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# Propositions

ACCOMPANYING THE THESIS

# Positive representations on ordered Banach spaces

BY

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(1) Chapter 2 of this thesis contains a proof of the following Open Mapping Theorem:

**Theorem.** *Let $X$ and $Y$ be Banach spaces and $C$ a closed cone in $X$. If $T : C \to Y$ is a continuous additive and positively homogeneous map, then $T$ is surjective if and only if $T$ is an open map.*

This theorem is somewhat of an embarrassment. Either to the author, because of being unable to find a reference for this result in the pre-2013 literature, or to the mathematical community, because there is no reference for this result in the pre-2013 literature.

(2) Chapter 2 of this thesis contains a proof the following:

**Theorem.** *If $X$ is a pre-ordered Banach space with a closed and generating cone $X_+$, then there exist a constant $\alpha > 1$ and continuous positively homogeneous maps $(\cdot)^{\pm} : X \to X_+$ such that, for all $x \in X$, $x = x^+ - x^-$ and $\|x^{\pm}\| \leq \alpha\|x\|$.*

The fact that the correspondence $\varphi : X \to 2^X$ defined by $\varphi(x) := X_+ \cap (x + X_+)$ is Lipschitz with respect to the Hausdorff distance $d_H$ (i.e., there exists a constant $\alpha > 0$ such that $d_H(\varphi(x), \varphi(y)) \leq \alpha\|x - y\|$ for all $x, y \in X$) suggests that the maps $(\cdot)^{\pm} : X \to X_+$ might be chosen to be Lipschitz rather than merely continuous. The fact that the lattice operations on Banach lattices are indeed Lipschitz lends credence to this suggestion.

(3) The specificity of Banach lattices far outweighs any pathology of quasi-lattices and pre-ordered Banach spaces.

(4) If $X$ is a Banach lattice, the algebra of all (regular) bounded linear operators $B(X)$ is not a Banach lattice in general. Therefore pre-ordered Banach algebras are more suitable abstractions of algebras of operators in an ordered context than Banach lattice algebras.

(5) Let $(A \rtimes_\alpha G)^{\mathcal{R}}$ be the Banach algebra crossed product corresponding to a Banach algebra dynamical system $(A, G, \alpha)$ and uniformly bounded class $\mathcal{R}$ of continuous covariant representations on Banach spaces. The Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ can always be isometrically embedded into the Banach algebra $B(X)$ for some Banach space $X$, where $X$ is determined by all the Banach spaces underlying the representations from $\mathcal{R}$ (Proposition 4.3.4 of this thesis).

In light of (4) above, when viewing crossed products in the ordered context, this fact already suggests that even with $A$ a Banach lattice algebra and $X$ a Banach lattice, $(A \rtimes_\alpha G)^{\mathcal{R}}$ need not necessarily be a Banach lattice.

(6) A straightforward argument will prove the following:

**Theorem.** *Let $X$ be a reflexive ordered Banach space with a closed and generating cone, and let $\alpha > 0$ be such that, for $x, y \in X$, $0 \leq x \leq y$ implies $\|x\| \leq \alpha\|y\|$. Then every pair of elements of $X$ has a minimal upper bound.*

It would seem that there is no straightforward argument that will (dis)prove this statement when dropping the assumption of reflexivity.

(7) Let $X$ be a pre-ordered Banach space with a closed cone and $\alpha > 0$. In [1] the following are proven to be equivalent:

(a) $X$ is approximately $\alpha$-sum-conormal, i.e., for every $x \in X$ and $\varepsilon > 0$, there exist $a, b \in X_+$ such that $x = a - b$ and $\|a\| + \|b\| \leq \alpha\|x\| + \varepsilon$.
(b) $X'$ is $\alpha$-max-normal, i.e., for $\phi, \varphi, \psi \in X'$, if $\varphi \leq \phi \leq \psi$ then $\|\phi\| \leq \alpha \max\{\|\varphi\|, \|\psi\|\}$.

Let $X$ be a Banach space and $\{C_i\}_{i\in I}$ a collection of closed cones in $X$ such that, for every $x \in X$ and $\varepsilon > 0$, there exist $c_i \in C_i$ $(i \in I)$ such that $x = \sum_{i\in I} c_i$ and $\sum_{i\in I} \|c_i\| \leq \alpha\|x\| + \varepsilon$. This notion generalizes approximate $\alpha$-sum-conormality as given above. There also exists a generalized notion of $\alpha$-max-normality and these generalized notions satisfy a similar duality relationship as above.

[1] A.J. Ellis, *The duality of partially ordered normed linear spaces,* J. London. Math. Soc. **39** (1964), 730–744.

(8) Given the power and beauty of Michael's Selection Theorem, it is regrettable that it has remained relatively obscure within the larger functional analysis community.

(9) Computer programming is a skill of basic literacy.

(10) Obstacles in mathematics are best overcome by removing assumptions, not by adding them.

(11) Research in pure mathematics has more in common with polar exploration than with other branches of science.

> Some will tell you that you are mad, and nearly all will say, 'What is the use?' For we are a nation of shopkeepers, and no shopkeeper will look at research which does not promise him a financial return within a year. And so you will sledge nearly alone, but those with whom you sledge will not be shopkeepers: that is worth a good deal. If you march your Winter Journeys you will have your reward, so long as all you want is a penguin's egg.

Apsley Cherry-Garrard, polar explorer
(2 January 1886 – 18 May 1959)