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Title: Group representations in Banach spaces and Banach lattices
Date: 2012-04-18

Chapter 1

Introduction

Motivated by quantum mechanics, amongst others, where there are many examples of group representation in Hilbert spaces, strongly continuous unitary representations of locally compact groups have been studied extensively. In particular, the decomposition theory is now well developed, which will now be illustrated by an example. Consider the group $[0, 2\pi)$ with addition modulo 2π, which is isomorphic to the circle group S^1, and the Hilbert space $L^2([0, 2\pi))$. We examine the left regular representation ρ of the group $[0, 2\pi)$ in $L^2([0, 2\pi))$ defined by $(\rho_s f)(x) := f(x - s)$, for $f \in L^2([0, 2\pi))$ and $s, x \in [0, 2\pi)$. The collection of functions $\{e^{in\cdot} := x \mapsto e^{inx}\}_{n\in\mathbb{Z}} \subset L^2([0, 2\pi))$ forms an orthogonal basis of $L^2([0, 2\pi))$, and the decomposition of a function $f \in L^2([0, 2\pi))$ into this orthogonal basis is its Fourier series $f = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\cdot}$, where

$$\hat{f}(n) := \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{e^{inx}}\,dx \tag{1.1}$$

denotes the n-th Fourier coefficient. Then, by the properties of the Fourier series, we obtain for $s \in [0, 2\pi)$,

$$\rho_s\left(\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\cdot}\right) = \sum_{n\in\mathbb{Z}} e^{-ins}\hat{f}(n)e^{in\cdot}. \tag{1.2}$$

If we fix $n \in \mathbb{Z}$, then the one dimensional subspace spanned by $e^{in\cdot}$ is invariant under ρ, and on this subspace the operator ρ_s is just a pointwise multiplication by e^{-ins}. In particular, the restriction of ρ to this subspace is an irreducible representation. Hence ρ is an orthogonal direct sum over $n \in \mathbb{Z}$ of irreducible representations. This example is a special case of the unitary representation theory of compact groups, which states that for any strongly continuous unitary representation of a compact group, the representation splits as an orthogonal direct sum of finite dimensional irreducible representations.

For non-compact locally compact groups such a direct sum decomposition is not always possible, but it is still possible to view the original representation as somehow "built up" from irreducible ones. The following example, which is similar to the above example, explains how this is done. Let G be any abelian locally compact group with a Haar measure μ, i.e., a left invariant regular measure on G which is finite on compact sets. Again we consider the left regular representation ρ of G on $L^2(G,\mu)$ defined by $(\rho_s f)(r) := f(r-s)$, for $s, r \in G$. Consider the dual group $\Gamma = \mathrm{Hom}(G, S^1)$. The dual group, equipped with the compact-open topology, is again a locally compact abelian group. An example is $G = \mathbb{R}$ with its natural topology, then $\Gamma \cong \mathbb{R}$ with its natural topology. The Fourier transform $f \mapsto \hat{f}$ from $L^1(G,\mu)$ to $C_0(\Gamma)$ defined by

$$\hat{f}(\gamma) := \int_G f(r)\overline{\gamma(r)}\, d\mu(r), \quad \gamma \in \Gamma$$

is a generalization of (1.1), and its restriction to $L^1(G,\mu) \cap L^2(G,\mu)$ maps isometrically into $L^2(\Gamma,\lambda)$, where λ is an appropriately chosen Haar measure on Γ, and it extends to an isometric isomorphism of Hilbert spaces between $L^2(G,\mu)$ and $L^2(\Gamma,\lambda)$. So we may transport our representation ρ from $L^2(G,\mu)$ to $L^2(\Gamma,\lambda)$, and there we obtain, using the properties of the Fourier transform, for $f \in L^2(G,\mu)$, $s \in G$ and almost every $\gamma \in \Gamma$,

$$(\rho_s \hat{f})(\gamma) = \overline{\gamma(s)}\hat{f}(\gamma),$$

which is similar to (1.2); here the representation corresponds to a pointwise almost everywhere multiplication by $\overline{\gamma(s)}$, some sort of "integral" of pointwise multiplications, in particular, of irreducible representations. This can be formalized using the notion of direct integrals of Hilbert spaces and direct integrals of representations, and using this notion, we can state the main theorem on decomposing strongly continuous unitary representations of locally compact groups in terms of irreducible representations, cf. [50, Corollary 14.9.5].

Theorem 1.1. *Let G be a separable locally compact group, H a separable Hilbert space and ρ a strongly continuous unitary representation of G on H. Then H is isometrically isomorphic to a direct integral of Hilbert spaces, such that under this isomorphism, the representation ρ corresponds to a direct integral of irreducible representations.*

Unitary representations often arise naturally whenever a group acts on a set, e.g., let $X \subset \mathbb{C}$ be the closed unit disc with Lebesgue measure, then S^1 acts naturally on X and a strongly continuous unitary representation ρ of S^1 in $L^2(X)$ is defined by $(\rho_s f)(x) := f(s^{-1}x)$, for $s \in S^1$, $f \in L^2(X)$ and $x \in X$. By the above such representations can be decomposed into irreducibles. However, the same formula yields a strongly continuous representation of S^1 in the Banach spaces $L^p(X)$, for $1 \leq p < \infty$, and in $C(X)$. Hence such representations on Banach spaces occur naturally - what about these representations? Do we have a similar decomposition theory as in the unitary case?

This thesis is a contribution to the theory of such representations. It consists of two parts. The first part, Chapter 2, is about the crossed product. The second part, Chapters 3 and 4, is about positive representations in partially ordered vector spaces. We will now discuss the first part.

Crossed products

When studying group representations, it is often useful to look at algebras. An example is the group algebra $k[G]$ of a finite group G over a field k. This algebra has the property that there is a bijection between representations of G on k-vector spaces and algebra representations of $k[G]$ on the same vector space, and so questions about group representations can be translated into questions about algebra representations.

In the theory of unitary representations, such an algebra also exists. Given a locally compact group G, this object is the group C^*-algebra $C^*(G)$, a C^*-algebra for which the nondegenerate algebra representations in a Hilbert space are in bijection with the strongly continuous unitary representations of the group in that Hilbert space. The C^*-algebra $C^*(G)$ is a crucial tool in proving Theorem 1.1. Indeed, all the hard work lies in proving a similar fact about representations of C^*-algebras on Hilbert spaces, and then the result about unitary representations follows immediately by applying this to $C^*(G)$.

In view of the above it is desirable, given a locally compact group G, to have a Banach algebra such that some of its algebra representations in certain Banach spaces are in bijection with some of the strongly continuous group representations of G in the same class of Banach spaces. The reason for not considering all representations is the following. In the Hilbert space case, there is up to isomorphism only one separable infinite dimensional Hilbert space. However, there is a great diversity of infinite dimensional separable Banach spaces, and to consider all representations in all these spaces seems a daunting task. It would be much better if the above Banach algebra can be specialized to specific situations. E.g, if one is interested in strongly continuous isometric group representations in L^p-spaces for some $p \geq 1$, then one would want a Banach algebra such that its algebra representations in L^p-spaces are in bijection with these group representations. Or, if one is interested in uniformly bounded representations in spaces of continuous functions, then one would want a different Banach algebra with a bijection concerning these representations in these spaces. In other words, the construction of the group C^*-algebra needs to be generalized such that it can be adapted to whatever representations one is interested in, instead of only the unitary group representations.

The group C^*-algebra is a special case of a more general object called a crossed product C^*-algebra, which is not only useful in translating unitary group representations to algebra representations, but also has applications concerning induced representations of subgroups. Hence we will generalize the crossed product C^*-algebra, which we will now briefly discuss.

A *C^*-dynamical system* is a triple (A, G, α), where A is a C^*-algebra, G is a locally compact group and $\alpha\colon G \to \mathrm{Aut}(A)$ is a strongly continuous action of G on A. This can be viewed as a generalization of a locally compact group, since if $A = \mathbb{C}$, the action α has to be trivial and G is the only nontrivial object. A *covariant representation* of (A, G, α) is a pair (π, U), where π is a representation of

A in a Hilbert space, and U is a strongly continuous unitary representation of G in the same Hilbert space, satisfying the covariance relation

$$U_s \pi(a) U_s^{-1} = \pi(\alpha_s(a)),$$

for all $s \in G$ and $a \in A$. Again, if $A = \mathbb{C}$, π is trivial and hence the covariant relation is always satisfied, and the group representation is the only nontrivial object. Hence studying covariant representations of $(\mathbb{C}, G, \alpha)$ is the same as studying strongly continuous unitary representations of G. The crossed product C^*-algebra is a C^*-algebra $A \rtimes_\alpha G$ with the property that the class of covariant representations of (A, G, α) is in bijection with the class of nondegenerate representations of $A \rtimes_\alpha G$ in Hilbert spaces. If $A = \mathbb{C}$, we recover the group C^*-algebra $C^*(G)$.

The above can be generalized to the Banach algebra case as follows. A Banach algebra dynamical system is a triple (A, G, α) with the same properties as a C^*-dynamical system, except that A is only assumed to be a Banach algebra. Covariant representations are generalized by allowing the representations to be in Banach spaces instead of Hilbert spaces. The main difference is in the class of covariant representations being considered; in the Hilbert space case, this class equals *all* covariant representations in Hilbert spaces. Since, as explained earlier, we want to vary the class of covariant representations being considered, this class is an additional variable, which will be called $\mathcal{R}$, going into the crossed product construction. It turns out that one cannot consider all classes $\mathcal{R}$. There has to be some uniform bound on the norm of the algebra representations, and the norm of the group representations has to be uniformly bounded by some fixed function $\nu\colon G \to [0, \infty)$ which is bounded on compact sets, i.e., $\|U_r\| \leq \nu(r)$ for all $(\pi, U) \in \mathcal{R}$. Note that these requirements are automatically satisfied in the C^*-algebra case, as C^*-algebra representations in Hilbert spaces are contractive and unitary representations are isometric. Given such a class $\mathcal{R}$, one also needs to define the class of $\mathcal{R}$-continuous covariant representations (Definition 2.5.1), which is in general a larger class than $\mathcal{R}$. In the C^*-algebra case, these classes coincide.

A technical obstacle that we encountered while generalizing the crossed product C^*-algebra is as follows. In the C^*-algebra case, it is at some point necessary to extend a bounded nondegenerate representation of a C^*-algebra to its multiplier algebra, which can be done easily using C^*-theory. The Banach algebra analogue of this is, given a nondegenerate bounded representation of a Banach algebra, to obtain an extension of this representation to centralizer algebras of the original Banach algebra. It turns out that this can be done in a satisfactory manner, which was worked out in [9]. After overcoming this technical obstacle, and incorporating the new features concerning the $\mathcal{R}$-continuous covariant representations, it turns out that the crossed product C^*-algebra can indeed be generalized, cf. Theorem 2.8.1, the main result of Chapter 2. It states that, given a Banach algebra dynamical system with a mild condition on A (it has to have a bounded approximate left identity), a class $\mathcal{R}$ as above and a class $\mathcal{X}$ of Banach spaces, there is a Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ such that its bounded nondegenerate algebra representations in spaces from $\mathcal{X}$ are in bijection with the class of $\mathcal{R}$-continuous covariant representations of (A, G, α) in spaces from $\mathcal{X}$.

We have specialized the above result to certain natural classes of group representations in Banach spaces, cf. Section 2.9, and this allows us to study such group representations by studying the representations of the specific Banach algebras thus obtained instead. Further investigation of these special Banach algebras is needed to optimally exploit this result, but at least the problem has become more tractable now from a functional analytical point of view, since a Banach algebra has more functional analytic structure than a group. Given the success of the archetypical transition from the group to the group C^*-algebra in the case of unitary group representations, the availability of such a transition, "tuned to the situation at hand", can be considered as a step forward towards a decomposition theorem for other classes of group representations than the unitary ones.

Another type of group representations one would like to understand better are the positive group representations, which we will discuss in the next section. For such representations, the appropriate Banach algebra of crossed product type will have to be a so-called Banach lattice algebra, and the construction of the crossed product needs further modification. We leave this for further research, noting that the results and techniques in Chapter 2 provide a concrete model to start from.

For unitary representation of compact groups, however, it is already possible to obtain the existence of a decomposition into irreducibles without the use of the group C^*-algebra. One might hope that a similar phenomenon occurs for positive representations of compact groups. As is shown in Chapters 3 and 4, which constitute the second part of this thesis, for certain spaces this is indeed the case.

We will now specialize our discusssion from general Banach space representations to positive representations.

Positive representations

We return again to our motivating example of the representation ρ of S^1 in $L^p(X)$ ($1 \leq p < \infty$) and $C(X)$, where $X \subset \mathbb{C}$ is the closed unit disc, defined by

$$(\rho_s f)(x) := f(s^{-1}x),$$

for $s \in S^1$ and $x \in X$. It is clear that, for $s \in S^1$, the maps ρ_s are positive, i.e., they map positive functions to positive functions. This positivity of the operators ρ_s is the context for Chapter 3 and Chapter 4.

The natural partial order on functions spaces such as $L^p(X)$, i.e., $f \geq g$ if and only if $f(x) \geq g(x)$ for almost every x, can be studied by the following abstraction. A *partially ordered vector space* is a real vector space equipped with a partial order that is compatible with the vector space structure, i.e., if the vectors x and y are positive, then $x + y$ is positive, and if λ is a positive scalar, then λx is positive. A *Riesz space* is a partially ordered vector space where each pair of vectors x and y has a supremum $x \vee y$ and an infimum $x \wedge y$. In a Riesz space one can define an absolute value as in function spaces, by $|x| := x \vee (-x)$, and x and y are called *disjoint* if $|x| \wedge |y| = 0$. If L is a subset of a Riesz space, then L^d denotes the disjoint complement of A, i.e., all vectors that are disjoint from all vectors from

A. A *Banach lattice* is a Riesz space, and a Banach space, such that the norm is compatible with the order structure, i.e., if x and y are vectors such that $|x| \leq |y|$, then $\|x\| \leq \|y\|$. Many function spaces considered in analysis, such as L^p-spaces and spaces of continuous functions, are Banach lattices.

By the above it makes sense to study positive representations in Banach lattices, as they appear naturally: whenever there is a group acting on some set, more often than not there is an induced positive representation in a Banach lattice of functions defined on that set. Very little is known about positive representations. The natural question in this case, similar to the unitary case, is whether a positive representation in a Banach lattice can be decomposed into order indecomposable subrepresentations. This will be the main theme in our investigations of positive representations.

In Chapter 3 we consider the simplest case: the finite dimensional case. Since vector space topologies are not interesting in the finite dimensional setting, we look at the more general setting of Riesz spaces, instead of Banach lattices. We are interested in decompositions, and a natural question is whether an order indecomposable positive representation of a finite group is finite dimensional. With an order indecomposable positive representation we mean a positive representations such that the Riesz space cannot be written as the order direct sum of two subspaces that are both invariant under the representation. An order direct sum of a Riesz space E means a direct sum $E = L \oplus M$, such that whenever $x = y + z \in E$ is positive, with $y \in L$ and $z \in M$, then y and z are positive. In this case L and M are automatically so-called projection bands which are each other's disjoint complement, so $E = L \oplus L^d$. This is similar to the Hilbert space case, where a closed linear subspace L of a Hilbert space H induces an orthogonal decomposition $H = L \oplus L^\perp$. Moreover, if a projection band is invariant under a positive representation of a group, its disjoint complement is also invariant, which is similar to the Hilbert space case where the orthogonal analogue holds for a unitary representation of a group. This easily implies that order indecomposability of a positive representation in a Riesz space is equivalent with its projection band irreducibility, i.e., that it does not possess a nontrivial proper invariant projection band. Again, this is similar to the case of unitary representations, where indecomposability is equivalent with irreducibility, i.e., with the absense of nontrivial proper invariant closed subspaces. Hence the above question can be reformulated as: is a projection band irreducible positive representation of a finite group finite dimensional?

The corresponding question in the unitary case is trivially true. Indeed, let ρ be an irreducible unitary representation of a finite group in a Hilbert space H. Take a nonzero vector $x \in H$ and consider the subspace generated by its orbit under ρ. This subspace is finite dimensional and hence closed, and by construction ρ-invariant, so it must equal the whole space H, hence H is finite dimensional. Unfortunately this proof breaks down in the ordered setting, as bands, in particular projection bands, are generally infinite dimensional, e.g., in $L^p([0,1])$, for any p, all nontrivial bands are infinite dimensional.

In the ordered case, the above question has a negative answer. Indeed, the representation of the trivial group in $C([0,1])$ is projection band irreducible. In this example the cause lies with the Riesz space, one might say, as $C([0,1])$ does not have any proper nontrivial projection bands at all: it is far from being what is called Dedekind complete. All L^p-spaces, on the other hand, are Dedekind complete.

If we assume that the Riesz space is Dedekind complete, then the situation improves. In this case we managed to show, with an unusual proof based on induction on the order of the group, that if a positive representation of a finite group in a Dedekind complete Riesz space is projection band irreducible, then the Riesz space is finite dimensional, cf. Theorem 3.3.14. In this theorem we actually prove a little bit more, but this is the most important consequence.

Having obtained this result, we then looked at the positive representations of finite groups in finite dimensional Archimedean Riesz space. These spaces are order isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$ with pointwise ordering. We first study the automorphism group of $\mathbb{R}^n$, i.e., the group of positive matrices with positive inverses, and show that it equals the semidirect product of the subgroup of diagonal matrices with strictly positive entries on the diagonal, and the subgroup of permutation matrices. Using some basic group cohomology methods, it turns out that every finite group of positive matrices equals a group of permutation matrices conjugated by a single diagonal matrix with strictly positive diagonal elements, and from this it follows easily that every positive representation equals a permutation representation conjugated by such a diagonal matrix. From this we obtain a nice characterization of the order dual of a finite group, i.e., the space of order equivalence classes of irreducible positive representations, in terms of group actions on finite sets, cf. Theorem 3.4.10, which is as follows.

Theorem 1.2. *Let G be a finite group. If $H \subset G$ is a subgroup, let E_H be the $|G : H|$-dimensional vector space of real-valued functions on G/H, equipped with the pointwise ordering. Let π^H be the canonical positive representation of G in E_H, corresponding to the action of G on G/H. Then, whenever H_1 and H_2 are conjugate, π^{H_1} and π^{H_2} are order equivalent, and the map*

$$[H] \mapsto [\pi^H]$$

is a bijection between the conjugacy classes of subgroups of G and the order equivalence classes of irreducible positive representations of G in nonzero finite dimensional Archimedean Riesz spaces.

Additionally, we obtain a unique decomposition of positive representations of finite groups in finite dimensional Archimedean Riesz spaces into band irreducibles.

We also show that characters do not, in general, determine representations, in the sense that there even exist band irreducible positive finite dimensional representations of finite groups, having the same character, which are not order isomorphic. Finally, we look at induction in the ordered setting, the categorical aspects of which are largely the same as in the nonordered setting, but for which the multiplicity version of Frobenius reciprocity turns out not to hold.

In Chapter 4 we take the above results to the next level: that of compact groups. As the image of a strongly continuous representation of a compact group in a Banach lattice is a group of positive operators which is compact in the strong operator topology, such compact groups of positive operators are investigated. We assume that these groups are contained in the product, which again is a semidirect product, of the group of central lattice automorphisms and the group of isometric lattice automorphisms. This is motivated by the above results on representations in $\mathbb{R}^n$ equipped with any of the p-norms, where the isometric lattice automorphisms are the permutation matrices and the central lattice automorphisms are the diagonal matrices with strictly positive elements on the diagonal, and so the whole automorphism group of $\mathbb{R}^n$ equals this semidirect product and hence every subgroup satisfies this assumption. This characterization of the automorphism group is also satisfied in many natural sequence spaces and spaces of continuous functions. However, not every Banach lattice has such a nice characterization of the automorphism group.

Under the additional technical Assumption 4.3.3 which is satisfied in many natural sequence spaces and spaces of continuous functions, we are able to obtain, as in the finite dimensional case, that such a compact group equals a group of isometric lattice automorphisms conjugated by a single central lattice automorphism. This is especially useful in the aforementioned spaces, as we have a nice description available of both the isometric lattice automorphisms and the central lattice automorphisms. Again this leads to a similar description of strongly continuous positive representations of compact groups with range in this product: it is a strongly continuous isometric positive representation conjugated by a single central lattice automorphism. Since everything depends only on the compactness in the strong operator topology of the image of the representation, we have the same result for arbitrary representations of arbitrary groups with compact image. Applying these results to the sequence space case, we obtain the following ordered analogue of the aforementioned theorem on the decomposition of strongly continuous unitary representations of compact groups, which is as follows, cf. Theorem 4.5.7.

Theorem 1.3. *Let E be a normalized symmetric Banach sequence space, let G be a group and let ρ be a positive representation of G in E. Then E splits into band irreducibles, in the sense that there exists an α with $1 \leq \alpha \leq \infty$ such that the set of invariant and band irreducible bands $\{B_n\}_{1\leq n\leq\alpha}$ (if $\alpha < \infty$) or $\{B_n\}_{1\leq n<\infty}$ (if $\alpha = \infty$) satisfies*

$$x = \sum_{n=1}^{\alpha} P_n x \quad \forall x \in E, \tag{1.3}$$

where $P_n \colon E \to B_n$ denotes the band projection, and the series is unconditionally order convergent, hence, in the case that E has order continuous norm, unconditionally convergent.

Moreover, if ρ has compact image and E has order continuous norm or $E = \ell^\infty$, then every invariant and band irreducible band is finite dimensional, and so $\alpha = \infty$.

Examples of normalized symmetric Banach sequence spaces with order continuous norm are the classical sequence spaces c_0 and ℓ^p for $1 \leq p < \infty$.

In general, one cannot expect a direct sum decomposition into band irreducibles as in the above theorem for positive representations of compact groups in arbitrary

Banach lattices. An example is the representation of the trivial group in $L^p([0,1])$, $1 \leq p \leq \infty$, where there are no nonzero band irreducible subrepresentations at all. In order to obtain some kind of decomposition in other spaces, some new ideas are needed. A result in this direction is [21], in which composition series of ordered structures are examined. Another result is [23], where positive representations of L^p-spaces associated with Polish transformation groups are considered. In that paper it is shown that for such representations, a decomposition into band irreducibles exists, in terms of Banach bundles, which is at least in spirit close to the direct integral of Hilbert spaces used in Theorem 1.1.

It is clear that there is still a lot of work to be done concerning decomposition theorems for positive representations, but the first steps have now been taken.