Collective Behaviour of Self-Propelling Particles with Conservative Kinematic Constraints

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"...then I saw all that God has done. No one can comprehend what goes on under the sun. Despite all his efforts to search it out, man cannot discover its meaning. Even if a wise man claims he knows, he cannot really comprehend it."

(Ecclesiastes 8:17)

To my Mother, Father and Sister

Contents

Chapter 1

General introduction

1.1 Systems of self-propelling particles

The dynamics of systems consisting of self-propelling particles (SPP) is of great interest for physicists as well as for biologists because of the complex and fascinating phenomenon of the emergence of ordered motion. Examples of such systems found in nature are: flocks of birds, schools of fishes, groups of bacteria, etc. (see Figures 1.1,1.2 and Refs. [1]-[5]).

A reason that these systems are interesting is that many aspects of the observed transition from disordered to ordered motion are as yet not fully understood.

One may distinguish SPP systems of two types. In the first type the systems consist of particles interacting via the background in which they are moving. The driving forces are due to the gradients of chemical or physical factors, such as concentrations, temperature, light, electric and magnetic fields, etc. (see Figures 1.3,1.4 and Refs. [6, 7]). The absence of conservation of translational and angular momentum (see, e.g. Refs. [8, 9]) is a direct consequence of these external factors.

The second type is formed by systems of particles, which interact via kinematic constraints imposed on their velocities. An example is a system where particles adjust the direction of their velocities to the direction of the average velocity in their neighbourhood. Realization of such constraints requires instant exchange of infor-

Figure 1.1: Flock of birds.

Figure 1.2: P. Nicklen, A diver descends into a vortex of 50000 farmed Salmon to check nets.

mation (visual or other sensorial) between the particles and their environment. These constraints induce coherent motion and the development of ordered patterns in the dynamics of the biological systems mentioned above, in the dynamics of crowds, and in the flocking in traffic-like systems [10, 11]. The clustering in these systems is driven by interparticle interaction of a non-potential character. The tendency of the particles to align their velocities with their neighbours is the crucial mechanism for the emergence of collective motion of the SPP. That is why it is often reasonable to assume that the frictional forces are balanced by the self-propelling forces so that the particles move with constant absolute speeds. Usual potential gradients or other physical forces are not so relevant for the collective behaviour, though the particles need some physical source of energy to sustain the constraints. It means that the system with such constraints is not closed and therefore its dynamics is not Hamiltonian. This does not necessarily mean that the energy dissipates; rather, a redistribution among the dynamic degrees of freedom takes place. The very form of the constraint is determined by the information exchange between particle, its neighbours

Figure 1.3: N. Pavloff, "White dune".

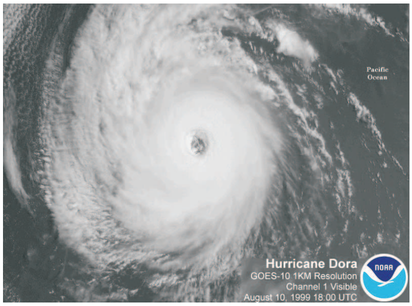

Figure 1.4: Live Science Image Gallery, Hurricane Dora in the Eastern Pacific.

and the background so that the particle moves accordingly.

The observed complex dynamics of systems of particles subjected to non-potential interactions remains poorly understood. That is why this phenomenon is presently of great interest. The absence of a Hamiltonian form of the equations of motion for such systems, which generally are far from equilibrium, hampers direct application of the machinery of statistical mechanics.

Numerous attempts have been made to find a model describing the collective behaviour of self-propelling particles. One may distinguish two main directions of research: numerical simulations (discrete description) and hydrodynamic (continuous) approaches. The use of continuum models and simulations to describe self-organization in biological and social systems is an active research field (for examples see [12]-[16] and references therein).

The first numerical model (discrete algorithm) for coherent motion of SPP was proposed by Vicsek et al. [17]. For shortness we will call it the Czirók-Vicsek automaton or algorithm (CV). In their work a simple kinematic updating rule for the directions of the velocities of the particles was proposed. Numerical evidence was given for a transition from a disordered state to ordered collective motion at a low noise amplitude and high density values. The model by Vicsek et al. describes such types of motion of SPP as linear flow, [17], and vortical flow depending on the imposed boundary conditions [18, 19]. Some of their results are shown in Figures 1.5,1.6.

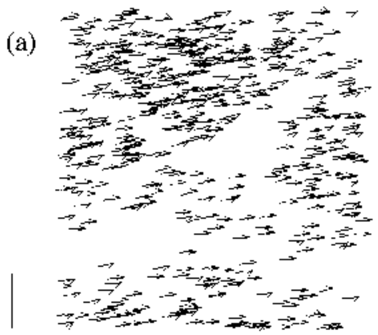

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Figure 1.5: Ordered motion of SPP at high density and low noise.

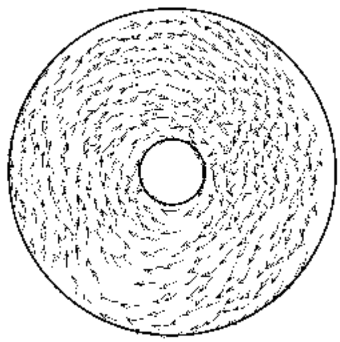
[1]

Figure 1.6: Stationary vortical state with reflective walls.

They seem to have been the first to realize that flocks fall into a broad category of nonequilibrium dynamical systems with many degrees of freedom and noted the analogy between flocking and ferromagnetism [20]. The model of Vicsek has become the "minimal" simulation model in the study of flocking, just like the Ising model is for systems undergoing a second order phase transition.

There is strong support that the transition from disordered to ordered motion of SPP observed in Vicsek's work [17] is of a continuous character [21]. However, in Ref. [22] it was shown by numerical simulation that the continuous character may be questioned, and that it depends on the size of the system (i.e. it is driven by finite size

[1]Reprinted figures with permission from A. Czirók, E. Ben-Jacob, I. Cohen, T. Vicsek, *Phys. Rev. E* **54**, 1791 (1996). Copyright © 2007 by the American Physical Society.

effects). The results in Ref. [22] demonstrate the difference between the influence of finite size effects for *angular* noise in the original algorithm, [17], and *vectorial* noise, when the random increment of the velocity is chosen and after that the direction is calculated. For vectorial noise it is distinctly discontinuous [22] even for rather small systems. For angular noise the discontinuity of the transition is slurred by finite size effects and it only becomes discontinuous when the size of the system is sufficiently large.

Recent investigations, [23], of the nature of the transition in the SPP systems confirm the results of Ref. [21] using the simulation parameters used in Ref. [22]. It was shown that the use of an updating rule, slightly different from the original Vicsek's model, led to a change in the nature of the transition obtained in the simulations, [22]. Moreover, it turns out that in the large velocity regime, due to emergence of a numerical artifact, it is not possible to make any conclusions about the nature of the transition.

Extensions of Vicsek's model have been proposed which consider particles with varying speeds and different types of noise. These extensions include external regular and stochastic force fields and/or interparticle attractive and repulsive forces [18, 22, 24, 25].

The dynamics of Vicsek's model was also investigated in the framework of graph theory. In Ref. [26] the spontaneous emergence of ordered motion has been studied in terms of so called control laws which govern the dynamics of the particles. Generalizations of the control laws were considered in Refs. [27, 28]. In Ref. [28] it was shown that the organized motion of SPP with the control laws depending on the relative orientations of the velocities and relative spacing, can be of two types only: parallel and circular motion. The stability properties of these discrete updating rules (including Vicsek's model) and the dynamics they describe were considered using Lyapunov's theory in Refs. [26, 27, 29, 30].

The discrete description by Vicsek et al. gave an impulse to the development of continuous approaches. One may distinguish two main classes of approaches. The first class consists of models which are based on the Navier-Stokes equation for a dissipative fluid formed by microorganisms floating in a medium. The models studied in Refs. [31, 32] include, in addition to the usual pressure term, phenomenological terms which generate a *spontaneous*, as opposed to *induced* by an external field, transition to a state of ordered motion observed in experiments and in simulations [21]. Terms describing the self-propelling nature of the system are also added. In Ref. [21] the pressure and viscous terms are incorporated into the model side by side with the driving force and the friction caused by the interaction with the environment and other self-propelling particles. The nature of the terms introduced in such a continuous description is not clear. The origin of the coherent motion is the alignment of the directions of particles' velocities. The driving force and the friction are of secondary importance in this context.

The inclusion of additional phenomenological terms in [32, 33] breaks the Gallilean invariance of the Navier-Stokes equation and generates the spontaneous transition to a state of ordered motion. Their inclusion od additional terms is based on the analogy with Landau's theory of equilibrium continuous phase transitions. In particular, the total momentum playes the role of an order parameter and is used to describe the noise-driven transition from an ordered to a disordered state of motion. A subsequent renormalization group treatment in Ref. [31] supports the existence of a nontrivial critical regime and gives the derivation of the critical exponents for the velocity correlation functions. The phenomenological terms included are based on symmetry considerations. The underlying physical arguments and a specific connection with the microscopic (discrete) interactions between the particles are not clearly established. An attempt to derive such phenomenological equations from the kinetic equation is made in [34].

One of the advantages of continuum models is that analytic solutions can often be constructed, while this is impossible for the discrete algorithms. This makes it possible to avoid difficulties in distinguishing physical factors from artificial numerical ones.

The second class of approaches contains models which describe the swarming behaviour of SPP by inclusion of attractive and repulsive interactions. Such a model, based on the diffusion-advection-reaction equation with nonlocal attractive and repulsive terms, is suggested in Ref. [35]. Their model gives a compact (with sharp edges) aggregation of SPP with a constant density. This, according to the authors, is biologically reasonable.

Another continuous model of this class of approaches for the behaviour of living organisms with nonlocal social interactions is proposed in Refs. [36, 37]. There the kinematic rule for the velocity field contains the density dependent drift and the nonlocal attraction and repulsion contributions. For the 2-dimensional case of an incompressible fluid with the motion of the particles normal to the population gradients, a flow of a constant density with a compact support is obtained.

All the proposed continuum models do not reflect the essence of the CV model with the proposed rule to update the directions of the velocities. They contain a lot of additional terms, which are not responsible for the collective behaviour of the self-propelling particles.

The aim of this thesis is to construct and to study hydrodynamic models for the dynamics of SPP which only contain terms needed to update the direction of the velocities, this in analogy with the updating rules used in the original CV model. As the self-propelling force is chosen such that it cancels frictional energy losses, there is no need to have either of these terms explicitly in the hydrodynamic model. Such models are expected to open new opportunities for the understanding of the collective behaviour of self-propelling particles.

1.2 Outline of the thesis

We have the following aims with the present investigation:

The general aim of the thesis is to construct models to describe the collective behaviour of self-propelling particles observed in nature using nonholonomic constraints. This is quite a challenging task because of the non-potential character of the interactions, which complicates the problem and excludes the possibility to use the standard methods of Hamiltonian dynamics.

As it is mentioned above, the pioneering work by Viscek et al. plays an essential role in a lot of existing approaches for systems of self-propelling particles. It is representative for all the models based on the discrete description.

In connection with this, a further aim is to obtain a continuous analogue of the CV algorithm. There are no known continuous approaches which reflect the essence of Vicsek's model.

In the present thesis a continuum model for the collective behaviour of self-propelling particles, based on the physical properties of the CV model, is proposed. In this model the number of particles and the magnitudes of the particle velocities is conserved. A rule to update the orientation of the velocities is formulated.

In this thesis a hydrodynamic model is proposed with conserved total number of particles and kinetic energy.

In Chapter 2 the hydrodynamic model is formulated and its properties are considered. In particular, the 2-dimensional stationary linear and vortical flows of self-propelling particles are obtained. The influence of the presence of non-potential interactions is shown by calculating the evolution of the vorticity and the velocity circulation, which is found to be not conserved. The stability of the ordered motion with respect to noise inclusion is investigated.

In Chapter 3 the class of all possible planar stationary flows is determined. Using the conformal representation, it is shown that the flows with translational and axial symmetry are the only flows possible in the model. Finite flocking behaviour for different models of the angular velocity field is obtained.

In Chapter 4 the stability properties of the obtained linear and vortical stationary flows with respect to small perturbations are considered.

In Chapter 5 a derivation of the continuous model proposed in Chapter 2 from the original discrete algorithm by Vicsek et al. is given. The averaging procedure of the discrete equations of motion is introduced and the corresponding hydrodynamic equations are obtained.

In Chapter 6 nonstationary flows of the self-propelling particles are considered. Linear and radial 2-dimensional flows are obtained and their properties are discussed.

In the last section a summary and conclusion are given.

Chapter 2

Hydrodynamics of systems of self-propelling particles

The dynamics of systems of self-propelling particles with kinematic constraints on the velocities is considered. A continuum model for a discrete algorithm used in works by Vicsek *et al.* (T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen and O. Shochet, Phys. Rev. Lett., **75**, 1226 (1995)) is proposed. For a case of planar geometry finite flocking behaviour is obtained. The circulation of the velocity field is found not to be conserved. The stability of ordered motion with respect to noise is discussed.

This chapter is an extension of the paper by V. L. Kulinskii , V. I. Ratushnaya, A. V. Zvelindovsky, D. Bedeaux, "Hydrodynamic model for a system of self-propelling particles with conservative kinematic constraints", *Europhys. Lett.* **71**, 207 (2005).

2.1 Introduction

In this chapter we formulate a hydrodynamic model which can be considered as a continuous extension of the discrete dynamic automaton proposed in Ref. [17] for the SPP system. It manifestly takes into account the local conservation laws for the number of particles and the kinetic energy. The self-propelling force and the frictional force are assumed to balance each other. In the first section of this chapter we give a brief description of Vicsek's model and discuss its results. Then we propose our continuous model with imposed constraints of conservation of kinetic energy and number of particles. We investigate its physical properties in the following sections. In particular, we introduce an angular velocity field, which describes the non-potential (nonholonomic) character of interactions between self-propelling particles. Using a model for this field, which is linear in the velocity and density fields and their gradients, we obtain two stationary regimes of motion of self-propelling particles. The first one is a linear flow with a density distribution changing along the normal to the flow direction. The second type of stationary flows is an axially symmetric or vortical flow. For this flow the velocity profile is found to be determined by the density distribution, which can be chosen relatively freely. In addition this solution describes finite flocking behaviour of SPP, when the density is small (zero) outside some region. In Section 5 we consider the vorticity and the velocity circulation in our model and discuss the differences between the classical ideal fluids and our hydrodynamic model with non-potential interactions between the particles. In the last section of this chapter we investigate the influence of noise.

2.2 Discrete description of the behaviour of self-propelling particles

2.2.1 Vicsek's model

In order to investigate flocking phenomenon in the biological systems Vicsek et al. have proposed a numerical algorithm for the velocities of the self-propelling particles. Originally their 2-dimensional model consisted of particles moving in a square box of linear size L with periodic boundary conditions [17]. Each ith particle is characterized by the position vector $\mathbf{r}_i(t)$ and the velocity $\mathbf{v}_i(t)$, whose direction is given by the angle $\theta_i = Arg(\mathbf{v}_i)$. The absolute value of the velocity of each particle is kept constant:

$$|\mathbf{v}_i| = v_0 = const. \tag{2.1}$$

According to Vicsek's rule at each time step each particle adjusts the direction of its velocity to the average velocity in the neighbourhood $S(i)$:

$$\theta_i(t + \Delta t) = \langle \theta(t) \rangle_{S(i)} + \xi, \tag{2.2}$$

where Δt is the time between the updates of the positions and velocity directions, and ξ is a noise contribution with uniform distribution in the interval $[-\eta/2, \eta/2]$. Here η is an amplitude of the noise and

$$\langle \theta \rangle_{S(i)} = Arg\left(\sum_{\substack{j \\ \mathbf{r}_j \in S(i)}} \mathbf{v}_j \right). \tag{2.3}$$

The position of ith particle at each time step is determined as follows:

$$\mathbf{r}_i(t + \Delta t) = \mathbf{r}_i(t) + \mathbf{v}_i(t)\,\Delta t. \tag{2.4}$$

The local surrounding $S(i)$ is defined either as a circle of radius R or as a lattice cell of length R, assuming that each particle interacts with all the particles situated in the same cell and in the eight neighbouring cells of the square lattice [17],[21],[38].

2.2.2 Results

Starting from a random distribution of the positions and the orientations of the velocities (see Figure 2.1 (a)) simulations for $N = 300$ [17], $N = 4000, 16000$ [38] and $N = 10^4 - 10^5$ [21] particles for varying values of the density and noise amplitude are performed. At small densities and noise the particles form groups coherently moving in random directions (Figure 2.1 (b)). For higher densities and noise there is some correlation between the particles (Figure 2.1 (c)). At higher density and small noise a transition to ordered motion is observed (Figure 2.1 (d)).

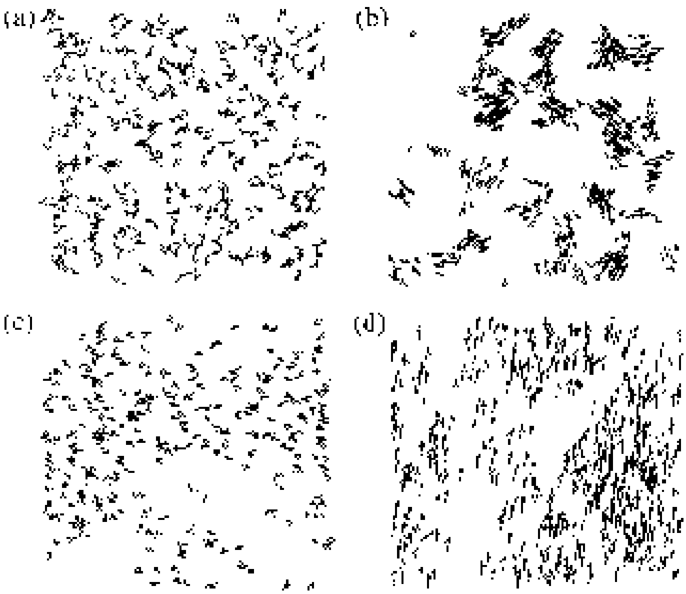

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Figure 2.1: Velocity fields obtained in the CV algorithm at different densities and noise, [17]. $(a)\, t = 0, L = 7, \eta = 2$; $(b)\, L = 25, \eta = 0.1$; $(c)\, L = 7, \eta = 2$; $(d)\, L = 5, \eta = 0.1$.

2.3 Hydrodynamic model

2.3.1 Formulation of the model

The discrete configuration updating algorithm described above works only on the directions of the particles' velocities but keeps their absolute value constant. When

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the time is continuous a particle subjected only to centripetal acceleration and also the number of particles is constant. Thus the proper equation of motion of a particle in the continuous time version of the CV algorithm is:

$$\frac{d}{dt}\mathbf{v}_i = \boldsymbol{\omega}_i \times \mathbf{v}_i, \tag{2.5}$$

where $\boldsymbol{\omega}_i$ is the "angular velocity" of i-th particle, which is chosen such that the velocity aligns with its neighbours.

We formulate a hydrodynamic model corresponding to the CV algorithm that is based on the following equations:

$$\begin{aligned} \frac{dN}{dt} = \frac{d}{dt}\int_V n(\mathbf{r},t)\; dV &= 0, & (2.6)\\ \frac{dT}{dt} = \frac{1}{2}\frac{d}{dt}\int_V n(\mathbf{r},t)\,\mathbf{v}^2(\mathbf{r},t)\; dV &= 0, & (2.7)\end{aligned}$$

where $n(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r},t)$ are the number density and the Eulerian velocity respectively. The volume V moves along with the velocity field. The first condition is the conservation of number of particles N. We can rewrite this condition in the differential form

$$\frac{\partial n(\mathbf{r},t)}{\partial t} + \mathrm{div}\,\left(n(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)\right) = 0. \tag{2.8}$$

This is a continuity equation. The second constraint, Eq. (2.7), means that the kinetic energy of a Lagrange particle is conserved. For the co-moving derivative Eq. (2.7), together with Eq. (2.6), leads to

$$\frac{d}{dt}|\mathbf{v}(\mathbf{r},t)|^2 = 0. \tag{2.9}$$

For the details see Appendix A.1.

Using Eq. (2.6) and Eq. (2.7), it can be shown that a field $\boldsymbol{\omega}(\mathbf{r},t)$ exists such that the Eulerian velocity, $\mathbf{v}(\mathbf{r},t)$, satisfies:

$$\frac{d}{dt}\mathbf{v}(\mathbf{r},t) = \boldsymbol{\omega}(\mathbf{r},t)\times\mathbf{v}(\mathbf{r},t)\,. \tag{2.10}$$

This equation can be considered as the continuous analogue of the conservative dynamic rule used by Vicsek et al. [17].

We propose the following "minimal" model for the field of the angular velocity $\boldsymbol{\omega}(\mathbf{r}, t)$ which is linear in spatial gradients of the fields $n(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$:

$$\boldsymbol{\omega}(\mathbf{r}, t) = \int K_1(\mathbf{r} - \mathbf{r}')\, n(\mathbf{r}', t)\ \text{rot}\ \mathbf{v}(\mathbf{r}', t)\, d\mathbf{r}'. \tag{2.11}$$

There are other possible choices like

$$\boldsymbol{\omega}(\mathbf{r}, t) = \int K_2(\mathbf{r} - \mathbf{r}')\, \nabla n(\mathbf{r}', t) \times \mathbf{v}(\mathbf{r}', t)\, d\mathbf{r}' \tag{2.12}$$

and combinations of the two. The $\boldsymbol{\omega}$-field has the proper pseudovector character. The averaging kernels $K_1(\mathbf{r} - \mathbf{r}')$ and $K_2(\mathbf{r} - \mathbf{r}')$ should naturally decrease with the distance in realistic models. They sample the density and the velocity around $\mathbf{r}$ in order to determine $\boldsymbol{\omega}(\mathbf{r}, t)$. The detailed derivation of the above continuous equations from the discrete model based on the automaton proposed by Vicsek et al. [17, 18, 21] will be given in Chapter 5. In this chapter we consider a case when the angular velocity is determined by the rotation of the velocity field only, i.e. $K_2(\mathbf{r} - \mathbf{r}') = 0$.

2.3.2 Solitary wave solutions

The models based on equations (2.8), (2.10) and (2.11) allow solutions of uniform motion in the form of a solitary packet:

$$n(\mathbf{r}, t) = n_0(\mathbf{r} - \mathbf{v}_0\, t) \tag{2.13}$$

with $\mathbf{v}_0$ independent of position and time and an arbitrary density distribution $n_0(\mathbf{r})$. Such solutions are obtained from a static solution, $\mathbf{v} = \mathbf{0}$, with an arbitrary $n_0(\mathbf{r})$ by a Gallilean transformation. The contribution to $\boldsymbol{\omega}$ due to K_1 is zero for an arbitrary density distribution n_0. The contribution due to K_2 is zero for density distributions n_0 which only depend on the position in the $\mathbf{v}_0$ direction. In this second case it follows from the continuity equation that n_0 should be everywhere constant. The density distribution n_0 should be chosen to ensure the finiteness of particle number and, correspondingly, the total kinetic energy.

Note that solutions of the same kind, with compact support, were found analytically in Ref. [36] and observed in simulations [22]. Only solutions with constant population density were discussed in Ref. [36] as a specific case in their nonlocal model. Note that such solutions exist not only in nonlocal case but also for the local model which we consider below. The existence of such solutions in our case is a direct consequence of the non-potential character of the model. Because of the correspondence with the CV algorithm, the pressure gradient term caused by the interparticle forces was not included in Eq. (2.10). The solitary solutions given by Eq. (2.13) have neutral stability with respect to number density perturbations. To first order in small deviations of density and velocity fields the static solutions mentioned above also show neutral stability: the deviations grow linearly for small t.

2.4 Stationary flows in the local hydrodynamic model

2.4.1 Stationary regimes of motion

In general stationary regimes of motion can be determined as follows. We can rewrite Eq. (2.10) in the following form:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \frac{\partial \mathbf{W}}{\partial t} = \mathrm{rot}\,(\mathbf{v} \times \mathbf{W})\,, \tag{2.14}$$

where $\mathbf{W}(\mathbf{r}, t) = \mathrm{rot}\,\mathbf{v} - \boldsymbol{\omega}$. For the details see Appendix A.2. Thus it follows that if $\mathbf{W}(\mathbf{r}, t) = \mathbf{0}$, then $\partial \boldsymbol{\omega}/\partial t = \mathbf{0}$ and therefore $\boldsymbol{\omega} = \mathrm{rot}\,\mathbf{v}$ is independent of the time. Such states are naturally interpreted as stationary translational $\boldsymbol{\omega} = \mathbf{0}$ or rotational $\boldsymbol{\omega} \neq \mathbf{0}$ regimes of motion. For model (2.11) together with $\mathbf{W} = \mathbf{0}$ we get the integral equation:

$$\int K(\mathbf{r} - \mathbf{r}')\, n(\mathbf{r}')\ \mathrm{rot}\ \mathbf{v}(\mathbf{r}')\, d\,\mathbf{r}' = \mathrm{rot}\,\mathbf{v}(\mathbf{r})\,, \tag{2.15}$$

which determines such states. Equation (2.15) gives stationary vortical motion, represented by a vector field with $|\mathbf{v}| = const$. From here it follows that the vorticity of the velocity field is an eigenstate of the integral operator with $n(\mathbf{r})$ as the corresponding weight factor. It should be noted that these stationary states do not exhaust all stationary vortical states.

2.4.2 Local hydrodynamic model

We further scale K_1 by dividing by $\left|\int K_1(\mathbf{r})\, d\mathbf{r}\right|$ and similarly scale the density by multiplying with the same factor. Furthermore we restrict our discussion to the simple case of a planar geometry with the averaging kernel in Eq. (2.11) as a δ-functional:

$$K_1(\mathbf{r}-\mathbf{r}') = s_1\, \delta(\mathbf{r}-\mathbf{r}'), \tag{2.16}$$

where K_1 is now the scaled kernel which integrates out to plus or minus one, i.e. $s_1 = \pm 1$. We will call this the *local hydrodynamic model (LHM)* . In such a case one may consider such a continuum model as the particular case of the general hydrodynamic model considered in Ref. [31] obeying the conservation rules of the CV algorithm. The viscous Navier-Stokes term is absent because of dissipative free character of the dynamics. In fact for the CV model the energy of the chaotic motion at low noise level still can be transformed into the ordered motion, while the viscosity for the ordinary fluid transmits the energy of the ordered motion into the heat. For this case of the local hydrodynamic model Eqs. (2.8) and (2.10) take the form:

$$\frac{\partial n(\mathbf{r},t)}{\partial t} + \mathrm{div}\,(n(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)) = 0, \tag{2.17}$$

$$\frac{d\mathbf{v}(\mathbf{r},t)}{dt} = s_1\, n(\mathbf{r},t)\,(\mathrm{rot}\,\mathbf{v}(\mathbf{r},t)) \times \mathbf{v}(\mathbf{r},t). \tag{2.18}$$

The last equation, Eq. (2.18), can be obtained as a special case of the corresponding one in Ref. [31]. Note that the local model with a δ-kernel in Eq. (2.12) may be identified with the *rotor chemotaxis* (i.e. caused by chemical field) force introduced in Ref. [18] if one takes into account a simple linear relation between the attractant density (field of food concentration) and the number density of particles.

2.4.3 Stationary vortical flow

The parameter s_1 of the local model (2.18) distinguishes different physical situations concerning the microscopic kinematic constraint. Using an electrodynamic analogy, this parameter can be interpreted as the sign of "charge" which determines the direction of the Lorentz force due to vorticity. From geometrical reasoning following from Eq. (2.5) and interpretation of rot $\mathbf{v}$ as the local rotational velocity for the vector field $\mathbf{v}(\mathbf{r}, t)$ we can conclude that $s_1 = +1$ corresponds to the CV algorithm, where the velocity tends to align with the local average. The case $s_1 = -1$ corresponds to a disalignment tendency at the microscopic level.

To see this we search for stationary axially symmetric (vortical) solutions of Eqs. (2.17) and (2.18) in the form $n = n(r)$, $\mathbf{v} = v_\varphi(r)\,\mathbf{e}_\varphi$. We obtain (see Appendix A.3):

$$v_\varphi(r) = \frac{C_{\mathrm{st}}}{2\pi r} \exp\left[s_1 \int_{r_0}^{r} \frac{1}{r'\, n(r')}\, dr' \right] \tag{2.19}$$

with the continuity equation (2.17) being satisfied trivially and r_0 being the cut-off radius of the vortex core. The constant C_{st} is determined by the circulation of the core

$$\oint_{r=r_0} \mathbf{v}\, d\mathbf{l} = C_{\mathrm{st}} \,. \tag{2.20}$$

The spatial character of the solution strongly depends on the parameter s_1. If $s_1 = -1$ infinitely extended distributions for $n(r)$ are allowed, e.g. $n(r) \propto r^{-\alpha}$, $\alpha > 0$. They lead to localized vortices with exponential decay of angular velocity. If $s_1 = +1$ only compact distributions, i.e. $n(r) \equiv 0$ outside some compact region, are consistent with the finiteness of the total kinetic energy and the number of particles. As an example we give

$$n(r) = \sqrt{\frac{r_0}{R - r}}\; \theta(r - r_0)\; \theta(R - r) \,, \tag{2.21}$$

where $\theta(x - x_0)$ is the Heaviside step function. Substituting Eq. (2.21) into Eq. (2.19), we obtain

$$v_\varphi(r) = \frac{C_{\mathrm{st}}}{2\pi r} \exp\left[2\sqrt{\frac{R}{r_0}} \left(\sqrt{1 - \frac{r}{R}} - \operatorname{arctanh} \sqrt{1 - \frac{r}{R}} \right)\Bigg|_{r_0}^{r} \right]. \tag{2.22}$$

The corresponding component of the velocity $\mathbf{v} = v_\varphi\, \mathbf{e}_\varphi$ and the density are shown in Figures 2.2, 2.3. The stability properties of the stationary solutions (2.19) are discussed in Chapter 4.

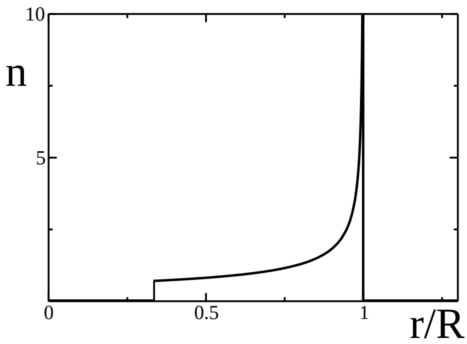

Figure 2.2: Density $n\,(r/R)$ given by Eq. (2.21) in the vortex of the local model at $r_0/R = 1/3$.

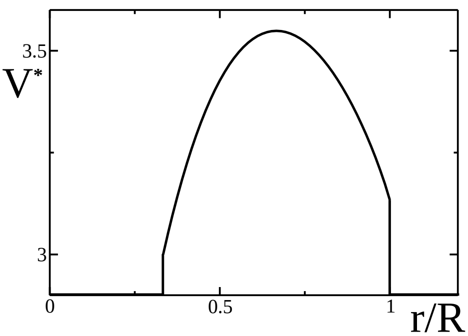

Figure 2.3: Velocity $V^*\,(r/R) = 2\pi R v_\varphi\,(r)\,/C_{st}$ in the vortex of the local model at $r_0/R = 1/3$.

2.5 Vorticity of the velocity field

The hydrodynamics of the model under consideration differs essentially from the potential dynamics of ideal fluids [39] due to the non-potential character of the equation of motion (2.10). The circulation in the fluid subjected to the equations (2.17) and

(2.18) is not conserved. This is not a case for classical ideal fluids where the circulation is an integral of motion. It can be shown that the momentum does not conserve either. Taking the rotation on both sides of Eq. (2.18), one obtains the following equation for the vorticity in a case of planar geometry (see Appendix A.4):

$$\operatorname{rot}\frac{d\mathbf{v}}{dt} = -s_1\left(\frac{\partial n}{\partial t}\operatorname{rot}\mathbf{v} - n\,(\mathbf{v}\cdot\nabla)\operatorname{rot}\mathbf{v}\right). \tag{2.23}$$

This implies that for $s_1 = +1$ the vorticity is damped by compression along the flow and therefore such a flow is stable with respect to vortical perturbations. For $s_1 = -1$ the vorticity is damped by expansion. The first term on the right-hand side of Eq. (2.23) shows the influence of compression on the evolution of the vorticity. The second term represents the modified spatial transfer of the vorticity along the flow. Since this term can be excluded locally by the choice of a local reference frame, which moves along with the flow, we consider the first term as the main source of the vorticity. In such an approximation we can write

$$\operatorname{rot}\frac{d\mathbf{v}}{dt} = -s_1\frac{\partial n}{\partial t}\operatorname{rot}\mathbf{v}. \tag{2.24}$$

The circulation is defined by

$$C\,(t) = \oint\limits_{\mathcal{L}}\mathbf{v}\cdot d\mathbf{l} = \int\limits_{S}\operatorname{rot}\mathbf{v}\cdot d\mathbf{S}, \tag{2.25}$$

where the integration contour $\mathcal{L}$ and the corresponding surface area S move along with the velocity field. The time derivative of the circulation is

$$\frac{d}{dt}C = \int\limits_{S}\operatorname{rot}\frac{d\mathbf{v}}{dt}\cdot d\mathbf{S}\,. \tag{2.26}$$

Thus the circulation does not conserve in contrast to the ideal fluid model. The details of the derivation are given in Appendix A.5.

The total momentum $\mathbf{P} = \int n\,\mathbf{v}\,dV$ does not conserve in our local model:

$$\frac{d}{dt}\mathbf{P} = \int s_1\,n^2\operatorname{rot}\mathbf{v}\times\mathbf{v}\,dV\,. \tag{2.27}$$

From Eq. (2.27) it follows that the damping of vortical part of the velocity leads to the formation of the states of uniform motion with $\mathbf{P}$ = const like those given above by Eq. (2.13) with corresponding kinetic energy.

2.6 Noise

Inclusion of stochastic noise in our model can be done in a way analogous to that used in Ref. [21]:

$$\omega(\mathbf{r},t) = \omega_0(\mathbf{r},t) + \delta\omega(\mathbf{r},t), \tag{2.28}$$

where $\omega_0 = s_1 n \operatorname{rot}\mathbf{v}$ is the same contribution as before and $\delta\omega$ is the stochastic contribution. These fluctuations lead to fluctuations of the density and velocity fields. Replacing $\partial n/\partial t$ by an average value $1/\tau$ plus a fluctuating contribution $\delta L(t)$ in Eq. (2.24) one obtains for the above described local model with $s_1 = +1$:

$$\frac{d}{dt}C = -\left(\frac{1}{\tau} + \delta L\right)C, \tag{2.29}$$

We consider the Gaussian white noise approximation:

$$\langle\, \delta L(t)\, \delta L(t')\, \rangle = 2\,\Gamma\,\delta(t - t'). \tag{2.30}$$

The stochastic equation (2.29) has the solution

$$C(t) = C_0 \exp\left(-\frac{t}{\tau}\right) \exp\left(-\mathcal{W}(t)\right), \tag{2.31}$$

where $\mathcal{W}(t) = \int\limits_0^t \delta L(t')\, dt'$ is the Wiener process [40]. Averaging over the realizations of the stochastic process, we get for the averaged evolution of the vorticity:

$$\langle\, C(t)\, \rangle = C_0 \exp\left(-\frac{t}{\tilde{\tau}}\right), \quad \tilde{\tau} = \frac{\tau}{1 - \tau\Gamma}, \tag{2.32}$$

where $\tilde{\tau}$ is the relaxation time of the circulation in the system. The details of the derivation are given in Appendix A.6. For large enough noise, $\tau\,\Gamma > 1$, the system becomes unstable. For $\tau\,\Gamma \leq 1$ the system is stable and the circulation decays to zero. When $\tau\Gamma \to 1$ the relaxation time $\tilde{\tau}$ goes to infinity, a result similar to critical slowing down near the critical point. The results of Ref. [22] and the results of the present chapter allow to conclude that the finite size effects are important when the CV dynamic updating rule is perturbed by noise. Apparently, the reason is the essential nonlinearity of the system.

2.7 Conclusions

In conclusion we have constructed a continuum SPP model with particle number and kinetic energy conservation. We found in 2D that vortical solutions exist for the model and that they show a finite flocking behaviour. The density profile could be chosen more or less freely. For one of such choices the velocity profiles were found to resemble those given in Ref. [19], using a different procedure. The kinematic constraints were found to lead to a circulation which was not conserved. The influence of noise on the stability of the ordered state of the system was discussed.

Chapter 3

Stationary regimes of motion in the local hydrodynamic model

In Chapter 2 we proposed a continuum model for the dynamics of systems of self-propelling particles with kinematic constraints on the velocities. In this chapter we prove that the only types of the stationary planar solutions in the model are either of translational or axial symmetry of the flow. Within the proposed model we differentiate between finite and infinite flocking behaviour by the finiteness of the kinetic energy and number of particles.

This chapter is based on a paper by V. I. Ratushnaya, V. L. Kulinskii, A. V. Zvelindovsky, D. Bedeaux, "Hydrodynamic model for the system of self-propelling particles with conservative kinematic constraints; Two dimensional stationary solutions", *Physica A* **366**, 107 (2006).

3.1 Introduction

In this chapter we determine class of possible stationary flows of the hydrodynamic model formulated in Chapter 2. Here we consider a general case for the angular velocity field given by

$$\boldsymbol{\omega}(\mathbf{r},t) = \int K_1(\mathbf{r}-\mathbf{r}')\; n(\mathbf{r}',t)\operatorname{rot}\mathbf{v}(\mathbf{r}',t)\; d\mathbf{r}' \\ + \int K_2(\mathbf{r}-\mathbf{r}')\,\nabla n(\mathbf{r}',t)\times\mathbf{v}(\mathbf{r}',t)\,d\mathbf{r}' \,. \tag{3.1}$$

As in the previous chapter we consider local hydrodynamic model determined by averaging kernels as δ-functionals:

$$K_j\,(\mathbf{r}-\mathbf{r}') = s_j\,\delta(\mathbf{r}-\mathbf{r}')\,, \qquad \text{where} \quad j = 1 \text{ or } 2\,. \tag{3.2}$$

In Chapter 2, where we only considered K_1, we scaled K_1 by dividing by $|s_1|$ and the density n by multiplying with $|s_1|$. This made it then possible to restrict the discussion to s_1 is plus or minus one. The disadvantage of this scaling procedure is that it changes the dimensionality of K_j and n. For two kernels it becomes impractical. We note that s_j is given by

$$s_j = \int K_j(\mathbf{r})\,d\mathbf{r}. \tag{3.3}$$

For the local model Eq. (3.1) reduces to:

$$\boldsymbol{\omega}(\mathbf{r},t) = s_1\, n(\mathbf{r},t)\operatorname{rot}\mathbf{v}(\mathbf{r},t) \; + s_2\nabla n(\mathbf{r},t)\times\mathbf{v}(\mathbf{r},t)\,. \tag{3.4}$$

and Eq. (2.10) for the velocity becomes

$$\frac{d}{dt}\mathbf{v}(\mathbf{r},t) = s_1\, n(\mathbf{r},t)\operatorname{rot}\mathbf{v}(\mathbf{r},t)\times\mathbf{v}(\mathbf{r},t) + s_2\,(\nabla n(\mathbf{r},t)\times\mathbf{v}(\mathbf{r},t))\times\mathbf{v}(\mathbf{r},t)\,. \tag{3.5}$$

In the following section we will show that the only stationary solutions in the local hydrodynamic model with $\boldsymbol{\omega}$ -field given by Eq. (3.4) are either the solutions of uniform motion (see Eq. (2.13)) or the axially symmetric planar solution which will be considered in detail in the following section.

In the third section we investigate the properties of the stationary axially symmetric solutions of the local hydrodynamic model for some special cases.

3.2 Possible types of stationary states in the local hydrodynamic model (LHM)

3.2.1 Conformal representation

The equations of motion to be solved are Eqs. (2.8) and (3.5). In order to find a class of 2D stationary solutions we consider this problem in a generalized curvilinear orthogonal coordinate system (u, v), which can be obtained from the Cartesian one (x, y) by some conformal transformation of the following form:

$$u + iv = F(z), \tag{3.6}$$

where $F(z)$ is an arbitrary analytical function of $z = x + iy$. In a curvilinear orthogonal coordinate system the fundamental tensor has a diagonal form, $g_{ik} = g_{ii}\delta_{ik}$, where the indices i, j are either u or v. The square of linear element in conformal coordinates is

$$d\mathbf{s}^2 = \frac{1}{D(u,v)}\left(du^2 + dv^2\right), \tag{3.7}$$

where

$$D(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \frac{1}{\sqrt{g}} \tag{3.8}$$

is the Jacobian of the inverse conformal transformation from the arbitrary curvilinear orthogonal to Cartesian coordinates, $(u, v) \to (x, y)$. Furthermore $g = g_{uu}g_{vv}$ is the determinant of the metric tensor g_{uv}. For a conformal transformation $g_{vv} = g_{uu} = 1/D$.

The differential operations are given by the following expressions [41]:

$$\nabla\phi = \sqrt{D}\left(\mathbf{e}_u\frac{\partial\phi}{\partial u} + \mathbf{e}_v\frac{\partial\phi}{\partial v}\right), \tag{3.9}$$

$$\operatorname{div}\mathbf{a} = D\left[\frac{\partial}{\partial u}\left(\frac{a_u}{\sqrt{D}}\right) + \frac{\partial}{\partial v}\left(\frac{a_v}{\sqrt{D}}\right)\right], \tag{3.10}$$

$$\operatorname{rot}\mathbf{a} \;=\; D\left[\frac{\partial}{\partial u}\left(\frac{a_v}{\sqrt{D}}\right)-\frac{\partial}{\partial v}\left(\frac{a_u}{\sqrt{D}}\right)\right], \tag{3.11}$$

$$\Delta\phi \;=\; D\left[\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}\right], \tag{3.12}$$

where ϕ and $\mathbf{a} = a_u\mathbf{e}_u + a_v\mathbf{e}_v$ are the arbitrary scalar and vector fields respectively. Here $\mathbf{e}_u$ and $\mathbf{e}_v$ are orthonormal base vectors in the directions of increasing u and v respectively. These base vectors are functions of the coordinates u and v. The projections of the vector field $\mathbf{a}$ on these directions are $a_u = \mathbf{a}\cdot\mathbf{e}_u$ and $a_v = \mathbf{a}\cdot\mathbf{e}_v$. Using Eqs. (3.9)-(3.12) for the velocity field given by $\mathbf{v} = v_u\,\mathbf{e}_u + v_v\,\mathbf{e}_v$, one obtains:

$$(\mathbf{v}\cdot\nabla)\;\mathbf{v} = \sqrt{D}\left[v_u\frac{\partial\,(v_u\mathbf{e}_u)}{\partial u}+v_v\frac{\partial\,(v_u\mathbf{e}_u)}{\partial v}+v_u\frac{\partial\,(v_v\mathbf{e}_v)}{\partial u}+v_v\frac{\partial\,(v_v\mathbf{e}_v)}{\partial v}\right], \tag{3.13}$$

$$(\operatorname{rot}\mathbf{v})\times\mathbf{v} = D\left[\frac{\partial}{\partial u}\left(\frac{v_v}{\sqrt{D}}\right)-\frac{\partial}{\partial v}\left(\frac{v_u}{\sqrt{D}}\right)\right](-v_v\mathbf{e}_u+v_u\mathbf{e}_v)\,, \tag{3.14}$$

$$(\nabla n\times\mathbf{v})\times\mathbf{v} = \sqrt{D}\left[\frac{\partial n}{\partial u}\,v_v-\frac{\partial n}{\partial v}\,v_u\right](-v_v\mathbf{e}_u+v_u\mathbf{e}_v)\,, \tag{3.15}$$

$$\operatorname{div}(n\mathbf{v}) = D\left[\frac{\partial}{\partial u}\left(\frac{n\,v_u}{\sqrt{D}}\right)+\frac{\partial}{\partial v}\left(\frac{n\,v_v}{\sqrt{D}}\right)\right] = 0. \tag{3.16}$$

3.2.2 Class of the stationary flows in the LHM

Substituting Eqs. (3.13)-(3.16) into Eqs. (2.8) and (3.5), we obtain the following system of equations, which determines all possible stationary flows for the LHM:

$$v_u\frac{\partial v_u}{\partial u}+(1-s_1n)\,v_v\frac{\partial v_u}{\partial v}+v_u\,v_v\left(f_3+\frac{s_1n}{2}\frac{\partial\ln D}{\partial v}-s_2\,\frac{\partial n}{\partial v}\right)$$

$$+v_v^2\left(f_4-\frac{s_1n}{2}\frac{\partial\ln D}{\partial u}+s_2\,\frac{\partial n}{\partial u}\right)+s_1n\,v_v\frac{\partial v_v}{\partial u} = 0, \tag{3.17}$$

$$v_v \frac{\partial v_v}{\partial v} + (1 - s_1 n) v_u \frac{\partial v_v}{\partial u} + v_u\, v_v \left(f_2 + \frac{s_1 n}{2} \frac{\partial \ln D}{\partial u} - s_2 \frac{\partial n}{\partial u} \right)$$

$$+ v_u^2 \left(f_1 - \frac{s_1 n}{2} \frac{\partial \ln D}{\partial v} + s_2 \frac{\partial n}{\partial v} \right) + s_1 n\, v_u \frac{\partial v_u}{\partial v} = 0, \tag{3.18}$$

$$\frac{\partial}{\partial u} \left(\frac{n\, v_u}{\sqrt{D}} \right) + \frac{\partial}{\partial v} \left(\frac{n\, v_v}{\sqrt{D}} \right) = 0, \tag{3.19}$$

where

$$\begin{aligned} f_1\,(u,v) &= \frac{\partial \mathbf{e}_u}{\partial u} \cdot \mathbf{e}_v \;\;,\;\; f_2\,(u,v) = \frac{\partial \mathbf{e}_u}{\partial v} \cdot \mathbf{e}_v, \\ f_3\,(u,v) &= \frac{\partial \mathbf{e}_v}{\partial u} \cdot \mathbf{e}_u \;\;,\;\; f_4\,(u,v) = \frac{\partial \mathbf{e}_v}{\partial v} \cdot \mathbf{e}_u. \end{aligned} \tag{3.20}$$

Now let us consider the case of "coordinate flows", when the flow is directed along one of the families of coordinate lines u, v for example along u-coordinate lines and is given by $\mathbf{v} = v_v\,(u,v)\,\mathbf{e}_v$, the density distribution is $n = n\,(u,v)$. The case of a velocity field $\mathbf{v} = v_u\,(u,v)\,\mathbf{e}_u$ is equivalent. From Eq. (3.17) we have:

$$v_v = C_0 \exp \left[\int I\,(u,v)\, du \right], \tag{3.21}$$

where C_0 is constant and

$$I\,(u,v) = \frac{1}{2} \frac{\partial \ln D}{\partial u} - \frac{f_4}{s_1\, n} - \frac{s_2}{s_1} \frac{\partial \ln n}{\partial u}\,. \tag{3.22}$$

Equations (3.18) and (3.19) take the form

$$\frac{\partial v_v}{\partial v} = 0, \tag{3.23}$$

$$\frac{\partial}{\partial v} \left(\frac{n\, v_v}{\sqrt{D}} \right) = 0 \tag{3.24}$$

and lead to

$$n\,(u,v) = h\,(u)\,\sqrt{D\,(u,v)}, \tag{3.25}$$

where $h(u)$ is an arbitrary function of u.
Taking into account that

$$f_4\,(u,v) = \left(\frac{\partial \mathbf{e}_v}{\partial v}\cdot \mathbf{e}_u\right) = -\frac{D}{2}\frac{\partial\,(1/D)}{\partial u} = \frac{1}{2}\frac{\partial \ln D}{\partial u} \tag{3.26}$$

and Eq. (3.25), we obtain:

$$I\,(u,v) = \frac{1}{2}\,\frac{\partial\,\ln D}{\partial\,u}\left[1 - \frac{1}{s_1\,h\,(u)\,\sqrt{D}} - \frac{s_2}{s_1}\left(2\frac{\partial\,\ln h\,(u)}{\partial\,u}\left(\frac{\partial\,\ln D}{\partial\,u}\right)^{-1} + 1\right)\right]. \tag{3.27}$$

Note that as it follows from Eq. (3.23)

$$v_v\,(u,v) = v_v\,(u)\,. \tag{3.28}$$

For the integrand in Eq. (3.21) this implies that

$$I\,(u,v) = I\,(u)\,. \tag{3.29}$$

Therefore from Eqs. (3.27) and (3.29) we can conclude that the function $D\,(u,v)$, which determines the coordinate system, depends only on one variable, $D = D\ (u)$.

In the case of conformal coordinates, defined by the metrics in Eq. (3.7), the Gaussian curvature of the surface is given by [42]:

$$K = \frac{1}{2}\,D\,\Delta\ln D\,. \tag{3.30}$$

For a planar flows the condition $K = 0$ leads to the following

$$\Delta\ln D = 0. \tag{3.31}$$

Using the expression for the Laplacian in the conformal representation (see Eq. (3.12)) and taking into account a fact that $D = D\,(u)$, one finds for (3.31):

$$D = \exp\left[c_1 u + c_2\right], \tag{3.32}$$

where $c_{1,2}$ are arbitrary constants. The case $c_1 = 0$ determines a Cartesian coordinate system, which is related to a linear class of stationary flow. The case $c_1 \neq 0$ determines a polar coordinate system [41], which corresponds to an axially symmetric (or vortical) type of flow.

Finally the velocity field for the LHM, with $s_1 \neq 0$ takes the form:

$$\begin{aligned} v_v(u) \quad = \quad C_0 \exp\left[\frac{1}{2}\int \frac{\partial \ln D}{\partial u}\left[1 - \frac{1}{s_1\, h(u)\sqrt{D}}\right.\right. \\ \left.\left. -\frac{s_2}{s_1}\left(2\,\frac{\partial \ln h(u)}{\partial u}\left(\frac{\partial \ln D}{\partial u}\right)^{-1} + 1\right)\right] du\right], \end{aligned} \tag{3.33}$$

Thus it is proved that for the case $s_1 \neq 0$ the only stationary solutions are those either with planar or axial symmetry of the flow.

The case $s_1 = 0$ is specific because, as it follows from Eqs. (3.17) and (3.18), the velocity field $v_v(u)$ is arbitrary while the density is given by

$$n = -\frac{1}{2\,s_2}\ \ln D + n_0\,. \tag{3.34}$$

The statement about the symmetry of the stationary solutions for such a model is the same as that proved above for the case $s_1 \neq 0$. Note that the parameter $\lambda = s_2/s_1$ can be considered as the weight factor of the rotor chemotaxis contribution.

3.3 The properties of axially symmetric stationary solutions

3.3.1 Finite flocking behaviour of self-propelling particles

In this section we investigate the stationary axially symmetric solutions for different cases of the local hydrodynamic models. In Chapter 2 we considered the $s_2 = 0$ case, which we called the local hydrodynamic model one (LHM1). Other models correspond to cases $s_1 = 0$ and $s_1 = s_2 = s$ which we will call the LHM2 and the LHM12 respectively. We distinguish finite and infinite flocking stationary states for these models. We differentiate between these two cases by the finiteness of two integrals of motion - the total number of particles

$$N = \int n(\mathbf{r}, t)\; dV \tag{3.35}$$

and the kinetic energy

$$T = \frac{1}{2} \int n(\mathbf{r}, t)\, \mathbf{v}^2(\mathbf{r}, t)\; dV. \tag{3.36}$$

The infinite flocking is associated with N infinite but finite T, while finite flocking corresponds to both N and T finite. It is natural to consider the finite flocking rather than infinite flocking behaviour. Note that in the finite flocking behaviour one may consider two cases with respect to the compactness of the $n(\mathbf{r}, t)$. Compactness means that the density has some upper cut-off beyond which it can be put zero.

3.3.2 The properties of the stationary solutions of the LHM1

For the scaling procedure defined in Section 3.1 the stationary vortical flow in the LHM1 is given by

$$v_\varphi(r) = \frac{C_1}{2\pi r} \exp\left(\frac{1}{s_1} \int\limits_{r_0}^{r} \frac{1}{r\, n(r')}\, dr' \right), \tag{3.37}$$

where C_1 has analogous to C_{st} physical meaning.

The spatial behaviour of the solution given in Eq. (3.37) strongly depends on the sign of the parameter s_1. The finiteness of integrals of motion Eq. (3.35) and Eq. (3.36) is guaranteed by either the fast enough decrease of the density $n(r)$ at $r \to \infty$ or its compactness ($n(r)$ as a function has finite support).

Let us consider the finite flocking behaviour (FFB), which is characterized by both N and T finite. If at $r \to \infty$ asymptotically $n(r) \sim r^{-\alpha}$, where $\alpha > 2$, the total number of particles N is finite. Then at such a behaviour of $n(r)$ the total kinetic energy is finite only if $s_1 < 0$.

In the case $s_1 > 0$ the condition of finiteness for the kinetic energy and the total number of particles is fulfilled only if $n(r)$ has finite support. The density profile considered in the previous chapter is one of the possibilities of the flow with FFB. For Eq. (3.37) it is given by

$$n(r) = n_1 \theta(r - r_0)\, \theta(R - r) \sqrt{\frac{r_0}{R - r}}, \quad n_1 > 0, \tag{3.38}$$

Substituting Eq. (3.38) into Eq. (3.37), one obtains:

$$v_\varphi(r) = \frac{C_1}{2\pi r} \exp\left[\frac{2}{s_1 n_1} \sqrt{\frac{R}{r_0}} \left(\sqrt{1 - \frac{r}{R}} - \operatorname{arctanh} \sqrt{1 - \frac{r}{R}} \right)\Bigg|_{r_0}^{r} \right]. \tag{3.39}$$

The corresponding profiles of the velocity $\mathbf{v} = v_\phi\, \mathbf{e}_\phi$ at different ratios R/r_0 are shown in Figure 3.1.

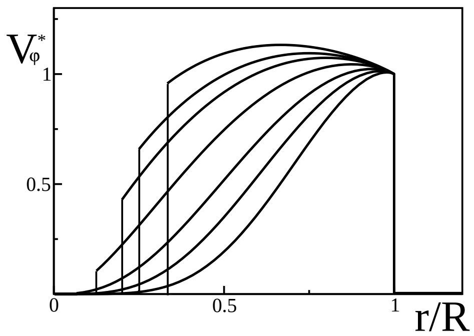

Figure 3.1: Velocity profiles for $\mathrm{V}_\varphi^* = 2\pi R v_\varphi(r/R)$ in the LHM1 at different ratios R/r_0 and $s_1 n_1 = 1$.

Note that for the case considered at $R/r_0 \geq 2$ we get monotonic profiles of the velocity which are similar to those observed in nature [4, 5], numerical experiments [18] and theory using a different approach [19].

The infinite flocking behaviour (IFB) is characterized by N infinite and T finite. For the case $s_1 > 0$ no physical solutions exist with such a behaviour. For the case $s_1 < 0$ slowly decaying density distributions $n(r) \propto r^{-\alpha}$ at $r \to \infty$ with $0 \leq \alpha \leq 2$ are consistent with the finiteness of T. These statements are summed up in Table 3.1.

LHM1	$s_1 > 0$	$s_1 < 0$
FFB ($N < \infty$, $T < \infty$)	compact support	$\alpha > 2$, no compact support
IFB ($N = \infty$, $T < \infty$)	no physical solutions	$0 \leq \alpha \leq 2$

Table 3.1: Properties of the stationary solutions of the LHM1.

3.3.3 The properties of the stationary solutions of the LHM2

For the case $s_1 = 0$ and s_2 finite one may also construct a stationary planar axially symmetric solution. In polar coordinates from Eq. (3.34) we obtain:

$$n(r) = \frac{1}{s_2} \ln \frac{r}{r_0}, \tag{3.40}$$

and one can choose the velocity field $v_\varphi(r)$ arbitrarily. For positive values of s_2 this density is positive for $r > r_0$ and for negative values of s_2 it is positive for $r < r_0$. Thus for positive values of s_2 the density profile becomes

$$n(r) = \frac{1}{s_2} \theta\,(R - r) \ln \frac{r}{r_0} \tag{3.41}$$

and for negative values of s_2 it is

$$n(r) = \frac{1}{|s_2|} \theta\,(r_0 - r) \ln \frac{r_0}{r}. \tag{3.42}$$

The results for finite and infinite flocking behaviour are given in Table 3.2.

LHM2	$s_2 > 0$	$s_2 < 0$
FFB ($N < \infty$, $T < \infty$)	no physical solution	compact support
IFB ($N = \infty$, $T < \infty$)	no physical solution	no physical solution

Table 3.2: Properties of the stationary solutions of the LHM2.

3.3.4 The properties of the stationary solutions of the LHM12

The third case which is expedient to consider is $s_1 = s_2 = s$. In this case, according to Eq. (3.4), the $\boldsymbol{\omega}$ -field is coupled to the number density flux $\mathbf{j} = n\,\mathbf{v}$:

$$\boldsymbol{\omega}(\mathbf{r}, t) = s \;\; \mathrm{rot}\,\mathbf{j} \tag{3.43}$$

so that Eq. (3.5) for the velocity is

$$\frac{d}{dt}\mathbf{v}(\mathbf{r}, t) = s \;\mathrm{rot}\,\mathbf{j} \times \mathbf{v}(\mathbf{r}, t)\,. \tag{3.44}$$

For the axially symmetric stationary planar solution $\mathbf{v} = v_\varphi(r)\,\mathbf{e}_\varphi$ this gives

$$v_\varphi(r) = s\frac{d}{dr}\left[r\,n(r)\,v_\varphi(r)\right]. \tag{3.45}$$

with

$$v_\varphi(r) = \frac{C_2}{2\pi r\,n(r)}\exp\left[\frac{1}{s}\int\limits_{r_0}^{r}\frac{1}{r'\,n\,(r')}\,dr'\right]. \tag{3.46}$$

as a solution. The constant C_2 is determined by the circulation of the core

$$\oint\limits_{r=r_0}\mathbf{v}\cdot d\mathbf{l} = \frac{C_2}{n(r_0)}\,. \tag{3.47}$$

The properties of finite and infinite flocking behaviour for this model are the same as those for the LHM1 (see Table 3.1).

3.4 Conclusions

In this chapter we considered the properties of the 2-dimensional stationary solutions of the LHM proposed in Chapter 2. We established that the only possible stationary solutions in the model are those with translational or axial symmetry. In this respect our continuum model gives results similar with those obtained in the discrete model of Vicsek [17, 18]. The cases of finite and infinite flocking behaviour are considered for different specific types of the LHMs. It is shown that the case $s_1 = 0$ (LHM2) is specific in a sense that there is only one density distribution, for which many velocity profiles can be realized. In general case ($s_1 \neq 0$) one is free to choose axially symmetric density distribution, which the velocity profile depends on (Eqs. (3.25) and (3.33)). Note that in this respect the general case is similar to the LHM1 considered earlier.

Chapter 4

Stability of hydrodynamic flows of self-propelling particles

In this chapter we analyze the stability of the stationary linear and vortical (axially symmetric) hydrodynamic flows, described in Chapter 3. The main result of this chapter is a neutral stability of the stationary solutions within the linear approximation. By comparison of such a situation with that for the Hamiltonian systems we conclude that the main reason for the neutral stability is the dissipative free character of the dynamics.

This chapter is based on a paper by V. I. Ratushnaya, D. Bedeaux, V. L. Kulinskii, A. V. Zvelindovsky, "Stability properties of the collective stationary motion of self-propelling particles with conservative kinematic constraints", *J. Phys. A: Math. Theor.* **40**, 2573 (2007).

4.1 Introduction

In this chapter we investigate the stability of the obtained regimes of motion with respect to small perturbations. In the next section we consider the stability of the planar stationary linear flow with respect to the velocity perturbation directed along the stationary flow and perpendicular to the flow. We show that in both cases the evolution of the perturbations has an oscillatory behaviour, which means that they neither grow nor decay with time. This can be interpreted as neutral stability of the corresponding stationary flow [44]. Also the external pressure term $-\nabla p/n$ can be included into Eq. (2.10) in order to account for potential external forces. In such a case with $s_2 = 0$ there exists the special case of the incompressible flows, $n = const$, when the equations of motion (2.8),(2.10) with (3.4) coincide with those for potential flow of ideal fluids. As it is known [44], in $2D$ geometry such motion is stable in the Lyapunov sense under rather weak restrictions on the initial velocity field profile.

In the third section we consider the stability of the planar vortical motion of SPP with constant velocity and density fields. We find that in this case the linear analysis does not lead to a conclusive answer about the stability of the solution.

4.2 Stability of planar stationary linear flow in the local hydrodynamic model

4.2.1 Stability with respect to a velocity perturbation along the flow

In this section we consider the stability properties of planar stationary linear flow for the local hydrodynamic model with $s_2 = 0$ (LHM1). At the end of the section we will shortly discuss how these results extend to the local hydrodynamic models with $s_1 = 0$ and $s_1 = s_2$. In the LHM1 the stationary linear flow is given by

$$\mathbf{v}_0(\mathbf{r}) = v_0\,\mathbf{e}_x \quad \text{and} \quad n_0(\mathbf{r}) = n_0, \tag{4.1}$$

where v_0 and n_0 are constants.

We consider velocity and the density perturbations of the following form:

$$\mathbf{v}_1(\mathbf{r},t) = v_0\, A_{||}\, e^{i\,\mathbf{k}\cdot\mathbf{r}}\, e^{\alpha_{||} t}\,\mathbf{e}_x \qquad \text{and} \qquad n_1(\mathbf{r},t) = n_0\, B_{||}\, e^{i\,\mathbf{k}\cdot\mathbf{r}}\, e^{\alpha_{||} t}, \tag{4.2}$$

The velocity perturbation chosen is directed along the stationary linear flow. Here $A_{||}, B_{||}$ are constants, $\mathbf{k} = k_x \mathbf{e}_x + k_y \mathbf{e}_y$ is the wave vector and $\alpha_{||}$ is an exponent, which determines the time evolution of the perturbation.
Substituting the solution $\mathbf{v}(\mathbf{r},t) = \mathbf{v}_0 + \mathbf{v}_1(\mathbf{r},t)$, $n(\mathbf{r},t) = n_0 + n_1(\mathbf{r},t)$ into Eqs. (2.8) and (2.10), we obtain the linearized system of equations:

$$\begin{aligned} \frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_0 \cdot \nabla)\,\mathbf{v}_1 &= s_1\, n_0\,(\mathrm{rot}\,\mathbf{v}_1) \times \mathbf{v}_0, &(4.3)\\ \frac{\partial n_1}{\partial t} + \nabla\cdot(n_0 \mathbf{v}_1) + \nabla\cdot(n_1 \mathbf{v}_0) &= 0. &(4.4) \end{aligned}$$

For the perturbations (4.2) this system reduces to

$$\frac{\partial v_1}{\partial t} + v_0\,\frac{\partial v_1}{\partial x} = 0, \tag{4.5}$$

$$\frac{\partial v_1}{\partial y} = 0, \tag{4.6}$$

$$\frac{\partial n_1}{\partial t} + v_0\,\frac{\partial n_1}{\partial x} + n_0 \frac{\partial v_1}{\partial x} = 0. \tag{4.7}$$

Using Eq. (4.2), one may obtain the relation between $\alpha_{||}$ and the wave number.
From Eq. (4.5) it follows that

$$\alpha_{||} = -ik_x\, v_0, \tag{4.8}$$

whereas from the linearized continuity equation (4.7) we have

$$\alpha_{||} = -ik_x\, v_0\, \frac{(A_{||} + B_{||})}{B_{||}}\,. \tag{4.9}$$

Both the equalities are satisfied only in the case when $A_{||} = 0$.

Thus, in the linear stability analysis with respect to small deviations of the velocity and density fields, we obtain the following perturbed solution

$$\mathbf{v} = v_0\,\mathbf{e}_x, \quad n = n_0\left[1 + B_{||}\, e^{ik_y y}\, e^{ik_x(x - v_0 t)}\right]. \tag{4.10}$$

Taking the real part of the density perturbation, we have

$$\mathbf{v} = v_0\,\mathbf{e}_x, \quad n = n_0\left[1 + B_{\|}\cos\left(\mathbf{k}\cdot\mathbf{r} - k_x v_0 t\right)\right]. \tag{4.11}$$

The corresponding density field is shown in Figure 4.1

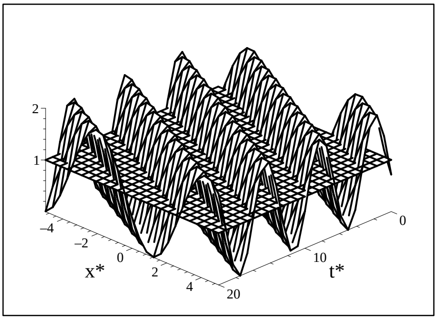

Figure 4.1: Total density field $n(\mathbf{r},t)/n_0$ and the stationary solution $n/n_0 = 1$ as a function of $x^* = k_x x$ and $t^* = k_x v_0 t$ for $k_y = 0$.

This flow, Eq. (4.11), should satisfy the linearized system of the constraints (conservation of the kinetic energy and the number of particles) which are imposed on any solution of our model. This implies that the following conditions must be fulfilled:

$$\int n_1(\mathbf{r},t)\,d\mathbf{r} = 0, \tag{4.12}$$

$$\int \left[2n_0\left(\mathbf{v}_0\cdot\mathbf{v}_1(\mathbf{r},t)\right) + n_1(\mathbf{r},t)\,v_0^2\right]d\mathbf{r} = 0. \tag{4.13}$$

Since $\mathbf{v}_1(\mathbf{r},t) = \mathbf{0}$ both conditions reduce to

$$\int n_1(\mathbf{r},t)\;d\mathbf{r} = n_0\,B_{\|}\int e^{ik_y y}dy\int e^{ik_x(x-v_0 t)}dx = 0. \tag{4.14}$$

If one integrates Eq. (4.14) over the period of the integrand, one may see that this condition is fulfilled.

The obtained perturbed flow is an oscillatory field (perturbation oscillates with a frequency $\alpha_{\|}$ as $t \to \infty$), which means that the corresponding stationary solution is neither stable nor unstable within the first order of the perturbation theory. The stability analysis of the other possible hydrodynamic models with $s_1 = 0$ or $s_1 = s_2$ gives qualitatively similar result.

Thus, we may conclude that in our local hydrodynamic model the stationary linear flow is neutrally stable with respect to a small density field perturbation.

4.2.2 Stability with respect to a velocity perturbation perpendicular to the flow

In this section we investigate the stability properties of the stationary linear flow in the LHM1, Eq. (4.1), with respect to a velocity perturbation normal to the stationary flow. We consider only a velocity perturbation, which we take in the form of a plane wave:

$$\mathbf{v}_1 = v_0 A_\perp e^{i\mathbf{k}\cdot\mathbf{r}} e^{\alpha_\perp t} \mathbf{e}_y, \quad n_1 = 0, \tag{4.15}$$

where $A_\perp$ is a constant, $\mathbf{k}$ is a wave vector and the exponent $\alpha_\perp$ describes the time evolution of the perturbation.

Substituting this perturbation into the linearized equations (4.3)-(4.4), we obtain

$$\frac{\partial v_1}{\partial t} + v_0 (1 - s_1 n_0) \frac{\partial v_1}{\partial x} = 0, \tag{4.16}$$

$$\frac{\partial v_1}{\partial y} = 0 \quad \text{and} \quad k_y = 0, \tag{4.17}$$

which implies that

$$\alpha_\perp = i k_x v_0 (s_1 n_0 - 1). \tag{4.18}$$

Thus the time evolution of the perturbed velocity field is determined by the purely imaginary exponent in Eq. (4.18):

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 (x, t) = v_0 \left[\mathbf{e}_x + A_\perp e^{i k_x (x + \mathbb{V} t)} \mathbf{e}_y \right], \quad n = n_0, \tag{4.19}$$

where the "phase speed" is given by

$$\mathbb{V} = v_0 (s_1 n_0 - 1). \tag{4.20}$$

Taking the real part of the velocity perturbation, we obtain as a final result:

$$\mathbf{v} = v_0 \left[\mathbf{e}_x + A_\perp \cos\left[k_x \left(x + \mathbb{V}t\right)\right] \mathbf{e}_y\right], \quad n = n_0. \tag{4.21}$$

The corresponding velocity profile is shown in Figure 4.2.

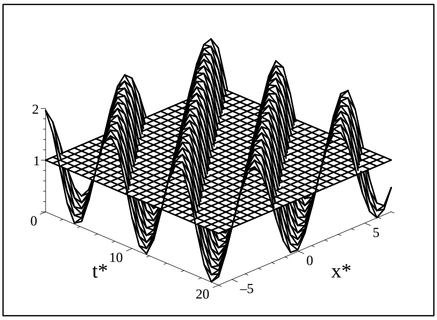

Figure 4.2: Total velocity field $v(x,t)/v_0$ and the stationary solution $v_0/v_0 = 1$ as a function of $x^* = k_x x$ and $t^* = k_x \mathbb{V} t$.

Since the velocity perturbation is taken to be normal to the unperturbed field and $n_1 = 0$, both of the constraints of the constancy of the kinetic energy and the number of particles, Eqs. (4.12) and (4.13), are satisfied.

As one may see the time dependent part of the velocity perturbation is a finite oscillatory function which means that the corresponding stationary solution is neutrally stable.

As in the previous section the stability analysis of the other possible hydrodynamic models with $s_1 = 0$ or $s_1 = s_2$ gives qualitatively similar result.

4.3 Stability of planar stationary vortical flow with constant velocity and density in the local hydrodynamic model

As we have shown in Chapter 3, there are two classes of the stationary flows in the LHM, linear and axially symmetric or vortical flow. The stationary vortical solution of the LHM1 (see Chapter2) is given by $\mathbf{v}_0(\mathbf{r}) = v_\varphi(r)\,\mathbf{e}_\varphi$, $n_0(\mathbf{r}) = n_0(r)$, where

$$v_\varphi(r) = \frac{C_{\rm st}}{2\pi r}\exp\left[s_1\int\limits_{r_0}^{r}\frac{dr'}{r'\,n_0(r')}\right]. \tag{4.22}$$

We consider small perturbations $\mathbf{v}_1(r,\varphi,t)$ of the velocity field and $n_1(r,\varphi,t)$ of the density field. Linearizing the system of equations of the LHM1, Eqs. (2.17),(2.18), we obtain

$$\begin{aligned}\frac{\partial\mathbf{v}_1}{\partial t} + (\mathbf{v}_1\cdot\nabla)\,\mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\,\mathbf{v}_1 &= s_1 n_0\left[(\mathrm{rot}\,\mathbf{v}_1)\times\mathbf{v}_0 + (\mathrm{rot}\,\mathbf{v}_0)\times\mathbf{v}_1\right]\\ &\quad + s_1 n_1(\mathrm{rot}\,\mathbf{v}_0)\times\mathbf{v}_0,\end{aligned} \tag{4.23}$$

$$\frac{\partial n_1}{\partial t} + \nabla\cdot(n_0\mathbf{v}_1) + \nabla\cdot(n_1\mathbf{v}_0) = 0. \tag{4.24}$$

In this section we consider the stability of a particular class of stationary vortical flow for which the density is constant and is given by $n_0 = 1/s_1$. Substitution of this density into Eq. (4.22) leads to a constant velocity field

$$\mathbf{v}_0 = v_\varphi\mathbf{e}_\varphi = \frac{C_{\rm st}}{2\pi r_0}\,\mathbf{e}_\varphi\,. \tag{4.25}$$

We write the small perturbations in a general form

$$\mathbf{v}_1 = a(r,\varphi,t)\,\mathbf{e}_r + b(r,\varphi,t)\,\mathbf{e}_\varphi \quad\text{and}\quad n_1 = n_0\,c_1(r,\varphi,t)\,. \tag{4.26}$$

For the projections of the velocity field $\mathbf{v} = \mathbf{v}_0(r) + \mathbf{v}_1(r,\varphi,t)$ together with the continuity equation for the density field $n = n_0 + n_1(r,\varphi,t)$ we have:

$$\frac{\partial a}{\partial t} - 2\frac{b\,v_\varphi}{r} + \frac{v_\varphi}{r}\frac{\partial a}{\partial\varphi} = -\frac{v_\varphi}{r}\left[\frac{\partial rb}{\partial r} - \frac{\partial a}{\partial\varphi}\right] - \frac{bv_\varphi}{r} - c_1\frac{v_\varphi^2}{r}, \tag{4.27}$$

$$\frac{\partial b}{\partial t} + \frac{v_\varphi}{r}\frac{\partial b}{\partial\varphi} = 0, \tag{4.28}$$

$$\frac{\partial c_1}{\partial t} + \frac{1}{r}\left[\frac{\partial ra}{\partial r} + \frac{\partial b}{\partial\varphi}\right] + \frac{v_\varphi}{r}\frac{\partial c_1}{\partial\varphi} = 0. \tag{4.29}$$

In order to simplify the problem we restrict our discussion to the case with the radial component of the velocity perturbation being constant, i.e. $a\,(r,\varphi,t) = const$.

Then one can transform equations (4.27)-(4.29) into

$$\frac{\partial b}{\partial t} + \frac{v_\varphi}{r}\frac{\partial b}{\partial \varphi} = 0, \tag{4.30}$$

$$\frac{\partial b}{\partial r} = -\frac{c_1 v_\varphi}{r}, \tag{4.31}$$

$$\frac{\partial c_1}{\partial t} + \frac{1}{r}\left(a + \frac{\partial b}{\partial \varphi}\right) + \frac{v_\varphi}{r}\frac{\partial c_1}{\partial \varphi} = 0. \tag{4.32}$$

The velocity perturbation must be a periodic functions of the angle φ and can therefore be written as

$$b\,(r,\varphi,t) = v_\varphi B\,(r)\,e^{im\varphi}e^{\beta t}, \tag{4.33}$$

where $B\,(r)$ is a function of r, m is an integer and β is a constant factor, which describes the time evolution of the perturbation, Eq. (4.26). Substituting this into Eq. (4.30), one obtains

$$\beta = -i\,m\frac{v_\varphi}{r} \tag{4.34}$$

and consequently

$$b\,(r,\varphi,t) = v_\varphi B\,(r)\exp\left[im\left(\varphi - \frac{v_\varphi}{r}\,t\right)\right]. \tag{4.35}$$

From Eq. (4.31) it follows that

$$c_1\,(r,\varphi,t) = -r\left(\frac{\partial B\,(r)}{\partial r} + im\frac{v_\varphi B\,(r)}{r^2}\,t\right)exp\left[im\left(\varphi - \frac{v_\varphi}{r}\,t\right)\right]. \tag{4.36}$$

Substituting this into Eq. (4.32), we obtain that $a\,(r,\varphi,t) = 0$.
The solutions (4.35) and (4.36) satisfy the linearized system of constraints, Eqs. (4.12) and (4.13), as one can see, by angular integration.

Thus, we see that the time evolution of the perturbation Eq. (4.26) is determined by the purely imaginary exponent Eq. (4.34). Taking the real part in Eqs. (4.35) and (4.36), we obtain

$$b\,(r,\varphi,t) = v_\varphi B\,(r)\cos\left[m\left(\varphi - \frac{v_\varphi}{r}\,t\right)\right], \tag{4.37}$$

$$n_1(r,\varphi,t) = n_0\left\{\frac{mv_\varphi B(r)}{r}\,t\sin\left[m\left(\varphi-\frac{v_\varphi}{r}t\right)\right]\right.$$
$$\left.-r\,\frac{\partial B(r)}{\partial r}\cos\left[m\left(\varphi-\frac{v_\varphi}{r}t\right)\right]\right\}. \tag{4.38}$$

As a result the whole solution for the velocity and the density profiles has the following form:

$$\mathbf{v}(r,\varphi,t) = v_\varphi\left\{1+B(r)\cos\left[m\left(\varphi-\frac{v_\varphi}{r}t\right)\right]\right\}\mathbf{e}_\varphi, \tag{4.39}$$

$$n(r,\varphi,t) = n_0\left\{1+\frac{mv_\varphi B(r)}{r}\,t\sin\left[m\left(\varphi-\frac{v_\varphi}{r}t\right)\right]\right.$$
$$\left.-r\,\frac{\partial B(r)}{\partial r}\cos\left[m\left(\varphi-\frac{v_\varphi}{r}t\right)\right]\right\}. \tag{4.40}$$

The velocity field, Eq. (4.39), is shown in Figure 4.3.

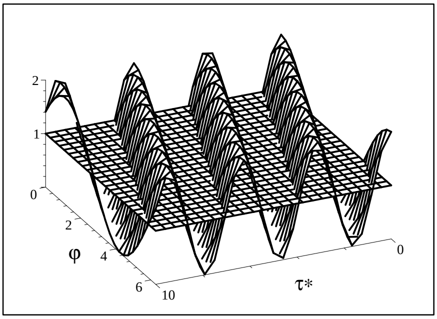

Figure 4.3: Total velocity field $v(r,\varphi,t)/v_\varphi$ and the stationary solution $v_\varphi/v_\varphi = 1$ as a function of φ and $\tau^* = v_\varphi t/r$ for $m = 1$ and $r = 5\,\mathrm{m}$.

Together with the oscillatory contributions we now also have the contribution proportional to t times an oscillating function. This does not necessarily mean that the stationary vortical flow is unstable. The linear analysis does not give the definitive answer regarding the stability of the stationary flow and further investigation of higher order terms is required. Note that such a situation is typical for Hamiltonian systems which are conservative by definition and therefore do not display an asymptotic type of stability [44]. Though the system under consideration is not Hamiltonian one may suppose that the reason for the neutral stability is the dissipative free character of the dynamics.

4.4 Conclusions

In this chapter we considered the stability properties of the planar stationary flows of the local hydrodynamic model constructed in Section 2 for a system of self-propelling particles. These flows are the linear flow and the axially symmetric flow. Our analysis shows for linear flow, using linear perturbation theory, that the time evolution of the imposed velocity and density perturbations is oscillatory. It follows that the linear flow is neutrally stable. For axially symmetric (vortical) flow linear perturbation theory does not lead to a conclusive result. Although exponential time dependence of the perturbation has purely imaginary factor, there appears an additional contribution proportional to t. This does not necessarily mean that the stationary solution is unstable. A definitive answer about the nature of the stability can only be given by considering also higher order terms in the perturbation expansion. Such an analysis is beyond the scope of the present thesis.

Chapter 5

Connection between discrete and continuous descriptions

In this chapter we derive the continuous hydrodynamic model constructed in Chapter 2 from the discrete description proposed by Vicsek et al. The averaging procedure is defined. The similarities and differences between the resulting model and our hydrodynamic model are discussed. The results clarify the assumptions used to obtain a continuous description.

This chapter is based on a paper by V. I. Ratushnaya, D. Bedeaux, V. L. Kulinskii, A. V. Zvelindovsky, "Collective behaviour of self-propelling particles with kinematic constraints; The relation between discrete and the continuous description", Physica A **381**, 39 (2007).

5.1 Introduction

In this chapter we obtain the continuous description by coarse-graining the discrete CV algorithm. In this respect our present results are meant to be a link between two existing groups of approaches: discrete and continuous. The importance of this analysis is that it clarifies which of the continuous models we proposed is closest to be the continuum analog of the CV model.

In the next section we will start with a rule for the velocities formulated by Vicsek et al. and obtain a discrete equation of motion for each particle. We introduce angular velocities associated with the rate of change of the direction of the linear velocity of the particles. These angular velocities contain the information about the non-potential interactions between a given particle and its local surrounding. We derive an expression for the angular velocities in the continuous time description. We show that to a first order in the velocity difference between the time steps the angular velocity for particle i depends on the average velocity in the neighbourhood of the ith particle and its rate of change.

In Section 5.3 we obtain the continuous description, with a conserved kinetic energy and number of particles, using a coarse-graining procedure. We obtain the angular velocity field which follows from the 2-dimensional CV model and compare it with the angular velocity fields we proposed in Chapter 2. It turns out that there are similarities and differences. Both the continuous description that follows from the CV model and our continuous model give stationary linear and vortical flow fields. The description of such flow fields is one of the aims of the model.

5.2 Continuum time limit

In this section we derive the equation of motion in continuous time from the CV algorithm. In the CV model the collective behaviour of self-propelling particles with respect to a change of the density and the strength of the noise was investigated. In our analysis the noise will not be considered. We focus on the systematic contribution. In Chapter 2 we discussed how noise can be added in our approach.

5.2.1 Angular velocity associated with the particle velocity

According to the CV rule at each discrete time step (labeled by n) the ith particle adjusts the direction of its velocity $\mathbf{v}_i(n)$ to the direction of the average velocity $\mathbf{u}_i(n)$ in its neighbourhood. The average is calculated over a region with a radius R around a given particle. Using this radius, we will call particle densities small compared to R^{-d}, where d is the dimensionality, small. When the density is larger we call it large. The CV rule implies that

$$\mathbf{v}_i(n+1) \times \mathbf{u}_i(n) = 0, \quad \forall\, i, n\,, \tag{5.1}$$

where the absolute value of the velocity of each particle is assumed to be constant, i.e.

$$|\, \mathbf{v}_i(n+1) \,|=|\, \mathbf{v}_i(n) \,|= v_i\,. \tag{5.2}$$

Together with Eq. (5.1) it follows that

$$\mathbf{v}_i(n+1) = v_i\, \mathbf{u}_i(n)\,, \quad \text{where} \quad |\, \mathbf{u}_i(n) \,|= 1\,. \tag{5.3}$$

Using the fact that $\mathbf{v}_i(n+1) - \mathbf{v}_i(n)$ is perpendicular to $\mathbf{v}_i(n+1) + \mathbf{v}_i(n)$, given the validity of Eq. (5.2), it can be shown that

$$\mathbf{v}_i(n+1) - \mathbf{v}_i(n) = \left[\widehat{\mathbf{v}}_i(n) \times \widehat{\mathbf{v}}_i(n+1)\right] \times \left[\frac{\mathbf{v}_i(n+1) + \mathbf{v}_i(n)}{1 + \widehat{\mathbf{v}}_i(n) \cdot \widehat{\mathbf{v}}_i(n+1)}\right], \tag{5.4}$$

where $\widehat{\mathbf{v}}_i(n) \equiv \mathbf{v}_i(n)/v_i$ is a unit vector in the direction of the velocity $\mathbf{v}_i(n)$. For the details see Appendix C.

It is important to realize that there is a difference between low density regions and high density regions. In high density regions the velocity of the particles is updated at every step. In the low density regions the average of the velocity of particles around and including particle i is equal to the velocity of particle i. It follows that $\mathbf{u}_i(n) = \mathbf{v}_i(n)/v_i$. As a consequence $\mathbf{v}_i(n+1) = \mathbf{v}_i(n)$. Substitution in Eq. (5.4) gives the equality zero equal to zero. The important conclusion is that in the low density regions the particles do not change their velocity. We will come back to this point when this is relevant.

In order to obtain a continuous description as a function of time, we assume the steps to be small so that

$$|\,\widehat{\mathbf{v}}_i\,(n+1) - \widehat{\mathbf{v}}_i\,(n)\,| \ll 1\,. \tag{5.5}$$

One may then write Eq. (5.4) to first order in the velocity difference as

$$\mathbf{v}_i\,(n+1) - \mathbf{v}_i\,(n) = [\widehat{\mathbf{v}}_i\,(n) \times \widehat{\mathbf{v}}_i\,(n+1)] \times \mathbf{v}_i\,(n) = [\widehat{\mathbf{v}}_i\,(n) \times \mathbf{u}_i\,(n)] \times \mathbf{v}_i\,(n)\,. \tag{5.6}$$

As we are interested in the rate of change of the velocity we divide this equation by the time step duration τ. This gives

$$\frac{\mathbf{v}_i\,(n+1) - \mathbf{v}_i\,(n)}{\tau} = \left[\frac{\widehat{\mathbf{v}}_i\,(n) \times \mathbf{u}_i\,(n)}{\tau}\right] \times \mathbf{v}_i\,(n) = \boldsymbol{\omega}_{\mathbf{v}_i} \times \mathbf{v}_i\,, \tag{5.7}$$

where

$$\boldsymbol{\omega}_{\mathbf{v}_i} = \frac{1}{\tau}\,\widehat{\mathbf{v}}_i \times \mathbf{u}_i \tag{5.8}$$

is an angular velocity associated with the particle velocity $\mathbf{v}_i$.

In view of Eq. (5.5) the left-hand side of Eq. (5.7) gives the continuous time derivative. In other words we may introduce the following definition:

$$\dot{\mathbf{v}}_i\,(n) \longleftrightarrow \frac{\mathbf{v}_i\,(n+1) - \mathbf{v}_i\,(n)}{\tau}\,. \tag{5.9}$$

5.2.2 Angular velocity associated with the average velocity

Using that $\mathbf{u}_i\,(n) = \widehat{\mathbf{v}}_i\,(n+1)$, it follows from Eq. (5.7) that

$$\mathbf{u}_i\,(n+1) - \mathbf{u}_i\,(n) = \tau\,\boldsymbol{\omega}_{\mathbf{u}_i}\,(n) \times \mathbf{u}_i\,(n)\,, \tag{5.10}$$

where the angular velocity $\boldsymbol{\omega}_{\mathbf{u}_i}\,(n)$ corresponding to the average velocity $\mathbf{u}_i\,(n)$ is defined as

$$\boldsymbol{\omega}_{\mathbf{u}_i}\,(n) = \boldsymbol{\omega}_{\mathbf{v}_i}\,(n+1)\,. \tag{5.11}$$

It can be shown that

$$\boldsymbol{\omega}_{\mathbf{u}_i}\,(n) = \frac{1}{\tau}\,[\widehat{\mathbf{v}}_i\,(n+1) \times \mathbf{u}_i\,(n+1)] = \frac{1}{\tau}\,[\mathbf{u}_i\,(n) \times \mathbf{u}_i\,(n+1)] = \mathbf{u}_i\,(n) \times \dot{\mathbf{u}}_i\,(n)\,, \tag{5.12}$$

where

$$\dot{\mathbf{u}}_i(n) = \frac{\mathbf{u}_i(n+1) - \mathbf{u}_i(n)}{\tau}. \tag{5.13}$$

Furthermore, one may show that

$$\omega_{\mathbf{u}_i}(n) - \omega_{\mathbf{v}_i}(n) = \tau\,\widehat{\mathbf{v}}_i(n+1) \times \ddot{\widehat{\mathbf{v}}}_i(n) = \tau\,\mathbf{u}_i(n) \times \ddot{\widehat{\mathbf{v}}}_i(n), \tag{5.14}$$

where the second order derivative is defined by

$$\ddot{\mathbf{v}}_i(n) = \frac{1}{\tau^2}\Big[\mathbf{v}_i(n+2) - 2\mathbf{v}_i(n+1) + \mathbf{v}_i(n)\Big]. \tag{5.15}$$

Combining Eqs. (5.12) and (5.14) results in

$$\omega_{\mathbf{v}_i}(n) = \omega_{\mathbf{u}_i}(n) - \tau\,\mathbf{u}_i(n) \times \ddot{\widehat{\mathbf{v}}}_i(n) = \mathbf{u}_i(n) \times \dot{\mathbf{u}}_i(n) - \tau\,\mathbf{u}_i(n) \times \ddot{\widehat{\mathbf{v}}}_i(n), \tag{5.16}$$

which to first order in the velocity difference implies that for the angular velocity associated with the particle velocity $\mathbf{v}_i$ we obtain the following expression:

$$\omega_{\mathbf{v}_i}(n) = \omega_{\mathbf{u}_i}(n) - \tau\,\mathbf{u}_i(n) \times \ddot{\widehat{\mathbf{u}}}_i(n) = \mathbf{u}_i(n) \times \dot{\mathbf{u}}_i(n). \tag{5.17}$$

The second equality follows from the fact that the second derivative $\ddot{\mathbf{u}}_i(n)$ is parallel to $\mathbf{u}_i(n)$ to first order in the difference.

Replacing n by the time t the resulting equation of motion becomes:

$$\frac{d\mathbf{v}_i(t)}{dt} = \omega_{\mathbf{u}_i}(t) \times \mathbf{v}_i(t) = [\mathbf{u}_i(t) \times \dot{\mathbf{u}}_i(t)] \times \mathbf{v}_i(t). \tag{5.18}$$

This equation is continuous in time and is derived from the discrete CV rule using Eq. (5.5). In order to obtain equations for the velocity and the density fields, which are continuous in space, we will coarse-grain Eqs. (5.17) and (5.18) in the next section. We note that both $d\mathbf{v}_i(t)/dt$ and $\dot{\mathbf{u}}_i(t)$ are zero in the low density regions.

5.3 Continuous transport equations

In this section we introduce the averaging procedure for the discrete model. We derive continuous equations for the velocity and the density fields by averaging the discrete equations. We will also discuss how to obtain the introduced in Chapter 2 angular velocity field in terms of the velocity and density fields.

5.3.1 Definition of the hydrodynamic quantities

In the CV algorithm the direction of the average velocity in the neighbourhood of particle i is given in the continuous time description by

$$\mathbf{u}_i(t) = \sum_j H\big(\mathbf{r}_{ij}(t)\big)\, \mathbf{v}_j(t) \Big| \sum_j H\big(\mathbf{r}_{ij}(t)\big)\, \mathbf{v}_j(t) \Big|^{-1}, \tag{5.19}$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. The dynamics of individual particles therefore reduces the difference between the direction of its velocity and that of the average velocity of the surrounding particles. $H(\mathbf{r})$ is an averaging kernel, which we assume to be normalized,

$$\int H(\mathbf{r})\, d\mathbf{r} = 1\,. \tag{5.20}$$

It has the characteristic averaging scale R, beyond which the kernel goes to zero [17, 21]. Usually one uses for H a normalized Heaviside step function.

In order to obtain a continuous description we define the average particle density (per unit of volume) and velocity fields by

$$\begin{aligned} n(\mathbf{r}, t) &= \sum_j H\big(\mathbf{r} - \mathbf{r}_j(t)\big), \\ n(\mathbf{r}, t)\, \mathbf{v}(\mathbf{r}, t) &= \sum_j H\big(\mathbf{r} - \mathbf{r}_j(t)\big)\, \mathbf{v}_j(t)\,. \end{aligned} \tag{5.21}$$

Using Eq. (5.21) in Eq. (5.19) for $\mathbf{r} = \mathbf{r}_i$, it follows that $\mathbf{u}_i(t) = \widehat{\mathbf{v}}(\mathbf{r}_i(t), t)$, where $\widehat{\mathbf{v}}(\mathbf{r}_i(t), t) = \mathbf{v}(\mathbf{r}_i(t), t) / |\mathbf{v}(\mathbf{r}_i(t), t)|$. Eq. (5.18) can therefore be written as

$$\frac{d\mathbf{v}_i(t)}{dt} = \left[\widehat{\mathbf{v}}(\mathbf{r}_i(t), t) \times \frac{d\widehat{\mathbf{v}}(\mathbf{r}_i(t), t)}{dt}\right] \times \mathbf{v}_i(t)\,. \tag{5.22}$$

By evaluating the time derivative between the square brackets on the right-hand side one obtains:

$$\begin{aligned}
\frac{d\mathbf{v}_i(t)}{dt} &= \left[\widehat{\mathbf{v}}(\mathbf{r},t) \times \left(\mathbf{v}_i(t) \cdot \nabla \widehat{\mathbf{v}}(\mathbf{r},t)\right)\right]_{\mathbf{r}=\mathbf{r}_i(t)} \times \mathbf{v}_i(t) \\
&+ \left[\widehat{\mathbf{v}}(\mathbf{r},t) \times \frac{\partial \widehat{\mathbf{v}}(\mathbf{r},t)}{\partial t}\right]_{\mathbf{r}=\mathbf{r}_i(t)} \times \mathbf{v}_i(t) \\
&= \left[\widehat{\mathbf{v}}(\mathbf{r},t) \times \left[\left(\mathbf{v}_i(t) - \mathbf{v}(\mathbf{r},t)\right) \cdot \nabla \widehat{\mathbf{v}}(\mathbf{r},t)\right)\right]_{\mathbf{r}=\mathbf{r}_i(t)} \times \mathbf{v}_i(t) \\
&+ \left[\widehat{\mathbf{v}}(\mathbf{r},t) \times \frac{d \widehat{\mathbf{v}}(\mathbf{r},t)}{dt}\right]_{\mathbf{r}=\mathbf{r}_i(t)} \times \mathbf{v}_i(t) \,. \qquad (5.23)
\end{aligned}$$

In view of the fact that the gradient of the direction of the average velocity is of the first order and that the velocity difference is also of the first order, the first contribution on the right-hand side is of the second order and can be neglected. Eq. (5.23) therefore reduces to

$$\frac{d\mathbf{v}_i(t)}{dt} = \left[\widehat{\mathbf{v}}(\mathbf{r},t) \times \frac{d \widehat{\mathbf{v}}(\mathbf{r},t)}{dt}\right]_{\mathbf{r}=\mathbf{r}_i(t)} \times \mathbf{v}_i(t) \,. \qquad (5.24)$$

When we now average this equation we can use the fact that the expression between the square brackets only depends on the coarse-graining functions and therefore varies slowly over the range of the averaging function.

5.3.2 Averaging procedure

Given the above definitions of the hydrodynamic density and velocity fields, Eq.(5.21), one can easily obtain the continuity equation (2.8). For the details see Appendix B. Averaging the left-hand side of Eq. (5.24) and using the continuity equation, we obtain:

$$
\begin{aligned}
&\sum_i \frac{d\mathbf{v}_i(t)}{dt} H(\mathbf{r}-\mathbf{r}_i(t)) \\
&= \frac{\partial}{\partial t}\sum_i \mathbf{v}_i(t) H(\mathbf{r}-\mathbf{r}_i(t)) - \sum_i \mathbf{v}_i(t) \frac{\partial}{\partial \mathbf{r}_i}\cdot \mathbf{v}_i(t) H(\mathbf{r}-\mathbf{r}_i(t)) \\
&= \frac{\partial(n(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t))}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\cdot \sum_i \mathbf{v}_i(t)\ \mathbf{v}_i(t)\ H(\mathbf{r}-\mathbf{r}_i(t)) \\
&= n(\mathbf{r},t)\ \frac{d\mathbf{v}(\mathbf{r},t)}{dt} - \nabla\cdot[n(\mathbf{r},t)\ \mathbf{v}(\mathbf{r},t)\otimes\mathbf{v}(\mathbf{r},t)] \\
&+ \nabla\cdot\sum_i \mathbf{v}_i(t)\,\mathbf{v}_i(t)\ H(\mathbf{r}-\mathbf{r}_i(t)) = n(\mathbf{r},t)\ \frac{d\mathbf{v}(\mathbf{r},t)}{dt} \\
&+ \nabla\cdot\sum_i (\mathbf{v}(\mathbf{r},t)-\mathbf{v}_i(t))(\mathbf{v}(\mathbf{r},t)-\mathbf{v}_i(t))\ H(\mathbf{r}-\mathbf{r}_i(t)) \\
&= n(\mathbf{r},t)\ \frac{d\mathbf{v}(\mathbf{r},t)}{dt}\,,
\end{aligned}
\tag{5.25}
$$

where we neglected the term of the second order in the velocity difference. This term would give a small contribution to the pressure tensor. It is therefore also a contribution which is in the formulation of the problem assumed to be cancelled by the self-propelling force.

On the right hand-side of Eq. (5.24) we have

$$
\begin{aligned}
&\sum_i \left[\widehat{\mathbf{v}}(\mathbf{r},t)\times\frac{d\widehat{\mathbf{v}}(\mathbf{r},t)}{dt}\right]_{\mathbf{r}=\mathbf{r}_i(t)} \times \mathbf{v}_i(t)\ H(\mathbf{r}-\mathbf{r}_i(t)) \\
&= \left[\widehat{\mathbf{v}}(\mathbf{r},t)\times\frac{d\widehat{\mathbf{v}}(\mathbf{r},t)}{dt}\right]\times n(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)\,.
\end{aligned}
\tag{5.26}
$$

This implies that the averaged equation of motion can be written as

$$
\frac{d\mathbf{v}(\mathbf{r},t)}{dt} = \left[\widehat{\mathbf{v}}(\mathbf{r},t)\times\frac{d\widehat{\mathbf{v}}(\mathbf{r},t)}{dt}\right]\times\mathbf{v}(\mathbf{r},t)\,. \tag{5.27}
$$

This gives

$$\frac{d\mathbf{v}(\mathbf{r},t)}{dt} = \boldsymbol{\omega}(\mathbf{r},t) \times \mathbf{v}(\mathbf{r},t), \tag{5.28}$$

where to first order in the velocity difference

$$\boldsymbol{\omega}(\mathbf{r},t) = \widehat{\mathbf{v}}(\mathbf{r},t) \times \frac{d\widehat{\mathbf{v}}(\mathbf{r},t)}{dt} = \frac{1}{\mathrm{v}^2(\mathbf{r},t)}\left[\mathbf{v}(\mathbf{r},t) \times \frac{d\mathbf{v}(\mathbf{r},t)}{dt}\right]. \tag{5.29}$$

where $\mathrm{v}(\mathbf{r},t) \equiv |\mathbf{v}(\mathbf{r},t)|$.

5.3.3 Averaged angular velocity field

Here we restrict our discussion by considering the 2-dimensional case in order to make a comparison with the results obtained in the previous chapters. By evaluating the time derivative in Eq. (5.29) we may rewrite this expression as follows:

$$\begin{aligned}
\boldsymbol{\omega}(\mathbf{r},t) &= \frac{1}{\mathrm{v}^2(\mathbf{r},t)}\left[\mathbf{v}(\mathbf{r},t) \times \left(\frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t} + (\mathbf{v}(\mathbf{r},t)\cdot\nabla)\,\mathbf{v}(\mathbf{r},t)\right)\right] \\
&= \frac{1}{\mathrm{v}^2(\mathbf{r},t)}\left[\mathbf{v}(\mathbf{r},t) \times \left(\frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t} + \nabla\frac{\mathrm{v}^2(\mathbf{r},t)}{2}\right)\right] \\
&- \frac{1}{\mathrm{v}^2(\mathbf{r},t)}\,\mathbf{v}(\mathbf{r},t) \times [\mathbf{v}(\mathbf{r},t) \times \mathrm{rot}\,\mathbf{v}(\mathbf{r},t)] \\
&= \frac{\mathbf{v}(\mathbf{r},t)}{\mathrm{v}^2(\mathbf{r},t)} \times \frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t} + \frac{\mathbf{v}(\mathbf{r},t)}{\mathrm{v}^2(\mathbf{r},t)} \times \nabla\frac{\mathrm{v}^2(\mathbf{r},t)}{2} + \mathrm{rot}\,\mathbf{v}(\mathbf{r},t)\ .
\end{aligned} \tag{5.30}$$

This is the angular velocity field obtained from the discrete algorithm used by Vicsek et al.

One may see that the continuous equation of motion, Eq. (5.28), with the angular velocity derived from the CV rule, Eq. (5.30), can be written as follows:

$$\frac{d\mathbf{v}}{dt} = (\mathbf{1} - \widehat{\mathbf{v}}\widehat{\mathbf{v}})\cdot\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{1} - \widehat{\mathbf{v}}\widehat{\mathbf{v}})\cdot\nabla\frac{\mathrm{v}^2}{2} + (\mathrm{rot}\,\mathbf{v}) \times \mathbf{v}. \tag{5.31}$$

where **1** is the unit tensor. All three terms on the right-hand side contribute to the co-moving derivative of the velocity which is orthogonal to the velocity field.

In the low density regions one obtains, as has been pointed out a number of times, $d\mathbf{v}/dt = 0$. As this is not so clearly visible in Eq. (5.31) it is appropriate to replace this equation by

$$\frac{d\mathbf{v}}{dt} = \left[(\mathbf{1} - \widehat{\mathbf{v}\,\mathbf{v}}) \cdot \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{1} - \widehat{\mathbf{v}\,\mathbf{v}}) \cdot \nabla \frac{\mathrm{v}^2}{2} + (\mathrm{rot}\,\mathbf{v}) \times \mathbf{v} \right] f\,(n\,(\mathbf{r}, t))\,, \tag{5.32}$$

where $f\,(n\,(\mathbf{r}, t))$ is the density dependent factor, which arises due to coarse-graining procedure. In low density limit it is natural that $f\,(n\,(\mathbf{r}, t)) \to 0$ as $n\,(\mathbf{r}, t) \to 0$. A more thorough analysis of this is beyond our present aim, however.

Before comparing this expression to the one we used in Chapters 2 and 3 we first verify that stationary linear and the vortical solutions are solutions of Eq. (5.31). In view of their stationarity the first contribution in Eq. (5.31) is equal to zero. For a linear flow $\mathbf{v} = \mathrm{v}_0\,\mathbf{e}_x$ the other two terms on the left hand-side of Eq. (5.31) are also zero. Stationary linear flow is therefore a solution. In case of stationary vortical flow, $\mathbf{v} = \mathrm{v}_\varphi\,(r)\,\mathbf{e}_\varphi\,(\varphi)$, the $(\mathbf{v} \cdot \nabla)\,\mathbf{v}$ term on the left-hand side of Eq. (5.31) cancels the terms due to the second and the third term. The continuity equation, Eq. (2.8), is satisfied for each density distribution which varies only in directions normal to the flow direction. It follows that the continuous analog of the CV model has stationary linear and vortical solutions.

In Chapters 2 and 3 we used an angular velocity field which was a linear combination of $n\,(\mathbf{r}, t)\,\mathrm{rot}\,\mathbf{v}\,(\mathbf{r}, t)$ and $\nabla n\,(\mathbf{r}, t) \times \mathbf{v}\,(\mathbf{r}, t)$. The resulting equation of motion was

$$\frac{d\mathbf{v}\,(\mathbf{r}, t)}{dt} = s_1 n\,(\mathbf{r}, t)\,(\mathrm{rot}\,\mathbf{v}\,(\mathbf{r}, t)) \times \mathbf{v}\,(\mathbf{r}, t) + s_2\,(\nabla n\,(\mathbf{r}, t) \times \mathbf{v}\,(\mathbf{r}, t)) \times \mathbf{v}\,(\mathbf{r}, t)\,. \tag{5.33}$$

The first term is analogous to the third term on the right-hand side of Eq. (5.31). Similar to the CV model this choice leads to stationary linear and vortical solutions. The linear dependence of our choice on the density leads to a dependence of the stationary velocity field on the density distribution. We refer to Chapters 2 and 3 for a detailed discussion of these solutions. For a small density the right-hand side of Eq. (5.33) makes $d\mathbf{v}/dt$ negligible. This is similar to the behaviour in Eq. (5.32).

When one modifies the updating rule in the CV model, as it was done in Refs. [18, 22, 24], this leads to a modification of the $\omega(\mathbf{r}, t)$ given in Eq. (5.30). Similarly, the choice of $\omega(\mathbf{r}, t)$ we used in Ref. [43, 45] can be modified to include such contributions. The freedom in the choice of $\omega(\mathbf{r}, t)$ in the continuous version of the CV model is one of its strength.

5.4 Conclusions

In this chapter we addressed the problem to derive our continuous description, proposed in Chapter 2, from the discrete model proposed by Vicsek et al. [17]. By coarse-graining the discrete equations we were able to derive expression for the angular velocity field used in the continuous model from the updating procedure used in their model. Modification of the updating rules in this model, as done in Refs. [18, 22, 24, 25], results in modifications of the resulting angular velocity field. The angular velocity field used in our work [43, 45] is one of such choices. One of the contributions in the continuous version of the CV model is very similar to one of the contributions which we have postulated in our hydrodynamic model, see Chapter2. Both the continuous CV model and our model lead to the linear and vortical flows of the self-propelling particles observed in nature and obtained in simulations and continuum approximations. This shows that they are appropriate for the description of flocking behaviour, which is one of the aims of the model. An interesting alternative choice of $\omega(\mathbf{r}, t)$ in the continuous description is, for instance, $\nabla c_A \times \mathbf{v}$ where c_A is the concentration of an attractant. As shown by Czirók et al. in Ref. [18], this choice can be used to describe *rotor chemotaxis*. Note that the term $\nabla n(\mathbf{r}, t) \times \mathbf{v}(\mathbf{r}, t)$, which we considered in our continuum model, is similar to the one considered by Czirók et al., when the concentration field is proportional to the concentration of the attractant.

Our analysis shows that one may coarse-grain the discrete updating rule and obtain the corresponding continuous description. This makes a direct comparison between discrete and continuous descriptions possible. For our own work it was found that our continuous description was similar to the continuous version of the original CV model but not identical. The analysis in this chapter makes it possible to extend our work on the continuous description such that it is either closer or more different from the original CV model.

Chapter 6

Nonstationary flows of self-propelling particles

In this chapter nonstationary hydrodynamic flows of self-propelling particles are obtained. These flows are found to be linear and radial. The obtained linear flow describes a collisionless regime of motion, which also takes place in the CV algorithm.

A paper based on this chapter is in preparation.

6.1 Introduction

In this chapter we consider 2-dimensional nonstationary flows of the local hydrodynamic model (LHM) described in the previous chapters.

Finding analytic solutions for nonlinear systems given by Eqs. (2.8),(2.10) is a very complicated task. We restrict our discussion to those solutions, which describe the collisionless regime of the CV algorithm. This regime describes inertial motion of particles with no interactions between each other. On the continuous level the collisionless medium is described by the equation $d\mathbf{v}/dt = \mathbf{0}$, which in our 2-dimensional LHM corresponds to $\omega = \mathbf{0}$. In particular, such a regime occurs for the linear flow of the CV algorithm when particles have different speeds (see Figure 6.1).

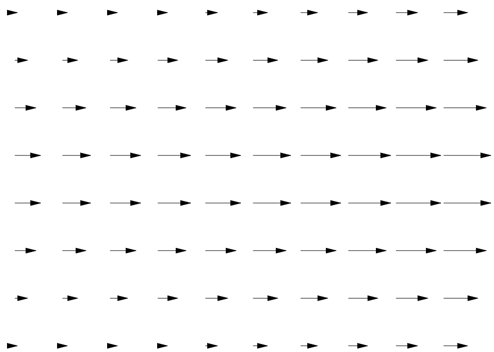

Figure 6.1: Collisionless regime of motion in the CV algorithm corresponding to the collisionless continuum medium.

Therefore one can expect that the continuous model analogous to the CV algorithm possesses solutions describing collisionless media. This is indeed the case for the LHM, when the right-hand side of Eq. (2.10) vanishes and the velocity field decouples from the density field.

In the second section we construct nonstationary linear flow and discuss some of its properties. In the third section of this chapter we construct nonstationary radial flow. The last section gives our conclusions.

6.2 Nonstationary linear flow in the LHM1

In this section we obtain 2-dimensional nonstationary linear flow in the LHM1. The general form of this flow is given by $\mathbf{v} = v(\mathbf{r}, t)\,\mathbf{e}_x$, $n = n(\mathbf{r}, t)$. Equations (2.17),(2.18) therefore reduce to

$$\begin{aligned} \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} &= 0, && (6.1)\\ -s_1\, n\, v\frac{\partial v}{\partial y} &= 0, && (6.2)\\ \frac{\partial n}{\partial t} + v\frac{\partial n}{\partial x} + n\frac{\partial v}{\partial x} &= 0. && (6.3) \end{aligned}$$

From Eq. (6.2) it follows that $v = v(x, t)$ and one may see that Eq. (6.1) is independent of the density field and therefore can be solved separately. This equation describes 1-dimensional linear flow of collisionless medium. We consider the following initial velocity profile

$$v(x, t = 0) = v_0(x), \tag{6.4}$$

where $v_0(x)$ is a continuous differentiable function. The system of characteristics

$$\frac{dt}{d\xi} = 1, \quad \frac{dx}{d\xi} = v, \quad \frac{dv}{d\xi} = 0, \tag{6.5}$$

where ξ is a parameter of the equation, together with the initial condition gives the solutions provided that $v(x, t)$ is a single valued function. This is true for all $t > 0$, if $v_0(x)$ is a nondecreasing function. Otherwise the solution of shock-wave type appears, since high-velocity regions will run down the low-velocity ones and overlap them.

As an example of nonstationary linear flow, which satisfies the imposed initial condition, Eq. (6.4), we consider the following solution:

$$v(x, t) = \frac{x - x_0}{t + t_0}, \tag{6.6}$$

where x_0, t_0 are constants and initial condition

$$v_0(x) = \frac{x - x_0}{t_0}. \tag{6.7}$$

Given this solution, one can find the density profile from the continuity equation, Eq. (6.3). The system of characteristics in this case is given by

$$\frac{dt}{d\xi} = 1, \quad \frac{dx}{d\xi} = v, \quad \frac{dn}{d\xi} = -\frac{n}{t+t_0}. \tag{6.8}$$

Taking as an initial condition a constant density profile, i.e. $n(x, t = 0) = n_0 = const$, one obtains:

$$n(t) = \frac{n_0 t_0}{t+t_0}. \tag{6.9}$$

The obtained solutions, Eqs. (6.6),(6.9), have to satisfy the conservation of the kinetic energy and number of particles:

$$T = \int n v^2 dx = \frac{n_0 t_0}{3(t+t_0)^3}(X - x_0)^3, \tag{6.10}$$

$$N = \int n dx = \frac{n_0 t_0}{t+t_0}(X - x_0), \tag{6.11}$$

where x_0 and X are the boundaries of the system.

One may see that the kinetic energy and number of particles are constant, when the boundary X moves uniformly along with the flow, i.e.

$$X(t) = x_0 + c_1 (t + t_0), \tag{6.12}$$

where c_1 is a velocity at the (of the) boundary.

Thus we obtain the following nonstationary linear flow:

$$\mathbf{v}(x,t) = \frac{x - x_0}{t+t_0}\mathbf{e}_x, \tag{6.13}$$

$$n(t) = \frac{n_0 t_0}{t+t_0} H(x - x_0) H(c_1 (t + t_0) + x_0 - x), \tag{6.14}$$

where $H(x)$ is a Heaviside step function. The corresponding plot of the velocity field $v(x,t) = | \mathbf{v}(x,t) |$ is shown in Figure 6.2. Velocity profiles at later time have smaller slopes.

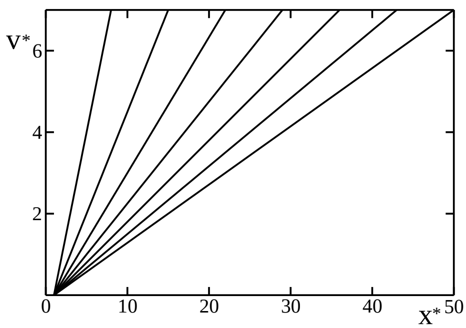

Figure 6.2: Nonstationary linear velocity field $v^*(x^*) = v\,t_0/x_0$ as a function of $x^* = x/x_0$ for different values of $t^* = t/t_0$.

6.3 Nonstationary radial flow in the LHM1

In this section we obtain nonstationary hydrodynamic flow of SPP with radial symmetry. System (2.17),(2.18) has the following form in the cylindrical coordinates for the velocity and density fields $\mathbf{v} = v(r,\varphi,t)\,\mathbf{e}_r, n(r,\varphi,t)$

$$\frac{\partial v_r}{\partial t} + v_r\,\frac{\partial v_r}{\partial r} = 0, \tag{6.15}$$

$$-\frac{s_1\,v_r\,n}{r}\,\frac{\partial v_r}{\partial \varphi} = 0, \tag{6.16}$$

$$\frac{\partial n}{\partial t} + \frac{1}{r}\,\frac{\partial(r\,n\,v_r)}{\partial r} = 0. \tag{6.17}$$

From the second equation, Eq. (6.16), it follows that $v_r = v_r(r,t)$. Eq. (6.15) is independent of the density and therefore can be solved separately. One may notice that it is similar to Eq. (6.1), thus, taking a similar initial condition, i.e. $v_r(r, t = 0) = v_0(r)$,we find

$$v_r(r,t) = \frac{r - r_0}{t + t_0}\,, \tag{6.18}$$

where r_0, t_0 are constants. With this solution we can solve the continuity equation (6.17)

$$\frac{\partial n}{\partial t} + \frac{r - r_0}{t + t_0}\frac{\partial n}{\partial r} = -\frac{1}{t + t_0}\left(2 - \frac{r_0}{r}\right)n \tag{6.19}$$

with the following initial density distribution:

$$n(r, t = 0) = n_0(r). \tag{6.20}$$

The corresponding system of the characteristics is given by

$$\frac{dt}{d\xi} = 1, \quad \frac{dr}{d\xi} = \frac{r - r_0}{t + t_0}, \quad \frac{dn}{d\xi} = -\frac{1}{t + t_0}\left(2 - \frac{r_0}{r}\right)n. \tag{6.21}$$

Solving this system, we obtain the density flow in the following form:

$$n(r, t) = \frac{c_{rad}}{r(t + t_0)}, \tag{6.22}$$

where $c_{rad} = const$.

Similarly to the linear flow, the front boundary of the system R moves together with the flow as

$$R(t) = r_0 + c_2(t + t_0), \tag{6.23}$$

where c_2 is a speed of its motion.

Thus the nonstationary radial flow in the LHM1 is given by

$$\mathbf{v}(r, t) = \frac{r - r_0}{t + t_0}\mathbf{e}_r, \tag{6.24}$$

$$n(r, t) = \frac{c_{rad}}{r(t + t_0)} H(r - r_0) H(c_2(t + t_0) + r_0 - r). \tag{6.25}$$

The corresponding plot of the density field $n(r, t)$ is shown in Figure 6.3.

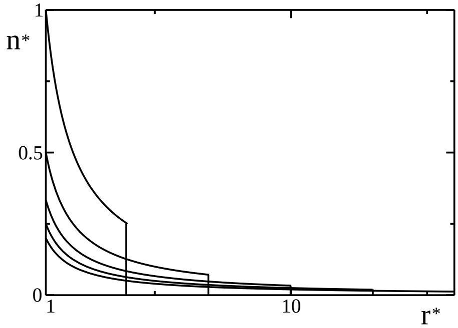

Figure 6.3: Nonstationary radial density field $n^*(r^*) = n\, r_0 t_0 / c_{rad}$ as a function of $r^* = r/r_0$ for different values of $t^* = t/t_0$.

6.4 Conclusions

In this chapter we obtained nonstationary regimes of motion of SPP, which correspond to the collisionless regime of the CV algorithm . For the local hydrodynamic model the linear and radial flows are obtained. The obtained solutions do not depend on a specific LHM, since the angular velocity field $\omega = \mathbf{0}$ in the collisionless regime of our 2-dimensional model. Depending on the initial velocity and density profiles, examples of these flows are considered. In both cases the front boundary was found to move uniformly along with the velocity field to maintain the constancy of the number of particles and kinetic energy. Additionally, a similar analysis shows that there exist 3-dimensional nonstationary radial flows. The results of this chapter give additional information about the dynamics of self-propelling particles side by side with the results obtained in the previous chapters for stationary flows. In particular, they demonstrate a qualitative correspondence between the obtained solutions of collisionless medium and the collisionless regimes of motion taking place in the CV model.

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Appendix A

Vorticity and circulation of the velocity field

A.1 Conservation of the kinetic energy

Imposing the kinematic constraint of the conservation of the kinetic energy, we have

$$\frac{d}{dt}\int n\mathbf{v}^2\,dV = 0, \tag{A.1}$$

where volume V moves along with the velocity field.
This equation can be rewritten as

$$\begin{aligned}\frac{d}{dt}\int n\mathbf{v}^2\,dV &= \int \frac{dn}{dt}\,n\mathbf{v}^2\,\delta V + \int n\frac{d\mathbf{v}^2}{dt}\,\delta V + \int n\mathbf{v}^2\,\frac{d\,\delta V}{dt} \\ &= \int \left[\frac{dn}{dt} + n\,(\nabla\cdot\mathbf{v})\right]\mathbf{v}^2\,dV + \int n\frac{d\mathbf{v}^2}{dt}\,dV = 0, \end{aligned}\tag{A.2}$$

where we use δ for the spatial variation. Due to the continuity equation, Eq. (2.8), the first contribution in Eq. (A.2) vanishes. This implies that the second contribution equals to zero as well for an arbitrary chosen volume V with $n > 0$ and we obtain

$$\frac{d\mathbf{v}^2}{dt} = 0. \tag{A.3}$$

A.2 Stationary regimes of motion

Taking the rotation on both sides of Eq. (2.10), one obtains:

$$\frac{\partial\,\mathrm{rot}\,\mathbf{v}}{\partial t} + \mathrm{rot}\left[(\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] = \mathrm{rot}\,(\boldsymbol{\omega}\times\mathbf{v})\,. \tag{A.4}$$

Using known result from vector analysis [47]

$$\mathbf{v}\times\mathrm{rot}\,\mathbf{v} = \frac{1}{2}\nabla\mathbf{v}^2 - (\mathbf{v}\cdot\nabla)\,\mathbf{v}, \tag{A.5}$$

one can transform Eq. (A.4) into

$$\frac{\partial\,\mathrm{rot}\,\mathbf{v}}{\partial t} = \mathrm{rot}\left[\mathbf{v}\times(\mathrm{rot}\,\mathbf{v}-\boldsymbol{\omega})\right]. \tag{A.6}$$

Introducing the quantity $\mathbf{W}(\mathbf{r},t) = \mathrm{rot}\,\mathbf{v} - \boldsymbol{\omega}$, one obtains

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \frac{\partial\mathbf{W}}{\partial t} = \mathrm{rot}\left[\mathbf{v}\times\mathbf{W}\right]. \tag{A.7}$$

A.3 Stationary vortical flow in the LHM1

In the polar coordinate system the differential operations are given by:

$$\nabla = \frac{\partial}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\varphi}\,\mathbf{e}_\varphi, \tag{A.8}$$

$$\mathrm{rot}\,\mathbf{v} = \frac{1}{r}\left(\frac{\partial\left(r\,v_\varphi\right)}{\partial r} - \frac{\partial v_r}{\partial\varphi}\right)\mathbf{e}_z. \tag{A.9}$$

For the vortical flow $\mathbf{v} = v_\varphi(r)\,\mathbf{e}_\varphi$ from Eq. (2.17) we obtain

$$\frac{dv_\varphi}{dr} = \frac{v_\varphi}{r}\left(\frac{1}{s_1 n} - 1\right), \tag{A.10}$$

which has the following solution:

$$v_\varphi = \frac{C_{\mathrm{st}}}{2\pi r}\exp\left[s_1\int\limits_{r_0}^{r}\frac{dr'}{n(r')\,r'}\right]. \tag{A.11}$$

A.4 Vorticity of the velocity field

In order to investigate the bahaviour of the vorticity of the velocity field in the LHM1, we apply rotation to Eq. (2.18):

$$\mathrm{rot}\,\frac{d\mathbf{v}}{dt} = s_1\,\mathrm{rot}\,(n\,\mathrm{rot}\,\mathbf{v}\times\mathbf{v})\ . \tag{A.12}$$

Using an expression from vector analysis [47]

$$\mathrm{rot}\,(\mathbf{a}\times\mathbf{b}) = (\mathbf{b}\cdot\nabla)\,\mathbf{a} - (\mathbf{a}\cdot\nabla)\,\mathbf{b} + \mathbf{a}\,(\nabla\cdot\mathbf{b}) - \mathbf{b}\,(\nabla\cdot\mathbf{a})\,, \tag{A.13}$$

one obtains

$$\begin{aligned}&\mathrm{rot}\,(n\,\mathrm{rot}\,\mathbf{v}\times\mathbf{v}) = \\ &\quad(\mathbf{v}\cdot\nabla)\,n\,\mathrm{rot}\,\mathbf{v} - (n\,\mathrm{rot}\,\mathbf{v}\cdot\nabla)\,\mathbf{v} + n\,\mathrm{rot}\,\mathbf{v}\,(\nabla\cdot\mathbf{v}) - \mathbf{v}\,(\nabla\cdot n\,\mathrm{rot}\,\mathbf{v})\,.\end{aligned} \tag{A.14}$$

In a case of a planar geometry the rotation of the velocity field $\mathrm{rot}\,\mathbf{v}$ is always perpendicular to the plane of the velocity field $\mathbf{v}$. Due to this fact the second and the fourth contributions on the right-hand side of Eq. (A.14) are equal to zero. Using the continuity equation (2.17) for the remaining nonzero contributions, we have:

$$\mathrm{rot}\,(n\,\mathrm{rot}\,\mathbf{v}\times\mathbf{v}) = n\,(\mathbf{v}\cdot\nabla)\,\mathrm{rot}\,\mathbf{v} - \frac{\partial n}{\partial t}\,\mathrm{rot}\,\mathbf{v}. \tag{A.15}$$

This results in the following expression for the evolution of the vorticity:

$$\mathrm{rot}\,\frac{d\mathbf{v}}{dt} = -s_1\left[\frac{\partial n}{\partial t}\,\mathrm{rot}\,\mathbf{v} - n\,(\mathbf{v}\cdot\nabla)\,\mathrm{rot}\,\mathbf{v}\right]. \tag{A.16}$$

A.5 Velocity circulation

The velocity circulation is defined by a line integral as follows:

$$C = \oint_{\mathcal{L}} \mathbf{v}\cdot d\mathbf{l}\,, \tag{A.17}$$

where the integration is taken along some closed "fluid contour" $\mathcal{L}$.
Differentiating this expression with respect to time, we have

$$\frac{dC}{dt} = \frac{d}{dt}\oint_{\mathcal{L}} \mathbf{v}\cdot d\mathbf{l} = \oint_{\mathcal{L}} \frac{d\mathbf{v}}{dt}\cdot\delta\mathbf{r} + \oint_{\mathcal{L}} \mathbf{v}\cdot\frac{d\,\delta\mathbf{r}}{dt}, \tag{A.18}$$

where the first term takes into account the change of the velocity field and the second contribution describes the change of the contour as it moves. Here d/dt and δ denote a substantional derivative and derivative with respect to the coordinates respectively. The element $d\mathbf{l}$ of the contour is performed as a difference $\delta\mathbf{r}$ between the radius-vectors $\mathbf{r}$ at the ends of the element $d\mathbf{l}$. The second contribution on the right-hand side of Eq. (A.18) can be simplified:

$$\mathbf{v}\cdot\frac{d\,\delta\mathbf{r}}{dt} = \mathbf{v}\cdot\delta\mathbf{v} = \frac{1}{2}\,\delta\left(v^2\right). \tag{A.19}$$

After integration this contribution gives zero as an integral of the total differential. From a physical point of view this is a kinetic energy which conserves in the fluid without dissipations.

Applying Stokes' theorem to Eq.(A.18), we obtain

$$\frac{dC}{dt} = \int_S \operatorname{rot}\frac{d\mathbf{v}}{dt}\cdot d\mathbf{S}, \tag{A.20}$$

which implies, according to Eq. (2.23), that circulation does not conserve in the fluid of self-propelling particles, Eqs. (2.17),(2.18).

A.6 Noise

The time evolution of the circulation has an exponential behaviour:

$$C(t) = C_0\,\exp\left[-\left(\frac{t}{\tau} + \mathcal{W}(t)\right)\right], \tag{A.21}$$

where

$$\mathcal{W}(t) = \int_0^t \delta L(t')\,dt' \tag{A.22}$$

is a Wiener process.

To calculate the average circulation we expand expression (A.21) into series:

$$\langle C(t)\rangle = C_0 \left\langle 1 - \left(\frac{t}{\tau} + \mathcal{W}\right) + \frac{1}{2!}\left(\frac{t}{\tau} + \mathcal{W}\right)^2 - \frac{1}{3!}\left(\frac{t}{\tau} + \mathcal{W}\right)^3 + \frac{1}{4!}\left(\frac{t}{\tau} + \mathcal{W}\right)^4 - \ldots \right\rangle \tag{A.23}$$

Using Wick's theorem for Gaussian averages, which states that the average of the product is equal to the sum of all possible pairings, we can calculate average values $\left\langle \mathcal{W}^{2n} \right\rangle$. In our case this theorem has the following form:

$$\left\langle \mathcal{W}^{2n} \right\rangle = \frac{(2n)!}{n!2^n} \left\langle \mathcal{W}^2 \right\rangle, \quad n \in Z, \tag{A.24}$$

where

$$\left\langle \mathcal{W}^2 \right\rangle = \int_0^t \int_0^{t_2} \langle \delta L(t')\, \delta L(t'') \rangle\, dt'\, dt'' = 2\Gamma \int_0^t \int_0^{t_2} \delta(t' - t'')\, dt'\, dt'' = 2\Gamma t. \tag{A.25}$$

This implies that the average circulation is given by

$$\langle C(t)\rangle = C_0 \exp\left[-\frac{t}{\tilde{\tau}}\right], \quad \text{where} \quad \tilde{\tau} = \frac{\tau}{1 - \Gamma\tau} \tag{A.26}$$

is a relaxation time of the circulation.

Appendix B

Continuity equation

The averaging procedure is define as follows:

$$n(\mathbf{r},t) = \sum_i H(\mathbf{r}-\mathbf{r}_i(t)), \tag{B.1}$$

$$n(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t) = \sum_i H(\mathbf{r}-\mathbf{r}_i(t))\,\mathbf{v}_i(t). \tag{B.2}$$

Differentiating (B.1) with respect to time, we obtain

$$\begin{aligned}\frac{\partial n(\mathbf{r},t)}{\partial t} &= \frac{\partial}{\partial t}\sum_i H(\mathbf{r}-\mathbf{r}_i(t)) = -\sum_i \frac{\partial}{\partial \mathbf{r}_i(t)}\cdot(-\mathbf{v}_i(t))\,H(\mathbf{r}-\mathbf{r}_i(t)) \\ &= -\frac{\partial}{\partial \mathbf{r}}\cdot\sum_i \mathbf{v}_i(t)\,H(\mathbf{r}-\mathbf{r}_i(t)) = -\operatorname{div}(n(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)).\end{aligned} \tag{B.3}$$

Thus we obtain the continuity equation:

$$\frac{\partial n(\mathbf{r},t)}{\partial t} + \operatorname{div}(n(\mathbf{r},t)\,\mathbf{v}(\mathbf{r},t)) = 0. \tag{B.4}$$

Appendix C

Velocity difference

Since the difference of two vectors of equal absolute values is perpendicular to their sum, we can write

$$\mathbf{v}_i(n+1) - \mathbf{v}_i(n) = \mathbf{X} \times (\mathbf{v}_i(n+1) + \mathbf{v}_i(n)), \tag{C.1}$$

where $\mathbf{X}$ is an unknown vector.
In order to find $\mathbf{X}$ we multiply this equation from the left by $\mathbf{v}_i(n+1) + \mathbf{v}_i(n)$ and obtain

$$\begin{aligned} 2(\mathbf{v}_i(n) \times \mathbf{v}_i(n+1)) &= \mathbf{X}(\mathbf{v}_i(n+1) + \mathbf{v}_i(n))^2 \\ &- (\mathbf{v}_i(n+1) + \mathbf{v}_i(n))(\mathbf{X} \cdot (\mathbf{v}_i(n+1) + \mathbf{v}_i(n)))\,. \end{aligned} \tag{C.2}$$

The second term on the right-hand side of Eq. (C.2) equals to zero due to a fact that $\mathbf{X}$ is perpendicular to the plane of 2-dimensional vectors $\mathbf{v}_i(n)$ and $\mathbf{v}_i(n+1)$.
This implies that

$$\mathbf{X} = 2\frac{\mathbf{v}_i(n) \times \mathbf{v}_i(n+1)}{(\mathbf{v}_i(n+1) + \mathbf{v}_i(n))^2} \tag{C.3}$$

and

$$\begin{aligned} \mathbf{v}_i(n+1) - \mathbf{v}_i(n) &= 2\frac{[\mathbf{v}_i(n) \times \mathbf{v}_i(n+1)]}{(\mathbf{v}_i(n+1) + \mathbf{v}_i(n))^2} \times [\mathbf{v}_i(n+1) + \mathbf{v}_i(n)] \\ &= \widehat{\mathbf{v}}_i(n) \times \widehat{\mathbf{v}}_i(n+1) \times \left[\frac{\mathbf{v}_i(n+1) + \mathbf{v}_i(n)}{1 + \widehat{\mathbf{v}}_i(n) \cdot \widehat{\mathbf{v}}_i(n+1)}\right], \end{aligned} \tag{C.4}$$

where $\widehat{\mathbf{v}}_i(n) = \mathbf{v}_i(n) \,/\, |\, \mathbf{v}_i(n)\,|$.

Summary

In this thesis I considered the dynamics of self-propelling particles (SPP). Flocking of living organisms like birds, fishes, ants, bacteria etc. is an area where the theory of the collective behaviour of SPP can be applied. One can often see how these animals develop coherent motion, amazing the observer by the diversity of its forms and shapes. Recently the complexity of the nonlinear dynamics of SPP has attracted a lot of attention. Further theoretical work to describe the observed biological phenomenon is clearly needed. The collective motion of SPP is driven by the presence of so-called kinematic (or nonholonomic) constraints, which are imposed on the orientations of the velocities of the particles. The tendency of the particles to adjust their velocity to the ones of the neighbours leads to the emergence of a coherent motion. In other words, these kinematic constraints can be considered as an interaction of a non-potential character. Due to these facts the usual methods of Hamiltonian dynamics cannot be applied to study the behaviour of SPP.

In the middle of 90's the investigation of SPP systems started to develop in several directions. The first numerical model describing flocking was proposed by Vicsek et al. [17]. We further call it the Czirók-Vicsek algorithm (CV algorithm). Introducing kinematic updating rules for the directions of the velocities of the particles, they obtained a transition from disordered to ordered motion of SPP at a low noise amplitude and high density.

This work initiated further investigations both on discrete and on continuous levels. These considered extensions of the CV model with different types of noise, varying absolute velocities, different boundary conditions, etc. [18, 22, 24, 25].

As mentioned above the dynamics of SPP differs from the dynamics of the classi-

cal systems due to nonholonomic constraints. That is why on a continuous level one cannot apply the Navier-Stokes equation usually used in hydrodynamics. However, some of the existing continuous models still use this approach, extending the equation with additional terms. In Ref. [21], which is proposed as a continuous analog of the discrete CV model, the self-propelling force and friction are added together with the pressure gradient and viscosity. The inclusion of the additional terms in Ref. [31] is based on symmetry consideration. Underlying arguments, taking into account the physical nature of the system, are not clearly given.

The aim of this thesis is to construct a hydrodynamic model to describe the flocking of SPP observed in nature and, in particular, to find a continuous analog of the discrete model of Vicsek et al. Our model is based on the physical properties of the CV algorithm, namely the conservation of the kinetic energy and the number of particles. In our analysis the constant nature of the absolute velocities of SPP is crucial. The self-propelling force of biological origin and the frictional forces balance each other. These factors are further not relevant to the swarming of self-propelling particles.

In Chapter 2 a hydrodynamic model with two integrals of motion (kinematic constraints) is proposed. The essential feature and importance of the model is that it has the essential properties of the CV algorithm. It contains the factors responsible for the collective motion of SPP and separates them from unnecessary contributions to the equations of motion proposed earlier in the literature. The imposed constraints determine the hydrodynamic equations. A simple model for the angular velocity field, appearing in the equations, is introduced. The average of the velocities in the surrounding of a given particle is contained in this angular velocity field. For a special case of the averaging kernels, which we called the *local hydrodynamic model (LHM)*, stationary planar flows of two types are obtained: linear flow and vortical hydrodynamic flow. A remarkable property of the vortical flow obtained is that it has finite flocking behaviour, where the density and the velocity fields are coupled. This is what one observes in nature. Moreover, qualitatively these results are similar with those obtained in Ref. [19]. Because of the lack of experimental data, it is hard to make any quantitative comparisons with the motion of self-propelling particles observed in nature.

In the second part of this chapter the properties of the LHM are investigated. It is shown that in contrast to the classical ideal fluid, the velocity circulation does not conserve in a fluid of self-propelling particles due to the presence of non-potential interactions. Using this result, the stability properties of the ordered state of the system with respect to noise are investigated in the last part of the second chapter. At large noise the circulation diverges and the system becomes unstable; for small noise the system is stable. A similar result was obtained by Vicsek et al. [17].

In Chapter 3 the class of stationary flows of the local model is determined. Using conformal representation, it is shown that in the local model the 2-dimensional stationary solutions can be of two types only: linear and vortical flow. Apart from being mathematically elegant, this result shows that other stationary flows are not possible in the model proposed in this thesis. Furthermore, some other properties of the vortical flows corresponding to different models of the angular velocity field are presented. For different choices of the LHM the finite flocking behaviour with compact support, in which the density is low (zero) outside some region, is obtained. From the physical point of view these flows are of interest because of their realization in nature.

In the next chapter, Chapter 4, the stability properties of the stationary linear and vortical flows are determined. For different types of velocity and density field perturbations the system with linear flow of SPP demonstrates a neutral stability. This means that the time evolution of the perturbation has an oscillatory behaviour, it neither grows nor decays. Regarding the stationary vortical flow, the flow with constant velocity and density profiles is considered. The linear analysis does not give a conclusive answer about the stability of the vortical flow and further stability analysis of the higher orders is required.

In Chapter 5 the connection between two approaches, the discrete and continuous, is analysed. The averaging procedure is introduced and is used to coarse-grain the discrete CV algorithm. The CV rule is rewritten in terms of the angular velocities associated with the particle velocities and the corresponding expression for the angular velocity in the continuous time description is derived. It is found to be proportional to the average velocity and acceleration in the neighbourhood of a given particle. The only approximation used is the smallness of the velocity difference between the time steps, i.e. $\mid \mathbf{v}_i\,(n+1) - \mathbf{v}_i\,(n) \mid \ll 1$. This is a reasonable assumption since we are interested in obtaining a continuous description. Then the continuous equations of motion are derived and the angular velocity field following from the discrete CV model is obtained.

The similarities and differences between the hydrodynamic model proposed in this thesis and the coarse-grained CV algorithm are discussed. Both models have as planar stationary solutions the linear and vortical flows. The analysis of this chapter shows that the freedom in the modeling of the angular velocity field in our model can be used to modify the model in such a way that it will be closer to the CV model. Or alternatively, one may see which of the discrete rules gives the corresponding continuous description. The importance of the results in this chapter is that they give a link between the discrete and continuous descriptions and make the comparison between both approaches possible. The results clarify which of the continuous models proposed in Chapter 2 is a hydrodynamic analog of the CV algorithm. The analysis in this chapter clarifies the freedom in the choice of the angular velocity field. The investigation of alternative choices is an important line of further work.

In the last chapter, Chapter 6, the nonstationary flows of the local hydrodynamic model are obtained. They are found to be linear and radial. These results demonstrate a qualitative correspondence between the obtained solutions of collisionless medium and the collisionless regimes of motion taking place in the CV algorithm.

As a future directions of the investigation, it is important to consider the noise influence on the ordered motion of self-propelling particles in more detail. One of the possibilities is to include noise in the angular velocity field. This will lead to a stochastic contributions in the velocity and the density. Another direction is a further analysis of the stability of the vortical flow. It would be interesting to obtain a more definite answer regarding its stability.

Samenvatting

In dit proefschrift heb ik de dynamica beschouwd van zogenaamde zelfaandrijvende deeltjes (we gebruiken hiervoor de afkorting SPP van "self-propelling particles"). Flocking (zwermen, kuddes) van levende organismen zoals vogels, vissen, mieren en bacteriën is een situatie waarvoor de theorie over het collectief gedrag van SPP van toepassing is. Men ziet vaak hoe deze dieren coherente beweging ontwikkelen en men verbaasd zich over de diversiteit van de patronen. Recent heeft de complexiteit van de niet-lineaire dynamica van SPP veel aandacht getrokken. Verder theoretisch werk is kennelijk nodig om de waargenomen biologische verschijnselen te beschrijven. De collectieve beweging van SPP wordt aangedreven door de aanwezigheid van zogenaamde kinematische (of niet-holonome) restricties voor de oriëntaties van de snelheden van de deeltjes. De neiging van de deeltjes om hun snelheid aan te passen aan die van naburige deeltjes leidt tot het ontstaan van coherente beweging. Met andere woorden, deze kinematische restricties kunnen beschouwd worden als een interactie van een niet-potentiaal karakter. Vanwege de bovengenoemde feiten kunnen de gebruikelijke methoden van de Hamilton dynamica niet toegepast worden om het gedrag van SPP te bestuderen.

In het midden van de jaren ’90 begon het onderzoek naar SPP systemen zich te ontwikkelen in meerdere richtingen. Het eerste numerieke model om flocking te beschrijven werd voorgesteld door Vicsek et al. [17]. Dit noemen we het Czirók-Vicsek algoritme (CV algoritme). Door kinematische regels voor de modificatie van de richtingen van de deeltjessnelheden te introduceren, verkregen zij een overgang van wanordelijke naar geordende beweging van SPP bij lage ruis en hoge dichtheid.

Dit werk heeft verder onderzoek op gang gebracht, zowel op discreet als continuüm niveau. Hierin worden uitbreidingen van het CV model bestudeerd met verschillende types ruis, variërende absolute snelheden, verschillende randvoorwaarden, etc. [18, 22, 24, 25].

Zoals boven vermeld onderscheidt de dynamica van SPP zich van de dynamica van klassieke systemen door de aanwezigheid van niet-holonome restricties. Dit is de reden dat men, voor een continuümbeschrijving, geen gebruik kan maken van de Navier-Stokes vergelijking die meestal gebruikt wordt in de hydrodynamica. Echter, sommige bestaande continuümmodellen gebruiken nog steeds deze benadering en breiden de vergelijking uit met aanvullende termen. In Ref. [21], voorgesteld als een continuüm analogon van het discrete CV model, zijn de zelfaandrijvende kracht en de wrijving bij elkaar opgeteld met de drukgradiënt en de viscositeit. De keuze van de aanvullende termen in Ref. [31] is gebaseerd op symmetrie overwegingen. De onderliggende argumenten met betrekking tot de fysische aard van het systeem worden niet duidelijk gegeven.

Het doel van dit proefschrift is om een hydrodynamische model te ontwikkelen om de flocking van SPP waargenomen in de natuur te beschrijven en met name een continuüm analogon te vinden voor het discrete model van Vicsek et al. Ons model is gebaseerd op de fysische eigenschappen van het CV algoritme, namelijk het behoud van de kinetische energie en het aantal deeltjes. In onze analyse is de constante aard van de absolute snelheden van de SPP essentiëel. De zelfaandrijvende kracht van biologische oorsprong en de wrijvingskrachten balanceren elkaar. Deze factoren zijn verder niet relevant voor de flocking van SPP.

In Hoofdstuk 2 is een hydrodynamische model met twee integralen van de beweging (kinematische restricties) voorgesteld. Het cruciale kenmerk en belang van dit model is het feit dat het de essentiële kenmerken heeft van het CV algoritme. Het bevat de factoren verantwoordelijk voor de collectieve beweging van SPP en scheidt deze van onnodige bijdrages aan de bewegingsvergelijkingen eerder voorgesteld in de literatuur. De opgelegde restricties bepalen de hydrodynamische vergelijkingen. Een simpel model voor het hoeksnelheidsveld, van belang in de vergelijkingen, wordt geïntroduceerd. Het gemiddelde van de snelheden in de omgeving van een deeltje bepaald dit hoeksnelheidsveld. Voor een speciaal geval van de integraalkernen, die we *het lokale hydrodynamische model (LHM)* noemen, zijn een tweetal soorten stationaire stromen verkregen: lineaire en roterende (draaiende) hydrodynamische stromen. Een opmerkelijke eigenschap van de verkregen roterende stroom is dat het eindig flocking gedrag vertoont, waarbij het dichtheids- en snelheidsveld zijn gekoppeld. Dit is wat men inderdaad waarneemt in de natuur. Bovendien zijn deze resultaten, kwalitatief gezien, gelijk aan die verkregen in Ref. [19]. Aangezien gedetaileerde experimentele data ontbreken, is het moeilijk om kwantitatieve vergelijkingen te maken met de waargenomen beweging van SPP in de natuur.

In het tweede gedeelte van dit hoofdstuk zijn de eigenschappen van het LHM onderzocht. Het blijkt dat in tegenstelling tot het klassieke ideale fluïdum, er geen behoud is van snelheidscirculatie in een fluïdum van SPP vanwege de aanwezigheid van niet-potentiële interacties. Gebruikmakend van dit resultaat zijn de stabiliteitseigenschappen van de geordende toestand van het systeem onderzocht met betrekking tot ruis in het laatste gedeelte van het tweede hoofdstuk. Bij veel ruis divergeert de circulatie en wordt het systeem onstabiel; bij weinig ruis is het systeem stabiel. Een vergelijkbare resultaat is verkregen door Vicsek et al. [17].

In Hoofdstuk 3 is de klasse van stationaire stromen van het lokale model bepaald. Gebruikmakend van conforme afbeeldingen is het gebleken dat er twee types zijn van de 2-dimensionale stationaire oplossingen in het lokale model: lineaire en roterende stromen. Afgezien van mathematische elegantie, toont dit resultaat aan dat andere stationaire stromen niet mogelijk zijn in het model voorgesteld in dit proefschrift. Verder, zijn andere karakteristieken van de roterende stromen geïntroduceerd die overeenkomstig zijn met verschillende modellen van het hoeksnelheidsveld. Voor verschillende keuzes van het LHM wordt het eindige flocking gedrag verkregen, waarbij de dichtheid laag (nul) is buiten een bepaald gebied. Vanuit een fysisch oogpunt zijn deze stromen van belang vanwege hun verwezenlijking in de natuur.

In het volgende hoofdstuk, Hoofdstuk 4, zijn de stabiliteitseigenschappen bepaald van de stationaire lineaire en roterende stromen. Voor verschillende typen verstoringen van het snelheids- en dichtheidsveld demonstreert het systeem met lineaire stroom van SPP een neutrale stabiliteit. Dit betekent dat de ontwikkeling van de verstoring in de loop van de tijd een oscillerend gedrag vertoont, met een constante amplitude. Ten opzichte van de stationaire roterende stroom, is de stroom met constante snelheids- en dichtheidsprofiel beschouwd. De lineaire analyse levert geen definitief antwoord over de stabiliteit van de roterende stroom. Hiervoor is een verdere stabiliteitsanalyse van de hogere ordes nodig.

In Hoofdstuk 5 is het verband tussen twee benaderingen, de discrete en de continue, geanalyseerd. De "coarse-graining" procedure wordt geïntroduceerd en gebruikt om een continue versie van het discrete CV algoritme te verkrijgen. De CV regel is herschreven in termen van de hoeksnelheden met betrekking tot de deeltjessnelheden. De daarbij behorende uitdrukking voor de hoeksnelheid in de continue tijdsbeschrijving wordt afgeleid. Gevonden wordt dat deze evenredig is met de gemiddelde snelheid en de versnelling in de omgeving van een bepaald deeltje. De enige gebruikte benadering is de kleinheid van het snelheidsverschil tussen de tijdsstappen, oftewel $| \mathbf{v}_i(n+1) - \mathbf{v}_i(n) | \ll 1$. Dit is een redelijke aanname aangezien we geïnteresseerd zijn in het verkrijgen van een continuümbeschrijving. Daarna zijn de continuüm bewegingsvergelijkingen afgeleid en het hoeksnelheidsveld volgend uit het discrete CV model verkregen.

De overeenkomsten en verschillen tussen het hydrodynamische model voorgesteld in dit proefschrift en het "coarse-grained" CV algoritme zijn behandeld. Beide modellen hebben als planaire stationaire oplossingen de lineaire en roterende stromen. De analyse van dit hoofdstuk toont aan dat de vrijheid in het modelleren van het hoeksnelheidsveld in ons model gebruikt kan worden om het model zodanig te modificeren dat het meer lijkt op het CV model. Of, men kan zien welke van de discrete regels de daarbij behorende continuümbeschrijving levert. De resultaten van dit hoofdstuk zijn belangrijk omdat ze een verband leggen tussen de discrete en continuümbeschrijvingen en een vergelijking tussen de twee benaderingen mogelijk maken. De resultaten maken duidelijk welke van de voorgestelde continuümmodellen in Hoofdstuk 2 een hydrodynamische analogon is van het CV algoritme. De analyse van dit hoofdstuk verduidelijkt de keuzevrijheid van het hoeksnelheidsveld. Het zoeken naar alternatieve keuzes is een belangrijke richting voor verdere onderzoek.

In het laatste hoofdstuk, Hoofdstuk 6, zijn de niet-stationaire stromen van het lokale hydrodynamische model verkregen. Deze blijken lineair en radiaal te zijn. Deze resultaten demonstreren een kwalitatieve overeenkomst tussen de verkregen oplossingen van een botsingsvrij medium en de botsingsvrije bewegingsregimes die plaatsvinden in het CV algoritme.

Als een richting voor onderzoek in de toekomst, is het van belang om de invloed van ruis op de geordende beweging van SPP nader te beschouwen. Een mogelijkheid is om ruis op te nemen in het hoeksnelheidsveld. Dit zal leiden tot stochastische bijdragen in de snelheid en de dichtheid. Een andere richting is een verdere analyse van de stabiliteit van de roterende stroming. Het zou interessant zijn om een meer compleet antwoord te krijgen omtrent haar stabiliteit.

Curriculum Vitae

I was born on 12th of November, 1981 in Odessa, Ukraine.

At the age of five I became a pupil of a musical school. After one year I went to a regular school. During my study there I discovered a particular interest to exact sciences, such as mathematics and physics. I graduated from the school with a gold medal in June 1998.

The same year I entered the Faculty of Physics of the I.I. Mechnikov Odessa National University and in September 2000 I became a student of the Faculty of Mathematics in the same university.

Also in 2000, I entered the Department of Theoretical Physics. There I started to work on critical phenomena in Coulombic systems under the supervision of Dr. V. L. Kulinskii. In July 2002 I received a *cum laude* Bachelor Degree in Theoretical Physics. In my Bachelor Thesis "Effective Hamiltonian of the System in the Vicinity of Tricritical Point" a magnetic system with interactions of different types of symmetry between the spins was considered and a canonical form of the Hamiltonian was obtained. In July 2003 I received a *cum laude* Master Degree in Theoretical Physics. A Hamiltonian of the dipolar phase of the Coulombic system was constructed and the stability of the system with respect to the addition of a small amount of ions was investigated in my Master Thesis entitled "Model Hamiltonian and Dipolar Phase Stability of Coulombic System".

In September 2003 I started a PhD research with Prof. Dr. D. Bedeaux in the group "Colloid and Interface Science" at the Leiden University.

I had poster presentations at the following meetings: International Workshop

"Brownian Motion 100 years after Einstein" (Oud Poelgeest, The Netherlands); 6th Liquid Matter Conference (Utrecht, The Netherlands); International Summer School "Fundamental Problems in Statistical Physics XI" (Leuven, Belgium); Workshop "Dynamics of Patterns" (Leiden, The Netherlands); Symposium on Dynamics of Patterns (Enschede, The Netherlands). I gave a talk at the "Vloeistoffen en grensvlakken" meeting in Lunteren, The Netherlands.

List of publications

1. "Effective Hamiltonian of the system in the vicinity of tricritical point", V. L. Kulinskii, V. I. Ratushnaya, Visnyk Lviv. Univ., Ser. Physic. **38**, 175 (2005).
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