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Chapter 4

Crossed products of Banach algebras

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4.1 Introduction and overview

This paper is an analytical continuation of [19] where, motivated by the theory of crossed products of C^*-algebras and its relevance for the theory of unitary group representations, a start was made with the theory of crossed products of Banach algebras. General Banach algebras lack the convenient rigidity of C^*-algebras where, e.g., morphisms are automatically continuous and even contractive, and this makes the task of developing the basics more laborious than it is for crossed products of C^*-algebras. Apart from some first applications, including the usual description of the non-degenerate (involutive) representations of the crossed product associated with a C^*-dynamical system (cf. [19, Theorem 9.3]), [19] is basically concerned with one theorem, the General Correspondence Theorem [19, Theorem 8.1], most of which is formulated as Theorem 4.2.1 below. If $\mathcal{R}$ is a non-empty class of non-degenerate continuous covariant representations of a Banach algebra dynamical system (A, G, α) – all notions will be reviewed in Section 4.2 – then Theorem 4.2.1 gives a bijection between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and the non-degenerate bounded representations of the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$, provided that A has a bounded approximate left identity. In the current paper, the basic theory is developed further and, in addition, a substantial part is concerned with generalized Beurling algebras $L^1(G, A, \omega; \alpha)$ and their representations. These are weighted Banach spaces of (equivalence classes) of A-valued functions that are also associative algebras with a multiplication that is continuous in both variables, but they are not Banach algebras in general, since the norm need not be submultiplicative. If A equals the scalars, they reduce to the ordinary Beurling algebras $L^1(G,\omega)$ (which are true Banach algebras) for a not necessarily abelian group G. We will describe the non-degenerate bounded representations of generalized Beurling algebras as a consequence of the General Correspondence Theorem, which is thus seen to be a common underlying principle for (at least) both crossed products of C^*-algebras and generalized Beurling algebras.

We will now briefly describe the contents of the paper.

In Section 4.2 we review the relevant definitions and results of [19]. In Section 4.3 it is investigated how the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ depends on $\mathcal{R}$, and it is also shown that there exists an isometric representation of this algebra on a Banach space. The latter result is used in Section 4.4. Loosely speaking, $(A \rtimes_\alpha G)^{\mathcal{R}}$ "generates" all non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α), and under two mild additional hypotheses it is shown to be the unique such algebra, up to isomorphism (cf. Theorem 4.4.4). This result parallels work of Raeburn's [38]. It is also shown (cf. Proposition 4.4.3) that the left regular representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ is a topological embedding into its left centralizer algebra $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$. Since $(A \rtimes_\alpha G)^{\mathcal{R}}$ need not have a bounded approximate right identity, this is not automatic.

Next, in Section 4.5 the generalized Beurling algebras $L^1(A, G, \omega; \alpha)$ make their appearance. These algebras can be defined for any Banach algebra dynamical system (A, G, α) and weight ω on G, provided that α is uniformly bounded. If A has a bounded approximate right identity, then it can be shown that $L^1(A, G, \omega; \alpha)$ is isomorphic to $(A \rtimes_\alpha G)^{\mathcal{R}}$, for a suitably chosen class $\mathcal{R}$ (cf. Theorem 4.5.17). Via this isomorphism the General Correspondence Theorem therefore predicts, if A has a bounded two-sided approximate identity, what the non-degenerate bounded representations of $L^1(A, G, \omega; \alpha)$ are, in terms of the non-degenerate continuous covariant representations of (A, G, α) (cf. Theorem 4.5.20), and some classical results are thus seen to be obtainable from the General Correspondence Theorem. As the easiest example, we retrieve the usual description of the non-degenerate left $L^1(G)$-modules in terms of the uniformly bounded strongly continuous representations of G. Naturally, there is a similar description of the non-degenerate right $L^1(G)$-modules, but an intermediate procedure is needed to obtain such a result from the General Correspondence Theorem, where one always ends up with left modules over the crossed product. This is taken up in Section 4.6, where we investigate all "reasonable" variations on the theme that $\pi : A \to B(X)$ and $U : G \to B(X)$ should be multiplicative, and that $U_r\pi(a)U_r^{-1} = \pi(\alpha_r(a))$ should hold for all $a \in A$ and $r \in G$. We argue that there are only three more "reasonable" requirements (cf. Table 4.1). One of these is, e.g., that π and U are anti-multiplicative and that $U_r\pi(a)U_r^{-1} = \pi(\alpha_{r^{-1}}(a))$ for all $a \in A$ and $r \in G$; for $A = \mathbb{K}$ and $\alpha = \text{triv}$ this covers the case of right G-modules. Moreover, we show that a pair (π, U) of each of the other three types can be reinterpreted as a covariant representation in the usual sense for a suitable "companion" Banach algebra dynamical system. The example (π, U) given above, where there are three "flaws" in the properties of (π, U), is a covariant representation for the opposite Banach algebra dynamical system (A^o, G^o, α^o). Therefore, if one seeks a Banach algebra of which the non-degenerate bounded (multiplicative) representations "encode" a family of such pairs (π, U), then a crossed product of type

$(A \rtimes_\alpha G)^{\mathcal{R}}$ is not what one should look at, but $(A^o \rtimes_{\alpha^o} G^o)^{\mathcal{R}^o}$ is to be considered.

Section 4.7 shows, as a particular case of Theorem 4.7.5, how the encoding for various types can be collected in one Banach algebra. For example, the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}} \hat{\otimes} (A^o \rtimes_{\alpha^o} G^o)^{\mathcal{R}^o}$ correspond to commuting non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $(A^o \rtimes_{\alpha^o} G^o)^{\mathcal{R}^o}$. These representations can then be respectively related to a usual covariant representation of (A, G, α) and a thrice "flawed" pair (π, U) as above, which again commute.

In the final Section 4.8 we combine the results from Sections 4.5, 4.6 and 4.7. Using the procedure from Section 4.6 and the results from Section 4.5, the relation between thrice "flawed" pairs (π, U) as above and the non-degenerate bounded representations of $L^1(G^o, A^o, \omega^o; \alpha^o)$ is easily established. Since coincidentally the generalized Beurling algebra $L^1(G^o, A^o, \omega^o; \alpha^o)$ turns out to be anti-isomorphic to $L^1(A, G, \omega; \alpha)$, these pairs (π, U) can then also be related to the non-degenerate right $L^1(A, G, \omega; \alpha)$-modules (cf. Theorem 4.8.3). It is then easy to describe the simultaneous left $L^1(A, G, \omega; \alpha)$– and right $L^1(B, H, \eta; \beta)$-modules, where the actions commute (cf. Theorem 4.8.4). In particular this describes the bimodules over a generalized Beurling algebra $L^1(A, G, \omega; \alpha)$. Specializing to the case where A equals the scalars yields a description of the non-degenerate bimodules over an ordinary Beurling algebra $L^1(G, \omega)$ in terms of G-bimodules. Specializing still further to $\omega = 1$ the classical description of the non-degenerate $L^1(G)$-bimodules in terms of a uniformly bounded G-bimodule is retrieved as the simplest case in the general picture.

4.2 Preliminaries and recapitulation

For the sake of self-containment we provide a brief recapitulation of definitions and results from earlier papers [18, 19].

Throughout this paper X and Y will denote Banach spaces. The algebra of bounded linear operators on X will be denoted by $B(X)$. By A and B we will denote Banach algebras, not necessarily unital, and by G and H locally compact groups (which are always assumed to be Hausdorff). We will always use the same symbol λ to denote the left regular representation of various Banach algebras instead of distinguishing between them, as the context will always make precise what is meant. If A is a Banach algebra, X a Banach space, and $\pi : A \to B(X)$ is a Banach algebra representation, when confusion could arise, we will write X_π instead of X to make clear that the Banach space X is related to the representation π. We do not assume that Banach algebra representations of unital Banach algebras are unital. Representations of algebras and groups are always multiplicative (so that we are considering left modules), unless explicitly stated otherwise.

Let A be a Banach algebra, G a locally compact Hausdorff group and $\alpha : G \to \mathrm{Aut}(A)$ a strongly continuous representation of G on A. Then the triple (A, G, α) is called a *Banach algebra dynamical system.*

Let (A, G, α) be a Banach algebra dynamical system, X a Banach space with $\pi : A \to B(X)$ and $U : G \to B(X)$ representations of the algebra A and group G on

X respectively. If (π, U) satisfies

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^{-1},$$

for all $a \in A$ and $s \in G$, the pair (π, U) is called a *covariant representation* of (A, G, α) on X. The pair (π, U) is said to be *continuous* if π is norm-bounded and U is strongly continuous. The pair (π, U) is called *non-degenerate* if π is non-degenerate (i.e., the span of $\pi(A)X$ lies dense in X).

Integrals of compactly supported continuous Banach space valued functions are, as in [19], defined by duality, following [40, Section 3]. Let $C_c(G, A)$ denote the space of all continuous compactly supported A-valued functions. For any $f, g \in C_c(G, A)$ and $s \in G$ defining the twisted convolution

$$[f * g](s) := \int_G f(r) \alpha_r(g(r^{-1}s))\, dr$$

gives $C_c(G, A)$ the structure of an associative algebra, where integration is with respect to a fixed left Haar measure on G.

If (π, U) is a continuous covariant representation of (A, G, α) on X, then, for $f \in C_c(G, A)$, we define $\pi \rtimes U(f) \in B(X)$, as in [19, Section 3], by

$$\pi \rtimes U(f)x := \int_G \pi(f(s)) U_s x\, ds \quad (x \in X).$$

The map $\pi \rtimes U : C_c(G, A) \to B(X)$ is a representation of the algebra $C_c(G, A)$ on X, and is called the *integrated form* of (π, U).

Let $\mathcal{R}$ be a class of covariant representations of (A, G, α). Then $\mathcal{R}$ is called a *uniformly bounded class of continuous covariant representations* if there exist a constant $C \geq 0$ and function $\nu : G \to [0, \infty)$ which is bounded on compact sets, such that, for any $(\pi, U) \in \mathcal{R}$, we have that $\|\pi\| \leq C$ and $\|U_r\| \leq \nu(r)$ for all $r \in G$. We will always tacitly assume that such a class $\mathcal{R}$ is non-empty. With $\mathcal{R}$ as such, it follows that $\|\pi \rtimes U(f)\| \leq C \left(\sup_{r \in \operatorname{supp}(f)} \nu(r)\right) \|f\|_1$ for all $(\pi, U) \in \mathcal{R}$ and $f \in C_c(G, A)$ [19, Remark 3.3].

We define the algebra seminorm $\sigma^{\mathcal{R}}$ on $C_c(G, A)$ by

$$\sigma^{\mathcal{R}}(f) := \sup_{(\pi, U) \in \mathcal{R}} \|\pi \rtimes U(f)\| \quad (f \in C_c(G, A)),$$

and denote the completion of the quotient $C_c(G, A)/\ker \sigma^{\mathcal{R}}$ by $(A \rtimes_\alpha G)^{\mathcal{R}}$, with $\|\cdot\|^{\mathcal{R}}$ denoting the norm induced by $\sigma^{\mathcal{R}}$. The Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ is called the *crossed product* corresponding to (A, G, α) and $\mathcal{R}$. The quotient homomorphism is denoted by $q^{\mathcal{R}} : C_c(G, A) \to (A \rtimes_\alpha G)^{\mathcal{R}}$.

A covariant representation of (A, G, α) is called *$\mathcal{R}$-continuous* if it is continuous and its integrated form is bounded with respect to the seminorm $\sigma^{\mathcal{R}}$. For any Banach space X and linear map $T : C_c(G, A) \to X$, if T is bounded with respect to the $\sigma^{\mathcal{R}}$ seminorm, we will denote the canonically induced linear map on $(A \rtimes_\alpha G)^{\mathcal{R}}$ by $T^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to X$, as detailed in [19, Section 3].

If A has a bounded approximate left (right) identity, then it can be shown that $(A \rtimes_\alpha G)^{\mathcal{R}}$ also has a bounded approximate left (right) identity, with estimates for its bound, [19, Theorem 4.4 and Corollary 4.6].

We will denote the left centralizer algebra of a Banach algebra B by $\mathcal{M}_l(B)$. Assuming B has a bounded approximate left identity (u_i), any non-degenerate bounded representation $T : B \to B(X)$ induces a non-degenerate bounded representation $\overline{T} : \mathcal{M}_l(B) \to B(X)$, by defining $\overline{T}(L) := \text{SOT-lim}_i T(Lu_i)$ for all $L \in \mathcal{M}_l(B)$, so that the following diagram commutes (cf. [18, Theorem 4.1]):

$$\begin{array}{ccc} B & \xrightarrow{T} & B(X) \\ & \searrow^{\lambda} & \uparrow{\scriptstyle \overline{T}} \\ & & \mathcal{M}_l(B) \end{array}$$

Moreover, $\overline{T}(L)T(a) = T(La)$ for all $a \in B$ and $L \in \mathcal{M}_l(B)$. We will often use this fact.

With (A, G, α) a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations, we define the homomorphisms $i_A : A \to \text{End}(C_c(G, A))$ and $i_G : G \to \text{End}(C_c(G, A))$ by

$$\begin{aligned} (i_A(a)f)(s) &:= af(s), \\ (i_G(r)f)(s) &:= \alpha_r(f(r^{-1}s)), \end{aligned}$$

for all $a \in A$, $f \in C_c(G, A)$ and $r, s \in G$. For each $a \in A$ and $r \in G$, the maps

$$i_A(a), i_G(r) : (C_c(G, A), \sigma^{\mathcal{R}}) \to (C_c(G, A), \sigma^{\mathcal{R}})$$

are bounded [19, Lemma 6.3], and

$$\begin{aligned} \|i_A(a)\|^{\mathcal{R}} &\leq \sup_{(\pi,U)\in\mathcal{R}} \|\pi(a)\|, \\ \|i_G(r)\|^{\mathcal{R}} &\leq \sup_{(\pi,U)\in\mathcal{R}} \|U_r\|. \end{aligned}$$

Defining $i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f) := q^{\mathcal{R}}(i_A(a)f)$ and $i_G^{\mathcal{R}}(r)q^{\mathcal{R}}(f) := q^{\mathcal{R}}(i_G(r)f)$ for all $a \in A$, $r \in G$ and $\in C_c(G, A)$, we obtain bounded maps

$$i_A^{\mathcal{R}}(a), i_G^{\mathcal{R}}(r) : (A \rtimes_\alpha G)^{\mathcal{R}} \to (A \rtimes_\alpha G)^{\mathcal{R}}.$$

Moreover, the maps $a \mapsto i_A^{\mathcal{R}}(a)$ and $r \mapsto i_G^{\mathcal{R}}(r)$ map A and G into $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$. If A has a bounded approximate left identity and $\mathcal{R}$ is a uniformly bounded class of non-degenerate continuous covariant representations, then $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on $(A \rtimes_\alpha G)^{\mathcal{R}}$ [19, Section 6] and the integrated form $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}$ equals the left regular representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ [19, Theorem 7.2].

The main theorem from [19] establishes, amongst others, a bijective relationship between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$, by letting (π, U) correspond to $(\pi \rtimes U)^{\mathcal{R}}$. This result will play a fundamental role throughout the rest of this paper, and the relevant part of [19, Theorem 8.1] can be stated as follows:

Theorem 4.2.1. (*General Correspondence Theorem, cf. [19, Theorem 8.1]*) *Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded approximate left identity. Let $\mathcal{R}$ be a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α). Then the map $(\pi, U) \mapsto (\pi \rtimes U)^{\mathcal{R}}$ is a bijection between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$.*

More precisely:

(1) If (π, U) is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α), then $(\pi \rtimes U)^{\mathcal{R}}$ is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, and

$$(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}, \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}) = (\pi, U),$$

where $\overline{(\pi \rtimes U)^{\mathcal{R}}}$ is the representation of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ as described above, cf. [19, Section 7].

(2) If T is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, then the pair $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α), and

$$(\overline{T} \circ i_A^{\mathcal{R}} \rtimes \overline{T} \circ i_G^{\mathcal{R}})^{\mathcal{R}} = T.$$

4.3 Varying $\mathcal{R}$

For a given Banach algebra dynamical system (A, G, α), one may ask what relationship exists between the crossed products $(A \rtimes_\alpha G)^{\mathcal{R}_1}$ and $(A \rtimes_\alpha G)^{\mathcal{R}_2}$ for two uniformly bounded classes $\mathcal{R}_1$ and $\mathcal{R}_2$ of possibly degenerate continuous covariant representations on Banach spaces. This section investigates this question.

Since uniformly bounded classes of covariant representations might be proper classes, we must take some care in working with them. Nevertheless, we can always choose a set from a uniformly bounded class $\mathcal{R}$ of covariant representations of a Banach algebra dynamical system (A, G, α) so that this set determines $\sigma^{\mathcal{R}}$. Indeed for every $f \in C_c(G, A)$, looking at the subset $\{\|\pi \rtimes U(f)\| : (\pi, U) \in \mathcal{R}\}$ of $\mathbb{R}$ (subclasses of sets are sets), we may choose a sequence from $\{\|\pi \rtimes U(f)\| : (\pi, U) \in \mathcal{R}\}$ converging to $\sigma^{\mathcal{R}}(f)$ and regard only those corresponding covariant representations. In this way, we can chose a set S from $\mathcal{R}$ of cardinality at most $|C_c(G, A) \times \mathbb{N}|$ such that $\sigma^S(f) = \sup_{(\pi,U)\in S} \|\pi \rtimes U(f)\| = \sigma^{\mathcal{R}}(f)$ for all $f \in C_c(G, A)$. Hence the following definition is meaningful; it will be required in Definition 4.3.3 and Proposition 4.3.4.

Definition 4.3.1. Let $\mathcal{R}$ be a uniformly bounded class of possibly degenerate continuous covariant representations of (A, G, α). We define $[\mathcal{R}]$ to be the collection of all uniformly bounded classes $\mathcal{S}$ that are actually sets and satisfy $\sigma^{\mathcal{R}} = \sigma^{\mathcal{S}}$ on $C_c(G, A)$. Elements of some $[\mathcal{R}]$ will be called *uniformly bounded sets of continuous covariant representations.*

Before addressing the question laid out in the first paragraph, we consider the following aside which will play a key role in Section 4.4.

Definition 4.3.2. Let I be an index set and $\{X_i : i \in I\}$ a family of Banach spaces. For $1 \leq p \leq \infty$, we will denote the ℓ^p-direct sum of $\{X_i : i \in I\}$ by $\ell^p\{X_i : i \in I\}$.

Definition 4.3.3. Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations. For $S \in [\mathcal{R}]$ and $1 \leq p < \infty$, suppressing the p-dependence in the notation, we define the representations $(\oplus_S \pi) : A \to B(\ell^p\{X_\pi : (\pi, U) \in S\})$ and $(\oplus_S U) : G \to B(\ell^p\{X_\pi : (\pi, U) \in S\})$ by $(\oplus_S \pi)(a) := \bigoplus_{(\pi,U)\in S} \pi(a)$ and $(\oplus_S U)_r := \bigoplus_{(\pi,U)\in S} U_r$ for all $a \in A$ and $r \in G$ respectively.

It is easily seen that $((\oplus_S \pi), (\oplus_S U))$ is a continuous covariant representation, that

$$((\oplus_S \pi) \rtimes (\oplus_S U))(f) = \bigoplus_{(\pi,U)\in S} \pi \rtimes U(f),$$

and that $\|((\oplus_S \pi) \rtimes (\oplus_S U))(f)\| = \sigma^S(f) = \sigma^{\mathcal{R}}(f)$, for all $f \in C_c(G, A)$.

We hence obtain the following (where the statement concerning non-degeneracy is an elementary verification).

Proposition 4.3.4. *Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations. For any $S \in [\mathcal{R}]$ and $1 \leq p < \infty$, there exists an $\mathcal{R}$-continuous covariant representation of (A, G, α) on $\ell^p\{X_\pi : (\pi, U) \in S\}$, denoted $((\oplus_S \pi), (\oplus_S U))$, such that its integrated form satisfies $\|((\oplus_S \pi) \rtimes (\oplus_S U))(f)\| = \sigma^{\mathcal{R}}(f)$ for all $f \in C_c(G, A)$ and hence induces an isometric representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on $\ell^p\{X_\pi : (\pi, U) \in S\}$.*

If every element of S is non-degenerate, then $((\oplus_S \pi), (\oplus_S U))$ is non-degenerate.

The previous theorem shows, in particular, that crossed products can always be realized isometrically as closed subalgebras of bounded operators on some (rather large) Banach space.

We now return to the original question. The following results examine relations that may exist between crossed products defined by using two different uniformly bounded classes of continuous covariant representations of a Banach algebra dynamical system.

Proposition 4.3.5. *Let (A, G, α) be a Banach algebra dynamical system. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be uniformly bounded classes of possibly degenerate continuous covariant representations of (A, G, α) and $M \geq 1$ a constant. Then the following are equivalent:*

(1) There exists a homomorphism $h : (A \rtimes_\alpha G)^{\mathcal{R}_2} \to (A \rtimes_\alpha G)^{\mathcal{R}_1}$ such that $\|h\| \leq M$ and $h \circ q^{\mathcal{R}_2}(f) = q^{\mathcal{R}_1}(f)$ for all $f \in C_c(G, A)$.

(2) The seminorms $\sigma^{\mathcal{R}_1}$ and $\sigma^{\mathcal{R}_2}$ satisfy $\sigma^{\mathcal{R}_1}(f) \leq M\sigma^{\mathcal{R}_2}(f)$ for all $f \in C_c(G, A)$.

(3) There exist uniformly bounded sets of continuous covariant representations $\mathcal{R}_1' \in [\mathcal{R}_1]$, $\mathcal{R}_2' \in [\mathcal{R}_2]$ and $\mathcal{R}_3'$ such that $\mathcal{R}_1' \cup \mathcal{R}_2' \subseteq \mathcal{R}_3'$ and $\sigma^{\mathcal{R}_2'}(f) \leq \sigma^{\mathcal{R}_3'}(f) \leq M\sigma^{\mathcal{R}_2'}(f)$ for all $f \in C_c(G, A)$.

(4) If (π, U) is an $\mathcal{R}_1$-continuous covariant representation of (A, G, α) and $M' \geq 0$ is such that $\|\pi \rtimes U(f)\| \leq M'\sigma^{\mathcal{R}_1}(f)$ for all $f \in C_c(G, A)$, then (π, U) is an $\mathcal{R}_2$-continuous covariant representation of (A, G, α), and $\|\pi \rtimes U(f)\| \leq M'M\sigma^{\mathcal{R}_2}(f)$ for all $f \in C_c(G, A)$.

(5) For any bounded representation $T : (A \rtimes_\alpha G)^{\mathcal{R}_1} \to B(X)$ there exists a bounded representation $S : (A \rtimes_\alpha G)^{\mathcal{R}_2} \to B(X)$ such that $T \circ q^{\mathcal{R}_1}(f) = S \circ q^{\mathcal{R}_2}(f)$ for all $f \in C_c(G, A)$ and $\|S\| \leq M\|T\|$.

Proof. We prove that (1) implies (5). Let $T : (A \rtimes_\alpha G)^{\mathcal{R}_1} \to B(X)$ be a bounded representation. Then $S := T \circ h : (A \rtimes_\alpha G)^{\mathcal{R}_2} \to B(X)$ satisfies $T \circ q^{\mathcal{R}_1}(f) = T \circ h \circ q^{\mathcal{R}_2}(f) = S \circ q^{\mathcal{R}_2}(f)$ for all $f \in C_c(G, A)$, and $\|S\| \leq \|T\|\|h\| \leq M\|T\|$.

We prove that (5) implies (4). Let (π, U) be $\mathcal{R}_1$-continuous and $M' \geq 0$ be such that $\|\pi \rtimes U(f)\| \leq M'\sigma^{\mathcal{R}_1}(f)$ for all $f \in C_c(G, A)$. Then, for the bounded representation $(\pi \rtimes U)^{\mathcal{R}_1} : (A \rtimes_\alpha G)^{\mathcal{R}_1} \to B(X_\pi)$, there exists a bounded representation $S : (A \rtimes_\alpha G)^{\mathcal{R}_2} \to B(X_\pi)$ such that

$$\pi \rtimes U(f) = (\pi \rtimes U)^{\mathcal{R}_1} \circ q^{\mathcal{R}_1}(f) = S \circ q^{\mathcal{R}_2}(f)$$

for all $f \in C_c(G, A)$, and $\|S\| \leq M\|(\pi \rtimes U)^{\mathcal{R}_1}\| \leq MM'$. Hence, (π, U) is $\mathcal{R}_2$-continuous, and $\|\pi \rtimes U(f)\| = \|S \circ q^{\mathcal{R}_2}(f)\| \leq MM'\sigma^{\mathcal{R}_2}(f)$ holds for all $f \in C_c(G, A)$.

We prove that (4) implies (2). Every $(\pi, U) \in \mathcal{R}_1$ is $\mathcal{R}_1$-continuous and satisfies $\|\pi \rtimes U(f)\| \leq \sigma^{\mathcal{R}_1}(f)$ for all $f \in C_c(G, A)$. Then, by hypothesis, (π, U) is $\mathcal{R}_2$-continuous and

$$\|\pi \rtimes U(f)\| \leq M\sigma^{\mathcal{R}_2}(f)$$

for all $f \in C_c(G, A)$. Taking the supremum over all $(\pi, U) \in \mathcal{R}_1$, we obtain $\sigma^{\mathcal{R}_1}(f) \leq M\sigma^{\mathcal{R}_2}(f)$ for all $f \in C_c(G, A)$.

We prove that (2) implies (1). Since $\ker \sigma^{\mathcal{R}_2} \subseteq \ker \sigma^{\mathcal{R}_1}$, a homomorphism

$$h : C_c(G, A)/\ker \sigma^{\mathcal{R}_2} \to C_c(G, A)/\ker \sigma^{\mathcal{R}_1}$$

can be defined by $h(q^{\mathcal{R}_2}(f)) := q^{\mathcal{R}_1}(f)$ for all $f \in C_c(G, A)$, and then satisfies $\|h\| \leq M$. The map h therefore extends to a homomorphism $h : (A \rtimes_\alpha G)^{\mathcal{R}_2} \to (A \rtimes_\alpha G)^{\mathcal{R}_1}$ with the same norm.

We prove that (2) implies (3). Let $\mathcal{R}_1' \in [\mathcal{R}_1]$ and $\mathcal{R}_2' \in [\mathcal{R}_2]$ and define $\mathcal{R}_3' := \mathcal{R}_1' \cup \mathcal{R}_2'$. By construction we have that $\sigma^{\mathcal{R}_2'}(f) \leq \sigma^{\mathcal{R}_3'}(f)$ for all $f \in C_c(G, A)$. By hypothesis we have that $\sigma^{\mathcal{R}_1'}(f) \leq M\sigma^{\mathcal{R}_2'}(f)$ for all $f \in C_c(G, A)$, as well as $M \geq 1$. Therefore,

$$\sigma^{\mathcal{R}_2'}(f) \leq \sigma^{\mathcal{R}_3'}(f) = \max\{\sigma^{\mathcal{R}_1'}(f), \sigma^{\mathcal{R}_2'}(f)\} \leq \max\{M\sigma^{\mathcal{R}_2'}(f), \sigma^{\mathcal{R}_2'}(f)\} = M\sigma^{\mathcal{R}_2'}(f).$$

We prove that (3) implies (2). Let $\mathcal{R}_1' \in [\mathcal{R}_1]$, $\mathcal{R}_2' \in [\mathcal{R}_2]$ and $\mathcal{R}_3'$ be such that $\mathcal{R}_1' \cup \mathcal{R}_2' \subseteq \mathcal{R}_3'$ and $\sigma^{\mathcal{R}_2'}(f) \leq \sigma^{\mathcal{R}_3'}(f) \leq M\sigma^{\mathcal{R}_2'}(f)$ for all $f \in C_c(G, A)$. Then

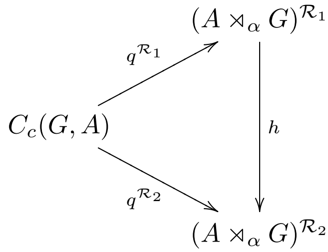
$$\sigma^{\mathcal{R}_1}(f) = \sigma^{\mathcal{R}_1'}(f) \leq \sigma^{\mathcal{R}_3'}(f) \leq M\sigma^{\mathcal{R}_2'}(f) \leq M\sigma^{\mathcal{R}_2}(f).$$

□

We can now describe the relationship between $\mathcal{R}$ and the isomorphism class of the pair $((A \rtimes_\alpha G)^{\mathcal{R}}, q^{\mathcal{R}})$.

Corollary 4.3.6. *Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}_1$ and $\mathcal{R}_2$ be uniformly bounded classes of possibly degenerate continuous covariant representations of (A, G, α). Then the following are equivalent:*

(1) There exists a topological algebra isomorphism $h : (A \rtimes_\alpha G)^{\mathcal{R}_1} \to (A \rtimes_\alpha G)^{\mathcal{R}_2}$ such that the following diagram commutes:



(2) The seminorms $\sigma^{\mathcal{R}_1}$ and $\sigma^{\mathcal{R}_2}$ on $C_c(G, A)$ are equivalent.

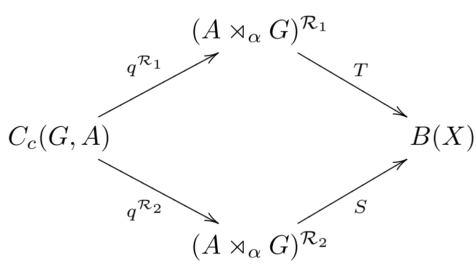
(3) There exist uniformly bounded sets of possibly degenerate continuous covariant representations $\mathcal{R}_1' \in [\mathcal{R}_1]$, $\mathcal{R}_2' \in [\mathcal{R}_2]$ and $\mathcal{R}_3'$ with $\mathcal{R}_1' \cup \mathcal{R}_2' \subseteq \mathcal{R}_3'$ and constants $M_1, M_2 \geq 0$, such that

$$\sigma^{\mathcal{R}_1'}(f) \leq \sigma^{\mathcal{R}_3'}(f) \leq M_1 \sigma^{\mathcal{R}_1'}(f),$$
$$\sigma^{\mathcal{R}_2'}(f) \leq \sigma^{\mathcal{R}_3'}(f) \leq M_2 \sigma^{\mathcal{R}_2'}(f),$$

for all $f \in C_c(G, A)$.

(4) The $\mathcal{R}_1$-continuous covariant representations of (A, G, α) coincide with the $\mathcal{R}_2$-continuous covariant representations of (A, G, α). Moreover, there exist constants $M_1, M_2 \geq 0$, with the property that, if $M' \geq 0$ and (π, U) is $\mathcal{R}_1$-continuous, such that $\|\pi \rtimes U(f)\| \leq M' \sigma^{\mathcal{R}_1}(f)$ for all $f \in C_c(G, A)$, then $\|\pi \rtimes U(f)\| \leq M_1 M' \sigma^{\mathcal{R}_2}(f)$ for all $f \in C_c(G, A)$, and likewise for the indices 1 and 2 interchanged.

(5) There exist constants $M_1, M_2 \geq 0$ with the property that, for every bounded representation $T : (A \rtimes_\alpha G)^{\mathcal{R}_1} \to B(X)$ there exists a bounded representation $S : (A \rtimes_\alpha G)^{\mathcal{R}_2} \to B(X)$ with $\|S\| \leq M_1 \|T\|$, such that the diagram



commutes, and likewise with the indices 1 and 2 interchanged.

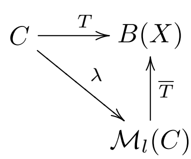
Proof. This follows from Proposition 4.3.5. □

4.4 Uniqueness of the crossed product

Theorem 4.2.1 asserts, amongst others, that all non-degenerate $\mathcal{R}$-continuous covariant representations of a Banach algebra dynamical system (A, G, α) can be generated from the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$, with the aid of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ and the pair $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$. In this section we show that, under mild additional hypotheses, $(A \rtimes_\alpha G)^{\mathcal{R}}$ is the unique Banach algebra with this generating property. These results are similar in nature as Raeburn's for the crossed product of a C^*-algebra, see [38] or [46, Theorem 2.61].

We start with the general framework of how to generate many non-degenerate $\mathcal{R}$-continuous covariant representations from a suitable basic one, on a Banach space that is a Banach algebra.

Lemma 4.4.1. *Let (A, G, α) be a Banach algebra dynamical system, and let $\mathcal{R}$ be a uniformly bounded class of continuous covariant representations of (A, G, α). Let C be a Banach algebra with a bounded approximate left identity, and let (k_A, k_G) be a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on the Banach space C, such that $k_A(A), k_G(G) \subseteq \mathcal{M}_l(C)$. Suppose $T : C \to B(X)$ is a non-degenerate bounded representation of C on a Banach space X. Let $\overline{T} : \mathcal{M}_l(C) \to B(X)$ be the non-degenerate bounded representation of $\mathcal{M}_l(C)$ such that the following diagram commutes:*



Then the pair $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α), and $(\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G) = \overline{T} \circ (k_A \rtimes k_G)$.

Proof. It is clear that $\overline{T} \circ k_A$ is a continuous representation of A on X. Since $\overline{T}$ is unital [18, Theorem 4.1], $\overline{T} \circ k_G$ is a representation of G on X. Using that $\overline{T}(L)T(c) = T(Lc)$ for $L \in \mathcal{M}_l(C)$ and $c \in C$, (cf. [18, Theorem 4.1]), we find, for $r \in G$, $c \in C$ and $x \in X$, that $(\overline{T} \circ k_G(r))T(c)x = T(k_G(r)c)x$. Since k_G is strongly continuous and T is continuous, we see that

$$\lim_{r \to e} (\overline{T} \circ k_G(r))T(c)x = \lim_{r \to e} T(k_G(r)c)x = T(c)x,$$

for all $c \in C$ and $x \in X$. The non-degeneracy of T, together with [19, Corollary 2.5] then imply that $T \circ k_G$ is strongly continuous. It is a routine verification that $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is covariant, so that $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is a continuous covariant representation of (A, G, α) on C.

We claim that $k_A \rtimes k_G : C_c(G, A) \to B(C)$ has its image in $\mathcal{M}_l(C)$, and that $(\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G) = \overline{T} \circ (k_A \rtimes k_G)$. The $\mathcal{R}$-continuity of (k_A, k_G) and the continuity of $\overline{T}$ then show that $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is $\mathcal{R}$-continuous. As to this claim, note that, for $f \in C_c(G, A)$, the integrand in $k_A \rtimes k_G(f) = \int_G k_A(f(r))k_G(r)\,dr$ takes values in the SOT-closed subspace $\mathcal{M}_l(C)$ of $B(C)$, hence the integral is likewise in this subspace. Hence $\overline{T} \circ (k_A \rtimes k_G) : C_c(G, A) \to B(X)$ is a meaningfully defined map. Using that that continuous operators can be pulled through the integral [40, Ch. 3, Exercise 24] and the definition of operator valued integrals [19, Proposition 2.19], we then have for all $x \in X$:

$$\begin{aligned}
\overline{T}\left(k_A \rtimes k_G(f)\right) T(c)x &= T\left(k_A \rtimes k_G(f)c\right) x \\
&= T\left(\int_G k_A(f(r))k_G(r)\,dr\ c\right) x \\
&= T\left(\int_G k_A(f(r))k_G(r)c\,dr\right) x \\
&= \int_G T\left(k_A(f(r))k_G(r)c\right) x\,dr \\
&= \int_G \overline{T}\left(k_A(f(r))k_G(r)\right) T(c)x\,dr \\
&= \int_G \overline{T} \circ k_A(f(r))\overline{T} \circ k_G(r)T(c)x\,dr \\
&= \left(\int_G \overline{T} \circ k_A(f(r))\overline{T} \circ k_G(r)\,dr\right) T(c)x \\
&= \big((\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G)(f)\big)\, T(c)x.
\end{aligned}$$

Since T is non-degenerate, this establishes the claim.

It remains to show that $\overline{T} \circ k_A$ is non-degenerate. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Since $\overline{T}$ is non-degenerate [18, Theorem 4.1], there exist finite sets $\{c_i\}_{i=1}^n \subseteq C$ and $\{x_i\}_{i=1}^n \subseteq X$ such that $\| \sum_{i=1}^n T(c_i)x_i - x\| < \varepsilon/2$. Since k_A is non-degenerate, for every $i \in \{1, \ldots, n\}$, there exist finite sets $\{a_{i,j}\}_{j=1}^{m_i} \subseteq A$ and $\{d_{i,j}\}_{j=1}^{m_i} \subseteq C$ such that $\|T\|\|x_i\|\|c_i - \sum_{j=1}^{m_i} k_A(a_{i,j})d_{i,j}\| < \varepsilon/2n$. Then

$$\begin{aligned}
&\left\| x - \sum_{i=1}^n \sum_{j=1}^{m_i} \left(\overline{T} \circ k_A(a_{i,j})\right) T(d_{i,j})x_i \right\| \\
= &\left\| x - \sum_{i=1}^n \sum_{j=1}^{m_i} T\left(k_A(a_{i,j})d_{i,j}\right) x_i \right\| \\
\leq &\left\| x - \sum_{i=1}^n T(c_i)x_i \right\| + \left\| \sum_{i=1}^n T(c_i)x_i - \sum_{i=1}^n \sum_{j=1}^{m_i} T\left(k_A(a_{i,j})d_{i,j}\right) x_i \right\| \\
\leq &\left\| x - \sum_{i=1}^n T(c_i)x_i \right\| + \sum_{i=1}^n \|T\| \left\| c_i - \sum_{j=1}^{m_i} k_A(a_{i,j})d_{i,j} \right\| \|x_i\| \\
< &\frac{\varepsilon}{2} + \frac{\varepsilon}{2}.
\end{aligned}$$

We conclude that $\overline{T} \circ k_A$ is non-degenerate. □

Naturally any Banach algebra C' isomorphic to C as in the previous lemma has a similar "generating pair" (k'_A, k'_G). The details are in the following result, the routine verification of which is left to the reader.

Lemma 4.4.2. *Let (A, G, α), $\mathcal{R}$, C and (k_A, k_G) be as in Lemma 4.4.1. Suppose C' is a Banach algebra and $\psi : C \to C'$ is a topological isomorphism. Then:*

(1) $\psi_l : \mathcal{M}_l(C) \to \mathcal{M}_l(C')$, defined by $\psi_l(L) := \psi L \psi^{-1}$ for $L \in \mathcal{M}_l(C)$, is a topological isomorphism.

(2) The pair $(k'_A, k'_G) := (\psi_l \circ k_A, \psi_l \circ k_G)$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on C', such that $k'_A(A), k'_G(G) \subseteq \mathcal{M}_l(C')$.

(3) If $T : C \to B(X)$ is a non-degenerate bounded representation, then so is $T' : C' \to B(X)$, where $T' := T \circ \psi^{-1}$.

(4) If $T : C \to B(X)$ is a non-degenerate bounded representation, and $\overline{T'} : \mathcal{M}_l(C') \to B(X)$ is the non-degenerate bounded representation of $\mathcal{M}_l(C')$ such that the diagram

$$\begin{array}{ccc} C' & \xrightarrow{T'} & B(X) \\ & \searrow^{\lambda} & \uparrow{\scriptstyle \overline{T'}} \\ & & \mathcal{M}_l(C') \end{array}$$

commutes, then $\overline{T} \circ k_A = \overline{T'} \circ k'_A$ and $\overline{T} \circ k_G = \overline{T'} \circ k'_G$.

Now let $\mathcal{R}$ be a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α), where A has a bounded approximate left identity. Then, according to [19, Theorem 7.2], $(A \rtimes_\alpha G)^{\mathcal{R}}$ has a bounded approximate left identity, and the maps $i_A^{\mathcal{R}} : A \to B((A \rtimes_\alpha G)^{\mathcal{R}})$ and $i_G^{\mathcal{R}} : G \to B((A \rtimes_\alpha G)^{\mathcal{R}})$ form a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on the Banach space $(A \rtimes_\alpha G)^{\mathcal{R}}$, with images in $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$. According to Lemma 4.4.1, the triple $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ can be used to produce non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) from non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$, and, according to Theorem 4.2.1, all non-degenerate $\mathcal{R}$-continuous covariant representations are thus obtained. According to Lemma 4.4.2, any Banach algebra isomorphic to $(A \rtimes_\alpha G)^{\mathcal{R}}$ has the same property. We will now proceed to show the converse: If (B, k_A, k_G) is a triple generating all non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α), then it can be obtained from $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ as in Lemma 4.4.2.

We start with a preliminary observation that is of some interest in its own right.

Proposition 4.4.3. *Let (A, G, α) be a Banach algebra dynamical system where A has a bounded approximate left identity. Let $\mathcal{R}$ be a uniformly bounded class of non-degenerate continuous covariant representations. Then the left regular representation $\lambda : (A \rtimes_\alpha G)^{\mathcal{R}} \to \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ is a topological embedding.*

Proof. According to Proposition 4.3.4, there exists a non-degenerate $\mathcal{R}$-continuous covariant representation (π, U) such that $(\pi \rtimes U)^{\mathcal{R}}$ is a non-degenerate isometric representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$. According to Theorem 4.2.1, $\pi = \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}$ and $U = \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}$. Furthermore, according to Lemma 4.4.1,

$$(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}) \rtimes (\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}) = \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ (i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}}).$$

We recall [19, Theorem 7.2] that $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}} = \lambda$. Combining all this, we see, with M denoting an upper bound for an approximate left identity of $(A \rtimes_\alpha G)^{\mathcal{R}}$ [19, Corollary 4.6], that, for $f \in C_c(G, A)$:

$$\begin{aligned}
\|q^{\mathcal{R}}(f)\|^{\mathcal{R}} &= \|(\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f))\| \\
&= \|\pi \rtimes U(f)\| \\
&= \|(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}) \rtimes (\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}})(f)\| \\
&= \|\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ (i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})(f)\| \\
&= \|\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ (i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}(q^{\mathcal{R}}(f))\| \\
&= \|\overline{(\pi \rtimes U)^{\mathcal{R}}}(\lambda(q^{\mathcal{R}}(f)))\| \\
&\leq M\|(\pi \rtimes U)^{\mathcal{R}}\|\|\lambda(q^{\mathcal{R}}(f))\| \\
&= M\|\lambda(q^{\mathcal{R}}(f))\|.
\end{aligned}$$

Since the inequality $\|\lambda(q^{\mathcal{R}}(f))\| \leq \|q^{\mathcal{R}}(f)\|$ is trivial, the result follows from the density of $q^{\mathcal{R}}(C_c(G, A))$ in $(A \rtimes_\alpha G)^{\mathcal{R}}$. □

Since $q^{\mathcal{R}}(C_c(G, A))$ is dense in $(A \rtimes_\alpha G)^{\mathcal{R}}$, we conclude that $\lambda \circ q^{\mathcal{R}}(C_c(G, A))$ is dense in $\lambda((A \rtimes_\alpha G)^{\mathcal{R}})$, i.e., that $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})(C_c(G, A))$ is dense in $\lambda((A \rtimes_\alpha G)^{\mathcal{R}})$. Together with Proposition 4.4.3 this gives the two additional hypotheses alluded to before, under which the following uniqueness theorem can now be established. As mentioned earlier, this should be compared with Raeburn's result for the crossed product of a C^*-algebra, see [38] or [46, Theorem 2.61].

Theorem 4.4.4. *Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded approximate left identity. Let $\mathcal{R}$ be a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α). Let B be a Banach algebra with a bounded approximate left identity, such that $\lambda : B \to \mathcal{M}_l(B)$ is a topological embedding. Let (k_A, k_G) be a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on the Banach space B, such that*

(1) $k_A(A), k_G(G) \subseteq \mathcal{M}_l(B)$

(2) $(k_A \rtimes k_G)(C_c(G, A)) \subseteq \lambda(B)$

(3) $(k_A \rtimes k_G)(C_c(G, A))$ is dense in $\lambda(B)$.

Suppose that, for each non-degenerate $\mathcal{R}$-continuous covariant representation (π, U) of (A, G, α) on a Banach space X, there exists a non-degenerate bounded representation $T : B \to B(X)$ such that the non-degenerate bounded representation

$\overline{T} : \mathcal{M}_l(B) \to B(X)$ in the commuting diagram

$$\begin{array}{ccc} B & \xrightarrow{T} & B(X) \\ & \searrow^{\lambda} & \uparrow{\scriptstyle \overline{T}} \\ & & \mathcal{M}_l(B) \end{array}$$

generates (π, U) as in Lemma 4.4.1, i.e., is such that $\overline{T} \circ k_A = \pi$ and $\overline{T} \circ k_G = U$.

Then there exists a unique topological isomorphism $\psi : (A \rtimes_\alpha G)^{\mathcal{R}} \to B$, such that the induced topological isomorphism $\psi_l : \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}}) \to \mathcal{M}_l(B)$, defined by $\psi_l(L) := \psi L \psi^{-1}$ for $L \in \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$, induces (k_A, k_G) as in Lemma 4.4.2 from $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$, i.e., is such that $k_A = \psi_l \circ i_A^{\mathcal{R}}$ and $k_G = \psi_l \circ i_G^{\mathcal{R}}$.

Proof. Proposition 4.3.4 provides a non-degenerate $\mathcal{R}$-continuous covariant representation (π, U) such that $(\pi \rtimes U)^{\mathcal{R}}$ is an isometric representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$. If $T : B \to B(X)$ is a non-degenerate bounded representation such that $\overline{T} \circ k_A = \pi$ and $\overline{T} \circ k_G = U$, then Lemma 4.4.1 shows that $(\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G) = \overline{T} \circ (k_A \rtimes k_G)$, i.e., that $\pi \rtimes U = \overline{T} \circ (k_A \rtimes k_G)$. Hence, for $f \in C_c(G, A)$:

$$\begin{aligned} \|q^{\mathcal{R}}(f)\|^{\mathcal{R}} &= \|(\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f))\| \\ &= \|\pi \rtimes U(f)\| \\ &= \|\overline{T} \circ (k_A \rtimes k_G)(f)\| \\ &\leq \|\overline{T}\| \|k_A \rtimes k_G(f)\| \\ &= \|\overline{T}\| \|(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\|. \end{aligned}$$

Since (k_A, k_G) is $\mathcal{R}$-continuous, $\|(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\| \leq \|(k_A \rtimes k_G)^{\mathcal{R}}\| \|(q^{\mathcal{R}}(f))\|^{\mathcal{R}}$, for all $f \in C_c(G, A)$, hence we can now conclude, using (2) and (3) and the fact that $\lambda(B)$ is closed, that $(k_A \rtimes k_G)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to \lambda(B)$ is a topological isomorphism of Banach algebras. Since $\lambda : B \to \mathcal{M}_l(B)$ is assumed to be a topological embedding,

$$\psi := \lambda^{-1} \circ (k_A \rtimes k_G)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B$$

is a topological isomorphism.

We proceed to show that ψ_l induces k_A and k_G. As a preparation, note that, since $(\pi \rtimes U)^{\mathcal{R}}$ is isometric and $(\pi \rtimes U)^{\mathcal{R}} = \overline{T} \circ (k_A \rtimes k_G)^{\mathcal{R}}$, the map $\overline{T} : \mathcal{M}_l(B) \to B(X)$ is injective on $(k_A \rtimes k_G)^{\mathcal{R}}((A \rtimes_\alpha G)^{\mathcal{R}}) = \lambda(B)$. Now by [19, Proposition 5.3], for $a \in A$ and $f \in C_c(G, A)$, we have

$$\begin{aligned} (\pi \rtimes U)^{\mathcal{R}}(i_A^{\mathcal{R}}(a) q^{\mathcal{R}}(f)) &= \pi \rtimes U(i_A(a) f) \\ &= \pi(a) \pi \rtimes U(f) \\ &= \overline{T} \circ k_A(a) (\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f)) \\ &= \overline{T} \circ k_A(a) \overline{T} \circ (k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f)) \\ &= \overline{T}\left(k_A(a)(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\right). \end{aligned}$$

Since $\lambda(B)$ is a left ideal in $\mathcal{M}_l(B)$ (as is the case for every Banach algebra), we note that $k_A(a)(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f)) \in \lambda(B)$. On the other hand, we also have

$$(\pi \rtimes U)^{\mathcal{R}}(i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f)) \quad = \quad \overline{T}\left((k_A \rtimes k_G)^{\mathcal{R}}(i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f))\right).$$

Hence the injectivity of $\overline{T}$ on $\lambda(B)$ shows that

$$k_A(a)(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f)) = (k_A \rtimes k_G)^{\mathcal{R}}(i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f)).$$

We now apply λ^{-1} to both sides, and use that $\lambda^{-1}(L \circ \lambda(b)) = L(b)$ for all $L \in \mathcal{M}_l(B)$ and $b \in B$, to see that

$$\begin{aligned}
\psi(i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f)) &= \lambda^{-1} \circ (k_A \rtimes k_G)^{\mathcal{R}}(i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f)) \\
&= \lambda^{-1}\left(k_A(a)(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\right) \\
&= \lambda^{-1}\left(k_A(a) \circ \lambda \circ \lambda^{-1} \circ (k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\right) \\
&= \lambda^{-1}\left(k_A(a) \circ \lambda \circ \psi(q^{\mathcal{R}}(f))\right) \\
&= k_A(a)\psi(q^{\mathcal{R}}(f)).
\end{aligned}$$

By density, we conclude that $\psi \circ i_A^{\mathcal{R}}(a) = k_A(a) \circ \psi$, for all $a \in A$, i.e., that $k_A = \psi_l \circ i_A^{\mathcal{R}}$. A similar argument yields $k_G = \psi_l \circ i_G^{\mathcal{R}}$.

As to uniqueness, suppose that $\phi : (A \rtimes_\alpha G)^{\mathcal{R}} \to B$ is a topological isomorphism such that $k_A = \phi_l \circ i_A^{\mathcal{R}}$ and $k_G = \phi_l \circ i_G^{\mathcal{R}}$. Remembering that $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}} = \lambda$ [19, Theorem 7.2], this readily implies that, for $f \in C_c(G, A)$,

$$\begin{aligned}
\phi \circ (\lambda(q^{\mathcal{R}}(f)) \circ \phi^{-1} &= \phi \circ (i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}(q^{\mathcal{R}}(f)) \circ \phi^{-1} \\
&= (k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f)),
\end{aligned}$$

hence

$$(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f)) \circ \phi = \phi \circ \lambda(q^{\mathcal{R}}(f)).$$

Applying this to $q^{\mathcal{R}}(g)$, for $g \in C_c(G, A)$, we find

$$\begin{aligned}
(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\phi(q^{\mathcal{R}}(g)) &= \phi\left(\lambda(q^{\mathcal{R}}(f))q^{\mathcal{R}}(g)\right) \\
&= \phi(q^{\mathcal{R}}(f)q^{\mathcal{R}}(g)) \\
&= \phi(q^{\mathcal{R}}(f))\phi(q^{\mathcal{R}}(g)) \\
&= \lambda(\phi(q^{\mathcal{R}}(f)))\phi(q^{\mathcal{R}}(g)).
\end{aligned}$$

By density, we conclude that $(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f)) = \lambda(\phi(q^{\mathcal{R}}(f)))$, for all $f \in C_c(G, A)$. Again by density, we conclude that

$$\phi = \lambda^{-1} \circ (k_A \rtimes k_G)^{\mathcal{R}} = \psi.$$

□

4.5 Generalized Beurling algebras as crossed products

In this section we give sufficient conditions for a crossed product of a Banach algebra to be topologically isomorphic to a generalized Beurling algebra (see Definition 4.5.4), cf. Theorem 4.5.13 and Corollary 4.5.14. Since these conditions can always be satisfied, all generalized Beurling algebras are topologically isomorphic to a crossed product (cf. Theorem 4.5.17). Through an application of the General Correspondence Theorem (Theorem 4.2.1) we then obtain a bijection between the non-degenerate bounded representations of such a generalized Beurling algebra and the non-degenerate continuous covariant representations of the Banach algebra dynamical system where the group representation is bounded by a multiple of the weight, cf. Theorem 4.5.20. When the Banach algebra in the Banach algebra dynamical system is taken to be the scalars, and the weight on the group G taken to be the constant 1 function, then Corollary 4.5.14 shows that $L^1(G)$ is isometrically isomorphic to a crossed product, and Theorem 4.5.20 reduces to the classical bijective correspondence between uniformly bounded strongly continuous representations of G and non-degenerate bounded representations of $L^1(G)$, cf. Corollary 4.5.22.

We start with the topological isomorphism between a generalized Beurling algebra and a crossed product.

Definition 4.5.1. For a locally compact group G, let $\omega : G \to [0,\infty)$ be a non-zero submultiplicative Borel measurable function. Then ω is called a *weight* on G.

Note that we do not assume that $\omega \geq 1$, as is done in some parts of the literature. The fact that ω is non-zero readily implies that $\omega(e) \geq 1$. More generally, if $K \subseteq G$ is a compact set, there exist $a, b > 0$ such that $a \leq \omega(s) \leq b$ for all $s \in K$ [26, Lemma 1.3.3].

Let (A, G, α) be a Banach algebra dynamical system, and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations of (A, G, α). We recall that $C^{\mathcal{R}} := \sup_{(\pi,U)\in\mathcal{R}} \|\pi\|$ and $\nu^{\mathcal{R}} : G \to \mathbb{R}_{\geq 0}$ is defined by $\nu^{\mathcal{R}}(r) := \sup_{(\pi,U)\in\mathcal{R}} \|U_r\|$ as in [19, Definition 3.1]. We note that the map $\nu^{\mathcal{R}}$ is a weight on G. It is clearly submultiplicative, and, being the supremum of a family of continuous maps $\{r \mapsto \|U_r x\| : (\pi, U) \in \mathcal{R},\ x \in X_\pi,\ \|x\| \leq 1\}$, the map $\nu^{\mathcal{R}}$ is lower semicontinuous, hence Borel measurable.

Let ω be a weight on G, such that $\nu^{\mathcal{R}} \leq \omega$. Then, for all $f \in C_c(G, A)$,

$$\begin{aligned}
\sigma^{\mathcal{R}}(f) &= \sup_{(\pi,U)\in\mathcal{R}} \left\| \int_G \pi(f(s)) U_s \, ds \right\| \\
&\leq \sup_{(\pi,U)\in\mathcal{R}} \int_G \|\pi(f(s))\| \|U_s\| \, ds \\
&\leq C^{\mathcal{R}} \int_G \|f(s)\| \nu^{\mathcal{R}}(s) \, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq C^{\mathcal{R}} \int_G \|f(s)\|\omega(s)\,ds \\
&= C^{\mathcal{R}} \|f\|_{1,\omega},
\end{aligned}
\tag{4.5.1}
$$

where $\|\cdot\|_{1,\omega}$ denotes the ω-weighted L^1-norm. In Theorem 4.5.13, we will give sufficient conditions under which a reverse inequality holds. Then $\sigma^{\mathcal{R}}$ is actually a norm on $C_c(G, A)$ and is equivalent to a weighted L^1-norm, so that $(A \rtimes_\alpha G)^{\mathcal{R}}$ will be isomorphic to a generalized Beurling algebra to be defined shortly.

Definition 4.5.2. Let X be a Banach space, $1 \leq p < \infty$, and $\omega : G \to [0, \infty)$ a weight on G. We define the weighted p-norm on $C_c(G, X)$ by

$$
\|h\|_{p,\omega} := \left(\int_G \|h(s)\|^p \omega(s)\,ds \right)^{1/p},
$$

and define $L^p(G, X, \omega)$ as the completion of $C_c(G, X)$ with this norm.

Remark 4.5.3. By definition $L^p(G, A, \omega)$ with $1 \leq p < \infty$ has $C_c(G, A)$ as a dense subspace. In view of the central role of $C_c(G, A)$ in our theory of crossed products of Banach algebras, this is clearly desirable, but it would be unsatisfactory not to discuss the relation with spaces of Bochner integrable functions. We will now address this and explain that for $p = 1$ (our main space of interest in the sequel), $L^1(G, A, \omega)$ is (isometrically isomorphic to) a Bochner space.

In most of the literature, the theory of the Bochner integral is developed for finite (or at least σ-finite) measures, and sometimes the Banach space in question is assumed to be separable. Since $\omega d\mu$ (where μ is the left Haar measure on G) need not be σ-finite and A need not be separable, this is not applicable to our situation. In [10, Appendix E], however, the theory is developed for an arbitrary measure μ on a σ-algebra $\mathcal{A}$ of subsets of a set Ω, and functions $f : \Omega \to X$ with values in an arbitrary Banach space X. Such a function f is called Bochner integrable if $f^{-1}(B) \in \mathcal{A}$ for every Borel subset of X, $f(\Omega)$ is separable, and $\int_\Omega \|f(\xi)\| d\mu(\xi) < \infty$ (the measurability of $\xi \mapsto \|f(\xi)\|$ is an automatic consequence of the Borel measurability of f). Identifying Bochner integrable functions that are equal μ-almost everywhere, one obtains a Banach space $L^1(\Omega, \mathcal{A}, \mu, X)$, where the norm is given by $\|[f]\| = \int_\Omega \|f(\xi)\|\,d\mu(\xi)$ with f any representative of the equivalence class $[f] \in L^1(\Omega, \mathcal{A}, \mu, X)$. Although it is not stated as such, it is in fact proved [10, p. 352] that the simple Bochner integrable functions (i.e., all functions of the form $\sum_{i=1}^n \chi_{A_i} \otimes x_i$, where $A_i \in \mathcal{A}$, $\mu(A_i) < \infty$ and $x_i \in X$) form a dense subspace of $L^1(\Omega, \mathcal{A}, \mu, X)$.

We claim that our space $L^1(G, A, \omega)$ is isometrically isomorphic to the Bochner space $L^1(G, \mathcal{B}, \omega d\mu, A)$, where $\mathcal{B}$ is the Borel σ-algebra of G, and μ is the left Haar measure on G again. To start with, if $f \in C_c(G, A)$, then certainly f is Bochner integrable, so that we obtain a (clearly isometric) inclusion map $j : C_c(G, A) \to L^1(G, \mathcal{B}, \omega d\mu, A)$, that can be extended to an isometric embedding of $L^1(G, A, \omega)$ into $L^1(G, \mathcal{B}, \omega d\mu, A)$. To see that the image is dense, it is, in view of the density of the simple Bochner integrable functions in $L^1(G, \mathcal{B}, \omega d\mu, A)$, sufficient to approximate $\chi_S \otimes a$ with elements from $C_c(G, A)$, for arbitrary $a \in A$ and $S \in \mathcal{B}$ with $\int_G \chi_S \omega d\mu < \infty$. Since $C_c(G)$ is dense in $L^1(G, \omega d\mu)$ [26, Lemma 1.3.5], we can choose a sequence

$(f_n) \subseteq C_c(G)$ such that $f_n \to \chi_S$ in $L^1(G, \omega d\mu)$, and then

$$\|f_n \otimes a - \chi_S \otimes a\| = \|a\| \int_G |f_n(r) - \chi_S(r)|\omega(r)\, d\mu(r) \to 0.$$

Hence the image of $j : C_c(G, A) \to L^1(G, \mathcal{B}, \omega d\mu, A)$ is dense and our claim has been established.

For the sake of completeness, we note that one cannot argue that $\omega d\mu$ is "clearly" a regular Borel measure on G, so that $C_c(G)$ is dense in $L^1(G, \omega d\mu)$ by the standard density result [22, Proposition 7.9]. Indeed, although [22, Exercises 7.2.7–9] give sufficient conditions for this to hold (none of which applies in our general case), the regularity is not automatic, see [22, Exercise 7.2.13]. The proof of the density of $C_c(G)$ in $L^1(G, \omega d\mu)$ in [26, Lemma 1.3.5] is direct and from first principles. It uses in an essential manner that ω is bounded away from zero on compact subsets of G, but not that the Haar measure should be σ-finite or that ω should be integrable.

With (A, G, α) a Banach algebra dynamical system and ω a weight on G, if α is uniformly bounded, say $\|\alpha_s\| \leq C_\alpha$ for some $C_\alpha \geq 0$ and all $s \in G$, then, using the submultiplicativity of ω, it is routine to verify that

$$\|f * g\|_{1,\omega} \leq C_\alpha \|f\|_{1,\omega} \|g\|_{1,\omega} \quad (f, g \in C_c(G, A)).$$

Hence the Banach space $L^1(G, A, \omega)$ can be supplied with the structure of an associative algebra, such that

$$\|u * v\|_{1,\omega} \leq C_\alpha \|u\|_{1,\omega} \|v\|_{1,\omega} \quad (u, v \in L^1(G, A, \omega)).$$

If $C_\alpha = 1$ (i.e., if α lets G act as isometries on A), then $L^1(G, A, \omega)$ is a Banach algebra, and when $C_\alpha \neq 1$, as is well known, there is an equivalent norm on $L^1(G, A, \omega)$ such that it becomes a Banach algebra. We will show below (cf. Theorem 4.5.17) that such a norm can always be obtained from a topological isomorphism between $L^1(G, A, \omega)$ and the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ for a suitable choice of $\mathcal{R}$.

Definition 4.5.4. Let (A, G, α) be a Banach algebra dynamical system with α uniformly bounded and ω a weight on G. The Banach space $L^1(G, A, \omega)$ endowed with the continuous multiplication induced by the twisted convolution on $C_c(G, A)$, given by

$$[f * g](s) := \int_G f(r)\alpha_r(g(r^{-1}s))\, dr \quad (f, g \in C_c(G, A),\ s \in G),$$

will be denoted by $L^1(G, A, \omega; \alpha)$ and called a *generalized Beurling algebra.*

As a special case, we note that for $A = \mathbb{K}$, the generalized Beurling algebra reduces to the classical Beurling algebra $L^1(G, \omega)$, which is a true Banach algebra.

Obtaining such a reverse inequality to (4.5.1) rests on inducing a covariant representation of (A, G, α) from the left regular representation $\lambda : A \to B(A)$ of A, analogous to [46, Example 2.14]. The key result is Proposition 4.5.12 and we will now start working towards it.

Definition 4.5.5. Let (A, G, α) be a Banach algebra dynamical system and let $\pi : A \to B(X)$ be a bounded representation of A on a Banach space X. We define the induced algebra representation $\tilde{\pi}$ and left regular group representation Λ on the space of all functions from G to X by the formulae:

$$\begin{aligned}
[\tilde{\pi}(a)h](s) &:= \pi(\alpha_s^{-1}(a))h(s), \\
(\Lambda_r h)(s) &:= h(r^{-1}s),
\end{aligned}$$

where $h : G \to X$, $r, s \in G$ and $a \in A$.

A routine calculation, left to the reader, shows that $(\tilde{\pi}, \Lambda)$ is covariant.

We need a number of lemmas in preparation for the proof of Proposition 4.5.12.

The following is clear.

Lemma 4.5.6. *If (A, G, α) is a Banach algebra dynamical system with α uniformly bounded by a constant C_α, and $\omega : G \to [0, \infty)$ a weight, then for any bounded representation $\pi : A \to B(X)$ on a Banach space X, both the maps $\tilde{\pi} : A \to B(C_0(G, X))$ (as defined in Definition 4.5.5) and $\tilde{\pi} : A \to B(L^p(G, X, \omega))$ for $1 \le p < \infty$ (the canonically induced representation $\tilde{\pi}$ of A on $L^p(G, X, \omega)$ as completion of $C_c(G, X)$ with the $\|\cdot\|_{p,\omega}$-norm) are representations with norms bounded by $C_\alpha \|\pi\|$. Moreover, $C_c(G, X)$ is invariant under both A-actions.*

Lemma 4.5.7. *If (A, G, α) is a Banach algebra dynamical system and X a Banach space and ω a weight on G, then both the left regular representations $\Lambda : G \to B(C_0(G, X))$ (as defined in Definition 4.5.5), and $\Lambda : G \to B(L^p(G, X, \omega))$ for $1 \le p < \infty$ (the canonically induced representation Λ of G on $L^p(G, X, \omega)$ as completion of $C_c(G, X)$ with the $\|\cdot\|_{p,\omega}$-norm) are strongly continuous group representations. The representation $\Lambda : G \to B(C_0(G, X))$ acts as isometries, and $\Lambda : G \to B(L^p(G, X, \omega))$ is bounded by $\omega^{1/p}$. Moreover, $C_c(G, X)$ is invariant under both G-actions.*

Proof. That $\Lambda : G \to B(C_0(G, X))$ acts on $C_0(G, X)$ as isometries is clear.

That $\Lambda : G \to B(L^p(G, X, \omega))$ is bounded by $\omega^{1/p}$ follows from left invariance of the Haar measure and submultiplicativity of ω: For any $h \in C_c(G, X)$ and $s \in G$,

$$\begin{aligned}
\|\Lambda_s h\|_{p,\omega}^p &= \int_G \|[\Lambda_s h](t)\|^p \omega(t)\, dt \\
&= \int_G \|h(s^{-1}t)\|^p \omega(t)\, dt \\
&= \int_G \|h(t)\|^p \omega(st)\, dt \\
&\le \omega(s) \int_G \|h(t)\|^p \omega(t)\, dt \\
&= \omega(s) \|h\|_{p,\omega}^p.
\end{aligned}$$

Therefore Λ_s induces a map on $L^p(G, X, \omega)$ with the same norm, denoted by the same symbol, and $\|\Lambda_s\| \le \omega(s)^{1/p}$.

To prove strong continuity of $\Lambda : G \to B(C_0(G,X))$ and $\Lambda : G \to B(L^p(G,X,\omega))$, it is sufficient to establish strong continuity at $e \in G$ on dense subsets of both $C_0(G,X)$ of $L^p(G,X,\omega)$ respectively [19, Corollary 2.5]. By the uniform continuity of elements in $C_c(G,X)$ [46, Lemma 1.88] and the density of $C_c(G,X)$ in $C_0(G,X)$, the result follows for $\Lambda : G \to B(C_0(G,X))$.

To establish the result for $L^p(G,X,\omega)$, let $\varepsilon > 0$ and $h \in C_c(G,X)$ be arbitrary. Let $K := \operatorname{supp}(h)$ and W a precompact neighbourhood of $e \in G$. By uniform continuity of h, there exists a symmetric neighbourhood $V \subseteq W$ of $e \in G$ such that $\|\Lambda_s h - h\|_\infty^p < \varepsilon^p / (\sup_{r\in WK} \omega(r))\, \mu(WK)$ for all $s \in V$. Then, for $s \in V$,

$$\begin{aligned}\|\Lambda_s h - h\|_{p,\omega}^p &= \int_{WK} \|h(s^{-1}r) - h(r)\|^p \omega(r)\, dr \\ &\leq \frac{\varepsilon^p}{(\sup_{r\in WK} \omega(r)^p)\, \mu(WK)} \int_{WK} \omega(r)\, dr \\ &\leq \varepsilon^p.\end{aligned}$$

By the density of $C_c(G,X)$ in $L^p(G,X,\omega)$, the result follows. □

Lemma 4.5.8. *Let (A,G,α) be a Banach algebra dynamical system where α is uniformly bounded by a constant C_α and ω a weight on G. Let $\pi : A \to B(X)$ be a non-degenerate bounded representation on a Banach space X. If $f \in C_c(G,X)$, then there exist a compact subset K of G, containing* $\operatorname{supp}(f)$*, and a sequence $(f_n) \subseteq$* $\operatorname{span}(\tilde{\pi}(A)(C_c(G) \otimes X))$ *such that* $\operatorname{supp}(f_n) \subseteq K$ *for all n, and (f_n) converges uniformly to f on G. Consequently the representations $\tilde{\pi} : A \to B(C_0(G,X))$ and $\tilde{\pi} : A \to B(L^p(G,X,\omega))$ for $1 \leq p < \infty$ (as yielded by Definition 4.5.5) are then non-degenerate.*

Proof. Let $f \in C_c(G,X)$ and $\varepsilon > 0$ be arbitrary. Since $\operatorname{supp}(f)$ is compact, we can fix some precompact open set U_f containing $\operatorname{supp}(f)$. Since π is non-degenerate, for every $s \in G$, there exist finite sets $\{a_{i,s}\}_{i=1}^{n_s}$ and $\{x_{i,s}\}_{i=1}^{n_s}$ such that $\|f(s) - \sum_{i=1}^{n_s} \pi(a_{i,s})x_{i,s}\| < \varepsilon$. Since α is strongly continuous, for each $s \in G$ and $i \in \{1,\ldots,n_s\}$, there exists some precompact neighbourhood $W_{i,s}$ of s, such that $t \in W_{i,s}$ implies $n_s\|\pi\|\|x_{i,s}\|\|a_{i,s} - \alpha_t^{-1} \circ \alpha_s(a_{i,s})\| < \varepsilon$. Furthermore, for any $s \in G$, we can choose a precompact neighbourhood V_s of s such that $t \in V_s$ implies $\|f(s) - f(t)\| < \varepsilon$. Define $W_s := \bigcap_{i=1}^{n_s} W_{i,s} \cap V_s \cap U_f$. Now $\{W_s\}_{s\in G}$ is an open cover of $\operatorname{supp}(f)$, hence let $\{W_{s_j}\}_{j=1}^m$ be a finite subcover. Let $\{u_j\}_{j=1}^m \subseteq C_c(G)$ be a partition of unity such that, for all $j \in \{1,\ldots,m\}$, $0 \leq u_j(t) \leq 1$ for $t \in G$, $\operatorname{supp}(u_j) \subseteq W_{s_j}$, $\sum_{j=1}^m u_j(t) = 1$ for $t \in \operatorname{supp}(f)$, and $\sum_{j=1}^m u_j(t) \leq 1$ for $t \in G$. Then, for $t \in G$,

$$\begin{aligned}&\left\| f(t) - \left(\sum_{j=1}^m \sum_{i=1}^{n_{s_j}} \tilde{\pi}(\alpha_{s_j}(a_{i,s_j}))u_j \otimes x_{i,s_j} \right)(t) \right\| \\ = &\left\| f(t) - \sum_{j=1}^m \sum_{i=1}^{n_{s_j}} u_j(t)\pi(\alpha_t^{-1} \circ \alpha_{s_j}(a_{i,s_j}))x_{i,s_j} \right\|\end{aligned}$$

$$
\begin{aligned}
&= \left\| \sum_{j=1}^{m} u_j(t) f(t) - \sum_{j=1}^{m} u_j(t) f(s_j) + \sum_{j=1}^{m} u_j(t) f(s_j) - \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \pi(a_{i,s_j}) x_{i,s_j} \right. \\
&\qquad \left. + \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \pi(a_{i,s_j}) x_{i,s_j} - \sum_{j=1}^{m} \sum_{i=1}^{n_{s_j}} u_j(t) \pi(\alpha_t^{-1} \circ \alpha_{s_j}(a_{i,s_j})) x_{i,s_j} \right\| \\
&\leq \sum_{j=1}^{m} u_j(t) \left\| f(t) - f(s_j) \right\| + \sum_{j=1}^{m} u_j(t) \left\| f(s_j) - \sum_{i=1}^{n_{s_j}} \pi(a_{i,s_j}) x_{i,s_j} \right\| \\
&\qquad + \left\| \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \pi(a_{i,s_j}) x_{i,s_j} - \sum_{j=1}^{m} \sum_{i=1}^{n_{s_j}} u_j(t) \pi(\alpha_t^{-1} \circ \alpha_{s_j}(a_{i,s_j})) x_{i,s_j} \right\| \\
&\leq \varepsilon + \varepsilon + \left\| \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \pi(a_{i,s_j}) x_{i,s_j} - \sum_{j=1}^{m} \sum_{i=1}^{n_{s_j}} u_j(t) \pi(\alpha_t^{-1} \circ \alpha_{s_j}(a_{i,s_j})) x_{i,s_j} \right\| \\
&\leq \varepsilon + \varepsilon + \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \left\| \pi(a_{i,s_j}) x_{i,s_j} - \pi(\alpha_t^{-1} \circ \alpha_{s_j}(a_{i,s_j})) x_{i,s_j} \right\| \\
&\leq \varepsilon + \varepsilon + \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \|\pi\| \|x_{i,s_j}\| \|a_{i,s_j} - \alpha_t^{-1} \circ \alpha_{s_j}(a_{i,s_j})\| \\
&\leq \varepsilon + \varepsilon + \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \frac{\varepsilon}{n_{s_j}} \\
&\leq \varepsilon + \varepsilon + \varepsilon.
\end{aligned}
$$

Since

$$
\sum_{j=1}^{m} \sum_{i=1}^{n_{s_j}} \tilde{\pi}(\alpha_{s_j}(a_{i,s_j})) u_j \otimes x_{i,s_j}
$$

is supported in the fixed compact set $\overline{U_f}$, the result follows. □

Combining the previous three lemmas yields:

Corollary 4.5.9. *If (A, G, α) is a Banach algebra dynamical system with α uniformly bounded by a constant C_α, ω a weight on G and $\pi : A \to B(X)$ a bounded representation on a Banach space X, then the pair $(\tilde{\pi}, \Lambda)$ (as yielded by Definition 4.5.5) is a continuous covariant representation of (A, G, α) on $C_0(G, X)$ or $L^p(G, X, \omega)$ for $1 \leq p < \infty$ respectively. Moreover:*

(1) Both representations $\tilde{\pi} : A \to B(C_0(G, X))$ and $\tilde{\pi} : A \to B(L^p(G, X, \omega))$ satisfy $\|\tilde{\pi}\| \leq C_\alpha \|\pi\|$.

(2) The left regular group representation $\Lambda : G \to B(C_0(G, X))$ acts as isometries on $C_0(G, X)$, and the left regular group representation $\Lambda : G \to B(L^p(G, X, \omega))$ is bounded by $\omega^{1/p}$ on G.

(3) *The space $C_c(G,X)$, seen as a subspace of $C_0(G,X)$ or $L^p(G,X,\omega)$, is invariant under actions of both A and G on $C_0(G,X)$ or $L^p(G,X,\omega)$ through the representations $\tilde{\pi} : A \to B(C_0(G,X))$ and $\Lambda : G \to B(C_0(G,X))$, or $\tilde{\pi} : A \to B(L^p(G,X,\omega))$ and $\Lambda : G \to B(L^p(G,X,\omega))$, respectively.*

(4) *If $\pi : A \to B(X)$ is non-degenerate, so are both representations $\tilde{\pi} : A \to B(C_0(G,X))$ and $\tilde{\pi} : A \to B(L^p(G,X,\omega))$.*

If α is uniformly bounded by $C_\alpha \geq 0$, Corollary 4.5.9 shows that the left regular representation $\lambda : A \to B(A)$ of A is such that the covariant representation $(\tilde{\lambda},\Lambda)$ of (A,G,α) on $L^1(G,A,\omega)$ (as yielded by Definition 4.5.5) is continuous with $\|\tilde{\lambda}\| \leq C_\alpha$ and $\|\Lambda_s\| \leq \omega(s)$. Moreover, if A has a bounded left or right approximate identity, then λ is non-degenerate, and hence $(\tilde{\lambda},\Lambda)$ is non-degenerate.

We need two more results before Proposition 4.5.12 can be established.

Lemma 4.5.10. *Let (A,G,α) be a Banach algebra dynamical system with α uniformly bounded. Let ω be a weight on G, and $\lambda : A \to B(A)$ the left regular representation of A. Let $(\tilde{\lambda},\Lambda)$ be the continuous covariant representation of (A,G,α) on $L^1(G,A,\omega)$ (as yielded by Definition 4.5.5). Then, for all $f \in C_c(G,A)$, $\tilde{\lambda} \rtimes \Lambda(f) \in B(L^1(G,A,\omega))$ leaves the subspace $C_c(G,A)$ of $L^1(G,A,\omega)$ invariant. In fact, if $h \in C_c(G,A) \subseteq L^1(G,A,\omega)$, then $\tilde{\lambda} \rtimes \Lambda(f)h \in L^1(G,A,\omega)$ is given by the pointwise formula*

$$[\tilde{\lambda} \rtimes \Lambda(f)h](s) = \int_G \alpha_s^{-1}(f(r))h(r^{-1}s)\,dr \quad (s \in G).$$

Proof. We proceed indirectly, via $C_0(G,A)$, and write $(\tilde{\lambda}_0,\Lambda_0)$ and $(\tilde{\lambda}_1,\Lambda_1)$ for the continuous covariant representations of (A,G,α) on $C_0(G,A)$ and $L^1(G,A,\omega)$, respectively. Let $f,h \in C_c(G,A)$ and consider the integral

$$\tilde{\lambda}_1 \rtimes \Lambda_1(f)h = \int_G \tilde{\lambda}_1(f(r))\Lambda_{1,r}h\,dr \in L^1(G,A,\omega).$$

Let $K := \mathrm{supp}(f) \cdot \mathrm{supp}(h)$, and put $C_0(G,A)_K := \{g \in C_0(G,A) : \mathrm{supp}(g) \subseteq K\}$. Then $C_0(G,A)_K$ is a closed subspace of $C_0(G,A)$ and the inclusion $j_K : C_0(G,A)_K \to L^1(G,A,\omega)$ is bounded, since ω is bounded on compact sets. Define $\psi : G \to C_0(G,A)_K$ by $\psi(r) := \tilde{\lambda}_0(f(r))\Lambda_{0,r}h$ for all $r \in G$. Then ψ is continuous and supported on the compact set $\mathrm{supp}(f) \subseteq G$. Now, by the boundedness of j_K,

$$\int_G \tilde{\lambda}_1(f(r))\Lambda_{1,r}h\,dr = \int_G j_K \circ \psi(r)\,dr = j_K\left(\int_G \psi(r)\,dr\right).$$

Since $\int_G \psi(r)\,dr \in C_0(G,A)_K$, we conclude that $\tilde{\lambda}_1 \rtimes \Lambda_1(f)h \in C_c(G,A)$.

Since the evaluations $\mathrm{ev}_s : C_0(G,A)_K \to A$, sending $g \in C_0(G,A)_K$ to $g(s) \in A$, are bounded for all $s \in G$, we find that, for all $s \in G$,

$$\begin{aligned}\left(\int_G \psi(r)\,dr\right)(s) &= \mathrm{ev}_s\left(\int_G \psi(r)\,dr\right)\\ &= \int_G \mathrm{ev}_s \circ \psi(r)\,dr\end{aligned}$$

$$
\begin{aligned}
&= \int_G \psi(r)(s)\,dr \\
&= \int_G \alpha_s^{-1}(f(r))h(r^{-1}s)\,dr.
\end{aligned}
$$

Therefore $[\tilde{\lambda}_1 \rtimes \Lambda_1(f)h](s) = \int_G \alpha_s^{-1}(f(r))h(r^{-1}s)\,dr$. □

Lemma 4.5.11. *Let A be a Banach algebra with bounded approximate right identity (u_i) and let $K \subseteq A$ be compact. Then, for any $\varepsilon > 0$, there exists an index i_0 such that $\|au_i\| \geq \|a\| - \varepsilon$ for all $a \in K$ and all $i \geq i_0$.*

Proof. Let $M \geq 1$ be an upper bound for (u_i) and $\varepsilon > 0$ be arbitrary. By compactness of K, there exist $a_1, \ldots, a_n \in K$ such that for all $a \in K$ there exists an index $k \in \{1, \ldots, n\}$ with $\|a - a_k\| < \varepsilon/3M \leq \varepsilon/3$. Let i_0 be such that $\|a_k u_i - a_k\| < \varepsilon/3$ for all $k \in \{1, \ldots, n\}$ and all $i \geq i_0$.

Now, for $a \in K$ arbitrary, let $k_0 \in \{1, \ldots, n\}$ be such that $\|a - a_{k_0}\| < \varepsilon/3$. For any $i \geq i_0$,

$$
\begin{aligned}
\|au_i\| &\geq \|a\| - \|au_i - a_{k_0}u_i\| - \|a_{k_0}u_i - a_{k_0}\| - \|a_{k_0} - a\| \\
&> \|a\| - \frac{\varepsilon}{3M}M - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \\
&= \|a\| - \varepsilon.
\end{aligned}
$$

□

Finally, we combine Lemmas 4.5.6–4.5.11 to obtain the following:

Proposition 4.5.12. *Let (A, G, α) be a Banach algebra dynamical system where A has an M-bounded approximate right identity and α is uniformly bounded by a constant C_α. Let ω be a weight on G, and $\lambda : A \to B(A)$ the left regular representation of A. Let $W \subseteq G$ be a precompact neighbourhood of $e \in G$. Then the non-degenerate continuous covariant representation $(\tilde{\lambda}, \Lambda)$ of (A, G, α) on $L^1(G, A, \omega)$ (as yielded by Definition 4.5.5) satisfies*

$$
\|\tilde{\lambda} \rtimes \Lambda(f)\| \geq \frac{1}{C_\alpha M \sup_{s \in W} \omega(s)} \|f\|_{1,\omega}
$$

for all $f \in C_c(G, A)$. Consequently $\tilde{\lambda} \rtimes \Lambda : C_c(G, A) \to B(L^1(G, A, \omega))$ is a faithful representation.

Proof. Let (u_i) be an M-bounded approximate right identity of A and $W \subseteq G$ any precompact neighbourhood of $e \in G$. Let $f \in C_c(G, A)$ and $\varepsilon > 0$ be arbitrary. By the uniform continuity of f, there exists a symmetric neighbourhood $V \subseteq W$ of $e \in G$ such that $\|f(r) - f(rs)\| < \varepsilon/2C_\alpha M$ for all $s \in V$ and $r \in G$. By continuity of all maps involved and the assumption that f is compactly supported, the set $\{\alpha_s^{-1}(f(s)) : s \in G\} \subseteq A$ is compact. Lemma 4.5.11 then asserts the existence of an index i_0, such that $\|au_{i_0}\| \geq \|a\| - \varepsilon/2$ for all $a \in \{\alpha_s^{-1}(f(s)) : s \in G\}$.

By Urysohn's Lemma, let $h_0 : G \to [0,1]$ be continuous with $h_0(e) = 1$ and $\mathrm{supp}(h_0) \subseteq V$, so that $h_0 \in C_c(G)$. We may assume $h_0(r) = h_0(r^{-1})$ for all $r \in G$, by replacing h_0 with $r \mapsto \max\{h_0(r), h_0(r^{-1})\}$. Define

$$h := \left(\int_G h_0(t)\,dt\right)^{-1} h_0 \otimes u_{i_0} \in C_c(G, A).$$

Then

$$\begin{aligned}
\|h\|_{1,\omega} &= \left(\int_G h_0(t)\,dt\right)^{-1} \int_G h_0(r)\|u_{i_0}\|\omega(r)\,dr \\
&\leq M \sup_{r\in V} \omega(r) \\
&\leq M \sup_{r\in W} \omega(r).
\end{aligned}$$

For every $s \in G$, we find, using the reverse triangle inequality, noting that $\|f(s)\| = \|\alpha_s \circ \alpha_{s^{-1}}(f(s))\| \leq C_\alpha \|\alpha_{s^{-1}}(f(s))\|$, remembering that h_0 is supported in V, and applying Lemma 4.5.10, that

$$\begin{aligned}
&\|[\tilde{\lambda} \rtimes \Lambda(f)h](s)\| \\
= &\left\|\int_G \alpha_s^{-1}(f(r))h(r^{-1}s)\,dr\right\| \\
= &\left\|\int_G \alpha_s^{-1}(f(sr))h(r^{-1})\,dr\right\| \\
= &\left(\int_G h_0(t)\,dt\right)^{-1} \left\|\int_G h_0(r^{-1})\alpha_s^{-1}(f(sr))u_{i_0}\,dr\right\| \\
\geq &\left(\int_G h_0(t)\,dt\right)^{-1} \left\|\int_G h_0(r^{-1})\alpha_s^{-1}(f(s))u_{i_0}\,dr\right\| \\
&\quad - \left(\int_G h_0(t)\,dt\right)^{-1} \left\|\int_G h_0(r^{-1})\alpha_s^{-1}(f(s) - f(sr))u_{i_0}\,dr\right\| \\
\geq &\left(\int_G h_0(t)\,dt\right)^{-1} \left(\int_G h_0(r)\,dr\right) \|\alpha_s^{-1}(f(s))u_{i_0}\| \\
&\quad - \left(\int_G h_0(t)\,dt\right)^{-1} \frac{\varepsilon C_\alpha M \left(\int_G h_0(r)\,dr\right)}{2C_\alpha M} \\
\geq &\|\alpha_{s^{-1}}(f(s))\| - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \\
\geq &\frac{1}{C_\alpha}\|f(s)\| - \varepsilon.
\end{aligned}$$

Hence, with $L := \sup_{s\in\mathrm{supp}(f)} \omega(s)$, which is finite since ω is bounded on compact sets,

$$\begin{aligned}\|\tilde{\lambda} \rtimes \Lambda(f)h\|_{1,\omega} &\geq \int_{\mathrm{supp}(f)} \|[\tilde{\lambda} \rtimes \Lambda(f)h](s)\|\omega(s)\,ds \\ &\geq \int_{\mathrm{supp}(f)} \left(\frac{1}{C_\alpha}\|f(s)\| - \varepsilon\right)\omega(s)\,ds \\ &\geq \frac{1}{C_\alpha}\|f\|_{1,\omega} - \varepsilon\mu(\mathrm{supp}(f))L.\end{aligned}$$

Now, since $\|h\|_{1,\omega} \leq M \sup_{r\in W} \omega(r)$, we obtain

$$\|\tilde{\lambda} \rtimes \Lambda(f)\| \geq \frac{1}{C_\alpha M \sup_{r\in W}\omega(r))}\|f\|_{1,\omega} - \frac{\varepsilon L}{M\sup_{r\in W}\omega(r)}\mu(\mathrm{supp}(f)).$$

Because $\varepsilon > 0$ was chosen arbitrarily, $\|\tilde{\lambda} \rtimes \Lambda(f)\| \geq (C_\alpha M \sup_{r\in W}\omega(r))^{-1}\|f\|_{1,\omega}$ now follows. ☐

We now combine our previous results, notably (4.5.1) and Proposition 4.5.12, to obtain sufficient conditions for a crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ to be isomorphic to a generalized Beurling algebra, and also collect a number of direct consequences in the following result. The desired reverse inequality to (4.5.1) is a consequence of Proposition 4.5.12, supplying the first inequality in (4.5.2) and the second inequality in (4.5.2), which follows from the assumption that $(\tilde{\lambda}, \Lambda)$ is $\mathcal{R}$-continuous.

Theorem 4.5.13. *Let (A, G, α) be a Banach algebra dynamical system where A has an M-bounded approximate right identity and α is uniformly bounded by a constant C_α. Let ω be a weight on G. Let $\mathcal{R}$ be a uniformly bounded class of continuous covariant representations of (A, G, α) with $C^{\mathcal{R}} = \sup_{(\pi,U)\in\mathcal{R}} \|\pi\| < \infty$ and satisfying $\nu^{\mathcal{R}}(r) = \sup_{(\pi,U)\in\mathcal{R}} \|U_r\| \leq \omega(r)$ for all $r \in G$. Let λ be the left regular representation of A, and suppose that the non-degenerate continuous covariant representation $(\tilde{\lambda}, \Lambda)$ of (A, G, α) on $L^1(G, A, \omega)$ (as yielded by Definition 4.5.5) is $\mathcal{R}$-continuous. Then, for all $f \in C_c(G, A)$, with $\mathcal{Z}$ denoting a neighbourhood base of $e \in G$ of which all elements are contained in a fixed compact set,*

$$\begin{aligned}\left(\frac{1}{C_\alpha M \inf_{W\in\mathcal{Z}}\sup_{r\in W}\omega(r)}\right)\|f\|_{1,\omega} &\leq \|\tilde{\lambda} \rtimes \Lambda(f)\| \qquad (4.5.2)\\ &\leq \|\tilde{\lambda} \rtimes \Lambda\|\sigma^{\mathcal{R}}(f) \leq \|\tilde{\lambda} \rtimes \Lambda\| C^{\mathcal{R}}\|f\|_{1,\omega}.\end{aligned}$$

In particular, $\sigma^{\mathcal{R}}$ is a norm on $C_c(G, A)$, so that $C_c(G, A)$ can be identified with a subspace of $(A \rtimes_\alpha G)^{\mathcal{R}}$. Since the norms $\sigma^{\mathcal{R}}$ and $\|\cdot\|_{1,\omega}$ on $C_c(G, A)$ are equivalent, there exists a topological isomorphism between the Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ and the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ that is the identity on $C_c(G, A)$.

The multiplication on the common dense subspace $C_c(G, A)$ of the spaces $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A, \omega; \alpha)$ is given by

$$[f * g](s) := \int_G f(r)\alpha_r(g(r^{-1}s))\,dr \quad (f, g \in C_c(G, A),\ s \in G).$$

The faithful representation $\tilde{\lambda} \rtimes \Lambda : C_c(G, A) \to B(L^1(G, A, \omega))$ extends to a topological embedding $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(L^1(G, A, \omega))$ of the Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ into $B(L^1(G, A, \omega))$.

Using Corollary 4.5.9, we have the following consequence of Theorem 4.5.13, where the isomorphism between $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A, \omega; \alpha)$ is isometric.

Corollary 4.5.14. *Let (A, G, α) be a Banach algebra dynamical system where A has a 1-bounded approximate right identity and α lets G act as isometries on A. Let ω be a weight on G, and λ the left regular representation of A. Then the non-degenerate continuous covariant representation $(\tilde{\lambda}, \Lambda)$ on $L^1(G, A, \omega)$ (as yielded by Definition 4.5.5) is such that $\tilde{\lambda}$ is contractive and Λ is bounded by ω.*

Suppose furthermore that $\inf_{W \in \mathcal{Z}} \sup_{r \in W} \omega(r) = 1$, with $\mathcal{Z}$ denoting a neighbourhood base of $e \in G$ of which all elements are contained in a fixed compact set, and that $\mathcal{R}$ is a uniformly bounded class of continuous covariant representations with $(\tilde{\lambda}, \Lambda) \in \mathcal{R}$, and satisfying

$$C^{\mathcal{R}} = \sup_{(\pi, U) \in \mathcal{R}} \|\pi\| \leq 1,$$

and

$$\nu^{\mathcal{R}}(r) = \sup_{(\pi, U) \in \mathcal{R}} \|U_r\| \leq \omega(r) \quad (r \in G).$$

Then $\sigma^{\mathcal{R}}(f) = \|f\|_{1,\omega}$ for $f \in C_c(G, A)$, and hence $(A \rtimes_\alpha G)^{\mathcal{R}}$ is isometrically isomorphic to the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$.

Moreover, $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(L^1(G, A, \omega))$ is an isometric embedding as a Banach algebra.

Proof. Since $(\tilde{\lambda}, \Lambda) \in \mathcal{R}$, we have $\|\tilde{\lambda} \rtimes \Lambda\| \leq 1$, and by hypothesis $C^{\mathcal{R}} \leq 1$. Therefore, by Theorem 4.5.13, for every $f \in C_c(G, A)$,

$$\|f\|_{1,\omega} \leq \|\tilde{\lambda} \rtimes \Lambda(f)\| \leq \|\tilde{\lambda} \rtimes \Lambda\| \sigma^{\mathcal{R}}(f) \leq C^{\mathcal{R}} \|\tilde{\lambda} \rtimes \Lambda\| \|f\|_{1,\omega} \leq \|f\|_{1,\omega}.$$

We conclude that $C^{\mathcal{R}} = \|\tilde{\lambda} \rtimes \Lambda\| = 1$, and the result now follows. ☐

Remark 4.5.15. Certainly if the weight $\omega : G \to [0, \infty)$ is continuous in $e \in G$ and $\omega(e) = 1$, then $\inf_{W \in \mathcal{Z}} \sup_{r \in W} \omega(r) = 1$, for example if ω is taken to be a continuous positive character of G.

Remark 4.5.16. We note that the representation

$$(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : L^1(G, A, \omega; \alpha) \to B(L^1(G, A, \omega; \alpha))$$

does not equal the left regular representation of $L^1(G, A, \omega; \alpha)$ in general, but they are always conjugate. To see this, define, for $h \in C_c(G, A)$ and $s \in G$, $\check{h}(s) := \alpha_{s^{-1}}(h(s))$, $\hat{h}(s) := \alpha_s(h(s))$. Then $\hat{\cdot} : C_c(G, A) \to C_c(G, A)$ and $\check{\cdot} : C_c(G, A) \to C_c(G, A)$ are mutual inverses and, since α is uniformly bounded, extend to mutually inverse Banach space isomorphisms of $L^1(G, A, \omega; \alpha)$ onto itself. Then $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}}$ and the left regular representation λ of $L^1(G, A, \omega; \alpha)$ are conjugate under $\hat{\cdot}$. Indeed by Lemma 4.5.10, for $f, h \in C_c(G, A)$ and $s \in G$,

$$\begin{aligned}
\left(\tilde{\lambda} \rtimes \Lambda(f)\check{h}\right)^{\wedge}(s) &= \alpha_s\left([\tilde{\lambda} \rtimes \Lambda(f)\check{h}](s)\right) \\
&= \alpha_s\left(\int_G \alpha_s^{-1}(f(r))\check{h}(r^{-1}s)\,dr\right) \\
&= \alpha_s\left(\int_G \alpha_s^{-1}(f(r))\alpha_{s^{-1}r}(h(r^{-1}s))\,dr\right) \\
&= \int_G f(r)\alpha_r(h(r^{-1}s))\,dr \\
&= [f * h](s) \\
&= [\lambda(f)h](s).
\end{aligned}$$

Hence $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}}$ and the left regular representation

$$\lambda : L^1(G, A, \omega; \alpha) \to B(L^1(G, A, \omega; \alpha))$$

of $L^1(G, A, \omega; \alpha)$ are conjugate as claimed. Note that $\hat{\cdot}$ is the identity if $\alpha = \text{triv}$, hence in that case $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} = \lambda$.

We continue the main line with the following trivial but important observation: If $(\tilde{\lambda}, \Lambda) \in \mathcal{R}$, for example, by taking $\mathcal{R} := \{(\tilde{\lambda}, \Lambda)\}$, then certainly $(\tilde{\lambda}, \Lambda)$ is $\mathcal{R}$-continuous, hence the conclusions in Theorem 4.5.13 hold, and in particular the algebras $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A, \omega; \alpha)$ are topologically isomorphic. A similar remark is applicable to Corollary 4.5.14, giving sufficient conditions for the mentioned topological isomorphism to be isometric. Hence we have the following:

Theorem 4.5.17. *Let (A, G, α) be a Banach algebra dynamical system where A has a bounded approximate right identity and α is uniformly bounded. Let ω be a weight on G and let the non-degenerate continuous covariant representation $(\tilde{\lambda}, \Lambda)$ of (A, G, α) on $L^1(G, A, \omega)$ be as yielded by Definition 4.5.5. Then the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ and the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ with $\mathcal{R} := \{(\tilde{\lambda}, \Lambda)\}$ are topologically isomorphic via an isomorphism that is the identity on $C_c(G, A)$.*

Furthermore, the map $\tilde{\lambda} \rtimes \Lambda : C_c(G, A) \to B(L^1(G, A, \omega))$ extends to a topological embedding of $L^1(G, A, \omega; \alpha)$ into $B(L^1(G, A, \omega))$.

If A has a 1-bounded two-sided approximate identity, α lets G act as isometries on A and $\inf_{W \in \mathcal{Z}} \sup_{r \in W} \omega(r) = 1$, with $\mathcal{Z}$ denoting a neighbourhood base of $e \in G$ of which all elements are contained in a fixed compact set, then the isomorphism between $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A, \omega; \alpha)$ is an isometry, and the embedding of $L^1(G, A, \omega; \alpha)$ into $B(L^1(G, A, \omega))$ is isometric.

Remark 4.5.18. As noted in Remark 4.5.16, when $\alpha = \text{triv}$, then $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}}$ equals the left regular representation $\lambda : L^1(G, A, \omega; \alpha) \to B(L^1(G, A, \omega; \alpha))$ of $L^1(G, A, \omega; \alpha)$.

Remark 4.5.19. We note that, for $(A, G, \alpha) = (\mathbb{K}, G, \text{triv})$, the second part of Theorem 4.5.17 asserts that $(\mathbb{K} \rtimes_{\text{triv}} G)^{\mathcal{R}}$ is isometrically isomorphic to the classical Beurling algebra $L^1(G, \omega)$, provided that $\inf_{W \in \mathcal{Z}} \sup_{r \in W} \omega(r) = 1$ (which is certainly true if ω is continuous at $e \in G$ and $\omega(e) = 1$). In particular $L^1(G)$ is isometrically isomorphic to a crossed product. Under the condition $\inf_{W \in \mathcal{Z}} \sup_{r \in W} \omega(r) = 1$, combining Remark 4.5.16 and Theorem 4.5.17 also shows that the left regular representation of $L^1(G, \omega)$ is an isometric embedding of $L^1(G, \omega)$ into $B(L^1(G, \omega))$.

Hence, provided that A has a bounded approximate right identity, the generalized Beurling algebras $L^1(G, A, \omega; \alpha)$, and in particular the classical Beurling algebras $L^1(G, \omega)$ for $A = \mathbb{K}$, are isomorphic to a crossed product associated with a Banach algebra dynamical system. Therefore, in the case where the algebra A has a two-sided identity, the General Correspondence Theorem (Theorem 4.2.1) determines the non-degenerate bounded representations of generalized Beurling algebras. This we will elaborate on in the rest of the section. In cases where the algebra is trivial, i.e., $A = \mathbb{K}$, we regain classical results on the representation theory of $L^1(G)$ and other classical Beurling algebras.

Assume, in addition to the hypothesis in Theorem 4.5.13, that A has an M-bounded two-sided approximate identity and that all continuous covariant representations in $\mathcal{R}$ are non-degenerate. In that case, we claim that the non-degenerate $\mathcal{R}$-continuous covariant representations are precisely the non-degenerate continuous covariant representations (π, U) of (A, G, α), with no further restriction on π, but with U such that $\|U_r\| \leq C_U \omega(r)$ for all $r \in G$ and a U-dependent constant C_U. To see this, we start by noting that, for $f \in C_c(G, A)$,

$$\begin{aligned} \|\pi \rtimes U(f)\| &\leq \int_G \|\pi(f(r))\| \|U_r\| \, dr \\ &\leq \int_G \|\pi\| \|f(r)\| C_U \omega(r) \, dr \\ &\leq C_U \|\pi\| \int_G \|f(r)\| \omega(r) \, dr \\ &= C_U \|\pi\| \|f\|_{1,\omega} \\ &\leq C'_{(\pi,U)} \sigma^{\mathcal{R}}(f) \end{aligned}$$

for some $C'_{(\pi,U)} \geq 0$, since $\| \cdot \|_{1,\omega}$ and $\sigma^{\mathcal{R}}$ are equivalent.

For the converse, we use that A has a bounded approximate left identity and that $\mathcal{R}$ consists of non-degenerate continuous covariant representations. If (π, U) is a non-degenerate $\mathcal{R}$-continuous representation of (A, G, α), then the General Correspondence Theorem (Theorem 4.2.1) asserts that

$$(\pi, U) = (\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}, \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}),$$

where $\overline{(\pi \rtimes U)^{\mathcal{R}}} : \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}}) \to B(X_\pi)$ is the non-degenerate bounded representation induced by the non-degenerate bounded representation $(\pi \rtimes U)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(X_\pi)$. However if $T : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(X)$ is any non-degenerate bounded representation, then [19, Proposition 7.1] asserts that there exists a constant $C_T := M_l^{\mathcal{R}}\|T\|$, with $M_l^{\mathcal{R}}$ a bound for a bounded approximate left identity in $(A \rtimes_\alpha G)^{\mathcal{R}}$, such that

$$\|\overline{T} \circ i_G^{\mathcal{R}}(r)\| \leq C_T \nu^{\mathcal{R}}(r) \leq C_T \omega(r) \quad (r \in G). \tag{4.5.3}$$

Therefore, $r \mapsto \|U_r\|$ is bounded by a multiple of ω, as claimed.

We now take $\mathcal{R} := \{(\tilde{\lambda}, \Lambda)\}$ as in Theorem 4.5.17. Theorem 4.5.17 shows that the non-degenerate bounded representations of $L^1(G, A, \omega; \alpha)$ can be identified with those of $(A \rtimes_\alpha G)^{\mathcal{R}}$. By the General Correspondence Theorem (Theorem 4.2.1) the latter are in natural bijection with the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and these we have just described. Hence the non-degenerate bounded representations of $L^1(G, A, \omega; \alpha)$ are in natural bijection with pairs (π, U) as above. Furthermore, slightly simplified versions of [19, Equations (8.1) and (8.2)] (cf. Remark 4.5.21) give explicit formulas for retrieving (π, U) from a non-degenerate bounded representation T of $(A \rtimes_\alpha G)^{\mathcal{R}} \simeq L^1(G, A, \omega; \alpha)$. Combining all this, we obtain the following correspondence between the non-degenerate continuous covariant representations of (A, G, α) and the non-degenerate bounded representations of the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$:

Theorem 4.5.20. *Let (A, G, α) be a Banach algebra dynamical system where A has a two-sided approximate identity and α is uniformly bounded by C_α. Let ω be a weight on G. Then the following maps are mutual inverses between the non-degenerate continuous covariant representations (π, U) of (A, G, α) on a Banach space X, satisfying $\|U_r\| \leq C_U \omega(r)$ for some $C_U \geq 0$ and all $r \in G$, and the non-degenerate bounded representations $T : L^1(G, A, \omega; \alpha) \to B(X)$ of the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ on X:*

$$(\pi, U) \mapsto \left(f \mapsto \int_G \pi(f(r))U_r \, dr\right) =: T^{(\pi,U)} \quad (f \in C_c(G, A)),$$

determining a non-degenerate bounded representation $T^{(\pi,U)}$ of the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$, and,

$$T \mapsto \begin{pmatrix} a \mapsto \text{SOT-lim}_{(V,i)} T(z_V \otimes a u_i), \\ s \mapsto \text{SOT-lim}_{(V,i)} T(z_V(s^{-1} \cdot) \otimes u_i) \end{pmatrix} =: (\pi^T, U^T),$$

where $\mathcal{Z}$ is a neighbourhood base of $e \in G$, of which all elements are contained in a fixed compact subset of G, $z_V \in C_c(G, A)$ is chosen such that $z_V \geq 0$, supported in $V \in \mathcal{Z}$, $\int_G z_V(r)dr = 1$, and (u_i) is any bounded approximate left identity of A.

Furthermore, if A has an M-bounded approximate left identity, then the following bounds for $T^{(\pi,U)}$ and (π^T, U^T) hold:

(1) $\|T^{(\pi,U)}\| \leq C_U \|\pi\|$,

(2) $\|\pi^T\| \leq (\inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r)) \, \|T\|$,

(3) $\|U_s^T\| \leq M \, (\inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r)) \, \|T\| \omega(s) \quad (s \in G)$.

Proof. Except for the claimed bounds for $\|T^U\|$, $\|\pi^T\|$ and $\|U^T\|$, all statements have been proven preceding the statement of the theorem. We will now establish these three bounds.

We prove (1). Let (π, U) be a non-degenerate continuous covariant representations of (A, G, α) on a Banach space X, satisfying $\|U_r\| \leq C_U \omega(r)$ for some $C_U \geq 0$ and all $r \in G$. Then, for any $f \in C_c(G, A)$,

$$\begin{aligned} \|T^{(\pi,U)}(f)\| &= \left\| \int_G \pi(f(r)) U_r \, dr \right\| \\ &\leq \int_G \|\pi\| \|f(r)\| \|U_r\| \, dr \\ &\leq \|\pi\| C_U \int_G \|f(r)\| \omega(r) \, dr \\ &= \|\pi\| C_U \|f\|_{1,\omega}. \end{aligned}$$

Therefore $\|T^{(\pi,U)}\| \leq \|\pi\| C_U$.

We prove (2). Let $T : L^1(G, A, \omega; \alpha) \to B(X)$ be a non-degenerate bounded representations of the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ on X. Choose a bounded two-sided approximate identity (u_i) of A. Then, for any $a \in A$,

$$\begin{aligned} \|T(z_V \otimes a u_i)\| &\leq \|T\| \|z_V \otimes a u_i\|_{1,\omega} \\ &\leq \|T\| \int_G z_V(r) \|a u_i\| \omega(r) \, dr \\ &= \|T\| \|a u_i\| \int_G z_V(r) \omega(r) \, dr \\ &\leq \|T\| \|a u_i\| \sup_{r \in V} \omega(r) \int_G z_V(r) \, dr \\ &= \|T\| \|a u_i\| \sup_{r \in V} \omega(r). \end{aligned}$$

Since, in particular, (u_i) is an approximate right identity of A, for any $\varepsilon_1 > 0$, there exists an index i_0 such that $i \geq i_0$ implies $\|a u_i\| \leq \|a\| + \varepsilon_1$. Also, for any $\varepsilon_2 > 0$, there exists some $V_0 \in \mathcal{Z}$ such that $\sup_{r \in V_0} \omega(r) \leq \inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r) + \varepsilon_2$. Now, if $(V, i) \geq (V_0, i_0)$, then $V_0 \supseteq V$ and $i \geq i_0$, and hence

$$\begin{aligned} \|T(z_V \otimes a u_i)\| &\leq \|T\| \|a u_i\| \sup_{r \in V} \omega(r) \\ &\leq \|T\| \|a u_i\| \sup_{r \in V_0} \omega(r) \\ &\leq \|T\| (\|a\| + \varepsilon_1) \left(\inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r) + \varepsilon_2 \right). \end{aligned}$$

Therefore, if $x \in X$, then

$$\begin{aligned}
\|\pi^T(a)x\| &= \lim_{(V,i)} \|T(z_V \otimes au_i)x\| \\
&= \lim_{(V,i)\geq(V_0,i_0)} \|T(z_V \otimes au_i)x\| \\
&\leq \|T\|(\|a\| + \varepsilon_1)\left(\inf_{V\in\mathcal{Z}} \sup_{r\in V} \omega(r) + \varepsilon_2\right)\|x\|.
\end{aligned}$$

Since ε_1 and ε_2 we chosen arbitrarily, $\|\pi^T\| \leq \|T\| \left(\inf_{V\in\mathcal{Z}} \sup_{r\in V} \omega(r)\right)$ now follows.

We prove (3). Let (u_i) be an M-bounded approximate left identity of A. Fix $s \in G$. Let $\varepsilon > 0$ be arbitrary and let $V_0 \in \mathcal{Z}$ be such that $\sup_{r\in V_0} \omega(r) \leq \inf_{V\in\mathcal{Z}} \sup_{r\in V} \omega(r) + \varepsilon$. Fix some index i_0, then, for every $(V,i) \geq (V_0, i_0)$,

$$\begin{aligned}
\|T(z_V(s^{-1}\cdot) \otimes u_i)\| &\leq \|T\|\|z_V(s^{-1}\cdot) \otimes u_i\|_{1,\omega} \\
&= \|T\| \int_G z_V(s^{-1}r)\|u_i\|\omega(r)\,dr \\
&\leq M\|T\| \int_G z_V(r)\omega(sr)\,dr \\
&\leq M\|T\| \int_G z_V(r)\omega(s)\omega(r)\,dr \\
&= M\|T\|\omega(s) \int_V z_V(r)\omega(r)\,dr \\
&\leq M\|T\| \left(\sup_{r\in V_0} \omega(r)\right)\omega(s) \int_V z_V(r)\,dr \\
&\leq M\|T\| \left(\inf_{V\in\mathcal{Z}} \sup_{r\in V} \omega(r) + \varepsilon\right)\omega(s).
\end{aligned}$$

Therefore, if $x \in X$, then

$$\begin{aligned}
\|U_s^T x\| &= \lim_{(V,i)} \|T(z_V(s^{-1}\cdot) \otimes u_i)x\| \\
&= \lim_{(V,i)\geq(V_0,i_0)} \|T(z_V(s^{-1}\cdot) \otimes u_i)x\| \\
&\leq M\|T\| \left(\inf_{V\in\mathcal{Z}} \sup_{r\in V} \omega(r) + \varepsilon\right)\omega(s)\|x\|.
\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily, $\|U_r^T\| \leq M\|T\| \left(\inf_{V\in\mathcal{Z}} \sup_{r\in V} \omega(r)\right)\omega(s)$ now follows. □

Remark 4.5.21. Our reconstruction formulas in Theorem 4.5.20 differs slightly from those given in [19, Equations (8.1) and (8.2)], where the reconstruction formula for U^T is given as

$$s \mapsto \text{SOT-lim}_{(V,i)} T(z_V(s^{-1}\cdot) \otimes \alpha_s(u_i)), \tag{4.5.4}$$

with (u_i) any bounded approximate left identity of A. However, if (u_i) is any bounded approximate left identity of A and $s \in G$ is fixed, then $(\alpha_{s^{-1}}(u_i))$ is also a bounded approximate left identity of A, and using this particular choice in (4.5.4) gives the formula in Theorem 4.5.20.

For the Banach algebra dynamical system $(\mathbb{K}, G, \mathrm{triv})$ and weight ω on G, Theorem 4.5.20 simplifies. We collect the statements from Theorem 4.5.20 concerning representations and some material from Remark 4.5.16, Corollary 4.5.14 in the following result, which contains a few classical results as special cases: For one-dimensional representations, the result reduces to the bijection between ω-bounded characters of G and multiplicative functionals of the Beurling algebra $L^1(G,\omega)$, see, e.g., [26, Theorem 2.8.2] (where, contrary to our general groups, G is assumed to be abelian). In the case where ω is the constant 1, the result reduces to the classical bijection between uniformly bounded strongly continuous representations of G and non-degenerate bounded representations of $L^1(G)$, see, e.g., [24, Assertion VI.1.32].

Corollary 4.5.22. *Let ω be a weight on G. With (z_V) as in Theorem 4.5.20, the maps*

$$U \mapsto \left(f \mapsto \int_G f(r)U_r\,dr\right) =: T^U \quad (f \in C_c(G)),$$

determining a non-degenerate bounded representation T^U of the Beurling algebra $L^1(G,\omega)$, and

$$T \mapsto \left(s \mapsto \text{SOT-lim}_V T(z_V(s^{-1}\cdot))\right) =: U^T$$

are mutual inverses between the strongly continuous group representations U of G on a Banach space X, satisfying $\|U_r\| \leq C_U\omega(r)$, for some $C_U \geq 0$ and all $r \in G$, and the non-degenerate bounded representations $T : L^1(G,\omega) \to B(X)$ of the Beurling algebra $L^1(G,\omega)$ on X).

If the weight satisfies $\inf_{W\in\mathcal{Z}} \sup_{r\in W} \omega(r) = 1$, where $\mathcal{Z}$ is a neighbourhood base of $e \in G$, of which all elements are contained in a fixed compact subset of G, then $\|T^U\| = \sup_{r\in G} \|U_r\|/\omega(r)$ and $\|U^T_r\| \leq \|T\|\omega(r)$ for all $r \in G$.

Proof. The only statement that does not follow directly from Theorem 4.5.20 is that $\|T^U\| = \sup_{r\in G} \|U_r\|/\omega(r)$, when $\sup_{W\in\mathcal{Z}}(\sup_{r\in W} \omega(r))^{-1} = 1$.

To establish this, we note that

$$\|U_r\| = \omega(r)\frac{\|U_r\|}{\omega(r)} \leq \left(\sup_{s\in G} \frac{\|U_s\|}{\omega(s)}\right)\omega(r).$$

Therefore, we can replace C_U with $\sup_{r\in G} \|U_r\|/\omega(r)$, and, by the bound (1) in Theorem 4.5.20, $\|T^U\| \leq \sup_{r\in G} \|U_r\|/\omega(r)$. The reverse inequality follows from (3) in Theorem 4.5.20, when noting that the maps $U \mapsto T^U$ and $T \mapsto U^T$ are mutual inverses. □

Remark 4.5.23. For one-dimensional representations, Corollary 4.5.22 implies that continuous characters $\chi : G \to \mathbb{C}^\times$ of G, such that $|\chi(r)| \leq C_\chi\omega(r)$ for some C_χ and all $r \in G$, are in natural bijection with the one-dimensional representations of

$L^1(G,\omega)$. Since this is a Banach algebra, such representations are contractive, and the final part of Corollary 4.5.22 then asserts that one can actually take $C_\chi = 1$ (cf. [26, Lemma 2.8.2] for abelian G). One can also verify this directly by noting that, if there exists some $s \in G$ for which $|\chi(s)| > \omega(s)$, then, for all $n \in \mathbb{N}$, by submultiplicativity of ω,

$$\left(\frac{|\chi(s)|}{\omega(s)}\right)^n = \frac{|\chi(s^n)|}{\omega(s)^n} \leq C_\chi \frac{\omega(s^n)}{\omega(s)^n} \leq C_\chi.$$

Therefore, since $|\chi(s)| > \omega(s)$, we must have that $C_\chi = \infty$, which is absurd. Hence $|\chi(r)| \leq \omega(r)$ for all $r \in G$.

4.6 Other types for (π, U)

For a given Banach algebra dynamical system (A, G, α) we have thus far been concerned with a uniformly bounded class of pairs (π, U), where $\pi : A \to B(X)$ and $U : G \to B(X)$ are multiplicative representations, U is strongly continuous, and satisfy the covariance condition

$$U_r \pi(a) U_r^{-1} = \pi(\alpha_r(a))$$

for all $r \in G$ and $a \in A$. On the other hand, in [19, Proposition 6.5], we have encountered an example of a pair (π, U) where π and U are both anti-multiplicative and satisfy the anti-covariance condition

$$U_r \pi(a) U_r^{-1} = \pi(\alpha_{r^{-1}}(a))$$

for all $r \in G$ and $a \in A$. Suppose one has a uniformly bounded class $\mathcal{R}$ of such pairs (π, U), with U strongly continuous, π non-degenerate and that A has a bounded "appropriately sided" approximate identity, can one then find a Banach algebra of crossed product type again, such that its non-degenerate bounded (perhaps anti-) representations are in natural bijection with the $\mathcal{R}$-continuous pairs (ρ, V), satisfying the aforementioned requirements for elements of $\mathcal{R}$? What about pairs (π, U) where π is multiplicative, U is anti-multiplicative and a covariance condition is satisfied? Can one, to ask a more fundamental question, expect a meaningful theory to exist for such pairs?

In this section we address these matters. We start by determining what appears to be the natural "reasonable" requirements in this vein on (π, U) for a meaningful theory to exist (and which are not met in the second-mentioned example). There turn out to be four cases. For each case we indicate a Banach algebra dynamical system (B, H, β) such that $B = A$ and $H = G$ as sets, and such that the given maps $\pi : B \to B(X)$ and $U : H \to B(X)$ are now multiplicative and satisfy a covariance condition. This brings us back into the realm of the correspondence as in Theorem 4.2.1 or [19, Theorem 8.1], but we leave it to the reader to formulate the resulting correspondence theorem for the other three types of uniformly bounded classes of non-degenerate continuous pairs (π, U).

After this, we turn to actions of A and G on $C_c(G,A)$. While this is not, in general, a Banach space, several Banach spaces are naturally obtained from $C_c(G,A)$ via quotients and/or completions, hence it is for this space that we list sixteen canonical pairs of actions, with each of the four "reasonable" properties occurring four times. We then explain that, even though the formulas look quite different, there is essentially only one pair, and the fifteen others can be derived from it. We conclude with natural pairs (π,U) of commuting actions on $C_c(G,A)$.

This section is, in a sense, elementary and almost entirely algebraic in nature. Nevertheless, we thought it worthwhile to make a systematic inventorization, once and for all, of the "reasonable" properties of pairs (π,U), the natural actions on A-valued function spaces on G, and the interrelations between the various formulas. A particular case of the results in the present section will be instrumental in Section 4.8 where we explain how non-degenerate right– and bimodules over generalized Beurling algebras fit into the general framework of crossed products of Banach algebras.

To start with, let (A,G,α) be a Banach algebra dynamical system. What are the "reasonable" properties of (π,U) that can lead to a meaningful theory? Let us assume that $\pi : A \to B(X)$ is linear and multiplicative or anti-multiplicative, that $U : G \to B(X)$ is a multiplicative or anti-multiplicative map of G into the group of invertible elements of $B(X)$, and that

$$U_r\pi(a)U_r^{-1} = \pi(\delta_r(a)) \tag{4.6.1}$$

for all $a \in A$ and $r \in G$, where δ is a multiplicative or anti-multiplicative map from G into the automorphisms or anti-automorphisms of A. This is "asking for the most general setup". We start by arguing that δ should map G into the automorphisms of A. Indeed, if π is multiplicative, $r \in G$ and $a_1, a_2 \in A$, then

$$\begin{aligned}
\pi(\delta_r(a_1a_2)) &= U_r\pi(a_1a_2)U_r^{-1} \\
&= U_r\pi(a_1)U_r^{-1}U_r\pi(a_2)U_r^{-1} \\
&= \pi(\delta_r(a_1))\pi(\delta_r(a_2)) \\
&= \pi(\delta_r(a_1)\delta_r(a_2)).
\end{aligned}$$

If π is anti-multiplicative, then again

$$\begin{aligned}
\pi(\delta_r(a_1a_2)) &= U_r\pi(a_1a_2)U_r^{-1} \\
&= U_r\pi(a_2)U_r^{-1}U_r\pi(a_1)U_r^{-1} \\
&= \pi(\delta_r(a_2))\pi(\delta_r(a_1)) \\
&= \pi(\delta_r(a_1)\delta_r(a_2)).
\end{aligned}$$

Hence one is led to assume that δ maps G into $\mathrm{Aut}(A)$, still leaving open the possible choice of $\delta : G \to \mathrm{Aut}(A)$ being multiplicative or anti-multiplicative.

To continue, if U is anti-multiplicative, then (4.6.1) implies, for $a \in A$ and $r_1, r_2 \in G$,

$$\pi(\delta_{r_1r_2}(a)) = U_{r_1r_2}\pi(a)U_{r_1r_2}^{-1}$$

$$\begin{aligned} &= U_{r_2} U_{r_1} \pi(a) U_{r_1}^{-1} U_{r_2}^{-1} \\ &= \pi(\delta_{r_2} \circ \delta_{r_1}(a)). \end{aligned}$$

Therefore, unless one imposes a further relation between π and U, it seems that only the possibility that δ is also anti-multiplicative will lead to a meaningful theory. Likewise, the multiplicativity of U "implies" that δ should be multiplicative. Using that δ_r is multiplicative on A for $r \in G$, it is easily seen that the covariance condition yields no implications on the nature of π.

With (A, G, α) given, the relevant non-trivial choice for a multiplicative δ is α, and for an anti-multiplicative δ it is α^o where $\alpha^o_r := \alpha_{r^{-1}}$ for all $r \in G$; the reason for this notation will become clear in a moment. We will consider these non-trivial choices for δ first, and return to $\delta = \mathrm{triv}$ later.

Hence we have to consider four meaningful possibilities for a pair (π, U) and the relation between π and U. If we let, e.g., (a, m) denote the case where π is anti-multiplicative and U is multiplicative, then, for (m, m) and (a, m), one should require

$$U_r \pi(a) U_r^{-1} = \pi(\alpha_r(a)),$$

and for (m, a) and (a, a), one should require

$$U_r \pi(a) U_r^{-1} = \pi(\alpha_{r^{-1}}(a)) = \pi(\alpha^o_r(a))$$

for all $a \in A$ and $r \in G$.

Now note that, with G^o denoting the opposite group, $\alpha^o : G^o \to \mathrm{Aut}(A)$ is a multiplicative strongly continuous map if α is. Therefore, if (A, G, α) is a Banach algebra dynamical system, then so is (A, G^o, α^o). Furthermore, if A^o is the opposite algebra, then $\mathrm{Aut}(A) = \mathrm{Aut}(A^o)$. Therefore, if (A, G, α) is a Banach algebra dynamical system, so is (A^o, G, α). Combining these two, a Banach algebra dynamical system has a third natural companion Banach algebra dynamical system, namely (A^o, G^o, α^o). In each of these three cases, the Banach algebra is A as a set, and the group is G as a set. Hence the given maps $\pi : A \to B(X)$ and $U : G \to B(X)$ can be viewed unaltered as maps for the new system, denoted by $\tilde{\pi}$ and $\tilde{U}$. The crux is, then, that anti-multiplicative representations of A correspond to multiplicative representations of A^o, and likewise for G and G^o. Hence, regardless of the type of (π, U), one can always pass to a suitable companion Banach algebra dynamical system to ensure that the same pair of maps is a pair of type (m, m) for the companion Banach algebra dynamical system. For example, if (π, U) is of type (a, a) for (A, G, α) and satisfies $U_r \pi(a) U_r^{-1} = \pi(\alpha_{r^{-1}}(a))$ for $a \in A$ and $r \in G$, then $\tilde{\pi} : A^o \to B(X)$ and $\tilde{U} : G^o \to B(X)$ form a pair of type (m, m) for (A^o, G^o, α^o), satisfying $\tilde{U}_r \tilde{\pi}(a) \tilde{U}_r^{-1} = \pi(\alpha^o_r(a))$ for $a \in A^o$ and $r \in G^o$. Hence, $(\tilde{\pi}, \tilde{U})$ is a covariant pair of type (m, m) for (A^o, G^o, α^o), and we are back at our original type of objects. One can argue similarly for the types (a, m) and (m, a), and this leads to Table 4.1.

We can now point out how classes of pairs (π, U) of other types than (m, m) can be related to representations of a crossed product of a Banach algebra. For

Type of (π, U) for (A, G, α)	Should require that $U_r \pi(a) U_r^{-1} =$	$(\tilde{\pi}, \tilde{U})$ is type (m, m) for	$\tilde{U}_r \tilde{\pi}(a) \tilde{U}_r^{-1} =$
(m, m)	$\pi(\alpha_r(a))$	(A, G, α)	$\tilde{\pi}(\alpha_r(a))$
(m, a)	$\pi(\alpha_{r^{-1}}(a))$	(A, G^o, α^o)	$\tilde{\pi}(\alpha_r^o(a))$
(a, m)	$\pi(\alpha_r(a))$	(A^o, G, α)	$\tilde{\pi}(\alpha_r(a))$
(a, a)	$\pi(\alpha_{r^{-1}}(a))$	(A^o, G^o, α^o)	$\tilde{\pi}(\alpha_r^o(a))$

Table 4.1

example, suppose that $\mathcal{R}$ is a uniformly bounded class (as in Section 4.2) of non-degenerate continuous pairs (π, U) where $\pi : A \to B(X)$ and $U : G \to B(X)$ are both anti-multiplicative satisfying $U_r \pi(a) U_r^{-1} = \pi(\alpha_{r^{-1}}(a))$. We pass to the system (A^o, G^o, α^o) and consider the class $\tilde{\mathcal{R}}$ consisting of all pairs $(\tilde{\pi}, \tilde{U}) = (\pi, U)$, for $(\pi, U) \in \mathcal{R}$. Then $\tilde{\mathcal{R}}$ is a uniformly bounded class of non-degenerate continuous covariant representations of (A^o, G^o, α^o), and the general correspondence theorem, Theorem 4.2.1 or [19, Theorem 8.1] furnishes a bijection between the non-degenerate bounded (multiplicative) representations of $(A^o \rtimes_{\alpha^o} G^o)^{\tilde{\mathcal{R}}}$ and the non-degenerate $\tilde{\mathcal{R}}$-continuous covariant representations of (A^o, G^o, α^o). It is then a matter of routine, left to the reader, to reformulate the latter class as pairs (π, U) of type (a, a) for (A, G, α) again, being aware that the Haar measure for G differs from that of G^o by the modular function. The remaining types (m, a) and (a, m) can be treated similarly and bring the non-degenerate bounded (always multiplicative) representations of $(A \rtimes_{\alpha^o} G^o)^{\tilde{\mathcal{R}}}$ and $(A^o \rtimes_{\alpha} G)^{\tilde{\mathcal{R}}}$, respectively, into play.

We now turn to what can perhaps be regarded as the sixteen canonical types of actions of A and G on the linear space $C_c(G, A)$ (and hence on many natural Banach spaces). They are listed in Table 4.2 and were originally obtained by judiciously experimenting with various candidate expressions. In this table $a \in A$, $r, s \in G$, $f \in C_c(G, A)$ and $\chi : G \to \mathbb{C}^\times$ is a continuous character. The possibility of inserting χ enables one to arrange, by choosing the modular function, that the group actions as in the lines 3, 8, 11 and 16 are isometric on L^p-type spaces for $1 \leq p < \infty$.

We will now explain why, essentially, there is only one canonical type of action from which all others can be derived. To start with, note that the spaces $C_c(G, A)$, $C_c(G^o, A)$, $C_c(G, A^o)$ and $C_c(G^o, A^o)$ can all be identified. This can be put to good use as follows: Suppose one has verified that the formulas in line 1 yield a pair (π, U) of type (m, m) for any Banach algebra dynamical system. Then one can apply this to (A, G^o, α^o) and view the resulting actions of A and G^o on $C_c(G^o, A)$, which are of type (m, m), as actions of A and G on $C_c(G, A)$. It is immediate that the resulting pair (π, U) will be of type (m, a) for (A, G, α). In fact, it is line 5 in the table. Likewise, line 1 for (A^o, G, α) and for (A^o, G^o, α^o) yields line 9 and 13 for (A, G, α), respectively. Similarly line 2 yields the lines 6, 10 and 14, line 3 yields the lines 7, 11 and 15, and line 4 yields the lines 8, 12 and 16. Thus the actions in lines 1 through 4 generate all others via passing to companion Banach algebra dynamical systems. These four actions of (A, G, α) of type (m, m) are, in turn, also essentially

No.	$(\pi(a)f)(s)$	$(U_r f)(s)$	Type (π, U)	$U_r\pi(a)U_r^{-1}$
1	$af(s)$	$\chi_r\alpha_r(f(r^{-1}s))$	(m, m)	$\pi(\alpha_r(a))$
2	$af(s)$	$\chi_r\alpha_r(f(sr))$	(m, m)	$\pi(\alpha_r(a))$
3	$\alpha_s(a)f(s)$	$\chi_r f(sr)$	(m, m)	$\pi(\alpha_r(a))$
4	$\alpha_{s^{-1}}(a)f(s)$	$\chi_r f(r^{-1}s)$	(m, m)	$\pi(\alpha_r(a))$
5	$af(s)$	$\chi_r\alpha_{r^{-1}}(f(sr^{-1}))$	(m, a)	$\pi(\alpha_{r^{-1}}(a))$
6	$af(s)$	$\chi_r\alpha_{r^{-1}}(f(rs))$	(m, a)	$\pi(\alpha_{r^{-1}}(a))$
7	$\alpha_{s^{-1}}(a)f(s)$	$\chi_r f(rs)$	(m, a)	$\pi(\alpha_{r^{-1}}(a))$
8	$\alpha_s(a)f(s)$	$\chi_r f(sr^{-1})$	(m, a)	$\pi(\alpha_{r^{-1}}(a))$
9	$f(s)a$	$\chi_r\alpha_r(f(r^{-1}s))$	(a, m)	$\pi(\alpha_r(a))$
10	$f(s)a$	$\chi_r\alpha_r(f(sr))$	(a, m)	$\pi(\alpha_r(a))$
11	$f(s)\alpha_s(a)$	$\chi_r f(sr)$	(a, m)	$\pi(\alpha_r(a))$
12	$f(s)\alpha_{s^{-1}}(a)$	$\chi_r f(r^{-1}s)$	(a, m)	$\pi(\alpha_r(a))$
13	$f(s)a$	$\chi_r\alpha_{r^{-1}}(f(sr^{-1}))$	(a, a)	$\pi(\alpha_{r^{-1}}(a))$
14	$f(s)a$	$\chi_r\alpha_{r^{-1}}(f(rs))$	(a, a)	$\pi(\alpha_{r^{-1}}(a))$
15	$f(s)\alpha_{s^{-1}}(a)$	$\chi_r f(rs)$	(a, a)	$\pi(\alpha_{r^{-1}}(a))$
16	$f(s)\alpha_s(a)$	$\chi_r f(sr^{-1})$	(a, a)	$\pi(\alpha_{r^{-1}}(a))$

Table 4.2

the same: They are, in fact, equivalent under linear automorphisms of $C_c(G, A)$. In order to see this, define, for a continuous character $\chi : G \to \mathbb{C}^\times$, the linear order 2 automorphism $T_\chi : C_c(G, A) \to C_c(G, A)$ by

$$(T_\chi f)(s) := \chi_s f(s^{-1})$$

for all $s \in G$ and $f \in C_c(G, A)$. Adding line numbers in brackets in the obvious way, one then verifies that

$$\pi_{(2)}(a) = T_{\chi(1)\chi(2)^{-1}}\pi_{(1)}(a)T^{-1}_{\chi(1)\chi(2)^{-1}}$$

for all $a \in A$, and

$$U_{(2),r} = T_{\chi(1)\chi(2)^{-1}}U_{(1),r}T^{-1}_{\chi(1)\chi(2)^{-1}}$$

for all $r \in G$. Thus the actions in the lines 1 and 2 are equivalent. Likewise,

$$\pi_{(4)}(a) = T_{\chi(4)\chi(3)^{-1}}\pi_{(3)}(a)T^{-1}_{\chi(4)\chi(3)^{-1}}$$

for all $a \in A$, and

$$U_{(4),r} = T_{\chi(4)\chi(3)^{-1}}U_{(3),r}T^{-1}_{\chi(4)\chi(3)^{-1}}$$

for all $r \in G$. Hence the actions in the lines 3 and 4 are equivalent. Furthermore, with $\chi : G \to \mathbb{C}^\times$ a continuous character as before, we let $S_\chi : C_c(G, A) \to C_c(G, A)$ be defined by

$$(S_\chi f)(s) := \chi_{s^{-1}}\alpha_{s^{-1}}(f(s))$$

for all $s \in G$ and $f \in C_c(G, A)$. Then S_χ is a linear automorphism of $C_c(G, A)$ and its inverse is given by

$$(S_\chi^{-1} f)(s) = \chi_s \alpha_s(f(s)).$$

It is then straightforward to check that

$$\pi_{(4)}(a) = S_{\chi(1)\chi(4)^{-1}} \pi_{(1)}(a) S^{-1}_{\chi(1)\chi(4)^{-1}}$$

for all $a \in A$, and

$$U_{(4),r} = S_{\chi(1)\chi(4)^{-1}} U_{(1),r} S^{-1}_{\chi(1)\chi(4)^{-1}}$$

for all $r \in G$. Thus the actions in the lines 1 and 4 are equivalent, and hence all actions of type (m, m) in the lines 1 through 4 are equivalent. Therefore, in spite of the different appearances, there is essentially only one type of canonical action in Table 4.2.

We conclude this section with a discussion of the remaining case $\delta = \text{triv}$ in (4.6.1), i.e., commuting actions of A and G. It is interesting to note that, given a Banach algebra dynamical system (A, G, α), we have eight canonical commuting actions of A and G on $C_c(G, A)$. They are listed in Table 4.3, with the same notational conventions as in Table 4.2.

No.	$(\pi(a)f)(s)$	$(U_r f)(s)$	Type (π, U)	$U_r \pi(a) U_r^{-1}$
1	$\alpha_s(a)f(s)$	$\chi_r \alpha_r(f(r^{-1}s))$	(m, m)	$\pi(a)$
2	$\alpha_{s^{-1}}(a)f(s)$	$\chi_r \alpha_r(f(sr))$	(m, m)	$\pi(a)$
3	$\alpha_{s^{-1}}(a)f(s)$	$\chi_r \alpha_{r^{-1}}(f(sr^{-1}))$	(m, a)	$\pi(a)$
4	$\alpha_s(a)f(s)$	$\chi_r \alpha_{r^{-1}}(f(rs))$	(m, a)	$\pi(a)$
5	$f(s)\alpha_s(a)$	$\chi_r \alpha_r(f(r^{-1}s))$	(a, m)	$\pi(a)$
6	$f(s)\alpha_{s^{-1}}(a)$	$\chi_r \alpha_r(f(sr))$	(a, m)	$\pi(a)$
7	$f(s)\alpha_{s^{-1}}(a)$	$\chi_r \alpha_{r^{-1}}(f(sr^{-1}))$	(a, a)	$\pi(a)$
8	$f(s)\alpha_s(a)$	$\chi_r \alpha_{r^{-1}}(f(rs))$	(a, a)	$\pi(a)$

Table 4.3

We employ a similar mechanism as before. Indeed, suppose we have verified that, for any Banach algebra dynamical system, the formulas in line 1 yield commuting actions of type (m, m). Applying this to (A, G^o, α^o) one obtains a commuting pair of type (m, a): line 3 in Table 4.3. Likewise, line 1 for (A^o, G, α) and for (A^o, G^o, α^o) yield line 5 and line 7, respectively. Similarly line 2 yields the lines 4, 6 and 8. Furthermore, with $\mathbf{1} : G \to \mathbb{C}^\times$ denoting the trivial character, one checks that

$$\pi_{(2)}(a) = T_{\mathbf{1}} \pi_{(1)}(a) T_{\mathbf{1}}^{-1}$$

for all $a \in A$, and

$$U_{(2),r} = T_{\mathbf{1}} U_{(1),r} T_{\mathbf{1}}^{-1}$$

for all $r \in G$. Thus the actions in lines 1 and 2 are equivalent, and again there is essentially only one pair of actions in Table 4.3. In this case, one can even go a bit further: Define

$$\begin{aligned}(\tilde{\pi}(a)f)(s) &:= af(s)\\ (\tilde{U}_r f)(s) &:= f(r^{-1}s)\end{aligned}$$

for all $a \in A$, $r \in G$ and $f \in C_c(G, A)$. Then $(\tilde{\pi}, \tilde{U})$ is "the" canonical covariant pair of type (m, m) for (A, G, triv), and one verifies that

$$\pi_{(1)}(a) = S_{\chi(1)}^{-1}\tilde{\pi}(a)S_{\chi(1)}$$

for all $a \in A$, and

$$U_{(1),r} = S_{\chi(1)}^{-1}\tilde{U}_r S_{\chi(1)}$$

for all $r \in G$. Hence all the commuting actions for A and G in Table 4.3 essentially originate from the canonical covariant pair $(\tilde{\pi}, \tilde{U})$ for (A, G, triv).

4.7 Several Banach algebra dynamical systems and classes

Suppose (A_i, G_i, α_i), with $i \in \{1, \ldots, n\}$, are finitely many Banach algebra dynamical systems, and that $\mathcal{R}_i$ is a non-empty uniformly bounded class of non-degenerate continuous covariant representations of (A_i, G_i, α_i). We will show (cf. Theorem 4.7.5) that, for a Banach space X, there is a natural bijection between the non-degenerate bounded representations of the projective tensor product $\widehat{\bigotimes}_{i=1}^{n}(A_i \rtimes_{\alpha_i} G_i)^{\mathcal{R}_i}$ on X and the n-tuples $((\pi_1, U_1), \ldots, (\pi_n, U_n))$, where, for each $i \in \{1, \ldots, n\}$, (π_i, U_i) is a non-degenerate $\mathcal{R}_i$-continuous covariant representation of (A_i, G_i, α_i) on X, and (π_i, U_i) commutes (to be defined below) with (π_j, U_j) for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$,. Such situations are quite common. For example if X is a G-bimodule (i.e., X is supplied with a left action U of G and a right action V of G that commute), then this can be interpreted as commuting non-degenerate continuous covariant representations (id, U) and (id, V) of $(\mathbb{K}, G, \mathrm{triv})$ and $(\mathbb{K}, G^o, \mathrm{triv})$, respectively (where G^o denotes the opposite group of G). In a similar vein, if (π, U) is a non-degenerate continuous covariant representation of (A, G, α) on X, and (ρ, V) is a non-degenerate continuous pair of type (a, a) (in the terminology of Section 4.6) and (π, U) and (ρ, V) commute, then (π, U) and (ρ, V) can be interpreted as a pair of commuting non-degenerate continuous covariant representations of (A, G, α) and (A^o, G^o, α^o), respectively (where A^o and G^o are, respectively, the opposite Banach algebra and group of A and G, with $\alpha_r^o := \alpha_{r^{-1}}$ for all $r \in G$ as in Section 4.6). Theorem 4.7.5 explains, as a special case, how such a pair of commuting non-degenerate covariant representations (π, U) and (ρ, V) can be related to a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}_1}\hat{\otimes}(A^o \rtimes_{\alpha^o} G^o)^{\mathcal{R}_2}$, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are uniformly bounded classes of non-degenerate continuous covariant representations of (A, G, α) and (A^o, G^o, α^o) respectively, and (π, U) and (ρ, V) are respectively $\mathcal{R}_1$-continuous and $\mathcal{R}_2$-continuous.

We will now proceed to establish Theorem 4.7.5, and start with a rather obvious definition.

Definition 4.7.1. Let X be a Banach space and let $\varphi_i : S_i \to B(X)$ be maps from sets S_i into $B(X)$ for $i \in \{1,2\}$. Then φ_1 and φ_2 are said to *commute* if $\varphi_1(s_1)\varphi_2(s_2) = \varphi_2(s_2)\varphi_1(s_1)$ for all $s_1 \in S_1$ and $s_2 \in S_2$.

Let (A_1, G_1, α_1) and (A_2, G_2, α_2) be Banach algebra dynamical systems with (π_1, U_1) and (π_2, U_2) pairs of maps $\pi_1 : A_1 \to B(X)$, $U_1 : G_1 \to B(X)$ and $\pi_2 : A_2 \to B(X)$, $U_2 : G_2 \to B(X)$. Then the pairs (π_1, U_1) and (π_2, U_2) are said to *commute* if each of π_1 and U_1 commutes with both π_2 and U_2.

We then have the following:

Lemma 4.7.2. *Let X be a Banach space. For $i \in \{1,2\}$, let (A_i, G_i, α_i) be a Banach algebra dynamical system and let (π_i, U_i) be a non-degenerate continuous covariant representation of (A_i, G_i, α_i) on X. Then the following are equivalent:*

(1) (π_1, U_1) *and* (π_2, U_2) *commute.*

(2) $\pi_1 \rtimes U_1 : C_c(G_1, A_1) \to B(X)$ *and* $\pi_2 \rtimes U_2 : C_c(G_2, A_2) \to B(X)$ *commute.*

If, for $i \in \{1,2\}$, $\mathcal{R}_i$ is a non-empty class of continuous covariant representations of (A_i, G_i, α_i), such that (π_i, U_i) is $\mathcal{R}_i$-continuous, then (1) *and* (2) *are also equivalent to*

(3) $(\pi_1 \rtimes U_1)^{\mathcal{R}_1} : (A_1 \rtimes_{\alpha_1} G_1)^{\mathcal{R}_1} \to B(X)$ *and* $(\pi_2 \rtimes U_2)^{\mathcal{R}_2} : (A_2 \rtimes_{\alpha_2} G_2)^{\mathcal{R}_2} \to B(X)$ *commute.*

Proof. That (1) implies (2) can be seen through repeated application of [19, Proposition 5.5.iii]. We note that non-degeneracy is not required in this step.

That (2) implies (1) follows again by repeated applications of [19, Propositions 5.5.iii], and relies on the non-degeneracy of (π_i, U_i) for $i \in \{1,2\}$.

That (2) is equivalent to (3) follows from the density of $q^{\mathcal{R}_i}(C_c(G_i, A_i))$ in $(A_i \rtimes_{\alpha_i} G_i)^{\mathcal{R}_i}$ and the fact that $(\pi_i \rtimes U_i)^{\mathcal{R}_i}(q^{\mathcal{R}_i}(f)) = \pi_i \rtimes U_i(f)$ for all $f \in C_c(G_i, A_i)$, for $i \in \{1,2\}$. We again note that non-degeneracy is not required in this step. ☐

The next step is to investigate the bounded representations of the projective tensor product $B_1 \hat{\otimes} B_2$ of two Banach algebras B_1 and B_2 (which will later be taken to be crossed products). We refer to [26, Section 1.5] for the details concerning the (canonical) algebra structure on the underlying projective tensor product $B_1 \hat{\otimes} B_2$ of the Banach spaces B_1 and B_2, and start with a lemma.

Lemma 4.7.3. *Let B_1 and B_2 be Banach algebras with commuting bounded representations $\pi_1 : B_1 \to B(X)$ and $\pi_2 : B_2 \to B(X)$ on the same Banach space X. Then the map $\pi_1 \odot \pi_2 : B_1 \otimes B_2 \to B(X)$ given by*

$$\pi_1 \odot \pi_2 \left(\sum_{i=1}^{n} b_1^{(i)} \otimes b_2^{(i)} \right) := \sum_{i=1}^{n} \pi_1(b_1^{(i)}) \pi_2(b_2^{(i)}),$$

where $b_j^{(i)} \in B_j$ for $j \in \{1,2\}$ and $i \in \{1, \ldots, n\}$, is well defined and extends uniquely to a bounded representation $\pi_1 \hat{\odot} \pi_2 : B_1 \hat{\otimes} B_2 \to B(X)$.

Furthermore,

(1) $\|\pi_1 \hat{\odot} \pi_2\| \leq \|\pi_1\|\|\pi_2\|$,

(2) $\pi_1 \hat{\odot} \pi_2 : B_1 \hat{\otimes} B_2 \to B(X)$ *is non-degenerate if and only if* $\pi_1 : B_1 \to B(X)$ *and* $\pi_2 : B_2 \to B(X)$ *are non-degenerate.*

Proof. It is routine to verify that $\pi_1 \odot \pi_2$ is well defined and that $\|\pi_1 \odot \pi_2\| \leq \|\pi_1\|\|\pi_2\|$. The fact that π_1 and π_2 commute implies that $\pi_1 \odot \pi_2$ is a representation of $B_1 \otimes B_2$, and then the existence of $\pi_1 \hat{\odot} \pi_2$ as a bounded representation of $B_1 \hat{\otimes} B_2$ is clear, as is (1).

Since obviously $\overline{\mathrm{span}(\pi_1 \hat{\odot} \pi_2(B_1 \hat{\otimes} B_2)X)} \subseteq \overline{\mathrm{span}(\pi_i(B_i)X)}$ for $i \in \{1,2\}$, the non-degeneracy of $\pi_1 \hat{\odot} \pi_2$ implies the non-degeneracy of both π_1 and π_2.

Conversely, assume that both π_1 and π_2 are non-degenerate, and let $x \in X$ and $\varepsilon > 0$ be arbitrary. Choose $b_1^{(j)} \in B_1$ and $x^{(j)} \in X$ with $j \in \{1, \ldots, n\}$ such that $\left\| x - \sum_{j=1}^n \pi_1(b_1^{(j)}) x^{(j)} \right\| \leq \varepsilon/2$. Next, choose $b_2^{(j,k)} \in B_2$ and $x^{(j,k)} \in X$ with $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m_j\}$ such that $\|\pi_1(b_1^{(j)})\| \left\| x^{(j)} - \sum_{k=1}^{m_j} \pi_2(b_2^{(j,k)}) x^{(j,k)} \right\| \leq \varepsilon/2n$ for all $j \in \{1, \ldots, n\}$. Then

$$
\begin{aligned}
& \left\| x - \sum_{j=1}^n \sum_{k=1}^{m_j} \pi_1 \odot \pi_2(b_1^{(j)} \otimes b_2^{(j,k)}) x^{(j,k)} \right\| \\
\leq\ & \left\| x - \sum_{j=1}^n \pi_1(b_1^{(j)}) x^{(j)} \right\| + \left\| \sum_{j=1}^n \pi_1(b_1^{(j)}) x^{(j)} - \sum_{j=1}^n \sum_{k=1}^{m_j} \pi_1 \odot \pi_2(b_1^{(j)} \otimes b_2^{(j,k)}) x^{(j,k)} \right\| \\
\leq\ & \frac{\varepsilon}{2} + \sum_{j=1}^n \|\pi_1(b_1^{(j)})\| \left\| x^{(j)} - \sum_{k=1}^{m_j} \pi_2(b_2^{(j,k)}) x^{(j,k)} \right\| \\
<\ & \varepsilon.
\end{aligned}
$$

Hence $\pi_1 \hat{\odot} \pi_2$ is non-degenerate. □

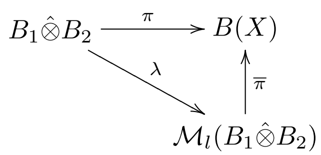
If both B_1 and B_2 have a bounded approximate left identity, then all non-degenerate bounded representations of $B_1 \hat{\otimes} B_2$ arise in this fashion for unique (necessarily non-degenerate, in view of Lemma 4.7.3) bounded π_1 and π_2. More precisely, we have the following result, for which we have not been able to find a reference.

Proposition 4.7.4. *Let* B_1 *and* B_2 *be Banach algebras both having a bounded approximate left identity, and let* X *be a Banach space. If* $\pi_1 : B_1 \to B(X)$ *and* $\pi_2 : B_2 \to B(X)$ *are commuting non-degenerate bounded representations, then* $\pi_1 \hat{\odot} \pi_2 : B_1 \hat{\otimes} B_2 \to B(X)$ *is a non-degenerate bounded representation, and all non-degenerate bounded representations of* $B_1 \hat{\otimes} B_2$ *are obtained in this fashion, for unique non-degenerate bounded representations* π_1 *and* π_2*. Then*

(1) $\|\pi_1 \hat{\odot} \pi_2\| \leq \|\pi_1\|\|\pi_2\|$

(2) If, for $i \in \{1,2\}$, B_i *has an* M_i*-bounded approximate left identity, then* $\|\pi_i\| \leq M_1 M_2 \|\lambda_{B_i}\| \|\pi_1 \hat{\odot} \pi_2\|$, *with* $\lambda_{B_i} : B_i \to B(B_i)$ *denoting the left regular representation of* B_i.

Proof. Part of the proposition, including (1), has already been established in Lemma 4.7.3. We start from a given non-degenerate bounded representation $\pi : B_1\hat{\otimes}B_2 \to B(X)$ and construct the non-degenerate bounded representations π_1 and π_2 such that $\pi = \pi_1\hat{\odot}\pi_2$. First, we note that $B_1\hat{\otimes}B_2$ has an approximate left identity bounded by M_1M_2 [26, Lemma 1.5.3]. Therefore, if we let $\overline{\pi} : \mathcal{M}_l(B_1\hat{\otimes}B_2) \to B(X)$ denote the non-degenerate bounded representations of $\mathcal{M}_l(B_1\hat{\otimes}B_2)$ such that the diagram



commutes, then $\|\overline{\pi}\| \leq M_1M_2\|\pi\|$ [18, Theorem 4.1]. We will now compose $\overline{\pi}$ with bounded homomorphisms of B_1 and B_2 into $\mathcal{M}_l(B_1\hat{\otimes}B_2)$ to obtain the sought representations π_1 and π_2. For $b_1 \in B_1$ consider $\lambda_{B_1}(b_1)\hat{\otimes}\mathrm{id}_{B_2} \in B(B_1\hat{\otimes}B_2)$, where $\lambda_{B_1}(b_1)$ is the image under the left regular representation $\lambda_{B_1} : B_1 \to B(B_1)$ of B_1. Clearly, $\|\lambda_{B_1}(b_1)\hat{\otimes}\mathrm{id}_{B_2}\| = \|\lambda_{B_1}(b_1)\| \leq \|\lambda_{B_1}\|\|b_1\|$, and one readily verifies that $\lambda_{B_1}(b_1)\hat{\otimes}\mathrm{id}_{B_2} \in \mathcal{M}_l(B_1\hat{\otimes}B_2)$. If we define $l_1 : B_1 \to \mathcal{M}_l(B_1\hat{\otimes}B_2)$ by $l_1(b_1) := \lambda_{B_1}(b_1)\hat{\otimes}\mathrm{id}_{B_2}$ for $b_1 \in B_1$, then l_1 is a bounded homomorphism, and $\|l_1\| \leq \|\lambda_{B_1}\|$. Likewise, $l_2 : B_2 \to \mathcal{M}_l(B_1\hat{\otimes}B_2)$, defined by $l_2(b_2) := \mathrm{id}_{B_1}\hat{\otimes}\lambda_{B_2}(b_2)$ for $b_2 \in B_2$, is a bounded homomorphism, and $\|l_2\| \leq \|\lambda_{B_2}\|$. Now, for $i \in \{1, 2\}$, define $\pi_i : B_i \to B(X)$ as $\pi_i := \overline{\pi} \circ l_i$. We note that $\|\pi_i\| \leq \|\overline{\pi}\|\|l_i\| \leq M_1M_2\|\lambda_{B_i}\|\|\pi\|$. Since l_1 and l_2 obviously commute, the same holds true for π_1 and π_2. Therefore $\pi_1\hat{\odot}\pi_2 : B_1\hat{\otimes}B_2 \to B(X)$ is a bounded representation.

We will proceed to show that $\pi_1\hat{\odot}\pi_2 = \pi$, and that π_1 and π_2 are uniquely determined. We compute, for $x \in X$, $b_1^{(1)}, b_1^{(2)} \in B_1$ and $b_2^{(1)}, b_2^{(2)} \in B_2$:

$$
\begin{aligned}
& \pi_1\hat{\odot}\pi_2(b_1^{(1)} \otimes b_2^{(1)})\pi(b_1^{(2)} \otimes b_2^{(2)})x \\
= \; & \pi_1(b_1^{(1)})\pi_2(b_2^{(1)})\pi(b_1^{(2)} \otimes b_2^{(2)})x \\
= \; & \overline{\pi}(\lambda_{B_1}(b_1^{(1)})\hat{\otimes}\mathrm{id}_{B_2})\overline{\pi}(\mathrm{id}_{B_1}\hat{\otimes}\lambda_{B_2}(b_2^{(1)}))\pi(b_1^{(2)} \otimes b_2^{(2)})x \\
= \; & \overline{\pi}(\lambda_{B_1}(b_1^{(1)})\hat{\otimes}\mathrm{id}_{B_2})\pi(\mathrm{id}_{B_1}\hat{\otimes}\lambda_{B_2}(b_2^{(1)})(b_1^{(2)} \otimes b_2^{(2)}))x \\
= \; & \pi(\lambda_{B_1}(b_1^{(1)})\hat{\otimes}\mathrm{id}_{B_2}(b_1^{(2)} \otimes b_2^{(1)}b_2^{(2)}))x \\
= \; & \pi(b_1^{(1)}b_1^{(2)} \otimes b_2^{(1)}b_2^{(2)})x \\
= \; & \pi(b_1^{(1)} \otimes b_2^{(1)})\pi(b_1^{(2)} \otimes b_2^{(2)})x.
\end{aligned}
$$

Since π is non-degenerate and $B_1 \otimes B_2$ is dense in $B_1\hat{\otimes}B_2$, the restriction of π to $B_1 \otimes B_2$ is also non-degenerate. Hence we conclude from the above that $\pi_1\hat{\odot}\pi_2(b_1 \otimes b_2) = \pi(b_1 \otimes b_2)$ for all $b_1 \in B_1$ and $b_2 \in B_2$, i.e., that $\pi_1\hat{\odot}\pi_2 = \pi$. It is now clear that $\|\pi_i\| \leq M_1M_2\|\lambda_{B_i}\|\|\pi_1\hat{\odot}\pi_2\|$. As already mentioned preceding the proposition, π_1 and π_2 are necessarily non-degenerate.

As to uniqueness, assume that $\rho_1 : B_1 \to B(X)$ and $\rho_2 : B_2 \to B(X)$ are commuting bounded representations such that $\rho_1\hat{\odot}\rho_2 = \pi$. Then, for $x \in X$, $b_1, b_1' \in$

B and $b_2' \in B_2$,

$$\begin{aligned}
\rho_1(b_1)\pi(b_1' \otimes b_2')x &= \rho_1(b_1)\rho_1 \hat{\odot} \rho_2(b_1' \otimes b_2')x \\
&= \rho_1(b_1)\rho_1(b_1')\rho_2(b_2')x \\
&= \rho_1(b_1 b_1')\rho_2(b_2')x \\
&= \rho_1 \hat{\odot} \rho_2(b_1 b_1' \otimes b_2')x \\
&= \pi(\lambda_{B_1}(b_1)\hat{\otimes}\mathrm{id}_{B_2}(b_1' \otimes b_2'))x \\
&= \overline{\pi}(\lambda_{B_1}(b_1)\hat{\otimes}\mathrm{id}_{B_2})\pi(b_1' \otimes b_2')x \\
&= \pi_1(b_1)\pi(b_1' \otimes b_2')x.
\end{aligned}$$

The non-degeneracy of π then implies that necessarily $\rho_1 = \pi_1$ and likewise that $\rho_2 = \pi_2$. □

The following is now simply a matter of combining the General Correspondence Theorem (Theorem 4.2.1), Lemma 4.7.3, Proposition 4.7.4, and an induction argument.

Theorem 4.7.5. *For $i \in \{1, \ldots, n\}$, let (A_i, G_i, α_i) be a Banach algebra dynamical system, where A_i has a bounded approximate left identity, and $\mathcal{R}_i$ is a non-empty uniformly bounded class of non-degenerate continuous covariant representations of (A_i, G_i, α_i). Let X be a Banach space. Let $((\pi_1, U_1), \ldots, (\pi_n, U_n))$ be an n-tuple where, for each $i \in \{1, \ldots, n\}$, the pair (π_i, U_i) is a non-degenerate $\mathcal{R}_i$-continuous covariant representation of (A_i, G_i, α_i) on X, and all (π_i, U_i) and (π_j, U_j) commute for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then the map sending $((\pi_1, U_1), \ldots, (\pi_n, U_n))$ to the representation*

$$\widehat{\bigodot}_{i=1}^{n} (\pi_i \rtimes U_i)^{\mathcal{R}_i} : \widehat{\bigotimes}_{i=1}^{n} (A_i \rtimes_{\alpha_i} G_i)^{\mathcal{R}_i} \to B(X),$$

is a bijection between the set of all such n-tuples and the set of all non-degenerate bounded representations of $\widehat{\bigotimes}_{i=1}^{n} (A_i \rtimes_{\alpha_i} G_i)^{\mathcal{R}_i}$ on X.

For the sake of completeness, we mention that the commutativity assumption applies only to the non-degenerate $\mathcal{R}_i$-continuous covariant representations (π_i, U_i), not to the elements of $\mathcal{R}_i$.

In Remark 4.8.5 we will apply Theorem 4.7.5 to relate bimodules over generalized Beurling algebras to left modules over a projective tensor product of the algebra acting on the left and the opposite algebra of the one acting on the right.

4.8 Right and bimodules over generalized Beurling algebras

Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded two-sided approximate identity and α is uniformly bounded, and let ω be a weight on G. In Section 4.5 we have seen that the Banach space $L^1(G, A, \omega)$ has the structure of an associative algebra, denoted $L^1(G, A, \omega; \alpha)$, with multiplication continuous in both variables, determined by

$$[f *_\alpha g](s) := \int_G f(r)\alpha_r(g(r^{-1}s))\, d\mu(r) \quad (f, g \in C_c(G, A),\ s \in G).$$

Here we have written $*_\alpha$ rather than $*$ to indicate the α-dependence of the multiplication (twisted convolution) on $C_c(G, A)$, as another multiplication will also appear. For the same reason we have now also written $d\mu$ for the chosen left Haar measure on G. Furthermore, we have seen in Section 4.5 that $L^1(G, A, \omega; \alpha)$ is isomorphic to the Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$, when $\mathcal{R}$ is chosen suitably. As a consequence of the General Correspondence Theorem (Theorem 4.2.1), it was then shown that if (π, U) is a non-degenerate continuous covariant representation of (A, G, α), such that $\|U_r\| \leq C_U\omega(r)$ for all $r \in G$, then $\pi \rtimes U(f) = \int \pi(f)U_r\, d\mu(r)$, for $f \in C_c(G, A)$, determines a non-degenerate bounded representation of $L^1(G, A, \omega; \alpha)$, and that all non-degenerate bounded representations of $L^1(G, A, \omega; \alpha)$ are uniquely determined in this way by such pairs (π, U).

In the current section we will explain how the non-degenerate bounded anti-representations of $L^1(G, A, \omega; \alpha)$ (i.e., non-degenerate right $L^1(G, A, \omega; \alpha)$-modules) are in natural bijection with the pairs (π, U), where $\pi : A \to B(X)$ is non-degenerate, bounded and anti-multiplicative, $U : G \to B(X)$ is strongly continuous and anti-multiplicative, satisfy

$$U_r\pi(a)U_r^{-1} = \pi(\alpha_{r^{-1}}(a)) \quad (a \in A,\ r \in G),$$

(i.e., with the non-degenerate continuous pairs (π, U) of type (a, a) as in Section 4.6, called thrice "flawed" in the introduction) and are such that $\|U_r\| \leq C_U\omega(r)$, for some $C_U \geq 0$ and all $r \in G$. This may look counterintuitive to the idea of Section 4.6, where it was argued that one can "always" reinterpret given data so as to end up with pairs of type (m, m) for a (companion) Banach algebra dynamical system, and then formulate a General Correspondence Theorem involving the non-degenerate bounded representations of a companion crossed product: anti-representations of the resulting crossed product never enter the picture. Yet this is precisely what we will do, but it is only the first step.

In this first step the relevant crossed product will, as in Section 4.5, turn out to be topologically isomorphic to $L^1(G^o, A^o, \omega^o; \alpha^o)$ (where ω^o equals ω, seen as a weight on G^o). As it happens, $L^1(G^o, A^o, \omega^o; \alpha^o)$ is topologically anti-isomorphic to $L^1(G, A, \omega; \alpha)$. Hence, in the second step, the non-degenerate bounded representations of $L^1(G^o, A^o, \omega^o; \alpha^o)$ are viewed as the non-degenerate bounded anti-representations of $L^1(G, A, \omega; \alpha)$, which are thus, in the end, related to pairs (π, U) of type (a, a) as above. For this result, therefore, one should not think of $L^1(G, A, \omega; \alpha)$ as being topologically isomorphic to a crossed product as in Section 4.5. Although this is also the case, its main feature here is that it is anti-isomorphic to the algebra $L^1(G^o, A^o, \omega^o; \alpha^o)$ which, in turn, is topologically isomorphic to the crossed product that "actually" explains the situation.

Once this has been completed, we remind ourselves again that $L^1(G, A, \omega; \alpha)$ itself is topologically isomorphic to a crossed product, and combine the results in the first part of this section with those in Sections 4.5 and 4.7 in Theorem 4.8.4, to describe for two Banach algebra dynamical systems (A, G, α) and (B, H, β) the non-degenerate simultaneously left $L^1(G, A, \omega; \alpha)$– and right $L^1(H, B, \eta; \beta)$-modules, and, in the special case where $(A, G, \alpha) = (B, H, \beta)$, the non-degenerate $L^1(G, A, \omega; \alpha)$-bimodules.

To start, recall that the canonical left invariant measure μ on the opposite group G^o of G is given by $\mu^o(E) := \mu(E^{-1})$, for E a Borel subset of G. Then, recalling that $\int_G f\, d\mu = \int_G f(r^{-1})\Delta(r^{-1})\, d\mu(r)$ [46, Lemma 1.67], for $f \in C_c(G)$, we have

$$\int_{G^o} f(r)\, d\mu^o(r) = \int_G f(r^{-1})\, d\mu(r) = \int_G f(r)\Delta(r^{-1})\, d\mu(r).$$

We recall from Section 4.6 if (A, G, α) is a Banach algebra dynamical system, then so is (A^o, G^o, α^o), where A^o is the opposite algebra of A, G^o is the opposite group of G, and $\alpha^o : G^o \to \mathrm{Aut}(A^o) = \mathrm{Aut}(A)$ is given by $\alpha^o_s = \alpha_{s^{-1}}$ for all $s \in G^o$. The vector spaces $C_c(G, A)$ and $C_c(G^o, A^o)$ can be identified, but there are two convolution structures on it. If $\circledcirc$ denotes the multiplication in A^o and G^o, then

$$[f *_\alpha g](s) = \int_G f(r)\alpha_r(g(r^{-1}s))\, d\mu(r) \quad (f, g \in C_c(G, A),\ s \in G),$$

and

$$[f *_{\alpha^o} g](s) = \int_G f(r) \circledcirc \alpha^o_r(g(r^{-1} \circledcirc s))\, d\mu^o(r) \quad (f, g \in C_c(G^o, A^o),\ s \in G^o).$$

Hence we have two associative algebras: $C_c(G, A)$ with multiplication $*_\alpha$, and $C_c(G^o, A^o)$ with multiplication $*_{\alpha^o}$, having the same underlying vector space. The first observation we need is then the following:

Lemma 4.8.1. *Let (A, G, α) be a Banach algebra dynamical system with companion opposite system (A^o, G^o, α^o), and let $\chi : G \to \mathbb{C}^\times$ be a continuous character of G. For $f \in C_c(G, A)$, define $\hat{f} \in C_c(G^o, A^o)$ by $\hat{f}(s) := \chi(s^{-1})\alpha_{s^{-1}}(f(s))$ for $s \in G^o$. Then the map $f \mapsto \hat{f}$ is an anti-isomorphism of the associative algebras $C_c(G, A)$ with multiplication $*_\alpha$, and $C_c(G^o, A^o)$ with multiplication $*_{\alpha^o}$. The inverse is given by $g \mapsto \check{g}$, where $\check{g}(s) := \chi(s)\alpha_s(g(s))$ for $g \in C_c(G^o, A^o)$ and $s \in G$.*

Proof. It is clear that $\hat{\cdot}$ and $\check{\cdot}$ are mutually inverse linear bijections. As to the multiplicative structures, we compute, for $f, g \in C_c(G, A)$ and $s \in G^o$,

$$\begin{aligned}
[\hat{f} *_{\alpha^o} \hat{g}](s) &= \int_{G^o} \hat{f}(r) \circledcirc \alpha^o_{r^{-1}}(\hat{g}(r^{-1} \circledcirc s))\, d\mu^o(r) \\
&= \int_G \hat{f}(r^{-1}) \circledcirc \alpha^o_{r^{-1}}(\hat{g}(r \circledcirc s))\, d\mu(r) \\
&= \int_G \alpha_r(\hat{g}(sr))\hat{f}(r^{-1})\, d\mu(r) \\
&= \int_G \alpha_r(\chi((sr)^{-1})\alpha_{(sr)^{-1}}g(sr))\chi(r)\alpha_r(f(r^{-1}))\, d\mu(r)
\end{aligned}$$

$$
\begin{aligned}
&= \chi(s^{-1}) \int_G \alpha_{s^{-1}}(g(sr))\alpha_r(f(r^{-1}))\, d\mu(r) \\
&= \chi(s^{-1})\alpha_{s^{-1}} \left(\int_G g(sr))\alpha_{sr}(f(r^{-1}))\, d\mu(r) \right) \\
&= \chi(s^{-1})\alpha_{s^{-1}} \left(\int_G g(r))\alpha_r(f(r^{-1}s))\, d\mu(r) \right) \\
&= (g *_\alpha f)^\wedge(s).
\end{aligned}
$$

□

Choosing χ suitably, we obtain a topological isomorphism in the next result.

Proposition 4.8.2. *Let (A, G, α) be a Banach algebra dynamical system, where α is uniformly bounded. Let ω be a weight on G and view $\omega^o := \omega$ also as a weight on G^o. Then the map $f \mapsto \hat{f}$, where $\hat{f}(s) := \Delta(s)\alpha_{s^{-1}}(f(s))$ for $f \in C_c(G, A)$ and $s \in G^o$ defines a topological anti-isomorphism between $L^1(G, A, \omega; \alpha)$ and $L^1(G^o, A^o, \omega^o; \alpha^o)$. The inverse map is determined by $g \mapsto \check{g}$ where $\check{g}(s) := \Delta(s^{-1})\alpha_s(g(s))$ for $g \in C_c(G^o, A^o)$ and $s \in G$.*

Proof. In view of Lemma 4.8.1, we need only show that $\hat{}$ and $\check{}$ are isomorphisms between the normed spaces $(C_c(G, A), \| \cdot \|_{1,\omega})$ and $(C_c(G^o, A^o), \| \cdot \|_{1,\omega^o})$. Let α be uniformly bounded by C_α. If $f \in C_c(G, A)$, then

$$
\begin{aligned}
\|\hat{f}\|_{1,\omega^o} &= \int_{G^o} \|\hat{f}(r)\|\omega^o(r)\, d\mu^o(r) \\
&= \int_{G^o} \|\Delta(r)\alpha_{r^{-1}}(f(r))\|\omega(r)\, d\mu^o(r) \\
&\leq C_\alpha \int_{G^o} \|f(r)\|\omega(r)\Delta(r)\, d\mu^o(r) \\
&= C_\alpha \int_G \|f(r^{-1})\|\omega(r^{-1})\Delta(r^{-1})\, d\mu(r) \\
&= C_\alpha \int_G \|f(r)\|\omega(r)\, d\mu(r) \\
&= C_\alpha \|f\|_{1,\omega}.
\end{aligned}
$$

Similarly $\|\check{f}\|_{1,\omega} \leq C_\alpha \|f\|_{1,\omega^o}$ for all $f \in C_c(G^o, A^o)$. □

It is now an easy matter to combine the ideas of Sections 4.5 and 4.6 with the above Proposition 4.8.2.

Let X be a Banach space and let (A, G, α) be a Banach algebra dynamical system, where A has a bounded two-sided approximate identity and α is uniformly bounded. As in Section 4.6, the pairs (π, U), where $\pi : A \to B(X)$ is non-degenerate, bounded and anti-multiplicative, $U : G \to B(X)$ is strongly continuous and anti-multiplicative, and $U_r^{-1}\pi(a)U_r = \pi(\alpha_{r^{-1}}(a))$ for $a \in A$ and $r \in G$, can be identified with the pairs (π^o, U^o), where $\pi^o : A^o \to B(X)$, with $\pi^o(a) := \pi(a)$ for $a \in A$, is non-degenerate, bounded and multiplicative, $U^o : G^o \to B(X)$, with $U^o_r = U_r$ for all $r \in G^o$, is strongly continuous and multiplicative, and $U^o_r \pi^o(a) U^{o-1}_r = \pi^o(\alpha^o_r(a))$ for $a \in A^o$ and $r \in G^o$. Furthermore, if ω is a weight on G, also viewed as a weight $\omega^o := \omega$ on G^o, then there exists a constant C_U such that $\|U_r\| \leq C_U \omega(r)$ for all $r \in G$ if and only if there exists a constant C_{U^o} such that $\|U^o_r\| \leq C_{U^o} \omega^o(r)$ for all $r \in G^o$: take the same constant. Now the collection of all such pairs (π^o, U^o) is, in view of Theorem 4.5.20, in natural bijection with the collection of all non-degenerate bounded representations of $L^1(G^o, A^o, \omega^o; \alpha^o)$ on X. As a consequence of Proposition 4.8.2, this can in turn be viewed as the collection of all non-degenerate bounded anti-representations of $L^1(G, A, \omega; \alpha)$ on X. Combining these three bijections, we can let pairs (π, U) as described above correspond bijectively to the non-degenerate bounded anti-representations of $L^1(G, A, \omega; \alpha)$ on X: If (π, U) is such a pair, we associate with it the non-degenerate bounded anti-representation of $L^1(G, A, \omega; \alpha)$ determined by sending $f \in C_c(G, A)$ to $\pi^o \rtimes U^o(\hat{f})$. Explicitly, for $f \in C_c(G, A)$,

$$
\begin{aligned}
\pi^o \rtimes U^o(\hat{f}) &= \int_{G^o} \pi^o(\hat{f}(r)) U^o_r \, d\mu^o(r) \\
&= \int_{G^o} \pi(\Delta(r) \alpha_{r^{-1}}(f(r))) U_r \, d\mu^o(r) \\
&= \int_G \pi(\alpha_r(f(r^{-1}))) U_{r^{-1}} \Delta(r^{-1}) \, d\mu(r) \\
&= \int_G \pi(\alpha_{r^{-1}}(f(r))) U_r \, d\mu(r) \\
&= \int_G U_r U_r^{-1} \pi(\alpha_{r^{-1}}(f(r))) U_r \, d\mu(r) \\
&= \int_G U_r \pi(\alpha_r \circ \alpha_{r^{-1}}(f(r))) \, d\mu(r) \\
&= \int_G U_r \pi(f(r)) \, d\mu(r).
\end{aligned}
$$

To retrieve the pair (π, U) from a non-degenerate bounded anti-representation T of $L^1(G, A, \omega; \alpha)$, we note that, by Proposition 4.8.2, $T \circ \check{\,}$ is a non-degenerate bounded representation of $L^1(G^o, A^o, \omega^o; \alpha^o)$, and hence, we can apply [19, Equations (8.1) and (8.2)] to $T \circ \check{\,}$. A bounded approximate left identity of A^o is then needed, and for this we take a bounded approximate right identity (u_i) of A. Furthermore, if V runs through a neighbourhood base $\mathcal{Z}$ of $e \in G$, of which all elements are contained in a fixed compact set of G, and $z_V \in C_c(G)$ is positive, supported in V, and $\int_G z_V(r^{-1}) \, d\mu(r) = \int_{G^o} z_V(r) \, d\mu^o(r) = 1$, then the $z_V \in C_c(G)$ are as required for [19, Equations (8.1) and (8.2)]. Hence, again taking Remark 4.5.21 into account, we have, for $a \in A$,

$$
\begin{aligned}
\pi(a) = \pi^o(a) &= \text{SOT-lim}_{(V,i)} T((z_V \otimes a \circledcirc u_i)^\vee) \\
&= \text{SOT-lim}_{(V,i)} T((z_V \otimes u_i a)^\vee),
\end{aligned}
$$

where $(z_V \otimes u_i a)^\vee(r) = \Delta(r^{-1}) z_V(r)\alpha_r(au_i)$ for $r \in G$, and, for $s \in G$,

$$\begin{aligned} U_s = U_s^o &= \text{SOT-lim}_{(V,i)} T((z_V(s^{-1} \circledcirc \cdot) \otimes u_i)^\vee) \\ &= \text{SOT-lim}_{(V,i)} T((z_V(\cdot s^{-1}) \otimes u_i)^\vee), \end{aligned}$$

where $(z_V(\cdot s^{-1}) \otimes u_i)^\vee(r) = \Delta(r^{-1}) z_V(rs^{-1})\alpha_r(u_i)$ for $r \in G$.

All in all, we have the following result in analogy to Theorem 4.5.20:

Theorem 4.8.3. *Let (A, G, α) be a Banach algebra dynamical system where A has a two-sided approximate identity and α is uniformly bounded by a constant C_α, and let ω be a weight on G. Let X be a Banach space. Let the pair (π, U) be such that $\pi : A \to B(X)$ is a non-degenerate bounded anti-representation, $U : G \to B(X)$ is a strongly continuous anti-representation satisfying $U_r\pi(\alpha)U_r^{-1} = \pi(\alpha_{r^{-1}}(a))$ for all $a \in A$ and $r \in G$, and with C_U a constant such that $\|U_r\| \leq C_U\omega(r)$ for all $r \in G$. Let $T : L^1(G, A, \omega; \alpha) \to B(X)$ be a non-degenerate bounded anti-representation of $L^1(G, A, \omega; \alpha)$ on X. Then the following maps are mutual inverses between all such pairs (π, U) and the non-degenerate bounded anti-representations T of $L^1(G, A, \omega; \alpha)$:*

$$(\pi, U) \mapsto \left(f \mapsto \int_G U_r\pi(f(r))\, dr \right) =: T^{(\pi,U)} \quad (f \in C_c(G, A)),$$

determining a non-degenerate bounded anti-representation $T^{(\pi,U)}$ of the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$, and,

$$T \mapsto \left(\begin{array}{l} a \mapsto \text{SOT-lim}_{(V,i)} T((z_V \otimes u_i a)^\vee), \\ s \mapsto \text{SOT-lim}_{(V,i)} T((z_V(\cdot s^{-1}) \otimes u_i)^\vee) \end{array} \right) =: (\pi^T, U^T),$$

where $\mathcal{Z}$ is a neighbourhood base of $e \in G$, of which all elements are contained in a fixed compact subset of G, $z_V \in C_c(G)$ is chosen such that $z_V \geq 0$, supported in $V \in \mathcal{Z}$, $\int_G z_V(r^{-1})\, dr = 1$, and (u_i) is any bounded approximate right identity of A.

Furthermore, if A has an M-bounded approximate right identity, then the following bounds for $T^{(\pi,U)}$ and (π^T, U^T) hold:

(1) $\|T^{(\pi,U)}\| \leq C_U\|\pi\|$,

(2) $\|\pi^T\| \leq (\inf_{V\in\mathcal{Z}} \sup_{r\in V} \omega(r))\, \|T\|$,

(3) $\|U_s^T\| \leq M\, (\inf_{V\in\mathcal{Z}} \sup_{r\in V} \omega(r))\, \|T\|\omega(s) \quad (s \in G)$.

Proof. Except for the bounds, all statements were proven in the discussion preceding the statement of the theorem. Establishing the bound (1) proceeds as in Theorem 4.5.20.

To establish (2), we choose a bounded two-sided approximate identity (u_i) of A. Let $a \in A$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ be arbitrary. There exists an index i_0 such that $\|u_i a\| \leq \|a\| + \varepsilon_1$ for all $i \geq i_0$. There exists some $W_1 \in \mathcal{Z}$ such that $\sup_{r\in W_1} \omega(r) \leq \inf_{V\in\mathcal{Z}} \sup_{r\in V} \omega(r) + \varepsilon_2$. Since $r \mapsto \|\alpha_r\|$ is lower semicontinuous and $\|\alpha_e\| = 1$, there exists some $W_2 \in \mathcal{Z}$ such that $\|\alpha_r\| \le 1+\varepsilon_3$ for all $r \in W_2$. Let $V_0 \in \mathcal{Z}$ be such that $V_0 \subseteq W_1 \cap W_2$. If $(V,i) \ge (V_0, i_0)$, then $V \subseteq V_0$ and $i \ge i_0$, hence

$$\begin{aligned}
\|T((z_V \otimes u_i a)^\vee)\| &\le \|T\|\,\|(z_V \otimes u_i a)^\vee\|_{1,\omega} \\
&= \|T\| \int_G \|(z_V \otimes u_i a)^\vee(r)\|\omega(r)\,dr \\
&= \|T\| \int_G \Delta(r^{-1}) z_V(r)\|\alpha_r(a u_i)\|\omega(r)\,dr \\
&\le \|T\|\|a u_i\|(1+\varepsilon_3)\left(\sup_{r\in V}\omega(r)\right)\int_G \Delta(r^{-1}) z_V(r)\,dr \\
&\le \|T\|(\|a\|+\varepsilon_1)(1+\varepsilon_3)\left(\sup_{r\in V_0}\omega(r)\right)\int_G z_V(r^{-1})\,dr \\
&\le \|T\|(\|a\|+\varepsilon_1)(1+\varepsilon_3)\left(\inf_{V\in\mathcal{Z}}\sup_{r\in V}\omega(r)+\varepsilon_2\right).
\end{aligned}$$

From this, the bound in (2) now follows as in the proof of Theorem 4.5.20.

As to (3), we fix $s \in G$. The operator $U_s^T = \text{SOT-lim}_{(V,i)} T((z_V(\cdot s^{-1}) \otimes u_i)^\vee)$ does not depend on the particular choice of the bounded approximate right identity (u_i) (see Remark 4.5.21). If (u_i) is an M-bounded approximate right identity of A, then $(\alpha_{s^{-1}}(u_i))$ is also a bounded approximate right identity of A, and hence $U_s^T = \text{SOT-lim}_{(V,i)} T((z_V(\cdot s^{-1}) \otimes \alpha_{s^{-1}}(u_i))^\vee)$. Let $\varepsilon_1, \varepsilon_2 > 0$ be arbitrary. Choose $W_1 \in \mathcal{Z}$ such that $\|\alpha_r\| \le 1+\varepsilon_1$ for all $r \in W_1$, and $W_2 \in \mathcal{Z}$ such that $\sup_{r\in W_2}\omega(r) \le \inf_{V\in\mathcal{Z}}\sup_{r\in V}\omega(r)+\varepsilon_2$. Let $V_0 \in \mathcal{Z}$ be such that $V_0 \subseteq W_1 \cap W_2$. If $(V,i) \ge (V_0,i_0)$, then $V \subseteq V_0$ and $i \ge i_0$, hence

$$\begin{aligned}
& \|T((z_V(\cdot s^{-1}) \otimes \alpha_{s^{-1}}(u_i))^\vee)\| \\
\le\ & \|T\|\,\left\|(z_V(\cdot s^{-1}) \otimes \alpha_{s^{-1}}(u_i))^\vee\right\|_{1,\omega} \\
=\ & \|T\| \int_G \|(z_V \otimes \alpha_{s^{-1}}(u_i))^\vee(r)\|\omega(r)\,dr \\
=\ & \|T\| \int_G \Delta(r^{-1}) z_V(rs^{-1})\|\alpha_{rs^{-1}}(u_i)\|\omega(r)\,dr \\
=\ & \|T\| \int_G z_V(r^{-1}s^{-1})\|\alpha_{r^{-1}s^{-1}}(u_i)\|\omega(r^{-1})\,dr \\
=\ & \|T\| \int_G z_V(r^{-1})\|\alpha_{r^{-1}}(u_i)\|\omega(r^{-1}s)\,dr \\
\le\ & \|T\| \int_G z_V(r^{-1})\|\alpha_{r^{-1}}(u_i)\|\omega(r^{-1})\omega(s)\,dr \\
\le\ & \|T\| \int_G z_V(r^{-1})(1+\varepsilon_1)\|u_i\|\left(\sup_{r\in V^{-1}}\omega(r^{-1})\right)\omega(s)\,dr \\
\le\ & \|T\|(1+\varepsilon_1)M\left(\sup_{r\in V^{-1}}\omega(r^{-1})\right)\omega(s)\int_G z_V(r^{-1})\,dr
\end{aligned}$$

$$\leq \|T\|(1+\varepsilon_1)M\left(\sup_{r\in V_0}\omega(r)\right)\omega(s)$$

$$\leq \|T\|(1+\varepsilon_1)M\left(\inf_{V\in\mathcal{Z}}\sup_{r\in V}\omega(r)+\varepsilon_2\right)\omega(s).$$

Once again, the bound in (3) now follows as in the proof of Theorem 4.5.20. □

We will now describe the non-degenerate bimodules over generalized Beurling algebras as a special case of a more general result. Let (A, G, α) and (B, H, β) be Banach algebra dynamical systems, where A and B have bounded two-sided approximate identities, and both α and β are uniformly bounded. Let ω be a weight on G, and η a weight on H. Remembering that $L^1(G, A, \omega; \alpha)$ and $L^1(H, B, \eta; \beta)$ are themselves also (isomorphic to) a crossed product of a Banach algebra dynamical system, Theorem 4.5.13, it is now easy to describe the non-degenerate simultaneously left $L^1(G, A, \omega; \alpha)$– and right $L^1(H, B, \eta; \beta)$-modules, as follows: Let X be a Banach space. Suppose that $T^m : L^1(G, A, \omega; \alpha) \to B(X)$ is a non-degenerate bounded representation of $L^1(G, A, \omega; \alpha)$ on X, and $T^a : L^1(H, B, \eta; \beta) \to B(X)$ is a non-degenerate bounded anti-representation, such that T^m and T^a commute. We know from Theorem 4.5.20 and Theorem 4.8.3 that T^m and T^a correspond to pairs (π^m, U^m) and (π^a, U^a), respectively, each with the appropriate properties. But then (π^m, U^m) and (π^a, U^a) must also commute in the sense of Definition 4.7.1. Indeed, (π^a, U^a) corresponds to T^a as being the pair such that the integrated form of $(\pi^{a,o}, U^{a,o})$ gives rise to the non-degenerate bounded representation T^a of $L^1(H^o, B^o, \eta^o; \beta^o)$ on X. But since $L^1(H^o, B^o, \eta^o; \beta^o)$ is (isomorphic to) a crossed product, and likewise for $L^1(G, A, \omega; \alpha)$, the fact that (π^m, U^m) and $(\pi^{a,o}, U^{a,o})$ commute then follows from Lemma 4.7.2 and the fact that T^m and T^a commute. Since $\pi^{a,o} = \pi^a$ and $U^{a,o} = U^a$ as set-theoretic maps, (π^m, U^m) and (π^a, U^a) also commute. The same kind of arguments show that the converse is equally true.

Combining these results, we obtain the following following description of the non-degenerate simultaneously left $L^1(G, A, \omega; \alpha)$– and right $L^1(H, B, \eta; \beta)$-modules. If $(A, G, \alpha) = (B, G, \beta)$ and $\omega = \eta$ it describes the non-degenerate $L^1(G, A, \omega; \alpha)$-bimodules.

Theorem 4.8.4. *Let (A, G, α) and (B, H, β) be a Banach algebra dynamical systems, where A and B have bounded two-sided approximate identities, and both α and β are uniformly bounded. Let ω be a weight on G, and η a weight on H. Let X be a Banach space.*

Suppose that (π^m, U^m) is a non-degenerate continuous covariant representation of (A, G, α) on X such that $\|U^m_r\| \leq C_{U^m}\omega(r)$ for some constant C_{U^m} and all $r \in G$. Suppose that the pair (π^a, U^a) is such that $\pi^a : B \to B(X)$ is a non-degenerate bounded anti-representation, that $U^a : H \to B(X)$ is a strongly continuous anti-representation, such that $U^a_s \pi^a(b) U^{a-1}_s = \pi^a(\alpha_{s^{-1}}(b))$ for all $b \in B$ and $s \in H$, and $\|U^a_s\| \leq C_{U^a}\eta(s)$ for some constant C_{U^a} and all $s \in H$. Furthermore, let (π^m, U^m) and (π^a, U^a) commute.

Then the map

$$T^m(f) := \int_G \pi^m(f(r))U_r^m \, d\mu_G(r) \quad (f \in C_c(G,A))$$

determines a non-degenerate bounded representation of $L^1(G,A,\omega;\alpha)$ *on* X*, and the map*

$$T^a(g) := \int_H U_s^a \pi^a(g(s)) \, d\mu_H(s) \quad (g \in C_c(H,B))$$

determines a non-degenerate bounded anti-representation of $L^1(H,B,\eta;\beta)$ *on* X*. Moreover,* $T^m : L^1(G,A,\omega;\alpha) \to B(X)$ *and* $T^a : L^1(H,B,\eta;\beta) \to B(X)$ *commute.*

All pairs (T^m, T^a)*, where* T^m *and* T^a *commute, are non-degenerate, bounded,* T^m *is a representation of* $L^1(G,A,\omega;\alpha)$ *on* X*, and* T^a *is an anti-representation of* $L^1(H,B,\eta;\beta)$ *on* X*, are obtained in this fashion from unique (necessarily commuting) pairs* (π^m, U^m) *and* (π^a, U^a) *with the above properties.*

For reasons of space, we do not repeat the formulas in Theorem 4.5.20 and Theorem 4.8.3 retrieving (π^m, U^m) from T^m and (π^a, U^a) from T^a, or the upper bounds therein.

Remark 4.8.5. The results of Section 4.6 make it possible to establish a bijection between the commuting pairs (π^m, U^m) and (π^a, U^a) as in Theorem 4.8.4 and the non-degenerate bounded representations of one single algebra (rather than two). To see this, note that, though $L^1(G,A,\omega;\alpha)$ and $L^1(H^o,B^o,\eta^o;\beta^o)$ are not Banach algebras in general, the continuity of the multiplication still implies that $L^1(G,A,\omega;\alpha)\hat{\otimes}L^1(H^o,B^o,\eta^o;\beta^o)$ can be supplied with the structure of an associative algebra such that multiplication is continuous. If $L^1(G,A,\omega;\alpha) \simeq C_1$ and $L^1(H^o,B^o,\eta^o;\beta^o) \simeq C_2$ as topological algebras, where C_1 and C_2 are crossed products of the relevant Banach algebra dynamical systems as in Section 4.5, then clearly

$$L^1(G,A,\omega;\alpha)\hat{\otimes}L^1(H,B,\eta;\beta)^o \simeq L^1(G,A,\omega;\alpha)\hat{\otimes}L^1(H^o,B^o,\eta^o;\beta^o) \simeq C_1\hat{\otimes}C_2$$

where Proposition 4.8.3 was used in the first step. From Theorem 4.7.5 we know what the non-degenerate bounded representations of $C_1\hat{\otimes}C_2$ are. Hence, combining all information, we see that the commuting pairs (π^m, U^m) and (π^a, U^a) as in Theorem 4.8.4 are in bijection with the non-degenerate bounded representations of $L^1(G,A,\omega;\alpha)\hat{\otimes}L^1(H,B,\eta;\beta)^o$, by letting (π^m, U^m) and (π^a, U^a) correspond to the non-degenerate bounded representation $T^m \odot T^a$, where T^m and T^a are as in Theorem 4.8.4 (the latter now viewed as a non-degenerate bounded representation of $L^1(H,B,\eta;\beta)^o$). Our notation is slightly imprecise here, since $L^1(G,A,\omega;\alpha)$ and $L^1(H,B,\eta;\beta)^o$ are not Banach algebras in general, but it is easily seen that Lemma 4.7.4 is equally valid when the norm need not be submultiplicative, but multiplication is still continuous.

Finally, we note that the special case where $(A,G,\alpha) = (B,H,\beta) = (\mathbb{K}, G, \mathrm{triv})$ in Theorem 4.8.4 states that the non-degenerate bimodules over $L^1(G,\omega)$ correspond naturally to the G-bimodules determined by a pair (U^m, U^a) of commuting maps U^m and U^a, where $U^m : G \to B(X)$ is a strongly continuous representation, U^a :

$G \to B(X)$ is a strongly continuous anti-representation, and $\|U_r^m\| \leq C_{U^m}\omega(r)$ and $\|U_r^a\| \leq C_{U^a}\omega(r)$ for some constants C_{U^m} and C_{U^a} and all $r \in G$. Specializing further by taking $\omega = 1$, we see that the non-degenerate bimodules over $L^1(G)$ correspond naturally to the G-bimodules determined by a commuting pair (U^m, U^a) as above, with now each of U^m and U^a uniformly bounded. This is a classical result, cf. [25, Proposition 2.1].