linear 2-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{F}_3)$ are listed in Table 5.1 below.

$\mathrm{SL}_2(\mathbb{F}_3)$	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&1\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&-1\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$
$\mathrm{Trace}(\tau_1)$	2	-2	-1	-1	1	1	0
$\mathrm{Trace}(\tau_2)$	2	-2	$1+\zeta$	$-\zeta$	ζ	$-1-\zeta$	0
$\mathrm{Trace}(\tau_3)$	2	-2	$-\zeta$	$1+\zeta$	$-1-\zeta$	ζ	0

Table 5.1: List of the traces of 2-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{F}_3)$ in characteristic zero, computed in PARI/GP. In the first row are listed representatives for the conjugacy classes. In the table ζ denotes a fixed 3-rd root of unity.

The representations τ_1, τ_2 and τ_3 are three 2-dimensional faithful irreducible representations of $\mathrm{SL}_2(\mathbb{F}_3)$ in characteristic zero and they are twist of each other by a character acting on C_3, the maximal cyclic group inside $\mathfrak{A}_4$. Hence, we have that $\tau_2(\mathrm{SL}_2(\mathbb{F}_3))$ and $\tau_3(\mathrm{SL}_2(\mathbb{F}_3))$ are contained in the scalar extension of $\tau_1(\mathrm{SL}_2(\mathbb{F}_3))$ by $\mathbb{Z}[\zeta]^*$.

Let us remark that the representations of $\mathrm{SL}_2(\mathbb{F}_3)$ in characteristic ℓ are reduction modulo ℓ of the representations in characteristic zero. The representation τ_1 is realized over $\mathbb{Z}$, while the representations τ_2 and τ_3 are realized over $\mathbb{Z}[\zeta]$.

Let λ be a maximal ideal over ℓ in the field of definition of the representation, composing with the projection to the quotient we have a representation in characteristic ℓ. Let us still denote by τ_1, τ_2 and τ_3 the three representation obtained fixing a maximal ideal over ℓ and reducing the characteristic zero representation modulo that ideal. We have that $\tau_2(\mathrm{SL}_2(\mathbb{F}_3))$ and $\tau_3(\mathrm{SL}_2(\mathbb{F}_3))$ are contained in the scalar extension of $\tau_1(\mathrm{SL}_2(\mathbb{F}_3))$ by $\mathbb{F}_{\ell^2}^*$. Moreover, let us remark that for an odd prime ℓ, the representations τ_1, τ_2 and τ_3 are three 2-dimensional faithful irreducible representations of $\mathrm{SL}_2(\mathbb{F}_3)$ in characteristic ℓ. Meanwhile, for $\ell = 2$ they are 2-dimensional representation of C_3, hence they are reducible.

The images of the projective representations given composing τ_1, τ_2 and τ_3 with the natural quotient map are isomorphic to $\mathfrak{A}_4$ in any characteristic different from 2. In characteristic 2 they are isomorphic to C_3 and their kernel is V_4, the Klein four group, given by double transpositions in $\mathfrak{A}_4$.

Proposition 5.2.4. *Let $\ell > 3$ be a prime and let $\mathbb{F}$ be a finite extension of $\mathbb{F}_\ell$. Let G be a subgroup of $\mathrm{GL}_2(\mathbb{F})$ with projective image $H \subset \mathrm{PGL}_2(\mathbb{F})$, and such that G acts irreducibly on $\mathbb{P}^1(\mathbb{F})$. Let us assume that H is isomorphic to $\mathfrak{A}_4$. Then -1 belongs to G and the group G, up to conjugation as subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, is given by*

$$G = (\tau_1(\mathrm{SL}_2(\mathbb{F}_3)) \cdot \mathbb{F}^* \xrightarrow{\det} (\mathbb{F}^*)^2)^{-1}(\det G),$$

where τ_1 is the reduction modulo ℓ of the representation in Table 5.1.

Proof. Under the hypotheses of the proposition we have the following diagram:

$$\begin{tikzcd}
\overline{\mathbb{F}}_\ell^* \arrow[r, tail] & \mathrm{GL}_2(\overline{\mathbb{F}}_\ell) \arrow[r, two heads, "\pi"] & \mathrm{PGL}_2(\overline{\mathbb{F}}_\ell) \\
G \cap \overline{\mathbb{F}}_\ell^* \arrow[r, tail] \arrow[u] & G \arrow[r, two heads, "\pi"] \arrow[u] & H \cong \mathfrak{A}_4. \arrow[u]
\end{tikzcd}$$

Let us recall that there is only a subgroup, up to conjugation, isomorphic to $\mathfrak{A}_4$ in $\mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$ by [Bea10, Proposition 4.1] or [Fab11, Proposition 4.14]. Since $\mathfrak{A}_4$ has no normal subgroup of order 2 then H is contained in $\mathrm{SL}_2(\overline{\mathbb{F}}_\ell)/\{\pm 1\}$. Therefore, for $\ell > 3$ the following diagram is exact:

$$\begin{tikzcd}
\{\pm 1\} \arrow[r, tail] & \mathrm{SL}_2(\overline{\mathbb{F}}_\ell) \arrow[r, two heads, "\pi"] & \mathrm{SL}_2(\overline{\mathbb{F}}_\ell)/\{\pm 1\} \\
\{\pm 1\} \arrow[r, tail] & \mathrm{SL}_2(\mathbb{F}_3) \arrow[r, two heads] \arrow[u, "\tau_1"] & \mathfrak{A}_4 \cong \mathrm{SL}_2(\overline{\mathbb{F}}_3)/\{\pm 1\} \cong H \cong \pi(\tau_1(\mathrm{SL}_2(\mathbb{F}_3))) \arrow[u]
\end{tikzcd}$$

where the map τ_1 is a 2-dimensional representation of $\mathrm{SL}_2(\mathbb{F}_3)$, given reducing the representation in Table 5.1 modulo ℓ. Let us remark that the field of definition of the representation τ_1 is $\mathbb{F}_\ell$.

Let us show that -1 belongs to G. Proceeding by contradiction, let us assume -1 is not in G. This means that $G \cap \overline{\mathbb{F}}_\ell^*$ is cyclic of odd order, by simple computation. Therefore, the determinant is a character of odd order once restricted to $G \cap \overline{\mathbb{F}}_\ell^*$. So, extending G by scalars, the intersection $G \cap \overline{\mathbb{F}}_\ell^*$ is trivial: this corresponds to twist the action of G on $\mathbb{P}^1(\mathbb{F})$ with a power of the determinant. Hence, there exists a non-trivial scalar extension of G which is isomorphic to $\mathfrak{A}_4$ and is a subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$. Since $\mathfrak{A}_4$ does not admit 2-dimensional non-trivial irreducible representations, we get a contradiction.

Since -1 belongs to G we have that the image of the determinant has 2-power order and also $G \cap \overline{\mathbb{F}}_\ell^*$ has 2-power order.

Let $\sigma \in \mathfrak{A}_4$ be an element of order 3, and let $\tilde{\sigma}$ be a preimage of σ. Then $\pi(\tilde{\sigma}^3) = 1$ so $\tilde{\sigma}^3 \in G \cap \overline{\mathbb{F}}_\ell^*$. This group is a cyclic group of 2-power order, hence, the order of $\tilde{\sigma}$ is $3 \cdot 2^t$ for $t \in \mathbb{Z}_{>0}$. Let $\tilde{\sigma}' = \tilde{\sigma}^{2^t}$, hence $\tilde{\sigma}'$ has order 3 and $\pi(\tilde{\sigma}') = \sigma^{\pm 1}$ has order 3.

Let G_3 be the subgroup of G generated by elements of order 3. The subgroup G_3 maps surjectively to $\mathfrak{A}_4$ via π, since $\mathfrak{A}_4$ is generated by 3-cycles. Moreover, G_3 is contained in $\mathrm{SL}_2(\mathbb{F})$ and its intersection with the scalar matrices in given by $\{\pm 1\}$, otherwise $\mathfrak{A}_4$ would have a 2-dimensional representation. This means that G_3 is a subgroup of $\mathrm{SL}_2(\mathbb{F})$ and by construction it is a central extension of $\mathfrak{A}_4$ by $\{\pm 1\}$:

$$\begin{tikzcd}
G \cap \overline{\mathbb{F}}_\ell^* \arrow[r, tail] & G \arrow[r, two heads, "\pi"] & H \cong \mathfrak{A}_4 \\
\{\pm 1\} \arrow[r, tail] \arrow[u] & G_3 \arrow[r] \arrow[u] \arrow[ur, two heads] & \mathrm{SL}_2(\mathbb{F}).
\end{tikzcd}$$

Therefore, G_3 is isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$ because $\mathrm{H}^2(\mathfrak{A}_4, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and the trivial extension has no 2-dimensional irreducible representations. So G_3 is the image of a 2-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{F}_3)$ in $\mathrm{SL}_2(\mathbb{F})$. From Table 5.1, it follows that $G_3 = \tau_1(\mathrm{SL}_2(\mathbb{F}_3))$ since the other representations are twist of τ_1 by a non-trivial character, hence are not defined over $\mathbb{F}_3$. The group G and the subgroup G_3 both surject to a group isomorphic to $\mathfrak{A}_4$. This implies that for all $g \in G$ there exists $g' \in G_3$ and $\lambda \in \det(G)$ such that $g = g'\lambda$, uniquely up to sign, by construction of G_3. Hence, we have that

$$G_3 \cong \tau_1(\mathrm{SL}_2(\mathbb{F}_3)) \subseteq G \subseteq \tau_1(\mathrm{SL}_2(\mathbb{F}_3)) \cdot \mathbb{F}^*.$$

Therefore, we have:

$$\begin{tikzcd}
G \arrow[r, "\det"] \arrow[d] & \det G \arrow[d] \\
\tau_1(\mathrm{SL}_2(\mathbb{F}_3)) \cdot \mathbb{F}^* \arrow[r, "\det"] & (\mathbb{F}^*)^2,
\end{tikzcd}$$

so the statement holds. $\square$

Remark 5.2.5. If the projective image is isomorphic to $\mathfrak{A}_4$, then it is contained in $\mathrm{SL}_2(\mathbb{F}')/\{\pm 1\}$, where $\mathbb{F}'$ is the field of definition of the projective representation. So, the set of determinant of the representation is a subset of the set of squares of $\mathbb{F}$, field of definition of the representation.

– $H \cong \mathfrak{S}_4$

The group $\mathfrak{S}_4$ has no non-trivial 2-dimensional irreducible linear representations.

Let r be a positive integer, the second cohomology group $\mathrm{H}^2(\mathfrak{S}_4, \mathbb{Z}/2^r\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by [Que95, Proposition 2.4 (i)]. Therefore, since this group classifies the central extensions of $\mathfrak{S}_4$ with kernel $\mathbb{Z}/2^r\mathbb{Z}$, we have that there exist three non-trivial extension by $\mathbb{Z}/2^r\mathbb{Z}$ up to isomorphism. Among them only one is odd: this follows from [Que95, Lemma 3.2]. For $r = 1$, this extension is isomorphic to the group $\mathrm{GL}_2(\mathbb{F}_3)$ since $\mathrm{PGL}_2(\mathbb{F}_3) \cong \mathfrak{S}_4$.

The complex linear 2-dimensional irreducible representations of $\mathrm{GL}_2(\mathbb{F}_3)$ are listed in the following table:

$\mathrm{GL}_2(\mathbb{F}_3)$	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0&1\\1&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&1\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$
$\mathrm{Trace}(\rho_1)$	2	2	2	0	0	-1	-1	0
$\mathrm{Trace}(\rho_2)$	2	-2	0	α	$-\alpha$	-1	1	0
$\mathrm{Trace}(\rho_3)$	2	-2	0	$-\alpha$	α	-1	1	0

Table 5.2: List of the traces of 2-dimensional irreducible representations of $\mathrm{GL}_2(\mathbb{F}_3)$ in characteristic zero, computed in PARI/GP. In the first row are listed representatives for the conjugacy classes. In the table α denotes $\sqrt{-2}$.

The representations ρ_1 is not faithful and it corresponds to a representation of $\mathfrak{S}_3$. The representations ρ_2 and ρ_3 are 2-dimensional faithful irreducible representations. They are twists of each other and they are defined over $\mathbb{Z}[\alpha]$. In particular, $\rho_3 \cong \rho_2 \otimes \det \rho_2$.

Proposition 5.2.6. *Let $\ell > 3$ be a prime and let $\mathbb{F}$ be a finite extension of $\mathbb{F}_\ell$. Let G be a subgroup of $\mathrm{GL}_2(\mathbb{F})$ with projective image $H \subset \mathrm{PGL}_2(\mathbb{F})$, and such that G acts irreducibly on $\mathbb{P}^1(\mathbb{F})$. Let us assume that H is isomorphic to $\mathfrak{S}_4$. Then -1 belongs to G and the group G, up to conjugation as subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, is given by*

$$G = (\rho_2(\mathrm{GL}_2(\mathbb{F}_3)) \cdot \mathbb{F}^* \xrightarrow{\det} \mathbb{F}^*)^{-1}(\det G),$$

where ρ_2 is the reduction modulo ℓ of the representation in Table 5.2.

Proof. We will use the same notation of the proof of Proposition 5.2.4. Since $\mathfrak{A}_4 \subset \mathfrak{S}_4$ we have that -1 belongs to G. Moreover, we have that:

$$\begin{tikzcd}
G \arrow[d] & G_3 \cong \mathrm{SL}_2(\mathbb{F}_3) \arrow[l, tail] \arrow[r, tail] \arrow[d] & \mathrm{GL}_2(\mathbb{F}_3) \cong \mathrm{SL}_2(\mathbb{F}_3) \rtimes \mathbb{F}_3^* \arrow[d] \\
\mathfrak{S}_4 & \mathfrak{A}_4 \arrow[l, tail] \arrow[r, tail] & \mathfrak{S}_4.
\end{tikzcd}$$

Let G' be the subgroup of G given by $G' := G_3 \rtimes \mathbb{F}_3^* \cong G_3 \rtimes \{\pm 1\}$. Then G' is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$ and it surjects to H:

$$\begin{tikzcd} G & G' \arrow[l, hook'] \arrow[r, hook] \arrow[d, two heads] & \mathrm{GL}_2(\mathbb{F}_3) \arrow[dl, two heads] \\ & H \cong \mathfrak{S}_4. \arrow[from=1-1, two heads] & \end{tikzcd}$$

This implies that for all $g \in G$ there exists $g' \in G'$ and $\lambda \in \det G$ such that $g = g'\lambda$ uniquely up to sign. Hence, we have that

$$G' \cong \rho_2(\mathrm{GL}_2(\mathbb{F}_3)) \subseteq G \subseteq \rho_2(\mathrm{GL}_2(\mathbb{F}_3)) \cdot \mathbb{F}^*,$$

and so the statement holds. $\square$

Remark 5.2.7. Let us remark that, since $\rho_3 = \rho_2 \otimes \det(\rho_2)$ and $\det(\rho_2)$ belongs to $\{\pm 1\}$, the choice of ρ_2 instead of ρ_3 does not change the image of the representation up to conjugation.

– $H \cong \mathfrak{A}_5$

The group $\mathfrak{A}_5$ has no non-trivial 2-dimensional irreducible linear representations.

Let r be a positive integer, the second cohomology group $\mathrm{H}^2(\mathfrak{A}_5, \mathbb{Z}/2^r\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ by [Que95, Proposition 2.1 (i)]. Therefore, there exists only one non-trivial extension up to isomorphism. For $r = 1$, this extension is isomorphic to the group $\mathrm{SL}_2(\mathbb{F}_5)$ since $\mathrm{PGL}_2(\mathbb{F}_5) \cong \mathfrak{S}_4$, and this group does admit 2-dimensional irreducible representations.

The complex linear 2-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{F}_5)$ are listed in the following table:

$\mathrm{SL}_2(\mathbb{F}_5)$	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0&-1\\1&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}2&0\\0&3\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}0&-1\\1&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&1\\0&-1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1&2\\0&-1\end{smallmatrix}\right)$
$\mathrm{Trace}(\iota_1)$	2	-2	-1	0	η	η^2	1	$-\eta$	$-\eta^2$
$\mathrm{Trace}(\iota_2)$	2	-2	-1	0	η^2	η	1	$-\eta^2$	$-\eta$

Table 5.3: List of the traces of 2-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{F}_5)$ in characteristic zero, computed in PARI/GP. In the first row are listed representatives for the conjugacy classes. In the table η denotes a fixed 5-th root of unity.

The representations ι_1 and ι_2 are 2-dimensional faithful irreducible representations and they are twist of each other. Moreover, there is an outer automorphism for which this two representations are conjugate (for example conjugation by $\left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{F}_5)$), hence up to conjugation in $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, these representation have the same image.

Proposition 5.2.8. *Let ℓ be an odd prime different from* 5 *and let* $\mathbb{F}$ *be a finite extension of* $\mathbb{F}_\ell$. *Let* G *be a subgroup of* $\mathrm{GL}_2(\mathbb{F})$ *with projective image* $H \subset \mathrm{PGL}_2(\mathbb{F})$, *and such that* G *acts irreducibly on* $\mathbb{P}^1(\mathbb{F})$. *Let us assume that* H *is isomorphic to* $\mathfrak{A}_5$. *Then* -1 *belongs to* G *and* G, *up to conjugation as subgroup of* $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, *is given by*

$$G = (\iota_1(\mathrm{SL}_2(\mathbb{F}_5)) \cdot \mathbb{F}^* \xrightarrow{\det} (\mathbb{F}^*)^2)^{-1}(\det G),$$

where ι_1 *is the reduction modulo* ℓ *of the representation in Table* 5.3.

Proof. There is only a subgroup, up to conjugation, isomorphic to $\mathfrak{A}_5$ in $\mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$ by [Bea10, Proposition 4.1] or [Fab11, Proposition 4.22]. Since $\mathfrak{A}_5$ has no normal subgroup of order 2 then H is contained in $\mathrm{SL}_2(\overline{\mathbb{F}}_\ell)/\{\pm 1\}$. Since $\mathfrak{A}_4 \subseteq \mathfrak{A}_5$, we can proceed as in Proposition 5.2.4 and conclude that -1 belongs to G.

Let $[G, G]$ be the commutator of G, let us recall that $\mathfrak{A}_5$ is a perfect group i.e. $\mathfrak{A}_5 = [\mathfrak{A}_5, \mathfrak{A}_5]$. By hypotheses G surjects to $\mathfrak{A}_5$ so also $[G, G]$ surject to $[\mathfrak{A}_5, \mathfrak{A}_5] = \mathfrak{A}_5$. Moreover, $[G, G] \in \mathrm{SL}_2(\overline{\mathbb{F}}_\ell)$ since elements of the form $ghg^{-1}h^{-1}$ have determinant 1. Hence, we have the following diagram:

$$\begin{tikzcd}
G \arrow[drr, two heads] & {[G,G]} \arrow[l, tail] \arrow[r, tail] \arrow[d, two heads] & \mathrm{SL}_2(\overline{\mathbb{F}}_\ell) \\
& \mathfrak{A}_5 & \mathrm{SL}_2(\mathbb{F}_5) \arrow[l, two heads] \arrow[u]
\end{tikzcd}$$

which implies that $\forall g \in G$ there exist $g' \in [G, G]$ and $\lambda \in (\det G)$ such that $g = g'\lambda$ uniquely up to sign. Since $[G, G] \in \mathrm{SL}_2(\overline{\mathbb{F}}_\ell)$ and $\mathrm{H}^2(\mathfrak{A}_5, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, then $[G, G] \cong \mathrm{SL}_2(\mathbb{F}_5)$. Hence, we conclude that

$$[G, G] \cong \iota_1(\mathrm{SL}_2(\mathbb{F}_5)) \subseteq G \subseteq \iota_1(\mathrm{SL}_2(\mathbb{F}_5)) \cdot \mathbb{F}^*.$$

And, since up to conjugation in $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ the representations ι_1 and ι_2 are equivalent, the statement follows. ☐

Remark 5.2.9. In Proposition 5.2.4 and 5.2.6 we assume the characteristic different from 2 and 3. Indeed, in the first case the representation is reducible by Theorem 5.1.2. In characteristic 3, projective image isomorphic to $\mathfrak{A}_4$ or to $\mathfrak{S}_4$ corresponds to have big image. In this last case we apply Proposition 5.2.1 and Proposition 5.2.2 to determine the image of the linear 2-dimensional representation: the set of determinant is known, hence we distinguish the two cases. Analogously, in Proposition 5.2.8 we assume the characteristic different from 2 and 5. Indeed, in both cases, projective image isomorphic to $\mathfrak{A}_5$ corresponds to have big image, therefore we apply Proposition 5.2.1 to determine the image of the linear 2-dimensional representation.